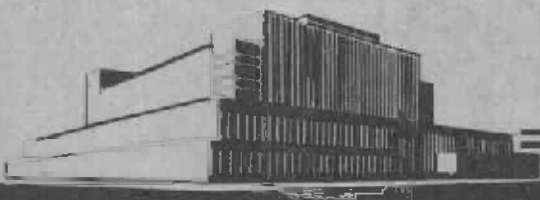


A METHOD OF CALCULATING INTERNAL STRESSES IN DRYING WOOD

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UNITED STATES DEPARTMENT OF AGRICULTURE
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In Cooperation with the University of Wisconsin

A METHOD OF CALCULATING INTERNAL STRESSES

IN DRYING WOOD

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Summary

A method has been developed for calculating the perpendicular-to-grain normal and shear stresses acting at any point on the cross section of a drying board. Required are measurements of instantaneous recovery of successive slices cut parallel to the face of the board from a short cross section of the board, information on the temperature and moisture content of the slices at the time they are cut off, and data on perpendicular-to-grain elastic properties under the temperature and moisture content conditions of the recovering slice. The method makes use of a stress function containing constants that are evaluated by combined use of simultaneous equations and the principle of least work.

The method is applied to calculation of drying stresses in a 2- by 7-inch flat-sawed plank of northern red oak (Quercus rubra L.) after four days of drying from the green condition at 80° F. and 85 percent relative humidity. The results appear to indicate quite satisfactorily the magnitude and distribution of normal and shear stresses at the stage of drying considered.

¹Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Introduction

As wood dries, it is strained by a complex pattern of internal stresses that develop as a result of restraints characteristic of normal shrinkage. Such stresses are found in all lumber during normal drying and are responsible for most of the defects associated with the drying process.

Although such stresses have been known and recognized for many years, no suitable method of calculating their magnitude and distribution has been available. As a result, the development of schedules for drying wood without excessive losses due to drying defects and without unduly prolonging the drying process has been almost entirely by empirical procedures.

In recent years, investigations of the stress behavior and perpendicular-to-grain mechanical properties of drying wood have laid the groundwork for a more fundamental approach to the problem of improved wood drying. However, effective use of such data requires a method of evaluating drying stresses at any point on the cross section of a drying board. Such a method has not been available up to this time.

This report describes a method for calculating the perpendicular-to-grain stresses associated with the drying process and illustrates the application of the method to one condition of wood drying.

Method

The "slicing" technique of drying strain analysis, as employed by McMillen^{2,3} in his investigations of the development of drying stresses in red oak, consists of cutting a cross section 1 inch along the grain from a drying board, slicing the section into several (usually 10) slices parallel to the broad faces of the board as shown in figure 1, and measuring the change in length of each slice as it is cut free. An increase in length indicates release from a compressive stress; a decrease in length indicates release from a tensile stress.

The stresses developed in wood as it dries are not, however, pure tension or pure compression in any one slice. Neither are stresses uniform along the

²McMillen, J. M. Drying stresses in red oak. Forest Products Journal 5:1:71-76. 1955.

³McMillen, J. M. Drying stresses in red oak. Effect of temperature. Forest Products Journal 5:4:230-241. 1955.

length of any slice. Rather, stresses are developed in gradient form and vary continuously from a surface of the drying board to the center of the board. The strain indicated in the recovery of any slice, therefore, represents a response to the integral of all stresses existing in the slice just before it is cut free. Any method for accurate estimation of stresses acting perpendicular to the grain in a drying board should then provide for calculation of stresses at any point on the cross section and should not be confined to determination of the average stress indicated by the recovery of a slice. This is especially important in determining stresses in a board early in drying, when tensile stress at the surface may be much greater than that indicated by the outside slice as a whole because of a very steep stress gradient in the outer portion of the wood.

The method now developed makes use of data on strain recovery by the "slicing" technique, such as those reported by McMillen,^{2, 3} and data on perpendicular-to-grain elastic properties of matched wood specimens, such as those presented by Youngs.⁴ Since the instantaneous recovery of such slices as they are cut free is an elastic phenomenon, knowledge of elastic properties under comparable conditions of temperature, moisture content, and growth ring orientation can be used to interpret strain data in terms of stress.

If it is assumed that perpendicular-to-grain stresses developed in drying are very great with respect to parallel-to-grain stresses, the analysis of drying stresses can be treated as a two-dimensional problem. This assumption is justified in dealing with normal straight-grained wood, since the longitudinal shrinkage of such material can be considered negligible as compared with shrinkage in either the tangential or radial direction. Thus, with reference to the axes shown in figure 1, analysis can be carried out in terms of the normal stress in the x direction (across the width of the board), the normal stress in the y direction (across the thickness of the board), and the shear stress associated with these directions.

Details of the development of the method are presented in the appendix. A brief description of the steps involved is presented in this section.

Two equilibrium equations must be satisfied at all points on the cross section. Each equilibrium equation equates to zero the sum of the forces acting at a point in one of the two principal directions. Since body forces may be neglected in this analysis, the only forces involved are those due to the normal stress in the direction under consideration and the associated shear stress.

⁴Youngs, R. L. The perpendicular-to-grain mechanical properties of red oak as related to temperature, moisture content, and time. U. S. Forest Products Laboratory Report No. 2079. 1957.

The surfaces of a drying board are free from tractions, which establishes boundary conditions of zero stress in the \underline{x} direction at the edges of the board, zero stress in the \underline{y} direction at the broad faces of the board, and zero shear stress at all points on the surface and along the central planes parallel to the surfaces through the midpoint of the cross section.

The internal stresses in the drying board can then be expressed in terms of a stress function, $\underline{\phi}$, which can be introduced as a second order differential in expressions for normal and shear stresses that satisfy the equilibrium equations and the boundary conditions. A suitable stress function is written in the appendix (equation 4) in terms of the ratio of \underline{x} to \underline{a} and the ratio of \underline{y} to \underline{b} , where \underline{a} represents the distance from the center of the cross section to an edge and \underline{b} the distance to the broad face, as shown in figure 1. The stress function is expanded as shown in equation (5) to give an array involving several constants, \underline{A}_{mn} , where \underline{m} and \underline{n} are even positive integers because of the assumed symmetry of stresses around the center of the cross section.

The strain recovery measured in the slicing technique represents instantaneously recoverable, or elastic, strain. Therefore, the recovered strains can be related to the corresponding stresses by means of equations (6) involving moduli of elasticity in the two principal directions, the corresponding Poisson's ratios, and the associated modulus of rigidity. Such relationships apply if the growth rings in the drying board are oriented essentially in one of the two principal directions; that is, if the board is either flatsawn or quartersawn and the growth rings have little curvature.

By applying Maxwell's relation between Poisson's ratios and moduli of elasticity along the orthotropic \underline{x} and \underline{y} axes, and by introducing the stress functions corresponding to the principal stresses, it is possible to derive expressions for the strains in the principal directions in terms of the modulus of elasticity in the \underline{x} direction and ratios of other moduli. The data of Youngs⁴ indicate that the ratios of moduli are substantially independent of temperature and moisture content, so the ratios could be assumed to be constant throughout the volume of the drying board.

The expression for the strain in the \underline{x} direction, across the width of the board, can then be integrated across the width to give an expression relating the observed recovery of any slice to the position of the centerline of the slice in the cross section of the board (\underline{y}). Substitution of suitable values for \underline{y} yields as many equations as there were slices taken. Since it is assumed, however, that the stresses are symmetrically distributed about the midpoint of the cross section, in practice the recovery of each pair of slices equidistant from the midthickness of the board is averaged and applied to one equation to give half as many equations as there have been slices cut.

These equations are linear and numerical, except for values of A_{mn} . They may be solved simultaneously to yield some values of A_{mn} . The remaining values of A_{mn} may be determined by an energy method applying the principle of least work (see appendix). Once these remaining values of A_{mn} have been calculated, it is possible to determine completely the stress function ϕ . The calculation of normal stresses σ_x and σ_y and the associated shearing stress σ_{xy} at any point on the cross section is then simply a matter of substituting the coordinates x and y of the point in the proper stress equation.

Application

The technique just described was applied to determine drying stresses perpendicular to the grain in 2- by 7-inch flat-sawed plank of northern red oak (*Quercus rubra* L.) while drying from the green condition. The section chosen for analysis was cut after 4 days of drying at 80° F. and 85 percent relative humidity. It was selected for the initial test of this method for several reasons. First, it represented a period early in drying, when the outer surface zone of the wood is under severe tensile stress. Second, the drying temperature was only slightly higher than the room temperature at which the slices were cut, thus practically eliminating errors due to temperature change during the slicing operations. Third, the portion of the plank at the surface having a moisture content below the fiber-saturation point was so small that no appreciable error would result from assuming a constant value of modulus of elasticity in any one direction throughout the piece, which considerably simplifies the calculations.

It was decided to use in this initial trial the first five terms of a power series such that the stress function would be expressed as

$$\phi = [1 - (x/a)^2]^2 [1 - (y/b)^2]^2 [A_{00} + A_{20} (x/a)^2 + A_{02} (y/b)^2 + A_{40} (x/a)^4 + A_{04} (y/b)^4]$$

in which x is the distance from center of plank toward the narrow edge, in inches; y is the distance from center of plank toward the broad face, in inches; a is one-half the width of the plank (3.5 inches); b is one-half the thickness of the plank (1.0 inch); and A_{mn} represents constants with m and n being even integers because of stress symmetry.

It was felt that this series would adequately fit the data without excessive calculation. Also, this series could be used conveniently with a constant value of modulus of elasticity, as assumed for the first trial.

An expression for σ_x , the stress in the x direction, was obtained as the second derivative of the above stress function with respect to y . That expression was then integrated with respect to x and multiplied by $1/E_x$ to yield a general expression involving the actual recovery of slices when cut free. A constant value of 64,000 pounds per square inch was assumed for E_x , based on the data of Youngs.⁴ The following values of y and corresponding values of slice recovery were then substituted to give five simultaneous equations with five unknowns:

<u>Slice</u>	<u>y</u>	<u>Recovery</u> <u>Inch</u>
1, 10	0.9	0.0195
2, 9	.7	-0.0045
3, 8	.5	-0.0055
4, 7	.3	-0.0055
5, 6	.1	-0.0040

In the above tabulation, a positive value indicates recovery from tension and a negative value indicates recovery from compression. The five equations were solved simultaneously to give:

$$A_{02} = 0.3147$$

$$A_{04} = 0.1060$$

$$A_{00} = 0.2639 - 1.7500 A_{20} - 7.1461 A_{40}$$

The constants A_{00} , A_{20} , and A_{40} could not be isolated from the simultaneous equations directly. Therefore, an energy method involving the principle of least work was used to determine those constants from the above relationship between A_{00} , A_{20} , and A_{40} . The constants thus calculated were:

$$A_{00} = 0.1677$$

$$A_{20} = 0.0223$$

$$A_{40} = 0.0080$$

Once these constants had been evaluated, it was simply a matter of substituting them in the expressions for the stresses σ_x , σ_y , and σ_{xy} . The values of these stresses at several points on the cross section were then determined by solving the appropriate equation with the values of x and y corresponding to the point under consideration.

Presentation of Data

Figures 2 through 6 present curves plotted from calculated values of stress and show the potentialities of the present method of calculation for indicating stress distribution and magnitude at any desired stage of drying. Figures 2 and 3 show the distribution of stress in the x direction (σ_x) along the y and x axes, respectively and are two plots of the same data. Figure 4 shows the distribution of stress in the y direction (σ_y) along the x axis. Figure 5 shows the distribution of shear stress associated with shear strain in the xy plane (σ_{xy}) along the x axis. Figure 6 shows the distribution of each of the three types of stress in one quadrant of the cross section of the drying board by means of lines connecting points of equal stress.

Discussion

The information derived by the technique presented in this report gives a much clearer insight into the stress behavior of drying wood than has heretofore been possible. The stress values themselves are limited in value, since they apply to only one stage of drying of one sample under one set of drying conditions. They do, however, illustrate what can be done by application of the new technique and should be considered as representative of the possibilities offered by the technique rather than as a detailed stress analysis.

The stresses that have been of primary interest in connection with drying defects in boards and planks are those directed across the width of the board -- that is, in the x direction in this analysis. Early in drying, the defect most likely to result from such stresses is surface checking. Figures 2 and 3 indicate that the highest tensile stress at the surface of the board was about 710 pounds per square inch. Assuming that the moisture content of the surface was approximately in equilibrium with the drying atmosphere, or 18 percent, that portion of the wood would be expected to have a static tensile strength perpendicular to the grain of about 850 pounds per square inch.⁴ Apparently the stress attained at the surface was nearly sufficient to produce surface checking,

especially when it is considered that the stress may be effective for a period of time great enough to produce appreciable weakening because of duration-of-loading effects.

Figures 2 and 3 also indicate that the highest compressive stress attained was about 135 pounds per square inch, which is only 20 percent of the tensile strength of green wood at 80° F.⁴ Such a stress would not be expected to result in any appreciable amount of nonrecoverable creep. McMillen's data³ for the drying run in question agree with this observation by showing development of very little compression set.

Figure 4 indicates that the maximum tensile stress in the thickness (y) direction is about 160 pounds per square inch, which is slightly less than a quarter of that in the width (x) direction. The ratio of maximum stresses in the two directions is less than would be expected on the basis of the 2:7 ratio of dimensions, which is probably due largely to the fact that shrinkage in the y direction is essentially radial and therefore less, percentagewise, than the tangential shrinkage in the x direction. The maximum compressive stress in the y direction is also about a quarter of that in the x direction. Neither the tensile stress nor the compressive stress indicated in figure 4 would be expected to lead to failure or appreciable amounts of nonrecovered creep in the thickness direction of the plank.

The role of shear stresses in the development of drying defects has been the subject of some concern, and values of shear strength associated with shear strain in the xy plane of oak have been reported by Kollmann⁵ and by Youngs.⁴ However, shear strength values are of no great value until more is known about the location and magnitude of shear stresses developed in drying wood.

Figure 5 illustrates the type of information on shear stress distribution that is obtainable by the present technique. The maximum shear stress indicated in figure 5 is about 45 pounds per square inch, which is only 12 percent of the shear strength value reported by Youngs⁴ for the temperature and moisture content conditions associated with the location of maximum shear stress in the specimen under consideration. Apparently the likelihood of shear failure is almost negligible for the stage of drying that is represented in the present analysis. However, this analysis does not take into account localized shear stress concentrations due to marked differences in shrinkage tendency between adjacent wood elements, such as wood ray cells and longitudinal fibers. Such

⁵Kollmann, F. Untersuchungen über die Ursachen von Schäden bei der Trocknung von grünem Eichenholz zugleich Mitteilung über Festigkeitsprüfungen an Holz in hochfrequenten Wechselfeld. Swedish Wood Research Institute, Wood Technical Department Bulletin No. 21. 1950.

localized shear stress concentrations could conceivably be greater than indicated in figure 5 for the cross section as a whole.

Figure 6 illustrates how the results of the present analytical technique can be plotted to show distribution of normal stresses in the x and y directions and associated shear stresses at the stage of drying under investigation. Each portion of the figure represents the upper righthand quadrant of the cross section with the assumption that the stresses are distributed symmetrically over the entire cross section of the board. The curves are based on the same data as figures 2, 3, 4, and 5, but the plotting in figure 6 emphasizes the distribution, rather than the magnitude, of the stresses.

The number of stress values calculated in this first trial of the technique is not sufficient to locate accurately all of the curves shown in figure 6. Consequently, the lines are based on the calculated points, but their exact locations are the result of some personal judgment coupled with a consideration of the assumed boundary conditions. It is felt that the lines represent with reasonable accuracy the stress conditions characteristic of the specimen investigated.

Figure 6, A points out the steep tensile stress gradients in the outer slices at the time of cutting, from 0 at the inner boundary of the slice ($y = 0.8$) to about 700 pounds per square inch at the surface, and from about 350 pounds per square inch at midthickness of the slice ($y = 0.9$) in the center of the section to 0 at the end of the slice. Figure 6, A also indicates quite clearly that the maximum compressive stress is not at the center of the cross section, but near the midlength at the boundary between slices 2, 9, and 3, 8 ($y = 0.6$).

Figure 6, B indicates that the normal stresses in the y direction are concentrated near the ends of the center slices and are negligible in the outer slices.

Figure 6, C locates the point of maximum shear stress at about 1/2 inch from the ends of slices 2, 9 and 3, 8 ($y = 0.6$). This figure also points out that shear stresses are quite small over most of the cross section at the stage of drying under investigation.

As mentioned previously, the principal object in presenting figures 2 through 6 is to point out the possibilities offered by the analytical procedure outlined in this report. The advantages of the new technique are readily apparent when the curves are compared with the points in figure 2 that were derived by the previous method of merely multiplying the average recovery strain of a pair of slices by the applicable value of modulus of elasticity. The points thus obtained fit into the array of curves, but make possible only a rough estimate of stresses in the x direction.

The next logical step in the application of the technique presented here would be to carry through some rather comprehensive analyses of drying stresses, using data now available on recovery strain and perpendicular-to-grain mechanical properties of red oak. In part of these analyses parallel calculations could be carried out, using different numbers of terms in the series to indicate the rate at which the series converges. In most of the subsequent analyses it would be necessary to take into account variation in modulus of elasticity as a function of moisture content and temperature gradients in the cross section, since the assumption of constant modulus of elasticity made in the initial application would become untenable when an appreciable portion of the wood has dried below the fiber-saturation point. A method of introducing modulus of elasticity as a variable is indicated in equation (9) of the appendix.

The present procedure presents imposing problems in carrying through the mathematics involved. However, the advent of the electronic computer has made such mathematical problems considerably less formidable than they were a few years ago. It seems likely that future work along this line will be dependent on the use of such a computer to perform the required mathematical operations.

It must be realized, of course, that any such analytical technique, however elegant, is no better than the basic data supplied for analysis. Therefore, attention must also be given to the techniques used for gathering data on strain recovery of slices and on the associated perpendicular-to-grain mechanical properties. If the results of such tests do not reflect accurately the properties and behavior of the drying wood, any subsequent analytical procedure will be limited in its usefulness and may even lead to erroneous conclusions.

APPENDIX

During the drying of a kiln load of lumber, a plank is removed from the kiln. A cross section about 1 inch long is cut from this plank and this section is sawn into slices as shown in figure 1. Before the cross section is sliced, the length of each slice to be cut is measured in the x direction. After the slices are cut, these lengths are measured again. The change in length is an indication of the internal stresses in the cross section before it was sliced. The following method is proposed for the complete determination of the stress distribution in the cross section from the measurements taken.

The equations of equilibrium apply to the internal stresses desired, thus:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Where σ_x and σ_y are the normal stresses in the x and y directions and σ_{xy} is the shear stress associated with these directions. Thus the stresses can be expressed in terms of a stress function ϕ as

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \sigma_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} \tag{2}$$

The surfaces of the slice are free from tractions so that the boundary conditions of these stresses are:

$$\sigma_x = 0 \text{ when } x = \pm a$$
$$\sigma_y = 0 \text{ when } y = \pm b \tag{3}$$

$$\sigma_{xy} = 0 \text{ when } x = \pm a \text{ and when } y = \pm b$$

A suitable stress function is:

$$\phi = [1 - (x/a)^2]^2 [1 - (y/b)^2]^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} (x/a)^m (y/b)^n \tag{4}$$

where m and n are even positive integers because of symmetry. The stresses determined from this function by means of equations (2) will exhibit the boundary conditions required by equations (3). The double summation is given by the array:

$$\begin{array}{r}
 A_{00} + A_{20} (x/a)^2 + A_{40} (x/a)^4 + A_{60} (x/a)^6 + \dots \\
 A_{02} (y/b)^2 + A_{22} (x/a)^2 (y/b)^2 + A_{42} (x/a)^4 (y/b)^2 + A_{62} (x/a)^6 (y/b)^2 + \dots \\
 A_{04} (y/b)^4 + A_{24} (x/a)^2 (y/b)^4 + A_{44} (x/a)^4 (y/b)^4 + A_{64} (x/a)^6 (y/b)^4 + \dots \\
 A_{06} (y/b)^6 + A_{26} (x/a)^2 (y/b)^6 + A_{46} (x/a)^4 (y/b)^6 + A_{66} (x/a)^6 (y/b)^6 + \dots \\
 \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\
 \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots
 \end{array} \tag{5}$$

The double summation represents the sum of all of these terms.

The strains that are immediately recoverable, and therefore responsible for the changes in width of the slices of the cross section, are elastic and are obtained from these stresses by the equations

$$\begin{aligned}
 e_x &= \frac{\sigma_x}{E_x} - \frac{\mu_{yx}}{E_y} \sigma_y \\
 e_y &= \frac{\sigma_y}{E_y} - \frac{\mu_{xy}}{E_x} \sigma_x \\
 e_{xy} &= \frac{\sigma_{xy}}{E_{xy}}
 \end{aligned} \tag{6}$$

where e_x and e_y are the normal strains in the x and y directions and e_{xy} is the shear strain associated with these directions; E_x and E_y are the moduli of elasticity in the x and y directions and E_{xy} the modulus of rigidity associated

with these directions; μ_{yx} and μ_{xy} are Poisson's ratios of contraction in the direction of the second subscript to expansion in that of the first subscript due to tension in the direction of the first subscript. These equations apply only if the plank is either flat sawn or quartersawn and the annual rings have little curvature.

Using Maxwell's relation:

$$\frac{\mu_{yx}}{E_y} = \frac{\mu_{xy}}{E_x}$$

equation (6) may be written

$$\begin{aligned} e_x &= \frac{1}{E_x} [\sigma_x - \mu_{xy} \sigma_y] \\ e_y &= \frac{1}{E_x} \left[\frac{E_x}{E_y} \sigma_y - \mu_{xy} \sigma_x \right] \\ e_{xy} &= \frac{1}{E_x} \frac{E_x}{E_{xy}} \sigma_{xy} \end{aligned} \tag{7}$$

Using equation (2), equations (7) may be written

$$\begin{aligned} e_x &= \frac{1}{E_x} \left[\frac{\partial^2 \phi}{\partial y^2} - \mu_{xy} \frac{\partial^2 \phi}{\partial x^2} \right] \\ e_y &= \frac{1}{E_x} \left[\frac{E_y}{E_y} \frac{\partial^2 \phi}{\partial x^2} - \mu_{xy} \frac{\partial^2 \phi}{\partial y^2} \right] \\ e_{xy} &= - \frac{1}{E_x} \frac{E_y}{E_{xy}} \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \tag{8}$$

The ratios of the elastic properties are substantially independent of the moisture content, so that they are constant throughout the volume of the sample taken from the dry kiln. The value of $1/E_x$ changes from point to point throughout the volume of the sample. Its value can be expressed by

$$\frac{1}{E_x} = \frac{1}{E_{x0}} + \left[\frac{1}{E_{xc}} - \frac{1}{E_{x0}} \right] \left[1 - (x/a)^p \right] \left[1 - (y/b)^q \right] \quad (9)$$

where E_{x0} is determined by the use of the equilibrium moisture content on the outside of the sample; E_{xc} is determined by the moisture content at the center of the sample; and p and q are even integers that allow the adjustment of equation (9) to approximate the variation of moisture content throughout the sample.

The change in length of each slice of the sample is given by the integral of the first of equations (8) as:

$$F_y = \int_{-a}^{+a} \frac{1}{E_x} \left[\frac{\partial^2 \phi}{\partial y^2} - \mu_{xy} \frac{\partial^2 \phi}{\partial x^2} \right] dx \quad (10)$$

This expression relates the change in length of each slice to the position (y) of the centerline of the slice in the cross section. The substitution of these positions for y in this expression yields a number of equations equal to the number of slices taken. (Because of the symmetry of the problem, the average changes in length of pairs of slices are taken and only half this number of equations are obtained.) The value of F_y for each equation is the measured value of the change in length of each slice. Thus, a number of linear equations relating the values of A_{mn} to each other for a particular sample are obtained.

These equations are numerical except for the values of A_{mn} because the symbol x disappears in the integration of equation (10), numerical values are substituted for the symbol y , and experimental values are substituted for the symbol F_y . If n pairs of slices of the specimen are cut, n equations are obtained and n values of A_{mn} may be expressed in terms of the remaining values. The remaining values of A_{mn} are determined by an energy method.

The immediately recoverable stored energy in the specimen is given by

$$V = \frac{1}{2} \int_{-a}^{+a} \int_{-b}^{+b} (\sigma_x e_x + \sigma_y e_y + \sigma_{xy} e_{xy}) dy dx \quad (11)$$

Using equations (2), (7), and (9), this energy expression becomes:

$$V = \frac{1}{2} \int_{-a}^{+a} \int_{-b}^{+b} \left\{ \frac{1}{E_{x_0}} + \left[\frac{1}{E_{x_c}} - \frac{1}{E_{x_0}} \right] \left[1 - \left(\frac{x}{a} \right)^p \right] \left[1 - \left(\frac{y}{b} \right)^q \right] \right\} \left(\frac{\partial^2 \phi}{\partial y^2} \right)^2 + \frac{E_x}{E_y} \left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 - 2\mu_{xy} \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{E_x}{E_y} \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \Bigg\} dx dy \quad (12)$$

When this expression for V is integrated and the known values of the various constants substituted for their symbols, it becomes a quadratic expression in the various values of A_{mn} . Certain values of A_{mn} are eliminated from this expression by using the n equations obtained from equation (10). The stresses will arrange themselves in the sample so that the energy, V , will be a minimum. Thus the remaining values of A_{mn} may be obtained by minimizing V with respect to them. Thus

$$\frac{\partial V}{\partial A_{mn}} = 0$$

yields as many equations as the number of remaining values of A_{mn} . These equations are linear and are solved simultaneously. The values of A_{mn} obtained from equation (10) are given in terms of these remaining values of A_{mn} and therefore may now be determined.

The values of A_{mn} are substituted in equation (4) to determine completely the stress function ϕ . Equations (2) then yield values of the stresses throughout the volume of the sample.

Pickett⁶ has integrated equation (12). He obtains:

⁶Professor of mechanics, University of Wisconsin, Madison, Wis.

$$\begin{aligned}
V = & \frac{1}{2ab E_{x0}} \sum_{m,n,r,s} A_{mn} A_{rs} \left[T_{mr} R_{ns} \frac{a^2}{b^2} + \frac{E_x}{E_y} R_{mr} T_{ns} \frac{b^2}{a^2} \right. \\
& + \left. \left(\frac{E_x}{E_{xy}} - 2\mu_{xy} \right) S_{mr} S_{ns} \right] + \frac{1}{2ab} \left[\frac{1}{E_{xc}} - \frac{1}{E_{x0}} \right] \sum_{m,n,r,s} A_{mn} A_{rs} \left[(T_{mr} \right. \\
& - T_{mrp}) (R_{ns} - R_{nsq}) \frac{a^2}{b^2} + \frac{E_x}{E_y} (R_{mr} - R_{mrp}) (T_{ns} - T_{nsq}) \frac{b^2}{a^2} \\
& \left. - 2\mu_{xy} (S_{mr} - S_{mrp})(S_{ns} - S_{nsq}) + \frac{E_x}{E_{xy}} (S_{mr} - S'_{mrp})(S_{ns} - S'_{nsq}) \right]
\end{aligned}$$

where

$$T_{ijk} = \frac{768}{(\delta + 1)(\delta + 3)(\delta + 5)(\delta + 7)(\delta + 9)}$$

$$\delta = i + j + k$$

$$S_{ijk} = 128 \frac{i^2 + j^2 - 4ij - 3(i + j) + 2 + k^2 - (i + j + 3)k}{(\delta - 1)(\delta + 1)(\delta + 3)(\delta + 5)(\delta + 7)}$$

$$S'_{ijk} = 128 \frac{i^2 + j^2 - 4ij - 3(i + j) + 2 - 2k^2 - (i + j)k}{(\delta - 1)(\delta + 1)(\delta + 3)(\delta + 5)(\delta + 7)}$$

$$R_{ijk} = 384 \frac{2i^2 j^2 + 2ij(i + j) - 8ij + i^2 + j^2 - 4(i + j) + 3}{(\delta - 3)(\delta - 1)(\delta + 1)(\delta + 3)(\delta + 5)} +$$

$$128 \frac{k^4 + (2i + 2j - 1)k^3 + [(i + j)^2 + 6ij + 3(i + j - 2)]k^2 + [5(i + j)^2 - 6ij - 11(i + j) - 6]k}{(\delta - 3)(\delta - 1)(\delta + 1)(\delta + 3)(\delta + 5)}$$

$$T_{ij} = T_{ijk} \Big|_{k=0} \quad S_{ij} = S_{ijk} \Big|_{k=0} = S'_{ijk} \Big|_{k=0} \quad R_{ij} = R_{ijk} \Big|_{k=0}$$

The integral of equation (10), using equation (9), is:

$$\begin{aligned}
 F_y = & \sum_m \sum_n A_{mn} \left\{ \frac{16a}{b^2} \left[n(n-1)(y/b)^{n-2} - 2(n+2)(n+1)(y/b)^n \right. \right. \\
 & \left. \left. + (n+4)(n+3)(y/b)^{n+2} \right] \left[\frac{1}{E_{xc}} \frac{1}{(m+1)(m+3)(m+5)} \right. \right. \\
 & \left. \left. - \left(\frac{1}{E_{xc}} - \frac{1}{E_{xo}} \right) \frac{1}{(m+p+1)(m+p+3)(m+p+5)} \right] \right. \\
 & \left. + \frac{2\mu_{xy}}{a} \left[\frac{1}{E_{xc}} - \frac{1}{E_{xo}} \right] \left[1 - (y/b)^q \right] \left[1 - (y/b)^2 \right]^2 (y/b)^n \left[\frac{m(m-1)}{m+p-1} \right. \right. \\
 & \left. \left. - \frac{2(m+2)(m+1)}{m+p+1} + \frac{(m+4)(m+3)}{m+p+3} \right] \right\}
 \end{aligned}$$

The stresses are given by equations (2) as:

$$\begin{aligned}
 \sigma_x = \frac{\partial^2 \phi}{\partial y^2} = & \frac{1}{b^2} \sum_m \sum_n A_{mn} (x/a)^m \left[1 - (x/a)^2 \right]^2 \left[n(n-1)(y/b)^{n-2} \right. \\
 & \left. - 2(n+2)(n+1)(y/b)^n + (n+4)(n+3)(y/b)^{n+2} \right] \\
 \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = & \frac{1}{a^2} \sum_m \sum_n A_{mn} (y/b)^n \left[1 - (y/b)^2 \right]^2 \left[m(m-1)(x/a)^{m-2} \right. \\
 & \left. - 2(m+2)(m+1)(x/a)^m + (m+4)(m+3)(x/a)^{m+2} \right]
 \end{aligned}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{1}{ab} \sum_m \sum_n A_{mn} \left[m(x/a)^{m-1} - 2(m+2)(x/a)^{m+1} \right. \\ \left. + (m+4)(x/a)^{m+3} \right] \left[n(y/b)^{n-1} - 2(n+2)(y/b)^{n+1} \right. \\ \left. + (n+4)(y/b)^{n+3} \right]$$

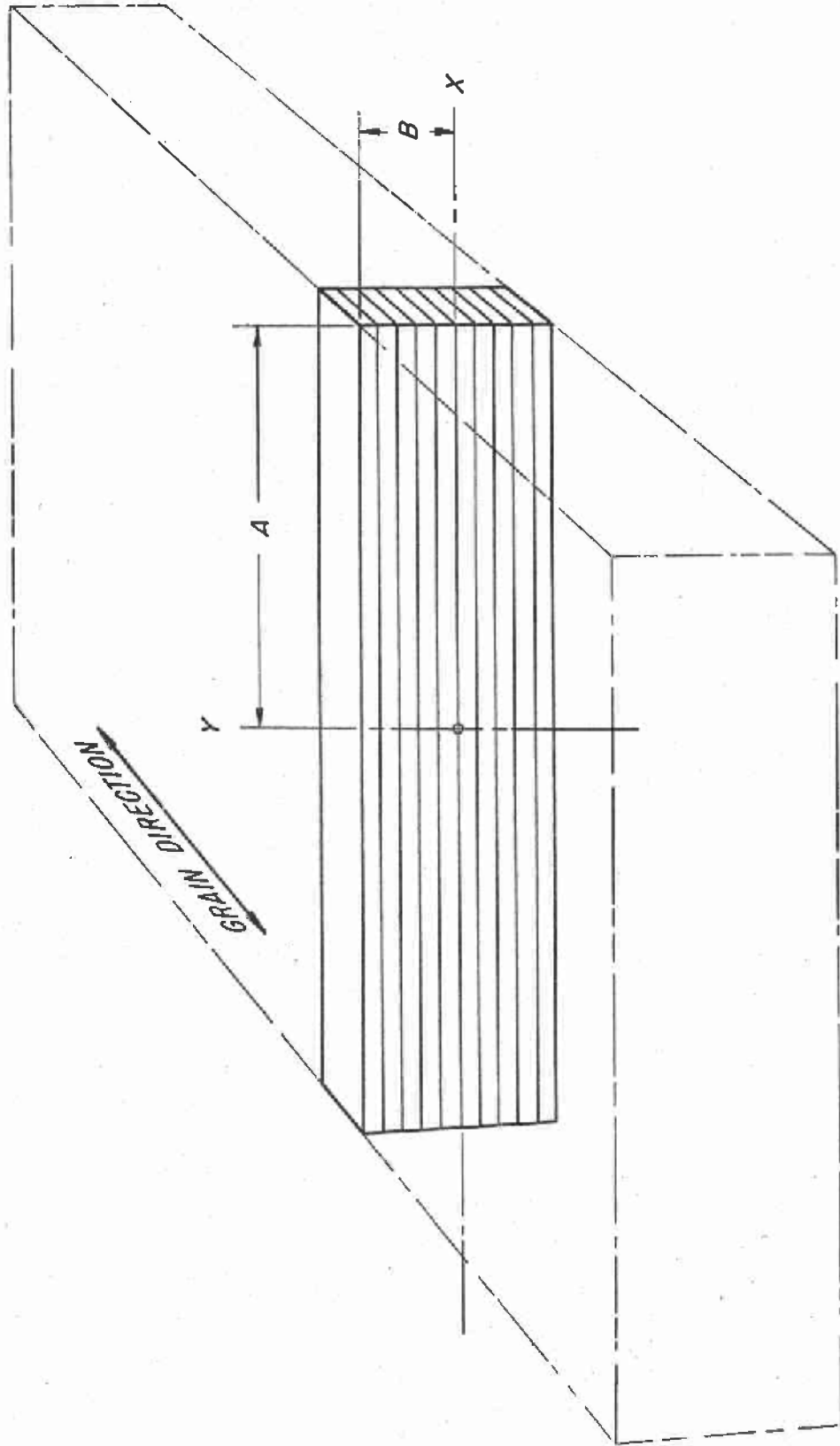


Figure 1. --Diagrammatic illustration of location of strain analysis section and slices cut for recovery measurement with respect to drying board and coordinate axes.

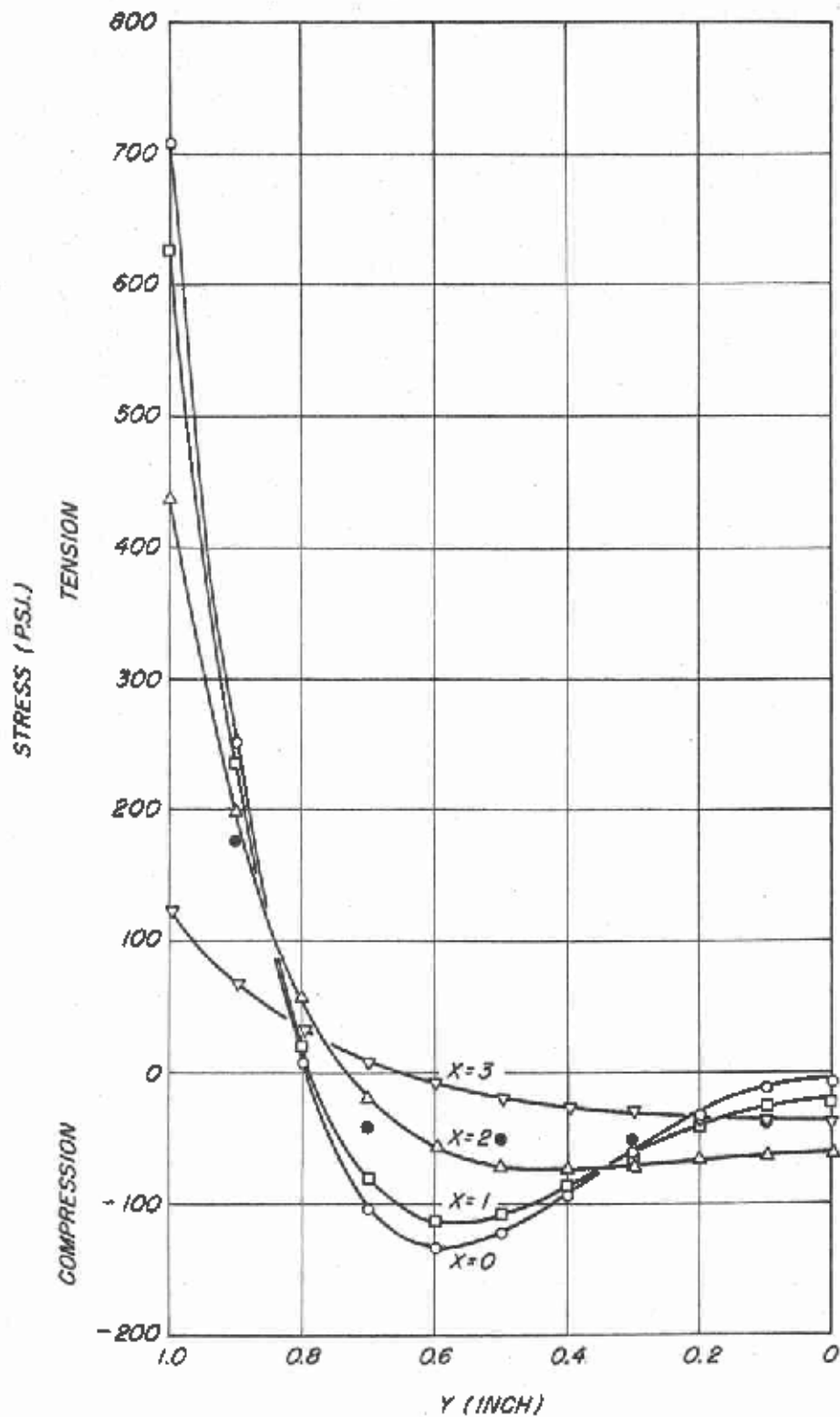


Figure 2. --Relationship of normal stress in the x direction to distance from the center in the y direction. Values indicated by open symbols were calculated for 4 points on the x axis from the derived formulas and comparative values indicated by closed symbol were calculated by multiplying recovery strain by modulus of elasticity.

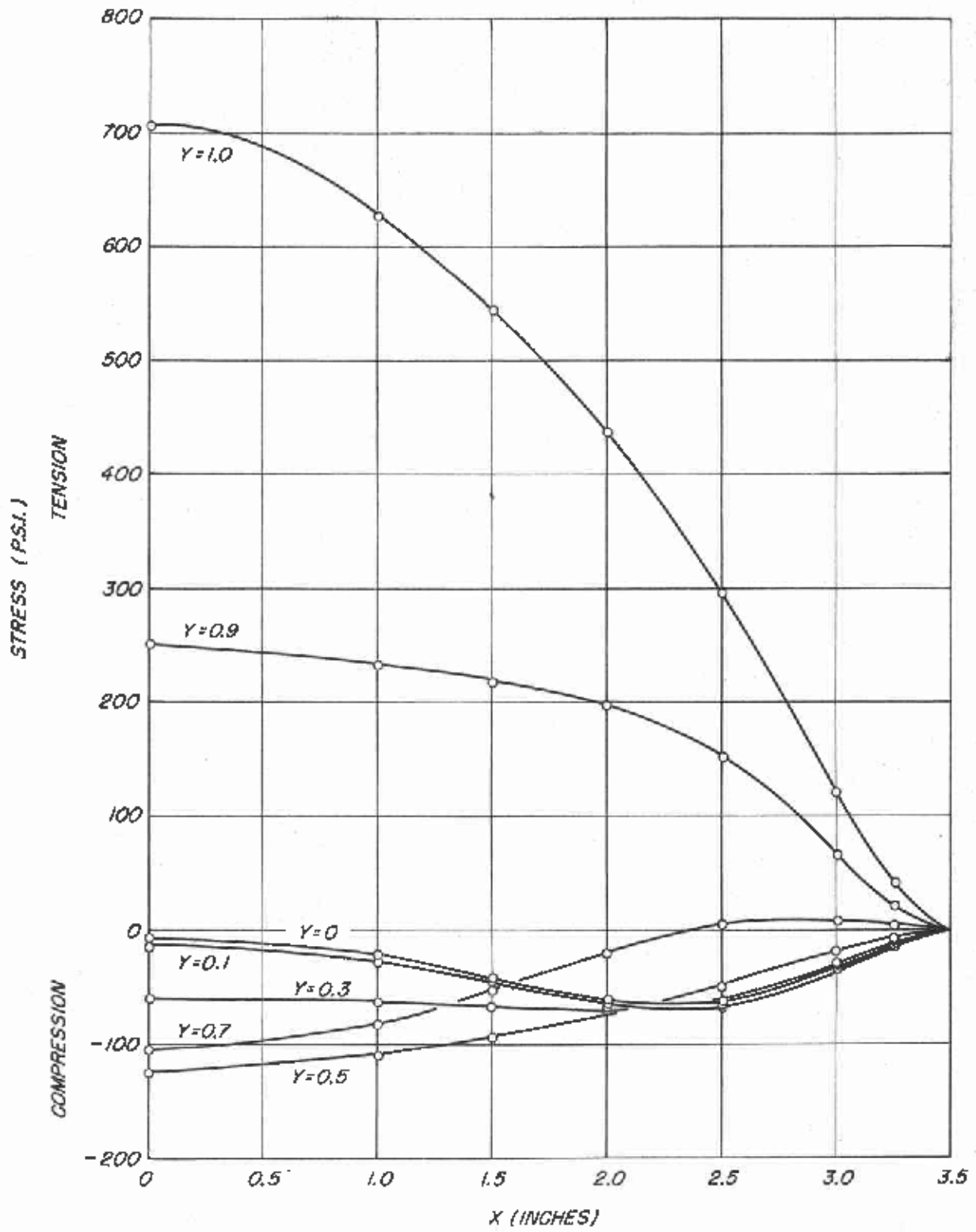


Figure 3. -- Relationship of normal stress in the x direction to distance from the center in the x direction at several points along the y axis. Indicated points were calculated from derived formulas.

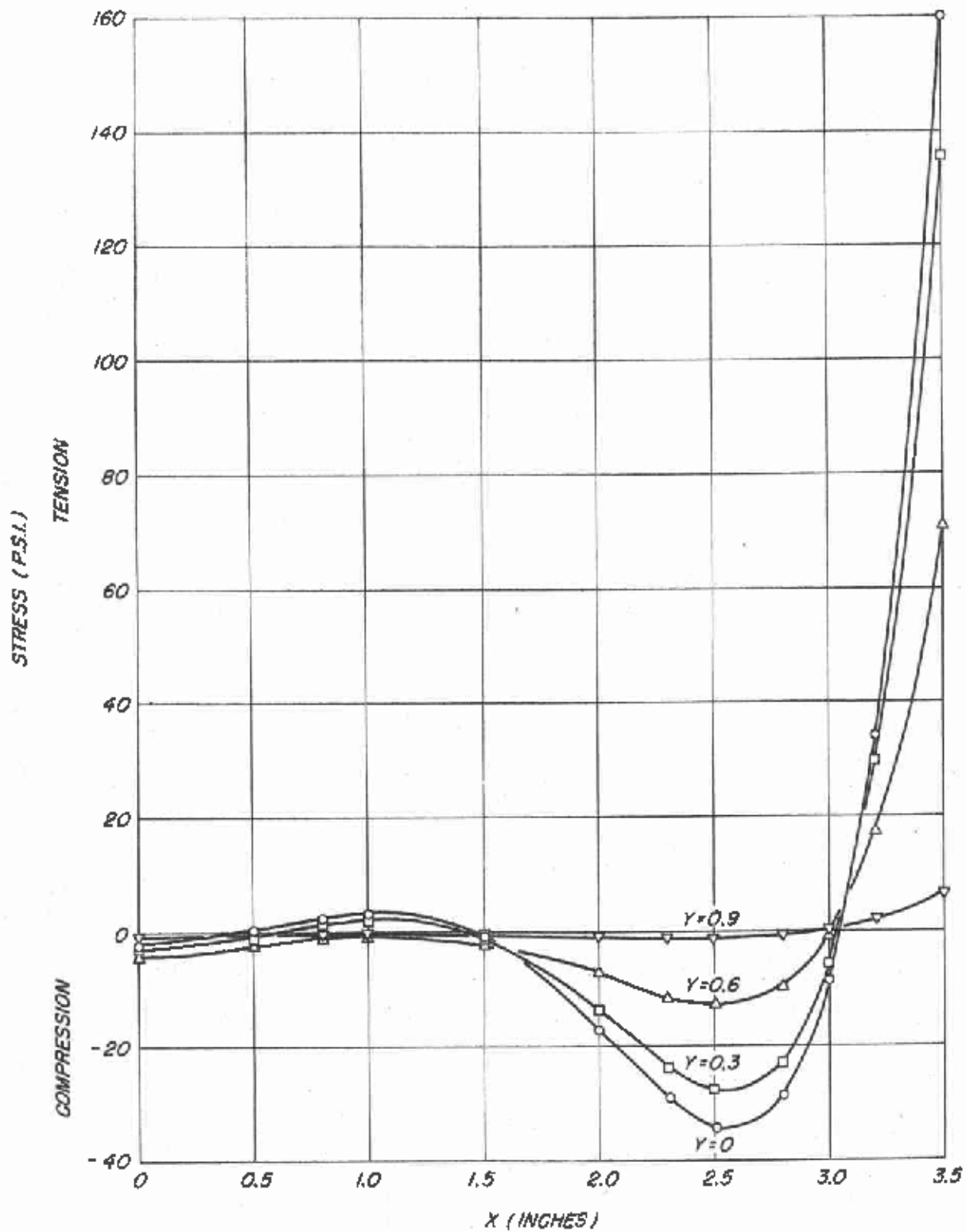


Figure 4. -- Relationship of normal stress in the y direction to distance from the center in the x direction. Indicated points were calculated from derived formulas.

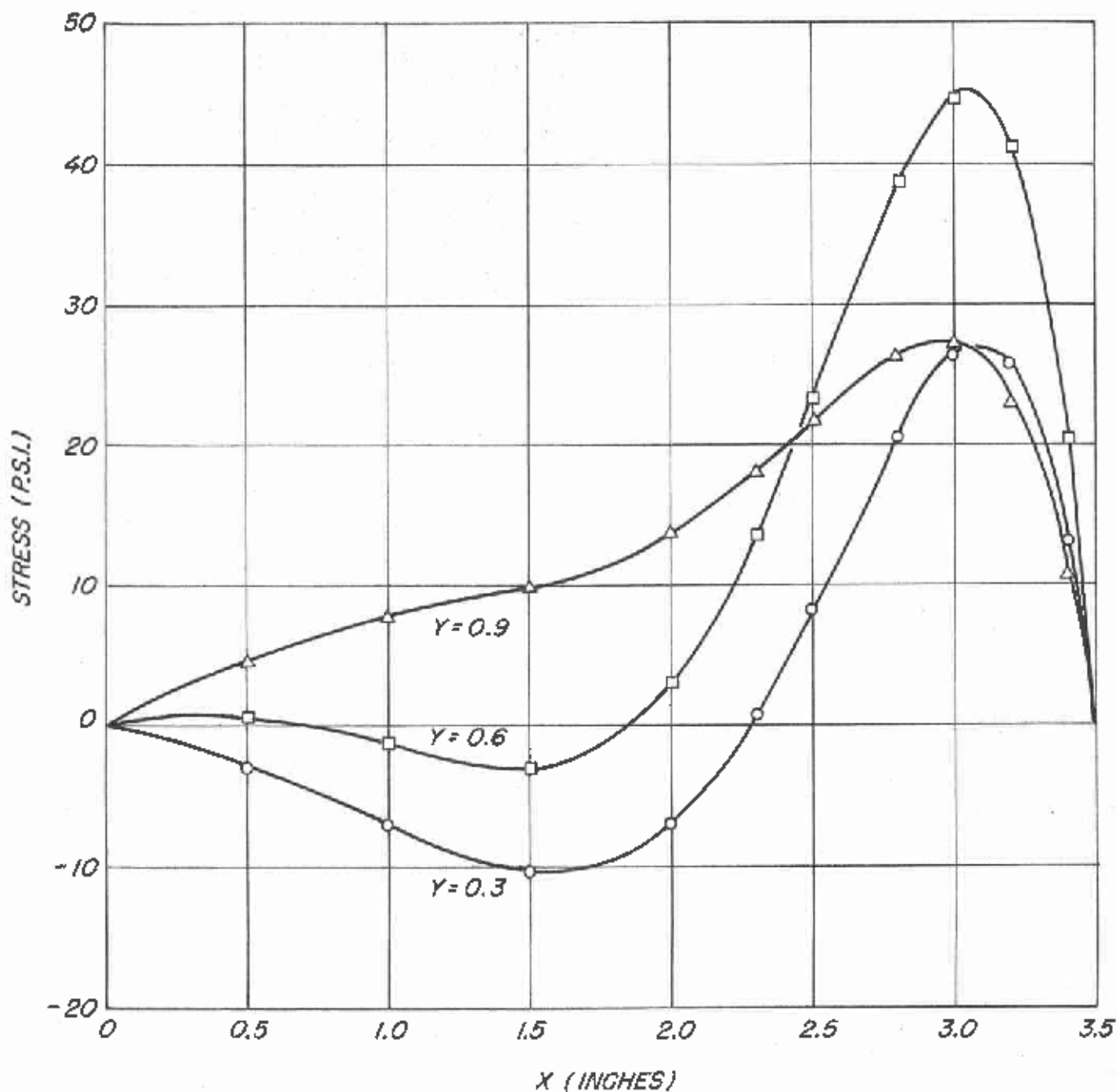


Figure 5. --Relationship of shear stress associated with shear strain in the xy plane to distance from the center in the x direction at three levels of y . Indicated points were calculated from derived formulas.

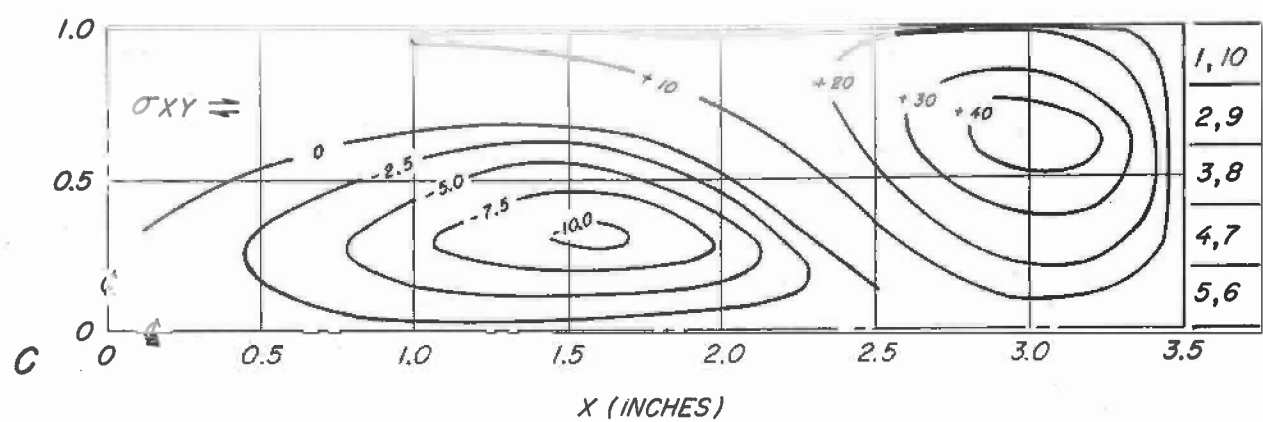
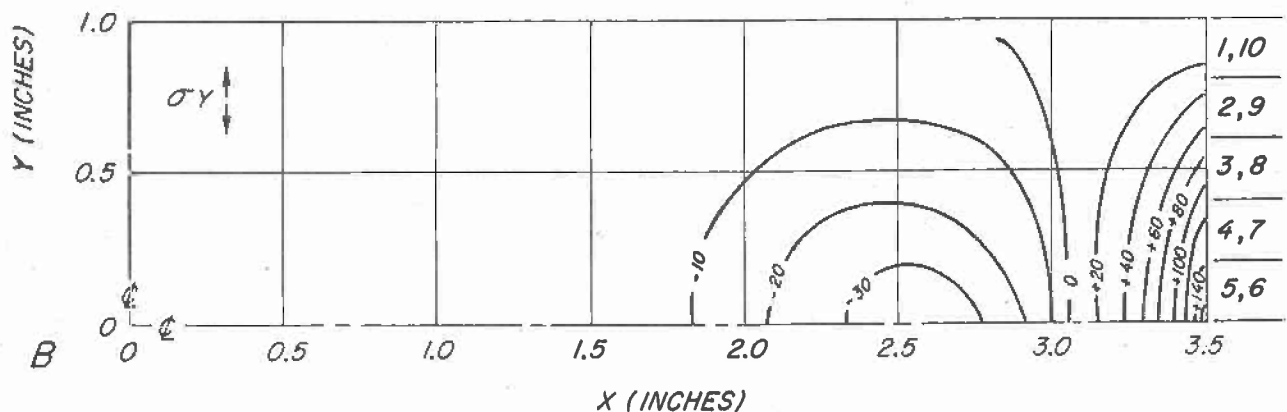
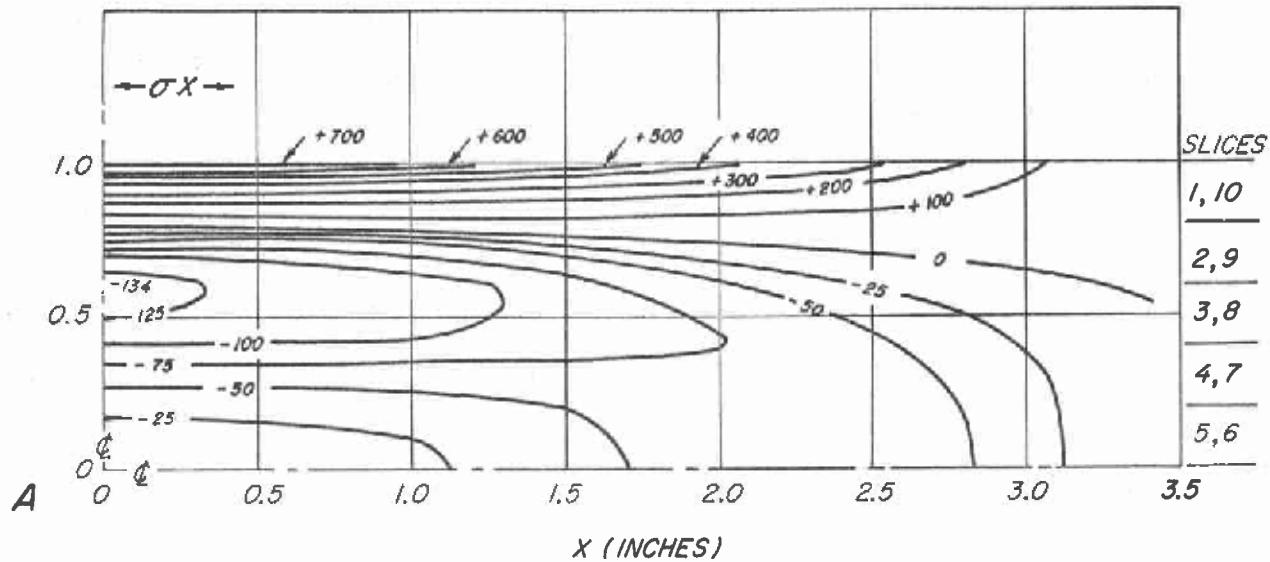


Figure 6. --Distribution of stress in the x direction (A), in the y direction (B), and associated shear stress (C) indicated by lines connecting points of equal stress.