The Fano Plane as an Octonionic Multiplication Table

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1 Introduction

When considering finite geometries, an obvious question to ask is what applications such geometries have. One use of the Fano geometry is as a multiplication table for the octonions. This table is shown in Figure 1.

The octonions, denoted $\mathbb{O}$, form an eight-dimensional number system. A number $z \in \mathbb{O}$ takes the form

$$z = x_1 + x_2 i + x_3 j + x_4 k + x_5 k\ell + x_6 j\ell + x_7 i\ell + x_8 \ell \tag{1}$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{R}$, and $i, j, k, k\ell, j\ell, i\ell,$ and ℓ are each a distinct square root of -1. That being the case, $i, j, k, k\ell, j\ell, i\ell,$ and ℓ all square to -1. The interesting question to ask, and the important one to answer, is what happens when you multiply $i, j, k, k\ell, j\ell, i\ell,$ and ℓ with each other. One would like to think that the product of any two distinct units would also be -1. However, this is not the case. The product of any two distinct units can be determined using the multiplication table given in Figure 1. In Figure 1, each point in the Fano geometry corresponds to an octonionic imaginary unit. To denote the products of i with ℓ, j with ℓ, and k with ℓ, we use the notation of Dray and Manogue and simply write $i\ell$, $j\ell$, and $k\ell$ respectively [1]. Each line in the Fano geometry contains three points labeled as units, and also has an orientation, indicated by an arrow. The product of two units on a line is the third unit on the line moving in the direction of the arrow on that line. Moving against the orientation introduces a negative sign in the product. Products are defined cyclicly on the lines so, for example,

$$(k\ell)j = i\ell, \tag{2}$$

$$j(i\ell) = k\ell, \tag{3}$$
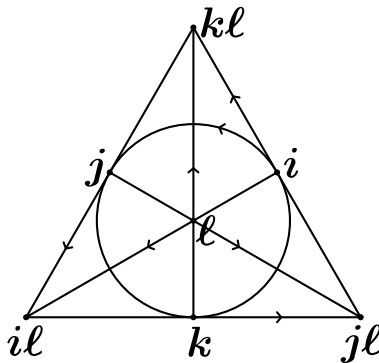
$$(i\ell)(k\ell) = j. \tag{4}$$

Figure 1: The Fano geometry labeled as to give products consistent with the multiplication table of the octonions.
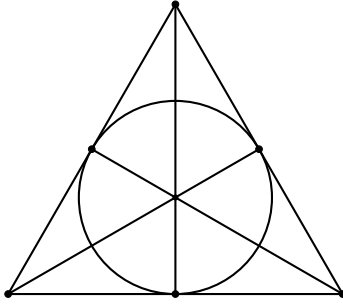
Figure 2: The 7-point projective plane, also known as the Fano geometry.

Two especially noteworthy facts about multiplication over the octonions are that multiplication is both non-associative and non-commutative. Below are a few examples that demonstrate these properties.

To see that multiplication is non-associative, observe

$$j(i\ell) = k\ell, \tag{5}$$

but

$$(ji)\ell = -k\ell. \tag{6}$$

To see that multipication is non-commutative, observe

$$(i\ell)k = j\ell, \tag{7}$$

but

$$k(i\ell) = -j\ell. \tag{8}$$

The question this paper will address is whether the labeling of the Fano geometry given in Figure 1 is unique in yielding products consistent with those defined on the octonions. If it is not, the goal is to determine how many ways the Fano geometry can be labeled such that it can be used as the octonionic multiplication table.

2 The Fano Plane

The Fano geometry is a finite geometry, that is, there are finitely many points and lines in it. Some basic facts about the Fano geometry are that **1)** there exist exactly seven points, **2)** there exist exactly seven lines, **3)** every line contains exactly three points, and **4)** three distinct lines intersect at any given point. The Fano geometry is shown in Figure 2.

For those not familiar with the Fano geometry, a few clarifications are in order. First, based on Figure 1, it may appear that the Fano geometry contains only six lines and one circle instead of seven lines. Since "line" is an undefined term in the axioms for the Fano geometry, the practice in this paper is to consider the circle as the seventh "line." The reason for this will become apparent in the course of this paper. Second, the terms Fano geometry and Fano plane are interchangeable. From this point on, Fano plane will be used most often in order to stress that this geometry is 2-dimensional.

3   Multiplication Properties of the Octonions

When labeling the Fano plane to obtain a multiplication table for the octonions, it must first be determined just how to construct a table that is consistent with the octonions. Such a construction is governed by a few multiplicative properties of the octonions, both of which can be observed in the multiplication table given in Figure 1. First and foremost is the fact that multiplication is cyclic within triples. In other words, given any triple of imaginary octonionic units, the product of multiplying any two of those units together is the third unit in the triple, with the sign depending on the order of multiplication. This is seen easily by looking at the multiplication table shown in Figure 1. The other property that must be taken into consideration is that multiplication is anticommutative on the units $i, j, k, k\ell, j\ell, i\ell,$ and ℓ. This determines how lines in the plane must be oriented once the points have been labeled. If a line does not have the correct orientation, the multiplication table will give products with the wrong sign.

Together, these two properties strongly dictate how the Fano plane can be labeled. Exploring their ramifications will help provide a method whereby to label the Fano plane such that it gives a multiplication table consistent with the octonions. The first consequence of these properties is that triples cannot be broken or manipulated aside from being reordered. For example, if i and j are both on one line, the third unit cannot be ℓ; it must be k. If a different unit is chosen for the third point on the line containing i and j, it will break the cyclic nature of the multiplication and thus yield a multiplication table that does not represent the octonions. Therefore, we see that each triple of octonionic imaginary units must go to one line on the Fano plane. The units can be listed in any order on a line in the Fano plane so long as all three units from a triple lie on one line in the Fano plane. The other requirement is that the line be oriented such that products have the correct sign. Since $3! = 6$, there are six different orderings for three units. The orderings for the triple $\{i, j, k\}$ are shown in Figure 3. Each triple is oriented so as to be consistent with the octonions

Given the seven lines, each with an orientation, and the seven octonionic imaginary units, it is clear that there are many possible ways to label the Fano plane. There are seven points on which to place units. Therefore, there are $7! = 5040$ ways to label the points in the Fano plane. Since each line can be oriented two different ways, there are 2^7 orientations of the Fano plane. That being the case, it would appear that there are

$$2^7 \cdot 7! = 645120 \tag{9}$$

different ways to label the Fano plane. However, not all of those labelings are truly distinct since any given labeling can be rotated three ways and reflected through three axes of sym-
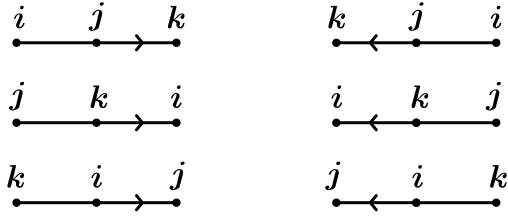
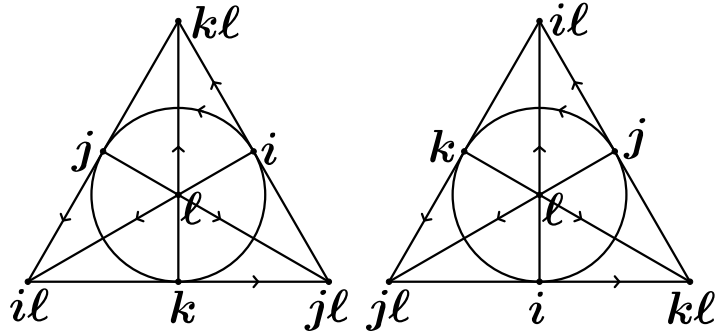Figure 3: The properly oriented orderings for the triple $\{i, j, k\}$.

Figure 4: An example of two labelings from the same congruence class.

metry. An example of clockwise rotation of a labeling is shown in Figure 4.

Even once congruence among tables has been accounted for, there still remains the question of just how many of the remaining tables correspond to the octonions. The table shown in Figure 5 is proof enough that not all labelings correspond to the octonions. This table gives the follwing products:

$$(k\ell)i = j\ell, \tag{10}$$

$$i(j\ell) = k\ell, \tag{11}$$

$$(j\ell)(k\ell) = i. \tag{12}$$

On the octonions, these products all have a negative sign:

$$(k\ell)i = -j\ell, \tag{13}$$

$$i(j\ell) = -k\ell, \tag{14}$$

$$(j\ell)(k\ell) = -i. \tag{15}$$

Rather than first determing how many different multiplication tables exist and then determining which multiplication tables corespond to the octonions, the goal is to determine a method for buildling all tables that correspond to the octonions and then count the number of such tables.
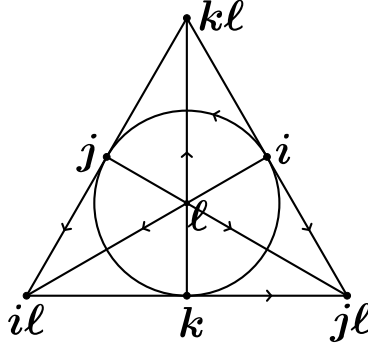
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Figure 5: An example of a labeling that does not correspond to the octonionic multiplication table.

$$\{i, j, k\} \quad \{i, \ell, i\ell\} \quad \{i\ell, k, j\ell\} \quad \{j, \ell, j\ell\}$$
$$\{j\ell, i, k\ell\} \quad \{k, \ell, k\ell\} \quad \{k\ell, j, i\ell\}$$

Figure 6: The seven octonionic triples.

4 Labeling the Fano Plane

In order to label the Fano plane as a multiplication table for the octonions, we first consider labeling only the perimeter. As has already been determined, triples of imaginary octonionic units must go to lines in the Fano plane. All seven triples are listed in Figure 6.

First, pick one of the seven triples listed above to place on a border line of the Fano plane. Each triple can be ordered six ways. Hence, we have $7 \cdot 6 = 42$ choices for how to pick and order the first triple. Next, a second triple must be picked to go on one of the adjacent sides of the Fano plane. This second triple must share exactly one unit with the first triple that was chosen. (This unit will be on the vertex of the two lines containing the two triples.) Simply reading through the list of triples makes it clear that each unit is contained in exactly three triples. Therefore, since the first triple already contains the shared unit, there are two possible choices for the second triple. Since the placement of the shared unit is fixed, the number of orderings of the second triple is limited. The only possible variation is that the placement of the other two units can be swapped. Therefore, there are two possible orderings for the second triple. Given seven choices for the first triple, six ways to order the first triple, two choices for the second triple and two ways to order the second triple, there are $7 \cdot 6 \cdot 2 \cdot 2 = 168$ ways to pick the first two triples to go on the border of the Fano plane. Figure 7 shows how the same two triples can be chosen in two non-equivalent ways.

Picking the third triple to go on the perimeter of the Fano is the most significant step in this process, mainly because there is only one way it can be picked. Since the third triple goes on the remaining perimeter line of the Fano plan, it intersects both the first and
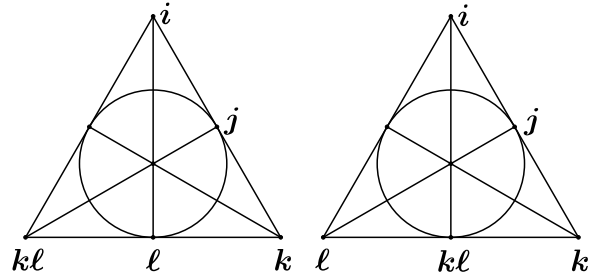
Figure 7: The second triple can be ordered in two distinct ways

the second triple. As a consequence, the third triple must share exactly one unit with the first triple and one unit with the second triple. Therefore, two units in the third triple are predetermined. However, given the cyclic nature of multiplication within triples, there can be only one triple that contains any two distinct units. Therefore, there is only one way the third triple can be chosen. Hence, there are

$$7 \cdot 6 \cdot 2 \cdot 2 = 168 \tag{16}$$

ways to label the perimeter.

Once the perimeter is labeled, it remains to label the point in the center of the Fano plane. However, there are six points on the perimeter of the Fano plane which have already been labeled. Since there are seven units that must be placed on the Fano plane, it is clear that only one unit is left unused. Therefore, the point in the center of the Fano plane must be labeled with the one unit that has not yet been assigned to a point. The question is whether or not this unit completes the unfinished triples that lie on axes of symmetry of the Fano plane. In fact, it does. To see this, assume that the last unit does not complete a triple lying on an axis of symmetry. For convenience, call the triple T. Then the unit missing from T must lie on the perimeter of the Fano plane. There are four possible points where the unit could be (It is obviously not on either endpoint of the line from which it is missing and the center point has no label yet). All four of these points are on a line containing one of the other units from T though since the four points are split over two sides of the Fano plane. Due to the cyclic nature of multiplication within triples, the third unit in T must then also be on the perimeter in the same triple. Then we have that T is the same as a triple on the perimeter. Call this triple A. Since A is on the perimeter, two units in A are on vertices. This means that one endpoint of T is on the same line as a vertex point from A. Then, again due to cyclic multiplication within triples, the third unit from T is in this second triple. Therefore, we have the same triple appearing twice on the perimeter. This contradicts how the perimeter was labeled though. Therefore, the unused unit does indeed complete all three triples that lie on axes of symmetry of the Fano plane.

Due to similarities between multiplicative properties of the octonions and the construction of the Fano plane, the three units on the circle also form a triple. Just as the Fano plane has seven points, there are seven octonionic units that must be placed on the Fano

plane. Similarly, the Fano plane contains seven lines and there are seven triples of octonionic units. Lastly, just as any point in the Fano plane lies on exactly three lines, any octonionic unit is contained in exactly three triples. Each unit on the circle is contained in a triple on the perimeter and in a triple on an axis of symmetry of the Fano plane. Therefore, each unit on the circle is contained in two triples. The other four units are already contained in three triples each. Moreover, six completed triples have already been placed on the Fano plane. Therefore, since there is one triple that has yet to be accounted for and all three of the units on the circle still need to be place in one triple each, it must be the case that the three triples on the circle indeed make up a triple. Therefore, all seven triples of octonionic units have been placed in the Fano plane.

It remains to provide an orientation for each line of the Fano plane. Each line has two possible orientations. However, due to the the non-commutativity of multiplication on the octonions, once the units on a line have been labeled, there is only one orientation that will give the products defined on the octonions. It has already been seen in Figure 3 that for any ordering of a triple, there is an orientation of the line on which they lie that yields octonionic products. Since the orientation of any one line has no effect on any other line, each line can be orientated such that the entire Fano plane yields the products defined on the octonions. Moreover, since any ordering of a triple has only one orientation that will produce the desired products, there is only one orientation for the Fano plane as a whole that will make a given labeling an octonionic multiplication table. Therefore, there are 168 ways to label the Fano plane such that it can be used as an octonionic multiplication table.

5 Accounting for Congruence

We have now shown that there are at most 168 different ways to label the Fano plane such that it can be used as a multiplication table for the octonions. What remains unanswered though is whether or not this number is counting some tables multiple times. In fact, it is. Consider the symmetries of the Fano plane. Clearly, there are three rotations. Since the Fano plane has three axes of symmetry, there are also three reflections. The question then is how many labelings are in a congruence class? Obviously, there are three due to rotation. Accounting for the reflections is slightly trickier since reflecting is a binary choice. It turns out that all three reflections can be generated by reflecting the table through any axis of symmetry and then rotating that labeling. This is shown for one labeling of the Fano plane in Figure 8. Note that table 1 can be reflected across its three axes of symmetry to obtain tables 2, 3, and 4. However, tables 2, 3, and 4 are all rotations of each other.

This being the case, there are six tables in each equivalence class. Given a table, it can be reflected through an axis of symmetry giving two equivalent tables. Each table can then be rotated one of three ways. This accounts for all rotations and all reflections of a table. Hence, for any given table, there are

$$3 \cdot 2 = 6 \tag{17}$$

equivalent ways of labeling the table. Therefore, we can divide 168 by 6 and this will account for all equivalent tables. Since
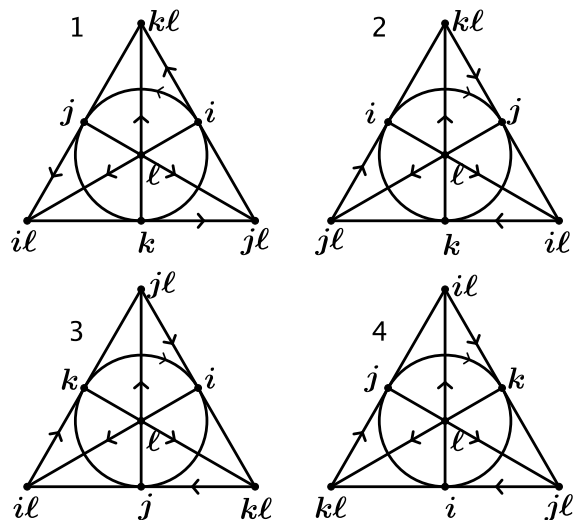
$$168/6 = 28, \tag{18}$$

Figure 8: The table in the upper left can be reflected through an axis of symmetry to obtain any of the other three tables. The other three tables are rotations of each other.

there are 28 distinct ways of labeling the Fano plane so that it can be used as a multiplication table for the octonions.

6   Conclusion

The fact that there exist 28 distinct ways of labeling the Fano plane as a multiplication table for the octonions motivates some further questions. For example, how many distinct ways of defining multiplication on seven objects exist? Though the answer to this question is not obvious based on the work from this paper, the answer can likely be determined using the main results presented herein and counting arguments similar to those employed for this paper. A related problem is determining the connection between 28 (the number of distinct ways to label the Fano plane as an octonionic multiplication table) and 14, the dimension of the automorphism group of the octonions.

   Yet another question motivated by this paper is whether or not multiplication tables for other number systems can be represented by finite geometries, or simply geometric figures in general. If so, the logical follow-up question would be how many nonequivalent ways such a table can be made. One example that is likely familiar to the reader is the multiplication mnemonic shown in Figure 9, which corresponds to the cross product defined on the basis vectors of $\mathbb{R}^3$, $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. This same multiplication table can be used for the quaternions, a number system similar to the octonions in that it has three imaginary units ($i, j,$ and k) making it four-dimensional as opposed to eight-dimensional. One important difference between the quaternions and the cross product in three dimensions is that while the cross product of any basis vector with itself is the zero-vector, the square of any quaternionic imaginary unit is -1.
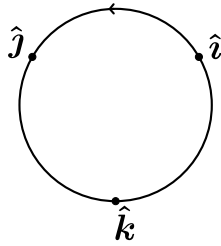
Figure 9: The multiplication table for the cross product defined on the basis vectors of $\mathbb{R}^3$.

Of particular interest to the author are the sedenions, a sixteen dimensional non-division algebra with fifteen imaginary units. Investigating the multiplicative properties of the sedenions and looking at different ways of representing their multiplication table are certainly future goals in the trajectory of the work presented in this paper.

References

[1] Dray, Tevian and Corinne A. Manogue *The Octonions*. Oregon State University, Corvallis, OR 2002