CONVERGENCE OF THE WICK-CHANDRASEKHAR APPROXIMATION TECHNIQUE IN RADIATIVE TRANSFER

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Chapter 1

INTRODUCTION

The distribution of radiation intensity in an emitting-absorbing-scattering atmosphere has concerned astrophysicists and applied mathematicians for over a century. The fundamental physical problem (11, pp. 65-88) leads quite naturally to the transfer equation, which is an integrodifferential equation with intensity as dependent variable and position, direction, time, and sometimes frequency, independent variables. We shall be concerned with the so-called classical transfer problem, which pertains to the distribution of intensity in a plane parallel atmosphere under the assumptions of time- and frequency-independence, isotropic scattering and no emission or absorption by the medium. This problem is described below. For more precise definitions of the physical terms involved and for details of arguments see (4, pp. 1-15) or (9, pp. 1-24).

§1.1 The Classical Transfer Problem

Suppose that the atmosphere under consideration is bounded by a plane surface S, that p is an arbitrary point in the medium, and that d is an arbitrary unit vector bound to p. Let τ be the normal optical distance from p to S and μ be the cosine of the angle between d and the outward unit normal to S. Then the intensity at p in the direction of d is defined as the energy per second per unit frequency (or in all frequencies) passing through a neighborhood of p per unit area perpendicular to d per unit solid angle about d. Because of the symmetry of the medium the intensity may be considered as a function of τ and μ; it is usually denoted by $I(\tau, \mu)$. The general transfer equation becomes, in this case,

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{1} I(\tau, \mu') \, d\mu' . \tag{1}$$

The above and all other integrals will be understood to have the sense of Lebesgue.

The integral term in (1) is denoted by $J(\tau)$ and is called the average intensity. Another important integral associated with $I(\tau, \mu)$ is the net flux,

$$F = 2 \int_{-1}^{1} I(\tau, \mu) \, \mu d\mu . \tag{2}$$

Since termwise integration of (1) yields $\frac{\partial F}{\partial \tau} = 0$, F is independent of τ and therefore is constant for each solution of (1).

In the classical transfer problem the atmosphere is of infinite optical depth, i. e., $0 \leq \tau < \infty$. The boundary conditions are

$$\begin{aligned} &I(0, \mu) = 0, && \mu < 0, \\ &\lim_{\tau \to 0} e^{-\tau/\mu} I(\tau, \mu) = 0, && \mu > 0. \end{aligned} \tag{3}$$

The first statement expresses the condition that no radiation enters the atmosphere through the surface S. The assumption of an everywhere finite non-negative solution of (1) leads to the condition (7, pp. 20-21) that $e^{-\tau/\mu} I(\tau, \mu) \to 0$ as $\tau \to \infty$ for almost all $\mu > 0$. It is customary to omit the phrase "almost all" since, as it turns out, a solution of the problem given by (1), the above boundary condition at $\tau = 0$, and the weaker condition at $\tau = \infty$ also satisfies the stronger condition at $\tau = \infty$ given in (3).

Since the integrodifferential equation and the boundary conditions are homogeneous, each multiple of a solution of (1) and (3) is also a solution. Because of this fact the net flux F in (2) may be considered as an arbitrary positive constant; thus a normalization condition is obtained.

Following is a complete mathematical statement of the classical transfer problem. We seek a function $I(\tau, \mu)$, defined for $0 \leqq \tau < \infty$, $-1 \leqq \mu \leqq 1$, such that

$$I(\tau, \mu) \geqq 0, \qquad I(\tau, \mu) \not\equiv 0, \tag{4}$$

which satisfies the integrodifferential equation,

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{1} I(\tau, \mu') \, d\mu', \tag{5}$$

the boundary conditions,

$$\begin{aligned} I(0, \mu) &= 0, & \mu < 0, \\ \lim_{\tau \to \infty} e^{-\tau/\mu} I(\tau, \mu) &= 0, & \mu > 0, \end{aligned} \tag{6}$$

and the normalization condition,

$$F = 2 \int_{-1}^{1} I(\tau, \mu) \, \mu \, d\mu, \tag{7}$$

where F is an arbitrary preassigned positive constant. Conditions on differentiability and integrability of $I(\tau, \mu)$ are implicit in (5).

§1.2 An Equivalent Formulation of the Classical Transfer Problem

From (5) and the equation for the average intensity,

$$J(\tau) = \frac{1}{2} \int_{-1}^{1} I(\tau, \mu)\, d\mu, \tag{8}$$

it follows that

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - J(\tau). \tag{9}$$

When μ is fixed, $\mu \neq 0$, (9) is a first order ordinary differential equation in $I(\tau, \mu)$ with constant coefficients. The general solution is

$$I(\tau, \mu) = I(\tau', \mu)\, e^{(\tau - \tau')/\mu} - \int_{\tau'}^{\tau} e^{(\tau - t)/\mu} J(t) \frac{dt}{\mu}, \quad \mu \neq 0, \tag{10}$$

where τ and τ' are arbitrary. It follows from (9), (10) and (6) that

$$\left.\begin{aligned} I(\tau, 0) &= J(\tau), \\ I(\tau, \mu) &= \int_{0}^{\tau} e^{(\tau - t)/\mu} J(t) \frac{dt}{-\mu}, \quad \mu < 0, \\ I(\tau, \mu) &= \int_{\tau}^{\infty} e^{(\tau - t)/\mu} J(t) \frac{dt}{\mu}, \quad \mu > 0. \end{aligned}\right\} \tag{11}$$

The existence of a function, $I(\tau, \mu)$, which satisfies (4), (5) and (6) implies the existence of the improper integral in (11).

From (4), (8) and (11),

$$J(\tau) \geqq 0, \qquad J(\tau) \not\equiv 0. \tag{12}$$

The following theorem provides an equivalent formulation of the classical transfer problem. Since the proof is almost immediate, it is omitted.

Theorem 1. The classical transfer problem has a solution if and only if there exist functions $I(\tau, \mu)$ and $J(\tau)$ which satisfy (7), (8), (11) and (12).

§1.3 Methods of Solution

The substitution of (11) into (8) yields

$$J(\tau) = \frac{1}{2} \int_{-1}^{0} \int_{0}^{\tau} e^{(\tau - t)/\mu} J(t) \frac{dt}{-\mu} d\mu + \frac{1}{2} \int_{0}^{1} \int_{\tau}^{\infty} e^{(\tau - t)/\mu} J(t) \frac{dt}{\mu} d\mu.$$

Reversing the order of integration in each of the above double integrals and then replacing μ by $-\mu$ in the first integral, we obtain

$$J(\tau) = \frac{1}{2} \int_{0}^{\tau} \int_{0}^{1} e^{(t - \tau)/\mu} \frac{d\mu}{\mu} J(t)\, dt + \frac{1}{2} \int_{\tau}^{\infty} \int_{0}^{1} e^{(\tau - t)/\mu} \frac{d\mu}{\mu} J(t)\, dt.$$

Similarly, the substitution of (11) into (7) yields

$$F = -2 \int_{0}^{\tau} \int_{0}^{1} e^{(t - \tau)/\mu} d\mu\, J(t)\, dt + 2 \int_{\tau}^{\infty} \int_{0}^{1} e^{(\tau - t)/\mu} d\mu\, J(t)\, dt.$$

These equations are expressed more compactly as

$$J(\tau) = \int_{0}^{\infty} K_1(\tau - t)\, J(t)\, dt, \tag{13}$$

$$F = -4 \int_0^{\tau} K_2(\tau - t)\, J(t)\, dt + 4 \int_{\tau}^{\infty} K_2(\tau - t)\, J(t)\, dt, \qquad (14)$$

where

$$K_r(x) = \frac{1}{2} \int_0^1 e^{-|x|/\mu} \mu^{r-2}\, d\mu, \qquad \begin{matrix} r = 1, & |x| > 0, \\ r \geqq 2, & |x| \geqq 0. \end{matrix} \qquad (15)$$

The functions $K_r(x)$, $r \geqq 3$, are introduced for later convenience. Equation (13) is known as the Schwarzschild-Milne integral equation for $J(\tau)$.

If (11) and (12) are assumed, (8) $\Longleftrightarrow$ (13) and (7) $\Longleftrightarrow$ (15). In view of Theorem 1, we have another formulation of the classical transfer problem.

Theorem 2. The classical transfer problem has a solution if and only if there exists a function $J(\tau)$, which satisfies (12), (13) and (14), in which case $I(\tau, \mu)$ is given by (11).

Hopf (5, p. 381) obtained a solution, $J(\tau)$, to the problem defined in Theorem 2. This solution is presented in Chapter 3 below. Hopf also proved (6, pp. 155-161) that his solution is unique. Hence, there is a unique solution, $I(\tau, \mu)$, to the classical transfer problem.

Since Hopf's expression for $J(\tau)$ is in the form of a slowly converging infinite series, the terms of which are successively more difficult to calculate, it is not used to obtain numerical results. A more important limitation of Hopf's method is that is generalizes to provide solutions of only a limited class of transfer problems. For these reasons and for their own intrinsic interest, other methods of attacking the classical transfer problem have been developed (9, pp. 86-225). A method of successive approximations due to Wick (16, pp. 702-710) and extended by Chandrasekhar (1, pp. 76-79 and 2, pp. 117-125) involves replacing the integrals in (5) and (7) by the sums corresponding to a particular quadrature formula and solving the resulting problem. If the number of subdivision points is varied a sequence of approximations, $I_n(\tau, \mu)$, may be obtained.

The convergence of the Wick-Chandrasekhar approximation technique is the main concern of this thesis. Chandrasekhar assumed (1, p. 84) that the sequence of which he derived converges to $I(\tau, \mu)$, but he apparently made no attempt to construct a proof. However, he did suggest (3, p. 189) a method for proving the convergence of $I_n(0, \mu)$ to $I(0, \mu)$. This restricted convergence question was also considered by Kourganoff (9, pp. 153-159) who obtained several minor results but no convergence theorem. Since the approach employed by Chandrasekhar and Kourganoff seems to involve insurmountable mathematical difficulties other methods have been devised by this author for dealing with the convergence problem.

§1.4 Summary of Later Chapters

The Wick-Chandrasekhar technique is generalized in Chapter 2 to apply to a certain class of quadrature formulas. The details of the analysis coincide in many respects to those of the treatment of Chandrasekhar (4, pp. 70-79).

In §1.3 above we formulated a problem which involved $J(\tau)$ but not $I(\tau, \mu)$. The same procedure is used in Chapter 3 to obtain a problem for the approximate average intensity, $J_n(\tau)$, which does not involve $I_n(\tau, \mu)$. Then a quite general problem, which includes these two problems as special cases, is defined and solved. The solution reduces to a pair of very similar expressions for $J(\tau)$ and $J_n(\tau)$.

Both Wick and Chandrasekhar used the Gauss quadrature formula for 2n subdivision points in the interval $-1 \leqq \mu \leqq 1$. Sykes (14, pp. 377-386) used the n point Gauss formula separately in each of the sub-intervals, $-1 \leqq \mu \leqq 0$ and $0 \leqq \mu \leqq 1$. Details of these quadrature formulas and their applications are recorded in Chapter 4.

In Chapter 5 it is proved that if either the Gauss or the double-Gauss quadrature formula is used, the corresponding sequences $\left\{J_n(\tau)\right\}$ and $\left\{I_n(\tau, \mu)\right\}$ converge to $J(\tau)$ and $I(\tau, \mu)$ respectively. The convergence is uniform for $-1 \leqq \mu \leqq 1$ and $0 \leqq \tau \leqq \tau_o < \infty$, where τ_o is arbitrary.

Chapter 2

GENERALIZATION OF THE WICK-CHANDRASEKHAR TECHNIQUE

Wick and Chandrasekhar replaced the integrals of the classical transfer problem by the sums corresponding to the 2n point Gauss quadrature formula, $n \geq 1$, and solved the resulting problem to obtain a sequence of approximations, $\{I_n(\tau, \mu)\}$, to $I(\tau, \mu)$. In the arguments establishing the existence and the uniqueness of the functions, $I_n(\tau, \mu)$, certain properties of the Gauss quadrature formulas are essential. Kourganoff and Pecker observed (8, p. 248) that the Wick-Chandrasekhar analysis goes through without change for the class of quadrature formulas having these properties. The class of quadrature formulas specified by Kourganoff and Pecker is generalized in the presentation given below. The derivation of the approximations, $I_n(\tau, \mu)$, turns out to be somewhat more complicated than that of Wick and Chandrasekhar in the special case they considered.

In the remainder of this chapter assume that n is an arbitrary fixed positive integer.

§2.1 The Problem for $I_n(\tau, \mu)$

Choose subdivision points, μ_{ni}, and coefficients, a_{ni}, for $i = \pm 1, \ldots, \pm n$, and define the following correspondence for an arbitrary Lebesgue integrable function, $f(\mu)$, defined for $-1 \leq \mu \leq 1$:

$$\int_{-1}^{1} f(\mu)\, d\mu \quad \sim \quad \sum_i a_{ni}\, f(\mu_{ni}). \qquad (16)$$

The range of the summation index in (16) is $i = \pm 1, \ldots, \pm n$. The same summation convention will be followed throughout this chapter. Similarly, whenever any statement involving the μ_{ni} or the a_{ni} appears it will be understood, unless otherwise stated, that it holds for $i = \pm 1, \ldots, \pm n$.

It is assumed that the μ_{ni} and the a_{ni} satisfy the conditions,

$$-1 \leqq \mu_{n,-n} < \ldots < \mu_{n,-1} < 0 < \mu_{n1} < \ldots < \mu_{nn} \leqq 1, \tag{17}$$

$$a_{ni} > 0, \tag{18}$$

$$\sum_i a_{ni}\,\mu_{ni}^m = \frac{1+(-1)^m}{m+1}, \quad m = 0,\ 1,\ 2. \tag{19}$$

Condition (19) is equivalent to the assumption that the correspondence in (16) is an equality for $f(\mu) = \mu^m$, $m = 0,\ 1,\ 2$. Examples of quadrature formulas for which (17), (18) and (19) are satisfied are given in Chapter 4.

Replace the integrals occurring in the basic problem by the sums corresponding to them by (16). It is shown below that the resulting problem has a unique solution, $I_n(\tau, \mu)$, which depends parametrically on the μ_{ni} and the a_{ni}. Thus, $I_n(\tau, \mu)$ is defined for $0 \leqq \tau < \infty$, $-1 \leqq \mu \leqq 1$, and satisfies the relations,

$$I_n(\tau, \mu) \geqq 0, \qquad\qquad I_n(\tau, \mu) \not\equiv 0, \tag{20}$$

$$\mu \frac{\partial I_n(\tau, \mu)}{\partial \tau} = I_n(\tau, \mu) - \tfrac{1}{2} \sum_j a_{nj}\, I_n(\tau, \mu_{nj}), \tag{21}$$

$$I_n(0, \mu) = 0, \qquad\qquad \mu < 0, \tag{22}$$

$$\lim_{\tau \to \infty} e^{-\tau/\mu} I_n(\tau, \mu) = 0, \qquad \mu > 0, \tag{23}$$

$$F = 2 \sum_i a_{ni} I_n(\tau, \mu_{ni}) \mu_{ni}. \tag{24}$$

The same constant, F, is used to normalize $I(\tau, \mu)$ and $I_n(\tau, \mu)$. The condition, implicit in (24), that $2 \sum_i a_{ni} I_n(\tau, \mu_{ni}) \mu_{ni}$ is independent of τ is also a consequence of (19) and (21). Thus, let $\mu = \mu_{ni}$ in (21), multiply termwise by $2a_{ni}$ and sum on i to obtain

$$2 \sum_i a_{ni} \mu_{ni} \frac{\partial I_n(\tau, \mu_{ni})}{\partial \tau} = 2 \sum_i a_{ni} I_n(\tau, \mu_{ni}) - \sum_j a_{nj} I_n(\tau, \mu_{nj}) \sum_i a_{ni}. \tag{25}$$

By (19), $\sum_i a_{ni} = 2$. Therefore, the right member of (25) is zero and

$$\frac{\partial}{\partial \tau} \left[2 \sum_i a_{ni} I_n(\tau, \mu_{ni}) \mu_{ni} \right] = 0. \tag{26}$$

§2.2 An Equivalent Formulation of the Problem for $I_n(\tau, \mu)$

By analogy with $J(\tau)$ we define

$$J_n(\tau) = \tfrac{1}{2} \sum_i a_{ni} I_n(\tau, \mu_{ni}). \tag{27}$$

Equation (21) can now be written as

$$\mu \frac{\partial I_n(\tau, \mu)}{\partial \tau} = I_n(\tau, \mu) - J_n(\tau). \tag{28}$$

Proceeding as in §1.2, we solve equation (28) to obtain

$$\left.\begin{aligned} & I_n(\tau, 0) = J_n(\tau), \\ & I_n(\tau, \mu) = \int_0^{\tau} e^{(\tau - t)/\mu} J_n(t) \frac{dt}{-\mu}, \qquad \mu > 0, \\ & I_n(\tau, \mu) = \int_{\tau}^{\infty} e^{(\tau - t)/\mu} J_n(t) \frac{dt}{\mu}, \qquad \mu > 0. \end{aligned}\right\} \tag{29}$$

The existence of a function, $I_n(\tau,\mu)$, satisfying (20)-(24) implies the existence of the improper integral in (29).

From (18), (20) and (27), $J_n(\tau) \geqq 0$. By (29), $J_n(\tau) \equiv 0$ implies that $I_n(\tau,\mu) \equiv 0$, which contradicts (20). Hence,

$$J_n(\tau) \geqq 0, \qquad J_n(\tau) \not\equiv 0. \tag{30}$$

The following theorem is stated without proof. It is a new result.

Theorem 3. The set of equations, $\{(24), (27), (29), (30)\}$, provides an equivalent formulation of the problem for $I_n(\tau,\mu)$.

§2.3 An Expression for $I_n(\tau,\mu_{ni})$

The substitution of $\mu = \mu_{ni}$ in (21) yields a system of 2n ordinary differential equations,

$$\mu_{ni} \frac{dI_n(\tau,\mu_{ni})}{d\tau} = I_n(\tau,\mu_{ni}) - \tfrac{1}{2} \sum_j a_{nj}\, I_n(\tau,\mu_{nj}), \tag{31}$$

for the 2n unknown functions of τ, $I_n(\tau,\mu_{ni})$, $i = \pm 1, \ldots, \pm n$. Since the coefficients are constants, we seek solutions of the form

$$I_n(\tau,\mu_{ni}) = C_{ni}\, e^{-k\tau}. \tag{32}$$

Substituting (32) into (31) and solving for C_{ni}, we find

$$C_{ni} = \frac{B_n}{1+\mu_{ni}k}, \tag{33}$$

where

$$B_n = \tfrac{1}{2} \sum_i a_{ni}\, C_{ni}. \tag{34}$$

From (33) and (34),

$$B_n = \frac{1}{2} \sum_i \frac{a_{ni} B_n}{1 + \mu_{ni} k},$$

$$1 = \frac{1}{2} \sum_i \frac{a_{ni}}{1 + \mu_{ni} k}. \qquad (35)$$

For each root, k, of the characteristic equation, (35), there is a corresponding solution,

$$I_n(\tau, \mu_{ni}) = \frac{B_n e^{-k\tau}}{1 + \mu_{ni} k}, \qquad (36)$$

of (31). Since the system, (31), is homogeneous, B_n is arbitrary.

Equation (35) is now expressed as $\Psi_n(k) = 0$, where

$$\Psi_n(k) = 1 - \frac{1}{2} \sum_i \frac{a_{ni}}{1 + \mu_{ni} k} \qquad (37)$$

If the fractions are cleared in (37), a polynomial of degree $\leq 2n$ results. Therefore, there are at most 2n roots of the characteristic equation.

From (37) and (19) with m = 0 we obtain

$$\Psi_n(k) = \frac{1}{2} \sum_i a_{ni} - \frac{1}{2} \sum_i \frac{a_{ni}}{1 + \mu_{ni} k} = \frac{1}{2} \sum_i a_{ni} \left[1 - \frac{1}{1 + \mu_{ni} k} \right],$$

$$\Psi_n(k) = \frac{1}{2} k \sum_i \frac{a_{ni} \mu_{ni}}{1 + \mu_{ni} k}. \qquad (38)$$

It follows from (38) and (19) with m = 1 that k = 0 is at least a double root of the characteristic equation. If n = 1, there is a double root, viz., k = 0, and no other roots.

For n > 1 consider $\Psi_n(k)$ expressed in the form

$$\Psi_n(k) = \tfrac{1}{2}k \sum_i \frac{a_{ni}}{(1/\mu_{ni}) + k}. \tag{39}$$

We see that $\Psi_n(k)$ is finite except for the points $k = -1/\mu_{ni}$ which, by (17), satisfy the inequalities,

$$-1/\mu_{n1} < \ldots < -1/\mu_{nn} \leqq -1, \qquad 1 \leqq -1/\mu_{n,-n} < \ldots < -1/\mu_{n,-1}. \tag{40}$$

It follows from (18) and (39) that $\Psi_n(k) \to \mp \infty$ as $k \to (-1/\mu_{ni}) \pm 0$, $i = 1, \ldots, n$, and that $\Psi_n(k) \to \pm \infty$ as $k \to (-1/\mu_{ni}) \pm 0$, $i = -1, \ldots, -n$.

Consider the open intervals,

$$\begin{aligned} O_{ni} &= (-1/\mu_{ni}, -1/\mu_{n,i+1}), \qquad i = 1, \ldots, n-1, \\ O_{ni} &= (-1/\mu_{n,i-1}, -1/\mu_{ni}), \qquad i = -1, \ldots, -(n-1). \end{aligned} \tag{41}$$

For k in O_{ni}, $\Psi_n(k)$ is continuous, tends to $+\infty$ as k approaches one end-point and tends to $-\infty$ as k approaches the other end-point. Therefore, $\Psi_n(k)$ vanishes for at least one point in each of the $2n-2$ intervals O_{ni}. Since $\Psi_n(k)$ has at most $2n$ roots and $k = 0$ is a double root, each interval O_{ni} contains exactly one root. Thus, $\Psi_n(k)$ has $2n$ roots. We denote the $2n-2$ non-zero roots by $k_{n\alpha}$, $\alpha = \pm 1, \ldots, \pm(n-1)$, where

$$k_{n,-(n-1)} < \ldots < k_{n,-1} < -1, \qquad 1 < k_{n1} < \ldots < k_{n,n-1}. \tag{42}$$

The $k_{n\alpha}$ and the numbers $-1/\mu_{ni}$ of (40) satisfy

$$(-1/\mu_{ni}) - k_{n\alpha} \begin{cases} > 0 \text{ if } \alpha < -(n-i), \\ < 0 \text{ if } \alpha \geqq -(n-i), \end{cases} \qquad i = 1, \ldots, n, \tag{43}$$

$$(-1/\mu_{ni}) - k_{n\alpha} \begin{cases} > 0 \text{ if } \alpha \leqq n+i, \\ < 0 \text{ if } \alpha > n+i, \end{cases} \qquad i = -1, \ldots, -n. \tag{44}$$

This completes the solution of the characteristic equation.

The $2n-1$ sets of functions given by (36) for $k = 0$ and $k = k_{n\alpha}$

comprise a set of 2n linearly independent solutions of (31). Since (36) reduces for $k = 0$ to $I_n(\tau, \mu_{ni}) = B_n$, B_n arbitrary, and since $k = 0$ is a double root of the characteristic equation, a solution of the form $I_n(\tau, \mu_{ni}) = \tau + d_{ni}$ is indicated. The substitution of this equation into (31) yields $d_{ni} = \mu_{ni}$. Thus, $I_n(\tau, \mu_{ni}) = \tau + \mu_{ni}$ satisfies (31).

The general solution of (31) is given by an arbitrary linear combination of the 2n linearly independent solutions which we have found. It is convenient to express the general solution in the form,

$$I_n(\tau, \mu_{ni}) = b_n \left[\tau + \mu_{ni} + Q_n + \sum_{\alpha=\pm 1}^{\pm(n-1)} \frac{L_{n\alpha}\, e^{-k_{n\alpha}\tau}}{1 + \mu_{ni}\, k_{n\alpha}} \right]. \tag{45}$$

Since the numbers $-1/\mu_{ni}$ and $k_{n\alpha}$ are interlaced, none of the denominators in (45) vanishes. If $n = 1$, the summation term in (45) and subsequent equations is not present.

§2.4 Determination of the Constants

The conditions (20), (22), (23) and (24) will now be imposed. From (45) and (24),

$$F = 2b_n \left[(\tau + Q_n) \sum_{i=\pm 1}^{\pm n} a_{ni}\, \mu_{ni} + \sum_{i=\pm 1}^{\pm n} a_{ni}\, \mu_{ni}^2 + \sum_{\alpha=\pm 1}^{\pm(n-1)} L_{n\alpha}\, e^{-k_{n\alpha}\tau} \sum_{i=\pm 1}^{\pm n} \frac{a_{ni}\, \mu_{ni}}{1 + \mu_{ni}\, k_{n\alpha}} \right].$$

This equation reduces by means of (19) and (38) to $F = 4b_n/3$.

Thus,

$$b_n = 3F/4, \tag{46}$$

where, it will be recalled, F is an arbitrary positive constant.

From (23), (45) and the fact that $b_n \neq 0$,

$$\lim_{\tau\to\infty}\left[(\tau+\mu_{ni}+Q_n)\, e^{-\tau/\mu_{ni}} + \sum_{\alpha=\pm 1}^{\pm(n-1)} \frac{L_{n\alpha}\, e^{\left[(-1/\mu_{ni})-k_{n\alpha}\right]\tau}}{1+\mu_{ni}\, k_{n\alpha}}\right] = 0 \tag{47}$$

for $i = 1, \ldots, n$. For $n = 1$, the sum is not present and (47) is satisfied for an arbitrary value of Q_1. Consider $n > 1$. The signs of the exponential terms in (47) are given by (43). For $i = 1$, (47) is satisfied with arbitrary Q_n and $L_{n\alpha}$. For $i = 2$, the exponential in (47) corresponding to $\alpha = -(n-1)$ tends to $+\infty$ and all other terms tend to zero as $\tau\to\infty$. Therefore, for $i = 2$, (47) is satisfied if and only if $L_{n,-(n-1)} = 0$. Considering in turn $i = 3, \ldots, n$, we obtain the result that (47) is satisfied if and only if

$$L_{n\alpha} = 0, \qquad \alpha < 0. \tag{48}$$

The constants Q_n and $L_{n\alpha}$, $\alpha > 0$, are still arbitrary.

Equation (22) implies that $I_n(0, \mu_{ni}) = 0$, $i = -1, \ldots, -n$. By (45), (48), and the fact that $b_n \neq 0$, this condition can also be expressed as

$$S_n(\mu_{ni}) = 0, \qquad i = -1, \ldots, -n, \tag{49}$$

where

$$S_n(\mu) \equiv \mu + Q_n + \sum_{\alpha=1}^{n-1} \frac{L_{n\alpha}}{1+\mu\, k_{n\alpha}} = \mu + Q_n + \sum_{\alpha=1}^{n-1} \frac{L_{n\alpha}/k_{n\alpha}}{\mu + (1/k_{n\alpha})}. \tag{50}$$

For $n = 1$, (49) and (50) yield

$$Q_1 = -\mu_{1,-1}, \qquad 0 < Q_1 \leq 1. \tag{51}$$

Consider $n > 1$. The function,

$$S_n(\mu) \prod_{\beta=1}^{n-1} (\mu + \frac{1}{k_{n\beta}}) =$$

$$(\mu + Q_n) \prod_{\beta=1}^{n-1} (\mu + \frac{1}{k_{n\beta}}) + \sum_{\alpha=1}^{n-1} \frac{L_{n\alpha}}{k_{n\alpha}} \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n-1} (\mu + \frac{1}{k_{n\beta}}), \tag{52}$$

is a polynomial in μ with μ^n as the term of highest degree. Therefore, (49) implies

$$S_n(\mu) \prod_{\beta=1}^{n-1} (\mu + \frac{1}{k_{n\beta}}) = \prod_{i=-1}^{-n} (\mu - \mu_{ni}), \tag{53}$$

$$S_n(\mu) = \prod_{i=-1}^{-n} (\mu - \mu_{ni}) \Big/ \prod_{\alpha=1}^{n-1} (\mu + \frac{1}{k_{n\alpha}}) . \tag{54}$$

Conversely, (54), implies (49). According to the partial fractions decomposition theorem, $S_n(\mu)$ as expressed by (54) has a unique representation in the form of the third member of (50). Thus, there exist unique values of Q_n and $L_{n\alpha}$ such that (54) and (49) are satisfied. Expressions for these values are obtained as follows.

From (52) and (53) equating coefficients of μ^{n-1} in the right members, we obtain

$$Q_n = - \sum_{i=-1}^{-n} \mu_{ni} - \sum_{\alpha=1}^{n-1} \frac{1}{k_{n\alpha}}. \tag{55}$$

Although (55) was derived for $n > 1$, it also yields the correct result for Q_1. Referring first to (50) and then to (54), we obtain

$$L_{n\alpha} = k_{n\alpha} \lim_{\mu \to (-1/k_{n\alpha})} \left[\mu + (1/k_{n\alpha}) \right] S_n(\mu),$$

$$L_{n\alpha} = k_{n\alpha} \prod_{i=-1}^{-n} \left(\frac{1}{k_{n\alpha}} + \mu_{ni} \right) \Bigg/ \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n-1} \left(\frac{1}{k_{n\alpha}} - \frac{1}{k_{n\beta}} \right), \quad \alpha > 0. \tag{56}$$

This completes the determination of the constants.

By (44) and (55),

$$0 < -\mu_{n,-1} < Q_n < -\mu_{n,-n} \leq 1, \qquad n > 1. \tag{57}$$

By (44), there are precisely α negative factors in the numerator in the right member of (56) and $\alpha - 1$ negative factors in the denominator. Therefore, since none of the factors is zero,

$$L_{n\alpha} < 0, \qquad \alpha > 0. \tag{58}$$

§2.5 Final Expressions for $J_n(\tau)$ and $I_n(\tau, \mu)$

Substituting (46) and (48) into (45), we obtain

$$I_n(\tau, \mu_{ni}) = \frac{3}{4} F \left[\tau + \mu_{ni} + Q_n + \sum_{\alpha=1}^{n-1} \frac{L_{n\alpha} e^{-k_{n\alpha} \tau}}{1 + \mu_{ni} k_{n\alpha}} \right], \tag{59}$$

where Q_n is given by (55) and the $L_{n\alpha}$ are given by (56). Substituting (59) into (27) and referring to (19) and (37), we obtain

$$J_n(\tau) = \frac{3}{4} F \left[\tau + Q_n + \prod_{\alpha=1}^{n-1} L_{n\alpha} e^{-k_{n\alpha} \tau} \right]. \tag{60}$$

Suppose that $I'_n(\tau, \mu)$ and $J'_n(\tau)$ are functions which satisfy (20) - (24) and (27). Since these conditions are used to determine the $I_n(\tau, \mu_{ni})$

of (59), we must have $I'_n(\tau, \mu_{ni}) = I_n(\tau, \mu_{ni})$. It then follows from (27) and (60) that $J'_n(\tau) = J_n(\tau)$.

From (60), (58) and (42),

$$J_n(\tau) \text{ is monotone increasing, uniformly continuous, asymptotic to } \tfrac{3}{4}F\left[\tau + Q_n\right] \text{ as } \tau \to \infty, \tag{61}$$

and from (60), (50), (51) and (54), $J_1(0) = \frac{3}{4}F\, Q_1 > 0$ and

$$J_n(0) = \tfrac{3}{4}F\, S_n(0) = \tfrac{3}{4}F \prod_{i=-1}^{-n} (-\mu_{ni}) \prod_{\alpha=1}^{n-1} k_{n\alpha} > 0, \quad n > 1.$$

Therefore, by (51), (57) and (61),

$$0 < J_n(\tau) \leqq \tfrac{3}{4}F\left[\tau + 1\right]. \tag{62}$$

There is equality in (62) only when $n = 1$ and $Q_1 = -\mu_{1,-1} = 1$.

The functions $I_n(\tau, \mu_{ni})$ and $J_n(\tau)$ derived above satisfy conditions (24), (27) and (30). It follows from Theorem 3 and the remarks following (60) that the substitution of (60) into (29) yields the unique solution to the original problem for $I_n(\tau, \mu)$. (By (62), the improper integral in (29) exists). In order to express $I_n(\tau, \mu)$ in compact form we introduce

$$G_n(\tau, \mu) = \tfrac{3}{4}F\left[\tau + \mu + Q_n + \sum_{\alpha=1}^{n-1} \frac{L_{n\alpha}\, e^{-k_{n\alpha}\tau}}{1 + \mu\, k_{n\alpha}}\right], \tag{63}$$

$$G_{n\alpha}(\tau, \mu) = \tfrac{3}{4}F\left[\tau + \mu + Q_n + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n-1} \frac{L_{n\beta}\, e^{-k_{n\beta}\tau}}{1 + \mu\, k_{n\beta}}\right]. \tag{64}$$

In terms of these functions, (29) yields

$$I_n(\tau, \mu) = G_n(\tau, \mu), \qquad \mu \geqq 0, \tag{65}$$

$$I_n(\tau, \mu) = G_n(\tau, \mu) - e^{\tau/\mu} G_n(0, \mu), \quad \mu < 0, \quad \mu \neq -1/k_{n\alpha}, \tag{66}$$

$$I_n(\tau, -1/k_{n\alpha}) = G_{n\alpha}(\tau, -1/k_{n\alpha}) - e^{-k_{n\alpha}\tau} G_{n\alpha}(0, -1/k_{n\alpha}) + \frac{3}{4} F\, k_{n\alpha}\, L_{n\alpha}\, e^{-k_{n\alpha}\tau}, \qquad \alpha > 0. \tag{67}$$

It follows easily from (60) and (29) that

$$I_n(\tau, \mu) = \int_0^{-\tau/\mu} e^{-x} J_n(\tau + \mu x)\, dx, \qquad \tau \geqq 0,\ \mu < 0, \quad \tau > 0,\ \mu = 0,$$

$$I_n(\tau, \mu) = \int_0^{\infty} e^{-x} J_n(\tau + \mu x)\, dx, \qquad \tau \geqq 0,\ \mu \geqq 0. \tag{68}$$

From (62) and (68), we have

$I_n(\tau, \mu)$ is continuous except for $\tau = 0$, $\mu = 0$. (69)

According to (22), $I_n(0, \mu) \to 0$ as $\mu \to 0-$, while, by (29), (61) and (62), $I_n(\tau, 0) = J_n(\tau) \to J_n(0) > 0$ as $\tau \to 0$. If μ is replaced by $-1/k_{n\alpha}$ in (66) the form $\infty - \infty$ results. In view of (69), the standard technique for evaluating this form yields the correct value for $I_n(\tau, \mu)$.

Let us verify that the quantities $I_n(\tau, \mu_{ni})$ obtained from (65) and (66) and the corresponding quantities of (59) are equal. By (59) and (63), $I_n(\tau, \mu_{ni}) = G_n(\tau, \mu_{ni})$, $i = \pm 1, \ldots, \pm n$. Equations (65) and (66) yield the same results since, by (63), (50) and (49), $G_n(0, \mu_{ni}) = \frac{3}{4} F\, S_n(\mu_{ni}) = 0$ for $i = -1, \ldots, -n$. This agreement was anticipated.

If we had not assumed condition (19) for $m = 2$, the preceding analysis would be unchanged except that $3F/4$ would be replaced by

$(2 \sum_i a_{ni}\mu_{ni}^2)^{-1}$ F in expressions (46) and following.

The main results of this chapter are summarized below.

Theorem 4. For each 2n-point quadrature formula of the form of (16) for which conditions (17), (18) and (19) are satisfied, there corresponds an approximation, $I_n(\tau, \mu)$, to $I(\tau, \mu)$. The function, $I_n(\tau, \mu)$, which is given by (65), (66) and (67), is the unique solution to the problem defined by the conditions (20) - (24).

Chapter 3

GENERALIZATION OF THE SCHWARZCHILD-MILNE INTEGRAL EQUATION

It is stated in §1.3 that the Schwarzchild-Milne integral equation and certain auxiliary conditions determine $J(\tau)$. An analogous problem for $J_n(\tau)$ is derived below. Then a generalization of these problems for $J(\tau)$ and $J_n(\tau)$ is defined and solved. The solution reduces in one case to the series expression for $J(\tau)$ derived by Hopf and in the other case it yields a very similar series expression for $J_n(\tau)$. These expressions are used in Chapter 5 to obtain convergence theorems for the sequences $\{J_n(\tau)\}$ and $\{I_n(\tau, \mu)\}$.

§3.1 The Integral Equations for $J_n(\tau)$

According to (30),

$$J_n(\tau) \geqq 0, \qquad\qquad J_n(\tau) \not\equiv 0. \tag{70}$$

The substitution of (29) into (27) yields

$$J_n(\tau) = \frac{1}{2}\int_0^\tau \sum_{i=-1}^{-n} a_{ni}\, e^{(\tau-t)/\mu_{ni}} (-\mu_{ni})^{-1} J_n(t)\, dt$$

$$+ \frac{1}{2}\int_\tau^\infty \sum_{i=1}^{n} a_{ni}\, e^{(\tau-t)/\mu_{ni}} (\mu_{ni})^{-1} J_n(t)\, dt.$$

Similarly, the substitution of (29) into (24) yields F =

$$-2\int_0^\tau \sum_{i=-1}^{-n} a_{ni} e^{(\tau-t)/\mu_{ni}} J_n(t)\, dt + 2\int_\tau^\infty \sum_{i=1}^{n} a_{ni}\, e^{(\tau-t)/\mu_{ni}} J_n(t)\, dt.$$

These equations are expressed equivalently as

$$J_n(\tau) = \int_0^{\infty} K_{n1}(\tau - t)\, J_n(t)\, dt, \tag{71}$$

$$F = -4\int_0^{\tau} K_{n2}(\tau - t)\, J_n(t)\, dt + 4\int_{\tau}^{\infty} K_{n2}(\tau - t)\, J_n(t)\, dt, \tag{72}$$

where

$$\left.\begin{aligned} K_{nr}(x) &= \frac{1}{2} \sum_{i=-1}^{-n} a_{ni}\, e^{x/\mu_{ni}}\, (-\mu_{ni})^{r-2}, && x > 0, \\ K_{nr}(x) &= \frac{1}{2} \sum_{i=1}^{n} a_{ni}\, e^{x/\mu_{ni}}\, (\mu_{ni})^{r-2}, && x < 0, \end{aligned}\right\} \quad r \geq 1. \tag{73}$$

Special cases of these equations were derived by Krook (10, p.496).

According to Theorem 3, conditions (24), (27), (29), and (30) provide a formulation of the problem for $I_n(\tau, \mu)$ and $J_n(\tau)$ which is equivalent to the original formulation given in §2.1. Since, as may easily be verified, the sets of equations $\{(24), (27), (29)\}$ and $\{(29), (71), (72)\}$ are equivalent and conditions (30) and (70) are identical, another formulation of the problem is given by conditions (29), (70), (71), (72). These considerations give us the following result.

Theorem 5. There exists a function $I_n(\tau, \mu)$ which satisfies the conditions of §2.1 if and only if there exists a function $J_n(\tau)$ which satisfies (70), (71), (72), in which case $I_n(\tau, \mu)$ is given in terms of $J_n(\tau)$ by (29).

The problem for $J_n(\tau)$ defined by (70), (71) and (72) is quite similar to the problem for $J(\tau)$ defined by (13), (14) and (15). The apparent similarity is increased by writing (73) in the form

$$K_{nr}(x) = \frac{1}{2} \sum_{i=1}^{n} a'_{ni}\, e^{-|x|/\mu'_{ni}} (\mu'_{ni})^{r-2}, \qquad x > 0,$$

$$r \geq 1, \quad (74)$$

$$K_{nr}(x) = \frac{1}{2} \sum_{i=1}^{n} a_{ni}\, e^{-|x|/\mu_{ni}} (\mu_{ni})^{r-2}, \qquad x < 0,$$

where $a'_{ni} = a_{n,-i}$ and $\mu'_{ni} = -\mu_{n,-i}$. It is now clear that, for fixed n, r and x, $K_{nr}(x)$ is a numerical integration type approximation to $K_r(x)$.

Following are some important properties of the functions $K_r(x)$ and $K_{nr}(x)$. From (15),

$$K_r(x) > 0, \qquad (75)$$

$$\left.\begin{aligned} K_r(x) &= \int_x^{\infty} K_{r-1}(y)\, dy, \qquad x > 0, \\ K_r(x) &= \int_{-\infty}^{x} K_{r-1}(y)\, dy, \qquad x < 0, \end{aligned}\right\} \qquad (76)$$

$$K_r(x) < K_{r-1}(x). \qquad (77)$$

From (73), (17) and (18),

$$K_{nr}(x) > 0, \qquad (78)$$

$$\left.\begin{aligned} K_{nr}(x) &= \int_x^{\infty} K_{n,r-1}(y)\, dy, \qquad x > 0, \\ K_{nr}(x) &= \int_{-\infty}^{x} K_{n,r-1}(y)\, dy, \qquad x < 0, \end{aligned}\right\} \qquad (79)$$

$$K_{nr}(x) < K_{n,r-1}(x). \qquad (80)$$

From (15) and (76),

$$\int_{-\infty}^{\infty} K_1(x)\, dx = 2 K_2(0) = 1. \tag{81}$$

From (73), (79) and (19) with m = 0,

$$\int_{-\infty}^{\infty} K_{n1}(x)\, dx = K_{n2}(0+) + K_{n2}(0-) = 1. \tag{82}$$

Since each function $K_r(x)$ is even,

$$\int_{-\infty}^{\infty} K_r(x)\, x\, dx = 0. \tag{83}$$

By (74), $K_{nr}(x)$ is even if and only if $a_{ni} = a_{n,-i}$ and $\mu_{ni} = -\mu_{n,-i}$.

However, even in the general case, we have from (73) and (19) that

$$\int_{-\infty}^{\infty} K_{n1}(x) x dx = -\tfrac{1}{2} \sum_{1} a_{ni}\, \mu_{ni} = 0. \tag{84}$$

§3. 2 The Functions $H_r(x)$

Hopf (7, pp. 35-37) generalized the Schwarzschild-Milne integral equation, (13), for $J(\tau)$ by replacing $K_1(x)$ by an arbitrary positive even function, $K(x)$, such that $\int_{-\infty}^{\infty} K(x)\, dx = 1$ and $\int_{x}^{\infty} K(y)\, dy \leqq C\, K(x)$, $x > 0$, for some positive constant C. We shall extend the work of Hopf and obtain several entirely new results.

Let $H_1(x)$ be any function defined for all real x except, possibly, for x = 0 such that

$$H_1(x) \geqq 0, \tag{85}$$

$$\int_{-\infty}^{\infty} H_1(x)\, dx = 1, \tag{86}$$

$$\int_{-\infty}^{\infty} H_1(x)\, x\, dx = 0, \tag{87}$$

$$\left.\begin{aligned} &\int_{x}^{\infty} H_1(y)\, dy \leqq C\, H_1(x), \quad x > 0, \\ &\int_{-\infty}^{x} H_1(y)\, dy \leqq C\, H_1(x), \quad x < 0, \end{aligned}\right. \tag{88}$$

where C is a positive constant.

It is clear from (75)-(84) that $K_1(x)$ and $K_{n1}(x)$ are special cases of $H_1(x)$ in which C may be taken as unity. By analogy with $\left\{K_r(x);\ r \geqq 2\right\}$ and $\left\{K_{nr}(x);\ r \geqq 2\right\}$, $n \geqq 1$, we introduce functions $H_r(x)$ such that

$$\left.\begin{aligned} &H_r(x) \int_{x}^{\infty} H_{r-1}(y)\, dy, \quad x > 0, \\ &H_r(x) = \int_{-\infty}^{x} H_{r-1}(y)\, dy, \quad x < 0, \end{aligned}\right\} \quad r \geqq 2. \tag{89}$$

By (85), (88) and induction on r, the functions defined by (89) exist for all $x \neq 0$ and have the following properties:

$$H_r(x) \leqq C\, H_{r-1}(x), \qquad H_r(x) \leqq C^{r-1} H_1(x), \qquad r \geqq 2; \tag{90}$$

$$\left.\begin{aligned} &H_r(x) \geqq 0; \quad H_r(x) \text{ is absolutely continuous,} \\ &\text{non-increasing for } x > 0, \text{ non-decreasing for } x < 0, \end{aligned}\right\} \quad r \geqq 2. \tag{91}$$

By (86) and (89), $H_2(0+)$ and $H_2(0-)$ exist and

$$H_2(0+) + H_2(0-) = 1. \tag{92}$$

Therefore, by (90) and (91), $H_r(0\pm)$ exist, are non-negative and

$$H_r(0+) + H_r(0-) \leqq C^{r-2}, \qquad r \geqq 2. \tag{93}$$

It follows from (85) and (89) that $H_2(0+) = 0$ if and only if $\int_0^\infty H_1(x)\ x\, dx = 0$ and $H_2(0-) = 0$ if and only if $\int_{-\infty}^0 H_1(x)\, x\, dx = 0$. In view of (87), $H_2(0+) = 0$ if and only if $H_2(0-) = 0$. This result, (92), and (91) give $H_2(0\pm) > 0$; induction on r yields $H_r(0\pm) > 0$, $r \geqq 2$. Hence, by (93),

$$0 < H_r(0\pm) < C^{r-2}, \qquad r \geqq 2. \tag{94}$$

It is well known and not difficult to prove by induction that

$$\int_x^\infty \int_{x_n}^\infty \cdots \int_{x_2}^\infty \int_{x_1}^\infty f(y)\, dy dx_1 \ldots dx_{n-1} dx_n = \int_x^\infty f(y) \frac{(y-x)^n}{n!}\, dy, \quad n \geqq 0,$$

whenever $f(y) \geqq 0$ and either of the two members exists. Applying this result to (89) we have

$$\left.\begin{aligned} H_{r+n+1}(x) &= \frac{1}{n!}\int_x^\infty H_r(y)\,(y-x)^n\,dy, \qquad x > 0, \\ H_{r+n+1}(x) &= \frac{1}{n!}\int_{-\infty}^x H_r(y)\,(x-y)^n\,dy, \qquad x < 0, \end{aligned}\right\} \quad r \geqq 1,\ n \geqq 0. \tag{95}$$

From (94), and (95),

$$\left.\begin{aligned} H_{r+n+1}(0+) &= \frac{1}{n!}\int_0^\infty H_r(x)\,x^n\,dx, \\ H_{r+n+1}(0-) &= \frac{(-1)^n}{n!}\int_{-\infty}^0 H_r(x)\,x^n\,dx, \end{aligned}\right\} \quad r \geqq 1,\ n \geqq 0, \tag{96}$$

and

$$0 < \int_0^\infty H_r(x)\, x^n\, dx < n! \quad C^{r+n-1},$$

$$r \geqq 1, \quad n \geqq 0. \tag{97}$$

$$0 < (-1)^n \int_{-\infty}^0 H_r(x)\, x^n\, dx < n! \quad C^{r+n-1},$$

From (87) and (96) with $r = n = 1$,

$$H_3(0+) = H_3(0-). \tag{98}$$

The next theorem indicates certain similarities among the functions $H_r(x)$, $r \geqq 1$. Let

$$E_+ = \left\{x;\ x > 0,\ H_2(x) > 0\right\}, \qquad \eta = \text{l.u.b. } E_+,$$
$$E_- = \left\{x;\ x < 0,\ H_2(x) > 0\right\}, \qquad \zeta = \text{g.l.b. } E_-. \tag{99}$$

where $0 < \eta \leqq \infty$ and $-\infty \leqq \zeta < 0$. By (91) and (94), the sets $E_{\pm}$ are non-void.

<u>Theorem 6.</u> $H_r(x) > 0$ for $x \in E_{\pm}$, $r \geqq 1$. If $\eta < \infty$, then $H_1(x) = 0$ for almost all $x \geqq \eta$ and $H_r(x) = 0$ for all $x \geqq \eta$, $r \geqq 2$. If $\zeta > -\infty$, then $H_1(x) = 0$ for almost all $x \leqq \zeta$ and $H_r(x) = 0$ for all $x \leqq \zeta$, $r \geqq 2$.

<u>Proof:</u> For $x \in E_{\pm}$, $H_2(x) > 0$ from (99) and $H_1(x) \geqq C^{-1} H_2(x) > 0$ from (90). Hence, by (85) and (89), $H_r(x) > 0$ for $x \in E_{\pm}$, $r \geqq 1$. By (91), $H_2(x) = 0$ for $x \geqq \eta$ and for $x \leqq \zeta$. The remainder of the theorem follows immediately from (85) and (89). ‖

The notation ‖ is used henceforth to signal the ends of proofs.

§3.3 The Integral Operator $\wedge$

We now introduce a linear integral operator with domain and range contained in the set of Lebesgue measurable functions defined on the non-negative real axis. For each function f such that the right member of (100) exists for all $\tau \geqq 0$, let Λf denote the function defined by

$$(\Lambda f)(\tau) = \int_0^\infty H_1(\tau - t)\, f(t)\, dt, \qquad \tau \geqq 0, \tag{100}$$

or, equivalently, by

$$(\Lambda f)(\tau) = \int_{-\infty}^{\tau} H_1(x)\, f(\tau - x)\, dx, \qquad \tau \geqq 0. \tag{101}$$

For $f(\tau) \equiv \tau^n$, $n \geqq 0$, (101) yields

$$(\Lambda f)(\tau) = \int_{-\infty}^{\tau} H_1(x)(\tau - x)^n\, dx = \int_{-\infty}^{\infty} H_1(x)(\tau - x)^n\, dx - \int_{\tau}^{\infty} H_1(x)(\tau - x)^n\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \tau^{n-k} \int_{-\infty}^{\infty} H_1(x) x^k\, dx - \int_{\tau}^{\infty} H_1(x)(\tau - x)^n\, dx.$$

By (97), each of the preceding improper integrals exists. It follows from (86), (87) and (95) that, for $n \geqq 0$,

$$f(\tau) \equiv \tau^n \Longrightarrow (\Lambda f)(\tau) = \tau^n + \sum_{k=2}^{n} \binom{n}{k} (-1)^k \tau^{n-k} \int_{-\infty}^{\infty} H_1(x) x^k\, dx + (-1)^n n!\, H_{n+2}(\tau). \tag{102}$$

When n = 0 or 1, the summation is not present and

$$(\Lambda 1)(\tau) = 1 - H_2(\tau), \tag{103}$$

$$f(\tau) \equiv \tau \Longrightarrow (\Lambda f)(\tau) = \tau + H_3(\tau), \tag{104}$$

where, in (103), 1 is an abbreviation for the function $f(\tau) \equiv 1$.

Define Λ^m, m = 0, 1, ..., in the usual manner: Λ^0 is the identity operator on the set of Lebesgue measurable functinns defined on the non-negative real axis; for $m \geqq 1$, $\Lambda^m f = \Lambda(\Lambda^{m-1} f)$ whenever the right member exists. Denote the domain of Λ^m by $D(\Lambda^m)$. The following theorem contains several properties of these operators.

Theorem 7. If f is measurable, $g \in D(\Lambda^m)$, and $|f| \leqq |g|$, then $f \in D(\Lambda^m)$. If f, $g \in D(\Lambda^m)$, then

$$f \geqq g \Longrightarrow \Lambda^m f \geqq \Lambda^m g; \tag{105}$$

$$f > g \Longrightarrow \Lambda^m f > \Lambda^m g. \tag{106}$$

If f is measurable and bounded, then $f \in D(\Lambda^m)$ for all $m \geqq 0$.

Proof: From (85) and (100) we get the first assertion of the theorem and, by induction, (105). Let $\psi \in D(\Lambda)$, $\psi > 0$. It follows from (89), (101) and the fact that $H_2(0-) > 0$ that $\Lambda\psi > 0$. Therefore, (106) holds for m = 1. By induction, it holds for $m \geqq 0$. Suppose that $g \in D(\Lambda)$, $g \geqq 0$, and $\Lambda g \leqq g$. Induction yields $g \in D(\Lambda^m)$ and $0 \leqq \Lambda^{m+1} g \leqq \Lambda^m g$, $m \geqq 0$. If $g > 0$ then $0 < \Lambda^m g$, and if $\Lambda g < g$ then $\Lambda^{m+1} g < \Lambda^m g$, $m \geqq 0$. By (91) and (94), $0 \leqq H_2(\tau) < 1$. By (103), $0 < \Lambda 1 \leqq 1$. Therefore, the preceding analysis yields $1 \in D(\Lambda^m)$ and

$$0 < \Lambda^{m+1} 1 \leqq \Lambda^m 1 \leqq 1, \qquad m \geqq 0. \tag{107}$$

Let f be measurable and bounded: $|f| \leqq M < \infty$. Let $g(\tau) \equiv M$. By (107), $g \in D(\Lambda^m)$ and $0 < \Lambda^m g \leqq M$ for $m \geqq 0$. By the first assertion of the theorem, $f \in D(\Lambda^m)$ for $m \geqq 0$. ‖

Theorems 8 and 9 state further properties of the operators Λ^m.

Theorem 8. Let $f \in D(\Lambda^m)$. Then

$$f \geqq 0,\ f \text{ non-decreasing} \Longrightarrow \Lambda^m f \text{ non-decreasing}; \tag{108}$$

$$f \text{ bounded, continuous} \Longrightarrow \Lambda^m f \text{ continuous}. \tag{109}$$

<u>Proof:</u> From (101), $(\Lambda f)(\tau) - (\Lambda f)(\tau') =$

$$\int_{-\infty}^{\tau'} H_1(x)\left[f(\tau-x) - f(\tau'-x)\right] dx + \int_{\tau'}^{\tau} H_1(x)\, f(\tau-x)\, dx.$$

This identity, (85), and (86) yield (108) and (109) for m = 1. Induction completes the proof. ‖

<u>Theorem 9.</u> Let $f \in D(\Lambda)$, $f \geqq 0$. Then, for $r \geqq 2$,

$$\int_0^{\infty} H_r(\tau-t)f(t)\, dt \leqq C^{r-1}(\Lambda f)(\tau); \tag{110}$$

$$\left.\begin{aligned} &\int_0^{\tau} H_r(\tau-t)f(t)dt = H_r(0+)\int_0^{\tau} f(t)dt - \int_0^{\tau}\int_0^{x} H_{r-1}(x-t)f(t)dtdx, \\ &\int_{\tau}^{\infty} H_r(\tau-t)f(t)\, dt = \\ &-H_r(0-)\int_0^{\tau} f(t)dt + \int_0^{\tau}\int_x^{\infty} H_{r-1}(x-t)f(t)dtdx + \int_0^{\infty} H_r(-t)f(t)dt. \end{aligned}\right\} \tag{111}$$

<u>Proof:</u> From (90) and (100), the left members in (110) and (111) exist and (110) is satisfied. Using (89), we obtain

$$H_r(0+)\int_0^{\tau} f(t)\, dt - \int_0^{\tau} H_r(\tau-t)f(t)\, dt = \int_0^{\tau}\int_0^{\tau-t} H_{r-1}(y)\, dy\; f(t)\, dt$$

$$= \int_0^{\tau}\int_t^{\tau} H_{r-1}(x-t)f(t)\, dx\, dt = \int_0^{\tau}\int_0^{x} H_{r-1}(x-t)f(t)\, dt\, dx.$$

This is the first equation of (111). Also by (89),

$$H_r(0+)\int_0^{\tau} f(t)\, dt - \int_0^{\infty} H_r(-t)f(t)\, dt + \int_{\tau}^{\infty} H_r(\tau-t)f(t)\, dt$$

$$= \int_0^\tau \left[H_r(0+) - H_r(-t) \right] f(t)\, dt + \int_\tau^\infty \left[H_r(\tau - t) - H_r(-t) \right] f(t)\, dt$$

$$= \int_0^\tau \int_0^t H_{r-1}(x-t) f(t)\, dx\, dt + \int_\tau^\infty \int_0^\tau H_{r-1}(x-t) f(t)\, dx\, dt$$

$$= \int_0^\tau \int_x^\infty H_{r-1}(x-t) f(t)\, dt\, dx.$$

This is the second equation of (111). ‖

§3.4 The Integral Equation $f = \Lambda f$

Consider the problem for f defined by

$$f = \Lambda f, \qquad f \geq 0, \qquad f \not\equiv 0. \tag{112}$$

Any function which satisfies (112) also satisfies the integral equations which are derived by repeatedly integrating $f = \Lambda f$. The first two of these equations are included in the following theorem.

<u>Theorem 10.</u> If f is a solution of (112), then

$$-\int_0^\tau H_2(\tau - t) f(t)\, dt + \int_\tau^\infty H_2(\tau - t) f(t)\, dt = \int_0^\infty H_2(-t) f(t)\, dt, \tag{113}$$

$$\int_0^\infty H_3(\tau - t) f(t)\, dt = \tau \int_0^\infty H_2(-t) f(t)\, dt + \int_0^\infty H_3(-t) f(t)\, dt. \tag{114}$$

<u>Proof:</u> From (92), (100), and (111) with $r = 2$,

$$-\int_0^\tau H_2(\tau - t) f(t)\, dt + \int_\tau^\infty H_2(\tau - t) f(t)\, dt - \int_0^\infty H_2(-t) f(t)\, dt$$

$$= -\int_0^\tau f(t)\, dt + \int_0^\tau (\Lambda f)(x)\, dx = \int_0^\tau \left[(\Lambda f)(t) - f(t) \right] dt = 0.$$

This establishes (113). From (98), (113), and (111) with $r = 3$,

$$\int_0^\infty H_3(\tau-t)f(t)\,dt - \int_0^\infty H_3(-t)f(t)\,dt$$

$$= \int_0^\tau \left[-\int_0^x H_2(x-t)f(t)\,dt + \int_x^\infty H_2(x-t)f(t)\,dt \right] dx$$

$$= \int_0^\tau \int_0^\infty H_2(-t)f(t)\,dt\,dx = \tau \int_0^\infty H_2(-t)f(t)\,dt. \quad \|$$

Since (113) reduces for $\tau = 0$ to an identity, an equivalent statement is that the left member of (113) is constant. According to the following theorem, this constant is positive.

Theorem 11. If f is a solution of (112), then $\int_0^\infty H_r(-t)f(t)\,dt > 0$, $r \geqq 2$.

Proof: Assume that

$$f = \Lambda f,\ f \geqq 0 \text{ and } \int_0^\infty H_r(-t)f(t)\,dt = 0 \text{ for some } r \geqq 2. \qquad (115)$$

It suffices to prove that $f \equiv 0$. It follows from (115) and Theorem 6 that $f(t) = 0$ for almost all t such that $0 < t < -\zeta$. If $\zeta = -\infty$, (100) and (115) give $\Lambda f \equiv 0$, $f \equiv 0$. Suppose that $\zeta > -\infty$. We shall prove by induction that

$$f(t) = 0 \text{ for almost all t such that } 0 < t < m\xi,\ m \geqq 1, \qquad (116)$$

where $\xi = -\zeta > 0$. We have already shown that (116) holds for $m = 1$. Assume (116) for $m = n \geqq 1$, shere n is fixed arbitrarily. Then $\int_0^{n\xi} H(\tau-t)f(t)\,dt = 0$ and, by Theorem 6, $\int_{\tau+\xi}^\infty H_1(\tau-t)f(t)\,dt = 0$.

Since $f = \Lambda f$, (100) yields

$$f(\tau) = \int_{n\xi}^{\tau+\xi} H_1(\tau - t) f(t)\, dt \quad \text{for } \tau + \xi \geqq n\xi, \text{ i.e., for } \tau \geqq (n-1)\xi.$$

Hence (116) with $m = n$ gives $\int_{(n-1)\xi}^{n\xi} \int_{n\xi}^{\tau+\xi} H_1(\tau - t) f(t)\, dt\, d\tau = \int_{(n-1)\xi}^{n\xi} f(\tau)\, d\tau = 0.$

We reverse the order of integration and refer to (89) to get

$$\int_{n\xi}^{(n+1)\xi} \int_{t-\xi}^{n\xi} H_1(\tau - t)\, d\tau\, f(t)\, dt = \int_{n\xi}^{(n+1)\xi} \Big[H_2(n\xi - t) - H_2(-\xi) \Big] f(t)\, dt = 0.$$

By Theorem 6, $H_2(-\xi) = 0$ and $H_2(n\xi - t) > 0$ for $-\xi < n\xi - t < 0$, i.e., for $n\xi < t < (n+1)\xi$. Therefore $f(t) = 0$ for almost all t such that $n\xi < t < (n+1)\xi$, and (116) holds for $m = n + 1$. By induction (116) holds for $m \geqq 1$, so that $f(t) = 0$ for almost all $t \geqq 0$. By (100) and (115), $\Lambda f \equiv 0$, $f \equiv 0$. ‖

Theorems 10 and 11 will be used to prove that a solution of (112) is necessarily positive. In fact, a stronger result is obtained.

<u>Theorem 12.</u> If f satisfies (112), then

$$f(\tau) > C^{-2} \Big[\tau + H_3(\tau) \Big] \int_0^{\infty} H_2(-t) f(t)\, dt, \tag{117}$$

$$\text{g.l.b.} \left\{ f(\tau);\ \tau \geqq 0 \right\} > 0, \tag{118}$$

$$\text{l.u.b.} \left\{ f(\tau);\ \tau \geqq 0 \right\} = \infty. \tag{119}$$

<u>Proof:</u> Assume that f satisfies (112). By (110), (114) and Theorem 11,

$$f(\tau) = (\Lambda f)(\tau) \geqq C^{-2} \int_0^{\infty} H_3(\tau - t) f(t)\, dt > C^{-2}\, \tau \int_0^{\infty} H_2(-t) f(t)\, dt.$$

This result and (104) and (106) yield (117). Then (118) and (119) follow by means of $H_3(\tau) \geqq 0$, $H_3(0+) > 0$ and Theorem 11.

We are now able to prove that if there exists a solution, f, of (112), then the set of positive multiples of f is the complete solution.

Theorem 13. Let f satisfy (112). Then g satisfies (112) if and only if $g = bf$ for some positive constant b.

Proof: Let g satisfy (112). By Theorem 12, $f > 0$, $g > 0$. Let $\psi = g - bf$, where $b = \text{g.l.b.}\left\{g(\tau)/f(\tau);\ \tau \geqq 0\right\}$. Then $\psi = \Lambda\psi$, $\psi \geqq 0$ and $\text{g.l.b.}\left\{\psi(\tau);\ \tau \geqq 0\right\} = 0$. By Theorem 12, $\psi \equiv 0$. Hence, $g \equiv bf$ and $b > 0$. Conversely, if $g = bf$ for some $b > 0$, then g satisfies (112). ||

The substitution of $\tau + g(\tau)$ for $f(\tau)$ in $f = \Lambda f$ yields, upon reference to (104), $g - \Lambda g = H_3$. Consequently there exists a solution to (112) if and only if there exists a solution to the problem for g defined by

$$g - \Lambda g = H_3, \quad g(\tau) \geqq -\tau, \qquad g(\tau) \not\equiv -\tau. \tag{120}$$

We shall show that the Neumann series, $\sum_0^\infty \Lambda^m H_3$, converges to a function which satisfies (120). In order to do this we need a few preliminary results.

By (103), $1 - \Lambda 1 = H_2$. According to the following theorem the Neumann series associated with the integral equation $g - \Lambda g = H_2$ converges to the solution $g(\tau) \equiv 1$.

Theorem 14. For each $\tau \geqq 0$,

$$\sum_{m=0}^{\nu-1} (\Lambda^m H_2)(\tau) = 1 - (\Lambda^\nu 1)(\tau), \tag{121}$$

$$\sum_{m=0}^{\infty} (\Lambda^m H_2)(\tau) = 1, \tag{122}$$

$$\lim_{\nu \to \infty} (\Lambda^\nu 1)(\tau) = 0. \tag{123}$$

Convergence in (122) and (123) is uniform in each finite τ-interval.

Proof: Since $H_2 = 1 - \Lambda 1$ and $\Lambda^m 1$ exists for $m \geqq 0$, $\Lambda^m H_2$ exists for

$m \geq 0$ and $\sum_0^\nu \Lambda^m H_2 = \sum_0^\nu \Lambda^m(1-\Lambda 1) = 1-\Lambda^{\nu+1}1$, which is equivalent to (121). It follows from (107) that the left member of (121) is non-negative, non-decreasing in ν, and bounded by 1. Therefore, for each $\tau \geq 0$, $\psi(\tau) = \sum_0^\infty (\Lambda^m H_2)(\tau)$ exists and $0 \leq \psi \leq 1$. Since $\Lambda 1$ exists, Lebesgue's dominated convergence theorem implies that $\Lambda\psi$ exists and $\Lambda\psi = \sum_0^\infty \Lambda^{m+1} H_2 = \psi - H_2$. Hence, $\psi - \Lambda\psi = H_2 = 1 - \Lambda 1$ and $(1-\psi) = \Lambda(1-\psi)$. Suppose that $1-\psi \not\equiv 0$. Then $1-\psi$ is a solution of (112) which, by Theorem 12, is unbounded. Since this is a contradiction, $1-\psi \equiv 0$ and (122) is established. Clearly, (121) and (122) imply (123).

Choose $x > 0$ and $\varepsilon > 0$ arbitrarily. Choose k such that $|(\Lambda^m 1)(x)| < \varepsilon$ for $m \geq k$. By (107) and (108), $(\Lambda^m 1)(\tau)$ is positive and non-decreasing in τ. Hence, $0 < (\Lambda^m 1)(\tau) < \varepsilon$ for $m \geq k$, $0 \leq \tau \leq x$, and the convergence in (123) is uniform in each finite τ-interval. In view of (121), so also is the convergence in (122). ‖

The next theorem concerns the existence of the Neumann series solution of (120).

<u>Theorem 15.</u> The series $\sum_0^\infty (\Lambda^m H_3)(\tau)$ converges uniformly in each finite τ-interval. The function ϕ defined by

$$\phi(\tau) = \sum_{m=0}^{\infty} (\Lambda^m H_3)(\tau), \quad \tau \geq 0, \tag{124}$$

is continuous and

$$\phi - \Lambda\phi = H_3, \tag{125}$$

$$0 < \phi \leq C. \tag{126}$$

<u>Proof:</u> Since $0 \leq H_3 \leq CH_2$ and $CH_2 \in D(\Lambda^m)$ for $m \geq 0$, it follows that $H_3 \in D(\Lambda^m)$ and $0 \leq \Lambda^m H_3 \leq C\, \Lambda^m H_2$, $m \geq 0$. Hence, by (122),

$\phi(\tau)$ exists and $0 \leq \phi(\tau) \leq C$. We have by induction from (100) and Theorem 6 that $(\Lambda^m H_3)(\tau) > 0$ for $m\eta \leq \tau < (m+1)\eta$, $m \geq 0$. Therefore, $\phi > 0$ and (126) is established. By Lebesgue's dominated convergence theorem, $\Lambda\phi = \sum_0^\infty \Lambda^{m+1} H_3 = \phi - H_3$, so that ϕ satisfies (125). By (126) and Theorem 7, $\phi \in D(\Lambda^m)$ and $0 < \Lambda^m \phi \leq C \Lambda^m 1$, $m \geq 0$. Hence, Theorem 14 gives

$$(\Lambda^m \phi)(\tau) \to 0 \text{ uniformly in each finite } \tau\text{-interval as } m \to \infty. \tag{127}$$

Since, by (125),

$$\sum_{m=0}^{\nu-1} \Lambda^m H_3 = \sum_{m=0}^{\nu-1} \Lambda^m(\phi - \Lambda\phi) = \phi - \Lambda^\nu \phi, \tag{128}$$

the convergence in (124) is uniform in each finite τ-interval. The continuity of ϕ is a consequence of (109), the boundedness and continuity of H_3, and the uniform convergence of (124). ‖

We can now express the complete solution of (112).

Theorem 16. A function f satisfies (112) if and only if

$$f(\tau) = b\left[\tau + \phi(\tau)\right] \tag{129}$$

for some positive constant b, where ϕ is given by (124).

Proof: Use (104) and Theorems 13 and 15. ‖

It is convenient to add a normalization condition of (112) in order to determine the constant b. Proceeding by analogy with the classical transfer problem, we shall use for this purpose the condition,

$$F = -4\int_0^\tau H_2(\tau - t) f(t)\, dt + 4\int_\tau^\infty H_2(\tau - t) f(t)\, dt, \tag{130}$$

where F is an arbitrary positive constant. For each function f which satisfies (112), the right member of (130) is constant by Theorem 10 and is positive by Theorem 11. Therefore the condition,

$$F = 4\int_0^\infty H_2(-t)f(t)\,dt \tag{131}$$

for an arbitrary $F > 0$, may be used in place of (130).

Theorem 17. The problem defined by (112) and (131) has a unique solution f which is given by

$$f(\tau) = (F/4\gamma)\left[\tau + \phi(\tau)\right], \tag{132}$$

where ϕ is given by (124) and

$$\gamma = \int_0^\infty H_2(-t)\left[t + \phi(t)\right]dt = \int_{-\infty}^\infty H_3(x)\,dx = H_4(0+) + H_4(0-). \tag{133}$$

Proof: By Theorem 16, f satisfies (112) and (131) if and only if $f(\tau) = b\left[\tau + \phi(\tau)\right]$ and $F = 4b\gamma$ where $\gamma = \int_0^\infty H_2(-t)\left[t + \phi(t)\right]dt$. By Theorem 11, $\gamma > 0$. Thus, $b = F/4\gamma$ and f is given by (132). Using (114) with $f(t) = t + \phi(t)$, we obtain

$$\gamma = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\infty H_3(\tau - t)\left[t + \phi(t)\right]dt$$

$$= \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\infty}^\tau H_3(x)\left[\tau - x + \phi(\tau - x)\right]dx$$

$$= \lim_{\tau\to\infty}\left[\int_{-\infty}^\tau H_3(x)\,dx - \frac{1}{\tau}\int_{-\infty}^\tau H_3(x)x\,dx + \frac{1}{\tau}\int_{-\infty}^\tau H_3(x)\phi(\tau - x)\,dx\right].$$

Equation (133) now follows by means of (89), (97) and (126). ‖

§3.5 Expressions for $J(\tau)$ and $J_n(\tau)$

Before the analysis of §4.4 can be applied to the problems for $J(\tau)$ and $J_n(\tau)$, some additional notation is required. When $H_1 = K_1$ we shall replace Λ by Γ and ϕ by q. When $H_1 = K_{n1}$ we shall replace Λ by Γ_n and ϕ by q_n. Thus, from (100) and (124),

$$(\Gamma f)(\tau) = \int_0^\infty K_1(\tau - t) f(t)\, dt, \qquad \tau \geqq 0, \tag{134}$$

$$(\Gamma_n f)(\tau) = \int_0^\infty K_{n1}(\tau - t) f(t)\, dt, \qquad \tau \geqq 0, \tag{135}$$

$$q(\tau) = \sum_{m=0}^{\infty} (\Gamma^m K_3)(\tau), \qquad \tau \geqq 0, \tag{136}$$

$$q_n(\tau) = \sum_{m=0}^{\infty} (\Gamma_n{}^m K_{n3})(\tau), \qquad \tau \geqq 0. \tag{137}$$

By the remark following (88), C may be taken as unity when $H_1 = K_1$ or $H_1 = K_{n1}$. So (126) yields

$$0 < q \leqq 1, \qquad 0 < q_n \leqq 1. \tag{138}$$

Except for change of notation, the second of these results is contained in (51) and (57).

We are now able to present the promised solutions to the problems for $I(\tau, \mu)$, $J(\tau)$, $I_n(\tau, \mu)$ and $J_n(\tau)$.

<u>Theorem 18.</u> There is a unique function $J(\tau)$ which satisfies (12)-(14) and a unique function $J_n(\tau)$ which satisfies (70)-(72):

$$J(\tau) = \frac{3}{4} F \left[\tau + q(\tau)\right], \qquad \tau \geqq 0, \tag{139}$$

$$J_n(\tau) = \frac{3}{4} F \left[\tau + q_n(\tau)\right], \qquad \tau \geqq 0. \tag{140}$$

The function $I(\tau, \mu)$ given by (11) and (139) is the unique solution to the classical transfer problem. The function given by (29) and (140) is the unique solution to the problem for $I_n(\tau, \mu)$ presented in §2. 1.

Proof: Theorem 17 applies. When $H_1 = K_1$, we have from (15) that $\gamma = K_4(0+) + K_4(0-) = 2K_4(0) = \int_0^1 \mu^2 \, d\mu = 1/3$. When $H_1 = K_{n1}$, (74) and (19) give $\gamma = K_{n4}(0+) + K_{n4}(0-) = \frac{1}{2} \sum_i a_{ni} u_{ni}^2 = 1/3$. Thus, (132) reduces in the corresponding cases to (139) and (140). The remainder of the proof consists of invoking Theorems 2 and 5.

Chapter 4

NUMERICAL INTEGRATION

Several different methods of numerical integration have been employed to obtain approximations to $I(\tau, \mu)$. We have already mentioned the Gauss and double-Gauss quadrature formulas. In addition, the Newton-Cotes and Tchebycheff formulas were investigated by Kourganoff and Pecker (8, pp. 247-263). Of all these methods, only the Gauss and double-Gauss satisfy condition (18) for an infinite number of positive integers, n. Hence, as far as is known, only these yield infinite sequences of approximations to $I(\tau, \mu)$. Both methods are described below.

§4.1 The Gauss Quadrature Formula

Let x_{ni}, $i = 1, \ldots, n$, be the zeros of the Legendre polynomial, $P_n(x)$, $n \geqq 1$, ordered such that $x_{ni} < x_{n,i+1}$. For each $n \geqq 1$ and each function $g(x)$ defined for $-1 < x < 1$, the Lagrange interpolation formula provides a correspondence,

$$g(x) \sim \sum_{i=1}^{n} g(x_{ni}) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_{nj}}{x_{ni} - x_{nj}} = \sum_{i=1}^{n} g(x_{ni}) \frac{P_n(x)}{P_n'(x_{ni})(x - x_{ni})}. \qquad (141)$$

Since this correspondence becomes an equality for $x = x_{ni}$, $i = 1, \ldots, n$, it becomes an identity for $g(x)$ an arbitrary polynomial of degree less than n. From (141), we obtain for each $n \geqq 1$ and each integrable function, $g(x)$, the correspondence

$$\int_{-1}^{1} g(x)\, dx \sim \sum_{i=1}^{n} \lambda_{ni}\, g(x_{ni}), \qquad (142)$$

where

$$\lambda_{ni} = \frac{1}{P_n'(x_{ni})} \int_{-1}^{1} \frac{P_n(x)}{x - x_{ni}} dx, \quad i = 1, \ldots, n. \tag{143}$$

This is one form of the Gauss quadrature formula.

It is well known that

$$x_{ni} = -x_{n,n-i}, \quad \lambda_{ni} = \lambda_{n,n-i}, \qquad i = 1, \ldots, n; \tag{144}$$

$$\lambda_{ni} > 0, \qquad i = 1, \ldots, n; \tag{145}$$

$$\int_{-1}^{1} g(x)\, dx = \sum_{i=1}^{n} \lambda_{ni}\, g(x_{ni}), \quad g(x) \text{ a polynomial of degree less than } 2n. \tag{146}$$

For a generalization of (145) see (15, p. 47). For $g(x) = x^m$, (146) yields

$$\sum_{i=1}^{n} \lambda_{ni}\, x_{ni}^m = \frac{1 + (-1)^m}{m + 1}, \qquad m = 0, 1, \ldots, 2n-1. \tag{147}$$

In their papers, Wick and Chandrasekhar used the Legendre polynomials, $P_{2n}(\mu)$, $n \geq 1$, and denoted the zeros by μ_{ni}, $i = \pm 1, \ldots, \pm n$, ordered according to (17). With the proper changes of variable, (142) and (145) assume the forms of (16) and (18), respectively, and (147) implies (19). Therefore, the Gauss method leads to an infinite sequence, $\left\{ I_n(\tau, \mu) \right\}$, of approximations to $I(\tau, \mu)$ and these approximations are given by (65) - (67). Some of the expressions occurring in Chapters 2 and 3 can be simplified for this case, since (144) implies

$$a_{ni} = a_{n,-i}, \qquad \mu_{ni} = -\mu_{n,-i}. \tag{148}$$

§4.2 The Double-Gauss Quadrature Formula

For each $n \geqq 1$ and each integrable function $f(\mu)$ defined for $0 < \mu < 1$ we have from (142),

$$\int_0^1 f(\mu)\, d\mu = \frac{1}{2}\int_{-1}^1 f\left(\tfrac{1}{2}(x+1)\right)\, dx \sim \tfrac{1}{2}\sum_{i=1}^n \lambda_{ni} f\left(\tfrac{1}{2}(x_{ni}+1)\right). \tag{149}$$

Thus, we have a correspondence,

$$\int_0^1 f(\mu)\, d\mu \sim \sum_{i=1}^n a_{ni} f(\mu_{ni}), \tag{150}$$

where

$$a_{ni} = \tfrac{1}{2}\lambda_{ni}, \qquad \mu_{ni} = \frac{(x_{ni}+1)}{2}, \qquad i = 1, \ldots, n. \tag{151}$$

For each integrable function $f(\mu)$ defined for $-1 < \mu < 0$, (150) gives a correspondence,

$$\int_{-1}^0 f(\mu)\, d\mu = \int_0^1 f(-\mu)\, d\mu \sim \sum_{i=1}^n a_{ni} f(-\mu_{ni}). \tag{152}$$

Define a_{ni} and μ_{ni} for $i = -1, \ldots, -n$ by

$$a_{ni} = a_{n,-i}, \qquad \mu_{ni} = -\mu_{n,-i}. \tag{153}$$

Then (152) becomes

$$\int_{-1}^0 f(\mu)\, d\mu \sim \sum_{i=-1}^{-n} a_{ni}\, f(\mu_{ni}). \tag{154}$$

For an integrable function $f(\mu)$ defined for $-1 < \mu < 0$ and $0 < \mu < 1$, we obtain from (150) and (154) the correspondence,

$$\int_{-1}^{1} f(\mu)\, d\mu \sim \sum_{i=\pm 1}^{\pm n} a_{ni}\, f(\mu_{ni}), \tag{155}$$

which is of the form of (16). For obvious reasons this correspondence is called the double-Gauss method of numerical integration.

It follows from (145), (151), (153) and the ordering of the x_{ni} that (17) and (18) are satisfied for $n \geqq 1$. To prove (19), consider the preceding expressions with $f(\mu) = \mu^m$, $m < 2n$. By (146) and (151),

$$\int_{0}^{1} \mu^m\, d\mu = \int_{-1}^{1} \tfrac{1}{2}\left(\frac{x+1}{2}\right)^m dx = \tfrac{1}{2}\sum_{i=1}^{n} \lambda_{ni}\left(\frac{x_{ni}+1}{2}\right)^m = \sum_{i=1}^{n} a_{ni}\mu_{ni}^m.$$

From this result and (153),

$$\left.\begin{aligned} \sum_{i=1}^{n} a_{ni}\mu_{ni}^m &= \frac{1}{m+1}, \\ \sum_{i=-1}^{-n} a_{ni}\mu_{ni}^m &= \frac{(-1)^m}{m+1}, \end{aligned}\right\} \quad m = 0, 1, \ldots, 2n-1. \tag{156}$$

Thus, (150) and (154) are equalities if $f(\mu)$ is a polynomial of degree less than $2n$.

Clearly, (156) implies (19) for $m = 0$ and 1, $n \geqq 1$, and for $m = 2$, $n \geqq 2$. Hence, the double-Gauss method yields an infinite sequence, $\left\{I_n(\tau,\mu)\right\}$, of approximations to $I(\tau,\mu)$. These approximations are given by equations (65) - (67), except that for $n = 1$ the coefficient $\frac{3}{4}F$ must be replaced by F. The result for $n = 1$,

$$I_1(\tau,\mu) = F\left[\tau + \mu + \tfrac{1}{2}\right], \qquad \mu \geqq 0,$$

$$I_1(\tau,\mu) = F\left[\tau + (\mu + \tfrac{1}{2})(1 - e^{\tau/\mu})\right], \qquad \mu < 0,$$

is the well-known approximation of Schuster (12, pp. 1-5) and Schwarzchild (13, pp, 41-53).

§4.3 Numerical Integration of Continuous Functions

Consider the Banach space B of continuous functions, $f(\mu)$, $0 \leqq \mu \leqq 1$, with the norm of uniform convergence,

$$\|f\| = \max \left\{ |f|;\ 0 \leqq \mu \leqq 1 \right\}.$$

Define linear functionals A and A_n on B such that

$$Af \equiv \int_0^1 f(\mu)\, d\mu, \tag{157}$$

$$A_n f \equiv \sum_{i=1}^{n} a_{ni} f(\mu_{ni}), \qquad n \geqq 1, \tag{158}$$

where the parameters a_{ni} and μ_{ni} correspond either to the Gauss or to the double-Gauss quadrature formula. Since, in either case, $a_{ni} = a_{n,-i}$ and $\mu_{ni} = -\mu_{n,-i}$, (157) and (158) can be replaced by

$$Af \equiv \frac{1}{2} \int_{-1}^{1} f(|\mu|)\, d\mu, \tag{159}$$

$$A_n f \equiv \frac{1}{2} \sum_{i=\pm 1}^{\pm n} a_{ni} f(|\mu_{ni}|), \qquad n \geqq 1. \tag{160}$$

The integrand in (159) is continuous in the interval $-1 \leqq \mu \leqq 1$. In view of (18) and (19), with $m = 0$, $A1 = A_n 1 = 1$ for $n \geqq 1$ and

$$\left.\begin{aligned} |Af| &\leqq \|f\|, \\ |A_n f| &\leqq \|f\|,\ n \geqq 1, \end{aligned}\right\} \quad f \in B. \tag{161}$$

In the remainder of this section we establish certain properties of A and A_n, $n \geq 1$, which will be needed in Chapter 5. These properties are well-known results in paraphrase.

Lemma 1. $A_n f \to Af$ for each $f \in B$.

Proof: Choose $f \in B$ and $\varepsilon > 0$ arbitrarily. Using the Weierstrass theorem, choose a polynomial $p \in B$ such that $\|f-p\| < \frac{1}{2}\varepsilon$. For $n > \frac{1}{2}$(degree of p), it follows from (146) that $A_n p = Ap$. So, by (161),

$$|A_n f - Af| \leq |A_n(f-p)| + |A_n p - Ap| + |A(p-f)| \leq 2\|f-p\| < \varepsilon$$

for all n sufficiently large. ‖

Lemma 2. Let $f(x, \mu)$ be defined for $0 \leq \mu \leq 1$, $x' \leq x \leq x''$, where $0 \leq x' < x'' < \infty$. Let $|\frac{\partial f}{\partial x}| \leq M < \infty$ and, for each fixed x, assume that $f \in B$. Then $A_n f \to Af$ uniformly in x.

Proof: For an arbitrary $\varepsilon > 0$, choose an integer m and x_j, $0 \leq j \leq m$, such that $x' = x_o < \ldots < x_m = x''$ and $x_j - x_{j-1} < \varepsilon/3M$. By Lemma 1, there exists n_o such that $|Af(x_j, \mu) - A_n f(x_j, \mu)| < \varepsilon/3$, $0 \leq j \leq m$, $n \geq n_o$. Choose x arbitrarily such that $x' \leq x \leq x''$ and j such that $|x-x_j| < \varepsilon/3M$. Since $|\frac{\partial f}{\partial x}| \leq M$, $|f(x, \mu) - f(x_j, \mu)| < \varepsilon/3$. Then $|Af(x, \mu) - A_n f(x, \mu)| \leq |Af(x, \mu) - Af(x_j, \mu)| + |Af(x_j, \mu) - A_n f(x_j, \mu)| + |A_n f(x_j, \mu) - A_n f(x, \mu)| < \varepsilon$ for $n \geq n_o$. ‖

Lemma 3. Let $f(x, \mu)$ be defined for $0 \leq \mu \leq 1$ and $x' \leq x < \infty$, where $x' \geq 0$. Let $|\frac{\partial f}{\partial x}| \leq M < \infty$, $f(x, \mu) \to 0$ uniformly in μ as $x \to \infty$, and, for each fixed x, assume that $f \in B$. Then $A_n f \to Af$ uniformly in x.

Proof: For an arbitrary $\varepsilon > 0$, choose $x'' > x'$ such that $|f(x, \mu)| < \varepsilon/2$ for $x > x''$. Then $|Af - A_n f| \leq |Af| + |A_n f| < \varepsilon$ for $n \geq 1$, $x > x''$. By Lemma 2, there exists n_o such that $|Af - A_n f| < \varepsilon$ for $n \geq n_o$. ‖

Chapter 5

FINAL CONVERGENCE THEOREMS

The principal results of this chapter and of the thesis are: if either the Gauss or the double-Gauss quadrature formula is used, then $J_n(\tau) \to J(\tau)$ and $I_n(\tau, \mu) \to I(\tau, \mu)$ uniformly for each bounded subset of the respective domains. The plan of the chapter is as follows. We first derive various convergence properties of the functions $K_{nr}(x)$ and $K_r(x)$. These are used to prove that the general term of the Neumann series (137) for $q_n(\tau)$ converges to the general term of the Neumann series (136) for $q(\tau)$. The series for the $q_n(\tau)$, $n \geqq 1$, are shown to be equiconvergent. Thus, the interchange of limits is justified and $q_n(\tau) \to q(\tau)$, which is equivalent to $J_n(\tau) \to J(\tau)$. Then $I_n(\tau, \mu) \to I(\tau, \mu)$ follows by means of the expressions for these functions in terms of $J_n(\tau)$ and $J(\tau)$.

Even when not explicitly stated, it should be understood that the quadrature formula involved is either the Gauss or the double-Gauss.

§5.1 Convergence Properties of the Functions $K_{nr}(x)$ and $K_r(x)$

For either of the quadrature formulas under consideration, $a_{ni} = a_{n,-i}$ and $\mu_{ni} = -\mu_{n,-i}$. Therefore, by (73),

$$K_{nr}(x) = \frac{1}{2} \sum_{i=1}^{n} a_{ni}\, e^{-x/\mu_{ni}} (\mu_{ni})^{r-2}, \tag{162}$$

$$K_{nr}(-x) = K_{nr}(x), \tag{163}$$

for $n \geqq 1$, $r \geqq 1$ and $x > 0$. Since $K_{nr}(0+)$ and $K_{nr}(0-)$ exist and are equal for $n \geqq 1$, $r \geqq 2$, we shall define $K_{nr}(0)$ for these cases by (162).

By (15),

$$K_r(x) = \frac{1}{2}\int_0^1 e^{-x/\mu}\,\mu^{r-2}\,d\mu, \tag{164}$$

$$K_r(-x) = K_r(x), \tag{165}$$

for $r = 1$, $x > 0$ and for $r \geqq 2$, $x \geqq 0$. We shall prove that $K_{nr}(x) \to K_r(x)$ as $n \to \infty$. In view of the symmetry relations, (163) and (165), it suffices to consider $x \geqq 0$.

In terms of the linear functionals A and A_n introduced in §4. 3,

$$K_r(x) = \tfrac{1}{2} A\, \Phi_r(x, \mu), \tag{166}$$

$$K_{nr}(x) = \tfrac{1}{2} A_n\, \Phi_r(x, \mu), \tag{167}$$

whenever the left members are defined, where

$$\Phi_r(x, \mu) = e^{-x/\mu}\,\mu^{r-2}, \qquad \begin{array}{l} r \leqq 1,\ x > 0,\ \mu \geqq 0, \\ r \geqq 2,\ x \geqq 0,\ \mu \geqq 0. \end{array} \tag{168}$$

It is understood in (168) that, for fixed r and x, $\Phi_r(x, 0)$ is defined by continuity. Thus, $\Phi_r(x, 0) = 0$ for $x > 0$, $\Phi_2(0, 0) = 1$, and $\Phi_r(0, 0) = 0$ for $r \geqq 3$. From (168),

$$\frac{\partial \Phi_r}{\partial x} = -\Phi_{r-1}, \tag{169}$$

$$\frac{\partial \Phi_r}{\partial \mu} = e^{-x/\mu}\,\mu^{r-4}\left[x + (r-2)\mu\right]. \tag{170}$$

Therefore, $\Phi_r(x, \mu)$ is:

non-increasing in x; (171)

non-decreasing in μ, $\quad r \leqq 1,\ 0 \leqq \mu \leqq x/2-r,$ $\quad r \geqq 2,\ \mu \geqq 0;$ (172)

non-increasing in μ, $\quad r \leqq 1,\ \mu \geqq x/2-r.$ (173)

It follows that, for fixed $r \leqq 1$ and $x > 0$, $\Phi_r(x, \mu)$ attains a maximum for $\mu = x/2-r$. Thus, by (168),

$$\Phi_r(x,\mu) \leq \left[(2-r)/ex\right]^{2-r}, \qquad r \leq 1,\ x > 0,\ \mu \geq 0. \tag{174}$$

In (166) and (167), $\Phi_r(x,\mu)$ is used only for $0 \leq \mu \leq 1$. We have from (168) and (172) that

$$\Phi_r(x,\mu) \leq \Phi_r(x,1) = e^{-x}, \qquad r \geq 2,\ x \geq 0,\ 0 \leq \mu \leq 1. \tag{175}$$

It follows from (169), (174) and (175) that

$$\begin{aligned} \left|\frac{\partial \Phi_r}{\partial x}\right| &\leq \left[(3-r)/ex\right]^{3-r}, \qquad r \leq 2,\ x > 0,\ \mu \geq 0, \\ \left|\frac{\partial \Phi_r}{\partial x}\right| &\leq e^{-x} \qquad\qquad\quad , \qquad r \geq 3,\ x \geq 0,\ 0 \leq \mu \leq 1. \end{aligned} \tag{176}$$

With these preliminary results established, we are ready to prove the convergence theorem for the functions $K_{nr}(x)$.

Theorem 19. Assume either the Gauss or the double Gauss quadrature formula. Then

$$K_{nr}(x) \to K_r(x) \text{ as } n \to \infty. \tag{177}$$

When $r = 1$ the convergence is uniform for $x_o \leq |x| < \infty$ for each $x_o > 0$. When $r \geq 2$ the convergence is uniform for $-\infty < x < \infty$.

Proof: We shall make use of Lemma 3 of Chapter 4. By (174) and (175),

$$\Phi_r(x,\mu) \to 0 \text{ uniformly in } \mu \text{ as } x \to \infty, \qquad 0 \leq \mu \leq 1.$$

By (176),

$$\begin{aligned} \left|\frac{\partial \Phi_r}{\partial x}\right| &\leq \left[(3-r)/ex_o\right]^{3-r}, \qquad r \leq 2,\ x \geq x_o > 0,\ \mu \geq 0, \\ \left|\frac{\partial \Phi_r}{\partial x}\right| &\leq 1, \qquad\qquad\qquad\qquad r \geq 3,\ x \geq 0,\ 0 \leq \mu \leq 1. \end{aligned}$$

Lemma 3, (166), (167) and the even character of the functions $K_{nr}(x)$

and $K_r(x)$ give us the statements of the theorem for $r = 1$ and $r \geq 3$ and the result that $K_{n2}(x) \to K_2(x)$ uniformly for $x_o \leq |x| < \infty$ for each $x_o > 0$.

The remainder of the proof will establish the uniformity of the convergence of $K_{n2}(x)$ for $-\infty < x < \infty$. We shall consider, in turn, $x \geq \varepsilon^2$, $0 < x < \varepsilon^2$ and $x = 0$, where ε is arbitrary such that $0 < \varepsilon < 1$. From the preceding analysis, there exists $n_o(\varepsilon)$ such that

$$|K_2(x) - K_{n2}(x)| < \varepsilon, \quad n \geq n_o(\varepsilon), \quad x \geq \varepsilon^2. \tag{178}$$

Now, in this paragraph, let $0 < x < \varepsilon^2$. Let

$$\Theta(x, \mu) = \max\left\{\Phi_2(x, \mu),\ e^{-\varepsilon}\right\}, \quad 0 < x < \varepsilon^2, \quad 0 \leq \mu \leq 1.$$

Note that Θ is continuous in μ for each fixed x and that, since $\Phi_2(x, \mu) = e^{-x/\mu}$,

$$0 \leq \Phi_2(x, \mu) \leq \Theta(x, \mu) = e^{-\varepsilon} < 1, \qquad 0 \leq \mu \leq x/\varepsilon,$$

$$0 < e^{-\varepsilon} \leq \Phi_2(x, \mu) = \Theta(x, \mu) < 1, \qquad x/\varepsilon \leq \mu \leq 1.$$

By the triangle inequality,

$$|A_n\Phi_2 - A\Phi_2| \leq$$

$$|A_n(\Phi_2 - \Theta)| + |A_n(\Theta - 1)| + |A_n 1 - A1| + |A(1-\Theta)| + |A(\Theta - \Phi_2)|, \tag{179}$$

where 1 denotes the function which is identically 1. Since $A_n 1 = A1$, $|A_n 1 - A1| = 0$. Since $0 < 1 - \Theta \leq 1 - e^{-\varepsilon} < \varepsilon$, it follows from (161) that $|A1 - A\Theta| < \varepsilon$ and $|A_n 1 - A_n\Theta| < \varepsilon$, $n \geq 1$. We also have (recalling the assumption that $0 < x < \varepsilon^2$),

$$0 < A(\Theta - \Phi_2) = \int_0^{x/\varepsilon} (e^{-\varepsilon} - e^{-x/\mu})\, d\mu < \int_0^{\varepsilon} 1\, d\mu = \varepsilon,$$

$$0 < A_n(\Theta - \Phi_2) = \sum_{\substack{i=1 \\ \mu_{ni} < x/\varepsilon}}^{n} a_{ni}\,(e^{-\varepsilon} - e^{-x/\mu_{ni}}) \leqq \sum_{\substack{i=1 \\ \mu_{ni} < \varepsilon}}^{n} a_{ni}. \qquad (180)$$

According to (15, p. 341), $A_n f \to Af$ as $n \to \infty$ for each Riemann integrable function f. Applying this to the case with $f(\mu) = 1$ for $0 \leqq \mu < \varepsilon$ and $f(\mu) = 0$ otherwise, we have

$$\sum_{\substack{i=1 \\ \mu_{ni} < \varepsilon}}^{n} a_{ni} \to \varepsilon \quad \text{as} \quad n \to \infty. \qquad (181)$$

This result can also be established by means of the separation theorem (15, pp. 49-52) and the fact (15, pp. 343-346) that $a_{ni} \to 0$ uniformly in i as $n \to \infty$. It follows from (180) and (181) that there exists $n_1(\varepsilon)$ such that $|A_n(\Theta - \Phi_2)| < 2\varepsilon$ for $n \geqq n_1(\varepsilon)$. All the terms of the right member of (179) have now been considered. Thus, from (179), (166) and (167),

$$|K_{n2}(x) - K_2(x)| < 5\varepsilon/2, \qquad n \geqq n_1(\varepsilon), \quad 0 < x < \varepsilon^2. \qquad (182)$$

It remains to consider $x = 0$. By (164), (166) and (168), $K_r(0) = \frac{1}{2}A(\mu^{r-2}) = 1/2(r-1)$, $r \geqq 2$. By (167) and (168), $K_{nr}(0) = \frac{1}{2}A_n(\mu^{r-2})$. As we remarked earlier (in the proof of Lemma 1), $A_n p = Ap$ for each polynomial p of degree less than 2n. Hence,

$$K_{nr}(0) = 1/2(r-1) = K_r(0), \qquad 2 \leqq r \leqq 2n + 1. \qquad (183)$$

The assertion of the theorem for $r = 2$ now follows from (178), (182), (183) and the even character of K_r and K_{nr}. ‖

For the purpose at hand we shall need Theorem 19 only for $r = 1$, 2 and 3. Since no additional work was involved in getting the results for $r \geqq 4$ they were included. Similarly, we shall state the following theorems in as much generality as is consistent with the desire not to go too far afield.

Theorem 20. Assume either the Gauss or the double-Gauss quadrature formula. Then, for each $r \geqq 2$,

$$K_{nr}(x) \to K_{nr}(0) \text{ uniformly in } n \text{ as } x \to 0. \tag{184}$$

For each $r \geqq 1$,

$$K_{nr}(x) \to 0 \text{ uniformly in } n \text{ as } x \to \pm\infty. \tag{185}$$

Proof: Fix $\varepsilon > 0$ and $r \geqq 2$ arbitrarily. Since $K_r(x)$ is continuous at $x = 0$, there exists $x^* > 0$ such that $|K_r(0) - K_r(x)| < \varepsilon/2$ for $0 \leqq |x| \leqq x^*$. According to Theorem 19 there exists n^* such that $|K_r(x) - K_{nr}(x)| < \varepsilon/2$ for $n \geqq n^*$. We shall insist that $n^* > r/2$. By (183) and the triangle inequality, $|K_{nr}(0) - K_{nr}(x)| = |K_r(0) - K_{nr}(x)| < \varepsilon$ for $0 \leqq |x| \leqq x^*$, $n \geqq n^*$. Since the functions $K_{nr}(x)$, $n \geqq 1$, are continuous at $x = 0$, there exists $x_n > 0$ such that

$$|K_{nr}(0) - K_{nr}(x)| < \varepsilon, \qquad 0 \leqq |x| \leqq x_n, \quad n \geqq 1.$$

In particular, let $x_n = x^*$ for $n \geqq n^*$. Then

$$|K_{nr}(0) - K_{nr}(x)| < \varepsilon, \qquad 0 \leqq |x| \leqq x_o, \; n \geqq 1,$$

where $x_o = \min \left\{ x_n ; n \geqq 1 \right\}$. This establishes (184). The proof of (185) is quite similar. ‖

§5.2 Convergence Properties of the Operators Γ_n and Γ

Let E denote the set of bounded measurable functions defined on the non-negative real axis. For each $f \in E$, let

$$\|f\| = \text{l. u. b.} \left\{|f(t)|;\ 0 \leqq t < \infty\right\}.$$

The set E is contained in the domain of Γ and in the domain of Γ_n for each $n \geqq 1$. By Theorem 7,

$$\|\Gamma f\| \leqq \|f\| \quad \text{and} \quad \|\Gamma_n f\| \leqq \|f\|, \quad n \geqq 1, \text{ for each } f \in E. \tag{186}$$

The following theorem asserts that the operators Γ_n, $n \geqq 1$, converge uniformly to Γ on E.

Theorem 21. Assume either the Gauss or the double-Gauss quadrature formula. For each $f \in E$, $\|\Gamma f - \Gamma_n f\| \to 0$ as $n \to \infty$.

Proof: Fix $\varepsilon > 0$ and $f \in E$ arbitrarily. By (164) and Theorem 20, with $r = 2$, there exist $\delta > 0$ and $z > \delta$ such that

$$\begin{aligned} &K_2(0) - K_2(\delta) < \varepsilon, && K_2(z) < \varepsilon, \\ &K_{n2}(0) - K_{n2}(\delta) < \varepsilon, && K_{n2}(z) < \varepsilon, \quad n \geqq 1. \end{aligned} \tag{187}$$

According to Theorem 19 there exists n_o such that

$$|K_1(x) - K_{n1}(x)| < \varepsilon/(z-\delta) \text{ for } \delta \leqq x \leqq z, \qquad n \geqq n_o. \tag{188}$$

From (134), (135), (163) and (165) we obtain $|(\Gamma f)(\tau) - (\Gamma_n f)(\tau)| \leqq$

$$\int_{-\infty}^{\tau} |K_1(x) - K_{n1}(x)|\,|f(\tau - x)|\,dx \leqq \|f\| \int_{-\infty}^{\infty} |K_1(x) - K_{n1}(x)|\,dx \leqq$$

$$2\|f\| \left[\int_0^{\delta} (K_1 + K_{n1})\,dx + \int_{\delta}^{z} |K_1 - K_{n1}|\,dx + \int_z^{\infty} (K_1 + K_{n1})\,dx\right].$$

Therefore, by (187), (188), and (89), $|(\Gamma f)(\tau) - (\Gamma_n f)(\tau)| \leqq 10\|f\|\varepsilon$ for $n \geqq n_o$. $\|$

It follows easily from Theorem 21 that, for each $m \geqq 0$, the operators Γ_n^m, $n \geqq 1$, converge uniformly to Γ^m on E. This is a special case of the next theorem.

Theorem 22. Assume either the Gauss or the double-Gauss quadrature formula. Suppose that $f \in E$, $f_n \in E$ for $n \geq 1$, and $\|f-f_n\| \to 0$ as $n \to \infty$. Then, for each $m \geq 0$, $\|\Gamma^m f - \Gamma_n^m f_n\| \to 0$ as $n \to \infty$.

Proof: The case with $m = 0$ is obvious. By (186) and Theorem 21, $\|\Gamma f - \Gamma_n f_n\| \leq \|\Gamma f - \Gamma_n f\| + \|\Gamma_n f - \Gamma_n f_n\| \leq \|\Gamma f - \Gamma_n f\| + \|f - f_n\| \to 0$ as $n \to \infty$. So the theorem is valid for $m = 1$. Since $\|\Gamma^{m+1} f - \Gamma_n^{m+1} f_n\| = \|\Gamma(\Gamma^m f) - \Gamma_n(\Gamma_n^m f)\|$, induction completes the proof. ‖

According to (136) and (137),

$$q = \sum_{m=0}^{\infty} \Gamma^m K_3, \qquad q_n = \sum_{m=0}^{\infty} \Gamma_n^m K_{n3}, \; n \geq 1. \tag{189}$$

The functions K_3 and K_{n3} belong to E and, by Theorem 19, $\|K_3 - K_{n3}\| \to 0$ as $n \to \infty$. Hence, Theorem 22 and the triangle inequality imply that

$$\Big\| \sum_{m=0}^{\nu} \Gamma^m K_3 - \sum_{m=0}^{\nu} \Gamma_n^m K_{n3} \Big\| \to 0 \text{ as } n \to \infty, \qquad \nu \geq 0, \tag{190}$$

$$q = \lim_{\nu \to \infty} \lim_{n \to \infty} \sum_{m=0}^{\nu} \Gamma_n^m K_{n3}. \tag{191}$$

If the limits can be interchanged in (191) it will result from (189) that $q_n \to q$ as $n \to \infty$. The next theorem provides a sufficient condition for the limit interchange, namely that the series for the q_n are equiconvergent.

Theorem 23. Assume either the Gauss or the double-Gauss quadrature formula. For each $n \geq 1$,

$$\Big| q_n(\tau) - \sum_{m=0}^{\nu} (\Gamma_n^m K_{n3})(\tau) \Big| \to 0 \text{ as } \nu \to \infty.$$

The convergence is uniform with respect to n and τ for $n \geq 1$ and $0 \leq \tau \leq \tau_o$, for each $\tau_o > 0$.

Proof: According to (128), (138) and Theorem 7 with the appropriate changes of notation,

$$0 \leq q_n - \sum_{m=0}^{\nu} \Gamma_n^m K_{n3} = \Gamma_n^{\nu+1} q_n \leq \Gamma_n^{\nu+1} 1, \quad \nu \geq 0. \quad (192)$$

Choose $\tau_o > 0$ and $\varepsilon > 0$ arbitrarily. By Theorem 14 there exists ν^* such that $|(\Gamma^{\nu^*}1)(\tau)| < \varepsilon/2$ for $0 \leq \tau \leq \tau_o$. By Theorem 22 with $f \equiv f_n \equiv 1$ there exists n_o such that $\|\Gamma_n^{\nu^*}1 - \Gamma^{\nu^*}1\| < \varepsilon/2$ for $n \geq n_o$. By the triangle inequality and (107), $|(\Gamma_n^{\nu}1)(\tau)| < \varepsilon$ for $n \geq n_o$, $0 \leq \tau \leq \tau_o$, $\nu \geq \nu^*$. For each $n \geq 1$ there exists, by Theorem 14, an integer ν_n such that $|(\Gamma_n^{\nu}1)(\tau)| < \varepsilon$ for $\nu \geq \nu_n$, $0 \leq \tau \leq \tau_o$. Let $\nu_n = \nu^*$ for $n \geq n_o$ and $\nu_o = \max\{\nu_n; n \geq 1\}$. Then $|(\Gamma_n^{\nu}1)(\tau)| < \varepsilon$ for $\nu \geq \nu_o$, $0 \leq \tau \leq \tau_o$, $n \geq 1$, and the theorem follows from (189). ‖

§5.3 The Convergence of $J_n(\tau)$ to $J(\tau)$ and of $I_n(\tau, \mu)$ to $I(\tau, \mu)$

With the aid of the machinery of §5.2 at hand the convergence theorem for the functions $J_n(\tau)$ is almost immediate.

Theorem 24. Assume either the Gauss or the double-Gauss quadrature formula. For each $\tau_o > 0$, $q_n(\tau) \to q(\tau)$ and $J_n(\tau) \to J(\tau)$ uniformly for $0 \leq \tau \leq \tau_o$ as $n \to \infty$.

Proof: From (136) and (137), $|q - q_n| \leq$

$$|q - \sum_{m=0}^{\nu} \Gamma^m K_3| + |\sum_{m=0}^{\nu} (\Gamma^m K_3 - \Gamma_n^m K_{n3})| + |\sum_{m=0}^{\nu} \Gamma_n^m K_{n3} - q_n|$$

for $\nu \geq 0$, $n \geq 1$. Choose $\varepsilon > 0$ and $\tau_o > 0$ arbitrarily. According to

Theorems 15 and 23 there exists ν^*, independent of n, such that

$$\left|q(\tau) - \sum_{m=0}^{\nu^*} (\Gamma^m K_3)(\tau)\right| < \varepsilon \text{ and } \left|\sum_{m=0}^{\nu^*} (\Gamma_n^m K_{n3})(\tau) - q_n(\tau)\right| < \varepsilon$$

for $0 \leqq \tau \leqq \tau_o$, $n \geqq 1$. By (190) there exists n^* such that

$\left\| \sum_{m=0}^{\nu^*} \Gamma^m K_3 - \Gamma_n^m K_{n3}) \right\| < \varepsilon$, $n \geqq n^*$. Therefore $|q(\tau) - q_n(\tau)| < 3\varepsilon$

and, by (139) and (140), $|J(\tau) - J_n(\tau)| < 3F\varepsilon$ for $0 \leqq \tau \leqq \tau_o$, $n \geqq n^*$. ‖

From (11), (138) and (139), we obtain

$$I(\tau, \mu) = \int_0^{-\tau/\mu} e^{-x} J(\tau + \mu x)\, dx, \qquad \tau \geqq 0,\ \mu < 0,\quad \tau > 0,\ \mu = 0,$$

$$I(\tau, \mu) = \int_0^{\infty} e^{-x} J(\tau + \mu x)\, dx, \qquad \tau \geqq 0,\ \mu \geqq 0. \tag{193}$$

We use (193) and the analogous expressions for $I_n(\tau, \mu)$, $n \geqq 1$, given by (68), to prove the final convergence theorem.

Theorem 25. If the Gauss or the double-Gauss quadrature formula is used, then for each $\tau_o > 0$, $I_n(\tau, \mu) \to I(\tau, \mu)$ uniformly for $0 \leqq \tau \leqq \tau_o$ and $-1 \leqq \mu \leqq 1$ as $n \to \infty$.

Proof: Choose $\varepsilon > 0$ and $\tau_o > 0$ arbitrarily, and $x^* > 0$ such that

$$\int_{x^*}^{\infty} e^{-x}(\tau_o + x + 1)\, dx < \varepsilon/F.$$

By (138), (139) and (140),

$J(\tau + \mu x) < F\left[\tau + x + 1\right]$ and $J_n(\tau + \mu x) < F\left[\tau + x + 1\right]$, $n \geqq 1$.

Therefore, by (193) and (68),

$$\left|I(\tau, \mu) - \int_0^{x^*} e^{-x} J(\tau + \mu x)\, dx\right| < \varepsilon, \quad \left|I_n(\tau, \mu) - \int_0^{x^*} e^{-x} J_n(\tau + \mu x)\, dx\right| < \varepsilon,$$

for $0 \leq \tau \leq \tau_o$, $0 \leq \mu \leq 1$, $n \geq 1$. By Theorem 24 there exists n^* such that $|J(\tau)-J_n(\tau)| < \varepsilon$ for $n \geq n^*$, $0 \leq \tau \leq \tau_o + x^*$. Then

$$\left| \int_o^{x^*} e^{-x} J(\tau+\mu x)\, dx - \int_o^{x^*} e^{-x} J_n(\tau+\mu x)\, dx \right| < \varepsilon, \quad n \geq n^*, \quad 0 \leq \tau \leq \tau_o, \quad 0 \leq \mu \leq 1.$$

By the triangle inequality, $|I(\tau,\mu) - I_n(\tau,\mu)| < 3\varepsilon$ for $n \geq n^*$, $0 \leq \tau \leq \tau_o$, $0 \leq \mu \leq 1$. The proof for the range $-1 \leq \mu < 0$ is identical except that x^* should be replaced by $\min(x^*, -\tau/\mu)$. ||

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