Separabilities for a New Class of Gray Codes

By

Jung Pil Park

Major Professor	:	Bella Bose
Minor Professor	:	Prasad Tadepalli
Committee Member	:	Paul Cull
Committee Member	:	Toshimi Minoura

Department of Computer Science

Oregon State University

Corvallis, OR 97331

November 8, 2002

Acknowledgements

I would like to thank my advisor, Dr.Bella Bose, for his support and guidance throughout my graduate career.

I would like to thank my committee, Dr.Prasad Tadepalli, Dr.Paul Cull and Dr.Toshimi Minoura, for their support in the completion of my master degree.

I would like to thank my family in Korea. Their love and support has made my life a success so far. I also thank my husband who has continued to give me endless love and too many encouragements.

Separabilities for a New Class of Gray Codes

Jung Pil Park

Department of Computer Science

Oregon State University

Corvallis, OR 97331

parkju@cs.orst.edu

Abstract

Recently new classes of binary Gray codes have been proposed for the Lee distance Gray codes over Z_4 with the mapping: $0 \leftrightarrow 00$, $1 \leftrightarrow 01$, $2 \leftrightarrow 11$ and $3 \leftrightarrow 10$. For these codes of length n if the Hamming distance between the Gray codes $g(i)$ and $g(j)$ is d, where i and j are integers, then it is proved that $|i-j| > \frac{4}{15}2^d$ and $|i-j| < 2^n - \frac{4}{15}2^d$ for d odd, and $|i-j| > \frac{7}{15}2^d$ and $|i-j| < 2^n - \frac{7}{15}2^d$ for d even.

Contents

1 Introduction

In a binary Gray code, the set of 2^n binary vectors of length n is arranged in a sequence such that the consecutive words differ by exactly one bit.

Over the last 5 decades binary reflected Gray codes have found applications in diverse areas [1] such as: VLSI testing [2], signal encoding [3], ordering of documents on the shelves [4], data compression [5], graphics and image processing [6], processor allocation in the hypercube [7], hashing [8], computing the permanent [9], information retrieval [10], puzzles (such as the chinese rings and towers of Hanoi) [11], efficient combinatorial algorithm designs [9, 12, 13], low power VLSI design [14, 15], etc.

The binary reflected Gray code can be defined in two equivalent ways: If L_n stands for the Gray binary sequence of n-bit strings, then L_n can be recursively defined using the two rules:

$$\begin{aligned} L_0 &= \epsilon \\ L_{n+1} &= 0L_n\ ,\ 1L_n^R \end{aligned}$$

Here ϵ denotes the empty string, $0L_n$ denotes the sequence L_n with 0 prefixed to each string, and $1L_n^R$ denotes the sequence L_n in reverse order with 1 prefixed to each string. For example, $L_1 = 0, 1$, $L_2 = 00, 01, 11, 10$, $L_3 = 000, 001, 011, 010, 110, 111, 101, 100$, etc.

Another way to define the binary reflected Gray code is to give a function g as follows: Let i be an integer in the range $0 \le i \le 2^n - 1$ with binary representation $i = (i_{n-1}\, i_{n-2}\, \cdots\, i_0)$, $i_k = 0, 1$ for $k = 0, 1, \cdots, n-1$; the i-th Gray code $g(i)$ has the binary representation:

$$g(i) = (g_{n-1}\, g_{n-2}\, \cdots\, g_0)$$

where

$$\begin{aligned} g_k &= i_k + i_{k+1} \quad mod\ 2, \quad k = 0,\ 1,\ \cdots, n-2 \\ g_{n-1} &= i_{n-1} \end{aligned}$$

For example, when $n = 8$ and $i = (0\,0\,0\,0\,0\,1\,0\,1)$, $g(i) = (0\,0\,0\,0\,0\,1\,1\,1)$.

Originally, the Gray code was introduced in [16, 17] for the purpose of minimizing the number of erroneous bits in the transmission of bit strings over analog channels. If the strings were coded arithmetically, then a small error in the analog signal could cause a large number of bits to be received incorrectly. However, if the Gray codes are used then a one-level error can only cause an error in one bit, since neighboring numbers in Gray code differ in only one binary digit. In general, a two-level error can generate nothing worse than a two-bit error and a three-level error can generate at most a three-bit error. On the other hand, the minimum analog error required to generate m-bit errors increases more rapidly with m than the above three examples indicate. [18] and [19] present the separabilities, i.e. the lower and upper bound on the signal error that produces a m-bit error in the reflected binary Gray code. That is, suppose i and j are encoded as the reflected binary Gray codewords $g(i)$ and $g(j)$; if $D_H(g(i), g(j)) = m$, $m \geq 1$, then $|i-j| > \frac{2m}{3}$ and $|i-j| < 2^n - \frac{2m}{3}$.

In this paper, we present lower and upper bounds on the analog error that generates a d-bit error in the Lee distance Gray code over Z_4^n and its corresponding binary Gray code given in [20, 21]. Section 2 describes the Lee distance Gray code over Z_k^n, where two consecutive codewords differ in exactly one position by ± 1. Then the new class of binary Gray code generated from Z_4 code is given. Lower and upper bounds are derived in Section 3. In Section 3 the main results of this project, the separabilities of these codes, are derived. Section 4 ends with the conclusion and discusses topics of future research.

2 Preliminaries

2.1 Lee Distance

Let $A = (a_{n-1}a_{n-2}\cdots a_0)$ be an $n-digit$ radix k vector i.e. each component a_i obeys $0 \le a_i \le k-1$. The Lee weight of A is defined as

$$W_L(A) = \sum_{i=0}^{n-1} |a_i|,$$
$$\text{where } |a_i| = min(a_i, k-a_i), \text{ for } i = 0, 1, \cdots, n-1.$$

The Lee distance between two vectors $A = (a_{n-1}a_{n-2}\cdots a_0)$ and $B = (b_{n-1}b_{n-2}\cdots b_0)$ is denoted by $D_L(A,B)$ and is defined to be $W_L(A-B)$. That is, the Lee distance between two vectors is the Lee weight of their digitwise difference *mod k*. In other words, $D_L(A,B) = \sum_{i=0}^{n-1} min(a_i - b_i, b_i - a_i)$, where $a_i - b_i$ and $b_i - a_i$ are computed *mod k*. For example, when $k=4$, $W_L(321) = min(3, 4-3) + min(2, 4-2) + min(1, 4-1) = 1+2+1 = 4$, and $D_L(123, 321) = W_L(202) = 4$.

2.2 Lee distance Gray code in Z_k

In a Lee distance Gray code C, the set of k^n vectors over Z_k^n are arranged in a sequence such that two adjacent vectors are at a Lee distance 1. Further, the first and the last vectors in this sequence are also at distance 1.

2.2.1 Code design

Let the radix k representation of an integer i, $0 \le i \le k^n - 1$, be $(i_{n-1}i_{n-2}\cdots i_1 i_0)$, where $i_t \in Z_k = \{0, 1, 2, \cdots, k-1\}$ for $t = 0, 1, \cdots, n-1$; the ith Gray code $g(i)$ has the representation [21, 22]:

$$g(i) = (g_{n-1}g_{n-2}\cdots g_0)$$

	Radix		Gray
0	00	$\longrightarrow$	00
1	01	$\longrightarrow$	01
2	02	$\longrightarrow$	02
3	10	$\longrightarrow$	12
4	11	$\longrightarrow$	10
5	12	$\longrightarrow$	11
6	20	$\longrightarrow$	21
7	21	$\longrightarrow$	22
8	22	$\longrightarrow$	20

Table 1: Lee distance Gray code in Z_3

where

$$\begin{cases} g_t &= i_t - i_{t+1} \quad mod\ k, \quad t = 0, 1, 2, ..., n-2, \\ g_{n-1} &= i_{n-1} \end{cases} \tag{1}$$

Example 1 A Lee distance Gray code is shown in Table 1 for $k = 3$ and $n = 2$.

Claim 1 Function g given in (1) generates a Lee distance Gray code in Z_k^n.

Proof Let $i = (i_{n-1}i_{n-2} \cdots i_0)$ and $j = (j_{n-1}j_{n-2} \cdots j_0)$ be the two consecutive integers in radix k number system and $g(i) = (g_{n-1}^i g_{n-2}^i \cdots g_0^i)$ and $g(j) = (g_{n-1}^j g_{n-2}^j \cdots g_0^j)$ be the corresponding Gray codewords of i and j, respectively. Then we need to prove that $D_L(g(i), g(j)) = 1$.

Case 1 : If $j_m = i_m + 1$ for some m, $0 \leq m \leq n-1$, then $i_t = k-1$ and $j_t = 0$ for all $t = 0, 1, \cdots, m-1$ and $i_t = j_t$ for all $t = m+1, m+2, \cdots, n-1$. Now

$$\begin{aligned} g_t^i &= g_t^j && \text{for } t = 0, 1, \cdots, m-2, m+1, m+2, \cdots, n-1 \\ g_m^i &= i_m - i_{m+1} && mod\ k \\ g_{m-1}^i &= (k-1) - i_m && mod\ k \\ g_m^j &= j_m - j_{m+1} && mod\ k \;=\; i_m + 1 - i_{m+1} \quad mod\ k \\ g_{m-1}^j &= 0 - j_m && mod\ k \;=\; k - 1 - i_m \quad mod\ k \end{aligned}$$

Thus $D_L(g(i), g(j)) = 1$.

Case 2 : If $i = (k-1\ k-1\ \cdots\ k-1)$ and $j = (0\ 0\ \cdots\ 0)$, then $g(i) = (k-1\ 0\ \cdots\ 0)$ and $g(j) = (0\ 0\ \cdots\ 0)$. Thus $D_L(g(i), g(j)) = 1$.

Therefore, this construction produces a Lee distance Gray code in Z_n^k. ∎

A concise way of writing the relation between $g(i)$ and i is

$$g(i) = i \ominus [i/k],$$

where $\ominus$ is the digitwise difference *mod* k, while $[x]$ denotes the largest integer not exceeding x.

We can recover i from $g(i)$ as follows:

$$\begin{aligned} i_{n-1} &= g_{n-1} \\ i_t &= i_{t+1} + g_t \quad mod\ k \end{aligned}$$

which gives

$$\begin{aligned} i_{n-2} &= g_{n-1} + g_{n-2} \quad mod\ k \\ i_{n-3} &= g_{n-1} + g_{n-2} + g_{n-3} \quad mod\ k \end{aligned}$$

Thus

$$i_t = \sum_{m=t}^{n-1} g_m \quad mod\ k$$

2.2.2 Binary Gray code

The Lee n-digit distance Gray code over Z_4 under the mapping f: $Z_4 \longrightarrow Z_2^2$ such that $0 \rightarrow 00, 1 \rightarrow 01, 2 \rightarrow 11$, and $3 \rightarrow 10$ gives a binary Gray code of length $2n$. Under this mapping f, the Hamming distance between two distinct codewords is equal to the Lee distance between them [20, 22].

Example 2 Table 2 shows the two-digit Lee distance Gray code over Z_4 and the corresponding binary code after applying the function f to this Lee distance Gray code.

Gray code in Z_4^2	Binary Gray code
00	0000
01	0001
02	0011
03	0010
13	0110
10	0100
11	0101
12	0111
22	1111
23	1110
20	1100
21	1101
31	1001
32	1011
33	1010
30	1000

Table 2: Lee distance Gray code in Z_4^2 and Binary Gray code

Note that this binary Gray code is not equivalent to the binary reflected Gray code. Thus, by designing a Lee distance Gray code C over Z_4^n and then applying the function f on the digits in C, we get a new class of binary Gray code C' over Z_2^{2n}.

[18] and [19] present the lower and upper bound on the signal error that produces a m-bit error in the reflected binary Gray code. That is, suppose i and j are encoded as the reflected binary Gray codewords $g(i)$ and $g(j)$: if $D_H(g(i), g(j)) = m$, $m \geq 1$, then $|i-j| > \frac{2^m}{3}$ and $|i-j| < 2^n - \frac{2^m}{3}$.

In the following section, we shall present the lower and upper bounds for the Lee distance Gray code over Z_4^n and the binary Gray code generated from Z_4 code.

3 Separability on Lee distance Gray code in Z_4

Claim 2 Let $i = (i_{n-1}i_{n-2}\cdots i_0)$ and $j = (j_{n-1}j_{n-2}\cdots j_0)$ be two integers in the radix k number system. Then $g(i) \ominus g(j) = g(i \ominus j)$.

Proof Let $g(i) \ominus g(j) = (x_{n-1}x_{n-2}\cdots x_0)$ and $g(i \ominus j) = (y_{n-1}y_{n-2}\cdots y_0)$. Then

$$
\begin{aligned}
x_{n-1} &= i_{n-1} - j_{n-1} && \mod k \;=\; y_{n-1} \\
x_t &= (i_t - i_{t+1}) - (j_t - j_{t+1}) && \mod k \\
&= (i_t - j_t) - (i_{t+1} - j_{t+1}) && \mod k \\
&= y_t && \text{for } \; t = 0, 1, \cdots n-2
\end{aligned}
$$

Thus $g(i) \ominus g(j) = g(i \ominus j)$. ■

The following theorem gives us the lower bound in the Lee distance Gray code in Z_4^n.

Theorem 1 *If the Lee distance between $g(i)$ and $g(j)$ is d, then $|i-j| > \frac{7}{15}2^d$ for d even and $|i-j| > \frac{4}{15}2^d$ for d odd.*

Proof Without loss of generality, assume $i > j$. Let $|i-j|_{min}$ be the minimum distance between i and j.

Since $D_L(g(i), g(j)) = d$, it implies that $W_L(g(i) \ominus g(j)) = W_L(g(i \ominus j)) = d$. Let $l = (l_{n-1}\, l_{n-2}\, \cdots\, l_0) = i \ominus j$ and $g(l) = (g_{n-1}\, g_{n-2}\, \cdots\, g_0)$; so $W_L(g(l)) = d$. Two cases, d even and d odd, are considered below.

Case 1: Suppose d is even.

Case 1.1: $d = 0 \bmod 4$, i.e. $d = 2m$ and m even.

Since $W_L(g(l)) = W_L(g(i \ominus j)) = d$, $i - j$ is minimized when

$$i_t = \begin{cases} 2 & \text{for } t = m-1 \\ 0 & \text{otherwise} \end{cases}$$

$$j_t = \begin{cases} 2 & \text{for } t = m-3,\, m-5,\, \cdots,\, 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case, $i \ominus j = (0\, \cdots\, 2\,0\,2\,0 \cdots\, 2\,0)$, $g(i \ominus j) = (0\, \cdots\, 2\,2\, \cdots\, 2)$ and $W_L(g(i \ominus j)) = 2m$. (For example, with $n = 8$ and $d = 12$, $i = (0\,0\,2\,0\,0\,0\,0\,0)$, $j = (0\,0\,0\,0\,2\,0\,2\,0)$, $i \ominus j = (0\,0\,2\,0\,2\,0\,2\,0)$, $g(i \ominus j) = (0\,0\,2\,2\,2\,2\,2\,2)$ and $D_L(g(i), g(j)) = W_L(g(i \ominus j)) = 12$.)

Thus, $|i-j|_{min} = 2 \cdot 4^{m-1} - 2(4^{m-3} + 4^{m-5} + \cdots + 4^1)$.

Let

$$A = 4^1 + 4^3 + \cdots + 4^{m-3} \qquad \text{and}$$
$$B = 4^0 + 4^2 + \cdots + 4^{m-4}$$

Note that

$$A + B = \frac{4^{m-2} - 1}{3} \qquad \text{and}$$
$$4B = A$$

It follows that

$$A = \frac{4(4^{m-2} - 1)}{15} \qquad \text{and}$$
$$B = \frac{4^{m-2} - 1}{15}$$

Consequently, we have

$$\begin{aligned} |i-j|_{min} &= 2 \cdot 4^{m-1} - 2(\frac{4^{m-1}}{15} - \frac{4}{15}) \\ &= \frac{28}{15}4^{m-1} + \frac{8}{15} \\ &= \frac{7}{15}4^m + \frac{8}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i-j &> \frac{7}{15}4^m \\ &= \frac{7}{15}2^d \qquad \text{for } d \text{ even.} \end{aligned}$$

Case 1.2: $d = 2 \bmod 4$, i.e. $d = 2m+2$ and m even.

In this case, $i-j$ is minimized when

$$i_t = \begin{cases} 2 & \text{for } t = m \\ 0 & \text{otherwise} \end{cases}$$

$$j_t = \begin{cases} 2 & \text{for } t = m-2,\ m-4,\ \cdots,\ 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $i \ominus j = (0\cdots02020\cdots202)$, $g(i\ominus j) = (0\cdots022\cdots2)$ and $W_L(g(i\ominus j)) = 2m+2$. (For example, if $n = 10$ and $d = 10$ then $i = (0\,0\,0\,0\,0\,2\,0\,0\,0\,0)$, $j = (0\,0\,0\,0\,0\,0\,0\,2\,0\,2)$, $i \ominus j = (0\,0\,0\,0\,0\,2\,0\,2\,0\,2)$, $g(i \ominus j) = (0\,0\,0\,0\,0\,2\,2\,2\,2\,2)$ and $D_L(g(i), g(j)) = W_L(g(i \ominus j)) = 10$.)

Thus, $|i-j|_{min} = 2 \cdot 4^m - 2(4^{m-2} + 4^{m-4} + \cdots + 4^0)$.

Let

$$\begin{aligned} A &= 4^0 + 4^2 + \cdots + 4^{m-2} \qquad \text{and} \\ B &= 4^1 + 4^3 + \cdots + 4^{m-1} \end{aligned}$$

Note that

$$\begin{aligned} A + B &= \frac{4^m - 1}{3} \qquad \text{and} \\ 4A &= B \end{aligned}$$

It follows that

$$A = \frac{4^m - 1}{15}$$

Consequently, we have

$$\begin{aligned} |i-j|_{min} &= 2 \cdot 4^m - 2\left(\frac{4^m - 1}{15}\right) \\ &= \frac{7}{15} 4^{m+1} + \frac{2}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i - j &> \frac{7}{15} 4^{m+1} \\ &= \frac{7}{15} 2^d \qquad \text{for } d \text{ even.} \end{aligned}$$

Case 2: Suppose d is odd.

Case 2.1: $d = 1 \bmod 4$, i.e. $d = 2m+1$ and m even.

Then $i - j$ is minimized if

$$i_t = \begin{cases} 1 & \text{for } t = m \\ 0 & \text{otherwise} \end{cases}$$

$$j_t = \begin{cases} 1 & \text{for } t = m-1,\ m-3,\ \cdots,\ 1 \\ 3 & \text{for } t = m-2,\ m-4,\ \cdots,\ 0 \\ 0 & \text{otherwise} \end{cases}$$

In this case, $i \ominus j = (0 \cdots 0\,1\,3\,1\,3 \cdots 1\,3\,1)$, $g(i \ominus j) = (0 \cdots 0\,1\,2\,2 \cdots 2)$ and $W_L(g(i \ominus j)) = 2m+1$. (For example, when $n = 8$ and $d = 9$, $i = (0\,0\,0\,1\,0\,0\,0\,0)$, $j = (0\,0\,0\,0\,1\,3\,1\,3)$, $i \ominus j = (0\,0\,0\,1\,3\,1\,3\,1)$, $g(i \ominus j) = (0\,0\,0\,1\,2\,2\,2\,2)$ and $D_L(g(i), g(j)) = W_L(g(i \ominus j)) = 9$.)

Thus, $|i-j|_{min} = 4^m - (1 \cdot 4^{m-1} + 3 \cdot 4^{m-2} + 1 \cdot 4^{m-3} + 3 \cdot 4^{m-4} + \cdots + 3 \cdot 4^0) = 4^m - A$ where

$$\begin{aligned} A &= 1 \cdot 4^{m-1} + 3 \cdot 4^{m-2} + 1 \cdot 4^{m-3} + 3 \cdot 4^{m-4} + \cdots + 3 \cdot 4^0 \\ &= 3(4^0 + 4^2 + \cdots + 4^{m-2}) + (4^1 + 4^3 + \cdots + 4^{m-1}). \end{aligned}$$

Let $B = (4^0 + 4^2 + \cdots + 4^{m-2}) + 3(4^1 + 4^3 + \cdots + 4^{m-3})$.

Note that

$$\begin{aligned} A + B &= \frac{7}{3}4^{m-1} - \frac{4}{3} \\ 4B + 3 &= A \end{aligned}$$

We obtain

$$A = \frac{7}{15}4^m - \frac{7}{15}$$

Consequently, we have

$$\begin{aligned} |i-j|_{min} &= 4^m - \frac{7}{15}(4^m - 1) \\ &= \frac{8}{15}4^m + \frac{7}{15} \\ &= \frac{4}{15}2^{2m+1} + \frac{7}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i - j &> \frac{4}{15}2^{2m+1} \\ &= \frac{4}{15}2^d \qquad \text{for } d \text{ odd.} \end{aligned}$$

Case 2.2: $d = 3 \bmod 4$, i.e. $d = 2m + 3$ and m even. Then $i - j$ is minimized if

$$i_t = \begin{cases} 1 & \text{for } t = m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$j_t = \begin{cases} 1 & \text{for } t = m, m-2, \cdots, 0 \\ 3 & \text{for } t = m-1, m-3, \cdots, 1 \\ 0 & \text{for other values of } t \end{cases}$$

In this case, $i \ominus j = (0 \cdots 01313 \cdots 13)$, $i = (0 \cdots 010 \cdots 0)$, $j = (0 \cdots 01313 \cdots 131)$, $g(i \ominus j) = (0 \cdots 0\,1\,2\,2 \cdots 2)$ and $W_L(g(i \ominus j)) = 2m + 3$. (For example, when $n = 8$ and $d = 7$, $i = (0\,0\,0\,0\,1\,0\,0\,0)$, $j = (0\,0\,0\,0\,0\,1\,3\,1)$, $i \ominus j = (0\,0\,0\,0\,1\,3\,1\,3)$, $g(i \ominus j) = (0\,0\,0\,0\,1\,2\,2\,2)$ and $D_L(g(i), g(j)) = W_L(g(i \ominus j)) = 7$.)

Thus, $|i-j|_{min} = 4^{m+1} - (1 \cdot 4^m + 3 \cdot 4^{m-1} + 1 \cdot 4^{m-2} + 3 \cdot 4^{m-3} + \cdots + 1 \cdot 4^0) = 4^{m+1} - A$ where

$$\begin{aligned} A &= 1 \cdot 4^m + 3 \cdot 4^{m-1} + 1 \cdot 4^{m-2} + 3 \cdot 4^{m-3} + \cdots + 1 \cdot 4^0 \\ &= (4^m + 4^{m-2} + \cdots + 4^0) + 3(4^{m-1} + 4^{m-3} + \cdots + 4^1). \end{aligned}$$

Let $B = 3(4^{m-2} + 4^{m-4} + \cdots + 4^0) + (4^{m-1} + 4^{m-3} + \cdots + 4^1)$.

Note that

$$\begin{aligned} A + B &= \frac{7}{3}4^m - \frac{4}{3} \qquad \text{and} \\ 4B + 1 &= A \end{aligned}$$

We obtain

$$A = \frac{7}{15}4^{m+1} - \frac{13}{15}$$

Consequently, we have

$$\begin{aligned} |i-j|_{min} &= 4^{m+1} - (\frac{7}{15}4^{m+1} - \frac{13}{15}) \\ &= \frac{8}{15}4^{m+1} + \frac{13}{15} \\ &= \frac{4}{15}2^{2m+3} + \frac{13}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i - j &> \frac{4}{15}2^{2m+3} \\ &= \frac{4}{15}2^d \qquad \text{for } d \text{ odd.} \quad \blacksquare \end{aligned}$$

Corollary 1 *Let f: $Z_4 \longrightarrow Z_2^2$ be the mapping such that $0 \to 00, 1 \to 01, 2 \to 11$, and $3 \to 10$ and let $f(g(i))$ and $f(g(j))$ be the binary Gray codewords of $g(i)$ and $g(j)$, respectively. If the Hamming distance between $f(g(i))$ and $f(g(j))$ is d, then $|i-j| > \frac{7}{15}2^d$ for d even and $|i-j| > \frac{4}{15}2^d$ for d odd.*

The following theorem shows the upper bound in the Lee distance Gray code in Z_4^n.

Theorem 2 *If the Lee distance between $g(i)$ and $g(j)$ is d, then $|i-j| < 4^n - \frac{7}{15}2^d$ for d even and $|i-j| < 4^n - \frac{4}{15}2^d$ for d odd.*

Proof Without loss of generality, assume $i > j$. Let $|i-j|_{max}$ be the maximum distance between i and j.

Since $D_L(g(i), g(j)) = d$, it implies that $W_L(g(i) \ominus g(j)) = W_L(g(i \ominus j)) = d$. Let $l = (l_{n-1} l_{n-2} \cdots l_0) = i \ominus j$ and $g(l) = (g_{n-1} g_{n-2} \cdots g_0)$; so $W_L(g(l)) = d$. Clearly $i-j$ is maximized if $i = l$ and $j = 0$. Two cases, d odd and d even, are considered below.

Case 1: Suppose d is odd

Case 1.1 : $d = 1 \bmod 4$, i.e. $d = 2m+1$ and m even.

Then l is maximized if

$$l_t = \begin{cases} 3 & \text{for} \quad t = n-1, n-2, \cdots, m \\ 1 & \text{for} \quad t = m-1, m-3, \cdots, 1 \\ 3 & \text{for} \quad t = m-2, m-4, \cdots, 0 \end{cases}$$

In this case, $l = (33 \cdots 31313 \cdots 13)$, $i = (33 \cdots 31313 \cdots 13)$, $j = (00 \cdots 00)$, $g(i \ominus j) = (300 \cdots 0222 \cdots 2)$ and $W_L(g(i \ominus j)) = 2m+1$. (For example, when $n = 8$ and $d = 9$, $l = i = (33331313)$, $j = (00000000)$, $g(i \ominus j) = (30002222)$, and $W_L(g(i \ominus j)) = 1+2+2+2+2 = 9$.)

Thus,

$$|i-j|_{max} = 4^n - 1 - A$$

$$\text{where } A = 2(4^{m-1} + 4^{m-3} + \cdots + 4^1).$$

Let $B = 2(4^{m-2} + 4^{m-4} + \cdots + 4^0)$.

Then

$$\begin{aligned} A + B &= 2(4^0 + 4^1 + \cdots + 4^{m-1}) = \frac{2}{3}(4^m - 1) \\ 4B &= 2(4^{m-1} + 4^{m-3} + \cdots + 4^1) = A \end{aligned}$$

We have

$$A = \frac{8}{15}(4^m - 1)$$

Finally, we obtain

$$\begin{aligned} |i-j|_{max} &= 4^n - 1 - \frac{8}{15}(4^m - 1) \\ &= 4^n - \frac{8}{15}4^m - \frac{7}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i - j &< 4^n - \frac{8}{15}4^m \\ &= 4^n - \frac{4}{15}2^{2m+1} \\ &= 4^n - \frac{4}{15}2^d \qquad \text{for } d \text{ odd.} \end{aligned}$$

Case 1.2 : $d = 3 \bmod 4$, i.e. $d = 2m+3$ and m even.

Then l is maximized if

$$l_t = \begin{cases} 3 & \text{for } t = m+1, m+2, \cdots, n-1 \\ 1 & \text{for } t = m, m-2, \cdots, 0 \\ 3 & \text{for } t = m-1, m-3, \cdots, 1 \end{cases}$$

In this case, $l = i \ominus j = (3 \cdots 3\,1\,3\,1\,3 \cdots 1\,3\,1)$, $i = (3 \cdots 3\,1\,3\,1\,3 \cdots 1\,3\,1)$, $j = (0\,0 \cdots 0\,0)$, $g(i \ominus j) = (3\,0\,0 \cdots 0\,2\,2 \cdots 2)$ and $W_L(g(i \ominus j)) = 2m+3$. (For example, when $n = 10$ and $d = 11$, $l = i = (3333313131)$, $j = (0000000000)$, $g(i \ominus j) = (3000022222)$, and $W_L(g(i \ominus j)) = 1+2+2+2+2+2 = 11$.)

Thus,

$$|i-j|_{max} = 4^n - 1 - A$$

where $A = 2(4^m + 4^{m-2} + \cdots + 4^0)$.

Let $B = 2(4^{m-1} + 4^{m-3} + \cdots + 4^1)$.

Note that

$$\begin{aligned} A + B &= 2(4^0 + 4^1 + \cdots + 4^m) = \frac{2}{3}(4^{m+1} - 1) \\ 4B + 2 &= 2(4^{m-1} + 4^{m-3} + \cdots + 4^0) = A \end{aligned}$$

We have

$$A = \frac{8}{15}4^{m+1} - \frac{2}{15}$$

Finally, we obtain

$$\begin{aligned} |i-j|_{max} &= 4^n - 1 - \frac{8}{15}4^{m+1} + \frac{2}{15} \\ &= 4^n - \frac{8}{15}4^{m+1} - \frac{13}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i-j &< 4^n - \frac{8}{15}4^{m+1} \\ &= 4^n - \frac{4}{15}2^{2m+3} \\ &= 4^n - \frac{4}{15}2^{d} \qquad \text{for } d \text{ odd.} \end{aligned}$$

Case 2: Suppose d is even.

Case 2.1 : $d = 0 \bmod 4$, i.e. $d = 2m$ and m even.

Then l is maximized if

$$l_t = \begin{cases} 3 & \text{for } t = n-1, n-2, \cdots, m \\ 2 & \text{for } t = m-1, m-3, \cdots, 1 \\ 0 & \text{for } t = m-2, m-4, \cdots, 0 \end{cases}$$

In this case, $l = (33\cdots32020\cdots20)$, $i = (33\cdots32020\cdots20)$, $j = (00\cdots00)$, $g(i\ominus j) = (300\cdots0322\cdots2)$ and $W_L(g(i\ominus j)) = 2m$. (For example, when $n = 8$ and $d = 8$, $l = i = (3\,3\,3\,3\,2\,0\,2\,0)$, $j = (0\,0\,0\,0\,0\,0\,0\,0)$, $g(i\ominus j) = (3\,0\,0\,0\,3\,2\,2\,2)$, and $W_L(g(i\ominus j)) = 1+1+2+2+2 = 8$.)

Thus,

$$|i-j|_{max} = 4^n - 1 - A$$

where $A = (4^{m-1} + 4^{m-3} + \cdots + 4^1) + 3(4^{m-2} + 4^{m-4} + \cdots + 4^0)$.

Let $B = (4^{m-2} + 4^{m-4} + \cdots + 4^0) + 3(4^{m-3} + 4^{m-5} + \cdots + 4^1)$.

Then

$$
\begin{aligned}
A+B &= \frac{4^m-1}{3}+\frac{3(4^{m-1}-1)}{3} \\
&= \frac{7\cdot 4^{m-1}-4}{3} \\
4B+3 &= A
\end{aligned}
$$

We have

$$A=\frac{7}{15}(4^m-1)$$

Finally, we obtain

$$
\begin{aligned}
|i-j|_{max} &= 4^n-1-\frac{7}{15}(4^m-1) \\
&= 4^n-\frac{7}{15}4^m-\frac{8}{15}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
i-j &< 4^n-\frac{7}{15}4^m \\
&= 4^n-\frac{7}{15}2^{2m} \\
&= 4^n-\frac{7}{15}2^d \qquad \text{for } d \text{ even.}
\end{aligned}
$$

Case 2.2 : $d=2 \bmod 4$, i.e. $d=2m+2$ and m even.

Then l is maximized if

$$
l_t = \begin{cases} 3 & \text{for } t=m+1,m+2,\cdots,n-1 \\ 2 & \text{for } t=m,m-2,\cdots,0 \\ 0 & \text{for } t=m-1,m-3,\cdots,1 \end{cases}
$$

In this case, $l=i\ominus j=(3\,\cdots\,3\,2\,0\,2\,0\cdots\,2\,0\,2)$, $i=(3\,\cdots\,3\,2\,0\,2\,0\cdots\,2\,0\,2)$, $j=(0\,0\,\cdots\,0\,0)$, $g(i\ominus j)=(3\,0\,0\cdots\,3\,2\,2\,\cdots\,2)$ and $W_L(g(i\ominus j))=2m+2$. (For example, when $n=10$ and $d=10$, $l=i=(3333320202)$, $j=(0000000000)$, $g(i\ominus j)=(3000032222)$, and $W_L(g(i\ominus j))=1+1+2+2+2+2=10$.)

Thus,

$$|i-j|_{max}=l=4^n-1-A$$

$$\text{where } A = (4^m + 4^{m-2} + \cdots + 4^0) + 3(4^{m-1} + 4^{m-3} + \cdots + 4^1).$$

Let $B = (4^{m-1} + 4^{m-3} + \cdots + 4^1) + 3(4^{m-2} + 4^{m-4} + \cdots + 4^0)$.

Then

$$\begin{aligned} A + B &= \frac{4^{m+1} - 1}{3} + \frac{3(4^m - 1)}{3} \\ &= \frac{7 \cdot 4^m - 4}{3} \\ 4B + 1 &= A \end{aligned}$$

We have

$$A = \frac{7}{15} 4^{m+1} - \frac{13}{15}$$

Finally, we obtain

$$\begin{aligned} |i - j|_{max} &= 4^n - 1 - \frac{7}{15} 4^{m+1} + \frac{13}{15} \\ &= 4^n - \frac{7}{15} 4^{m+1} - \frac{2}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} i - j &< 4^n - \frac{7}{15} 4^{m+1} \\ &= 4^n - \frac{7}{15} 2^{2m+2} \\ &= 4^n - \frac{7}{15} 2^d \qquad \text{for } d \text{ even.} \quad \blacksquare \end{aligned}$$

Corollary 2 *Let $f\colon Z_4 \longrightarrow Z_2^2$ be the mapping such that $0 \to 00, 1 \to 01, 2 \to 11$, and $3 \to 10$ and let $f(g(i))$ and $f(g(j))$ be the binary Gray codewords of $g(i)$ and $g(j)$, respectively. If the Hamming distance between $f(g(i))$ and $f(g(j))$ is d, then $|i - j| < 2^{2n} - \frac{7}{15}2^d$ for d even and $|i - j| < 2^{2n} - \frac{4}{15}2^d$ for d odd.*

4 Conclusion and Future Research

In this paper, we have presented the lower and upper bounds on the signal error that produces a d-bit error in the Lee distance Gray code over Z_4^n and the binary Gray code generated from this Lee distance Gray code. Our future research will include extending these results to the Lee distance Gray code over Z_k^n.

In [20, 22, 23], another Lee distance Gray code in Z_k^n is introduced: The function $h : Z_k^n \longrightarrow Z_k^n$ which generates a Gray code can be obtained as follows:

$$\begin{cases} h(a) & = a & \text{for } n = 1 \\ h(a_{2m-1}\, a_{2m-2} \cdots a_0) & = h(a_{2m-1}\, a_{2m-2} \cdots a_m) h(d_{m-1}\, d_{m-2} \cdots d_0) & \text{for } n = 2m \end{cases} \quad (2)$$

where $d_{m-1}\, d_{m-2} \cdots d_0 = (a_{m-1}\, a_{m-2} \cdots a_0) \ominus_{k^n} (a_{2m-1}\, a_{2m-2} \cdots a_m)$ and $\ominus_{k^n}$ is the minus operator in Z_k^n.

Example 3 Let $A = (21132301)$ over Z_4. The Gray codeword of A can be computed as follows.

$$\begin{aligned} h(21132301) &= h(2113)h(2301 \ominus_{4^4} 2113) \\ &= h(2113)h(0122) \\ &= h(21)h(13 \ominus_{4^2} 21)h(01)h(22 \ominus_{4^2} 01) \\ &= h(21)h(32)h(01)h(21) \\ &= h(2)h(1 \ominus_4 2)h(3)h(2 \ominus_4 3)h(0)h(1 \ominus_4 0)h(2)h(1 \ominus_4 2) \\ &= h(2)h(3)h(3)h(3)h(0)h(1)h(2)h(3) \\ &= (23330123) \end{aligned}$$

The separabilities of these codes are still not known and this is an open research problem.

References

[1] C. Savage, "A Survey of Combinatorial Gray Codes", SIAM REV., 39(4):605-629, December 1997.

[2] J. Robinson and M. Cohn, "Counting sequences", *IEEE Transactions on Computers*, C-30:17-23, 1981.

[3] J. E. Ludman, "Gray code generation for MPSK signals", *IEEE Transactions on Communications*, 29:1519-1522, 1981.

[4] R. M. Losse, "A Gray code based ordering for documents of shelves: Classification for browsing and retrieval", *Journal of the American Society for Information Science*, 43(4):312-322, 1992.

[5] D. Richards, "Data compression and Gray-code sorting", it Information Processing Letters, 22:210-205, 1986.

[6] D. J. Amalraj, N. Sundararajan, and G. Dhar, "A data structure based on Gray code encoding for graphics and image processing", In *SPIE: International Society for Optical Engineering*, pp.65-76, 1990.

[7] M. Chen and K. G. Shin, "Subcube allocation and task migration in hypercube machines", *IEEE Transactions of Computers*, 39(9):1146-1155, 1990.

[8] C.Faloutsos, "Gray codes for partial match and range queries", *IEEE Transactions of Software Engineering*, 14(10):1381-1393, 1988.

[9] A.Nijenhuis and . S. Wilf, "Combinatorial Algorithms for Computers and Calculators", Academic Press, 1978.

[10] C. C. Chang, H. Y. Chen, and C. Y. Chen, "Symbolic Gray codes as a data allocation scheme for two disc systems", *Computer Journal*, 35(3):299-305, 1992.

[11] Martin Gardner, "Curious properties of the Gray code and how it can be used to solve puzzles", *Scientific American*, 227(2):106-109, 1972.

[12] E.Reingold, J. Nievergelt, and N.Deo, "Combinatorial Algorithms - Theory and Practise", Prentice Hall, 1977.

[13] H. S. Wilf, "Combinatorial Algorithms: An Update", *Society for Industrial and Applied Mathematics*, Philadelphia, Pennsylvania, 1989.

[14] C. Su, C. Tsui, and A. Despain, "Low power architecture design and compilation techniques for high performance processors", *IEEE Symposium on Low Power Electronics*, pp.49-54, 1994.

[15] H. Mehta, R. M. Owens, and M. J. Irwin, "Some issues in Gray code addressing", *IEEE Symposium on Low Power Electronics*, pp.178-181, 1996.

[16] E. N. Gilbert, "Gray codes and paths on the n-cube", *Bell Systems Technical Journal*, 37:815-826, 1958.

[17] F. Gray, "Pulse cose Communication", U.S. Patent 2632058, March 1953.

[18] C. K. Yuen, "The separability of Gray code", IEEE *Trans. Inform Theory*, vol.IT-20, pp.668, Sept. 1974.

[19] Stephan R. Cavior, "An Upper Bound Associated with Errors in Gray Code", *IEEE Trans. Inform Theory*, vol IT-21, p.596, September 1975.

[20] B. Bose, M. Bae, and B. AlMohammad, "Lee Distance Gray Codes", preprint.

[21] B. Bose, B. Broeg, Y. Kwon, and Y. Ashir, "Lee distance and topological properties of k-ary n-cubes", *IEEE Transactions on Computers*, vol 44, No 8, pp. 1021-1030, August 1995.

[22] B. Bose and M. Bae, "Gray Codes and Edge Disjoint Hamiltonian Cycles in Torus and Hypercube", *IEEE International Symposium on Information Theory*, June-July 2002.

[23] M. Bae and B. Bose, "Edge Disjoint Hamiltonian Cycles in k-ary n-cubes and Hypercubes", *IEEE Transactions on Computers*, to be published

[24] C. K. Yuen, "Comments on correction of errors in multilevel Gray coded data", IEEE *Trans. Inform Theory*, vol.IT-20, pp.283-284, Mar. 1974.