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Title THE CONVERGENCE OF THE DISCRETE ORDINATES METHOD FOR INTEGRAL EQUATIONS OF ANISOTROPIC RADIATIVE TRANSFER

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Let $I(\tau, \mu)$, $J(\tau, \mu)$ and $\Gamma(\tau, \mu)$ denote intensity, average intensity and source function for radiation in a plane-parallel, anisotropically scattering medium with optical thickness $\tau_0 \leq \infty$ and albedo $\omega_0 \leq 1$. Let $I_m(\tau, \mu)$, $J_m(\tau, \mu)$ and $\Gamma_m(\tau, \mu)$, $m \geq 1$, denote discrete ordinates approximations to $I(\tau, \mu)$, $J(\tau, \mu)$ and $\Gamma(\tau, \mu)$ respectively. The transfer problem and its associated approximate problem are studied in integral form using Neumann series techniques and positive operator theory. It is proved that, under suitable auxiliary conditions, $I_m(\tau, \mu) \rightarrow I(\tau, \mu)$, $J_m(\tau, \mu) \rightarrow J(\tau, \mu)$ and $\Gamma_m(\tau, \mu) \rightarrow \Gamma(\tau, \mu)$. The convergence is uniform if $\tau_0 < \infty$ and $\omega_0 \leq 1$ or if $\tau_0 = \infty$ and $\omega_0 < 1$. Error bounds are obtained when $\tau_0 < \infty$ and $\omega_0 \leq 1$. The convergence is uniform on each finite τ interval if $\tau_0 = \infty$ and $\omega_0 = 1$.

THE CONVERGENCE OF THE DISCRETE ORDINATES METHOD FOR INTEGRAL EQUATIONS OF ANISOTROPIC RADIATIVE TRANSFER

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CHAPTER 1

INTRODUCTION

§1.1 Preliminary Remarks

Historically, transfer theory was concerned with the transfer of radiation through the atmosphere of a star. In connection with this, many interesting mathematical problems have arisen. Hopf (1934) was one of the first to consider transfer theory from a serious mathematical point of view. He was primarily concerned with problems involving conservative isotropic scattering.

It was later discovered that the same laws governing stellar transfer theory could be applied, with some modifications, to the scattering of neutrons and other particles. In these applications (Davison, 1957) the term transport theory is used. For an excellent review of the literature of transfer theory, we refer the reader to a recent paper by Mullikin (1964a). An excellent bibliography is contained in Wing (1962). For more precise definitions of physical terms and derivations of equations used below, we refer the reader to the standard works of Chandrasekhar (1950), Kourganoff (1952), or Busbridge (1960).

Most of the completely solved problems in transfer theory either are concerned with media which are stratified in plane-parallel layers or can be reduced to such cases. In practice, radiation fields are usually assumed to be stationary, i. e., time independent. Time dependent problems present mathematical difficulties requiring sophisticated techniques and have only recently been studied in any detail; see for example Case (1960) or Lehner and Wing (1956).

Problems in transfer theory are usually posed in terms of the so-called transfer equation, which is an integro-differential equation. By certain standard techniques, these problems are often reduced to the discussion of pure integral equations.

We shall be concerned with transfer problems which pertain to the distribution of radiation in a plane-parallel atmosphere under the assumptions of time and frequency independence. The energy flow at a point in a radiation field is specified by the intensity. This is a function of the geometrical depth of the point below the upper surface of the medium and of the direction of the flow considered. A medium is called isotropic if the probability density of the direction of a scattered particle does not depend on its original direction. In this dissertation, we shall assume that the medium is anisotropic, i. e., not necessarily isotropic. Anisotropic transfer problems seldom have been studied in detail, although in physical reality, scattering is usually anisotropic. Busbridge (1960) and Chandrasekhar (1950) have contributed to this part of the theory and, more recently, Mullikin (1961, 1964a,b) has investigated some aspects of anisotropic scattering.

In the so-called discrete-ordinates method of Wick and Chandrasekhar, numerical integration is used to replace the transfer equation by a system of ordinary differential equations. The solution of such a system provides an approximate solution of the transfer problem. Under fairly general conditions on the quadrature formula, Anselone (1957, 1958, 1961) proved for various isotropic problems that the approximate solutions converge uniformly to the desired intensity as the number of subdivision points increases. It is our main objective to generalize these results to the anisotropic case.

§1.2 The Transfer Problem

Consider a medium which is either a half-space or is bounded by two planes. The radiation field is assumed to be in equilibrium. Let τ denote the optical distance from the outer surface of the medium measured in the direction normal to the surface. Let μ be the cosine of the angle θ which a given direction makes with the outward normal to the surface $\tau = 0$, and let ϕ denote the azimuth of this direction referred to a fixed plane normal to the τ-axis. At a depth τ below the surface, the intensity of radiation is a function of τ, μ, and ϕ, and is customarily denoted by $I(\tau, \mu, \phi)$. We shall henceforth assume axial symmetry and denote the intensity by $I(\tau, \mu)$. The non-axially symmetric case can be treated similarly, but with additional notational complexity.

The interaction of radiation with the medium results in the absorption as well as the scattering of some of the energy in other directions. We shall assume that the radiation fields do not interact with each other and that the frequency of the radiation is unchanged upon scattering. Radiation absorbed may be either lost to the radiation field or reemitted as scattered radiation.

If a photon traveling with direction (μ', ϕ') is absorbed, the probability density (per unit solid angle) that it will be scattered with direction (μ, ϕ) is the phase function $\mathcal{P}(\mu, \phi; \mu', \phi')$, which depends on the cosine of the angle Θ between the two directions, i.e., upon

$$\cos \Theta = \mu\mu' + (1-\mu^2)^{\frac{1}{2}} (1-\mu'^2)^{\frac{1}{2}} \cos(\phi-\phi').$$

In the case of axial symmetry, the quantity

$$p(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(\mu, \phi; \mu', \phi') d\phi',$$

also called a phase function, plays an important role. We observe that

(1.2.1) $$p(\mu,\mu') = p(\mu',\mu), \quad p(-\mu,-\mu') = p(\mu,\mu') .$$

We assume, in addition, that $p(\mu,\mu')$ is continuous and positive. In the isotropic case, both $p(\mu,\mu')$ and $\mathcal{P}(\mu,\phi;\mu',\phi')$ are constant.

The probability that a photon which is absorbed will be scattered rather than lost to the radiation field is the albedo

(1.2.2) $$\omega_0 = \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')d\mu' .$$

Note that $0 < \omega_0 \leq 1$. In the so-called conservative case of perfect scattering, $\omega_0 = 1$. A related quantity, ω_1, is defined by

(1.2.3) $$\frac{1}{3}\omega_1\mu = \frac{1}{2}\int_{-1}^{1} \mu' p(\mu,\mu')d\mu' .$$

The significance of ω_1 will be indicated later. In the isotropic case $\omega_1 = 0$. In general, it follows readily from (1.2.2) and (1.2.3) that $\frac{1}{3}\omega_1 < \omega_0 \leq 1$.

Let $I(\tau,\mu)$, $J(\tau,\mu)$, and $\Gamma(\tau,\mu)$ denote, respectively, the intensity, average intensity, and source function of diffuse radiation in the medium except reduced incident radiation if any is present. Let $\Gamma^0(\tau,\mu)$ be the sum of the components of $\Gamma(\tau,\mu)$ due to scattering of reduced incident radiation and to emission in the medium. It will be assumed that $\Gamma^0(\tau,\mu)$ is given. These quantities are related by

$$J(\tau,\mu) = \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' ,$$

(1.2.4)

$$\Gamma(\tau,\mu) = J(\tau,\mu) + \Gamma^0(\tau,\mu) .$$

According to their physical definitions, $I(\tau,\mu)$ and $\Gamma^0(\tau,\mu)$ are non-negative; furthermore, by (1.2.4), $\Gamma(\tau,\mu)$ is also non-negative. We assume all of these functions to be continuous in the interior of the medium.

Denote the total optical thickness of the medium by τ_0 $(0 < \tau_0 \leq \infty)$. If $\tau_0 < \infty$, then $I(\tau,\mu)$ will usually be known when $\tau = 0$ and $\tau = \tau_0$. If $\tau_0 = \infty$, there will be a condition on the asymptotic behavior of $I(\tau,\mu)$, or of $\Gamma(\tau,\mu)$, as $\tau \to \infty$. In addition, the net flux through the bounding surfaces may be given.

We are now ready to make a complete mathematical statement of the transfer problem. We seek a function $I(\tau,\mu)$, defined for $0 \leq \tau \leq \tau_0$, $\tau < \infty$, $-1 \leq \mu \leq 1$, such that

$$I(\tau,\mu) \geq 0, \quad I(\tau,\mu) \not\equiv 0 ,$$

(1.2.5) $$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' - \Gamma^0(\tau,\mu) ,$$

with the boundary conditions

$$(1.2.6) \qquad \begin{cases} I(0,\mu) = 0, & \mu < 0, \\ I(\tau_0,\mu) = 0, & \mu > 0, \text{ if } \tau_0 < \infty, \end{cases}$$

$$(1.2.7) \qquad \lim_{\tau \to \infty} e^{-\frac{\tau}{\mu}} I(\tau,\mu) = 0, \quad \mu > 0, \text{ if } \tau_0 = \infty .$$

For the physical significance of (1.2.7) see Hopf's (1934, Chapter 1) monograph.

§ 1.3 An Equivalent Formulation of the Transfer Problem

In view of (1.2.4), the transfer equation (1.2.5) can also be written in the form

$$(1.3.1) \qquad \mu \frac{\partial I(\tau,\mu)}{\partial \tau} = I(\tau,\mu) - \Gamma(\tau,\mu) .$$

If μ is fixed, $\mu \neq 0$, then (1.3.1) is a first order linear ordinary differential equation for $I(\tau,\mu)$ in terms of $\Gamma(\tau,\mu)$. By elementary methods, the solution of (1.3.1), subject to the boundary conditions, can be written

$$(1.3.2) \qquad I(\tau,0) = \Gamma(\tau,0) ,$$

$$
(1.3.3) \qquad I(\tau,\mu) = \begin{cases} \displaystyle\int_0^{\tau} \Gamma(\tau',\mu)\, e^{\frac{\tau-\tau'}{\mu}} \frac{d\tau'}{-\mu}, & \mu < 0, \\[2ex] \displaystyle\int_{\tau}^{\tau_0} \Gamma(\tau',\mu)\, e^{\frac{\tau-\tau'}{\mu}} \frac{d\tau'}{\mu}, & \mu > 0. \end{cases}
$$

The substitution of (1.3.3) into (1.2.4) yields

$$
(1.3.4) \qquad \Gamma(\tau,\mu) = \frac{1}{2}\int_0^1 \int_{\tau}^{\tau_0} p(\mu,\mu') \frac{e^{\frac{\tau-\tau'}{\mu'}}}{\mu'} \Gamma(\tau',\mu')\, d\tau'\, d\mu'
$$

$$
+ \frac{1}{2}\int_{-1}^0 \int_0^{\tau} p(\mu,\mu') \frac{e^{\frac{\tau-\tau'}{\mu'}}}{-\mu'} \Gamma(\tau',\mu')\, d\tau'\, d\mu' + \Gamma^0(\tau,\mu).
$$

The above is an integral equation for $\Gamma(\tau,\mu)$. If it can be solved, then (1.3.3) gives $I(\tau,\mu)$. Hence, we have an equivalent formulation for the transfer problem. Note that $\tau_0 = \infty$ in (1.3.4) if the medium is semi-infinite. In this case (1.2.7) implies the existence of the improper integral.

§ 1.4 The Net Flux

The rate of energy flow in the normal direction per unit area of the plane at depth τ is

$$\pi F(\tau) = 2\pi \int_{-1}^{1} \mu I(\tau, \mu) d\mu \, ; \tag{1.4.1}$$

$F(\tau)$ is called the net flux at depth τ. Note that $F(\tau)$ is twice the first moment of $I(\tau, \mu)$ with respect to μ. The substitution of (1.3.3) into (1.4.1) yields $F(\tau)$ in terms of $\Gamma(\tau, \mu)$:

$$F(\tau) = 2\int_{0}^{1} \int_{\tau}^{\tau_0} e^{\frac{\tau-\tau'}{\mu}} \Gamma(\tau', \mu) d\tau' d\mu \tag{1.4.2}$$
$$-2\int_{-1}^{0} \int_{0}^{\tau} e^{\frac{\tau-\tau'}{\mu}} \Gamma(\tau', \mu) d\tau' d\mu \quad .$$

Another useful relation involving $F(\tau)$ is obtained as follows. Define $\bar{I}(\tau)$ and $\bar{\Gamma}^0(\tau)$, respectively, by

$$\bar{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau, \mu) d\mu \, , \quad \bar{\Gamma}^0(\tau) = \frac{1}{2}\int_{-1}^{1} \Gamma^0(\tau, \mu) d\mu \quad . \tag{1.4.3}$$

Integration of (1.3.1) with respect to μ gives

$$\frac{d}{d\tau} \int_{-1}^{1} \mu I(\tau, \mu) d\mu = \int_{-1}^{1} I(\tau, \mu) d\mu - \int_{-1}^{1} \Gamma(\tau, \mu) d\mu \, ,$$

or

$$\frac{d}{d\tau}F(\tau) = 2\bar{I}(\tau) - \frac{1}{2}\int_{-1}^{1}\left[\frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' - \Gamma^0(\tau,\mu)\right]d\mu$$

$$= 2\bar{I}(\tau) - 2\omega_0\bar{I}(\tau) - 2\bar{\Gamma}^0(\tau) = 2\bar{I}(\tau)(1-\omega_0) - 2\,\bar{\Gamma}^0(\tau)\ .$$

If $\omega_0 = 1$ and $\bar{\Gamma}^0(\tau) = 0$, then $\frac{d}{d\tau}F(\tau) = 0$, so $F(\tau) \equiv F$, a constant. By (1.4.1) with $\tau = 0$,

(1.4.4) $$F = 2\int_{-1}^{1} \mu I(0,\mu)d\mu\ .$$

By (1.4.2) with $\tau = 0$,

(1.4.5) $$F = 2\int_0^1\int_0^{\tau_0} e^{-\frac{\tau'}{\mu}}\,\Gamma(\tau',\mu)d\tau'\,d\mu\ .$$

If $\omega_0 = 1$, $\tau_0 = \infty$, and $\Gamma^0(\tau,\mu) = 0$, it turns out that the corresponding (homogeneous) transfer problem has a non-trivial solution (cf. Chapter 5). The general solution is an arbitrary multiple of this function. It is customary to make the solution unique by specifying the net flux. Thus, in the integral equation formulation (1.3.4) of this homogeneous transfer problem, the equation

(1.4.6) $$F = 2\int_0^1\int_0^{\infty} e^{-\frac{\tau'}{\mu}}\,\Gamma(\tau',\mu)d\tau'\,d\mu$$

provides an appropriate normalization condition.

The integral equation (1.4.2) can also be obtained by integrating (1.2.4) with respect to τ and μ.

§ 1.5 The K-integral

The second moment of $I(\tau,\mu)$ with respect to μ also plays an important role in our analysis. Let

(1.5.1)
$$K(\tau) = \frac{1}{2}\int_{-1}^{1} \mu^2 I(\tau,\mu)d\mu .$$

Returning to the transfer equation (1.3.1), we multiply both sides by μ and integrate over $[-1,1]$. Using (1.2.3), this gives

$$2\frac{d}{d\tau}K(\tau) = \int_{-1}^{1} \mu I(\tau,\mu)d\mu - \int_{-1}^{1} \mu \Gamma(\tau,\mu)d\mu$$

$$= \frac{1}{2}F(\tau) - \int_{-1}^{1} \mu \left[\frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu'\right] d\mu$$

$$= \frac{1}{2}F(\tau) - \frac{1}{3}\int_{-1}^{1} \omega_1 \mu' I(\tau,\mu')d\mu'$$

$$= \frac{1}{2}F(\tau) - \frac{\omega_1}{3}\frac{1}{2}F(\tau) = \frac{1}{2}(1-\frac{\omega_1}{3})F(\tau) .$$

Therefore

(1.5.2)
$$\frac{d}{d\tau}K(\tau) = \frac{1}{4}(1-\frac{\omega_1}{3})F(\tau) .$$

If $\omega_0 = 1$ and $\overline{\Gamma}^0(\tau) = 0$, then $F(\tau)$ is constant and

$$K(\tau) = \frac{1}{4}\left(1-\frac{\omega_1}{3}\right)F\tau + K(0), \tag{1.5.3}$$

where $1 - \frac{\omega_1}{3} > 0$. This is known as the K-integral in the conservative case. If we substitute for $I(\tau, \mu)$ in (1.5.1), we obtain the following integral equation for $\Gamma(\tau, \mu)$:

$$K(\tau) = \frac{1}{2}\int_0^1 \int_\tau^{\tau_0} \mu\, e^{\frac{\tau-\tau'}{\mu}}\, \Gamma(\tau', \mu)\,d\tau'\,d\mu \tag{1.5.4}$$

$$- \frac{1}{2}\int_{-1}^0 \int_0^\tau \mu\, e^{\frac{\tau-\tau'}{\mu}}\, \Gamma(\tau', \mu)\,d\tau'\,d\mu \ .$$

In particular,

$$K(0) = \frac{1}{2}\int_0^1 \int_0^{\tau_0} \mu\, e^{-\frac{\tau'}{\mu}}\, \Gamma(\tau', \mu)\,d\tau' \ . \tag{1.5.5}$$

The K-integral will appear later in the discussion of the solution of the homogeneous integral equation.

The equations (1.5.4) and (1.5.5) can also be obtained from (1.3.4) by integration.

CHAPTER 2

THE WICK-CHANDRASEKHAR TECHNIQUE

§ 2.1 Preliminary Remarks

A method of successive approximations due to Wick (1943) and considerably exploited by Chandrasekhar (1950) is based on the Gauss quadrature formula. For each $m = 1, 2, \cdots,$ the Gauss formula of order $2m$ is expressed by a correspondence of the form

$$\sum_j a_{mj} f(\mu_{mj}) \sim \int_{-1}^{1} f(\mu) d\mu , \tag{2.1.1}$$

where (here and henceforth in this chapter) $j = \pm 1, \pm 2, \cdots, \pm m$. The μ_{mj} are the zeros of the Legendre polynomials $P_{2m}(\mu)$. Formulas for the coefficients are given in Szegö (1939). The coefficients a_{mj} and subdivision points μ_{mj} satisfy

$$a_{m,-j} = a_{mj} > 0 , \tag{2.1.2}$$

$$\mu_{m,-j} = -\mu_{mj}, \quad 0 < \mu_{m1} < \cdots < \mu_{mm} \leq 1 . \tag{2.1.3}$$

The correspondence (2.1.1) is an equality for any polynomial of order at most $4m-1$.

If the integrals in the transfer problem are replaced by the corresponding sums then this replacement yields a system of $2m$ ordinary differential equations for approximations $I_m(\tau, \mu)$ to $I(\tau, \mu)$. Anselone (1957, 1958, 1961) has proven the convergence of $I_m(\tau, \mu)$ to $I(\tau, \mu)$ for various isotropic transfer problems. As was previously mentioned, it is our purpose to extend his work to the anisotropic case. This chapter is devoted to the derivation of the approximate solutions. For some further details see Chandrasekhar (1950). We feel that the derivation, as given below, is more efficient in several respects. Convergence questions are treated in later chapters.

A quadrature formula more general than the Gauss can be used; the analysis is similar, but notationally more complex. We will point out below where the Gauss formula simplifies the analysis.

§2.2 The Problem for $I_m(\tau, \mu)$

We now write out the equations which define the approximate problem. Replace the integrals which occur in the transfer problem by their corresponding sums. We seek $I_m(\tau, \mu)$ such that

$$I_m(\tau, \mu) \geq 0, \qquad I_m(\tau, \mu) \not\equiv 0,$$

(2.2.1) $$\mu \frac{\partial I_m(\tau, \mu)}{\partial \tau} = I_m(\tau, \mu) - \frac{1}{2} \sum_j a_{mj} p(\mu, \mu_{mj}) I_m(\tau, \mu_{mj}) - \Gamma^0(\tau, \mu),$$

$$(2.2.2)\qquad \begin{cases} I_m(0,\mu) = 0, & \mu < 0, \\ I_m(\tau_0,\mu) = 0, & \mu > 0, \text{ if } \tau_0 < \infty, \\ \lim\limits_{\tau\to\infty} e^{-\frac{\tau}{\mu}} I_m(\tau,\mu) = 0, & \mu > 0, \text{ if } \tau_0 = \infty. \end{cases}$$

By analogy with (1.4.1), we define

$$(2.2.3)\qquad F_m(\tau) = 2\sum_j a_{mj}\mu_{mj} I_m(\tau,\mu_{mj}).$$

Equations analogous to (1.4.4)-(1.4.6) are easily derived. The right member of (2.2.3) is constant if $\omega_0 = 1$ and $\Gamma^0(\tau,\mu) \equiv 0$. The proof is very similar to that in § 1.4. This provides a normalization condition when $\omega_0 = 1$, $\tau_0 = \infty$ and $\Gamma^0(\tau,\mu) \equiv 0$. In this case, we specify $F_m = F$ as the same arbitrary positive constant in (1.4.4) and (2.2.3).

By analogy with the exact problem, we define $J_m(\tau,\mu)$ and $\Gamma_m(\tau,\mu)$ in terms of $I_m(\tau,\mu)$ by

$$(2.2.4)\qquad \begin{cases} J_m(\tau,\mu) = \frac{1}{2}\sum\limits_j a_{mj} p(\mu,\mu_{mj}) I_m(\tau,\mu_{mj}), \\ \Gamma_m(\tau,\mu) = J_m(\tau,\mu) + \Gamma^0(\tau,\mu). \end{cases}$$

For convenience, we suppose that the phase function $p(\mu, \mu')$ can be represented by a finite Legendre sum

$$(2.2.5) \qquad p(\mu, \mu') = \sum_{\ell=0}^{N} \omega_\ell P_\ell(\mu) P_\ell(\mu'),$$

where $2N < 4m-1$. This will simplify the analysis below. If $p(\mu, \mu')$ is sufficiently smooth and m is large enough, then $p(\mu, \mu')$ can be uniformly approximated by such a Legendre sum to within a small error. The effect of replacing $p(\mu, \mu')$ by a good uniform approximation introduces only a small error in $I_m(\tau, \mu)$. We will say more about this in later chapters.

We now let $\mu = \mu_{mi}$ and substitute (2.2.5) into (2.2.1) to obtain

$$(2.2.6) \qquad \mu_{mi} \frac{\partial I_m(\tau, \mu_{mi})}{\partial \tau} = I_m(\tau, \mu_{mi}) - \frac{1}{2} \sum_j \sum_{\ell=0}^{N} a_{mj} \omega_\ell I_m(\tau, \mu_{mj}) P_\ell(\mu_{mi}) P_\ell(\mu_{mj}) - \Gamma^0(\tau, \mu_{mi}),$$

for $i = \pm 1, \pm 2, \cdots, \pm m$. This is a system of $2m$ linear ordinary differential equations with constant coefficients. Following the usual procedure, we shall seek a solution of the associated homogeneous system (i.e., with $\Gamma^0(\tau, \mu_{mi}) \equiv 0$) and then add to it a particular integral of the inhomogeneous system. This particular integral can be found by the method of variation of parameters.

Once $I_m(\tau, \mu_{mi})$ is obtained, then $I_m(\tau, \mu)$ for all μ can be derived from (2.2.1). This is done in § 2.5.

§ 2.3 Solution of the Homogeneous Differential Equations

Consider the homogeneous system associated with (2.2.6). Following standard procedure, we seek solutions of the form

$$I_m(\tau, \mu_{mi}) = C_{mi} e^{-k\tau} \qquad i = \pm 1, \pm 2, \cdots, \pm m, \tag{2.3.1}$$

where k will depend on m. Substitute (2.3.1) into (2.2.6) with $\Gamma^0(\tau, \mu) = 0$ to obtain

$$-\mu_{mi} k C_{mi} = C_{mi} - \frac{1}{2} \sum_j \sum_{\ell=0}^{N} a_{mj} C_{mj} \omega_\ell P_\ell(\mu_{mi}) P_\ell(\mu_{mj}) . \tag{2.3.2}$$

Solving for C_{mi}, we obtain

$$C_{mi} = \frac{\sum_{\ell=0}^{N} \omega_\ell P_\ell(\mu_{mi}) \xi_{m\ell}}{1 + \mu_{mi} k} , \tag{2.3.3}$$

where

(2.3.4) $$\xi_{m\ell} = \frac{1}{2}\sum_j a_{mj} C_{mj} P_\ell(\mu_{mj}) \qquad (\ell = 0, 1, 2, \cdots, N).$$

Substituting (2.3.3) into (2.3.4), we have

(2.3.5) $$\xi_{m\ell} = \sum_{\lambda=0}^{N} D_{\ell\lambda}\,\omega_\lambda\,\xi_{m\lambda},$$

where

(2.3.6) $$D_{\ell\lambda} = \frac{1}{2}\sum_j a_{mj} \frac{P_\ell(\mu_{mj}) P_\lambda(\mu_{mj})}{1+\mu_{mj}k}.$$

Although obscured by the notation, the foregoing amounts to a simple change of variable in a system of linear equations.

We now examine some properties of $D_{\ell\lambda}$. An equivalent form of $D_{\ell\lambda}$ is

(2.3.7) $$D_{\ell\lambda} = \frac{1}{2}\sum_j a_{mj} P_\ell(\mu_{mj}) P_\lambda(\mu_{mj}) \left[1 - \frac{k\mu_{mj}}{1+\mu_{mj}k} \right].$$

From $0 \le \ell \le N$, $0 \le \lambda \le N$, and (2.2.5)ff, we have $\lambda + \ell \le 2N < 4m - 1$. It follows that

$$(2.3.8)\quad \frac{1}{2}\sum_j a_{mj} P_\ell(\mu_{mj}) P_\lambda(\mu_{mj}) = \frac{1}{2}\int_{-1}^{1} P_\ell(\mu) P_\lambda(\mu)\, d\mu = \frac{\delta_{\ell\lambda}}{2\ell+1} ,$$

where $\delta_{\ell\lambda}$ is the Kronecker delta. If a more general quadrature were used and the phase function were arbitrary, then the left member of (2.3.8) would be a constant which we could calculate, but not necessarily in such a convenient form.

We now have

$$D_{\ell\lambda} = \frac{\delta_{\ell\lambda}}{2\ell+1} - \frac{1}{2} k \sum_j \frac{a_{mj} P_\lambda(\mu_{mj})}{1+\mu_{mj} k} \mu_{mj} P_\ell(\mu_{mj})$$

$$= \frac{\delta_{\ell\lambda}}{2\ell+1} - \frac{k}{2(2\ell+1)} \sum_j a_{mj} P_\lambda(\mu_{mj}) \frac{[(\ell+1) P_{\ell+1}(\mu_{mj}) + \ell P_{\ell-1}(\mu_{mj})]}{1+\mu_{mj} K}$$

$$= \frac{\delta_{\ell\lambda}}{2\ell+1} - \frac{k}{2\ell+1} [(\ell+1) D_{\ell+1,\lambda} + \ell D_{\ell-1,\lambda}] .$$

We write this as

$$(2.3.9)\qquad (2\ell+1) D_{\ell\lambda} = \delta_{\ell\lambda} - k[(\ell+1) D_{\ell+1,\lambda} + \ell D_{\ell-1,\lambda}] .$$

By (2.3.5) and (2.3.9),

$$(2.3.10)\qquad (2\ell+1)\xi_{m\ell} = \omega_\ell \xi_{m\ell} - k[(\ell+1)\xi_{m,\ell+1} + \ell \xi_{m,\ell-1}]$$

or

$$(2.3.11) \quad \xi_{m,\ell+1} = -\frac{2\ell+1-\omega_\ell}{k(\ell+1)}\xi_{m,\ell} - \frac{\ell}{\ell+1}\xi_{m,\ell-1} \qquad (0 \leq \ell < N),$$

where we define $\xi_{m,-1} = 0$. This equation enables us to determine all of the constants $\xi_{m\ell}$ in terms of ξ_{m0}. Without loss of generality, assume $\xi_{m0}=1$. Then, in particular,

$$\xi_{m1} = -\frac{1-\omega_0}{k}, \qquad \xi_{m2} = \frac{(1-\omega_0)(3-\omega_1)}{2k^2} - \frac{1}{2} .$$

Note that the $\xi_{m\ell}$, $\ell > 0$, are all functions of k. With this in mind, we write $\xi_{m\ell}(k)$, $\ell = 1, 2, \cdots, N$. By (2.3.11) $k^\ell \xi_{m\ell}(k)$ is a polynomial in k of order $\leq \ell$.

Let $\ell = 0$ in (2.3.5) and (2.3.6) to obtain the so-called characteristic equation for k:

$$(2.3.12) \quad 1 = \sum_{\lambda=0}^{N} D_{0\lambda}\,\omega_\lambda\,\xi_{m\lambda} = \frac{1}{2}\sum_j \frac{a_{mj}}{1+\mu_{mj}k} \sum_{\lambda=0}^{N} \omega_\lambda \xi_{m\lambda} P_\lambda(\mu_{mj}) .$$

For each root k of the above equation, we have a solution of the original homogeneous differential equation. Equation (2.3.12) arises since we have a system of linear homogeneous equations and the determinant of the system must vanish in order to have a non-trivial solution.

By (2.1.2) and (2.1.3) and the fact that the $\xi_{m\lambda}$'s are functions of k, it can be seen that clearing of fractions in (2.4.12) results in a polynomial equation of order m in k^2. Hence, it has $2m$ roots, and these roots occur in pairs, say $\pm k_{m\alpha}$, $\alpha = 1, 2, \cdots, m$.

If $\omega_0 < 1$, then these roots are all distinct and we obtain $2m$ linearly independent solutions of the homogeneous system. Thus, the general solution of the homogeneous system associated with (2.2.6) is

$$(2.3.13)\quad {}^{h}I_m(\tau, \mu_{mi}) = \sum_{\alpha=\pm 1}^{\pm m} M_{m\alpha} \frac{\sum_{\ell=0}^{N} \omega_0 \xi_{m\ell}(\pm k_{m\alpha}) P_\ell(\mu_{mi})}{1 \pm \mu_{mi} k_{m\alpha}} e^{\mp k_{m\alpha}\tau},$$

where $i = \pm 1, \pm 2, \cdots, \pm m$. The $M_{m\alpha}$ are $2m$ arbitrary constants and the superscript h on the left indicates the homogeneous problem.

The case with $\omega_0 = 1$ is exceptional since $k = 0$ is then a double root of the characteristic equation. We have, (cf. (2.3.10)), for example

$$\xi_{m0} = 1, \qquad \xi_{m1} = 0, \qquad \xi_{m2} = -\frac{1}{2}, \qquad \xi_{m3} = \frac{5-\omega_2}{6k}$$

and

$$\frac{1}{2}\sum_{j}\sum_{\lambda=0}^{N} a_{mj}\omega_\lambda \xi_{m\lambda} P_\lambda (\mu_{mi}) = \frac{1}{2}\sum_{j} a_{mj}\omega_0 = 1 ;$$

so there will be only $(2m-2)$ distinct non-vanishing roots $\pm k_{m\alpha}$, $\alpha = 1, 2, \cdots, m-1$. The root $k = 0$ corresponds to a constant solution of the homogeneous differential equation. In this case, we also seek a solution which is linear in τ. It is possible to express this solution in the convenient form (Chandrasekhar, 1950, p. 14)

$$I_m^h(\tau, \mu_{mi}) = b_m\left(\tau + \frac{\mu_{mi}}{1-\frac{\omega_1}{3}} + Q_m\right), \tag{2.3.14}$$

where b_m and Q_m are arbitrary constants. A particularly neat verification is as follows (recall $\omega_0 = 1$):

$$\mu_{mi} = \tau + \frac{\mu_{mi}}{1-\frac{\omega_1}{3}} + Q_m - \tau\omega_0 - Q_m\omega_0 - \frac{\frac{\omega_1\mu_{mi}}{3}}{1-\frac{\omega_1}{3}}$$

$$= \frac{1}{b_m} I_m(\tau,\mu_{mi}) - (\tau + Q_m)\ \frac{1}{2} \sum_j \sum_{\ell=0}^{N} a_{mj}\omega_\ell P_\ell(\mu_{mi})P_\ell(\mu_{mj})$$

$$- \frac{1}{1-\frac{\omega_1}{3}}\ \frac{1}{2} \sum_j \sum_{\ell=0}^{N} a_{mj}\mu_{mj}\omega_\ell P_\ell(\mu_{mi})P_\ell(\mu_{mj})$$

$$= \frac{1}{b_m} I^h_m(\tau,\mu_{mi}) - \frac{1}{2b_m} \sum_j \sum_{\ell=0}^{N} a_{mj} I^h_m(\tau,\mu_{mi})\omega_\ell P_\ell(\mu_{mi})P_\ell(\mu_{mj}),$$

whence it follows that (2. 3. 14) satisfies the homogeneous differential equation associated with (2. 2. 1).

Hence, when $\omega_0 = 1$, the general solution of the homogeneous differential equation is given by

$$I^h_m(\tau,\mu_{mi}) = b_m\left[\tau + \frac{\mu_{mi}}{1-\frac{\omega_1}{3}} + Q_m + \sum_{\alpha=\pm 1}^{\pm(m-1)} M_{m\alpha} \frac{\sum_{\ell=0}^{N} \omega_\ell \xi_{m\ell}(k_{m\alpha})P_\ell(\mu_{mi})}{1 \pm \mu_{mi}k_{m\alpha}}\ e^{\mp k_{m\alpha}\tau}\right],$$

where $i = \pm 1, \pm 2, \cdots, \pm m$ (if $m = 1$, there is no summation term present). This completes the solution of the homogeneous system.

§ 2. 4 The Inhomogeneous Problem

In the foregoing section, we obtained the general solution of the homogeneous system without the boundary conditions. As mentioned previously, the method of variation of parameters yields a particular solution of the inhomogeneous problem. Thus the general solution of the inhomogeneous problem has the form

$$I_m(\tau, \mu_{mi}) = I_m^h(\tau, \mu_{mi}) + I_m^p(\tau, \mu_{mi}), \tag{2.4.1}$$

where $I_m^h(\tau, \mu_{mi})$ is given by (2. 3. 13) or (2. 3. 15), and $I_m^p(\tau, \mu_{mi})$ is a particular solution.

The constants appearing in $I_m^h(\tau, \mu_{mi})$ can be determined from the boundary and auxiliary conditions. They, and hence, $I_m(\tau, \mu_{mi})$ are determined uniquely except when $\omega_0=1$ and $\tau_0=\infty$. The justification of this assertion will become apparent in the following chapters.

In particular, when $\tau_0=\infty$, then the asymptotic condition in (2. 2. 2) implies that $M_{m\alpha} = 0$ for $\alpha < 0$. When $\omega_0=1$ and $\Gamma^0(\tau, \mu) = 0$, the normalization condition yields (cf, Chandrasekhar, 1950, p. 14)

(2.4.2) $$b_m = \frac{3}{4}\left(1 - \frac{\omega_1}{3}\right) F,$$

where F is defined as in (2.2.3)ff.

Further determination of the constants is for the most part a computational problem and this is not our primary concern.

§ 2.5 An Equivalent Formulation of the Problem for $I_m(\tau, \mu)$

We conclude this chapter with a reformulation of the approximate transfer problem in integral equation form. From (2.2.1) and (2.2.4), we have

(2.5.1) $$\mu \frac{\partial I_m(\tau, \mu)}{\partial \tau} = I_m(\tau, \mu) - \Gamma_m(\tau, \mu) .$$

Formally solving this problem subject to the boundary conditions gives

(2.5.2) $$I_m(\tau, 0) = \Gamma_m(\tau, 0),$$

(2.5.3) $$I_m(\tau, \mu) = \begin{cases} \displaystyle\int_0^{\tau} \Gamma_m(\tau', \mu) e^{\frac{\tau - \tau'}{\mu}} \frac{d\tau'}{-\mu} & \mu < 0 , \\ \displaystyle\int_{\tau}^{\tau_0} \Gamma_m(\tau', \mu) e^{\frac{\tau - \tau'}{\mu}} \frac{d\tau'}{\mu} & \mu > 0 . \end{cases}$$

From (2. 2. 4),

$$\Gamma_m(\tau, \mu) = \frac{1}{2} \sum_j a_{mj} p(\mu, \mu_{mj}) I_m(\tau, \mu_{mj}) - \Gamma^0(\tau, \mu) \quad . \tag{2. 5. 4}$$

Once the system of differential equations has been solved for $I_m(\tau, \mu_{mj})$, then (2. 5. 4) yields $\Gamma_m(\tau, \mu)$ and (2. 5. 2), (2. 5. 3) yield $I_m(\tau, \mu)$. Thus we obtain the desired solution of the approximate problem.

The substitution of (2. 5. 3) into (2. 5. 4) gives

$$\Gamma_m(\tau, \mu) = \frac{1}{2} \sum_{j=1}^{m} a_{mj} \int_{\tau}^{\tau_0} p(\mu, \mu_{mj}) \frac{e^{\frac{\tau - \tau'}{\mu_{mj}}}}{\mu_{mj}} \Gamma(\tau', \mu_{mj}) d\tau'$$

$$+ \frac{1}{2} \sum_{j=-1}^{-m} a_{mj} \int_{0}^{\tau} p(\mu, \mu_{mj}) \frac{e^{\frac{\tau - \tau'}{\mu_{mj}}}}{-\mu_{mj}} \Gamma(\tau', \mu_{mj}) d\tau'$$

$$+ \Gamma^0(\tau, \mu) \ . \tag{2. 5. 4}$$

This integral equation for $\Gamma_m(\tau, \mu)$ is analogous to (1. 3. 4). It provides an equivalent formulation for the approximate problem.

If $\omega_0 = 1$, $\tau_0 = \infty$ and $\Gamma^0(\tau, \mu) = 0$, then a suitable normalization condition is given by the equation

$$F = 2\sum_{j=1}^{m} a_{mj} \int_0^\infty e^{-\frac{\tau'}{\mu_{mj}}} \Gamma(\tau', \mu_{mj}) d\tau' , \tag{2.5.5}$$

which is analogous to (1.4.6).

CHAPTER 3

INTEGRAL EQUATIONS FOR FINITE τ_0

§ 3.1 Preliminary Remarks

In the first two chapters, we expressed the exact and approximate problems in integral equation form. These integral equations are studied in detail in this and the following chapters. Henceforth, $p(\mu, \mu')$ satisfies the conditions given in § 1.2, but no longer is assumed to have the special form (2.2.5). Throughout this chapter, τ_0 is assumed to be finite.

We now suppose a general quadrature rule of the form (2.1.1), where the coefficients and subdivision points satisfy (2.1.2) and (2.1.3).

Let $C(R)$ denote the Banach space of continuous real-valued functions $f(\tau, \mu)$, defined on $R: \{0 \le \tau \le \tau_0; -1 \le \mu \le 1\}$, with the uniform norm

$$||f|| = \max_{(\tau,\mu)\in R} |f(\tau,\mu)|.$$

The norm of a bounded linear operator L which maps $C(R)$ into $C(R)$ will be denoted by $||L||$ where

$$||L|| = \sup_{||f|| \le 1} ||Lf||.$$

§ 3.2 Operator Equations for $\Gamma(\tau,\mu)$ and $\Gamma_m(\tau,\mu)$

Motivated by (1.3.4), we define the linear integral operator Λ, for $f \in C(R)$, by

(3.2.1) $$(\Lambda f)(\tau,\mu) = \int_{-1}^{1} \int_{0}^{\tau_0} K(\tau,\mu;\tau',\mu')f(\tau',\mu')d\tau'd\mu',$$

where

(3.2.2) $$K(\tau,\mu;\tau',\mu') = \frac{1}{2}p(\mu,\mu')k(\tau'-\tau,\mu'),$$

(3.2.3) $$k(x,\mu) = \begin{cases} \frac{1}{|\mu|}e^{-\frac{x}{\mu}}, & \frac{x}{\mu} > 0, \\ 0, & \frac{x}{\mu} < 0. \end{cases}$$

It is easy to verify that the integral in (3.2.1) exists for each (τ,μ) in R. We will show in § 3.3 that Λ maps $C(R)$ into $C(R)$ and is bounded.

Equivalent forms of (3.2.1) are

$$(3.2.4)\qquad (\Lambda f)(\tau,\mu) = \frac{1}{2}\int_0^1\int_0^{\tau} p(\mu,-\mu')\frac{e^{\frac{\tau-\tau'}{-\mu'}}}{\mu'} f(\tau',-\mu')d\tau' d\mu'$$

$$+\frac{1}{2}\int_0^1\int_{\tau}^{\tau_0} p(\mu,\mu')\frac{e^{\frac{\tau-\tau'}{\mu'}}}{\mu'} f(\tau',\mu')d\tau' d\mu',$$

and

$$(3.2.5)\qquad (\Lambda f)(\tau,\mu) = \frac{1}{2}\int_0^1\int_0^{\frac{\tau}{\mu'}} p(\mu,-\mu')e^{-x}f(\tau-\mu'x,-\mu')dxd\mu'$$

$$+\frac{1}{2}\int_0^1\int_0^{\frac{\tau_0-\tau}{\mu'}} p(\mu,\mu')e^{-x}f(\tau+\mu'x,\mu')dxd\mu' .$$

Similarly, motivated by (2.5.4), we define the linear integral operators Λ_m, $m = 1, 2, \cdots$, for $f \in C(R)$, by

$$(3.2.6)\qquad (\Lambda_m f)(\tau,\mu) = \sum_{j=\pm 1}^{\pm m}\int_0^{\tau_0} a_{mj}K(\tau,\mu;\tau',\mu_{mj})f(\tau,\mu_{mj})d\tau'$$

where $K(\tau,\mu;\tau',\mu_{mj})$ is given by (3.2.2).

It will be shown that each Λ_m is a bounded operator on $C(R)$ into $C(R)$. Forms analogous to (3.2.4) and (3.2.5) are easily obtained.

We now express the integral equations (1,3.4) and (2.5.4) for Γ and Γ_m in the form

$$(I-\Lambda)\Gamma = \Gamma^0 , \tag{3.2.7}$$

$$(I-\Lambda_m)\Gamma_m = \Gamma^0 , \tag{3.2.8}$$

where I is the identity operator on $C(R)$. Since the intensity $I(\tau, \mu)$ is always displayed with arguments, there should be no confusion.

§ 3.3 Properties of the Operators Λ and Λ_m

We show next that Λ and Λ_m, $m \geq 1$ are bounded linear operators which map $C(R)$ into $C(R)$. First consider Λ. Since

$$|(\Lambda f)(\tau,\mu)-(\Lambda f)(\bar{\tau},\bar{\mu})| \leq ||f|| \int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau,\mu;\tau',\mu') - K(\bar{\tau},\bar{\mu};\tau',\mu')| d\tau' d\mu' , \tag{3.3.1}$$

it suffices to show that

$$\int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau,\mu;\tau',\mu')-K(\bar{\tau},\bar{\mu};\tau',\mu')| d\tau' d\mu' \to 0 \tag{3.3.2}$$

as $(\tau,\mu) \to (\bar{\tau},\bar{\mu})$.

In order to establish (3.3.2), we shall construct a continuous kernel $K^{\varepsilon}(\tau, \mu; \tau', \mu')$ for each $\varepsilon > 0$ such that

$$(3.3.3) \quad \int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau, \mu; \tau', \mu') - K^{\varepsilon}(\tau, \mu; \tau', \mu')| d\tau' d\mu' \to 0$$

as $\varepsilon \to 0$

uniformly for all (τ, μ) in R. Then the desired result will follow from

$$(3.3.4) \quad \int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau, \mu; \tau', \mu') - K(\bar{\tau}, \bar{\mu}; \tau', \mu') d\tau' d\mu' \leq$$

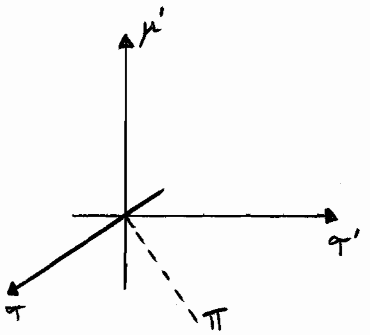
$$\int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau, \mu; \tau', \mu') - K^{\varepsilon}(\tau, \mu; \tau', \mu')| d\tau' d\mu'$$

$$+ \int_{-1}^{1} \int_{0}^{\tau_0} |K^{\varepsilon}(\tau, \mu; \tau', \mu') - K^{\varepsilon}(\bar{\tau}, \bar{\mu}; \tau', \mu')| d\tau' d\mu'$$

$$+ \int_{-1}^{1} \int_{0}^{\tau_0} |K^{\varepsilon}(\bar{\tau}, \bar{\mu}; \tau', \mu') - K(\bar{\tau}, \bar{\mu}; \tau', \mu')| d\tau' d\mu' .$$

We now proceed to the construction of $K^{\varepsilon}(\tau, \mu; \tau', \mu')$. For convenience, let $P = (\tau, \mu; \tau', \mu')$ in four-space. Then $K(P) = K(\tau, \mu; \tau', \mu')$ is a function with domain $\mathcal{U} = R \times R$.

Discontinuities of $K(P)$ are confined to points of the hyperplane $\pi : \{\tau = \tau', \mu = 0\}$. The projection of π in the μ', τ, τ'-space is indicated in the accompanying figure. For each



$\varepsilon > 0$, let $\mathcal{U}^{\varepsilon} = \{P: \rho(P, \pi) < \varepsilon\}$,

where $\rho(P, \pi) = \inf_{Q \in \pi} \|P - Q\|$,

with $\| \quad \|$ the maximum norm.

Then $\mathcal{U}^{\varepsilon}$ is an open neighborhood of π. Define $K^{\varepsilon}(P) = K(P)$ on the closed set $\mathcal{U} - \mathcal{U}^{\varepsilon}$ and extend it to a continuous function on $\mathcal{U}$ such that $0 < K^{\varepsilon}(P) \le 2K(P)$ by means of the Tietze Extension Theorem (Bartle, 1964, p. 187). Since $K(P) = K^{\varepsilon}(P)$ on $\mathcal{U} - \mathcal{U}^{\varepsilon}$,

$$(3.3.5) \quad \iint_R |K(P) - K^{\varepsilon}(P)|\, d\tau'\, d\mu' \le \int_{-\varepsilon}^{\varepsilon} \int_{\tau_{\varepsilon}}^{\tau^{\varepsilon}} K(P)\, d\tau'\, d\mu'$$

$$+ \int_{-\varepsilon}^{\varepsilon} \int_{\tau_{\varepsilon}}^{\tau^{\varepsilon}} K^{\varepsilon}(P)\, d\tau'\, d\mu' \le 3 \int_{-\varepsilon}^{\varepsilon} \int_{\tau_{\varepsilon}}^{\tau^{\varepsilon}} K(P)\, d\tau'\, d\mu' ,$$

where $\tau_{\varepsilon} = \max(0, \tau - \varepsilon)$ and $\tau^{\varepsilon} = \min(\tau_0, \tau + \varepsilon)$.

Since $p(\mu, \mu')$ is bounded, say by W, (3.2.2) and (3.2.3) yield

$$\int_{-\varepsilon}^{\varepsilon}\int_{T_\varepsilon}^{T^\varepsilon} K(P)dT'd\mu' \leq \frac{1}{2}\int_{-\varepsilon}^{\varepsilon}\int_{T-\varepsilon}^{T+\varepsilon} p(\mu,\mu')k(T'-T,\mu')dT'd\mu'$$

$$\leq \frac{W}{2}\int_{-\varepsilon}^{\varepsilon}\int_{T-\varepsilon}^{T+\varepsilon} k(T'-T,\mu')dT'd\mu'$$

$$= W\int_{-1}^{1}(1-e^{-\frac{\varepsilon}{|\mu'|}})\,d\mu'$$

$$(3.3.6) \qquad = 2W\varepsilon\int_{0}^{1}(1-e^{x})dx \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Now (3.3.3) follows from (3.3.5) and (3.3.6).

It is a consequence of (3.3.1), (3.3.4) and the uniform continuity of $K^{\varepsilon}(P)$ that $(\Lambda f)(T,\mu)$ is continuous for each $f \in C(R)$. Thus, Λ maps $C(R)$ into $C(R)$.

It is much easier to see that each Λ_m maps $C(R)$ into $C(R)$. Note that $K_m(T,\mu;T',\mu_{mj})$ in (3.2.6) is continuous since every $u_{mj} \neq 0$.

The function $(\Lambda 1)(T,\mu)$, where 1 denotes a constant function, will be useful in showing that Λ is a bounded operator on $C(R)$. By (3.2.1)

$$(\Lambda 1)(\tau,\mu) = \int_{-1}^{1}\int_{0}^{\tau_0} K(\tau,\mu;\tau',\mu')d\tau'\,d\mu' = \frac{1}{2}\int_{-1}^{1}\int_{0}^{\tau_0} p(\mu,\mu')k(\tau'-\tau,\mu')d\tau' d\mu'$$

$$= \frac{1}{2}\int_{-1}^{0}\int_{0}^{\tau} p(\mu,\mu')\frac{e^{\frac{\tau-\tau'}{\mu'}}}{-\mu'}d\tau'\,d\mu' + \frac{1}{2}\int_{0}^{1}\int_{\tau}^{\tau_0} p(\mu,\mu')\frac{e^{\frac{\tau-\tau'}{\mu'}}}{\mu'}d\tau' d\mu'$$

$$= \frac{1}{2}\int_{-1}^{0} p(\mu,\mu')(1-e^{\frac{\tau}{\mu'}})d\mu' + \frac{1}{2}\int_{0}^{1} p(\mu,\mu')(1-e^{\frac{\tau-\tau_0}{\mu'}})\,d\mu' ,$$

and by (1.2.2),

$$(3.3.7) \qquad (\Lambda 1)(\tau,\mu) = \omega_0 - \frac{1}{2}\int_{0}^{1}\left[p(\mu,-\mu')e^{-\frac{\tau}{\mu'}} + p(\mu,\mu')e^{\frac{\tau-\tau_0}{\mu'}}\right]d\mu' .$$

Elementary calculus yields

$$(3.3.8) \qquad (\Lambda 1)(\tau,\mu) \le \omega_0 - \delta \int_{0}^{1} e^{-\frac{\tau_0}{2\mu'}}\,d\mu' ,$$

where $\delta = \min p(\mu,\mu')$. Similarly

$$(3.3.9) \qquad (\Lambda_m 1)(\tau,\mu) = \omega_{m0} - \frac{1}{2}\sum_{j=1}^{m} a_{mj}\left[p(\mu,-\mu_{mj})e^{-\frac{\tau}{\mu_{mj}}} \right.$$

$$\left. + \; p(\mu,\mu_{mj})\,e^{\frac{\tau-\tau_0}{\mu_{mj}}} \right]$$

and

(3.3.10) $$(\Lambda_m 1)(\tau,\mu) \leq \omega_{m0} - \delta \sum_{j=1}^{m} a_{mj} e^{-\frac{\tau_0}{2\mu_{mj}}},$$

where

(3.3.11) $$\omega_{m0} = \frac{1}{2} \sum_j a_{mj} p(\mu, \mu_{mj}) \to \omega_0 \quad \text{as} \quad m \to \infty .$$

We note that $K(\tau, \mu; \tau', \mu') > 0$. Hence, Λ and Λ_m, $m \geq 1$ are positive operators. We have the following consequences: if $f, g \in C(R)$, then

(3.3.12)
$$f \geq 0 \Rightarrow \Lambda f \geq 0 ; \qquad f > 0 \Rightarrow \Lambda f > 0 ;$$
$$f \geq g \Rightarrow \Lambda f \geq \Lambda g ; \qquad f > g \Rightarrow \Lambda f > \Lambda g ;$$
$$|\Lambda f| \leq \Lambda |f| ,$$

and similarly for Λ_m.

It now follows that

$$||\Lambda f|| \leq ||f|| \; ||\Lambda 1||$$

for all $f \in C(R)$. Hence, Λ is a bounded linear operator and

$$||\Lambda|| \leq ||\Lambda 1|| .$$

However, $||\Lambda|| = ||\Lambda 1||$ since $||1|| = 1$. By (3.3.8),

(3.3.13) $$||\Lambda|| = ||\Lambda 1|| < \omega_0 \leq 1 .$$

Similarly,

(3.3.14) $$||\Lambda_m|| = ||\Lambda_m 1|| < \omega_{m0} .$$

By (3.3.10), (3.3.11) and (3.3.14),

(3.3.15) $$||\Lambda_m|| < \omega_0 \leq 1 ,$$

for m sufficiently large.

§ 3.4 Solution of the Integral Equations for Finite τ_0

We now return to the solution of the integral equations (3.2.7) and (3.2.8). Since $||\Lambda|| < 1$ and $||\Lambda_m|| < 1$ for m sufficiently large, the operators $I-\Lambda$ and $I-\Lambda_m$ have unique bounded inverses defined on $C(R)$ which are given by the Neumann series

(3.4.1) $$(I-\Lambda)^{-1} = \sum_{n=0}^{\infty} \Lambda^n,$$

(3.4.2) $$(I-\Lambda_m)^{-1} = \sum_{n=0}^{\infty} \Lambda_m^n ,$$

which converge in the operator norms. The bounds of these operators satisfy

$$\|(I-\Lambda)^{-1}\| \leq \frac{1}{1-\|\Lambda\|}, \tag{3.4.3}$$

$$\|(I-\Lambda_m)^{-1}\| \leq \frac{1}{1-\|\Lambda_m\|}, \tag{3.4.4}$$

for m sufficiently large.

Hence, the integral equations (3.2.7) and (3.2.8) have the unique solutions given by the uniformly convergent Neumann series

$$\Gamma = \sum_{n=0}^{\infty} \Lambda^n \Gamma^0, \tag{3.4.5}$$

$$\Gamma_m = \sum_{n=0}^{\infty} \Lambda_m^n \Gamma^0, \tag{3.4.6}$$

and

$$\|\Gamma\| \leq \frac{\|\Gamma^0\|}{1-\|\Lambda\|}, \tag{3.4.7}$$

$$\|\Gamma_m\| \leq \frac{\|\Gamma^0\|}{1-\|\Lambda_m\|}, \tag{3.4.8}$$

again for m sufficiently large.

Under the condition $\Gamma^0(\tau, \mu) \geq 0$, it follows from (3.3.12) (3.4.5) and (3.4.6) that $\Gamma(\tau, \mu) \geq 0$ and $\Gamma_m(\tau, \mu) \geq 0$. This completes the solution of the integral equations (3.2.7) and (3.2.8) for the case $\tau_0 < \infty$.

CHAPTER 4

CONVERGENCE THEOREMS FOR FINITE τ_0

§ 4.1 Preliminary Remarks

Having solved the integral equations (3.2.7) and (3.2.8) for the case $\tau_0 < \infty$, we now turn to convergence questions. We continue to use a general quadrature rule of the form (2.1.1) which for convenience satisfies (2.1.2) and (2.1.3) Moreover, suppose

$$(4.1.1) \qquad \sum_{j=\pm 1}^{\pm m} a_{mj} f(\mu_{mj}) \to \int_{-1}^{1} f(\mu)d\mu \qquad \text{as } m \to \infty,$$

for $f \in C(R)$, and that

$$(4.1.2) \qquad \sum_{j=\pm 1}^{\pm m} a_{mj} \leq B, \quad m \geq 1,$$

for some $B < \infty$. (Actually, (4.1.1) implies (4.1.2) by the principle of uniform boundedness.) The rectangular, trapezoidal, Simpson and Gauss (but not Newton-Cotes) quadrature rules have these properties. The convergence in (4.1.1) is uniform on any bounded equicontinuous family of functions and also on the characteristic functions of intervals (cf. Anselone, 1965).

We recall some definitions from functional analysis. A linear operator L which maps a normed linear space X into a normed linear space Y is said to be compact (or completely continuous) if the image of any bounded set B in X has compact closure (equivalently, is totally bounded). The following idea, due to Anselone and Moore (1964), is also used. A set of operators $\{L_a\}$ is said to be collectively compact if, for each bounded set B,

$$\bigcup_{a} \{L_a f : f \in B\} \tag{4.1.3}$$

has compact closure (is totally bounded). It suffices to take $B:\{||f|| \leq 1\}$.

It will be shown that Λ is compact, $\{\Lambda_m; m \geq 1\}$ is collectively compact, and $||\Lambda f - \Lambda_m f|| \to 0$ as $m \to \infty$ for each $f \in C(R)$. Results of Anselone and Moore (1964) will then be used to obtain convergence theorems and error bounds in the case $\tau_0 < \infty$.

§ 4.2 Further Properties of Λ and Λ_m

Define Λ^ε and Λ^ε_m, $\varepsilon > 0$, $m \geq 1$, on $C(R)$, by

$$(\Lambda^\varepsilon f)(\tau, \mu) = \int_{-1}^{1} \int_{0}^{\tau_0} K^\varepsilon(\tau, \mu; \tau', \mu') f(\tau', \mu') d\tau' d\mu', \tag{4.2.1}$$

$$(\Lambda^\varepsilon_m f)(\tau, \mu) = \sum_j a_{mj} \int_0^{\tau_0} K^\varepsilon(\tau, \mu; \tau', \mu_{mj}) f(\tau', \mu_{mj}) d\tau', \tag{4.2.2}$$

where $K^\varepsilon(\tau, \mu; \tau', \mu')$ is defined in §3.3. We will use these operators to establish the desired properties of Λ and Λ_m. First of all, observe that since $K^\varepsilon(\tau, \mu; \tau', \mu')$ is continuous, Λ^ε and Λ_m^ε are compact operators on $C(R)$.

From

(4.2.3) $$||\Lambda - \Lambda^\varepsilon|| = \sup_{(\tau,\mu)\in R} \int_{-1}^{1} \int_{0}^{\tau_0} |K(\tau, \mu; \tau', \mu') - K^\varepsilon(\tau, \mu; \tau', \mu')| d\tau' d\mu',$$

and (3.3.3) ff it follows that

(4.2.4) $$||\Lambda - \Lambda^\varepsilon|| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Since Λ is a limit of compact operators, Λ is compact.

Since every $\mu_{mj} \neq 0$, the kernel in (3.2.6) is continuous in τ, μ and τ'. It follows that each Λ_m, $m \geq 1$ is a compact operator on $C(R)$. In view of the definition of K^ε, for each m there exists $\varepsilon(m) > 0$ such that $K(\tau, \mu; \tau', \mu_{mj}) \equiv K^\varepsilon(\tau, \mu; \tau', \mu_{mj})$ if $\varepsilon \leq \varepsilon(m)$. Hence,

(4.2.5) $$\Lambda_m = \Lambda_m^\varepsilon, \qquad \varepsilon \leq \varepsilon(m) .$$

By (3.3.11) and (3.3.14), the operators Λ_m, $m \geq 1$ are uniformly bounded.

<u>Theorem 4.2.1.</u> $||\Lambda f - \Lambda_m f|| \to 0$ for all $f \in C(R)$ as $m \to \infty$.

Proof: By the triangle inequality,

$$(4.2.6) \quad \|\Lambda f-\Lambda_m f\| \leq \|\Lambda f-\Lambda^\varepsilon f\| + \|\Lambda^\varepsilon f-\Lambda_m^\varepsilon f\| + \|\Lambda_m^\varepsilon f-\Lambda_m f\| .$$

We examine each of the terms of the right member. First choose $\delta > 0$. Then fix ε such that $\varepsilon < \delta$ and, by (4.2.4), $\|\Lambda f-\Lambda^\varepsilon f\| < \delta$. Now consider $\|\Lambda^\varepsilon f-\Lambda_m^\varepsilon f\|$. By (4.2.1) and (4.2.2),

$$\|\Lambda^\varepsilon f-\Lambda_m^\varepsilon f\| \leq \sup_{(\tau,\mu)\epsilon R} \int_0^{\tau_0} \Big| \int_{-1}^{1} K^\varepsilon(\tau,\mu;\tau',\mu')f(\tau',\mu')d\mu'$$

$$- \sum_j a_{mj} K^\varepsilon(\tau,\mu;\tau',\mu_{mj})f(\tau',\mu_{mj}) \Big| d\tau' .$$

Since $K^\varepsilon(\tau,\mu;\tau',\mu')$ and $f(\tau',\mu')$ are uniformly continuous in their arguments, $K^\varepsilon(\tau,\mu;\tau',\mu')f(\tau',\mu')$ may be regarded as a bounded equicontinuous family of functions of μ' parametrized by τ,μ,τ'. Since the convergence in (4.1.1) is uniform on such a family, $\|\Lambda^\varepsilon f-\Lambda_m^\varepsilon f\| \to 0$ as $m \to \infty$. Thus, there exists $m_1 = m_1(\varepsilon)$ such that $\|\Lambda^\varepsilon f-\Lambda_m^\varepsilon f\| < \delta$ for $m \geq m_1$.

Finally, consider $\|\Lambda_m^\varepsilon f-\Lambda_m f\|$. By (3.2.6), (4.2.2) and the definition of K^ε,

$$\|\Lambda_m-\Lambda_m^\varepsilon\| = \sup_{(\tau,\mu)\epsilon R} \sum_j a_{mj} \int_0^{\tau_0} |K(\tau,\mu;\tau',\mu_{mj})-K^\varepsilon(\tau,\mu;\tau',\mu_{mj})| d\tau' .$$

Proceeding as in (3. 3. 5)ff, we have

$$\| {}_m - \Lambda^{\varepsilon}_m \| = \sup_{\substack{(\tau,\mu)\in R \\ |\mu_{mj}| \leq \varepsilon}} \sum_j a_{mj} \int_{\tau_\varepsilon}^{\tau^\varepsilon} | K(\tau,\mu;\tau',\mu_{mj}) - K^{\varepsilon}(\tau,\mu;\tau',\mu_{mj}) | \, d\tau'$$

$$\leq 3 \sup_{\substack{(\tau,\mu)\in R \\ |\mu_{mj}| \leq \varepsilon}} \sum_j a_{mj} \int_{\tau_\varepsilon}^{\tau^\varepsilon} K(\tau,\mu;\tau',\mu_{mj}) d\tau' ,$$

$$\leq 3W \sum_j a_{mi} \left[1 - e^{-\frac{\varepsilon}{|\mu_{mj}|}} \right] \chi_{[-\varepsilon,\varepsilon]} (\mu_{mj}),$$

where $\chi_{[-\varepsilon,\varepsilon]}$ is the characteristic function of the interval $[-\varepsilon,\varepsilon]$. By (4. 1. 1)ff, the last sum converges to

$$\int_{-\varepsilon}^{\varepsilon} (1 - e^{-\frac{\varepsilon}{|\mu|}}) \, d\mu = \frac{2\varepsilon}{e} < \varepsilon < \delta$$

as $m \to \infty$. Hence, there exists $m_2 = m_2(\varepsilon)$ such that $||\Lambda_m - \Lambda^{\varepsilon}_m|| < 3W\delta$ for $m \geq m_2$. The desired result now follows from (4. 2. 6).

<u>Lemma 4. 2. 1.</u> The set $\{\Lambda_m, m \geq 1\}$ is collectively compact.

<u>Proof:</u> Fix $\delta > 0$ and let $\varepsilon = \delta/3W$. Then (cf. p. 44), there exists $m(\delta)$ such that

$$||\Lambda_m f - \Lambda_m^\varepsilon f|| < \delta \quad \text{for} \quad m > m(\delta), \quad ||f|| \leq 1.$$

Thus, $\bigcup_{m > m(\delta)} \{\Lambda_m^\varepsilon f : ||f|| \leq 1\}$ forms a δ-net for $\bigcup_{m > m(\delta)} \{\Lambda_m f : ||f|| \leq 1\}$. It follows directly from (4.1.2), (4.2.2) and the fact that $K^\varepsilon(\tau, \mu; \tau', \mu_{mj})$ is continuous that $\bigcup_{m > m(\delta)} \{\Lambda_m^\varepsilon f : ||f|| \leq 1\}$ is bounded and equicontinuous or, equivalently, totally bounded. By a standard theorem (Liusternik and Sobolev, 1961, p. 136) $\bigcup_{m > m(\delta)} \{\Lambda_m f : ||f|| \leq 1\}$ is totally bounded. For each $m = 1, 2, \cdots, m(\delta)$, Λ_m is compact, so $\{\Lambda_m f : ||f|| \leq 1\}$ is totally bounded. It follows that $\bigcup_{m=1}^{\infty} \{\Lambda_m f : ||f|| \leq 1\}$ is totally bounded. Hence, $\{\Lambda_m : m \geq 1\}$ is collectively compact.

Recapitulating, we have shown that

(4.2.7)
(i) Λ is compact
(ii) $\{\Lambda_m : m \geq 1\}$ is collectively compact,
(iii) $||\Lambda f - \Lambda_m f|| \to 0$ for each $f \in C(R)$.

Moreover (cf. § 3.4) $(I-\Lambda)^{-1}$ exists and $(I-\Lambda_m)^{-1}$ exists and is uniformly bounded for sufficiently large m.

§ 4.3 Convergence Theorems for Finite T_0

Anselone and Moore (1964) have studied operators which satisfy the conditions (4.2.7). A direct application of their results yields the following theorems.

Theorem 4.3.1. As $m \to \infty$,

$$||\Lambda_m \Lambda - \Lambda^2|| \to 0 \quad . \tag{4.3.1}$$

If

$$||\Lambda_m \Lambda - \Lambda^2|| < \frac{1}{||(I-\Lambda_m)^{-1}||} \quad , \tag{4.3.2}$$

which holds for m sufficiently large, then

$$||(I-\Lambda)^{-1}|| \leq \frac{1+||(I-\Lambda_m)^{-1}||\ ||\Lambda||}{1-||(I-\Lambda_m)^{-1}||\ ||\Lambda_m \Lambda - \Lambda^2||} \tag{4.3.3}$$

and

$$||\Gamma - \Gamma_m|| \leq ||(I-\Lambda_m)^{-1}|| \frac{||\Lambda \Gamma^0 - \Lambda_m \Gamma^0|| + ||\Lambda_m \Lambda - \Lambda^2||\ ||\Gamma_m||}{1-||(I-\Lambda_m)^{-1}||\ ||\Lambda_m \Lambda - \Lambda^2||} . \tag{4.3.4}$$

We remark that (4.3.3) and (4.3.4) provide a means of estimating Γ and $\Gamma - \Gamma_m$ since $\Gamma = (I-\Lambda)^{-1}\Gamma^0$. In this connection (4.3.1) is a consequence of (4.1.1) and hence can be estimated in terms of an error formula for the quadrature formula.

Theorem 4.3.2. As $m \to \infty$,

$$\|\Lambda\Lambda_m - \Lambda_m^2\| \to 0 . \tag{4.3.5}$$

If

$$\|\Lambda\Lambda_m - \Lambda_m^2\| < \frac{1}{\|(I-\Lambda)^{-1}\|} , \tag{4.3.4}$$

then

$$\|(I-\Lambda_m)^{-1}\| \le \frac{1+\|(I-\Lambda)^{-1}\| \; \|\Lambda_m\|}{1-\|(I-\Lambda)^{-1}\| \; \|\Lambda\Lambda_m - \Lambda_m^2\|} \tag{4.3.5}$$

and

$$\|\Gamma - \Gamma_m\| \le \|(I-\Lambda)^{-1}\| \frac{\|\Lambda\Gamma^0 - \Lambda_m\Gamma^0\| + \|\Lambda\Lambda_m - \Lambda_m^2\| \; \|\Gamma\|}{1-\|(I-\Lambda)^{-1}\| \; \|\Lambda\Lambda_m - \Lambda_m^2\|} . \tag{4.3.6}$$

Moreover

$$\|\Gamma - \Gamma_m\| \to 0 \quad \text{as} \quad m \to \infty . \tag{4.3.7}$$

We are now able to state a general convergence theorem for the discrete ordinates method in the case $\tau_0 \to \infty$.

Theorem 4.3.3. If $\Gamma^0 \in C(R)$, then $\Gamma_m(\tau, \mu) \to \Gamma(\tau, \mu)$,

$I_m(\tau, \mu) \to I(\tau, \mu)$, and $J_m(\tau, \mu) \to J(\tau, \mu)$ uniformly as $m \to \infty$.

Proof: Apply (4. 3. 7) for the first conclusion. The others follow immediately from (1. 3. 3) and (2. 5. 3), (1. 2. 4) and (2. 2. 4).

In conclusion, we remark that $\Gamma_m(\tau, \mu)$, $I_m(\tau, \mu)$ and $J_m(\tau, \mu)$ can be calculated at least if $p(\mu, \mu')$ is sufficiently smooth. As suggested in the second chapter, replace $p(\mu, \mu')$ by a good uniform approximation in the form of a finite Legendre sum. This causes small perturbations in Λ_m, $(I-\Lambda_m)^{-1}$ and hence in $\Gamma_m(\tau, \mu)$, $I_m(\tau, \mu)$ and $J_m(\tau, \mu)$. Error estimates for this approximation scheme are obtainable from functional analysis theory.

CHAPTER 5

INTEGRAL EQUATIONS FOR INFINITE τ_0

§ 5.1 Preliminary Remarks

We now turn to the solution of the integral equations (3.2.7) and (3.2.8) for the case $\tau_0 = \infty$. In order to do this, it is necessary to define the operators Λ and Λ_m, $m \geq 1$, on a more general class of functions than $C(R)$. To avoid proliferation of new symbols, the notation for Λ and Λ_m will be carried over to this chapter with the change $\tau_0 = \infty$ in the integrals concerned.

We again assume a general quadrature rule of the form (2.1.1) where the coefficients and subdivision points satisfy (2.1.2) and (2.1.3). As before, weaker conditons would suffice.

§ 5.2 Properties of the Operators Λ and Λ_m

Let $D(\Lambda)$ consist of the set of all Lebesgue integrable functions $f(\tau, \mu)$ for which the right member of

$$(5.2.1) \qquad (\Lambda f)(\tau, \mu) = \int_{-1}^{1} \int_{0}^{\infty} K(\tau, \mu; \tau', \mu') f(\tau', \mu') d\tau' d\mu'$$

exists for each fixed τ and μ; $K(\tau, \mu; \tau', \mu')$ is defined as before by (3. 2. 2). More explicitly,

(5. 2. 2)
$$(\Lambda f)(\tau, \mu) = \frac{1}{2}\int_{-1}^{0}\int_{0}^{\tau} p(\mu, \mu')\frac{e^{\frac{\tau-\tau'}{\mu'}}}{-\mu'} f(\tau', \mu')d\tau' d\mu'$$
$$+ \frac{1}{2}\int_{0}^{1}\int_{\tau}^{\infty} p(\mu, \mu')\frac{e^{\frac{\tau-\tau'}{\mu'}}}{\mu'} f(\tau', \mu')d\tau' d\mu' .$$

Similar relations hold for Λ_m, $m \geq 1$.

Using (5. 2. 1) or (5. 2. 2), we can indicate various functions which are contained in $D(\Lambda)$. If $f(\tau, \mu) = \mu$, then $f \in D(\Lambda)$ and (with this function denoted by μ)

(5. 2. 3)
$$(\Lambda \mu)(\tau, \mu) = \frac{1}{3}\omega_1\mu + \frac{1}{2}\int_0^1 \mu' p(\mu, -\mu')e^{-\frac{\tau}{\mu'}} d\mu' .$$

If $f(\tau, \mu) = \tau^n$, $n \geq 0$, then $f \in D(\Lambda)$, and

(5. 2. 4)
$$(\Lambda 1)(\tau, \mu) = \omega_0 - \frac{1}{2}\int_0^1 p(\mu, -\mu')e^{-\frac{\tau}{\mu'}} d\mu' ,$$

(5. 2. 5)
$$(\Lambda \tau)(\tau, \mu) = \omega_0\tau + \frac{1}{3}\omega_1\mu + \frac{1}{2}\int_0^1 p(\mu, -\mu')\mu' e^{-\frac{\tau}{\mu'}} d\mu' ,$$

(5. 2. 6)
$$(\Lambda \tau^n)(\tau, \mu) = \omega_0\tau^n + Q_{n-1}(\tau) + o(\tau) ,$$

where $Q_{n-1}(\tau)$ is a polynomial of degree $\leq n-1$, and $o(1)$ denotes a function of τ and μ which tends to zero uniformly in μ as $\tau \to \infty$. Similar expressions hold for Λ_m with ω_0 replaced by ω_{m0}, given by (3.3.11), and with the integrals replaced by corresponding sums. It follows that $D(\Lambda)$ and $D(\Lambda_m)$ contain all polynomials.

For $f, g \in D(\Lambda)$, and from the definition (5.2.1),

(5.2.7)

$$\text{(a)} \quad f \geq 0 \Rightarrow \Lambda f \geq 0, \quad f > 0 \Rightarrow \Lambda f > 0,$$

$$\text{(b)} \quad f \geq g \Rightarrow \Lambda f \geq \Lambda g, \quad f > g \Rightarrow \Lambda f > \Lambda g.$$

If f is measurable and $|f| \leq g$ for some $g \in D(\Lambda)$, then $|f| \in D(\Lambda)$, $f \in D(\Lambda)$, and $|\Lambda f| \leq \Lambda|f| \leq \Lambda g$. Hence, if $f(\tau, \mu) = O(\tau^n)$ uniformly in μ as $\tau \to \infty$, then $f \in D(\Lambda)$ and $\Lambda f = O(\tau^n)$. In particular, $D(\Lambda)$ contains all bounded measurable functions and

(5.2.8) $$|f| \leq M \Rightarrow |\Lambda f| \leq \omega_0 M \leq M.$$

We assert that if f is continuous and $f = O(\tau^n)$ for some $n \geq 0$, then Λf and $\Lambda_m f$ are continuous. By (5.2.1), for any $y > 0$ we have

$$|(\Lambda f)(\tau,\mu)-(\Lambda f)(\bar{\tau},\bar{\mu})| \leq \int_{-1}^{1}\int_{0}^{y} |K(\tau,\mu;\tau',\mu')-K(\bar{\tau},\bar{\mu};\tau',\mu')|\,|f(\tau',\mu')|\,d\tau' d\mu'$$

$$+ \int_{-1}^{1}\int_{y}^{\infty} [K(\tau,\mu;\tau',\mu')+K(\bar{\tau},\bar{\mu};\tau',\mu')]\,|f(\tau',\mu')|\,d\tau' d\mu'.$$

The second integral can be made arbitrarily small for all τ, $\bar{\tau}$, μ, $\bar{\mu}$ such that $0 \leq \tau \leq \frac{y}{2}$, $0 \leq \bar{\tau} \leq \frac{y}{2}$, $-1 \leq \mu \leq 1$, $-1 \leq \mu' \leq 1$, by choosing y sufficiently large. To see this, put the integral in the form analagous to (3. 2. 5) (with $\tau_0 = \infty$). With y thus fixed, it follows from (3. 3. 3) that the first integral is arbitrarily small if, in addition, $|\tau-\bar{\tau}|$ and $|\mu-\bar{\mu}|$ are sufficiently small. The proof for Λ_m is similar and is omitted.

In this chapter, we will let $C(R)$ denote the Banach space of bounded, continuous functions defined on the set $R: \{-1 \leq \mu \leq 1,\ 0 \leq \tau < \infty\}$, with the supremum norm. Then Λ and Λ_m map $C(R)$ into $C(R)$ and, by (5. 2. 8) and (5. 2. 4),

(5. 2. 10) $$||\Lambda|| = \omega_0 \leq 1 \quad \text{on} \quad C(R),$$

(5. 2. 11) $$||\Lambda_m|| = \omega_{m0} \quad \text{on} \quad C(R),$$

where $\omega_{m0} \to \omega_0$ as $m \to \infty$.

If $\omega_0 < 1$, then $(I-\Lambda)^{-1}$ exists on $C(R)$ and is given by a Neumann series which converges in the operator norm. A similar statement holds for $(I-\Lambda_m)^{-1}$ for m sufficiently large. Hence, we again have the formulas and results of §3.4.

The situation is more difficult if $\omega_0 = 1$ or if we seek unbounded solutions. These questions will be discussed in the following sections.

Two closed subspaces of $C(R)$ will be of interest. Let

$$C_0(R) = \{f: f(\tau, \mu) \to 0 \text{ uniformly in } \mu \text{ as } \tau \to \infty\}$$

$$C_1(R) = \{f: f(\tau, \mu) \text{ converges uniformly in } \mu \text{ as } \tau \to \infty\}.$$

Lemma 5.2.1. $f \in C_0(R) \Rightarrow \Lambda f \in C_0(R)$, $\Lambda_m f \in C_0(R)$, $m \geq 1$.

Proof: By (5.2.2), $f \to 0 \Rightarrow \Lambda f \to 0$ (uniformly in μ as $\tau \to \infty$), and similarly for Λ_m. The lemma follows.

Lemma 5.2.2. $f \in C_1(R) \Rightarrow \Lambda f \in C_1(R)$. Moreover,

$$f \to a \Rightarrow \begin{cases} \Lambda f \to a\,\omega_0, \\ \Lambda_m f \to a\,\omega_{m0}, \quad m \geq 1, \end{cases}$$

(uniformly in μ as $\tau \to \infty$).

Proof: Let $g = f - a$. Then $g \to 0$, so $\Lambda g \to 0$, and by (5.2.4)

$$\Lambda f = \Lambda g + a \Lambda 1 \to a \omega_0 ,$$

and similarly for Λ_m. The lemma follows.

§ 5.3 Uniqueness Questions

In this section, we consider the uniqueness of solutions of the equations $f - \Lambda f = g$ and $f - \Lambda_m f = g$. Thus, we examine the associated homogeneous equations $f = \Lambda f$ and $f = \Lambda_m f$.

First suppose $\omega_0 < 1$.

Theorem 5.3.1. Let $\omega_0 < 1$. If $f = \Lambda f$ where $f = O(\tau^n)$ for some $n \geq 0$, then $f \equiv 0$.

Proof: For $n = 0$, the desired result means that $f = \Lambda f$ and f bounded implies $f \equiv 0$. This follows from the existence of $(I-\Lambda)^{-1}$ on $C(R)$. Now let $n = 1$. If $|f| \leq M\tau$, then by (5.2.5)

$$|f| = |\Lambda f| \leq \Lambda |f| \leq M \Lambda \tau \leq M\omega_0 \tau + Mb,$$

where b is a constant. Apply the operator again to obtain

$$|f| = |\Lambda^2 f| \leq M(\Lambda^2 \tau) \leq M\omega_0^2 \tau + Mb(\omega_0 + 1).$$

Repeated application of Λ gives

$$|f| = |\Lambda^n f| \le M\omega_0^n \tau + Mb(\omega_0^{n-1} + \omega_0^{n-2} + \cdots + \omega_0 + 1) < M\omega_0^n + \frac{Mb}{1-\omega_0} .$$

Let $n \to \infty$ to obtain

$$|f| \le \frac{Mb}{1-\omega_0} .$$

Thus $f \equiv 0$ since the theorem holds for $n = 0$.

In a similar manner, it can be shown that

$$f = O(\tau^n) \Rightarrow f = O(\tau^{n-1}).$$

Hence, induction yields the desired result.

Corollary 5.3.1. If $f - \Lambda f = g$ and $f = O(\tau^n)$, then f is unique. Similar results hold for Λ_m with m sufficiently large.

Now consider $\omega_0 = 1$.

Theorem 5.3.2. Let $\omega_0 = 1$. If $f = \Lambda f$ where $f \ge 0$, $f \not\equiv 0$, then $f(\tau, \mu) \ge M\tau + N$ for some positive M and N.

Proof: From (1.5.4), (5.2.2) and for $\delta = \min p(\mu, \mu')$, we have

$$\frac{1}{4}(1 - \frac{1}{3}\omega_1)F\tau + K(0) = \frac{1}{2}\int_0^1 \int_\tau^\infty \mu' e^{\frac{\tau-\tau'}{\mu'}} f(\tau', \mu')d\tau'$$

$$- \frac{1}{2}\int_{-1}^0 \int_0^\tau \mu' e^{\frac{\tau-\tau'}{\mu'}} f(\tau', \mu')d\tau' \le \frac{(\Lambda f)(\tau, \mu)}{\delta} = \frac{f(\tau, \mu)}{\delta} .$$

Hence,

(5.3.1) $$f(\tau, \mu) \geq \frac{\delta}{4}(1 - \frac{1}{3}\omega_1)F\tau + K(0)\delta = M\tau + N.$$

Application of this theorem gives the following corollary.

Corollary 5.3.2. Let $\omega_0 = 1$. If $f = \Lambda f$, where $f \geq 0$, $f(\tau, \mu) = o(\tau)$ uniformly in μ as $\tau \to \infty$, then $f(\tau, \mu) \equiv 0$.

Similar results hold for Λ_m with m sufficiently large.

§ 5.4 Existence of Neumann Series Solutions

We now turn to the solution of the equation $f - \Lambda f = g$. For the analogous problem in the isotropic case Hopf (1934) showed that a Neumann series solution exists if and only if the right member is integrable in which case the Neumann series converges almost everywhere to the solution. We shall obtain similar results for the anisotropic case. Primary attention shall be given to convergence which is uniform at least on finite intervals.

The following lemmas will be useful in the discussion of the Neumann series solution $f = \sum_{n=0}^{\infty} \Lambda^n g$ of the equation $f - \Lambda f = g$.

Lemma 5.4.1. Let $\omega_0 \leq 1$. If $f = \sum_{n=0}^{\infty} \Lambda^n g$ exists and $g \geq 0$, then $f - \Lambda f = g$.

Proof: Since $f \geq 0$, Lebesgue's monotone convergence theorem implies that

$$\Lambda f = \sum_{n=0}^{\infty} \Lambda^{n+1} g = f - g, \quad \text{so} \quad f - \Lambda f = g.$$

Lemma 5.4.2. Let $\omega_0 \leq 1$. If $f - \Lambda f = g$ and $g \geq 0$, $f \geq 0$, with f bounded, then

$$f = \sum_{n=0}^{\infty} \Lambda^n g \quad \text{and} \quad \Lambda^n f \to 0 \ \text{monotonically.}$$

Proof: Now $0 \leq \Lambda f = f - g \leq f$, and $0 \leq \Lambda^{n+1} f \leq \Lambda^n f \leq f$. Hence,

$$\sum_{n=0}^{N-1} \Lambda^n g = \sum_{n=0}^{N-1} \Lambda^n (f - \Lambda f) = \sum_{n=0}^{N-1} \Lambda^n f - \sum_{n=0}^{N-1} \Lambda^{n+1} f$$

$$= \Lambda^0 f - \Lambda^N f = f - \Lambda^N f \leq f .$$

Therefore,

$$\psi = \sum_{n=0}^{\infty} \Lambda^n g$$

exists and $0 \leq \psi \leq f$, $\Lambda^n f \rightarrow f - \psi$ monotonically. By Lemma 5.4.1, $\psi - \Lambda\psi = g$, and therefore $f - \psi = \Lambda(f-\psi)$ and $0 \leq f - \psi \leq f$.

Assume $f - \psi \not\equiv 0$. Then, by Theorem 5.3.2, $f - \psi$ and f are unbounded, a contradiction. Hence, $f \equiv \psi$ and the desired results follow.

We now exhibit the Neumann series solution for the special case $f(\tau, \mu) \equiv 1$. Note that by (5.2.4), for $\omega_0 = 1$,

$$1 - \Lambda 1 = v(\tau, \mu), \tag{5.4.1}$$

where

$$v(\tau, \mu) \equiv \frac{1}{2} \int_0^1 p(\mu, -\mu') \, e^{-\frac{\tau}{\mu'}} \, d\mu' . \tag{5.4.2}$$

<u>Lemma 5.4.3.</u> If $\omega_0 = 1$, then $\sum_{n=0}^{\infty} \Lambda^n v(\tau, \mu) = 1$ and $\Lambda^n 1 \rightarrow 0$ monotonically. Moreover, the convergence is uniform in each finite interval.

<u>Proof:</u> The first two assertions follow from Lemmas 5.4.1 and 5.4.2. Since $\Lambda^n 1$ is continuous, and

$$\sum_{n=0}^{N-1} \Lambda^n v = 1 - \Lambda^N 1 ,$$

the uniform convergence follows from Dini's theorem.

Lemma 5.4.4. Let $\omega_0 = 1$. If $f - \Lambda f = g$ and f is bounded, then $f = \sum_{n=0}^{\infty} \Lambda^n g$ and $\Lambda^n g \to 0$. In both statements the convergence is uniform on each finite interval.

Proof: Let $|f| \leq M$, then

$$|f - \sum_{n=0}^{N-1} \Lambda^n g| = |\Lambda^n f| \leq M\Lambda^N 1 .$$

The desired results now follow from Lemma 5.4.3.

Lemma 5.4.5. Let $\omega_0 = 1$. If $0 \leq g(\tau, \mu) \leq Mv(\tau, \mu)$ for some $M > 0$, then the equation $f - \Lambda f = g$ has the Neumann series solution $f = \sum_{n=0}^{\infty} \Lambda^n g$, where the series converges uniformly on each finite interval. Moreover, $f \leq M$.

Proof: We have

$$\Lambda^n g \leq M\Lambda^n v.$$

Apply Lemmas 5.4.1 and 5.4.3 to obtain the desired convergence.

We point out that $g(\tau, \mu) = O(e^{-\tau})$ implies $g(\tau, \mu) = O(v(\tau, \mu))$.

Lemma 5.4.6. If $f = \Lambda f$ and $f \geq -M$, $M > 0$, then $f \geq 0$. If, furthermore $f \not\equiv 0$ a.e., then $f > 0$.

Proof: $f \geq -M \Rightarrow f = \Lambda^n f \geq -M\Lambda^n 1 \to 0$ as $n \to \infty$ by Lemma 5.4.3. Hence, $f \geq 0$. If $f \not\equiv 0$ a.e., then $f > 0$ since $f = \Lambda f$ and the kernel $K(\tau, \mu; \tau', \mu')$ is positive.

§ 5.5 The General Solution in the Conservative Case

Following Busbridge (1960), and motivated partly by (5.3.1), we seek a solution of the equation $f = \Lambda f$, $f \geq 0$, in the case $\omega_0 = 1$, which has the form

$$f(\tau, \mu) = \tau + B\mu + q(\tau, \mu), \tag{5.5.1}$$

where $B = \dfrac{\omega_1}{3-\omega_1}$, and $q(\tau, \mu)$ is bounded.

If $\omega_0 = 1$, formally substitute (5.5.1) into $f = \Lambda f$ to obtain

$$q - \Lambda q = w(\tau, \mu), \tag{5.5.2}$$

where

$$w(\tau, \mu) \equiv \frac{B+1}{2} \int_0^1 \mu' p(\mu, -\mu') e^{-\frac{\tau}{\mu'}} d\mu' . \tag{5.5.3}$$

Since $w(\tau, \mu) = O(v(\tau, \mu))$, it follows from Lemma 5.4.5 that (5.5.2) has the Neumann series solution

(5. 5. 4)
$$q = \sum_{n=0}^{\infty} \Lambda^n w(\tau, \mu),$$

where the series converges uniformly on each finite interval, and

(5. 5. 5)
$$0 < q(\tau, \mu) \leq B + 1 \quad .$$

Thus (5. 5. 1) provides a solution of $f = \Lambda f$ when $\omega_0 = 1$. Note that $f(\tau, \mu) \geq -B$. Hence, by Lemma 5. 4. 6, $f > 0$.

Theorem 5. 5. 1. Let $\omega_0 = 1$. The general solution of $\Gamma = \Lambda\Gamma$ with $\Gamma \geq 0$, $\Gamma \not\equiv 0$ is given by

(5. 5. 6)
$$\Gamma(\tau, \mu) = Af(\tau, \mu) = A[\tau + B\mu + q(\tau, \mu)]$$

where $B = \frac{\omega_1}{3 - \omega_1}$, $A > 0$, and q is given by (5. 5. 4). Moreover, $\Gamma > 0$.

Proof: It is clear by the preceding remarks that (5. 5. 6) will give such a desired solution. The uniqueness is proved in the same way Anselone (1958, p. 563) proved the analogous result in the isotropic case. In view of this, we omit the details .

Once again an analogous result holds for $\Gamma_m = \Lambda_m \Gamma_m$ with $\Gamma_m \geq 0$, $\Gamma_m \not\equiv 0$. The general solution is of the form

$$\Gamma_m(\tau, \mu) = A[\tau + B_m \mu + q_m(\tau, \mu)],$$

where B_m is constant, $A > 0$ and q_m is the Neumann series solution of

(5.5.7) $$q_m - \Lambda q_m = w_m ,$$

where

(5.5.8) $$w_m(\tau, \mu) \equiv \frac{B_m + 1}{2} \sum_j a_{mj} \mu_{mj} p(\mu, -\mu_{mj}) e^{-\frac{\tau}{\mu_{mj}}} .$$

Thus,

(5.5.9) $$q_m = \sum_{n=0}^{\infty} \Lambda_m^n w_m .$$

As previously mentioned, the flux F can be used as a normalization condition in the case of Theorem 5.5.1. Hence, we can express A in terms of F as follows: Using (1.5.3) and (1.5.4) with $\tau_0 = \infty$, dividing by τ and passing to the limit, we obtain

$$\lim_{\tau \to \infty} \frac{\frac{1}{4}(1 - \frac{1}{3}\omega_1) F \tau + K(0)}{\tau} = \lim_{\tau \to \infty} \frac{A}{2\tau} \int_0^1 \int_0^{\infty} [\tau' + B\mu' + q(\tau', \mu')] \mu' e^{-\frac{|\tau - \tau'|}{\mu'}} d\tau' d\mu'$$

or

$$\frac{1}{4}(1-\frac{1}{3}\omega_1)F = \lim_{\tau \to \infty} \frac{A}{2\tau} \int_0^1 \int_0^\infty \tau'\mu' \, e^{-\frac{|\tau-\tau'|}{\mu'}} \, d\tau' \, d\mu'$$

$$= \lim_{\tau \to \infty} \frac{A}{2\tau} [\frac{2}{3}\tau + E_5(\tau)] = \frac{A}{3} \quad .$$

Here

$$E_n(\tau) = \int_1^\infty x^{-n} e^{-\tau x} dx \qquad (n = 1, 2, \cdots)$$

is the well-known exponential integral. Hence,

$$A = \frac{3}{4}(1 - \frac{1}{3}\omega_1) F \, , \tag{5.5.10}$$

and

$$\Gamma(\tau, \mu) = \frac{3}{4}(1-\frac{1}{3}\omega_1) F [\tau + (B+1)\mu + q(\tau, \mu)] \, . \tag{5.5.11}$$

A similar result holds for $\Gamma_m(\tau, \mu)$.

We note the similarity of (5.5.11) to (2.3.14) (if b_m is replaced by (2.4.2).

CHAPTER 6

CONVERGENCE THEOREMS FOR INFINITE τ_0

§ 6. 1 Preliminary Remarks

We have discussed the solution of the integral equations (3. 2. 7) and (3. 2. 8) for the case $\tau_0 = \infty$. Hence, we turn our attention to convergence questions. Again, assume a quadrature rule which has the convergence properties given in § 4. 1.

Convergence results for the discrete ordinates method with $\tau_0 = \infty$ are obtained. The convergence is uniform on each finite τ interval if $\omega_0 = 1$ and is uniform for all τ and μ if $\omega_0 < 1$.

§ 6. 2 Convergence Theorems for Infinite τ_0

Before turning to the statement of convergence theorems for the case $\tau_0 = \infty$, we give a few preliminary results which will be useful.

Lemma 6. 2. 1. If $f \epsilon C(R)$, then $\Lambda_m f \rightarrow \Lambda f$ uniformly on $-1 \leq \mu \leq 1$, $0 \leq \tau \leq \bar{\tau} < \infty$ as $m \rightarrow \infty$, for each fixed $\bar{\tau}$.

Proof: The device used to prove that f continuous $f = O(\tau^n)$ implies Λf continuous, reduces this question to that for $\tau_0 < \infty$. This case was treated in Theorem 4. 2. 1.

Lemma 6. 2. 2. If $f \in C_1(R)$, then $||\Lambda_m f - \Lambda f|| \to 0$ as $m \to \infty$.

Proof: Choose $\varepsilon > 0$. Fix $\bar{T}$ such that

$$\left.\begin{array}{l} |(\Lambda f)(T,\mu)| < \frac{\varepsilon}{2}, \quad |(\Lambda_m f)(T,\mu)| < \frac{\varepsilon}{2}, \\ \\ |(\Lambda_m f)(T,\mu) - (\Lambda f)(T,\mu)| < \varepsilon, \end{array}\right\} \text{ for } T > \bar{T},\ m \geq 1 .$$

If $T < \bar{T}$, then apply the previous lemma. Hence, the result follows.

Lemma 6.2.3. Let $n \geq 0$. As $m \to \infty$, $||f - f_m|| \to 0$ implies that

$$(6.2.1) \quad \Lambda_m^n f_m \to \Lambda^n f \quad \begin{cases} \text{uniformly on any bounded set if } f \in C(R), \\ \\ \text{uniformly for all } T, \mu \text{ if } f \in C_1(R). \end{cases}$$

Proof: The case with $n = 0$ follows from the two preceding lemmas. Since

$$||\Lambda f - \Lambda_m f_m|| \leq ||\Lambda f - \Lambda_m f|| + ||\Lambda_m (f - f_m)||$$

$$\leq ||\Lambda f - \Lambda_m f|| + ||\Lambda_m|| \; ||f - f_m|| \to 0,$$

(6. 2. 1) holds when $n = 1$. Since

$$||\Lambda^{n+1}f-\Lambda_m^{n+1}f_m|| = ||\Lambda^n(\Lambda f)-\Lambda_m^n(\Lambda_m f_m)||,$$

induction completes the proof.

We note that

$$||\Lambda_m^n 1 - \Lambda^n 1|| \to 0 \quad \text{as} \quad m \to \infty$$

is a special case of (6.2.1).

Theorem 6.2.1. Let f and f_m be the Neumann series solutions of

$$f - \Lambda f = g, \qquad f_m - \Lambda_m f_m = g_m,$$

where $0 \le g \le Mv$, $0 \le g_m \le Mv$ and $||g_m - g|| \to 0$ as $m \to \infty$. Thus

$$f = \sum_{n=0}^{\infty} \Lambda^n g, \qquad f_m = \sum_{n=0}^{\infty} \Lambda_m^n g_m .$$

Then, for each fixed $\overline{\tau}$

$$f_m(\tau,\mu) \to f(\tau,\mu) \begin{cases} \text{uniformly for} \quad 0 \le \tau < \infty,\ -1 \le \mu \le 1 \ \text{if} \ \omega_0 < 1, \\ \text{uniformly for} \quad 0 \le \tau \le \overline{\tau} < \infty,\ -1 \le \mu \le 1 \ \text{if} \ \omega_0 = 1 . \end{cases}$$

Proof: Note that

$$f = \sum_{n=0}^{N-1} \Lambda^n g + \Lambda^N f, \qquad f_m = \sum_{n=0}^{N-1} \Lambda_m^n g_m + \Lambda_m^N f,$$

for $m \geq 1$, $N \geq 1$. Now $0 < f(\tau, \mu) \leq M$ and $0 < f_m(\tau, \mu) \leq M$ by Lemma 5.4.5. Then

Write

$$|f-f_m| \leq \left|\sum_{n=0}^{N-1} \Lambda^n g - \sum_{n=0}^{N-1} \Lambda_m^n g_m\right| + M\Lambda^N 1 + M\Lambda_m^N 1,$$

(6.2.2) $$|f-f_m| \leq \left|\left|\sum_{n=0}^{N-1} \Lambda^n g - \sum_{n=0}^{N-1} \Lambda_n^m g_m\right|\right| + M||\Lambda_m^N 1 - \Lambda^N 1|| + 2M\Lambda^N 1,$$

for $N \geq 1$. By Lemma 5.4.3, $\Lambda^N 1 \to 0$ uniformly on each finite τ interval if $\omega_0 = 1$. If $\omega_0 < 1$, then $||\Lambda^N 1|| \leq \omega_0^N \to 0$ as $N \to \infty$. Choose $\overline{\tau} > 0$ and $\varepsilon > 0$ arbitrarily. Fix N such that $\Lambda^N 1 < \varepsilon$ on $[0, \overline{\tau}]$ if $\omega_0 = 1$ and $||\Lambda^N 1|| < \varepsilon$ if $\omega_0 < 1$. For m sufficiently large,

$$||\Lambda_m^N 1 - \Lambda^N 1|| < \varepsilon,$$

$$\left|\left|\sum_{n=0}^{N-1} \Lambda^n g - \sum_{n=0}^{N-1} \Lambda_m^n g_m\right|\right| < \varepsilon .$$

The result now follows from (6.2.2).

An important application of this theorem is when $\Gamma(\tau, \mu)$ is the bounded source function corresponding to a given function $\Gamma^0(\tau, \mu) = O(e^{-\tau})$. For example, $\Gamma^0(\tau, \mu)$ may be due to reduced incident radiation. In this case we have the integral equation

(6.2.3) $$\Gamma - \Lambda\Gamma = \Gamma^0 ,$$

and the discrete ordinates approximation equation

(6.2.4) $$\Gamma_m - \Lambda_m \Gamma_m = \Gamma^0 .$$

The following theorem is a direct application of Theorem 6.2.1.

<u>Theorem 6.2.2.</u> Let $\Gamma(\tau, \mu)$, $I(\tau, \mu)$ and $J(\tau, \mu)$ be the unique bounded solutions to the given transfer problem (6.2.3) with (1.3.3) and (1.2.4). Furthermore, let $\tau_0 = \infty$ and $\Gamma^0(\tau, \mu) = O(e^{-\tau})$. If $\Gamma_m(\tau, \mu)$ satisfies (6.2.4), $I_m(\tau, \mu)$ satisfies (2.5.3) and $J_m(\tau, \mu)$ satisfies (2.2.4), then $\Gamma_m(\tau, \mu) \to \Gamma(\tau, \mu)$, $I_m(\tau, \mu) \to I(\tau, \mu)$ and $J_m(\tau, \mu) \to J(\tau, \mu)$. If $\omega_0 < 1$, the convergence is uniform for all τ and μ. If $\omega_0 = 1$, the convergence is uniform on each finite τ interval for all μ.

Another important application of Theorem 6.2.1 is to the equations for q and q_m, (5.5.2) and (5.5.7) respectively.

<u>Theorem 6.2.3.</u> Let $\omega_0 = 1$ and $\tau_0 = \infty$. Let Γ, I, J, q be the unique non-negative functions which satisfy the homogeneous transfer problem $(\Gamma^0 \equiv 0)$. Let Γ_m, I_m, J_m, q_m be the corresponding discrete ordinates approximations. Then

$$q_m \to q,$$

$$\Gamma_m \to \Gamma,$$

$$I_m \to I,$$

$$J_m \to J,$$

uniformly on each finite τ interval for all μ .

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