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Transmission over wireless networks presents multiple technical challenges due to noise, interference, fading, power constraints and bandwidth limitation. Different solutions have been propposed to overcome these issues and some of them are treated here. Cooperative diversity has been proposed as an implementation for networks where terminals are restricted to using physical arrays; this technique implements space diversity by creating virtual antennas arrays with cooperating nodes in order to combat multipath fading. Network Coding recently has been presented as a technique to increase the throughput in multicast networks. Most of the work done on the topic considers an error free transmission and few works have taken into account the errors due to the nature of the wireless channel. This thesis proposes the use of network coding over some scenarios in relay networks, in order to obtain diversity. It also addresses some cooperative protocols and their performance in terms of bit error rate. Reliability criteria

based on channel information are established for a practical network implementation. In short, we propose a scheme for a wireless network using ideas based on network coding.

Network Coding in Relay Networks

by

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Eduardo Alban, Author

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DEDICATION

To my beloved family: Eduardo, Marianita and Gabi.

Chapter 1 – Introduction

Information and communications technologies play prominent roles in our daily lives. Since the 1980's they have become the engines of economic development in the actual Kondratieff cycle [36]. Nikolai Kondratieff established that economic development is determined by long waves, i.e. Kondratieff cycle, that last 45 to 60 years. It has been established that the source of this development is a main innovative technology that predominates in each cycle. The five Kondratieff cycles that have been identifying and the main technologies in each period are [36]:

- 1780 - 1849 Industrialization, steam engine
- 1849 - 1890 Second Industrial Revolution, railway
- 1890 - 1940 Electrical Engineering, heavy industry, chemicals
- 1940 - 1980 Single-purpose automation, transistor, computer
- 1980 - Information and communications technology

At present, wireless communications is one of the most relevant components of information and communications technology. Wireless communication is a broad field that encompasses all the technologies that allow the transmission of information over a wireless medium.

The development of wireless communication technologies has made possible the communication between people in spite of the great distances that could separate them; an activity that happens every second and turns out to be daily and ordinary for people living in developed countries or where the basic infrastructure exists. Nowadays, a person as far as Australia can communicate with a person in London, whether by cellphone, by Internet or by a satellite phone. Moreover, people do not just communicate with each other, they share and look for information, e.g., people browsing on their wireless laptop, palm or cellphones. Wireless technology is one of the main players in the present technological era.

Wireless technology is less expensive and easy to deploy [35] than the wired one, which makes it a suitable technology to reach areas that are inaccessible or have poor infrastructure as in undeveloped countries. As a consequence, the number of people with access to the services provided has been considerably incremented. Therefore, the importance of wireless communications is not only economical, the main reason for companies to invest in research, but also social. The access to educational information that once was restricted to a minority group with access to libraries and expensive literature, is an enough reason to consider the technology as a social actor of development.

Wireless communications have their origins in 1864 when James C. Maxwell predicted electromagnetic radiation. But it was not until 1893 that it become a reality with the invention of radio communication by Nicolas Tesla [35]. Satellite communications and cellular networks are the most prominent wireless technologies that have had a great success. Since its first deployment, cellular technology has evolved from a voice service, to a voice-data service with an increasing demand for services, this is a proof of the success that wireless communications has had in the last years.

The increasing demand for wireless services has resulted in the development of new technologies such as Wifi which is based on the IEEE 802.11 standard, WIMAX which is based on the IEEE 802.16 standard , bluetooth, ZigBee, etc.. These new technologies are found in a variety of devices like electronic agendas, notebook computers, wireless headphones, that have started becoming part of daily life. As a consequence, the demand for more services related to wireless devices has increased considerably.

There are still many technical challenges that have to be addressed in order to satisfy the demand for wireless services and the future requirements of wireless applications. Improvements in energy efficiency, higher throughput, quality of service, spectrum efficiency and processing time are among them. The success of the technology and the aspects mentioned before have captured the imagination of many researchers in the field who will continue to develop the next generation of wireless communications.

This thesis is focused on the communication process at the physical layer, where the transmission of bits over the wireless medium takes place. Some of the ideas applied here are base on the concepts of information theory developed by Claude Shannon around sixty years ago. Shannon was the creator of information theory and in his work [34] he established concepts such as mutual information and channel capacity. Specifically, we apply these ideas to a specific network configuration called a relay network.

The relay channel was first introduced and studied by Van der Meulen in [38] and [37]. Cover and El Gamal extended the work in [7] and found some bounds of channel capacity. Telatar in [2] established some limits on the channel capacity of multi antenna systems. The concept of cooperative diversity was introduced by Laneman et al. in [23]. Laneman in [21] treats the concept in more depth. Network coding was formally stated by Ahlswede et al. in [1]. In [24], Li et al. restricted their attention to linear network codes. Koetter and Medard presented algebraic network coding in [18]. The concept passed from the theory to pioneering practical applications in COPE [25] or in Avalanche [11]. In the same line, the research community has started to address the practicality and performance of network coding in noisy environments. The work of Xiao et al. in [39] presented an algorithm to compute the performance for algebraic network coding.

Specifically, this thesis treats some fading mitigation techniques, protocols and presents some performance results in terms of the bit error rate on the specific network configuration known as relay networks. A transmission strategy is established as well as some performance criteria. In spite of the fact that most of the results and conclusions obtained here are more practical than theoretical, this thesis is just a start for establishing a formal mathematical framework of the impact of network coding in the performance of realistic wireless networks. The transmission over larger networks could be addressed using an expanded version of the model studied here.

This thesis is organized as follows: Chapter 2 gives a description of relay networks. A diversity technique, i.e., cooperative diversity, that mitigates fading in this networks is treated as well. Next, the mathematical model for the system studied is stated. Then, the protocols and mutual information for some network configurations are described. Finally, the derivation of reliability thresholds and the results of simulations for the different protocols are presented.

Chapter 3 gives a brief description of receiver diversity, a technique that mitigates the multipath effects of the transmission over a wireless medium. Specially, the focus is on Maximal Ratio Combining.

Chapter 4 describes the theory of network coding as defined in [1]. An example that clarifies the concept and its benefits is presented.

Chapter 5 presents the wireless model proposed that puts together cooperative diversity with network coding. The network configuration and the network coding scheme used is stated too. Simulation results are presented.

Finally, conclusions and future work are also presented.

Chapter 2 – Relay Network

A network with intermediate nodes, called relays, can help overcome the effects of multipath fading in wireless transmissions between sources and destinations. This type of network is called a relay network. An example of a relay network with one source, one destination and one relay is shown in Figure 2.1.

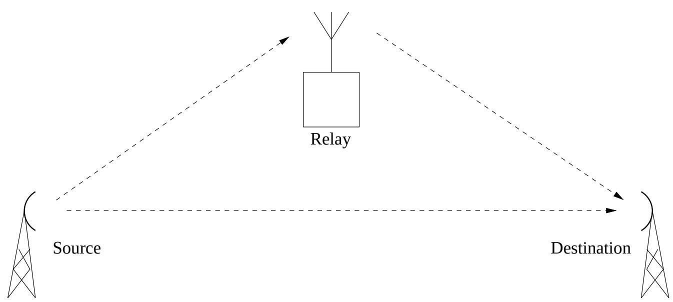


Figure 2.1: Relay Network

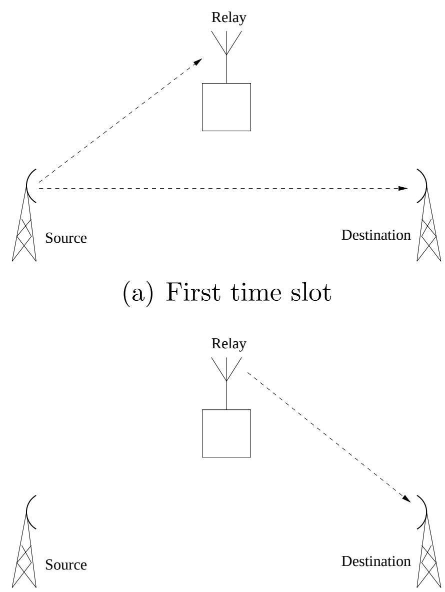
The network is similar to a multiple-input multiple-output (MIMO) system that employs several replicas of a signal over independently fading channels to improve the performance [32, pg. 821]. While in MIMO networks the number of transmitting and receiving antennas is fixed, the number of antennas in relay networks is not fixed given that it depends on how many terminals cooperate. Moreover the information at the relays comes from transmissions over noisy channels [3]. The transmission to the destination is done using a collection of distributed antennas from different terminals that create virtual antennas arrays. This form of spatial diversity is called ”cooperative diversity” [21]. This type of diversity takes advantage of the broadcast nature of the wireless medium in which a transmitted signal can be received by any terminal inside the source coverage area.

Conceptually, cooperative diversity can be explained briefly using the scenario shown in Figure 2.2, where the transmission is performed in two time slots. In the first time slot the source broadcasts a signal; this is received by the destination and a relay that is in the proximity of both source and destitnation nodes and has decided to cooperate in the transmission process. In the second time slot, the relay processes the signal and transmits it to the destination. As the last step, the received signals are combined under some criteria and passed onto a decision device. The logic behind this type of redundancy will improve the performance of the transmission compared with a point-to-point transmission.

Cooperative Diversity has to address, among other things, the following issues:

- Modulation schemes
- Relay processing schemes
- Detection techniques
- Power allocation
- Coding strategies

Innovative solutions and implementations on the aspects mentioned above have been presented by the research community. Modulation using orthogonal schemes are treated in [29], [44] and [28]; detection techniques such as selection combining,



(a) First time slot

(b) Second time slot

Figure 2.2: Cooperative Diversity scheme

maximum likelihood for noncoherent demodulation and a minimum mean square scheme are treated in [17], [5] and [4].Relay processing schemes are found in [21], [46] and [19]. Optimal power allocation algorithms under different constraints are presented in [26] and [27]. Space-Time coded protocols are analyzed in [22] and coded cooperation is examined in [15].

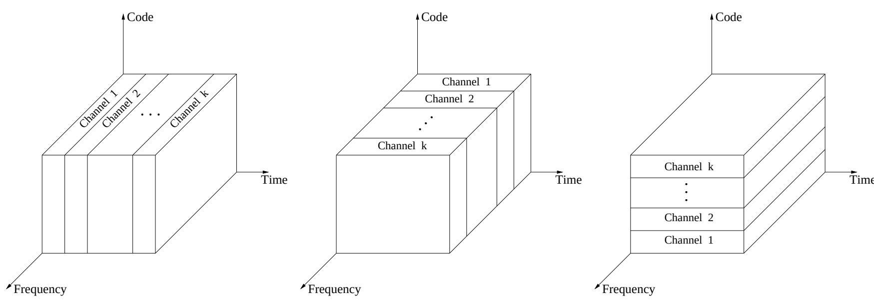
The interested reader can consult any of the documents mentioned before as well as its corresponding references for more details. The present document takes into consideration some of the important results and conclusions, from the mentioned work, that are suitable for the model presented here.

In the following section, different relay processing protocols are presented and discussed.

2.1 System Model

The wireless network architecture studied in this thesis and the assumptions made are presented next.

The model consists of a single source, a single destination and multiple terminals that act as relays, similar to the model in Figure 2.1 where only one relay is considered.



(a) TDM (b) FDM (c) CDM

Figure 2.3: Multiplexing types

We consider a single communication channel where multiple transmissions occur at the same time. As a result, interference among the signals occurs. The multiplexing techniques shown in Figure 2.3 have been proposed to overcome this. A brief description of the techniques folows [45].

Time Division Multiplexing (TDM):

> In this technique, each terminal takes turns to transmit. The time line is divided in cyclic time slots. Each time slot corresponds to a subchannel where the corresponding terminal transmits.

Frequency Division Multiplexing (FDM):

Multiple signals are transmitted in different frequencies. Under this scheme, the transmissions occur at the same time. If k messages have to be sent, the available bandwith is divided in k subchannels.

Code Division Multiplexing (CDM):

Each message signal is assigned to a specific code that spreads the signal over a common available bandwith. The codes are independent to each other thus the signals can be individually recovered at the receiver.

It is assumed that perfect syncronization exists between the terminals and each receiver is capable of coherent detection. Moreover, the narrowband transmissions over the wireless channel are affected by non selective frequency fading, i.e. the coherence bandwith of the channel is much larger than the signal bandwith [32, pg. 815]. Consequently, a transmission between two terminals at the discrete time k is represented by the discrete-time model for the continuous-time channel as following [12, pg. 102]:

$$y_j[k] = a_{i,j}[k]x_i[k] + z_j[k] \qquad i \neq j, \tag{2.1}$$

where $x_i[k]$ is the transmitted signal sent by terminal i at time k. The received signal at the j destination is represented by y_j. The term $a_{i,j}$ represents the time varying characteristic of the channel between terminal i and teminal j. The noise introduced at the receiver j by the electronic components, i.e. thermal noise, is represented by z_j and it is statistically characterized as an additive white Gaussian noise(AWGN)[32, pg. 11].

In order to represent the effects of fading and shadowing due to multipath, the channel $a_{i,j}$ is modeled as a circularly symmetric zero mean complex Gaussian random variable with variance σ^2 [20]. The resulting baseband channel is a complex random process of the form

$$a_{i,j}(t) = \alpha(t)e^{-j\theta(t)}, \tag{2.2}$$

where $\alpha(t)$ is the envelope that is Rayleigh distributed. The phase is represented by $\theta(t)$ which is uniform distributed over $(-\pi, \pi)$ [32, page 815]. The noise z_j at the terminal j is modelled as zero-mean, circularly-symmetric complex Gaussian random sequences with variance N_0.

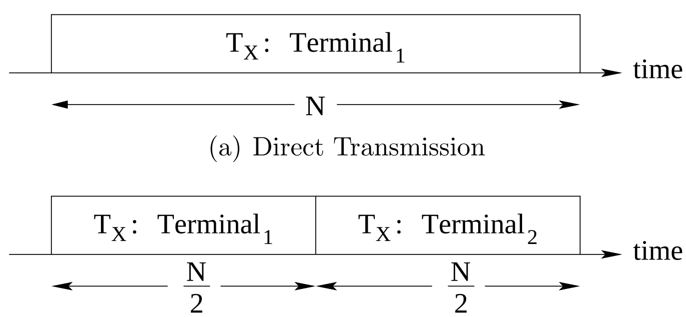
Without loss of generality, we assumed that the available bandwith B for transmission is equal to unity, i.e., $B = 1$Hertz. We also keep the same spectral efficiency for all the scenenarios. Therefore, when one of the diversity techniques explained before is used, modulation with higher data rate is employed in the transmission.

At the receivers, we consider detection techniques where Channel Side Information(CSI) is available, i.e., channel coefficients are perfectly estimated. In practical scenarios, this can be done using training signals with each transmitted signal affected by different channel conditions [33]. In the model studied here no error correcting or detecting technique is used.

2.2 Cooperative Protocols

Laneman in [21] outlines some low-complexity protocols for cooperative diversity that take into consideration the current physical limitations of the radios that cannot transmit and receive at the same time, *half duplex* transmission. As mention in last section, coherent detection and CSI at the receivers is assumed, with each channel independent of each other.

For the purpose of the model studied here, time division is used, therefore each transmission channel is split into two time slots as is seen in Figure 2.4. Using the same example of Figure 2.2 from the previous section, we have that the source broadcasts a signal to the destination and the relay in the first time slot and in the second time slot the relay forwards the received signal using one of the cooperative diversity protocols described below.



(a) Direct Transmission

(b) Time Division Multiplexing

Figure 2.4: Time Allocation

The protocols can be categorized as in [46] by their forwarding strategy at the relays and by their dynamic behavior. The protocols under the former criterion are:

Amplify and Forward:

The signal received at the relay is amplified by a factor β before retransmission. The value of β is restricted to a power constraint that has to be fulfilled by any transmitting terminal.

Decode and Forward:

The received signal is first demodulated and decoded. The obtained binary sequence is encoded, modulated and finally transmitted.

The protocols under the dynamic behavior criterion are:

Fixed Protocol:

The relays always transmit their received signals.

Adaptive Protocol

At each terminal, the received signal could occasionally be discarded using a strategy that produces the best performance. This scheme can be implemented, based on the knowdledge of the channel state information, by testing if the effective signal to noise ratio (SNR_{eff}) falls below some threshold, at which the received signal could be considered unreliable[21].

We want to maintain the characteristics of the link between source and relay for the decoding of the signal at the final destination. Therefore, we focus our attention in the amplify and forward scheme implemetation. Moreover, since an adaptive protocol achieves higher diversity gain [21], we employ what we called an "Adaptive Amplify and Forward" protocol in this work.

The adaptive technique can be described as:

- First the signal is received along with the CSI.
- The received signal is amplified and sent only if the SNR_{eff} is above some threshold.

Figure 2.5 shows a flow diagram of the algorithm at the relay.

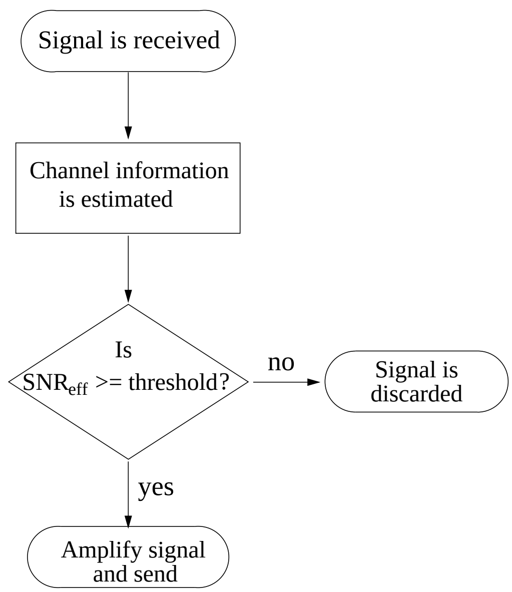


Figure 2.5: Algorithm at the relay

The chosen threshold depends on reliability requirements. Next some thresholds are obtained, but first the necessary background to establish those is presented.

2.3 Mutual information

The mutual information for multiples scenarios is described here. The derivations and results obtained in this section are useful in later ones.

By definition the mutual information between the random variable X and the random variable Y in terms of their corresponding entropies, i.e. the amount of uncertainty of the random variables, is defined as [8]

$$\mathcal{I}(X;Y) = \mathcal{H}(Y) - \mathcal{H}(Y|X). \tag{2.3}$$

where the mutual information $\mathcal{I}(X;Y)$ represents the amount of uncertainty that remains from the random variable Y after knowing the random variable X [8].

Now, having $Y = aX + Z$ with X and Z independent of each other, we have that

$$\begin{aligned} \mathcal{I}(X;Y) &= \mathcal{H}(Y) - \mathcal{H}(Y|X) \\ &= \mathcal{H}(Y) - \mathcal{H}(aX + Z|X). \ \cdot \\ &= \mathcal{H}(Y) - \mathcal{H}(Z) \end{aligned} \tag{2.4}$$

By definition, the channel capacity is given by [8]:

$$C = \max_{p(x)} \mathcal{I}(X;Y). \tag{2.5}$$

Consequently, the capacity is obtained maximizing the difference between the entropies given by the expression $\mathcal{H}(Y)-\mathcal{H}(Z)$. However, the noise is self generated by the receiver, therefore it has an entropy independent of the signal transmitted.

Thus, the maximization of 2.4 is been done maximizing $\mathcal{H}(Y)$. Telatar proved in [2, pag. 4] that the entropy of a random variable Y, i.e $\mathcal{H}(Y)$, is maximized when the random variable Y is zero-mean circularly symmetric complex Gaussian. Therefore, in this section, we restrict our attention to circularly symmetric complex Gaussian random variables in order to obtain the results for the best possible scenario.

Thus if Y is a circularly symmetric complex gaussian randdom variable with variance σ^2, the differential entropy for a band-limited channel with an available bandwith B is given by [12]

$$\mathcal{H}(Y) = B\log(2\pi e\sigma^2). \tag{2.6}$$

2.3.1 Direct Transmission

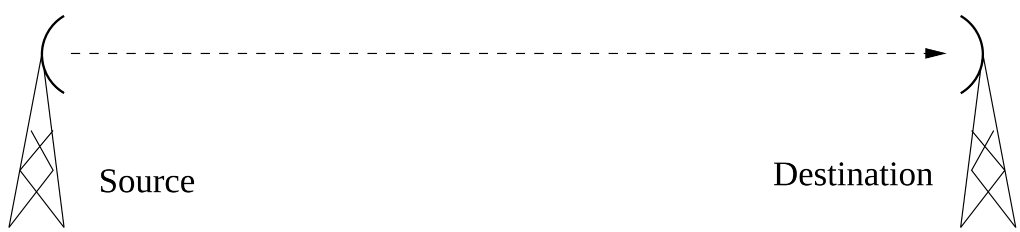


Figure 2.6: Direct Transmission

A direct transmission between terminal i and terminal j is described by the relation $y_j = a_{i,j}x_i + z_j$, which has a power constraint for the trasmitted signal of $E\{x^2\} = P$ and an available bandwith of $B = 1$ Hertz. Therefore, with CSI at the receiver the received power is given by:

$$\begin{aligned}
E\{y_j y_j^*\} &= E\{(a_{i,j}x_i + z_j)\,(a_{i,j}x_i + z_j)^*\} \\
&= E\{|a_{i,j}|^2 x^2 + a_{i,j}x_i z_j^* + z_j a_{i,j}^* x_i^* + |z_j|^2\} \\
&= |a_{i,j}|^2 E\{x^2\} + E\{|z_j|^2\} \\
&= |a_{i,j}|^2 P + N_0
\end{aligned} \quad . \tag{2.7}$$

The mutual information is given by:

$$\begin{aligned}
\mathcal{I}_D &= \mathcal{H}(y) - \mathcal{H}(z) \\
&= \log(2\pi e E\{yy^*\}) - \log(2\pi e N_0), \\
&= \log(1 + SNR|a_{i,j}|^2)
\end{aligned}$$

which can be generalized for m transmitters as:

$$\mathcal{I}_D = \frac{1}{m}\log\left(1 + SNR\sum_{i}^{m}|a_{si,d}|^2\right) \tag{2.8}$$

2.3.2 Multihop transmission

In a multihop transmission, as the one showed in Figure 2.7, there is no direct link between the source and the destination. In this configuration, the source transmits a packet to the relay. Before the relay transmits to the destination, it can amplify the received signal or it can fully decode the signal. In the first case, the received

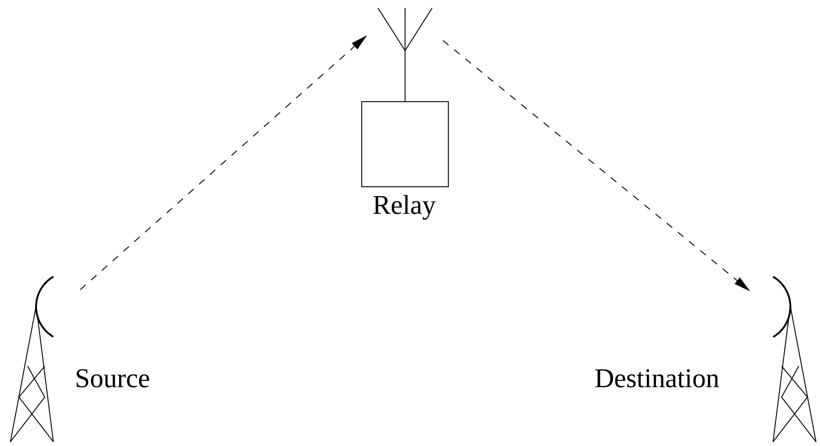


Figure 2.7: Tranmission between through relay

signal $y_r = a_{s,r}x_s + z_r$ is amplified by a factor β as

$$x_r = \beta y_r, \tag{2.9}$$

where the transmited power is under the power constraint $E\{x_r^2\} \leq P$. The autocorrelation of the signal to be trasmitted is given by:

$$\begin{aligned}
E\{x_r^2\} &= E\{(\beta y_r)(\beta y_r)^*\} \\
&= E\{(\beta\,(a_{s,r}x_s + z_r))\,(\beta\,(a_{s,r}X_s + Z_r))^*\} \\
&= E\{\beta^2|a_{s,r}|^2x_s^2 + \beta a_{s,r}x_s z_r^* \\
&\qquad +\beta a_{s,r}^* x_s^* z_r + \beta^2 z_r^2)\} \qquad . \\
&= \beta^2|a_{s,r}|^2E\{x^2\} + \beta a_{s,r}E\{x_s\}E\{z_r^*\} \\
&\qquad +\beta a_{s,r}^*E\{x_s^*\}E\{Z_r\} + \beta^2E\{z_r^2\} \\
&= \beta^2(|a_{s,r}|^2P + N_0) \leq P
\end{aligned}$$

Therefore,

$$\beta \leq \sqrt{\frac{P}{|a_{i,j}|^2 P + N_0}}. \tag{2.10}$$

Now, the signal received at the destination is given by the expression:

$$y_d(t) = a_{r,d}(t)\beta a_{s,r}(t)x(t) + \beta z_r(t) + z_d(t).$$

The autocorrelation at the receiver, see Appendix 1 , is given by:

$$E\{y \cdot y^*\} = |a_{r,d}|^2 |a_{s,r}|^2 \beta^2 E\{x^2\} + |a_{r,d}|^2 \beta^2 N_0 + N_0$$

The mutual information when the transmission is in a multihop fashion with an amplifying scheme at the relay is given by:

$$\mathcal{I} = \frac{1}{2} \log(1 + \beta^2 |a_{r,d}|^2 + SNR\beta^2 |a_{r,d}|^2 |a_{s,r}|^2) \tag{2.11}$$

When the relay decodes the signal before transmitting it, the mutual information can be obtained as [21]

$$\mathcal{I}_{DF} = \frac{1}{2} \min\{\log(1 + |a_{s,r}|^2), \log(1 + |a_{r,d}|^2)\} \tag{2.12}$$

The first term is the maximum rate at which the relay can decode and the other one is the rate at which the destination can decode.

2.3.3 Amplify and Forward Relay Transmission

The amplify and forward scheme, Figure 2.2, differs from the multihop transmission because there is a link between the source and the destinaton. In this scheme, the relay always scales the received signal by factor β under the same power constraint as before.

In this case the mutual information is given by [20]

$$\mathcal{I}_{AF} = \frac{1}{2}\log(1 + SNR|a_{s,d}|^2 + f(SNR|a_{s,r}|^2, SNR|a_{r,d}|^2)), \tag{2.13}$$

where

$$f(x, y) := \frac{xy}{x + y + 1}.$$

Which can be generalized for m-1 relays as

$$\mathcal{I}_{AF} = \frac{1}{m}\log\left(1 + SNR|a_{s,d}|^2 + \sum_r f(SNR|a_{s,r}|^2, SNR|a_{r,d}|^2)\right). \tag{2.14}$$

2.3.4 Adaptive Amplify and Forward Relaying

In an adaptive scheme using an amplify and forward scheme we have:

$$\mathcal{I}_{AAF} = \begin{cases} \frac{1}{2}\log(1 + SNR|a_{s,d}|^2), & |a_{s,r}|^2 < \text{reliable threshold} \\ \frac{1}{2}\log(1 + SNR(|a_{s,d}|^2 + |a_{r,d}|^2)), & |a_{s,r}|^2 \geq \text{reliable threshold} \end{cases}. \tag{2.15}$$

2.3.5 Selection Relaying

In selection relaying, using repetition codes, the mutual information is given by [21]

$$\mathcal{I}_{SDF} = \begin{cases} \frac{1}{2}\log(1+2SNR|a_{s,d}|^2), & |a_{s,r}|^2 < g(SNR) \\ \frac{1}{2}\log(1+SNR(|a_{s,d}|^2+|a_{r,d}|^2)), & |a_{s,r}|^2 \geq g(SNR) \end{cases} . \qquad (2.16)$$

where $g(SNR)$ is some reliable criteria base on the effective SNR.

2.4 Thresholds

The adaptive protocols, described before, require the set up of a threshold above which the transmission between two terminals is considered reliable. The derivation of this threshold for different scenarios is described in this section.

The performance of a wireless communication system is characterized by the bit error rate (BER), i.e., the probability of error at a specific signal-to-noise ratio. On the other hand, the reliability of a system is based on the BER requirements of a specific application; for example BER requirement for voice applications is 10^{-3} and data transmissions is as low as 10^{-8} [12]. Therefore, the optimum threshold is established from the performance view point, taking into consideration the technical requirements of the specific wireless application.

The threshold can be established by means of simulation using a trial-and-error method, i.e., varying the threshold until the wanted BER is obtained. Though this method is not very efficient, it is still a valid approach. A more technical approach using information theory is described in the following paragraphs. It is important to point out that the threshold obtained with the last approach could not be the optimum for an specific application, therefore it can be combined with the trial an error approach to obtain a better one.

From the information theory point of view, an outage event is defined as the event where the mutual information between the transmitted signal and a received one $\mathcal{I}(X;Y)$ goes below some given rate R [30]. Therefore the outage event is expressed as $\mathcal{I}(X;Y) < R$. On the other hand, a reliable communication can be considered to occur when

$$\mathcal{I}(X;Y) \geq R. \tag{2.17}$$

From the mutual information results obtained in the last section, the thresholds can be obtained for each scenario. As example, the threshold for a direct transmission is obtained as follows:

$$\begin{aligned} \mathcal{I}(X;Y) &\geq R \\ \log_2(1 + kSNR) &\geq R \\ k &\geq \tfrac{(2)^R - 1}{(E_b/N_0)} \end{aligned} .$$

Therefore, for the network in Figure 2.1 only half of the bandwidth is utilized and the threshold is given by

$$k \geq \frac{2^{2R} - 1}{(E_b/N_0)}. \tag{2.18}$$

Where the outage event for spectral efficiency R is given by $I_D < R$, therefore

$$|a_{s,d}|^2 < \frac{2^R - 1}{SNR}. \tag{2.19}$$

And the outage event for spectral effiency R is given by $I_{AF} < R$, therefore

$$|a_{s,d}|^2 + \frac{1}{SNR} f(SNR|a_{s,r}|^2, SNR|a_{r,d}|^2) < \frac{2^{2R} - 1}{SNR}. \tag{2.20}$$

2.5 Simulation

Monte Carlo simulations for a traditional non cooperative model and the one in Figure 2.8 are presented in order to compare the performance of the protocols described before.

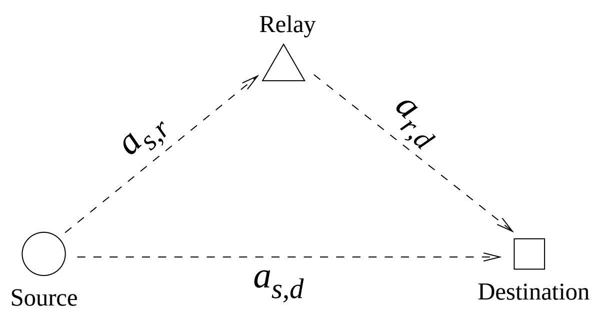


Figure 2.8: Simulation model

All the simulations have been parametrized to have an spectral efficiency of R = 1bit/s/Hz. Moreover, a TDM transmission scheme is used for the cooperative protocols. Therefore, the transmission rate has to be increased in order to maintain the same spectral efficiency and make it fully comparable with the non cooperative scheme. Consequently, a BPSK modulation scheme is used for the last one, i.e., direct transmission, and a QPSK modulation scheme is been used for the cooperative schemes.

We model the network of Figure 2.8 for a statistically symmetric case, i.e. $a_{s,d} = a_{s,r} = a_{d,r}$ with $\sigma^2 = 1$. Also, the amplification factor at the relays is set to $\beta = 1$ to obtain the best performance as it is shown in [46].

The performance of the cooperative protocol is conditioned by the quality of the transmission between source and relay; the more reliable the link is, the better the overall performance. Therefore, a lower bound for the BER is obtained assuming a perfect transmission between source and relay. The bound called Orthogonal Transmit diversity [21] is obtained by assuming that two transmissions of the same signal are made from source to destination. Each transmission uses half of the available channel for transmission. In this case under TDM scheme, the transmission between the source and the relay at the first time slot is expressed as

$$y_r[k] = a_{i,r}[k]x_i[k] + z_r[k] \qquad k = 1, \ldots, N/2, \tag{2.21}$$

while at the second time slot the trasmission between the relay and the destination is given by

$$y_j[k] = a_{r,j}[k]x_r[k] + z_j[k] \qquad k = N/2 + 1, \ldots, N. \tag{2.22}$$

From Figure 2.9 it is evident that the adaptive protocol performs better and gets closer to the transmit diversity bound.

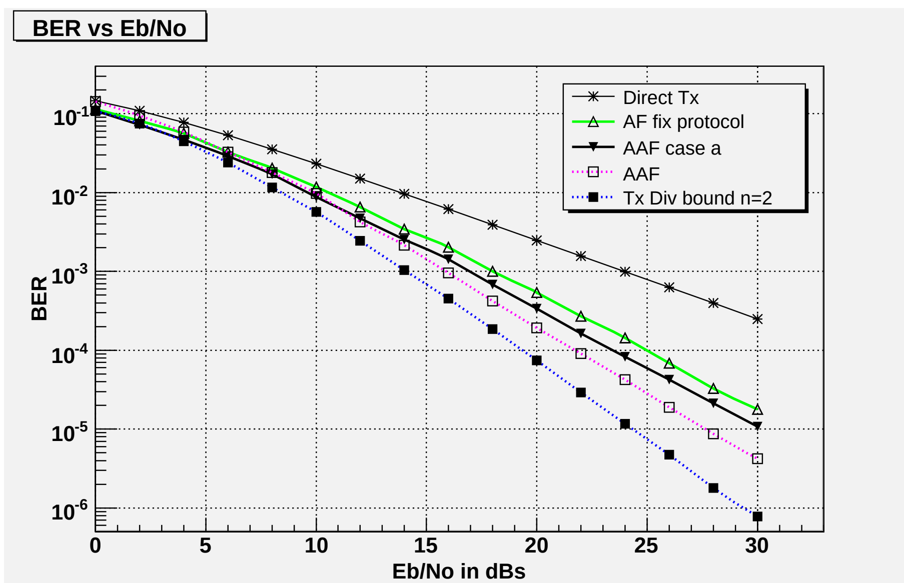


Figure 2.9: Cooperative diversity performance

Chapter 3 – Receiver Diversity

Wireless Communications are affected by the dynamics of the environment. This as a result of the time varying nature of the channel and the mobility of users and objects in between. This challenging scenario that becomes unpredictable causes the transmissions on a wireless environment to be affected by multipath fading and shadowing.

Diversity techniques have been proposed as solutions to mitigate the channel effects by sending the same signal over independent paths, as it is shown in Figure 3.1. The techniques take advantage of the low probability that independent channels are affected by a large fade at the same time [12]. The diversity techniques are classified as microdiversity if it deals with path loss and macrodiversity if it deals with shadowing.

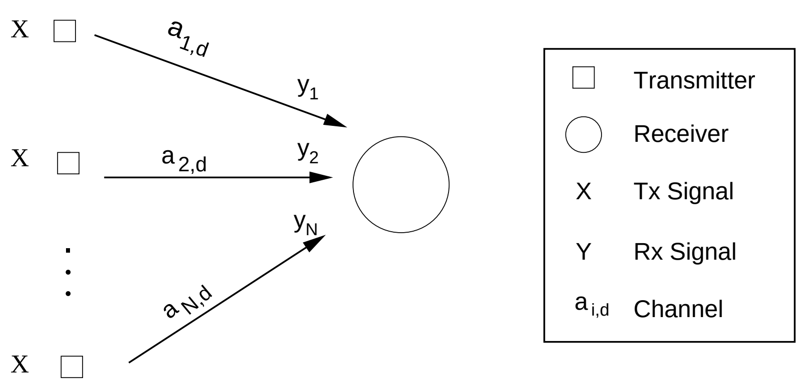


Figure 3.1: Independent transmitted signals

For the former work, we focus our attention on a microdiversity technique called space diversity. Space diversity is a well known technique that uses multiple antennas at the transmitter or at the receiver. We have to point out that in our model each of the terminals has just a single antenna. However, as it was explained before, a cooperative diversity scheme can be used when more terminals are involved in the transmission. This creates a virtual antenna array [21] which allows to have multiple realizations of a same signal at the receiver. Therefore, the diversity techniques explained below are still applicable will small modifications

Receiver diversity allows us to combine independent fading realizations of a signal, into another that is passed to the demodulator. Between the different techniques that have been proposed we have [12]:

Selection Combining

The incoming signals are passed to a switching device. It scans through all the signals and chooses the one with the highest SNR_{eff} as output. Finally, the output signal is passed to the decision device.

Threshold Combining

This diversity technique scans sequentially the received signals and compares the SNR_{eff} with some threshold. As soon as one of them is above the threshold, the scanning is stopped and the corresponding signal is sent to the output. The device keeps tracks of the SNR_{eff} of the signal, so when it falls down the threshold the scanning starts again. This technique is similar to the one described but with a faster response given that it does not scan the whole incoming signals all the time.

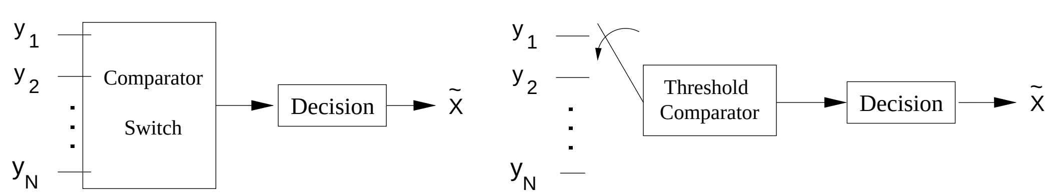
Maximum-Ratio Combining

In this technique all the received signals are multiplied by a weighted factor obtained from channel estimation and then added together. The result is passed to the output for further processing. Before any combination is done the received signals have to be coherently detected. The probability of error for L received signals is [12]:

$$P_b = \left(\frac{1-\Gamma}{L}\right)^L \sum_{m=0}^{L-1} \binom{L-1+m}{m} \left(\frac{1+\Gamma}{2}\right)^m \tag{3.1}$$

where Γ is given by:

$$\Gamma = \sqrt{\frac{SNR_{avg}}{1+SNR_{avg}}}$$



(a) Receiver Combining (b) Threshold Combining

Figure 3.2: Combining techniques

In our model a Maximum Ratio Combining with a threshold criterion is used. Figure 3.3 shows a model of the technique.

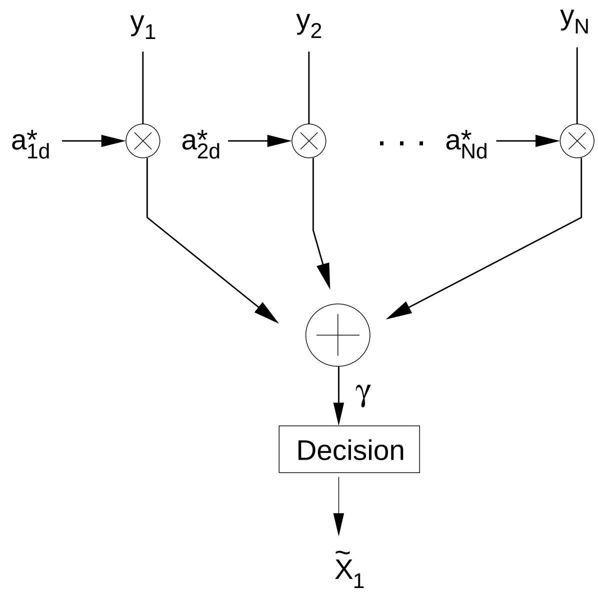


Figure 3.3: Receiver Combiner

The decision variable is obtained as a weighted sum of all the received signals. With the CSI known at the receiver, the weighted value is given by the conjugate of the channel, $a^*_{i,j}$, which co-phase the received signals and achieve the best reduction of fading from any known linear combining technique [33]. Then, the decision variable is given by

$$\gamma = \sum_i a_i^* y_i. \tag{3.2}$$

This signal is passed through a decision device which converts it in binary data.

$$\tilde{X} = decision(\gamma). \tag{3.3}$$

Chapter 4 – Network Coding

Network Coding was formally stated by Ahlswede et al. in [1]. This came up as a result of a previous work for satellite communications by Yeung et al. in [42]. Network coding has been proposed as a scheme that improves the throughput in a network [1] comparing to the traditional store and forward.

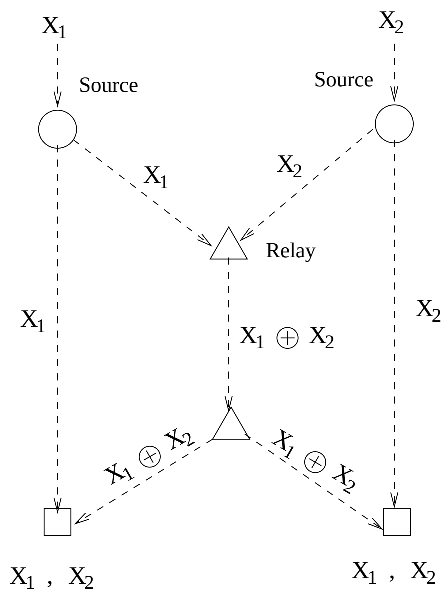
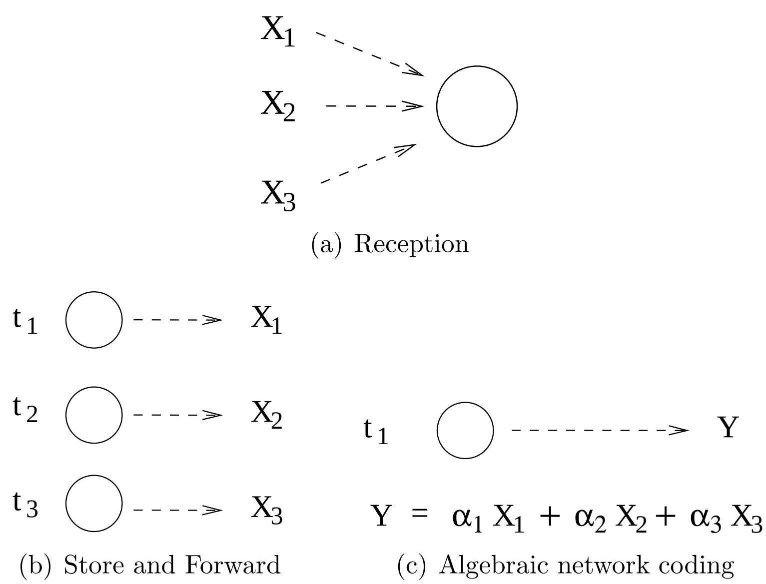


Figure 4.1: Network Coding

The following example illustrates the concept of network coding. Consider that two terminals request packets X_1 and X_2 to be sent to them. In the vicinity, two terminals become the sources of the packets; these are linked with the destinations by direct connections and through other terminals, i.e. relays. The network created is the well known butterfly network [1] shown in Figure 4.1. The transmission proceeds as follows. First the packets X_1 and X_2 are transmitted by the sources. Packet X_1 and X_2 are received by the terminals that are directly linked to it, in this case the packets are received by the corresponding destinations and the relay. Until now nothing has change from the traditional transmission scheme; the differences and the advantages of network coding become evident from this point. Next, the packets X_1 and X_2 at the relay have to be transmitted. In the store and forward scheme, only one packet can be transmitted at a time; therefore, one of the packets stays in the queue while the relay transmits the other one. The stored packet is transmitted later to complete the delivery of the packets X_1 and X_2 to the final destinations. With network coding, there is no need to store a packet at the relay given that the incoming packets can be mapped into a single packet by a network code[1], in this case this is done with a simple XOR operation, i.e. $X_1 \oplus X_2$, and transmitted. The advantages of network coding are evident, it takes less number of transmissions to deliver the packets than the store and forward scheme. As a result the network throughput is increased when network coding is used.

From the previous example, we can see that some important aspects of network coding stand out. First, the example deals with an ideal scenario where it is assumed that all links work all the time. Also, it assumes that the links are error free. Next, the encoding operation $X_1 \oplus X_2$ at the relay is a suitable encoding scheme for this particular network topology given that one of the encoded packets will be available at each destination. Consequently, each destination terminal has enough information to decode the packet send by the relay. It is evident that the failure of one of the links could cause that no packet can be decoded at a destination terminal. Moreover, network coding is not applicable to some topologies. Thus, network coding can be applied in particular network topologies under certain reliability conditions.



(a) Reception

(b) Store and Forward

(c) Algebraic network coding

Figure 4.2: Combining techniques

Following the work by Ahlswede et al. in [1], Li et al. in [24] established that a linear mapping, i.e. linear network coding, is sufficient to achieve the capacity bounds established in [1]. Generally speaking, linear network coding states the possibility to send as an output a linear combination over a finite field $\mathcal{F}$ of multiple incoming packets [18]. Figure 4.2 illustrates this concept and makes a comparison with the classical transmission scheme of store and forward. The illustration shows some specific node in a network where independent data, X_1, X_2 and X_3, from three different sources are received. In the classical transmission scheme, the node sends each of the packets one by one, which is reflected in the time that the node employs to transmit all the received packets. The node takes three packet transmission time units to send all the inputs. On the other hand, the independent data can be combined into a unique packet using linear network coding, namely, under the assumption that given the broadcast nature of the network the recipient of the packet has enough information to decode it. With the later scheme, the transmission process takes only one packet transmission time unit, consequently network resources are saved.

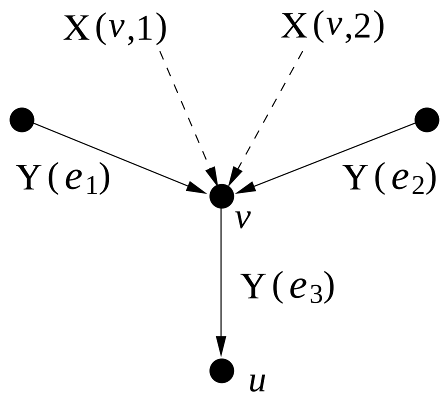
Koetter and Medard in [18] presented an algebraic framework for linear network code and explore some network coding solutions under links failure scenarios. Thus, having a graph $\mathcal{G}(V, E)$, that represents a communication network, where V is the set of vertex or nodes and E is the set of edges or links e, the authors showed that the input X and the output Z in a graph where linear network coding is given by the following linear operation

$$Z = X \cdot M \tag{4.1}$$

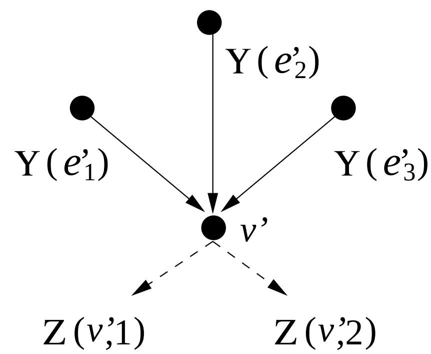
where M is a transfer matrix that contains the overall encoding factors and can be decomposed as

$$M = A(I - F)^{-1} B^T \tag{4.2}$$

In order to maintain the same notation from the original work, when we refer to network coding, we represent $X(v, l)$ as the input l at node v, the output signal at node v' as $Z(v', l)$ and $Y(e_i)$ as the information flowing on the edge i, i.e, e_i. The elements of matrices A, B, and F were defined to be the encoding factors along the graph. Figure 4.3 shows the relationship between $X(v,l)$, $Y(e_i)$, $Z(v',l)$ and the



(a) At source node



(b) At destination node

Figure 4.3: Algebraic Network Coding

encoding factors α_{e_i,e_j}, $\omega_{l,j}$ and $\varepsilon_{e,j}$ over a finite field $\mathcal{F}$ at source and destination nodes. Therefore, the signal $Y(e)$ transmitted over the link e is obtained as

$$Y(e) = \sum_{l=1} \alpha_l X(v,l) + \sum_{i:head(e')=tail(e)} \omega_{e'_i,e} Y(e'_i), \tag{4.3}$$

where α_l is the encoding factor for the l input signal generated by the source and $\omega_{e'_i,e}$ is the encoding one for the incoming signal from the e'_i link. On the other hand, the output j at node v' is given by

$$Z(v',j) = \sum_{l} \varepsilon_{e'_l,j} Y(e'_l), \tag{4.4}$$

where $\varepsilon_{e'_l,j}$ is the decoding factor of the received signal from the lth link.

By definition, A is a matrix with dimensions $u \times |E|$, where $|E|$ is the number of edges in the graph that represents the network and its elements are obtained as

$$A_{i,j} = \begin{cases} \alpha_{l,e_j} & x_i = X(\mathrm{tail}(e_j), l) \\ 0, & \text{otherwise} \end{cases}, \tag{4.5}$$

and the elements of matrix B are

$$B_{i,j} = \begin{cases} \varepsilon_{e_j,l} & z_i = Z(\mathrm{head}(e_j), l) \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

The elements for matrix F are obtained from the corresponding directed labeled line graph $\mathfrak{G}(\mathbb{V}, \mathbb{E})$. This graph is formed using nodes that represent the edges of the actual graph $\mathcal{G}(V, E)$ and with its vertex representing the transitions between the links of the actual graph. The factor $\omega_{e'_i,e}$ represents this transition. An example is shown in Figure 4.4. Consequently, by definition in [18], the elements

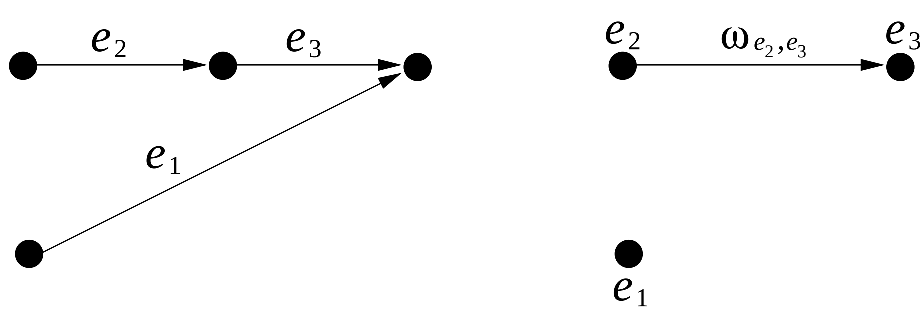


Figure 4.4: A network with its labeled representation

of matrix F are

$$F_{i,j} = \begin{cases} \omega_{e_i,e_j} & \text{head}(e_i) = \text{tail}(e_j) \\ 0, & \text{otherwise} \end{cases} \tag{4.7}$$

The three aspects that have to be taken into account in the construction of a r-dimensional linear network code in an acyclic network, i.e. a network without loops, over some finite field $\mathcal{F}$ [41, page 25] are:

- The dimension r of the code
- The topology of the network
- The base field $\mathcal{F}$

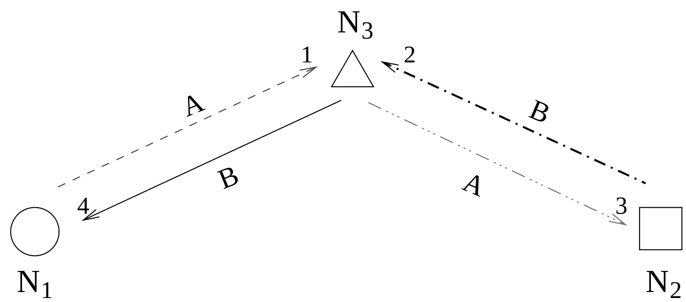
Considering codewords of length r and a binary transmission, the linear encoding is performed over the finite field, $\mathcal{F}$, $GF(2^r)$. A network code can be found using the algorithm in [18]. This can be a complicated and time consuming task because of the necessity for the knowledge of the entire network topology. An alternative to this algorithm has been proposed by Ho in [14]. The authors give a distributed scheme where each node encoded independently of each other using random codes. With this scheme, the destination only needs the overall linear combination coefficient to decode the received codeword. They demonstrate the sufficiency of random codes when the network coding scheme is implemented over a sufficiently large finite field.

The work by Lun et al. in [25] gives a brief review of some relevant aspects of network coding in wireless environments. The necessity of energy efficiency implementations and distributed algorithms are the aspects pointed by the authors. The dynamics of the network makes it evident that the use of a centralized algorithm could become an inefficient and energy consuming solution.

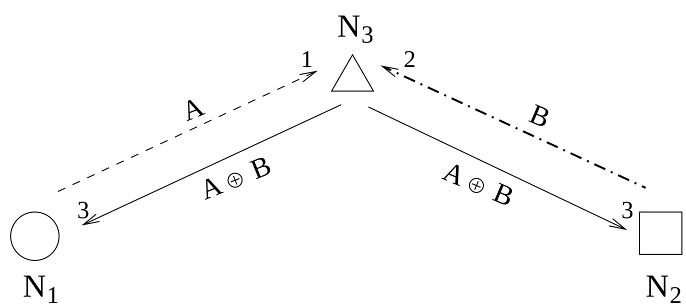
The relay network studied here does not require the construction of any sophisticated code. Thus, the design of the network coding scheme reduces its complexity to a simple XOR operation. This particularity brings our attention to the practicality of network coding over realistic scenarios, noisy environments, and its consequences on the overall performance rather than the aspects and requirements that involves the design of an optimum and suitable linear network code.

In the last few years, network coding has gone from being a theoretical concept, where the benefits proclaimed by the research community were found just in papers, to practical applications in real scenarios. An example of this is Avalanche. Avalanche is a peer-to-peer application for the distribution of large files using network coding [11]. The system involves a limited-capacity single server and multiple clients requesting a file. In order to serve all the clients, the server splits the file into smaller packets that are transmitted on a peer-to-peer basis to some clients. Then, they share its bandwidth to reach other clients. The distribution of the smaller packets is done using a linear combination of the packets; the network encoding is done at the server and at the collaborating nodes. The authors demonstrate an improvement of 20-30% in the file download time compared with a system without network coding. Network coding has been proposed also for [10] network tomography, network security, sensor networks and wireless networks.

The necessity of conserving resources, e.g., battery life and the inherent property of sharing resources such as bandwidth on wireless networks makes it a suitable



(a) Traditional scheme



(b) Network coding

Figure 4.5: Wireless application of network coding

scenario where network coding can be employed [9]. A reason for this is the reduction in the number of transmissions compared with a traditional scheme. An example of this is shown in Figure 4.5 where the network coding scheme requires just three transmissions instead of four from the traditional one.

Most of the work in this area has been done under the assumption of error-free links without considering the issues presented at the physical layer . The error occurrence due to the transmission over the wireless medium is a fact that cannot be ignored for practical applications. Lately, the research community has tried to address these issues. In the work of Xiao et al. in [39], an algorithm to obtain the performance of a network in terms of the bit error probability, i.e., BEP, is derived using transfer matrix M obtained in [18]. The algorithm in [39] can be summarized as follows:

1. Matrices A, B and F are found for the corresponding network following [18].
2. Matrix H is obtained as
$$H = A(1 - F)^{-1} \tag{4.8}$$
3. The encoding factors of matrix H, e.g., α_{1,e_3} and ω_{e_4,e_5}, are replaced by their second subscript, e.g., e_3 and e_5. These subscripts correspond to the edges of the graph $\mathcal{G}(V, E)$ where the information flows.
4. In each column of matrix H the redundant elements are replaced with ones.
5. The edge or edges assumed to be in error are marked with the variable T, which help us to keep track of the error propagation in the network.

6. Matrix M is found as $M = HB^T$, then all the unknown variables, with the exception of T, are replaced with ones.

7. In a binary transmission, a channel error occurs when 0 is received instead of 1 or 1 instead of 0. But when two channel errors occur in the same path, the received bit is the same as the one transmitted, e.g., 0 -¿ 1 -¿ 0. This is expressed using the tracking variable T as

$$T \cdot T = 1. \tag{4.9}$$

   Likewise, an error is cancel if two erroneous bits are added together which is expressed as

$$T + T = 0 \tag{4.10}$$

   Using these equations we simplify matrix M.

8. The number of columns that have an odd number of T's are counted. This number represents the weight of the error.

9. Steps 5 to 8 are repeated for all the possible erroneous scenarios.

10. Finally, the bit error probability is obtained summing the weights of each event multiplied by their corresponding probability.

The results presented in [39] for AWGN channels show our intuitive idea that transmission errors propagate on the network and the overall performance drops. The same authors extended their work in [40] for hard and soft decision decoding.

In that work the method of maximum likelihood decoding is treated and some bounds are derived.

With respect to the relay network, the main topic in this thesis, the authors in [13] propose a join scheme using Turbo codes and network coding and present some results. Wireless diversity through network coding is investigated in [6]. The paper proposes network coding to be applied in distributed antenna systems where the users collaborate with each other in the transmission of their corresponding packets to the base station. A scheme that maps modulated received signals at the relay into another, obtaining an equivalence physical layer operation of the $GF(2)$ summation, is presented in [43]. The scheme is called PNC, i.e., physical-layer network coding. Popovski et al. in [31] study some transmission methods for the two-way relay, e.g., Figure 4.5, and the rate that each one can achieve.

An application that is of interest in this thesis is COPE [16]. COPE is a forwarding architecture that improves the throughput of wireless mesh networks and works on the MAC and IP layers, network layer and Internet layer respectively in the TCP/IP model. In this scheme multiple packets are sent in a single transmission with a simple XOR encoding. The main idea behind the application and functionality of this architecture can be explained with the example of Figure 4.5 b. Packet A and B are broadcast in two transmission time units from terminal N_1 and terminal N_2 respectively to terminal N_3, which is the only destination for both transmitters. In the third transmission time unit an XORed packet of A and B is broadcasted by terminal N_3. We notice that N_3 sends and XORed packet that the destinations are capable of decode. COPE is responsible for finding the optimal combination of packets to be encoded. This is done by first making a scan of the packets available at the queue of the receiving terminals. Then, the encoder uses this information to select the packets that can be delivered in a single transmission. In the next section, we clarify the encoding process for cope.

In this thesis we are interested in the study of COPE architecture in relay networks just at the physical layer level. We are not interested in the upper layer design, implementation, packet format or control flow treated in [16]. In the next section we present the model studied here, some simulation results and our proposed transmission scheme. These are our main contributions in this thesis.

Chapter 5 – Cooperative Diversity Using Network Coding

This thesis proposes the use of network coding in wireless environments where relays are present. The source can deliver packets to the relays in less transmissions, using network coding, in order to improve the performance applying cooperative diversity techniques. In this section we employ the concepts presented in previous sections. Our study investigates the application of network coding in the network architecture presented in Chapter 2. We concentrate our attention on the transmission of bits over a wireless medium. Our study is applied at the physical layer of the communication process, where noise, multipath fading and errors propagation must be addressed. Specifically, we consider a one-directional transmission of information from a source to a destination with neighboring terminals cooperating with the transmission. We start this section with a general example that describes the transmission process that is of our interest. Also, we explain the schemes that we propose to be applied at the different parts of the process. Then, we present the specific characteristics of the network studied and the assumptions made. We also investigate some strategies that can be employed in order to mitigate the effects caused by the hostility of the wireless channel. Finally, we establish the performance of the schemes over the relay network by means of simulations. The simulation results are presented at the end.

We separate the entire transmission process in three stages for a better com-

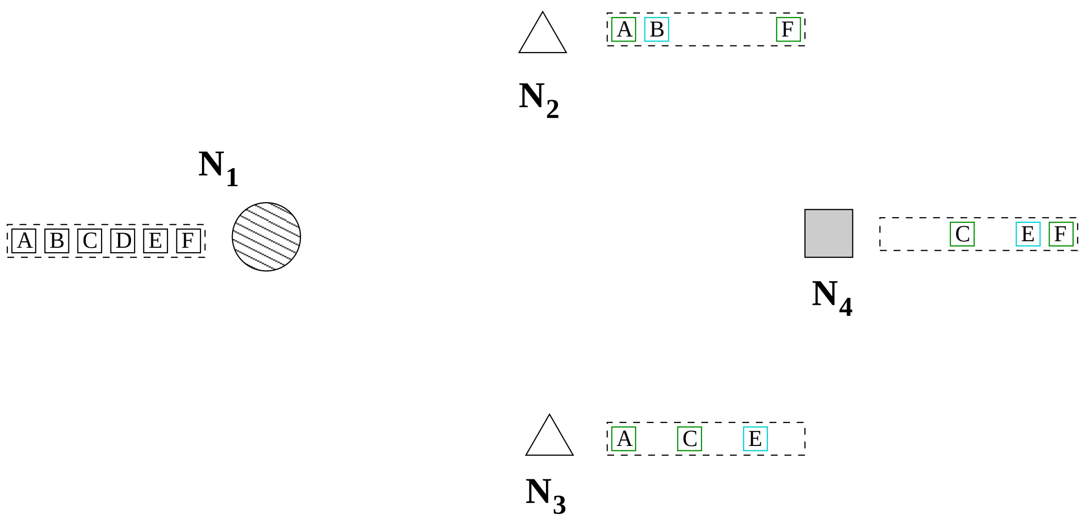


Figure 5.1: Queue at the terminals

prehension of the different concepts employed. The following example illustrates the idea behind our model and the concepts applied at the different stages. This example consists of a single source, a single destination and two relays. We utilize more relays than the above model to show how the concept can be extended to more complex configurations.

In the initial stage, each of the terminals generates packets or acquires them from other terminals. We point out the fact that given the broadcast nature of the wireless channel, the terminals in the transmission range of a source could receive the entire transmitted packets. However, some of the packets transmitted over a wireless medium are lost due to collisions, interference, fading or limited queues. In fact, in practical scenarios, a source transmits the same packet multiple times until there is the certainty that the packet has been received. In what follows we present a scheme that takes advantage of the knowledge of the packets successfully received by the neighboring terminals. In this specific example, we are not interested in how each terminal acquires its own packets. We start with a specific scenario where packets are stored in the terminals queues as it is shown in Figure 5.1.

In the second stage, the source selects multiple packets and encodes them in a single one using network coding. The selection of the packets to be encoded is based on the encoding strategy presented in the forwarding architecture known as COPE [16]. COPE was developed and tested for 802.11 networks. The authors in [16] established some coding algorithm, packet format and implementation with TCP and UDP flows. Contrary to this architecture, our focus is on the physical layer and we are only interested in which packets are sent using this scheme. We only borrow the encoding scheme but we are not concerned with routing protocols, packet format or how the flow is managed by upper layers. In COPE, all the terminals broadcast a report of the packets stored in their queues. Then, a queue table is built by each terminal with the information received in the reports. Consequently, each terminal knows the packets that are stored in the neighboring terminals. We assume that the table is already known at the source, because it is not of our interest to know how the table is filled. The queue table for the example in Figure 5.1 is shown in Table 5.1. With this information, the source proceeds to select the

Terminal	Packets in queue					
N_1	A	B	C	D	E	F
N_2	A	B				F
N_3	A		C		E	
N_4			C		E	F

Table 5.1: Queue Table

packets to be encoded. This selection of the packets is called by the authors in [16] as "opportunistic coding", the reason for this is that it exploits the information in the queue table in order to maximize the number of packets that are sent in a single transmission. The procedure is as follows:

- First, the source lists the packets that have to be sent to other terminals.

Terminal	Packets to be sent
N_2	C, E
N_3	B, F
N_4	A, D

Table 5.2: List of packets to be sent

- Then, the encoder chooses from the list the maximum number of packets that can be decoded by all the receiving terminals and then uses just an XOR operation, which reduces the encoding complexity to a minimum to form a single packet. To clarify this idea, suppose three packets are encoded, e.g., $A \oplus B \oplus C$. At the receiver, the decoding operation for obtaining packet A is done by performing another XOR operation as

$$A = A \oplus B \oplus C \oplus B \oplus C. \tag{5.1}$$

  Clearly we see that without packets B and C at the receiver, it is impossible to decode the packet. In general, with an XOR encoding, there should be $n - 1$ of the encoded packets stored at the receiving terminals in order to decode n packets [16]. For the present example, we use the packets listed in

Table 5.2 in order to find the encoding options for this system; some of these options are listed in Table 5.3. From table 5.3, we see that the best option

Encoding	Observations
$A \oplus B$	Cannot be decoded by N_4
$A \oplus C$	Good option, two packets
$A \oplus D$	Cannot be decoded by N_4
$A \oplus E$	Good option, two packets
$A \oplus F$	Good option, two packets
$A \oplus B \oplus C$	Cannot be decoded by N_4
$A \oplus B \oplus D$	Cannot be decoded by N_3 and N_4
$A \oplus B \oplus E$	Cannot be decoded by N_4
$A \oplus C \oplus F$	Good option, three packets
$A \oplus B \oplus C \oplus E$	Cannot be decoded by N_2 and N_4
$A \oplus B \oplus C \oplus F$	Cannot be decoded by N_3 and N_4

Table 5.3: Coding Options

is $A \oplus C \oplus F$ because it can be decoded by all the receivers and it encodes the highest number of packages.

After encoding, the source broadcast the encoded packet to the neighboring terminals as it is shown in Figure 5.2. Finally, the packet is decoded at each terminal and the new packets are stored in the queues. With this encoding scheme, the packets are sent to the terminals in less transmissions than in the traditional scheme, i.e., store and forward.

The transmission over a wireless medium involves many issues that have to be addressed. Among them are fading, shadowing and noise. Due to this fact, the decoded packets at the receiver terminals are noisy versions of their original ones. In the third and final stage of the transmission process studied in this thesis, we

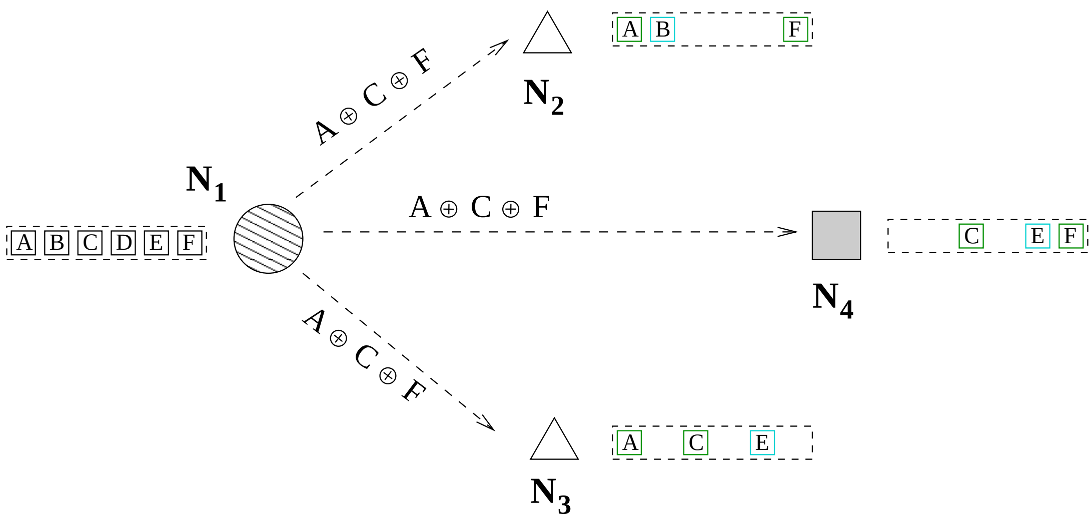


Figure 5.2: Network coding transmission

focus on the mitigation of the effects of fading by using diversity techniques [12, page 204]. Cooperative diversity has been proposed as a suitable technique for combating fading using distributed antennas [20]. Now, the relays N_2 and N_3 are capable of cooperating with the destination sending packets A, C, F and even E to improve the performance of the system by creating diversity. Specifically we used the Adaptive Amplify and Forward (AAF) protocol treated in a previous chapter. Briefly, this protocol states that a relay only retransmits a signal if its SNR_{eff} is above some threshold k derived in chapter 2. This stage is shown in Figure 5.3. Therefore, when the relay retransmits, independent fading signal paths are available at the destination. This last one takes advantage of the independent paths and combines them to achieve diversity. As a consequence, the performance of the system is improved.

The previous example shows a general application of the studied model which

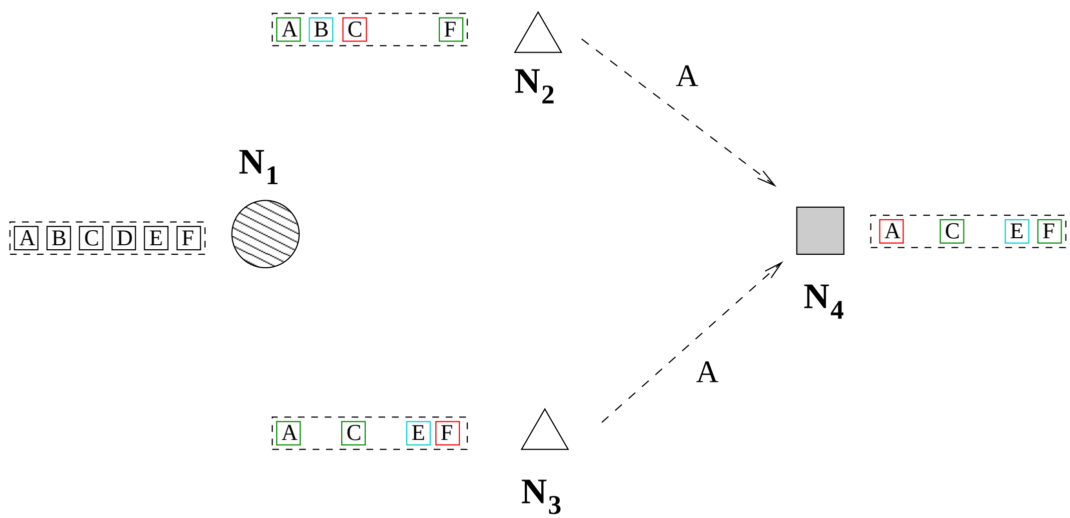


Figure 5.3: Cooperative diversity

allows us to visualize how our model can be extended to larger queues, more terminals and a larger multihop network. Now, we present the conditions of our model and the assumptions made at the different stages. Our study focuses on the transmission of two packets X_1 and X_2 where each transmission follows the model presented in section 2.1.

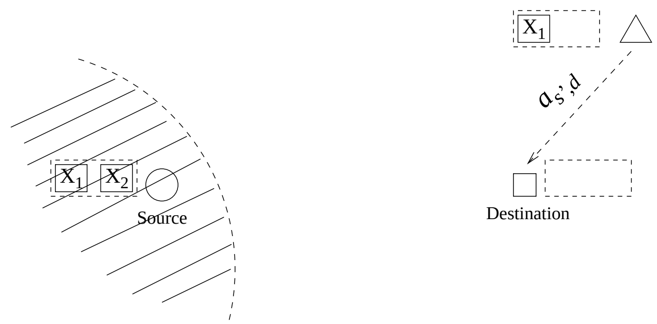
In the first stage, packets X_1 and X_2 are generated by the source and packet X_2 is also generated by the relay. We assume in this stage that the source is out of reach of the relay and the destination. Our initial assumption is that packet X_1 from a previous transmission is already stored in the destination queue as well as its CSI, we assume that the packet was obtained in a direct transmission with another terminal s'. The signal received at the destination is expressed as

$$y_d(t) = a_{r,d}(t)x_1(t) + z_d(t), \tag{5.2}$$

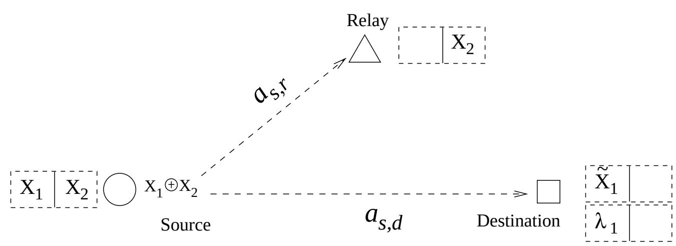
where a noisy version of X_1 is obtained by

$$\tilde{X}_1 = \text{decision}\{a_{r,d}(t)^* y_d(t)\}. \tag{5.3}$$

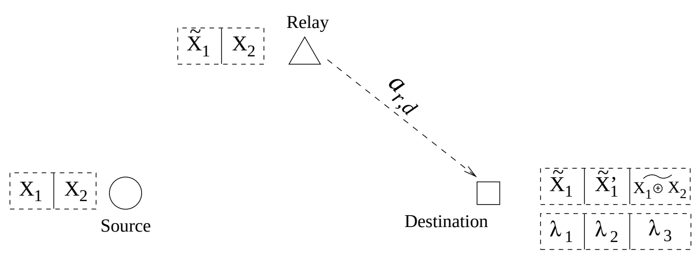
The packet $\tilde{X}_1$ is stored as well as the CSI as $\lambda_1 = |a_{s',d}|^2$.



(a) First Stage



(b) Second Stage



(c) Third Stage

Figure 5.4: Cooperative Diversity scheme

For the second and third stages, the transmission is done using TDM with each one occurring in a single time slot. In the second stage, the source and the relay are in the destination range area. The destination requests packet X_1 to the source, which is stored in its queue. The source applies the encoding scheme previously described and finds that the best encoding option, the one that can be decoded by all terminals and maximizes the number of packets sent in a single transmission, is

$$X_1 \oplus X_2 \tag{5.4}$$

which is transmitted by the source and received at the relay and the destination. This encoding scheme gives the relay the opportunity to obtain packet X_1 and the destination to obtain packet X_2 . In the classical transmission scheme, we would need two transmissions in order to do the same.

In the third stage, using the AAF cooperative protocol, the signal received at the relay is retransmitted only if its SNR_{eff} is over the threshold obtained in chapter 2. This is expressed as

$$\begin{aligned} |a_{s,r}|^2 &\geq k \\ &\geq \tfrac{2^2-1}{SNR} = \tfrac{3}{SNR} \end{aligned} \tag{5.5}$$

Therefore, at this stage, there are three packets available with their corresponding CSI at the destination, i.e.,

- Packet $\tilde{X}_1$ and λ_1.
- $\tilde{X}_1{}'$ transmitted by the relay and λ_2.

- The encoded packet $X_1 \tilde{\oplus} X_2$ and λ_3.

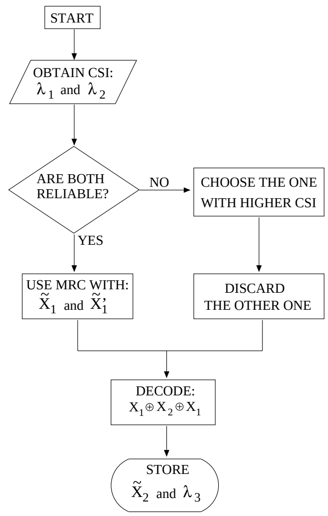


Figure 5.5: Proposed scheme

The scheme that we proposed here is presented in Figure 5.5 and works as follows:

- First, the reliability of the signals $\tilde{X}_1$ and $\tilde{X}_1{}'$ is tested. The stored signals are considered reliable if their corresponding CSI's are over a threshold that are derived based on their corresponding type of transmissions. Therefore, the threshold for packet $\tilde{X}_1$ is obtained by considering that it was sent in a direct transmission. Consequently, this threshold is:

$$\begin{aligned} \lambda &\geq k \\ &\geq \tfrac{2-1}{SNR} \end{aligned} \tag{5.6}$$

On the other hand, the signal $\tilde{X}_1{}'$ was transmitted by the relay after receiving it from the source, multihop transmission, we derive the threshold as follows: From the mutual information given in chapter 2,

$$\mathcal{I} = \frac{1}{2}\log(1 + |a_{r,d}|^2 + SNR|a_{r,d}|^2|a_{s,r}|^2),$$

we assume reliable conditions on the channel between source and relay, i.e.,$|a_{s,r}|^2 = 1$. We define $|a_{r,d}|^2 = k$, then

$$\begin{array}{rcl} \frac{1}{2}\log(1 + |a_{r,d}|^2 + SNR|a_{r,d}|^2|a_{s,r}|^2) & \geq & R \\ \frac{1}{2}\log(1 + k + SNRk) & \geq & R \\ 1 + k + SNRk & \geq & 2^{2R} \\ k(1 + SNR) & \geq & 2^{2R} - 1 \\ k & \geq & \frac{2^2 - 1}{(1+SNR)} \end{array} . \tag{5.7}$$

- If both signals are reliable, then the destination combines them using the MRC technique. The result is passed to a decision device to obtain $\hat{X}$. On the other hand, if one or both of the signals are not reliable, then the one with the highest CSI is chosen to go to the decision device to obtain $\hat{X}$.

- Then, the destination proceeds to decode the packet $\tilde{X_2}$ as

$$\tilde{X_2} = X_1 \tilde{\oplus} X_2 \oplus \hat{X}_1 \tag{5.8}$$

- Finally, we replace $X_1 \tilde{\oplus} X_2$ with the decoded signal $\tilde{X_2}$.

In order to compare our scheme, we use the procedure developed in [39] to obtain the performance of network coding in noisy environments. With the purpose of following the same convention as the work of Koetter et al. in [18], we consider the following equivalences for the transmitted signals:

$$X_1 = X(\nu, 1) \text{ and } X_2 = X(\nu, 2),$$

where ν represents the node at which both signals are generated. First we consider the case where there is no relay; this network can be represented as a graph $\mathcal{G}(V, E)$ shown in Figure 5.6. The relationships that govern the network are:

$$\begin{array}{lcl} Y_{e_1} & = & \alpha_{1,e_1} X(\nu, 1) \\ Y_{e_2} & = & \alpha_{1,e_2} X(\nu, 1) \\ Y_{e_3} & = & \alpha_{2,e_3} X(\nu, 2) \\ Y_{e_4} & = & \omega_{e_2,e_4} Y_{e_2} + \omega_{e_3,e_4} Y_{e_3} \\ Z(\nu', 1) & = & \varepsilon_{e_1,1} Y_{e_1} + \varepsilon_{e_4,1} Y_{e_4} \\ Z(\nu', 2) & = & \varepsilon_{e_1,2} Y_{e_1} + \varepsilon_{e_4,2} Y_{e_4} \end{array},$$

we want the received signals to be the same as the transmitted ones. Therefore, we have that

$$X(\nu', 1) = Z(\nu', 1)$$

and

$$X(\nu', 2) = Z(\nu', 2).$$

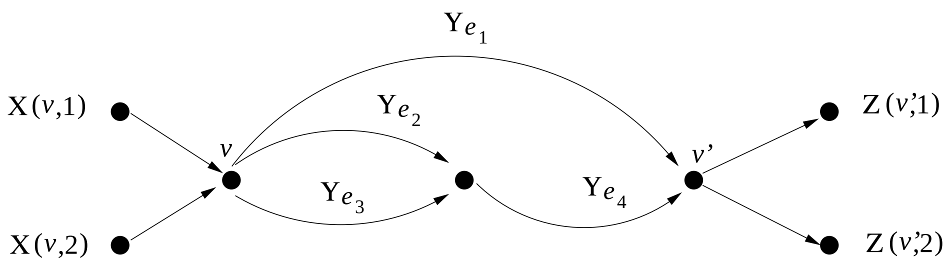


Figure 5.6: Network representation in a graph

The reduced transfer matrix M, see Appendix 2 for the derivation, obtained from the previous relations is found to be

$$M = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 \\ 0 & 0 & \alpha_{2,e_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \omega_{e_2,e_4} \\ 0 & \omega_{e_3,e_4} \end{pmatrix} \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & \varepsilon_{e_4,2} \end{pmatrix}. \tag{5.9}$$

Following the algorithm in [39] we find that

$$H = A \cdot P = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2}\omega_{e_2,e_4} \\ 0 & \alpha_{2,e_3}\omega_{e_3,e_4} \end{pmatrix}. \tag{5.10}$$

Then, replacing the variables by their second subscript, we get

$$H = \begin{pmatrix} e_1 & e_2e_4 \\ 0 & e_3e_4 \end{pmatrix}. \tag{5.11}$$

The redundant channels are removed by replacing them with 1, i.e.,

$$H = \begin{pmatrix} e_1 & e_2 e_4 \\ 0 & e_3 \end{pmatrix}. \tag{5.12}$$

In this network, we consider two links that can be erroneous, which are the physical wireless connection between source and destination, the links are e_1 and e_4, therefore the number of erroneous situations that have to be checked are: $\binom{2}{1} + \binom{2}{2} = 3$. To illustrate, how the algorithm in [39] works, we make the procedure for one of the erroneous situations. We start assuming that e_1 is a link in error. Therefore, we replace the erroneous channel by the variable T and the other variables, which are considered without errors with ones, i.e.,

$$H = \begin{pmatrix} T & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.13}$$

Next, we replace the encoding factors in Matrix B by ones as

$$B^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.14}$$

Matrices H and B are multiplied together and applying the rules $T \cdot T = 1$ and

$T + T = 0$ to the result, we obtain

$$HB^T = \begin{pmatrix} T & T+1 \\ 0 & 1 \end{pmatrix}. \tag{5.15}$$

Finally, we find the weight of the erroneous situation for the corresponding receiving signal by counting if the number of T's in the column is an odd number. In this case the weight for the received signal $X(\nu', 1)$ is 1 and for the received signal $X(\nu', 2)$ is also 1. Therefore, the probability for this case is given by:

$$P_b(X(\nu', 1)) = P_{e_1}\overline{P}_{e_4}$$

and

$$P_b(X(\nu', 2)) = P_{e_1}\overline{P}_{e_4},$$

where P_{e_i} is the probability of error in the i Rayleigh channel given by [32, page 818]

$$P_e = P_{e_i} = \frac{1}{2}\sqrt{\frac{SNR_i}{1 + SNR_i}}. \tag{5.16}$$

The total bit error probability for this network, see the appendix 2 for the whole derivation, is given by

$$P_b(X(\nu', 1)) = P_{e_1}\overline{P}_{e_4} + P_{e_1}P_{e_4} \tag{5.17}$$

and

$$P_b(X(\nu', 2)) = P_{e_1}\overline{P}_{e_4} + P_{e_4}\overline{P}_{e_1}. \tag{5.18}$$

Now, assuming that the two channels are identically distributed and that only direct transmissions are performed between the source and the destination, we have that

$$P_b(X(\nu', 1)) = P_e\overline{P_e} + P_e^2 \tag{5.19}$$

and

$$P_b(X(\nu', 2)) = 2P_e\overline{P_e}. \tag{5.20}$$

We performed a simulation for direct transmissions to the destination. First X_1 was transmitted from a terminal to the source; then $X_1 \oplus X_2$ was transmitted to the destination as well. Finally, the packet X_2 was decoded. The results are presented in Figure 5.7. The figure clearly shows the agreement between the theoretical results and the ones obtained by a Monte Carlo simulation.

We use graph $\mathcal{G}(V, E)$, shown in Figure 5.6, to obtain some bounds on the performance of the network with a relay. Considering that the relay decodes the signal $X_1 \oplus X_2$ and retransmits signal X_1, we compute the performance of this scheme assuming that the channel between source and relay is perfect. Therefore, the destination performs and MRC technique with the two independent realizations of the signal X_1 obtained, one from the relay and one from the source. Having the

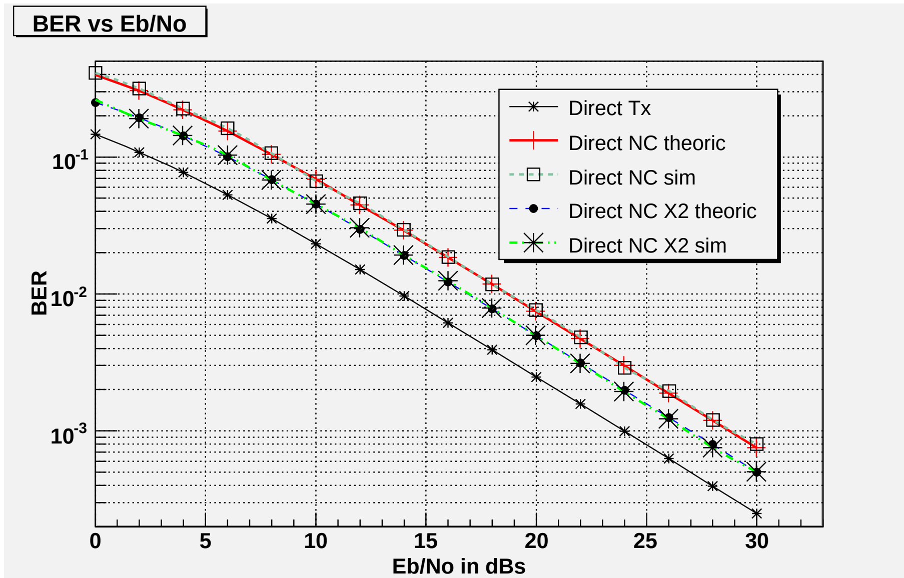


Figure 5.7: Simulation results for direct transmissions

probability of error for MRC with L independent realization given by

$$P_{b_{MRC}} = \left(\frac{1-\Gamma}{L}\right)^{L} \sum_{m=0}^{L-1} \binom{L-1+m}{m} \left(\frac{1+\Gamma}{2}\right)^{m} \tag{5.21}$$

where Γ is obtained as:

$$\Gamma = \sqrt{\frac{SNR_{avg}}{1+SNR_{avg}}}$$

we have that

$$P_b(X(\nu',1)) = P_{b_{MRC}}\overline{P}_e + P_e P_{b_{MRC}} \tag{5.22}$$

and

$$P_b(X(\nu',2)) = P_{b_{MRC}}\overline{P}_e + P_e \overline{P}_{b_{MRC}}. \tag{5.23}$$

where $P_b(X(\nu',1))$ is the probability of error for packet X_1 and $P_b(X(\nu',2))$ is the probability of error for packet X_2 at the destination.

The scheme proposed over the relay network was simulated using the model discussed in chapter 2. In order to maintain the same spectral efficiency, we use a BPSK modulation for the non-cooperative schemes and QPSK for the cooperative ones. Figure 5.8 shows the simulation results obtained for packet X_1. From the

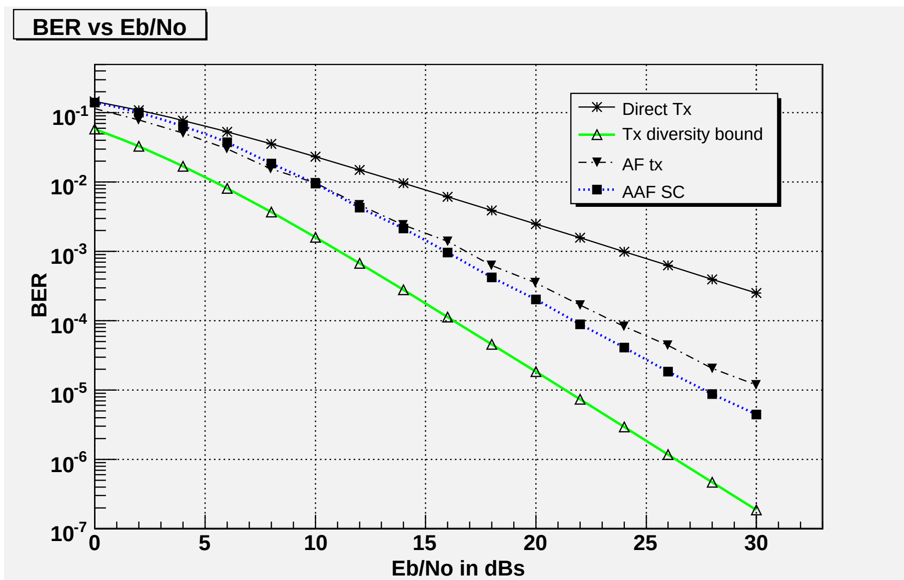


Figure 5.8: Simulation results packet X_1

results we have that the adaptive scheme performs better than a simple amplify and forward scheme because the relay does not transmit unreliable packets. The use of the relay gives diversity but still is not close to the transmit diversity bound. This last one was obtained assuming an ideal channel between source and relay.

Figure 5.9 shows the simulation results obtained for packet X_2. It shows two scenarios. The first one when X_2 is obtained from a direct transmission of X_1 and $X_1 \oplus X_2$, and another one when the relay is used to give diversity to X_1. There is

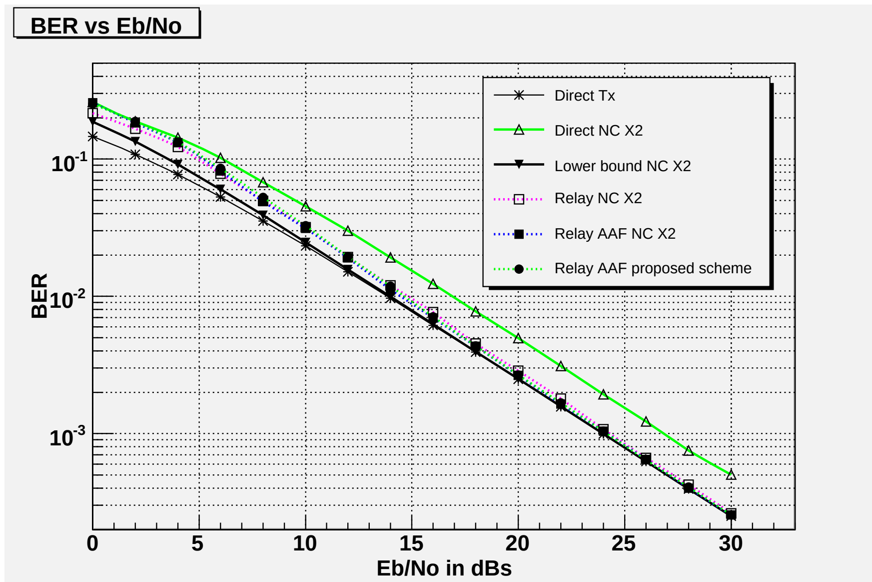


Figure 5.9: Simulation results packet X_2

a difference of 3 dB between the two cases. The adaptive scheme starts performing close to the first scenario, but at high SNR the adaptive scheme performs as well as the predicted bound. It is obvious that an increase in performance of X_1 contributes in an increase in the performance of X_2

At the relay, the number of arrived packets that are reliable increases with high SNR, this is reflected in Figure 5.10. The ratio of packets dropped gives us a measure of the consumption of energy of the system. The reason is that if the number of reliable packets increases, the power consumption at the relay increases as well because it performs more retransmissions.

Finally, we investigate another scenario for the transmission of packet X_2. First, we incorporate the relay to the graph in Figure 5.6. Figure 5.11 shows a more detailed graph that represents the network with a relay channel. The aim of this is

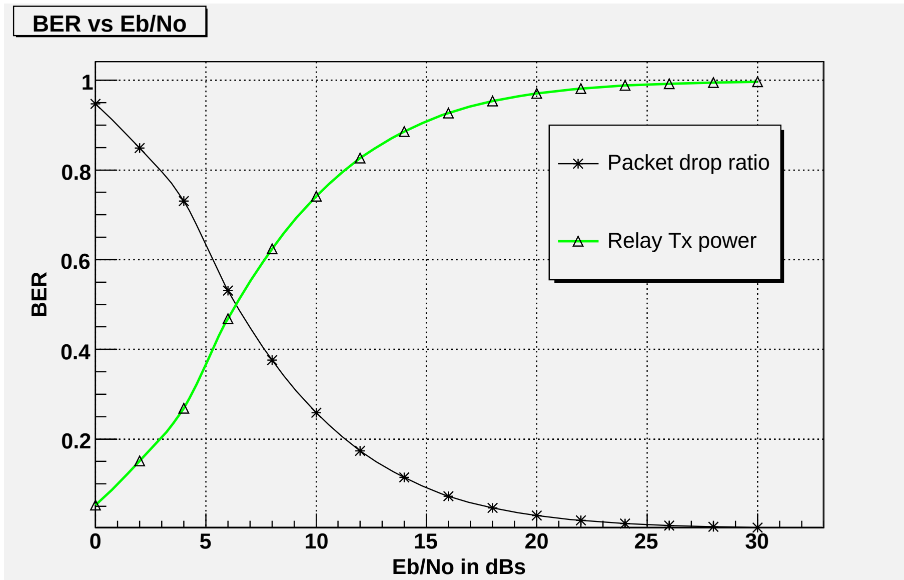


Figure 5.10: Packets dropped at relay

to show how the errors propagate through the network and how without a physical layer strategy, the performance of the network deteriorates.

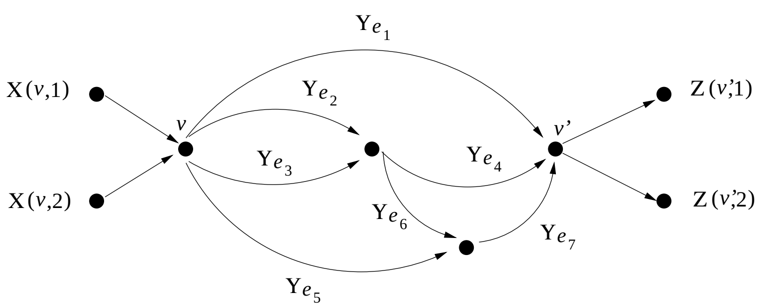


Figure 5.11: Graph representing a network with relay

For this graph, the transfer matrix, see appendix 3, is given by

$$M = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & \alpha_{2,e_5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{e_2,e_4} & \omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & \omega_{e_3,e_4} & \omega_{e_3,e_6}\omega_{e_6,e_7} \\ 0 & 0 & \omega_{e_5,e_7} \end{pmatrix} \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & \varepsilon_{e_4,2} \\ \varepsilon_{e_7,1} & \varepsilon_{e_7,2} \end{pmatrix}. \tag{5.24}$$

The matrix to compute the bit error probability is obtained as

$$H = A \cdot P = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2}\omega_{e_2,e_4} & \alpha_{1,e_2}\omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & \alpha_{2,e_3}\omega_{e_3,e_4} & \omega_{e_3,e_6}\omega_{e_6,e_7} + \alpha_{2,e_5}\omega_{e_5,e_7} \end{pmatrix}. \tag{5.25}$$

In this network, we consider four links that could be in error, which are the physical wireless connection between source and destination, the links are e_1, e_4, e_6 and e_7 therefore the number of erroneous situations that have to be checked are: $\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 15$. The derivation of the bit error probability for this network is shown in appendix 3. The final expression is given by

$$P_b(X(\nu', 1)) = 3P_e\overline{P}_e^3 + 3P_e^2\overline{P}_e^2 + P_e^3\overline{P}_e + P_e^4, \tag{5.26}$$

and

$$P_b(X(\nu', 2)) = 4P_e\overline{P}_e^3 + 2P_e^3\overline{P}_e. \tag{5.27}$$

Another scenario that we consider is the classical one using repetition codes. In this scenario, the destination checks for the reliability of the transmission of packet $X_1 \oplus X_2$. If the SNR_{eff} of that transmission goes below the threshold, that was obtained before, the source retransmits the packet to the destination, we consider the case when only one retransmission is allowed. Figure 5.12 shows the results for the different schemes considered to get packet X_2. It is evident that the

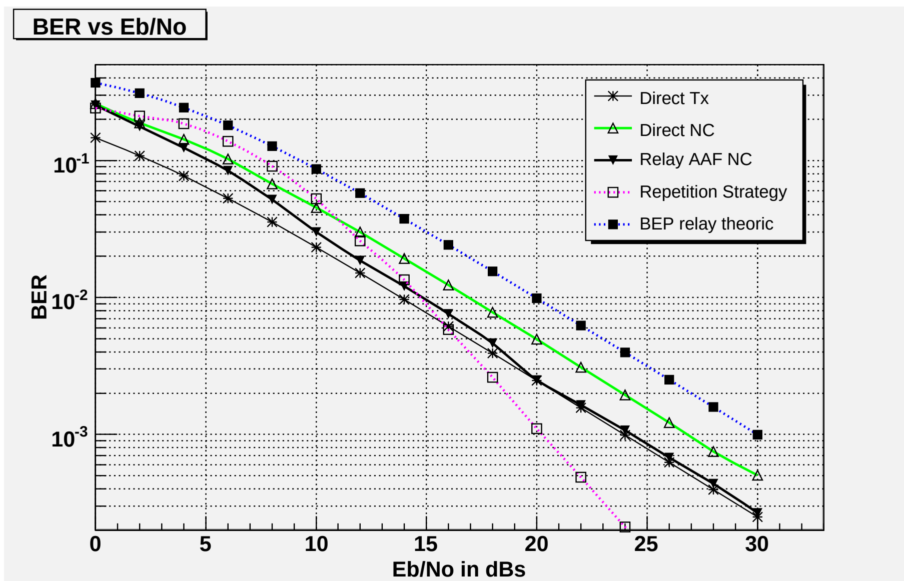


Figure 5.12: Simulation results for different transmission schemes

use of more cooperative relays would contribute to a better performance.

Chapter 6 – Conclusions and Future Work

This thesis has presented a survey of different concepts that have helped us to investigate a practical application of network coding in wireless applications.

We have investigated a diversity technique called cooperative diversity that increases the performance with the help of relay terminals. This is a useful technique to be applied in existing terminals that are limited in the number of antennas without any large increase in the complexity of its hardware. The cooperating terminals and the source creates a virtual array that provides diversity in wireless environments. Also, we have shown that an adaptive scheme, depending on the conditions of the channel, is the more desirable strategy for a wireless scenarios due to the hostility of the channel.

We have proposed the use of network coding using and opportunistic approach in order to provided of diversity to the system using the broadcast nature of the wireless channel. Taking advantage of the knowledge of the packets stored in the terminals queues, we can encode a packet that provides diversity to packets sent to a destination using less transmissions. The propagation of errors in a network is an issue that has to be taken into consideration when network coding is applied in wireless applications. We have shown how the performance is affected by this fact. Specially, the increment in complexity and nodes in the network decreases the performance of the system.

The benefits of network coding depend on the increasing number of cooperating terminals and the knowledge of previous transmissions. Cooperative diversity is a technique that makes network coding a more suitable and practical application to wireless networks. Combining both, we can take advantage of the available resources.

There are still open questions that have to be addressed before the deployment of network coding in real networks. As a future work, the results can be extended to larger network configurations. The study can be done integrating upper layer protocols or developing new protocols suitable for this scenarios. Also, it can be extended to cross layer design. A full mathematical description of the performance of network coding, considering the physical layer and the strategies applied there, is something that has to be addressed too.

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APPENDICES

Appendix A – Variance of One Hop Transmission

The signal received by the relay is given by

$$y_r = a_{s,r}x_s + z_r,$$

where x_s is the transmitted signal, $a_{s,r}$ is the fading coefficient of the source-relay path, and z_r is AWGN produced by the relay. At the relay, the signal is scaled by a factor β as

$$x_r = \beta y_r.$$

Therefore, the signal received at the destination is

$$\begin{aligned} y_d &= a_{r,d}x_r + z_d \\ &= a_{r,d}\beta y_r + z_d \\ &= a_{r,d}\beta(a_{s,r}x_s + z_r) + z_d \\ &= a_{r,d}\beta a_{s,r}x_s + a_{r,d}\beta z_r + z_d \end{aligned}$$

The variance $\sigma_d = E\{yy^*\}$ of the receiver signal can be obtained as

$$
\begin{aligned}
E\{y \cdot y^*\} &= E\{(a_{r,d}\beta a_{s,r}x_s + \beta a_{r,d}z_r + z_d)\,(a_{r,d}\beta a_{s,r}x_s + \beta a_{r,d}z_r + z_d)^*\} \\
&= E\{\beta^2 a_{s,r}a_{s,r}^* a_{r,d}a_{r,d}^* x_s x_s^* + \beta^2 a_{s,r}a_{r,d}a_{r,d}^* x_s z_r^* \\
&\qquad +\beta a_{s,r}a_{r,d}x_s z_d^* + \beta^2 z_r a_{r,d}a_{r,d}^* a_{s,r}^* x_s^* \\
&\qquad +\beta^2 a_{r,d}a_{r,d}^* z_r z_r^* + \beta a_{r,d}z_r z_d^* \\
&\qquad +\beta x_s^* a_{s,r}^* a_{r,d}^* z_d + \beta a_{r,d}^* z_d z_r^* + z_d z_d^*\} \\
&= \beta^2 a_{s,r}a_{s,r}^* a_{r,d}a_{r,d}^* E\{x_s x_s^*\} + \beta^2 a_{s,r}a_{r,d}a_{r,d}^* E\{x_s z_r^*\} \\
&\qquad +\beta a_{s,r}a_{r,d}E\{x_s z_d^*\} + \beta^2 a_{r,d}a_{r,d}^* a_{s,r}^* E\{z_r x_s^*\} \\
&\qquad +\beta^2 a_{r,d}a_{r,d}^* E\{z_r z_r^*\} + \beta a_{r,d}E\{z_r z_d^*\} \\
&\qquad +\beta a_{s,r}^* a_{r,d}^* E\{x_s^* z_d\} + \beta a_{r,d}^* E\{z_d z_r^*\} + E\{z_d z_d^*\} \\
&= \beta^2 |a_{s,r}|^2 |a_{r,d}|^2 E\{|x_s|^2\} + \beta^2 a_{s,r}|a_{r,d}|^2 E\{x_s\}E\{z_r^*\} \\
&\qquad +\beta a_{s,r}a_{r,d}E\{x_s\}E\{z_d^*\} + \beta^2 |a_{r,d}|^2 a_{s,r}^* E\{z_r\}E\{x_s^*\} \\
&\qquad +\beta^2 |a_{r,d}|^2 E\{|z_r|^2\} + \beta a_{r,d}E\{z_r\}E\{z_d^*\} \\
&\qquad +\beta a_{s,r}^* a_{r,d}^* E\{x_s^*\}E\{z_d\} + \beta a_{r,d}^* E\{z_d\}E\{z_r^*\} + E\{|z_d|^2\} \\
&= \beta^2 |a_{s,r}|^2 |a_{r,d}|^2 E\{|x_s|^2\} + \beta^2 |a_{r,d}|^2 E\{|z_r|^2\} + E\{|z_d|^2\} \\
&= \beta^2 |a_{s,r}|^2 |a_{r,d}|^2 E\{|x_s|^2\} + \beta^2 |a_{r,d}|^2 N_0 + N_0
\end{aligned}
$$

With $\beta = 1$, the expression is simplified as

$$E\{y \cdot y^*\} = |a_{r,d}|^2 |a_{s,r}|^2 E\{|x_s|^2\} + |a_{r,d}|^2 N_0 + N_0$$

Appendix B – Direct Network

In this appendix, we derive the transfer matrix for the network represented by the graph $\mathcal{G}(V, E)$ of Fig. B.1 and using the algorithm in [39].

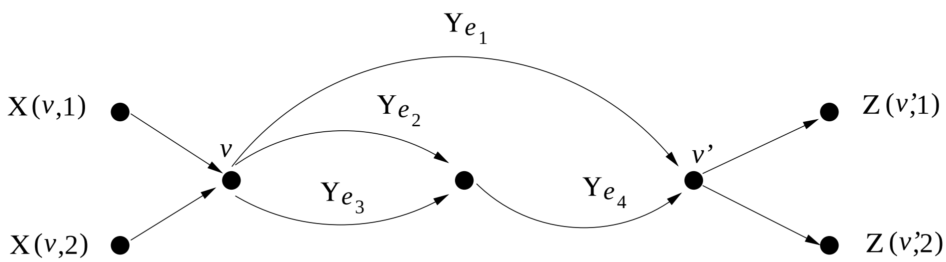

Figure B.1: Network representation in a Graph

The equations that define the graph are

$$\begin{aligned}
Y_{e_1} &= \alpha_{1,e_1} X(\nu, 1) \\
Y_{e_2} &= \alpha_{1,e_2} X(\nu, 1) \\
Y_{e_3} &= \alpha_{2,e_3} X(\nu, 2) \\
Y_{e_4} &= \omega_{e_2,e_4} Y_{e_2} + \omega_{e_3,e_4} Y_{e_3} \\
Z(\nu', 1) &= \varepsilon_{e_1,1} Y_{e_1} + \varepsilon_{e_4,1} Y_{e_4} \\
Z(\nu', 2) &= \varepsilon_{e_1,2} Y_{e_1} + \varepsilon_{e_4,2} Y_{e_4}
\end{aligned}$$

From Fig. B.1, we derive matrix A, using the definition given in chapter 4, as follows:

$$A = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & 0 \end{pmatrix}. \tag{B.1}$$

On the other hand, matrix B is obtained as

$$B = \begin{pmatrix} \varepsilon_{e_1,1} & 0 & 0 & 0 \\ \varepsilon_{e_1,2} & 0 & 0 & \varepsilon_{e_4,2} \end{pmatrix}. \tag{B.2}$$

In order to obtain matrix F, we use the directed labeled line graph $\mathfrak{G}(\mathbb{V}, \mathbb{E})$ on Fig. B.2 from the graph $\mathcal{G}(V, E)$

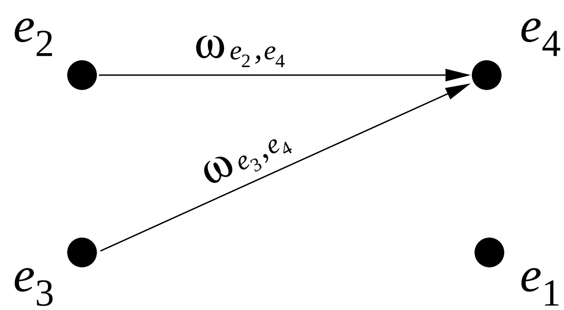

Figure B.2: Directed labeled line graph

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{e_2,e_4} \\ 0 & 0 & 0 & \omega_{e_3,e_4} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.3}$$

From where we get $(I - F)$ as

$$I - F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\omega_{e_2,e_4} \\ 0 & 0 & 1 & -\omega_{e_3,e_4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.4}$$

and its inverse

$$(I - F)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{e_2,e_4} \\ 0 & 0 & 1 & \omega_{e_3,e_4} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.5}$$

Therefore, matrix M is given by

$$M = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{e_2,e_4} \\ 0 & 0 & 1 & \omega_{e_3,e_4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & \varepsilon_{e_4,2} \end{pmatrix}, \tag{B.6}$$

which is equivalent to the reduced form

$$M = \begin{pmatrix} \omega_{1,e_1} & \omega_{1,e_2} & 0 \\ 0 & 0 & \omega_{2,e_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \omega_{e_2,e_4} \\ 0 & \omega_{e_3,e_4} \end{pmatrix} \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & \varepsilon_{e_4,2} \end{pmatrix}. \tag{B.7}$$

The computation of the errors requires the computation of matrices H and B, i.e.,

$$H = A \cdot P = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2}\omega_{e_2,e_4} \\ 0 & \alpha_{2,e_3}\omega_{e_3,e_4} \end{pmatrix}, \tag{B.8}$$

$$H = \begin{pmatrix} e_1 & e_2e_4 \\ 0 & e_3e_4 \end{pmatrix}, \tag{B.9}$$

$$H = \begin{pmatrix} e_1 & e_2e_4 \\ 0 & e_3 \end{pmatrix}. \tag{B.10}$$

In this network there are only two links where errors can occur, which are the physical wireless connections between source and destination, i.e., the links e_1 and e_4. Therefore the number of erroneous situations that have to be checked are: $\binom{2}{1} + \binom{2}{2} = 3$.

If e_1 is in error, then

$$H = \begin{pmatrix} T & 1 \\ 0 & 1 \end{pmatrix}. \tag{B.11}$$

Encoding variables in Matrix B are replaced by one's

$$B^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{B.12}$$

We multiply H with B^T and simplify using $T \cdot T = 1$ and $T + T = 0$ to obtain

$$HB^T = \begin{pmatrix} T & T+1 \\ 0 & 1 \end{pmatrix}. \tag{B.13}$$

The weights for $X(\nu', 1)$ and $X(\nu', 2)$ are 1 and 1, respectively, with probability given by $P(e_1 \text{in error}) = P_{e_1}\overline{P_{e_4}}$.

If e_4 is in error, then

$$HB^T = \begin{pmatrix} 1 & T+1 \\ 0 & 1 \end{pmatrix} \tag{B.14}$$

the weight for $X(\nu', 1)$ is 0 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_4 \text{ in error}) = P_{e_4}\overline{P_{e_1}}$.

If e_1 and e_4 is an error:

$$HB^T = \begin{pmatrix} T & T+T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \tag{B.15}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_1 \text{ and } e_4 \text{ in error}) = P_{e_1}P_{e_4}$. Hence,

$$P_b(X(\nu', 1)) = P_{e_1}\overline{P}_{e_4} + P_{e_1}P_{e_4} \tag{B.16}$$

and

$$P_b(X(\nu', 2)) = P_{e_1}\overline{P}_{e_4} + P_{e_4}\overline{P}_{e_1}. \tag{B.17}$$

Moreover, given that all the channels are symmetric, we have that $P_e = P_{e_1} = P_{e_4}$, therefore

$$P_b(X(\nu', 1)) = P_e\overline{P_e} + P_e^2 \tag{B.18}$$

and

$$P_b(X(\nu', 2)) = 2P_e\overline{P_e}. \tag{B.19}$$

Appendix C – Relay Network with Algebraic Network Coding

The transfer matrix for the relay network represented by the graph of Fig. C.1 is derived as well as the bit error probability using the algorithm in [39].

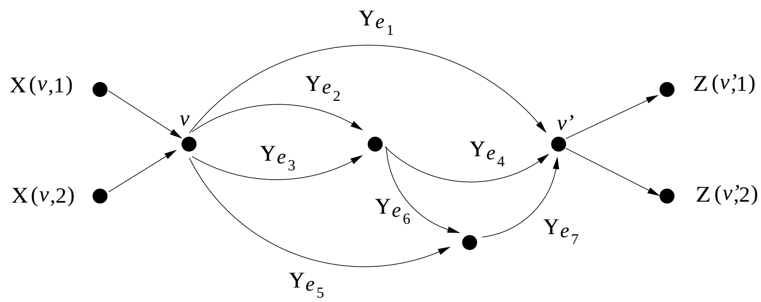

Figure C.1: Network relay representation in a Graph

The equations that define the graph are

$$
\begin{aligned}
Y_{e_1} &= \alpha_{1,e_1} X(\nu,1) \\
Y_{e_2} &= \alpha_{1,e_2} X(\nu,1) \\
Y_{e_3} &= \alpha_{2,e_3} X(\nu,2) \\
Y_{e_4} &= \omega_{e_2,e_4} Y_{e_2} + \omega_{e_3,e_4} Y_{e_3} \\
Y_{e_5} &= \alpha_{2,e_5} X(\nu,2) \\
Y_{e_6} &= \omega_{e_2,e_6} Y_{e_2} + \omega_{e_3,e_6} Y_{e_3} \\
Y_{e_7} &= \omega_{e_5,e_7} Y_{e_5} + \omega_{e_6,e_7} Y_{e_6} \\
Z(\nu',1) &= \varepsilon_{e_1,1} Y_{e_1} + \varepsilon_{e_7,1} Y_{e_7} \\
Z(\nu',2) &= \varepsilon_{e_1,2} Y_{e_1} + \varepsilon_{e_4,2} Y_{e_4} + \varepsilon_{e_7,2} Y_{e_7}
\end{aligned}
\quad .
$$

By definition, the matrix A is given by

$$A = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & 0 & \alpha_{2,e_5} & 0 & 0 \end{pmatrix}. \tag{C.1}$$

Looking at the output from the graph and all the links that are connected to the destination node, we can established that matrix B is given by

$$B = \begin{pmatrix} \varepsilon_{e_1,1} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{e_7,1} \\ \varepsilon_{e_1,2} & 0 & 0 & \varepsilon_{e_4,2} & 0 & 0 & \varepsilon_{e_7,2} \end{pmatrix}. \tag{C.2}$$

From the graph $\mathcal{G}(V, E)$, we obtain the directed labeled line graph $\mathfrak{G}(\mathbb{V}, \mathbb{E})$ shown on Fig. C.2. From where we obtain matrix F as

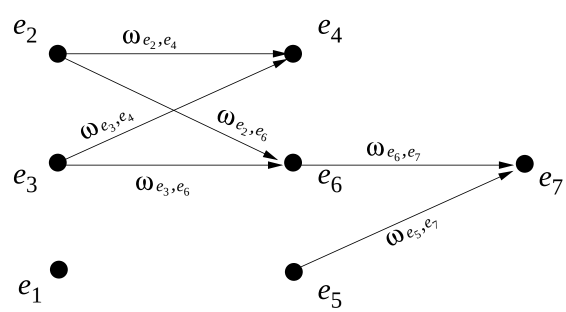

Figure C.2: Directed labeled line graph

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{e_2,e_4} & 0 & \omega_{e_2,e_6} & 0 \\ 0 & 0 & 0 & \omega_{e_3,e_4} & 0 & \omega_{e_3,e_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_{e_5,e_7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_{e_6,e_7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.3}$$

and

$$(I-F)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{e_2,e_4} & 0 & \omega_{e_2,e_6} & \omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & 0 & 1 & \omega_{e_3,e_4} & 0 & \omega_{e_3,e_6} & \omega_{e_3,e_6}\omega_{e_6,e_7} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \omega_{e_5,e_7} \\ 0 & 0 & 0 & 0 & 0 & 1 & \omega_{e_6,e_7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C.4}$$

Therefore, matrix M is

$$
\begin{aligned}
M \;=\; & \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & 0 & \alpha_{2,e_5} & 0 & 0 \end{pmatrix} \\
& \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{e_2,e_4} & 0 & \omega_{e_2,e_6} & \omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & 0 & 1 & \omega_{e_3,e_4} & 0 & \omega_{e_3,e_6} & \omega_{e_3,e_6}\omega_{e_6,e_7} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \omega_{e_5,e_7} \\ 0 & 0 & 0 & 0 & 0 & 1 & \omega_{e_6,e_7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{(C.5)} \\
& \times \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & \varepsilon_{e_4,2} \\ 0 & 0 \\ 0 & 0 \\ \varepsilon_{e_7,1} & \varepsilon_{e_7,2} \end{pmatrix}
\end{aligned}
$$

which is equivalent to the reduced form

$$M = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 \\ 0 & 0 & \alpha_{2,e_3} & \alpha_{2,e_5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{e_2,e_4} & \omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & \omega_{e_3,e_4} & \omega_{e_3,e_6}\omega_{e_6,e_7} \\ 0 & 0 & \omega_{e_5,e_7} \end{pmatrix} \begin{pmatrix} \varepsilon_{e_1,1} & \varepsilon_{e_1,2} \\ 0 & \varepsilon_{e_4,2} \\ \varepsilon_{e_7,1} & \varepsilon_{e_7,2} \end{pmatrix}. \tag{C.6}$$

The computation of the errors, requires the computation of matrices H and B, i.e.,

$$H = A \cdot P = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2}\omega_{e_2,e_4} & \alpha_{1,e_2}\omega_{e_2,e_6}\omega_{e_6,e_7} \\ 0 & \alpha_{2,e_3}\omega_{e_3,e_4} & \omega_{e_3,e_6}\omega_{e_6,e_7} + \alpha_{2,e_5}\omega_{e_5,e_7} \end{pmatrix}, \tag{C.7}$$

$$H = \begin{pmatrix} e_1 & e_2e_4 & e_2e_6e_7 \\ 0 & e_3e_4 & e_6e_7 + e_5e_7 \end{pmatrix}, \tag{C.8}$$

$$H = \begin{pmatrix} e_1 & e_2e_4 & e_2e_6e_7 \\ 0 & e_3 & 1 + e_5 \end{pmatrix}. \tag{C.9}$$

The encoding factor in Matrix B are replaced by one's, i.e,

$$B^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \tag{C.10}$$

$$HB^T = \begin{pmatrix} e_1 + e_2e_6e_7 & e_1 + e_2e_4 + e_2e_6e_7 \\ 1 + e_5 & 1 + e_3 + e_5 \end{pmatrix}. \tag{C.11}$$

In this network there are only four links with errors, which are the physical wireless connection between source and destination, the links are e_1, e_4, e_6 and e_7 therefore the number of erroneous situations that have to be checked are: $\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + +\binom{4}{4} = 15$.

If e_1 is in error, then

$$HB^T = \begin{pmatrix} T + 1 & T + 1 + 1 \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix}, \tag{C.12}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_1 \text{ in error}) = P_{e_1}\overline{P}_{e_4}\overline{P}_{e_6}\overline{P}_{e_7}$.

If e_4 is in error, then

$$HB^T = \begin{pmatrix} 1 + 1 & 1 + T + 1 \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix}, \tag{C.13}$$

the weight for $X(\nu', 1)$ is 0 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_4 \text{ in error}) = P_{e_4}\overline{P}_{e_1}\overline{P}_{e_6}\overline{P}_{e_7}$.

If e_6 is in error, then

$$HB^T = \begin{pmatrix} 1+T & 1+1+T \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.14}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_6 \text{ in error}) = P_{e_6}\overline{P}_{e_1}\overline{P}_{e_4}\overline{P}_{e_7}$.

If e_7 is in error, then

$$HB^T = \begin{pmatrix} 1+T & 1+1+T \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.15}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_7 \text{ in error}) = P_{e_7}\overline{P_{e_1}P_{e_4}P_{e_6}}$.

If e_1 and e_4 are in error, then

$$HB^T = \begin{pmatrix} T+1 & T+T+1 \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} T+1 & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.16}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_1 \text{ and } e_4 \text{ in error}) = P_{e_1}P_{e_4}\overline{P}_{e_6}\overline{P}_{e_7}$.

If e_1 and e_6 are in error, then

$$HB^T = \begin{pmatrix} T+T & T+1+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.17}$$

the weight for $X(\nu', 1)$ is 0 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_1 \text{ and } e_6 \text{ in error}) = P_{e_1}P_{e_6}\overline{P}_{e_4}\overline{P}_{e_7}$.

If e_1 and e_7 are in error, then

$$HB^T = \begin{pmatrix} T+T & T+1+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.18}$$

the weight for $X(\nu', 1)$ is 0 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_1 \text{ and } e_7 \text{ in error}) = P_{e_1}P_{e_7}\overline{P}_{e_4}\overline{P}_{e_6}$.

If e_4 and e_6 are in error, then

$$HB^T = \begin{pmatrix} 1+T & 1+T+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 1+T & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.19}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_4 \text{ and } e_6 \text{ in error}) = P_{e_4}P_{e_6}\overline{P}_{e_1}\overline{P}_{e_7}$.

If e_4 and e_7 are in error, then

$$HB^T = \begin{pmatrix} 1+T & 1+T+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 1+T & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.20}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_4 \text{ and } e_7 \text{ in error}) = P_{e_4}P_{e_7}\overline{P}_{e_1}\overline{P}_{e_6}$.

If e_6 and e_7 are in error, then

$$HB^T = \begin{pmatrix} 1+T\cdot T & 1+1+T\cdot T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 1+1 & 1+1+1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.21}$$

the weight for $X(\nu',1)$ is 0 and the weight for $X(\nu',2)$ is 0, with probability given by $P(e_6 \text{ and } e_7 \text{ in error}) = P_{e_6}P_{e_7}\overline{P}_{e_1}\overline{P}_{e_4}$.

If e_1, e_4 and e_6 are in error, then

$$HB^T = \begin{pmatrix} T+T & 1+T+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.22}$$

the weight for $X(\nu',1)$ is 0 and the weight for $X(\nu',2)$ is 0, with probability given by $P(e_1, e_4 \text{ and } e_6 \text{ in error}) = P_{e_1}P_{e_4}P_{e_6}\overline{P}_{e_7}$.

If e_1, e_4 and e_7 are in error, then

$$HB^T = \begin{pmatrix} T+T & 1+T+T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.23}$$

the weight for $X(\nu',1)$ is 0 and the weight for $X(\nu',2)$ is 0, with probability given by $P(e_1, e_4 \text{ and } e_7 \text{ in error}) = P_{e_1}P_{e_4}P_{e_7}\overline{P}_{e_6}$.

If e_1, e_6 and e_7 are in error, then

$$HB^T = \begin{pmatrix} T+T\cdot T & T+1+T\cdot T \\ 1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} T+1 & T+1+1 \\ 1+1 & 1+1+1 \end{pmatrix}, \tag{C.24}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_1, e_6 \text{ and } e_7 \text{ in error}) = P_{e_1} P_{e_6} P_{e_7} \overline{P}_{e_4}$.

If e_4, e_6 and e_7 are in error, then

$$HB^T = \begin{pmatrix} 1 + T \cdot T & 1 + T + T \cdot T \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix} = \begin{pmatrix} 1 + 1 & 1 + T + 1 \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix}, \tag{C.25}$$

the weight for $X(\nu', 1)$ is 0 and the weight for $X(\nu', 2)$ is 1, with probability given by $P(e_4, e_6 \text{ and } e_7 \text{ in error}) = P_{e_4} P_{e_6} P_{e_7} \overline{P}_{e_1}$.

If e_1,e_4, e_6 and e_7 are in error, then

$$HB^T = \begin{pmatrix} T + T \cdot T & T + T + T \cdot T \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix} = \begin{pmatrix} T + 1 & 1 \\ 1 + 1 & 1 + 1 + 1 \end{pmatrix}, \tag{C.26}$$

the weight for $X(\nu', 1)$ is 1 and the weight for $X(\nu', 2)$ is 0, with probability given by $P(e_1, e_4, e_6 \text{ and } e_7 \text{ in error}) = P_{e_1} P_{e_4} P_{e_6} P_{e_7}$.

Adding all the probabilities with their respective weights, we get

$$\begin{aligned} P_b(X(\nu', 1)) \;\; = \;\; & P_{e_1} \overline{P}_{e_4} \overline{P}_{e_6} \overline{P}_{e_7} + P_{e_6} \overline{P}_{e_1} \overline{P}_{e_4} \overline{P}_{e_7} + P_{e_7} \overline{P}_{e_1} \overline{P}_{e_4} \overline{P}_{e_6} \\ & + P_{e_1} P_{e_4} \overline{P}_{e_6} \overline{P}_{e_7} + P_{e_4} P_{e_6} \overline{P}_{e_1} \overline{P}_{e_7} + P_{e_4} P_{e_7} \overline{P}_{e_1} \overline{P}_{e_6} \;, \\ & + P_{e_1} P_{e_6} P_{e_7} \overline{P}_{e_4} + P_{e_1} P_{e_4} P_{e_6} P_{e_7} \end{aligned} \tag{C.27}$$

and

$$P_b(X(\nu',2)) = P_{e_1}\overline{P}_{e_4}\overline{P}_{e_6}\overline{P}_{e_7} + P_{e_4}\overline{P}_{e_1}\overline{P}_{e_6}\overline{P}_{e_7} + P_{e_6}\overline{P}_{e_1}\overline{P}_{e_4}\overline{P}_{e_7} + P_{e_7}\overline{P}_{e_1}\overline{P}_{e_4}\overline{P}_{e_6} + P_{e_1}P_{e_6}P_{e_7}\overline{P}_{e_4} + P_{e_4}P_{e_6}P_{e_7}\overline{P}_{e_1} . \quad (C.28)$$

Moreover, given that all the channels are symmetric, we have that $P_e = P_{e_1} = P_{e_4} = P_{e_6} = P_{e_7}$, therefore

$$P_b(X(\nu',1)) = 3P_e\overline{P}_e^3 + 3P_e^2\overline{P}_e^2 + P_e^3\overline{P}_e + P_e^4, \quad (C.29)$$

and

$$P_b(X(\nu',2)) = 4P_e\overline{P}_e^3 + 2P_e^3\overline{P}_e. \quad (C.30)$$