ELASTIC BUCKLING OF A SIMPLY SUPPORTED RECTANGULAR SANDWICH PANEL SUBJECTED TO COMBINED EDGEWISE BENDING AND COMPRESSION

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ELASTIC BUCKLING OF A SIMPLY SUPPORTED RECTANGULAR SANDWICH PANEL SUBJECTED TO COMBINED EDGEWISE BENDING AND COMPRESSION*

By

W. R. KIMEL, Engineer

Forest Products Laboratory,** Forest Service
U. S. Department of Agriculture

I. SUMMARY AND CONCLUSIONS

A theoretical analysis is made of the problem of the elastic buckling of simply supported rectangular sandwich panels acted upon by any combination of edgewise bending and compression on opposite

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edges. The solution is based on the assumption that the sandwich panel is composed of isotropic plate facings of unequal thickness and an orthotropic core subjected only to anti-plane stress. The mathematical solution of the problem is based upon a Rayleigh-Ritz energy method using a double Fourier series with configuration parameters which are constants of integration obtained from solution of the core equilibrium equations. The specific method of approach is thought not to have been previously applied in sandwich analyses. The solution is in the form of a characteristic determinant of order infinity, except in the special case of pure edgewise compression, in which case the determinant is of order six. Evaluation of an order eighteen principal minor from the determinant of order infinity is made to obtain data for design curves.

Design curves based on the additional assumptions of membrane facings and infinite transverse modulus of elasticity of the core are compiled for the case of pure edgewise compression and for the case of pure edgewise bending. These design curves are believed sufficiently accurate for use in the design of a great many panels of modern design.

Equations are presented from which the critical load on sandwich panels composed of plate facings and orthotropic cores with finite transverse moduli of elasticity can be obtained. Unfortunately, however, considerable computational labor will be involved in the determination of critical load for this more general case. Design curves can be constructed from these more general equations, if the time and expense involved in their preparation can be justified.

II. INTRODUCTION

An elastic sandwich is a structural component consisting of two relatively thin external members called facings separated by and bonded to a relatively thick internal member called the core. The facings are commonly a material with comparatively high strength and stiffness, whereas the core is commonly a material of lighter density and relatively low strength and stiffness. The resulting layered-type structure is characterized by an extremely high strength-weight ratio as compared to that obtainable with the use of a single homogeneous material. For this reason, its primary field of application has been in guided missile and airframe assemblies, for example, wings, wall panels, webs of beams, and so forth. The thin facings of the sandwich, if not bonded to the core, are incapable of resisting reasonable design loads in their own plane because of their inability to resist becoming elastically unstable. A primary function of the core is thus seen to be to maintain the stability of the sandwich. A typical elastic sandwich panel is composed of aluminum facings and aluminum honeycomb core. The aluminum facings are commonly bonded to the core by epoxy or vinyl phenolic resins. Recently improved methods of fabrication and the development of improved bonding agents have made practical the use of sandwich construction in an increasing number of different fields of application.

Continually increasing applications of the elastic sandwich as a structural component have made necessary the analytical development of equations defining the stability criteria for different geometrical configurations of panels subjected to various combinations of loading. Because of the many variables which enter the problem, a study of experimental data alone cannot be expected to yield all the information which rational design procedures require. A panel of a committee composed of representatives of the Department of the Air Force, Department of the Navy, and Department of Commerce has been organized to unify, interpret, and present known rational approaches and data for sandwich structure design. It is known as the ANC-23 Panel.[1] The problem chosen for analysis in this thesis is one that has been clearly indicated by the ANC-23 Panel as being unsolved, and one which is encountered in the design of structures using sandwich plates in their construction.

The purpose of this thesis is, therefore, to present a rigorous derivation of the stability criteria for the simply supported rectangular

[1] See reference 1.

sandwich panel when it is subjected to combinations of edgewise bending and compression loadings.

Anti-plane stress[2] may be defined as that state of stress that exhibits stress components which are zero in a state of plane stress. Conversely, a state of plane stress exhibits stress components which are zero for anti-plane stress. Because sandwich cores have such low load carrying capacities in the direction of the plane of the panel as compared to the relatively stiff facings, the normal stresses and shear stresses in the core in the direction of the plane of the panel are assumed to be negligible. Thus, the sandwich core in this analysis is assumed to be subjected to a state of anti-plane stress. This assumption has been used in many previous analyses and is known to represent actual sandwich construction very well. The facings are treated by isotropic thin plate theory, that is, plane sections initially perpendicular to the median plane of the plate remain plane during deformation in accordance with the Bernoulli-Navier hypothesis.

The specific method of approach used in the solution of this problem is believed not to have been previously applied to sandwich analysis, but an analogous method has been commonly applied to nonlayered systems.

Equations for the three rectangular components of core displacement are found which satisfy the core equilibrium equations and the

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[2] See reference 5 and figure 2.

boundary conditions of the simply supported panel. By evaluating the displacements of the core at the interfaces (the junctions of the core and the facings) and equating these core displacements to displacements of the facings at these interfaces, displacements at any point in the facings may be found. From these displacement equations, strains and subsequently elastic energy of both the facings and the core may be expressed. It is noteworthy that the displacement functions written from a solution of the core equilibrium equations are, in this particular analysis, equal to zero until the sandwich starts to buckle. The edge loads are applied to the facings in the conventional manner as in the related ordinary plate problem presented by Timoshenko.[3]

Now, let the elastic energy of a general elastic system with respect to an undeflected configuration of that system be $\underline{V}$ and the potential energy of the external loads with respect to their undeflected positions immediately prior to buckling be $\underline{T}$. Then the total energy of the system, $\underline{U}$, with respect to the undeflected state may be expressed as

$$U = V - T.$$

For a condition of instability, $\underline{U}$ must have a stationary value with respect to any arbitrary change in configuration of the system.

The configuration of the system is defined in this thesis by parameters $\underline{A_{mn}}$, $\underline{B_{mn}}$, . . . , $\underline{G_{mn}}$ which are actually constants of

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[3] See reference 14, page 351.

integration obtained from solutions of the core equilibrium equations.

Thus,

$$\frac{\partial U}{\partial A_{mn}} dA_{mn} + \frac{\partial U}{\partial B_{mn}} dB_{mn} + \ldots + \frac{\partial U}{\partial G_{mn}} dG_{mn} = 0,$$

from which it is clearly evident that the buckling criteria are:

$$\frac{\partial U}{\partial A_{mn}} = \frac{\partial V}{\partial A_{mn}} - \frac{\partial T}{\partial A_{mn}} = 0$$

$$\frac{\partial U}{\partial B_{mn}} = \frac{\partial V}{\partial B_{mn}} - \frac{\partial T}{\partial B_{mn}} = 0 \tag{1}$$

$$\vdots$$

$$\frac{\partial U}{\partial G_{mn}} = \frac{\partial V}{\partial G_{mn}} - \frac{\partial T}{\partial G_{mn}} = 0.$$

This solution is seen to be a Rayleigh-Ritz method, but it differs from the Rayleigh-Ritz procedure as applied to the conventional stability analyses of ordinary plate problems in that certain of the differential equations of equilibrium associated with the problem (the core equilibrium equations) are herein satisfied exactly, while the facings are mathematically attached to the core by requiring displacement continuity between core and facings at the interfaces. The facings are then treated by the conventional energy method for plates.

This method of approach to sandwich stability analysis was prompted by, and is believed to be complementary to, a sandwich stability analysis by Raville[4] who solved all the differential equations of equilibrium in the facings as well as in the core. It is believed that the method of analysis presented in this thesis, together with the method presented by Raville, exhibit fundamental methods of approach whereby it may be expected that if a problem has been solved in the thin plate or shell literature, its counterpart in the sandwich panel can be solved.

In this thesis, literal solutions are derived within the scope of the aforementioned assumptions; however, the reduction of these equations for use in preparing usable design curves seemed prohibitive in view of the many parameters involved. Therefore, in the preparation of design curves, the additional assumptions of membrane facings[5] and infinite transverse modulus of elasticity of the core are made. The transverse modulus of elasticity of the core refers to the modulus of elasticity of the core in a direction perpendicular to the facings. These assumptions have been previously used[6] and are known to represent actual sandwich construction very well. It is to be emphasized that literal equations are presented from which the numerical calculation of

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[4] See reference 12.

[5] In using the design curves, the actual flexural rigidity of the spaced facings, D, may be used as a good approximation. See Use of Design Curves and Discussion of Results.

[6] See references 1, 2, 3, 8, 11, 13, and 16. It is shown in this thesis that the "tilting" method involves this assumption. See Discussion of Results.

critical load for any one particular sandwich may be computed without the latter two assumptions. It is believed that design curves can be prepared from these more exact literal equations if the time and expense involved in their preparation can be justified. Such design curves would be of limited usefulness, however, in view of the many complex parameters that would of necessity be involved in this case.

III. NOTATION

x, y, z	rectangular coordinates (fig. 1)
a	length of sandwich in direction of loading
b	width of sandwich in direction perpendicular to loading
c	thickness of core
t	thickness of upper facing
t'	thickness of lower facing
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
E_c	modulus of elasticity of core in _z_ direction -- transverse modulus of elasticity of core
G_{xz}	modulus of rigidity of core in _xz_ plane
G_{yz}	modulus of rigidity of core in _yz_ plane
N_b	maximum value of loading (load per unit width, _b_, of panel) due to pure edgewise bending (fig. 4)
N_c	value of loading (load per unit width, _b_, of panel) due to pure edgewise compression (fig. 4)

N_o	$\underline{N_b} + \underline{N_c}$ (fig. 4)
N_x	value of $\underline{N_o}$ at any location on loaded edge (fig. 4)
N_{ocr}	value of $\underline{N_o}$ at buckling
a	$\frac{2N_b}{N_o}$ (fig. 4)
ϵ_{zc}	normal strain in core in $\underline{z}$ direction
$\gamma_{xzc}, \gamma_{yzc}$	shear strains in core
$\epsilon_x, \epsilon_y, \gamma_{xy}$	normal and shear strains in upper facing
$\epsilon'_x, \epsilon'_y, \gamma'_{xy}$	normal and shear strains in lower facing
$\epsilon_{xM}, \epsilon_{yM}, \gamma_{xyM}$	membrane strains in upper facing
$\epsilon'_{xM}, \epsilon'_{yM}, \gamma'_{xyM}$	membrane strains in lower facing
$\epsilon_{xB}, \epsilon_{yB}, \gamma_{xyB}$	bending strains in upper facing
$\epsilon'_{xB}, \epsilon'_{yB}, \gamma'_{xyB}$	bending strains in lower facing
u, v, w	displacements of upper facing in $\underline{x}$, $\underline{y}$, and $\underline{z}$ directions, respectively
u', v', w'	displacement of lower facing in $\underline{x}$, $\underline{y}$, and $\underline{z}$ directions, respectively
u_c, v_c, w_c	displacements of core in $\underline{x}$, $\underline{y}$, and $\underline{z}$ directions, respectively
m, n, p, q, i, j	integers
$A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}, G_{mn}$	configuration parameters
τ_{xzc}, τ_{yzc}	shear stresses in core in $\underline{xz}$ and $\underline{yz}$ planes, respectively

σ_{zc}	normal stress in core in z direction
σ_{xM}, σ_{yM}, τ_{xyM}	membrane stresses in upper facing
σ'_{xM}, σ'_{yM}, τ'_{xyM}	membrane stresses in lower facing
σ_{xB}, σ_{yB}, τ_{xyB}	bending stresses in upper facing
σ'_{xB}, σ'_{yB}, τ'_{xyB}	bending stresses in lower facing
z'	distance from middle surface of upper facing
z''	distance from middle surface of lower facing
V_c	elastic energy of core
V_{MF}, V'_{MF}	elastic energy of upper and lower facing, respectively, associated with membrane strains
V_{BF}, V'_{BF}	elastic energy of upper and lower facing, respectively, associated with bending strains
V	total elastic energy of sandwich
δ	$\dfrac{E\pi^2}{c(1-\mu^2)}$
ρ_{mn}	$1 + \dfrac{G_{yz}}{G_{xz}} \dfrac{a^2}{m^2} \dfrac{n^2}{b^2}$
θ_{mn}	$\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}$
Φ_{mn}	$m^2 + n^2 \dfrac{a^2}{b^2}$
$k_{m\alpha}$	critical load factor corresponding to loading defined by α

I — moment of inertia of spaced plate facings,

$$\frac{(c + 2t)^3 - c^3}{12} \quad \text{for } t = t'$$

I_M — moment of inertia of spaced membrane facings,

$$\frac{c^2 t}{2}$$

I_{BM} — fictitious moment of inertia of spaced plate facings,

$$\frac{t(c + t)^2}{2} \quad \text{for } t = t'$$

D — flexural rigidity of spaced plate facings, $\dfrac{EI}{1 - \mu^2}$

D_M — flexural rigidity of spaced membrane facings,

$$\frac{EI_M}{1 - \mu^2}$$

D_{BM} — fictitious flexural rigidity of spaced plate facings,

$$\frac{EI_{BM}}{1 - \mu^2}$$

S — $$\frac{\pi^2 Ect}{2Ga^2 (1 - \mu^2)} \quad \text{for } t = t'$$

W — $$\frac{\pi^2 Ect}{2Gb^2 (1 - \mu^2)} \quad \text{for } t = t'$$

β_{mn} — $$\frac{\Phi^2_{mn}}{1 + S\Phi_{mn}}$$

λ — $$N_{ocr} \frac{m^2 a^2}{\pi^2} \frac{1}{D_M} (1 - \frac{a}{2})$$

$H_{p,q}$, $J_{p,q}$, $K_{p,q}$ — elements of determinants

IV. MATHEMATICAL ANALYSIS

The sandwich panel and its relation to the coordinate system are shown in figure 1. Figure 3 illustrates different combinations of loading which are defined by different values of α. Specifically, as shown in figure 4,

$$\alpha = \frac{2N_b}{N_b + N_c} = \frac{2N_b}{N_o} \tag{2}$$

where N_b is the maximum loading on the sandwich (pounds per inch) due to pure edgewise bending and N_c is the loading on the sandwich (pounds per inch) due to pure edgewise compression. This scheme makes possible a general solution for determination of the critical load of a simply supported rectangular sandwich panel subjected to any combination of edgewise bending and compression.

1. Determination of Displacement Functions which Satisfy Equilibrium of the Core

In accordance with the assumptions outlined in the Introduction, σ_{xc}, σ_{yc}, and τ_{xyc} are assumed equal to zero. The core is therefore in a state of anti-plane stress.[7] A differential element of the core is shown in figure 2. Summations of forces in the x, y, and z directions, respectively, give the following equations:

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[7] Anti-plane stress is defined in the Introduction.

$$\frac{\partial \tau_{xzc}}{\partial z} = 0 \tag{3}$$

$$\frac{\partial \tau_{yzc}}{\partial z} = 0 \tag{4}$$

and

$$\frac{\partial \tau_{xzc}}{\partial x} + \frac{\partial \tau_{yzc}}{\partial y} + \frac{\partial \sigma_{zc}}{\partial z} = 0. \tag{5}$$

Relations between stress and strain are applicable, that is:

$$\sigma_{zc} = E_c \, \epsilon_{zc} \tag{6}$$

$$\tau_{xzc} = G_{xz} \, \gamma_{xzc} \tag{7}$$

and

$$\tau_{yzc} = G_{yz} \, \gamma_{yzc}. \tag{8}$$

In addition, the strains and displacements are related by the following equations:

$$\epsilon_{zc} = \frac{\partial w_c}{\partial z} \tag{9}$$

$$\gamma_{xzc} = \frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x} \tag{10}$$

and

$$\gamma_{yzc} = \frac{\partial v_c}{\partial z} + \frac{\partial w_c}{\partial y} \tag{11}$$

Equations (6) through (11) allow the equilibrium equations, (3), (4), and (5), to be written as follows:

$$\frac{\partial^2 u_c}{\partial z^2} + \frac{\partial^2 w_c}{\partial x \partial z} = 0 \tag{12}$$

$$\frac{\partial^2 v_c}{\partial z^2} + \frac{\partial^2 w_c}{\partial y \partial z} = 0 \tag{13}$$

$$G_{xz} \left(\frac{\partial^2 u_c}{\partial x \partial z} + \frac{\partial^2 w_c}{\partial x^2}\right) + G_{yz} \left(\frac{\partial^2 v_c}{\partial y \partial z} + \frac{\partial^2 w_c}{\partial y^2}\right) + E_c \frac{\partial^2 w_c}{\partial z^2} = 0. \tag{14}$$

In order to satisfy the boundary conditions of the simply supported panel, as well as the core equilibrium equations, the core displacements are assumed to be of the forms

$$u_c = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}^{(1)}(z) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{15}$$

$$v_c = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}^{(2)}(z) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{16}$$

and

$$w_c = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}^{(3)}(z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{17}$$

where $\underline{f_{mn}^{(1)}(z)}$, $\underline{f_{mn}^{(2)}(z)}$, and $\underline{f_{mn}^{(3)}(z)}$ denote separate functions of $\underline{z}$ alone. These functions of $\underline{z}$ alone will be determined by the requirement that equations (15), (16), and (17) satisfy the core equilibrium equations (12), (13), and (14). It is to be noted that equations (15), (16), and (17) satisfy the boundary conditions,

$$u_c\Big|_{\substack{y=0\\y=b}} = 0, \qquad v_c\Big|_{\substack{x=0\\x=a}} = 0, \qquad w_c\Big|_{\substack{x=0,\ y=0\\x=a,\ y=b}} = 0,$$

$$M_x\Big|_{\substack{y=0\\y=b}} = 0, \quad \text{and} \quad M_y\Big|_{\substack{x=0\\x=a}} = 0,$$

where M_x and M_y signify edge moments on the panel about axes parallel to the x and y axes respectively. Substitution of equations (15), (16), and (17) into equations (12), (13), and (14) and requiring that the resulting equations be valid for all values of x and y as well as for all values of m and n results in the equations

$$\frac{d^2 f^{(1)}_{mn}(z)}{dz^2} + \frac{m\pi}{a}\,\frac{df^{(3)}_{mn}(z)}{dz} = 0 \tag{18}$$

$$\frac{d^2 f^{(2)}_{mn}(z)}{dz^2} + \frac{n\pi}{b}\,\frac{df^{(3)}_{mn}(z)}{dz} = 0 \tag{19}$$

and

$$-G_{xz}\left[\frac{m\pi}{a}\,\frac{df^{(1)}_{mn}(z)}{dz} + \frac{m^2\pi^2}{a^2}\,f^{(3)}_{mn}(z)\right] - G_{yz}\left[\frac{n\pi}{b}\,\frac{df^{(2)}_{mn}(z)}{dz}\right.$$

$$\left. + \frac{n^2\pi^2}{b^2}\,f^{(3)}_{mn}(z)\right] + E_c\,\frac{d^2 f^{(3)}_{mn}(z)}{dz^2} = 0. \tag{20}$$

The functions $f^{(1)}_{mn}(z)$, $f^{(2)}_{mn}(z)$, and $f^{(3)}_{mn}(z)$ will now be found which satisfy equations (18), (19), and (20). Differentiation of equation (20) with respect to z gives

$$-G_{xz}\left[\frac{m\pi}{a}\frac{d^2f^{(1)}_{mn}(z)}{dz^2}+\frac{m^2\pi^2}{a^2}\frac{df^{(3)}_{mn}(z)}{dz}\right]-G_{yz}\left[\frac{n\pi}{b}\frac{d^2f^{(2)}_{mn}(z)}{dz^2}\right.$$

$$\left.+\frac{n^2\pi^2}{b^2}\frac{df^{(3)}_{mn}(z)}{dz}\right]+E_c\frac{d^3f^{(3)}_{mn}(z)}{dz^3}=0. \tag{21}$$

Substitution of equations (18) and (19) into (21) yields

$$\frac{d^3f^{(3)}_{mn}(z)}{dz^3}=0 \tag{22}$$

which can be integrated directly to the form

$$f^{(3)}_{mn}(z)=A_{mn}\frac{z^2}{2}+B_{mn}z+C_{mn}. \tag{23}$$

Substitution of equation (23) into equation (19) with subsequent integration results in

$$f^{(2)}_{mn}(z)=-\frac{n\pi}{b}\left[A_{mn}\frac{z^3}{6}+B_{mn}\frac{z^2}{2}+D_{mn}z+E_{mn}\right]. \tag{24}$$

Substitution of equation (23) into equation (18) gives an equation which is integrable to

$$f^{(1)}_{mn}(z)=-\frac{m\pi}{a}\left[A_{mn}\frac{z^3}{6}+B_{mn}\frac{z^2}{2}+F_{mn}z+G_{mn}\right]. \tag{25}$$

$\underline{A_{mn}}$, $\underline{B_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, $\underline{E_{mn}}$, $\underline{F_{mn}}$, and $\underline{G_{mn}}$ in equations (23), (24), and (25) are constants of integration which are not all independent. To determine relations between these integration constants, equations (23), (24), and (25) are substituted into equations (18), (19), and (20). Equations (18) and (19) are seen to be satisfied identically by this substitution, but equation (20) yields the relation

$$\frac{m^2\pi^2}{a^2} G_{xz} (F_{mn} - C_{mn}) + \frac{n^2\pi^2}{b^2} G_{yz} (D_{mn} - C_{mn})$$

$$+ E_c A_{mn} = 0. \tag{26}$$

By substituting equations (23), (24), and (25) into equations (15), (16), and (17) there results the equations

$$u_c = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\pi}{a} \left[A_{mn} \frac{z^3}{6} + B_{mn} \frac{z^2}{2} + F_{mn} z \right.$$

$$\left. + G_{mn} \right] \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{27}$$

$$v_c = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n\pi}{b} \left[A_{mn} \frac{z^3}{6} + B_{mn} \frac{z^2}{2} + D_{mn} z \right.$$

$$\left. + E_{mn} \right] \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{28}$$

and

$$w_c = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \frac{z^2}{2} + B_{mn} z + C_{mn} \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} . \tag{29}$$

The above equations for core displacements satisfy the equations of equilibrium of the core if the constants $\underline{A_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, and $\underline{F_{mn}}$ are related by equation (26). Thus, it is seen that there are only six independent constants of integration in the expressions for the core displacements.

2. Core Strains

The core strains, ϵ_{zc}, γ_{xzc} and γ_{yzc} are now obtained by substituting equations (27), (28), and (29) into equations (30), (31), and (32) which follow directly.

$$\epsilon_{zc} = \frac{\partial w_c}{\partial z} \tag{30}$$

$$\gamma_{xzc} = \frac{\partial w_c}{\partial x} + \frac{\partial u_c}{\partial z} \tag{31}$$

and

$$\gamma_{yzc} = \frac{\partial w_c}{\partial y} + \frac{\partial v_c}{\partial z} \tag{32}$$

This substitution gives

$$\epsilon_{zc} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} z + B_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{33}$$

$$\gamma_{xzc} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (C_{mn} - F_{mn}) \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{34}$$

and

$$\gamma_{yzc} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (C_{mn} - D_{mn}) \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}. \tag{35}$$

The core shear strains are thus seen to remain constant with variations in $\underline{z}$ alone. This was evident, of course, from the original core equilibrium equations (3) and (4). Again it is emphasized that $\underline{A_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, and $\underline{F_{mn}}$ are not independent.

3. Facing Strains Associated with Deformations of the Middle Surfaces of the Facings (Membrane Strains)

The expressions for the displacement components of points in either facing may be obtained by requiring displacement continuity between the core and facings at their bonding surfaces (interfaces), that is, the interface displacements of the facings must be equal to the interface displacements of the core. The middle surface displacements of the facings may be expressed in terms of these interface displacements by assuming:

1. $\underline{w}$ and $\underline{w'}$, the $\underline{z}$-direction displacements of the upper and lower facings respectively, are constant through the facing thicknesses. In analyses, where displacements associated with incipient buckling are under consideration (as in this paper), this assumption seems particularly valid.

2. $\underline{u}$ and $\underline{u'}$, the $\underline{x}$-direction displacements in the upper and lower facings respectively, vary linearly through the facing thicknesses.

3. $\underline{v}$ and $\underline{v'}$, the $\underline{y}$-direction displacements in the upper and lower facings respectively, vary linearly through the facing thicknesses.

The facing strains are arbitrarily divided into two distinct parts. One part is that associated with strains in the middle surfaces of the facings, that is, the membrane strains. The other part is that associated with strains caused by bending of the facings about their own middle surfaces.

a. Upper facing.--The displacements at the middle surface of the upper facing may therefore be evaluated as follows:

$$u\Big|_{z = -\frac{t}{2}} = u_c\Big|_{z=0} + \frac{t}{2}\frac{\partial w_c}{\partial x}\Big|_{z=0} \tag{36}$$

$$v\Big|_{z = -\frac{t}{2}} = v_c\Big|_{z=0} + \frac{t}{2}\frac{\partial w_c}{\partial y}\Big|_{z=0} \tag{37}$$

and

$$w\Big|_{z = -\frac{t}{2}} = w_c\Big|_{z=0} \tag{38}$$

The membrane strains in the upper facing may be written as follows:

$$\epsilon_{xM} = \frac{\partial u\Big|_{z = -\frac{t}{2}}}{\partial x} = \frac{\partial u_c}{\partial x}\Big|_{z=0} + \frac{t}{2}\frac{\partial^2 w_c}{\partial x^2}\Big|_{z=0} \tag{39}$$

$$\epsilon_{yM} = \frac{\partial v\Big|_{z = -\frac{t}{2}}}{\partial y} = \frac{\partial v_c}{\partial y}\Big|_{z=0} + \frac{t}{2}\frac{\partial^2 w_c}{\partial y^2}\Big|_{z=0} \tag{40}$$

and

$$\gamma_{xyM} = \frac{\partial u \Big|_{z = -\frac{t}{2}}}{\partial y} + \frac{\partial v \Big|_{z = -\frac{t}{2}}}{\partial x}$$

$$= \frac{\partial u_c}{\partial y}\Big|_{z=0} + t\,\frac{\partial^2 w_c}{\partial x \partial y}\Big|_{z=0} + \frac{\partial v_c}{\partial x}\Big|_{z=0} \; . \tag{41}$$

By substituting equations (27), (28), and (29) into equations (39), (40), and (41), the following relations are obtained for the membrane strains in the upper facing:

$$\epsilon_{xM} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (G_{mn} - \frac{t}{2} C_{mn}) \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{42}$$

$$\epsilon_{yM} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (E_{mn} - \frac{t}{2} C_{mn}) \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{43}$$

and

$$\gamma_{xyM} = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (G_{mn} + E_{mn}$$

$$- t\,C_{mn}) \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{44}$$

b. Lower facing.--The displacements in the middle surface of the lower facing may be evaluated as follows:

$$u' \Big|_{z = c + \frac{t'}{2}} = u_c \Big|_{z = c} - \frac{t'}{2} \frac{\partial w_c}{\partial x}\Big|_{z = c} \tag{45}$$

$$v'\Big|_{z = c + \frac{t'}{2}} = v_c\Big|_{z = c} - \frac{t'}{2}\frac{\partial w_c}{\partial y}\Bigg|_{z = c} \tag{46}$$

and

$$w'\Big|_{z = c + \frac{t'}{2}} = w_c\Big|_{z = c} \ . \tag{47}$$

The membrane strains in the lower facing may be written as follows:

$$\epsilon'_{xM} = \frac{\partial u'\Big|_{z = c + \frac{t'}{2}}}{\partial x} = \frac{\partial u_c}{\partial x}\Bigg|_{z = c} - \frac{t'}{2}\frac{\partial^2 w_c}{\partial x^2}\Bigg|_{z = c} \tag{48}$$

$$\epsilon'_{yM} = \frac{\partial v'\Big|_{z = c + \frac{t'}{2}}}{\partial y} = \frac{\partial v_c}{\partial y}\Bigg|_{z = c} - \frac{t'}{2}\frac{\partial^2 w_c}{\partial y^2}\Bigg|_{z = c} \tag{49}$$

and

$$\gamma'_{xyM} = \frac{\partial u'\Big|_{z = c + \frac{t'}{2}}}{\partial y} + \frac{\partial v'\Big|_{z = c + \frac{t'}{2}}}{\partial x}$$

$$= \frac{\partial u_c}{\partial y}\Bigg|_{z = c} - t'\frac{\partial^2 w_c}{\partial x \partial y}\Bigg|_{z = c} + \frac{\partial v_c}{\partial x}\Bigg|_{z = c} \ . \tag{50}$$

By substituting equations (27), (28), and (29) into equations (48), (49), and (50), the following relations are obtained for the membrane strains in the lower facing:

$$\epsilon'_{xM} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t' c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t' c}{2}\right) + \frac{C_{mn} t'}{2} \right.$$

$$\left. + F_{mn} c + G_{mn} \right] \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (51)$$

$$\epsilon'_{yM} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t' c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t' c}{2}\right) + \frac{C_{mn} t'}{2} \right.$$

$$\left. + D_{mn} c + E_{mn} \right] \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (52)$$

and

$$\gamma'_{xyM} = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \left(\frac{c^3}{3} + \frac{t' c^2}{2}\right) + B_{mn} (c^2 + t' c) + C_{mn} t' \right.$$

$$\left. + (D_{mn} + F_{mn}) c + E_{mn} + G_{mn} \right] \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \qquad (53)$$

4. Facing Strains Associated with Bending of the Facings About Their Own Middle Surfaces

(a) Upper facing. --The facing strains due to bending of the facings about their own middle surfaces may be determined from a knowledge of the slopes and curvatures of each facing in the xz and yz planes. These slopes and curvatures may be evaluated by differentiations of equation (38).

Thus, it is seen that the facing strains due to bending of the upper facing about its own middle surface may be derived from equation (38), and hence from equation (29), as follows:

$$\epsilon_{xB} = z' \frac{\partial^2 w}{\partial x^2}\bigg|_{z=-\frac{t}{2}} = -z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{54}$$

$$\epsilon_{yB} = z' \frac{\partial^2 w}{\partial y^2}\bigg|_{z=-\frac{t}{2}} = -z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{55}$$

and

$$\gamma_{xyB} = \frac{\partial}{\partial y}\left(z' \frac{\partial w}{\partial x}\bigg|_{z=-\frac{t}{2}} \right) + \frac{\partial}{\partial x}\left(z' \frac{\partial w}{\partial y}\bigg|_{z=-\frac{t}{2}} \right) = 2z' \frac{\partial^2 w}{\partial x \partial y}\bigg|_{z=-\frac{t}{2}}$$

$$= 2z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{56}$$

where $\underline{z'}$ is measured from the center plane of the upper facing so that

$$-\frac{t}{2} \leq z' \leq \frac{t}{2} .$$

<u>(b) Lower facing</u>.--Similarly, the facing strains due to bending of the lower facing about its own middle surface may be derived from equation (38), and hence from equation (29), as follows:

$$\epsilon'_{xB} = z'' \left.\frac{\partial^2 w'}{\partial x^2}\right|_{z = c + \frac{t'}{2}} = -z'' \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\frac{c^2}{2} + B_{mn}c + C_{mn}\right)\frac{m^2\pi^2}{a^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (57)$$

$$\epsilon'_{yB} = z'' \left.\frac{\partial^2 w'}{\partial y^2}\right|_{z = c + \frac{t'}{2}} = -z'' \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\frac{c^2}{2} + B_{mn}c + C_{mn}\right)\frac{n^2\pi^2}{b^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (58)$$

and

$$\gamma'_{xyB} = 2z'' \left.\frac{\partial^2 w'}{\partial x \partial y}\right|_{z = c + \frac{t'}{2}} = 2z'' \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\frac{c^2}{2} + B_{mn}c + C_{mn}\right)\frac{m\pi}{a}\frac{n\pi}{b} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \quad (59)$$

where z'' is measured from the middle plane of the lower facing so that

$$-\frac{t'}{2} \leq z'' \leq \frac{t'}{2} .$$

5. Elastic Energy of the Core

The elastic energy of the core due to the anti-plane stress components may be expressed as

$$V_c = \frac{1}{2}\int_0^a\int_0^b\int_0^c (\sigma_{zc}\,\epsilon_{zc} + \tau_{yzc}\,\gamma_{yzc} + \tau_{xzc}\,\gamma_{xzc})\,dx\,dy\,dz. \qquad (60)$$

An energy expression due to elastic energy of the core may be expressed in terms of core strains by substituting equations (6), (7), and (8) into equation (60). This substitution gives

$$V_c = \frac{1}{2}\int_0^a\int_0^b\int_0^c (E_c\,\epsilon_{zc}^2 + G_{xz}\,\gamma_{xz}^2 + G_{yz}\,\gamma_{yz}^2)\,dx\,dy\,dz. \qquad (61)$$

If equations (9), (10), and (11) are substituted into equation (61), there results

$$V_c = \frac{1}{2}\int_0^a\int_0^b\int_0^c \left[E_c\left(\frac{\partial w_c}{\partial z}\right)^2 + G_{xz}\left(\frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x}\right)^2 + G_{yz}\left(\frac{\partial v_c}{\partial z} + \frac{\partial w_c}{\partial y}\right)^2\right] dx\,dy\,dz \qquad (62)$$

From equations (27), (28), and (29) the following equations may be written:

$$\frac{\partial u_c}{\partial z} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\left[A_{mn}\frac{z^2}{2} + B_{mn}\,z + F_{mn}\right]\cos\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b} \qquad (63)$$

$$\frac{\partial v_c}{\partial z} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{n\pi}{b}\left[A_{mn}\frac{z^2}{2} + B_{mn}\, z + D_{mn}\right] \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{64}$$

$$\frac{\partial w_c}{\partial x} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\frac{z^2}{2} + B_{mn}\, z + C_{mn}\right]\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{65}$$

$$\frac{\partial w_c}{\partial y} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\frac{z^2}{2} + B_{mn}\, z + C_{mn}\right]\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{66}$$

and

$$\frac{\partial w_c}{\partial z} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\, z + B_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\ . \tag{67}$$

It is convenient to write equation (62) as

$$V_c = V_{c1} + V_{c2} + V_{c3} \tag{68}$$

where

$$V_{c1} = \frac{1}{2}\int_0^a\int_0^b\int_0^c E_c\left(\frac{\partial w_c}{\partial z}\right)^2 dx\, dy\, dz \tag{69}$$

$$V_{c2} = \frac{1}{2}\int_0^a\int_0^b\int_0^c G_{xz}\left(\frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x}\right)^2 dx\, dy\, dz \tag{70}$$

and

$$V_{c3} = \frac{1}{2} \int_0^a \int_0^b \int_0^c G_{yz} \left(\frac{\partial v_c}{\partial z} + \frac{\partial w_c}{\partial y} \right)^2 dx\, dy\, dz. \tag{71}$$

Substitution of equations (63), (64), (65), (66), and (67) into equations (69), (70), and (71) yields respectively,

$$V_{c1} = \frac{E_c}{2} \int_0^a \int_0^b \int_0^c \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} z + B_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx\, dy\, dz \tag{72}$$

$$V_{c2} = \frac{G_{xz}}{2} \int_0^a \int_0^b \int_0^c \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (C_{mn} - F_{mn}) \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx\, dy\, dz \tag{73}$$

and

$$V_{c3} = \frac{G_{yz}}{2} \int_0^a \int_0^b \int_0^c \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (C_{mn} - D_{mn}) \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right]^2 dx\, dy\, dz \tag{74}$$

which, upon integration[8] may be expressed respectively as

$$V_{c1} = E_c \frac{abc}{24} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^2 c^2 + 3A_{mn} B_{mn} c + 3B_{mn}^2) \qquad (75)$$

$$V_{c2} = G_{xz} \frac{abc}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} (C_{mn} - F_{mn})^2 \qquad (76)$$

and

$$V_{c3} = G_{yz} \frac{abc}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{b^2} (C_{mn} - D_{mn})^2 . \qquad (77)$$

A complete expression for the elastic energy of the core which is subjected to anti-plane stress may now be written by substituting equations (75), (76), and (77) into equation (68). Thus, it is seen that

$$V_c = \frac{abc}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{E_c}{3} (A_{mn}^2 c^2 + 3A_{mn} B_{mn} c + 3B_{mn}^2) + G_{xz} \frac{m^2 \pi^2}{a^2} (C_{mn} - F_{mn})^2 + G_{yz} \frac{n^2 \pi^2}{b^2} (C_{mn} - D_{mn})^2 \right]. \qquad (78)$$

6. Elastic Energy of the Facings Associated with Membrane Strains

(a) Upper facing. --The elastic energy of the upper facing associated with strains in its middle surface (membrane strains) may be written as follows:

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[8] See Appendix C for further details.

$$V_{MF} = \frac{1}{2} \int_0^a \int_0^b \int_0^t (\sigma_{xM}\,\epsilon_{xM} + \sigma_{yM}\,\epsilon_{yM} + \tau_{xyM}\,\gamma_{xyM})\, dx\, dy\, dz. \tag{79}$$

This energy expression may be written in terms of strains by substituting

$$\sigma_{xM} = \frac{E}{1-\mu^2}(\epsilon_{xM} + \mu\epsilon_{yM}) \tag{80}$$

$$\sigma_{yM} = \frac{E}{1-\mu^2}(\epsilon_{yM} + \mu\epsilon_{xM}) \tag{81}$$

and

$$\tau_{xyM} = \frac{E}{2(1+\mu)}\gamma_{xyM} \tag{82}$$

into equation (79) to give

$$V_{MF} = \frac{E}{2} \int_0^a \int_0^b \int_0^t \left[\frac{1}{1-\mu^2}(\epsilon^2_{xM} + \epsilon^2_{yM} + 2\mu\epsilon_{xM}\,\epsilon_{yM}) + \frac{1}{2(1+\mu)}\gamma^2_{xyM} \right] dx\, dy\, dz. \tag{83}$$

It is convenient to write equation (83) as

$$V_{MF} = V_{MF1} + V_{MF2} + V_{MF3} + V_{MF4} \tag{84}$$

where

$$V_{MF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_0^t \epsilon^2_{xM}\, dx\, dy\, dz \tag{85}$$

$$V_{MF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_0^t \epsilon_{yM}^2 \, dx\, dy\, dz \tag{86}$$

$$V_{MF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_0^t \epsilon_{xM}\, \epsilon_{yM} \, dx\, dy\, dz \tag{87}$$

and

$$V_{MF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_0^t \gamma_{xyM}^2 \, dx\, dy\, dz. \tag{88}$$

Substitution of equations (42), (43), and (44) into equations (85), (86), (87), and (88) yields, respectively,

$$V_{MF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_0^t \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (G_{mn} - \frac{t}{2} C_{mn}) \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx\, dy\, dz \tag{89}$$

$$V_{MF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_0^t \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (E_{mn} - \frac{t}{2} C_{mn}) \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \right]^2 dx\, dy\, dz \tag{90}$$

$$V_{MF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_0^t \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (G_{mn} - \frac{t}{2} C_{mn}) \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \right] \left[\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} (E_{p\,q} - \frac{t}{2} C_{p\,q}) \frac{q^2 \pi^2}{b^2} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} \right] dx\, dy\, dz \quad (91)$$

and

$$V_{MF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_0^t \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (G_{mn} + E_{mn} - t\, C_{mn}) \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right]^2 dx\, dy\, dz \quad (92)$$

which, upon integration may be expressed, respectively, as

$$V_{MF1} = \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^4 \pi^4}{a^4} (G_{mn} - \frac{t}{2} C_{mn})^2 \quad (93)$$

$$V_{MF2} = \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^4 \pi^4}{b^4} (E_{mn} - \frac{t}{2} C_{mn})^2 \quad (94)$$

$$V_{MF3} = \frac{abt\mu E}{4(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (G_{mn} - \frac{t}{2} C_{mn})(E_{mn} - \frac{t}{2} C_{mn}) \tag{95}$$

and

$$V_{MF4} = \frac{abtE}{16(1+\mu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (G_{mn} + E_{mn} - t\, C_{mn})^2 . \tag{96}$$

A complete expression for the elastic energy of the upper facing due to the membrane strains in the upper facing may now be written by substituting equations (93), (94), (95), and (96) into equation (84). Thus, it is seen that

$$V_{MF} = \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{m^4\pi^4}{a^4} (G_{mn} - \frac{t}{2} C_{mn})^2 + \frac{n^4\pi^4}{b^4} (E_{mn} - \frac{t}{2} C_{mn})^2 + 2\mu \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (G_{mn} - \frac{t}{2} C_{mn}) (E_{mn} - \frac{t}{2} C_{mn}) + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (G_{mn} + E_{mn} - t\, C_{mn})^2 \right] . \tag{97}$$

As shown in Appendix A, equation (97) may be algebraically simplified to the form

$$V_{MF} = \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[\frac{m^2\pi^2}{a^2} (G_{mn} - \frac{t}{2} C_{mn}) + \frac{n^2\pi^2}{b^2} (E_{mn} - \frac{t}{2} C_{mn}) \right]^2 + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \left[G_{mn} - E_{mn} \right]^2 \right\} . \tag{98}$$

(b) Lower facing. --The elastic energy of the lower facing associated with strains in the middle surface of the lower facing (membrane strains) may be written as follows:

$$V'_{MF} = \frac{1}{2} \int_0^a \int_0^b \int_c^{c+t'} (\sigma'_{xM} \epsilon'_{xM} + \sigma'_{yM} \epsilon'_{yM} + \tau'_{xyM} \gamma'_{xyM}) \, dx \, dy \, dz \, . \tag{99}$$

As was shown in the case of the upper facing, this energy expression may be written in terms of membrane strains alone, as follows:

$$V'_{MF} = \frac{E}{2} \int_0^a \int_0^b \int_c^{c+t'} \left\{ \frac{1}{1-\mu^2} \left[(\epsilon'_{xM})^2 + (\epsilon'_{yM})^2 + 2\mu \, \epsilon'_{xM} \, \epsilon'_{yM} \right] + \frac{1}{2(1+\mu)} (\gamma'_{xyM})^2 \right\} dx \, dy \, dz \tag{100}$$

It is convenient to write equation (100) as

$$V'_{MF} = V'_{MF1} + V'_{MF2} + V'_{MF3} + V'_{MF4} \tag{101}$$

where

$$V'_{MF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_c^{c+t'} (\epsilon'_{xM})^2 \, dx \, dy \, dz \tag{102}$$

$$V'_{MF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_c^{c+t'} (\epsilon'_{yM})^2 \, dx \, dy \, dz \tag{103}$$

$$V'_{MF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_c^{c+t'} \epsilon'_{xM} \, \epsilon'_{yM} \; dx\, dy\, dz \tag{104}$$

and

$$V'_{MF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_c^{c+t'} (\gamma'_{xyM})^2 \; dx\, dy\, dz \tag{105}$$

Substitution of equations (51), (52), and (53) into equations (102), (103), (104), and (105) yields, respectively,

$$V'_{MF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_c^{c+t'} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + F_{mn}\, c + G_{mn} \right] \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\}^2 dx\, dy\, dz \qquad (106)$$

$$V'_{MF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_c^{c+t'} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + D_{mn}\, c + E_{mn} \right] \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\}^2 dx\, dy\, dz \qquad (107)$$

$$V'_{MF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_c^{c+t'} \Bigg\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2}$$

$$+ F_{mn}\, c + G_{mn} \Bigg] \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Bigg[A_{pq} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{pq} \left(\frac{c^2}{2} + \frac{t'c}{2}\right)$$

$$+ C_{pq} \frac{t'}{2} + D_{pq}\, c + E_{pq} \Bigg] \frac{q^2 \pi^2}{b^2} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} \Bigg\} dx\, dy\, dz \qquad (108)$$

and

$$V'_{MF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_c^{c+t'} \Bigg\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[A_{mn} \left(\frac{c^3}{3} + \frac{t'c^2}{2}\right) + B_{mn} (c^2 + t'c) + C_{mn}\, t' + (D_{mn}$$

$$+ F_{mn})\, c + E_{mn} + G_{mn} \Bigg] \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \Bigg\}^2 dx\, dy\, dz \qquad (109)$$

which upon integration may be expressed, respectively, as

$$V'_{MF1} = \frac{abt'E}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^4 \pi^4}{a^4} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + F_{mn} c + G_{mn} \right]^2 \tag{110}$$

$$V'_{MF2} = \frac{abt'E}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^4 \pi^4}{b^4} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + D_{mn} c + E_{mn} \right]^2 \tag{111}$$

$$V'_{MF3} = \frac{abt'\mu E}{4(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \frac{n^2 \pi^2}{b^2} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} \right.$$
$$\left. + F_{mn} c + G_{mn} \right] \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + D_{mn} c + E_{mn} \right] \tag{112}$$

and

$$V'_{MF4} = \frac{abt'E}{16(1+\mu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \Bigg[A_{mn} \left(\frac{c^3}{3} + \frac{t'c^2}{2}\right) + B_{mn} (c^2 + t'c) + C_{mn} t'$$

$$+ (D_{mn} + F_{mn}) c + E_{mn} + G_{mn} \Bigg]^2 . \qquad (113)$$

A complete expression for the elastic energy of the lower facing due to the membrane strains in the lower facing may now be written by substituting equations (110), (111), (112), and (113) into equation (101). Thus, it is seen that

$$V'_{MF} = \frac{abt'E}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ \frac{m^4\pi^4}{a^4} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + F_{mn} c + G_{mn} \right]^2$$

$$+ \frac{n^4\pi^4}{b^4} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + D_{mn} c + E_{mn} \right]^2 + 2\mu \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \Bigg[A_{mn} \Bigg(\frac{c^3}{6}$$

$$+ \frac{t'c^2}{4}\Bigg) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2} + F_{mn} c + G_{mn} \Bigg] \Bigg[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn} \frac{t'}{2}$$

$$+ D_{mn} c + E_{mn} \Bigg] + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \Bigg[A_{mn} \left(\frac{c^3}{3} + \frac{t'c^2}{2}\right) + B_{mn} (c^2 + t'c) + C_{mn} t'$$

$$+ (D_{mn} + F_{mn}) c + E_{mn} + G_{mn} \Bigg]^2 \Bigg\} . \qquad (114)$$

As shown in Appendix A, equation (114) may be algebraically simplified to the form

$$V'_{MF} = \frac{abt'E}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\left\{ \frac{m^2\pi^2}{a^2} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4} \right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2} \right) + C_{mn} \frac{t'}{2} + F_{mn} c \right. \right. \right.$$

$$\left. \left. + G_{mn} \right] + \frac{n^2\pi^2}{b^2} \left[A_{mn} \left(\frac{c^3}{6} + \frac{t'c^2}{4} \right) + B_{mn} \left(\frac{c^2}{2} + \frac{t'c}{2} \right) + C_{mn} \frac{t'}{2} + D_{mn} c + E_{mn} \right] \right\}^2$$

$$\left. + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \left[c\left(F_{mn} - D_{mn}\right) + G_{mn} - E_{mn} \right]^2 \right] . \qquad (115)$$

7. Elastic Energy of the Facings Associated with Bending of the Facings About Their Own Middle Surfaces

(a) Upper facing. --The elastic energy of the upper facing associated with strains caused by bending of the upper facing about its own middle surface may be written as follows:

$$V_{BF} = \frac{1}{2} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} (\sigma_{xB}\, \epsilon_{xB} + \sigma_{yB}\, \epsilon_{yB} + \tau_{xyB}\, \gamma_{xyB})\; dx\, dy\, dz' \tag{116}$$

where z' is measured from the center plane of the upper facing. This energy expression may be written in terms of strains by substituting

$$\sigma_{xB} = \frac{E}{1-\mu^2} (\epsilon_{xB} + \mu\, \epsilon_{yB}) \tag{117}$$

$$\sigma_{yB} = \frac{E}{1-\mu^2} (\epsilon_{yB} + \mu\, \epsilon_{xB}) \tag{118}$$

and

$$\tau_{xyB} = \frac{E}{2(1+\mu)}\, \gamma_{xyB} \tag{119}$$

into equation (116) to give

$$V_{BF} = \frac{E}{2} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \left[\frac{1}{1-\mu^2} (\epsilon_{xB}^2 + \epsilon_{yB}^2 + 2\mu\, \epsilon_{xB}\, \epsilon_{yB}) + \frac{1}{2(1+\mu)}\, \gamma_{xyB}^2 \right] dx\, dy\, dz' \tag{120}$$

It is convenient to write equation (120) as

$$V_{BF} = V_{BF1} + V_{BF2} + V_{BF3} + V_{BF4} \tag{121}$$

where

$$V_{BF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \epsilon_{xB}^2 \; dx\, dy\, dz' \tag{122}$$

$$V_{BF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \epsilon_{yB}^2 \; dx\, dy\, dz' \tag{123}$$

$$V_{BF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \epsilon_{xB}\, \epsilon_{yB} \; dx\, dy\, dz' \tag{124}$$

and

$$V_{BF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \gamma_{xyB}^2 \; dx\, dy\, dz' \; , \tag{125}$$

Substitution of equations (54), (55), and (56) into equations (122), (123), (124), and (125) yields, respectively,

$$V_{BF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \left[z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx\, dy\, dz' \qquad (126)$$

$$V_{BF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \left[z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{n^2 \pi^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx\, dy\, dz' \qquad (127)$$

$$V_{BF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \left[z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m^2 \pi^2}{a^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]$$

$$\times \left[z' \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} C_{pq} \frac{q^2 \pi^2}{b^2} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} \right] dx\, dy\, dz' \qquad (128)$$

and

$$V_{BF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_{-\frac{t}{2}}^{+\frac{t}{2}} \left[2z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right]^2 dx\, dy\, dz' \qquad (129)$$

which, upon integration may be expressed, respectively, as

$$V_{BF1} = \frac{abt^3 E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^4 \pi^4}{a^4} C_{mn}^2 \qquad (130)$$

$$V_{BF2} = \frac{abt^3 E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^4 \pi^4}{b^4} C_{mn}^2 \qquad (131)$$

$$V_{BF3} = \frac{abt^3 \mu E}{48(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \frac{n^2 \pi^2}{b^2} C_{mn}^2 \qquad (132)$$

and

$$V_{BF4} = \frac{abt^3 E}{48(1+\mu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \frac{n^2 \pi^2}{b^2} C_{mn}^2 \; . \qquad (133)$$

A complete expression for the elastic energy of the upper facing associated with strains caused by bending of the upper facing about its own middle surface may now be written by substituting equations (130), (131), (132), and (133) into equation (121). Thus, it is seen that

$$V_{BF} = \frac{abt^3E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{m^4\pi^4}{a^4} + \frac{n^4\pi^4}{b^4} + 2\mu \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} + 2(1-\mu) \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \right] C_{mn}^2 \qquad (134)$$

and this equation can be algebraically simplified to the form

$$V_{BF} = \frac{abt^3E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 C_{mn}^2 \ . \qquad (135)$$

(b) Lower facing. --The elastic energy of the lower facing associated with strains caused by bending of the lower facing about its own middle surface may be written as

$$V'_{BF} = \frac{1}{2} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} (\sigma'_{xB} \epsilon'_{xB} + \sigma'_{yB} \epsilon'_{yB} + \tau'_{xyB} \gamma'_{xyB}) \, dx \, dy \, dz'' \qquad (136)$$

where z'' is measured from the middle plane of the lower facing. As was shown in the case of the upper facing, this energy expression, equation (136), may be written in terms of bending strains alone, as follows:

$$V'_{BF} = \frac{E}{2} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} \left\{ \frac{1}{1-\mu^2} \left[(\epsilon'_{xB})^2 + (\epsilon'_{yB})^2 + 2\mu \epsilon'_{xB} \epsilon'_{yB} \right] + \frac{1}{2(1+\mu)} (\gamma'_{xyB})^2 \right\} dx\, dy\, dz'' \quad . \tag{137}$$

It is convenient to write equation (137) as

$$V'_{BF} = V'_{BF1} + V'_{BF2} + V'_{BF3} + V'_{BF4} \tag{138}$$

where

$$V'_{BF1} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} (\epsilon'_{xB})^2 \; dx\, dy\, dz'' \tag{139}$$

$$V'_{BF2} = \frac{E}{2(1-\mu^2)} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} (\epsilon'_{yB})^2 \; dx\, dy\, dz'' \tag{140}$$

$$V'_{BF3} = \frac{\mu E}{1-\mu^2} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} \epsilon'_{xB}\, \epsilon'_{yB} \; dx\, dy\, dz'' \tag{141}$$

and

$$V'_{BF4} = \frac{E}{4(1+\mu)} \int_0^a \int_0^b \int_{-\frac{t'}{2}}^{+\frac{t'}{2}} (\gamma'_{xyB})^2 \; dx\, dy\, dz'' \quad . \tag{142}$$

Substitution of equations (57), (58), and (59) into equations (139), (140), (141), and (142) yields, respectively,

$$V'_{BF1} = \frac{ab(t')^3E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^4\pi^4}{a^4} \left(A_{mn} \frac{c^2}{2} + B_{mn}\, c + C_{mn}\right)^2 \quad (143)$$

$$V'_{BF2} = \frac{ab(t')^3E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^4\pi^4}{b^4} \left(A_{mn} \frac{c^2}{2} + B_{mn}\, c + C_{mn}\right)^2 \quad (144)$$

$$V'_{BF3} = \frac{ab(t')^3\mu E}{48(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \left(A_{mn} \frac{c^2}{2} + B_{mn}\, c + C_{mn}\right)^2 \quad (145)$$

and

$$V'_{BF4} = \frac{ab(t')^3E}{48(1+\mu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \left(A_{mn} \frac{c^2}{2} + B_{mn}\, c + C_{mn}\right)^2 . \quad (146)$$

A complete expression for the elastic energy of the lower facing associated with strains caused by bending of the lower facing about its own middle surface may now be written by substituting equations (143), (144), (145), and (146) into equation (138). Thus, it is seen that

$$V'_{BF} = \frac{ab(t')^3E}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 \left(A_{mn} \frac{c^2}{2} + B_{mn}\, c + C_{mn}\right)^2 \quad (147)$$

8. Total Elastic Energy of the Simply Supported Sandwich

The total elastic energy of a simply supported rectangular sandwich may now be written as the sum of the elastic energy of the core and the elastic energy of the facings. Thus,

$$V = V_c + V_{MF} + V'_{MF} + V_{BF} + V'_{BF} . \tag{148}$$

Substitution of equations (78), (98), (115), (135), and (147) into equation (148) yields

$$\begin{aligned}
V = \frac{abc}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[& \frac{E_c}{3} (A_{mn}^2 c^2 + 3A_{mn} B_{mn} c + 3B_{mn}^2) + G_{xz} \frac{m^2\pi^2}{a^2} (C_{mn} - F_{mn})^2 + G_{yz} \frac{n^2\pi^2}{b^2} (C_{mn} - D_{mn})^2 \Bigg] \\
& + \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[\frac{m^2\pi^2}{a^2} (G_{mn} - \frac{t}{2} C_{mn}) + \frac{n^2\pi^2}{b^2} (E_{mn} - \frac{t}{2} C_{mn}) \right]^2 + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (G_{mn} - E_{mn})^2 \right\} \\
& + \frac{abt'E}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[\Bigg\{ \frac{m^2\pi^2}{a^2} \left[A_{mn} (\frac{c^3}{6} + \frac{t'c^2}{4}) + B_{mn} (\frac{c^2}{2} + \frac{t'c}{2}) + C_{mn} \frac{t'}{2} + F_{mn} c + G_{mn} \right] \\
& + \frac{n^2\pi^2}{b^2} \left[A_{mn} (\frac{c^3}{6} + \frac{t'c^2}{4}) + B_{mn} (\frac{c^2}{2} + \frac{t'c}{2}) + C_{mn} \frac{t'}{2} + D_{mn} c + E_{mn} \right] \Bigg\}^2 + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} \Big[c (F_{mn} - D_{mn}) \\
& + G_{mn} - E_{mn} \Big]^2 \Bigg\} + \frac{abE}{96(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 \left[C_{mn}^2 t^3 + (A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn})^2 (t')^3 \right] .
\end{aligned} \tag{149}$$

It will be recalled that the parameters $\underline{A_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, and $\underline{F_{mn}}$ in equation (149) are not all independent. The relation between these parameters is given in equation (26), which when solved for $\underline{F_{mn}}$ yields

$$F_{mn} = C_{mn}\,\rho_{mn} - \frac{1}{G_{xz}}\,\frac{a^2}{m^2}\left(\frac{E_c}{\pi^2}\,A_{mn} + G_{yz}\,\frac{n^2}{b^2}\,D_{mn}\right) \tag{150}$$

where

$$\rho_{mn} = 1 + \frac{G_{yz}}{G_{xz}}\,\frac{a^2}{m^2}\,\frac{n^2}{b^2}\ . \tag{151}$$

Introduction of the additional notations

$$\theta_{mn} = \frac{m^2}{a^2} + \frac{n^2}{b^2} \tag{152}$$

and

$$\delta = \frac{E\pi^2}{c(1-\mu^2)} \tag{153}$$

and substitution of equation (150) into equation (149) permits the writing of an expression for the total elastic energy of the simply supported sandwich in which all of the parameters, $\underline{A_{mn}}$, $\underline{B_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, $\underline{E_{mn}}$, and $\underline{G_{mn}}$ are now independent. Thus,

$$
\begin{aligned}
V = \frac{abc}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ & \frac{E_c}{3} (A_{mn}^2 c^2 + 3A_{mn} B_{mn} c + 3B_{mn}^2) + G_{yz} \frac{n^2 \pi^2}{b^2} (C_{mn} - D_{mn})^2 \\
& + \frac{1}{G_{xz}} \frac{a^2}{m^2 \pi^2} \left[G_{yz} \frac{n^2 \pi^2}{b^2} (D_{mn} - C_{mn}) + E_c A_{mn} \right]^2 \Bigg\} + \frac{abct\delta\pi^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[\Big(\frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} \\
& - \frac{t}{2} \theta_{mn} C_{mn} \Big)^2 + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} (G_{mn} - E_{mn})^2 \Bigg] + \frac{abct'\delta\pi^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ \Bigg[\theta_{mn} \Big(\frac{c^3}{6} + \frac{c^2 t'}{4} \Big) A_{mn} \\
& + \theta_{mn} \Big(\frac{c^2}{2} + \frac{ct'}{2} \Big) B_{mn} + \theta_{mn} \frac{t'}{2} C_{mn} + \frac{m^2}{a^2} \rho_{mn} c\, C_{mn} - \frac{E_c}{G_{xz}} \frac{c}{\pi^2} A_{mn} + \frac{n^2}{b^2} c \Big(1 - \frac{G_{yz}}{G_{xz}} \Big) D_{mn} \\
& + \frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} \Bigg]^2 + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} \Bigg[c\, \rho_{mn} (C_{mn} - D_{mn}) - \frac{E_c}{G_{xz}} \frac{a^2}{m^2 \pi^2} c\, A_{mn} + G_{mn} - E_{mn} \Bigg]^2 \Bigg\} \\
& + \frac{abc\delta\pi^2}{96} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{mn}^2 \Bigg[C_{mn}^2 t^3 + \Big(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \Big)^2 (t')^3 \Bigg] \qquad (154)
\end{aligned}
$$

9. Potential Energy of the Edge Loads

The potential energy of the edge loads with respect to the undeflected configuration of the panel, immediately prior to buckling, is denoted by T, and may be derived in exactly the same manner as in the analogous plate analysis.[9] The edge loads are defined in terms of N_x pounds per inch of sandwich acting on edges perpendicular to the x axis as shown in figure 4. Since the core was assumed initially to be incapable of carrying loads in directions parallel to the facings, all the edge loads must be mathematically applied to the facings. It is of interest to note that in actual sandwich applications, the edges must be designed so that these same edge loads are actually applied to the facings. It is recalled that the core displacement functions given in equations (27), (28), and (29) are compatible with zero edge moments about the edges of the sandwich panel. This necessitates that the strains in both facings be the same in the direction of loading. This means that the stresses in both facings will be the same for facings of like materials. Thus,

$$T = \frac{1}{2} \int_0^a \int_0^b \left\{ \frac{N_x}{t + t'} \left[t \left(\frac{\partial w}{\partial x}\right)^2 + t' \left(\frac{\partial w'}{\partial x}\right)^2 \right] \right\} dx\, dy \qquad (155)$$

where $\frac{\partial w}{\partial x}$ and $\frac{\partial w'}{\partial x}$ refer to the slopes in the xz plane of the upper and lower facings, respectively. Because of the nature of the application of the edge loads shown in figure 4,

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[9] See reference 14, page 351.

$$N_x = N_o\left(1 - \frac{ay}{b}\right). \tag{156}$$

Substitution of equation (156) into equation (155) gives

$$T = \frac{1}{2}\int_0^a\int_0^b \left\{\frac{1}{t + t'} N_o\left(1 - \frac{ay}{b}\right)\left[t\left(\frac{\partial w}{\partial x}\right)^2 + t'\left(\frac{\partial w'}{\partial x}\right)^2\right]\right\} dx\, dy \tag{157}$$

which may be written as

$$T = \frac{N_o}{2(t + t')}\int_0^a\int_0^b \left[t\left(\frac{\partial w}{\partial x}\right)^2 + t'\left(\frac{\partial w'}{\partial x}\right)^2\right] dx\, dy$$

$$- \frac{N_o a}{2(t + t')b}\int_0^a\int_0^b y\left[t\left(\frac{\partial w}{\partial x}\right)^2 + t'\left(\frac{\partial w'}{\partial x}\right)^2\right] dx\, dy. \tag{158}$$

Because of displacement continuity between the core and facings at the interfaces and with reference to equation (65), it is seen that

$$\frac{\partial w}{\partial x} = \left.\frac{\partial w_c}{\partial x}\right|_{z=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{159}$$

and

$$\frac{\partial w'}{\partial x} = \left.\frac{\partial w_c}{\partial x}\right|_{z=c} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\frac{c^2}{2} + B_{mn}c\right.$$

$$\left. + C_{mn}\right)\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{160}$$

It follows that[10]

[10]See Appendix C for further details.

$$\int_0^a \int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dx\,dy = \frac{ab}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} C_{mn}^2 \tag{161}$$

$$\int_0^a \int_0^b \left(\frac{\partial w'}{\partial x}\right)^2 dx\,dy = \frac{ab}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn}\right)^2 \tag{162}$$

$$\int_0^a \int_0^b y \left(\frac{\partial w}{\partial x}\right)^2 dx\,dy = \frac{ab^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left[C_{mn}^2 - \frac{16}{\pi^2} C_{mn} \sum_{i}^{\infty} C_{mi} \frac{ni}{(n^2 - i^2)^2} \right] \tag{163}$$

and

$$\int_0^a \int_0^b y \left(\frac{\partial w'}{\partial x}\right)^2 dx\,dy = \frac{ab^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left[\left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn}\right)^2 - \frac{16}{\pi^2} \left(A_{mn} \frac{c^2}{2} \right.\right.$$

$$\left.\left. + B_{mn} c + C_{mn}\right) \sum_{i}^{\infty} \left(A_{mi} \frac{c^2}{2} + B_{mi} c + C_{mi}\right) \frac{ni}{(n^2 - i^2)^2} \right] \tag{164}$$

where in equations (163) and (164), the integers denoted by $\underline{i}$ must be selected so that $\underline{i} \pm \underline{n}$ is always odd and so that $\underline{i} \neq \underline{n}$. Substitution of equations (161), 162), (163), and (164) into equation (158) gives

$$T = \frac{N_o ab}{8(t + t')} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left[t\, C_{mn}^2 + t' \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \right)^2 \right] - \frac{N_o aab}{16(t+t')} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \right.$$

$$\times \left[t\, C_{mn}^2 + t' \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \right)^2 \right] - \frac{16}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \sum_{i}^{\infty} \left[t\, C_{mn} C_{mi} \right.$$

$$\left. \left. + t' \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \right) \left(A_{mi} \frac{c^2}{2} + B_{mi} c + C_{mi} \right) \right] \frac{ni}{(n^2 - i^2)^2} \right\} \tag{165}$$

where $\underline{i} \pm \underline{n}$ must always be odd and $\underline{i} \neq \underline{n}$. Equation (165) may be written in the form

$$T = \frac{N_o ab}{8(t + t')} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} \left\{ \left[t\, C_{mn}^2 + t' \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \right)^2 \right] \left[1 - \frac{a}{2} \right] \right.$$

$$\left. + \frac{8a}{\pi^2} \sum_{i}^{\infty} \left[t\, C_{mn} C_{mi} + t' \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn} \right) \left(A_{mi} \frac{c^2}{2} + B_{mi} c + C_{mi} \right) \right] \frac{ni}{(n^2 - i^2)^2} \right\} . \tag{166}$$

10. Equations Which Define Instability of the Sandwich

As indicated in the introduction of this paper, the total energy of the system, (V - T), must possess a stationary value at instability, that is,

$$\frac{\partial V}{\partial A_{mn}} - \frac{\partial T}{\partial A_{mn}} = 0 \tag{167}$$

$$\frac{\partial V}{\partial B_{mn}} - \frac{\partial T}{\partial B_{mn}} = 0 \tag{168}$$

$$\frac{\partial V}{\partial C_{mn}} - \frac{\partial T}{\partial C_{mn}} = 0 \tag{169}$$

$$\frac{\partial V}{\partial D_{mn}} - \frac{\partial T}{\partial D_{mn}} = 0 \tag{170}$$

$$\frac{\partial V}{\partial E_{mn}} - \frac{\partial T}{\partial E_{mn}} = 0 \tag{171}$$

$$\frac{\partial V}{\partial G_{mn}} - \frac{\partial T}{\partial G_{mn}} = 0 \ . \tag{172}$$

Substitution of equations (154) and (166) into equation (167) and division by factor $\frac{abc}{4}$ gives

$$E_c \left(A_{mn}\frac{c^2}{3} + B_{mn}\frac{c}{2}\right) + \frac{E_c}{G_{xz}}\frac{a^2}{m^2\pi^2}\left[G_{yz}\frac{n^2\pi^2}{b^2}(D_{mn} - C_{mn}) + E_c A_{mn}\right] + \frac{c^2}{24}\delta\pi^2\theta_{mn}^2 (t')^3 \Big/ \left(A_{mn}\frac{c^2}{2}\right.$$

$$\left. + B_{mn} c + C_{mn}\right) + t'\delta\pi^2\left\{\left[\theta_{mn}\left(\frac{c^3}{6} + \frac{c^2 t'}{4}\right) A_{mn} + \theta_{mn}\left(\frac{c^2}{2} + \frac{ct'}{2}\right)B_{mn} + \theta_{mn}\frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c\rho_{mn} C_{mn}\right.\right.$$

$$\left. - \frac{E_c}{G_{xz}}\frac{c}{\pi^2} A_{mn} + \frac{n^2}{b^2} c\left(1 - \frac{G_{yz}}{G_{xz}}\right) D_{mn} + \frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn}\right]\left[\theta_{mn}\left(\frac{c^3}{6} + \frac{c^2 t'}{4}\right) - \frac{E_c c}{G_{xz}\pi^2}\right]$$

$$\left. - \frac{n^2}{b^2\pi^2}\frac{E_c}{G_{xz}}\frac{1-\mu}{2}\left[c^2\rho_{mn}(C_{mn} - D_{mn}) - \frac{E_c}{G_{xz}}\frac{a^2}{m^2\pi^2} c^2 A_{mn} + c(G_{mn} - E_{mn})\right]\right\} - \frac{N_o ct'}{2(t+t')}\frac{m^2\pi^2}{a^2}$$

$$\times \left[\left(\frac{A_{mn}c^2}{2} + B_{mn} c + C_{mn}\right)\left(1 - \frac{a}{2}\right) + \frac{8a}{\pi^2}\sum_{1}^{\infty}\left(A_{mi}\frac{c^2}{2} + B_{mi} c + C_{mi}\right)\frac{ni}{(n^2 - i^2)^2}\right] = 0 \qquad (173)$$

where $\underline{i} \pm \underline{n}$ is odd and $\underline{i} \neq \underline{n}$.

Substitution of equations (154) and (166) into equation (168) and division by factor $\frac{abc}{4}$ gives

$$E_c \left(A_{mn} \frac{c}{2} + B_{mn}\right) + \frac{c\delta\pi^2}{12} \theta_{mn}^2 (t')^3 \left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn}\right) + t'\delta\pi^2 \left\{ \left[\theta_{mn} \left(\frac{c^3}{6} + \frac{c^2 t'}{4}\right) A_{mn} \right. \right.$$

$$+ \theta_{mn} \left(\frac{c^2}{2} + \frac{ct'}{2}\right) B_{mn} + \theta_{mn} \frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c \rho_{mn} C_{mn} - \frac{E_c}{G_{xz}} \frac{c}{\pi^2} A_{mn} + \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) D_{mn}$$

$$\left. \left. + \frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} \right] \left[\theta_{mn} \left(\frac{c^2}{2} + \frac{ct'}{2}\right) \right] \right\} - \frac{N_o t'}{t + t'} \frac{m^2 \pi^2}{a^2} \left[\left(A_{mn} \frac{c^2}{2} + B_{mn} c + C_{mn}\right) \left(1 - \frac{a}{2}\right) \right.$$

$$\left. + \frac{8a}{\pi^2} \sum_i^{\infty} \left(A_{mi} \frac{c^2}{2} + B_{mi} c + C_{mi}\right) \frac{ni}{(n^2 - i^2)^2} \right] = 0 \qquad (174)$$

where $i \pm n$ is odd and $i \neq n$.

Substitution of equations (154) and (166) into equation (169) and division by factor $\frac{abc}{4}$ gives

$$G_{yz} \frac{n^2\pi^2}{b^2} \rho_{mn} (C_{mn} - D_{mn}) - \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{n^2}{b^2} E_c A_{mn} + \frac{\delta\pi^2\theta^2_{mn}}{12} \Big[t^3 C_{mn} + (t')^3 (A_{mn} \frac{c^2}{2} + B_{mn} c$$

$$+ C_{mn}) \Big] - \frac{t^2\delta\pi^2\theta_{mn}}{2} (\frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} - \frac{t}{2} \theta_{mn} C_{mn}) + t'\delta\pi^2 \Big\{ \Big[\theta_{mn} (\frac{c^3}{6} + \frac{c^2 t'}{4}) A_{mn}$$

$$+ \theta_{mn} (\frac{c^2}{2} + \frac{ct'}{2}) B_{mn} + \theta_{mn} \frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c \rho_{mn} C_{mn} - \frac{E_c c}{G_{xz} \pi^2} A_{mn} + \frac{n^2}{b^2} c (1 - \frac{G_{yz}}{G_{xz}}) D_{mn}$$

$$+ \frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} \Big] \Big[\theta_{mn} \frac{t'}{2} + \frac{m^2}{a^2} c \rho_{mn} \Big] + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} \Big[c^2 \rho^2_{mn} (C_{mn} - D_{mn})$$

$$- \frac{E_c}{G_{xz}} \frac{a^2}{m^2\pi^2} c^2 \rho_{mn} A_{mn} + c \rho_{mn} (G_{mn} - E_{mn}) \Big] \Big\} - \frac{N_o}{c(t + t')} \frac{m^2\pi^2}{a^2} \Big\{ \Big[t C_{mn} + t' (A_{mn} \frac{c^2}{2}$$

$$+ B_{mn} c + C_{mn}) \Big] \Big[1 - \frac{a}{2} \Big] + \frac{8a}{\pi^2} \sum_{i}^{\infty} \Big[t C_{mi} + t' (A_{mi} \frac{c^2}{2} + B_{mi} c + C_{mi}) \Big] \frac{ni}{(n^2 - i^2)^2} \Big\} = 0 \quad (175)$$

where $i \pm n$ is odd and $i \neq n$.

Substitution of equations (154) and (166) into equation (170) and division by factor $\frac{abc}{4}$ gives

$$
-G_{yz} \frac{n^2 \pi^2}{b^2} \rho_{mn} (C_{mn} - D_{mn}) + \frac{G_{yz}}{G_{xz}} \frac{a^2 n^2}{m^2 b^2} E_c A_{mn} + t' \delta \pi^2 \Bigg\{ \Bigg[\theta_{mn} \left(\frac{c^3}{6} + \frac{c^2 t'}{4}\right) A_{mn} + \theta_{mn} \left(\frac{c^2}{2} + \frac{ct'}{2}\right) B_{mn}
$$

$$
+ \theta_{mn} \frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c \rho_{mn} C_{mn} - \frac{E_c}{G_{xz}} \frac{c}{\pi^2} A_{mn} + \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) D_{mn} + \frac{m^2}{a^2} G_{mn}
$$

$$
+ \frac{n^2}{b^2} E_{mn} \Bigg] \left[\frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)\right] + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} \Bigg[c^2 \rho_{mn}^2 (D_{mn} - C_{mn}) + \frac{E_c}{G_{xz}} \frac{a^2}{m^2 \pi^2} c^2 \rho_{mn} A_{mn}
$$

$$
+ c \rho_{mn} (E_{mn} - G_{mn}) \Bigg] \Bigg\} = 0. \qquad (176)
$$

Substitution of equations (154) and (166) into equation (171) and division by factor $\frac{abc}{4}$ gives

$$t\delta\pi^2 \left[\frac{n^2}{b^2}\left(\frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} - \frac{t}{2}\theta_{mn} C_{mn}\right) + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} (E_{mn} - G_{mn}) \right] + t'\delta\pi^2 \left\{ \frac{n^2}{b^2} \left[\theta_{mn} \left(\frac{c^3}{6} \right.\right.\right.$$

$$\left. + \frac{c^2 t'}{4}\right) A_{mn} + \theta_{mn} \left(\frac{c^2}{2} + \frac{ct'}{2}\right) B_{mn} + \theta_{mn} \frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c\, \rho_{mn} C_{mn} - \frac{E_c}{G_{xz}} \frac{c}{\pi^2} A_{mn}$$

$$+ \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) D_{mn} + \frac{m^2}{a^2} G_{mn} + \left. \frac{n^2}{b^2} E_{mn} \right] + \frac{1-\mu}{2} \frac{m^2}{a^2} \frac{n^2}{b^2} \left[c\, \rho_{mn} (D_{mn} - C_{mn}) \right.$$

$$\left.\left. + \frac{E_c}{G_{xz}} \frac{a^2}{m^2 \pi^2} c\, A_{mn} - G_{mn} + E_{mn} \right] \right\} = 0 \,. \qquad (177)$$

Substitution of equations (154) and (166) into equation (172) and division by factor $\frac{abc}{4}$ gives

$$t\delta\pi^2 \left[\frac{m^2}{a^2}\left(\frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn} - \frac{t}{2}\theta_{mn} C_{mn}\right) + \frac{1-\mu}{2}\frac{m^2}{a^2}\frac{n^2}{b^2}(G_{mn} - E_{mn})\right] + t'\delta\pi^2 \left\{\frac{m^2}{a^2}\left[\theta_{mn}\left(\frac{c^3}{6}\right.\right.\right.$$

$$\left. + \frac{c^2 t'}{4}\right) A_{mn} + \theta_{mn}\left(\frac{c^2}{2} + \frac{ct'}{2}\right) B_{mn} + \theta_{mn}\frac{t'}{2} C_{mn} + \frac{m^2}{a^2} c\,\rho_{mn} C_{mn} - \frac{E_c}{G_{xz}}\frac{c}{\pi^2} A_{mn}$$

$$\left. + \frac{n^2}{b^2} c\left(1 - \frac{G_{yz}}{G_{xz}}\right) D_{mn} + \frac{m^2}{a^2} G_{mn} + \frac{n^2}{b^2} E_{mn}\right] + \frac{1-\mu}{2}\frac{m^2}{a^2}\frac{n^2}{b^2}\left[c\,\rho_{mn}(C_{mn} - D_{mn})\right.$$

$$\left.\left. - \frac{E_c}{G_{xz}}\frac{a^2}{m^2\pi^2} c\,A_{mn} + G_{mn} - E_{mn}\right]\right\} = 0 \,. \qquad (178)$$

Equations (173), (174), (175), (176), (177), and (178) constitute a linear, homogeneous set which will be satisfied by a value of $\underline{N_o}$ equal to $\underline{N_{ocr}}$, the buckling load. If all of the parameters, $\underline{A_{mn}}$, $\underline{B_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, $\underline{E_{mn}}$, and $\underline{G_{mn}}$ are identically equal to zero, this set of equations is satisfied, but this is a trivial solution associated with the nonbuckled state of the sandwich. The solution of interest is that which satisfies the set of equations (173) through (178) when at least one of the parameters $\underline{A_{mn}}$ through $\underline{G_{mn}}$ assumes a value other than zero. Such a solution can be obtained by equating to zero the determinant of the coefficients of the parameters $\underline{A_{mn}}$, $\underline{B_{mn}}$, $\underline{C_{mn}}$, $\underline{D_{mn}}$, $\underline{E_{mn}}$, and $\underline{G_{mn}}$ in the set of equations (173), (174), (175), (176), (177), and (178). Since there is an infinite number of these parameters in the infinite number of equations which constitute this set, the resulting determinant is of order infinity unless $\underline{\alpha} = 0$, in which case the determinant is of order six. Fortunately, however, for $\underline{\alpha} \neq 0$ a relatively small part of the determinant of order infinity will yield a satisfactory approximation to the critical load.

V. NUMERICAL COMPUTATIONS

1. General Case

A solution of equations (173) through (178) will provide the critical load for the most general case of sandwich panel and loading. For a specific panel, a numerical solution for critical load can be obtained.

For further details of such a general case solution, see the section entitled Discussion of Results.

2. Case $\alpha = 0$, E_c Finite

The equation which defines the critical load for the case $\alpha = 0$ (pure edgewise compression, see fig. 3), is found by equating to zero the determinant of the coefficients of A_{mn}, B_{mn}, . . . , G_{mn} in equations (173) through (178). This equation is

$$\begin{vmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,6} \\ H_{2,1} & H_{2,2} & \cdots & H_{2,6} \\ \vdots & & & \\ H_{6,1} & H_{6,2} & \cdots & H_{6,6} \end{vmatrix} = 0 \tag{179}$$

where the elements $H_{p,q}$, after some reduction, are given in Appendix D. For the determination of the critical load of any specific panel loaded in edgewise compression ($\alpha = 0$), equation (179) may be solved numerically for N_{ocr}.

3. Case $\alpha = 0$, $E_c = \infty$

The assumption of infinite transverse modulus of elasticity of the core ($E_c = \infty$) introduces welcome simplifications into equation (179). This assumption is believed valid in analyses relating to modern sandwich construction.[11] In this case, equation (179) can be written as

$$\begin{vmatrix} J_{1,1} & J_{1,2} & \cdots & J_{1,6} \\ J_{2,1} & J_{2,2} & \cdots & J_{2,6} \\ \vdots & & & \\ J_{6,1} & J_{6,2} & \cdots & J_{6,6} \end{vmatrix} = 0 \tag{180}$$

where the elements $\underline{J_{p,q}}$ are given in Appendix E. For this case, when $\underline{G_{yz}} = \underline{G_{xz}} = \underline{G}$ and $\underline{t'} = \underline{t}$, the determinant in equation (180) may be simplified by judicious additions and subtractions of multiples of rows and columns. Thus, equation (180) reduces to

$$N_{ocr} = \frac{\pi^2}{m^2 a^2} D \Phi_{mn}^2 \left[1 - \frac{\frac{t(c+t)^2}{2I}}{\frac{1}{S\Phi_{mn}} + 1} \right] \tag{181}$$

where

$$I = \frac{(c+2t)^3 - c^3}{12} \tag{182}$$

$$\Phi_{mn} = m^2 + n^2 \frac{a^2}{b^2} \tag{183}$$

[11] See Discussion of Results.

$$D = \frac{EI}{1-\mu^2} \tag{184}$$

and

$$S = \frac{\pi^2 Ect}{2Ga^2 (1-\mu^2)} \tag{185}$$

Equation (181) can be written as

$$N_{ocr} = \frac{\pi^2 D}{b^2} k'_{m0} \tag{186}$$

where

$$k'_{m0} = \frac{b^2}{m^2 a^2} \Phi^2_{mn} \left[1 - \frac{\frac{t(c+t)^2}{2I}}{\frac{1}{S\Phi_{mn}} + 1} \right] \tag{187}$$

Further simplifications result when the facings are sufficiently thin to be considered membranes. In such cases $t \lesssim c$, so that terms involving either the square or cube of t are small enough to be neglected. This latter assumption is equivalent to the assumption of membrane facings, and is known to represent many actual sandwich constructions very well. This assumption of membrane facings permits equations (186) and (187) to be written, respectively, as follows:

$$N_{ocr} = \frac{\pi^2 D_M}{b^2} k_{m0} \tag{188}$$

and

$$k_{m0} = \frac{b^2}{m^2 a^2} \frac{\Phi_{mn}^2}{1 + S\,\Phi_{mn}} \tag{189}$$

where[12]

$$D_M = \frac{E I_M}{1 - \mu^2} \tag{190}$$

and

$$I_M = \frac{tc^2}{2} \; . \tag{191}$$

Instead of the parameter, S, a related parameter, W, may be used in equation (189). Thus, equation (189) may be expressed as

$$k_{m0} = \frac{b^2}{m^2 a^2} \frac{\Phi_{mn}^2}{1 + \frac{b^2}{a^2} W \Phi_{mn}} \tag{192}$$

---

[12] The simplification from equation (187) to equations (189) and (192) requires a modification (in equation (186)) of D, the flexural rigidity of the spaced plate facings. The assumption of membrane facings, for which $D_M = \frac{EI_M}{1-\mu^2}$ will effect this simplification. A more accurate assumption which will produce this same simplification is to use $D_{BM} = \frac{EI_{BM}}{1-\mu^2}$. This point is discussed further in Discussion of Results.

where

$$W = \frac{\pi^2 Ect}{2Gb^2 (1-\mu^2)} \,. \qquad (193)$$

Equation (192) is presented graphically in figure 5.

4. Case $\alpha = 2$, $E_c = \infty$

For the determination of critical load in all cases where α is other than zero, a determinant of order eighteen is solved.[12] Thus, the equation defining the critical load of the sandwich panel for the case $\alpha = 2$ (pure edgewise bending) is found by equating to zero the determinant of the coefficients of A_{mn}, B_{mn}, . . . , G_{mn} in equations (173) through (178). This equation is

$$\begin{vmatrix} K_{1,1} & K_{1,2} & \cdots & K_{1,18} \\ K_{2,1} & K_{2,2} & \cdots & K_{2,18} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ K_{18,1} & K_{18,2} & \cdots & K_{18,18} \end{vmatrix} = 0 \qquad (194)$$

where the elements $K_{p,q}$ are given in Appendix F. For the case where $G_{xz} = G_{yz} = G$ and $t = t'$, the determinant in equation (194) may be

[12] See Discussion of Results for further details relating to choice of order of the determinant used here.

simplified by a scheme similar to that used in simplifying equation (180). A reduced form of equation (194) is

$$
\begin{vmatrix}
\Phi_{m1}^2 \left[1 - \dfrac{\dfrac{t\,(c+t)^2}{2I}}{\dfrac{1}{S\,\Phi_{m1}} + 1} \right] & -\dfrac{32}{9} N_o \dfrac{m^2 a^2}{\pi^4} \dfrac{1}{D} & 0 \\
-\dfrac{32}{9} N_o \dfrac{m^2 a^2}{\pi^4} \dfrac{1}{D} & \Phi_{m2}^2 \left[1 - \dfrac{\dfrac{t\,(c+t)^2}{2I}}{\dfrac{1}{S\,\Phi_{m2}} + 1} \right] & -\dfrac{96}{25} N_o \dfrac{m^2 a^2}{\pi^4} \dfrac{1}{D} \\
0 & -\dfrac{96}{25} N_o \dfrac{m^2 a^2}{\pi^4} \dfrac{1}{D} & \Phi_{m3}^2 \left[1 - \dfrac{\dfrac{t\,(c+t)^2}{2I}}{\dfrac{1}{S\,\Phi_{m3}} + 1} \right]
\end{vmatrix} = 0 \qquad (195)
$$

The further assumption of membrane facings introduces additional simplification[13] so that equation (195) may be written as

$$\begin{vmatrix} \Phi_{m1}^2\left[1-\dfrac{1}{\dfrac{1}{S\,\Phi_{m1}}+1}\right] & -\dfrac{32}{9}N_o\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & 0 \\ -\dfrac{32}{9}N_o\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & \Phi_{m2}^2\left[1-\dfrac{1}{\dfrac{1}{S\,\Phi_{m2}}+1}\right] & -\dfrac{96}{25}N_o\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} \\ 0 & -\dfrac{96}{25}N_o\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & \Phi_{m3}^2\left[1-\dfrac{1}{\dfrac{1}{S\,\Phi_{m3}}+1}\right] \end{vmatrix} = 0 \qquad (196)$$

[13] This same simplification (from equation (195) to equation (196)) may be effected by introducing the more accurate $\underline{D_{BM}}$ instead of $\underline{D_M}$ for $\underline{D}$. This point is discussed further in Discussion of Results.

Expansion of the determinant in equation (196) leads to the equation

$$N_{ocr} = \frac{D_M \pi^4}{2m^2 a^2} \sqrt{\frac{\left(\frac{\Phi^2_{m1}}{1 + S\Phi_{m1}}\right)\left(\frac{\Phi^2_{m2}}{1 + S\Phi_{m2}}\right)\left(\frac{\Phi^2_{m3}}{1 + S\Phi_{m3}}\right)}{\left(\frac{48}{25}\right)^2 \frac{\Phi^2_{m1}}{1 + S\Phi_{m1}} + \left(\frac{16}{9}\right)^2 \frac{\Phi^2_{m3}}{1 + S\Phi_{m3}}}} \qquad (197)$$

which may be expressed as

$$N_{ocr} = \frac{\pi^2 D_M}{b^2} k_{m2} \qquad (198)$$

where

$$k_{m2} = \frac{\pi^2}{m^2} \frac{b^2}{a^2} \sqrt{\frac{\left(\frac{\Phi^2_{m1}}{1 + S\Phi_{m1}}\right)\left(\frac{\Phi^2_{m2}}{1 + S\Phi_{m2}}\right)\left(\frac{\Phi^2_{m3}}{1 + S\Phi_{m3}}\right)}{\left(\frac{96}{25}\right)^2 \frac{\Phi^2_{m1}}{1 + S\Phi_{m1}} + \left(\frac{32}{9}\right)^2 \frac{\Phi^2_{m3}}{1 + S\Phi_{m3}}}} \qquad (199)$$

or in terms of the parameter $\underline{W}$ as

$$k_{m2} = \frac{\pi^2}{m^2}\frac{b^2}{a^2}\sqrt{\frac{\left(\dfrac{\Phi_{m1}^2}{1 + W\dfrac{b^2}{a^2}\Phi_{m1}}\right)\left(\dfrac{\Phi_{m2}^2}{1 + W\dfrac{b^2}{a^2}\Phi_{m2}}\right)\left(\dfrac{\Phi_{m3}^2}{1 + W\dfrac{b^2}{a^2}\Phi_{m3}}\right)}{\left(\dfrac{96}{25}\right)^2\dfrac{\Phi_{m1}^2}{1 + W\dfrac{b^2}{a^2}\Phi_{m1}} + \left(\dfrac{32}{9}\right)^2\dfrac{\Phi_{m3}^2}{1 + W\dfrac{b^2}{a^2}\Phi_{m3}}}} \tag{200}$$

Equation (199) is presented graphically in figure 7. Equation (200) is presented graphically in figure 6.

5. Case $\underline{a}$ General (See Figure 4), $E_c = \infty$

A reduced form of the determinant in the equation which defines the critical load for the case with membrane facings, $\underline{E_c} = \infty$, $\underline{G_{yz}} = \underline{G_{xz}} = \underline{G}$, $\underline{t} = \underline{t'}$, and $\underline{a}$ is any value defined by equation (2), is

$$\begin{vmatrix} \dfrac{\Phi_{m1}^2}{1 + S\Phi_{m1}} - N_{ocr}\dfrac{m^2a^2}{\pi^2}\dfrac{1}{D_M}\left(1 - \dfrac{a}{2}\right) & -\dfrac{16}{9}\,a\,N_{ocr}\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & 0 \\ -\dfrac{16}{9}\,a\,N_{ocr}\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & \dfrac{\Phi_{m2}^2}{1 + S\Phi_{m2}} - N_{ocr}\dfrac{m^2a^2}{\pi^2}\dfrac{1}{D_M}\left(1 - \dfrac{a}{2}\right) & -\dfrac{48}{25}\,a\,N_{ocr}\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} \\ 0 & -\dfrac{48}{25}\,a\,N_{ocr}\dfrac{m^2a^2}{\pi^4}\dfrac{1}{D_M} & \dfrac{\Phi_{m3}^2}{1 + S\Phi_{m3}} - N_{ocr}\dfrac{m^2a^2}{\pi^2}\dfrac{1}{D_M}\left(1 - \dfrac{a}{2}\right) \end{vmatrix} = 0. \qquad (201)$$

Expansion of equation (201) yields the following equation:

$$\beta_{m1}\beta_{m2}\beta_{m3} - \lambda(\beta_{m1}\beta_{m2} + \beta_{m1}\beta_{m3} + \beta_{m2}\beta_{m3}) + \lambda^2(\beta_{m1} + \beta_{m2} + \beta_{m3}) - \lambda^3 - \left(\frac{aN_{ocr}}{D_M}\frac{m^2a^2}{\pi^4}\right)^2\left[\left(\frac{48}{25}\right)^2(\beta_{m1} - \lambda) + \left(\frac{16}{9}\right)^2(\beta_{m3} - \lambda)\right] = 0 \qquad (202)$$

where

$$\beta_{mn} = \frac{\Phi^2_{mn}}{1 + S\Phi_{mn}} \qquad (203)$$

and

$$\lambda = N_{ocr}\frac{m^2a^2}{\pi^2}\frac{1}{D_M}\left(1 - \frac{a}{2}\right). \qquad (204)$$

No design curves were made for values of $\underline{a}$ other than $\underline{a} = 0$ and $\underline{a} = 2$. Equation (202) is presented with the thought that design curves for various values of $\underline{a}$ can be prepared if desired. Until such curves are compiled, the designer can numerically solve equation (202) for the critical load of any particular panel.

VI. USE OF DESIGN CURVES

In using either figure 5, 6, or 7 to obtain the critical load of a particular sandwich panel, select the correct member of the family of curves[14] by calculating the value of the parameter $\underline{S}$ (or $\underline{W}$ if using either figure 5 or 6) from the physical properties of the particular sandwich under consideration. Then, read the lowest value of $\underline{k_{m\alpha}}$ corresponding to the ratio $\frac{a}{b}$ of the panel. The integer $\underline{m}$ associated with the particular curve from which $\underline{k_{m\alpha}}$ is selected, indicates the number of half sine waves into which the panel will buckle if its critical load is applied. The critical load of the panel, $\underline{N_{ocr}}$, can now be computed by substituting this value of $\underline{k_{m\alpha}}$ into either equation (188) or equation (198), depending on whether the panel is to be subjected to pure edgewise compression ($\underline{\alpha} = 0$) or pure edgewise bending ($\underline{\alpha} = 2$). It should be noted that figure 5 applies for $\underline{\alpha} = 0$, whereas figures 6 and 7 are both applicable for $\underline{\alpha} = 2$.

In the curves shown in figures 5 and 6, the parameter $\underline{W}$ is used to separate each member of the family of curves, whereas in figure 7, the parameter $\underline{S}$ is used for this purpose. Curves 6 and 7 use parameters $\underline{W}$ and $\underline{S}$ respectively, to define the $\underline{k_{m2}}$ for the same identical case of loading ($\underline{\alpha} = 2$). It is believed that the designer will find the use of figures

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[14] A family of curves is here defined as that set of curves corresponding to a particular value of $\underline{S}$ (or $\underline{W}$).

6 and 7 together will aid in the interpolation to a correct value of k_{m2} for any specific sandwich.

If the facings are very thin, then the assumption of membrane facings is sufficiently accurate, and the designer may use a flexural rigidity factor denoted by D_M. For sandwich panels with facings of thicknesses such that the assumption of membrane facings is deemed inaccurate, the designer is advised to use the flexural rigidity denoted by D. A further discussion of the different approximations for flexural rigidity is contained in the section entitled Discussion of Results.

VII. DISCUSSION OF RESULTS

1. General Case

The general case discussed here refers to a sandwich panel composed of elements with the following properties:

1. The core is orthotropic and is capable of resisting only antiplane stress. The transverse modulus of elasticity of the core, E_c, is finite.

2. The facings are isotropic and are of unequal thickness.

The loading on the panel for this general case is any combination of edgewise bending and compression on opposite edges of the panel (denoted by α from equation 2).

The solution for critical load in such a general case may be effected by equating to zero the determinant of the coefficients of A_{mn},

$\underline{B}_{mn}$, ... , $\underline{G}_{mn}$ in equations (173) through (178). This characteristic determinant is of order infinity, except in the special case of pure edgewise compression ($\underline{\alpha} = 0$), in which case the determinant is of order six. A close approximation to the critical load may be obtained by replacing this determinant of order infinity by its first principal minor of order eighteen. Specifically, this first principal minor of order eighteen is composed of the coefficients of the configuration parameters $\underline{A}_{m1}$, $\underline{A}_{m2}$, $\underline{A}_{m3}$, $\underline{B}_{m1}$, $\underline{B}_{m2}$, $\underline{B}_{m3}$, ... , $\underline{G}_{m1}$, $\underline{G}_{m2}$, $\underline{G}_{m3}$ in equations (173) through (178). The equation formed by setting this first principal minor to zero is believed to yield a solution which is sufficiently accurate[15] for design. For combinations of loading defined by values of $\underline{\alpha}$ close to zero, smaller principal minors will yield sufficient accuracy for design.[16] It is to be emphasized that this method of solution will involve

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[15] For the analogous homogeneous plate problem with $\underline{\alpha} = 2$ (see reference 14, page 355), Timoshenko asserts, "the difference between the third and fourth approximation is only about one-third of one percent." These third and fourth approximations to which Timoshenko refers are analogous to the principal minors of order eighteen and twenty-four, respectively, referred to in this thesis. Therefore, for $\underline{\alpha} = 2$, it is believed that the principal minor of order eighteen will provide accuracy within about one-third of one percent of that obtainable from the principal minor of order twenty-four.

[16] The error in any given calculation for critical load may always be estimated by again solving for critical load using the next larger principal minor. However, the principal minors which must be considered are (in order of increasing accuracy) of order 6, 12, 18, 24, 30, ... , so that this error is not easily determined.

a considerable expenditure of computational labor,[17] and is not recommended unless the designer believes the core-flattening effect which accompanies incipient buckling is appreciable. This flattening effect is believed negligible for a majority of panels. Following this idea, equation (202) has been derived assuming that $\underline{E_c} = \infty$. Equation (202) is applicable for any combination of edgewise bending and compression (any value of $\underline{\alpha}$), for $\underline{t'} = \underline{t}$, and $\underline{G_{yz}} = \underline{G_{xz}}$.

2. Case $\underline{\alpha} = 0$, Otherwise General

For the case of pure edgewise compression ($\underline{\alpha} = 0$), equation (179) can be solved numerically for the critical load of any particular panel. This equation includes the effect of core-flattening on the critical load, and is sufficiently general to accomodate sandwich panels with orthotropic cores and plate facings of unequal thickness. This solution has not been reduced to design curves.

3. Case $\underline{\alpha} = 0$, $\underline{E_c} = \infty$, $\underline{G_{yz}} = \underline{G_{xz}} = \underline{G}$, $\underline{t'} = \underline{t}$

The case discussed here refers to a sandwich panel composed of elements with the following properties:

1. The core is isotropic and is capable of resisting only antiplane stress. The transverse modulus of elasticity of the core, $\underline{E_c}$, is infinite.

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[17] A numerical solution for the critical load of a particular panel involves the solution of an order eighteen numerical determinant and is therefore possible. A general literal solution for $\underline{N_{ocr}}$ seems impossible.

2. The facings are isotropic and are of equal thickness.

The loading on the panel for this case is pure edgewise compression.

The critical load for this case is defined by equation (186). Equation (186) may be reduced to equation (188) by a modification of the flexural rigidity factor, D, of the spaced plate facings. The design curves shown in figure 5 were constructed from equation (188) and are therefore applicable to this case subject to the aforementioned modification. The modification here referred to is that involved in assuming

$$\frac{t\,(c+t)^2}{2I} \doteq 1 . \qquad (205)$$

For facings with negligible flexural rigidity, the assumption of membrane facings is valid, that is, $t \lessapprox c$, so that all terms involving squares or cubes of the facing thickness, t, may be assumed negligible and hence $I = I_M$ (see equation (191)). Therefore, with the assumption of membrane facings, equation (205) is seen to be identically satisfied. A much closer approximation which will give this identical simplication occurs if I in equation (205) is taken equal to a fictitious area moment of inertia denoted by I_{BM}, where

$$I_{BM} = \frac{t}{2}\,(c+t)^2 . \qquad (206)$$

To further emphasize the fact that the use of I_{BM} in place of I in equation (205) is a much better approximation than is the use of I_M in place of $\underline{I}$, a comparison of these approximations to $\underline{I}$ follows:

$$I = \frac{(c + 2t)^3 - c^3}{12} \tag{207}$$

$$I_{BM} + \frac{t^3}{6} \doteq I \tag{208}$$

$$I_M + ct^2 + \frac{2}{3} t^3 = I. \tag{209}$$

Associated with $\underline{I}$, $\underline{I_{BM}}$, and $\underline{I_M}$ are the flexural rigidity factors $\underline{D}$, $\underline{D_{BM}}$, and $\underline{D_M}$ of the spaced facings. For example,

$$D = \frac{EI}{1-\mu^2} .$$

$\underline{I}$ is not significantly different from $\underline{I_{BM}}$, therefore, the designer is advised to use the exact value of $\underline{D}$ with figures 5, 6, and 7 in cases where the facings are believed too thick to be treated as membranes. For many applications, however, the assumption of membrane facings and, therefore, the use of $\underline{I_M}$ will be sufficiently accurate.

Equation (186), which gives the critical load for this case when the facings are considered as plates, reduces to

$$\lim_{G \to \infty} N_{ocr} = \frac{\pi^2 D}{m^2 a^2} \Phi_{mn}^2 \tag{210}$$

when the modulus of rigidity of the core is infinite. Equation (210) is identical to the result obtained from the analogous homogeneous plate analysis,[18] where D is the flexural rigidity of the spaced plate facings. When the facings are considered as membranes (see equation 188), this critical load is again expressed by equation (210), except that D becomes D_M, the flexural rigidity of the spaced membrane facings.

Equation (186) reduces to

$$\lim_{G \to 0} N_{ocr} = \frac{\pi^2}{m^2 a^2} \frac{Et^3}{6(1-\mu^2)} \Phi^2_{mn} \tag{211}$$

when the modulus of rigidity of the core is zero. Equation (211) yields the critical load[18] for two simply-supported rectangular homogeneous plates of thickness t. When the facings are considered as membranes (see equation (188), this critical load reduces to zero as would be expected.

The critical load for a sandwich strip with plate facings in plane strain may be obtained from equation (186). It is

$$\lim_{b \to \infty} N_{ocr} = \frac{\pi^2 m^2}{a^2} D \left[1 - \frac{\frac{t(c+t)^2}{2I}}{\frac{1}{m^2 S} + 1} \right] . \tag{212}$$

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[18] See reference 14, page 328.

Where the facings are considered as membranes (see equation 188), the critical load of a sandwich strip is again expressed by equation (212), except that $\underline{D}$ becomes $\underline{D_M}$, the flexural rigidity of the spaced membrane facings.

For infinite modulus of rigidity of the core, equation (212) reduces to the Euler column equation for two homogeneous spaced strips in plane strain, that is,

$$\lim_{\substack{b \to \infty \\ G \to \infty}} N_{ocr} = \frac{\pi^2 m^2}{a^2} D \ . \tag{213}$$

With infinite modulus of rigidity of the core and with membrane facings, the critical load of a sandwich strip may be expressed by equation (213), wherein $\underline{D}$ is replaced by $\underline{D_M}$.

With $\underline{G} = 0$, equation (212) yields the familiar Euler column formula for two homogeneous strips in plane strain, that is,

$$\lim_{\substack{b \to \infty \\ G \to 0}} N_{ocr} = \frac{\pi^2 m^2}{a^2} \frac{E}{1-\mu^2} \frac{t^3}{6} \ . \tag{214}$$

An additional limiting case of interest is that which occurs when either $\underline{m}$ is made infinite or $\underline{a}$ is made zero. In this case, the critical load on a sandwich panel with plate facings (defined in equation (186)) is infinite, that is,

$$\lim_{\substack{m \to \infty \\ \text{or} \\ a \to 0}} N_{ocr} = \infty . \tag{215}$$

However, this critical load for a sandwich panel with membrane facings (see equation 188) is finite, that is,

$$\lim_{\substack{m \to \infty \\ \text{or} \\ a \to 0}} N_{ocr} = cG . \tag{216}$$

The finite limiting value of critical load given in equation (216) is characteristic of sandwich analyses wherein the assumptions of $\underline{E_c} = \infty$ and of membrane facings are made. This limiting value of the critical load may be attributed to the shear instability of the core.[19]

The value of $\frac{a}{b}$ corresponding to the minimum $\underline{k_{m0}}$ for each of the individual curves in figure 5 is found by setting

$$\frac{\partial k_{m0}}{\partial \frac{a}{b}} = 0 . \tag{217}$$

Thus,

$$\left.\frac{a}{b}\right|_{k_{m0}\ \min} = m \sqrt{\frac{1 - W}{1 + W}} . \tag{218}$$

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[19] See reference 1.

Therefore, the equation of the horizontal tangent to each family of curves[20] is

$$k_{m0_{min}} = \frac{4}{(1 + W)^2} , \qquad (W < 1) . \qquad (219)$$

Also, note that

$$k_{m0} \Big|_{\frac{a}{b} = 0} = \frac{1}{W} . \qquad (220)$$

Equation (219) is of particular interest to the designer because it can be used to compute the critical load factor, $\underline{k_{m0}}$, of a particular panel whenever $\frac{a}{b} > 1$, provided $\underline{W} < 1$. For values of $\underline{W} \geq 1$, equation (219) is not valid. The designer may use the equation

$$k_{m0} = \frac{1}{W} , \qquad (W \geq 1) \qquad (221)$$

for the determination of critical load factor, $\underline{k_{m0}}$, for all $\frac{a}{b}$ ratios when the parameter $\underline{W} \geq 1$.

The result from this limiting case analysis, that is, equation (181), is identical with the result obtained from the so-called "tilting" method of analysis.[21] This comparison seems to reveal the fundamental nature of the assumptions involved in the "tilting" method,

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[20] A family of curves is here defined as the set of curves corresponding to a particular value of $\underline{W}$ (or $\underline{S}$).

[21] See references 2, 3, 16.

that is, the core is in anti-plane stress and the transverse modulus of elasticity of the core, E_c, is infinite.

4. Case $\alpha = 2$, $E_c = \infty$, $G_{yz} = G_{xz} = G$, $t' = t$

The case discussed here refers to a sandwich panel composed of elements with the following properties:

1. The core is isotropic and is capable of resisting only anti-plane stress. The transverse modulus of elasticity of the core, E_c, is infinite.

2. The facings are isotropic and are of equal thickness.

The loading on the panel for this case is pure edgewise bending.

The critical load for this case is defined by equation (195). Equation (195) may be simplified to give equation (198) by a modification of the flexural rigidity factor, D, of the spaced plate facings. This modification is based on the same assumption given in the approximate equation (205). The design curves shown in figure 6 and in figure 7 give k_{m2} versus $\frac{a}{b}$ for the same case of sandwich panel and loading; however, the parameter W is used to distinguish between families of curves in figure 6, whereas, the parameter S is used to distinguish between families of curves in figure 7.

Equation (195), which gives the critical load for this case when the facings are considered as plates, reduces to

$$\lim_{G \to \infty} N_{ocr} = \frac{D\pi^4}{2m^2a^2} \sqrt{\frac{\Phi_{m1}^2 \Phi_{m2}^2 \Phi_{m3}^2}{(\frac{48}{25})^2 \Phi_{m1}^2 + (\frac{16}{9})^2 \Phi_{m3}^2}} \qquad (222)$$

when the modulus of rigidity of the core is infinite. Equation (222) is identical to the result obtained from the homogeneous plate analysis,[22] where D, is the flexural rigidity of the spaced plate facings. When the facings are considered as membranes (see equation (198)), this critical load is again expressed by equation (222), except that D becomes D_M, the flexural rigidity of the spaced membrane facings.

Equation (195) reduces to

$$\lim_{G \to 0} N_{ocr} = \frac{Et^3}{6(1-\mu^2)} \frac{\pi^4}{2m^2a^2} \sqrt{\frac{\Phi_{m1}^2 \Phi_{m2}^2 \Phi_{m3}^2}{(\frac{48}{25})^2 \Phi_{m1}^2 + (\frac{16}{9})^2 \Phi_{m3}^2}} \qquad (223)$$

when the modulus of rigidity of the core is zero. Equation (223) yields the critical load[22] for two simply supported rectangular homogeneous plates of thickness t subjected to pure edgewise bending. When the facings are considered as membranes (see equation (198)), this critical load reduces to zero as would be expected.

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[22] See reference 14, page 355.

When either m is taken equal to infinity or a is taken equal to zero, the critical load on a sandwich panel with plate facings (see equation (195)) is infinite, that is,

$$\lim_{\substack{m \to \infty \\ \text{or} \\ a \to 0}} N_{ocr} = \infty. \tag{224}$$

However, this critical load for a sandwich panel with membrane facings (see equation (198)) is finite, that is,

$$\lim_{\substack{m \to \infty \\ \text{or} \\ a \to 0}} N_{ocr} = 1.886\, c\, G. \tag{225}$$

The finite limiting value of critical load given in equation (225) is characteristic of sandwich analyses in which the assumptions of $E_c = \infty$ and of membrane facings are made. As stated previously, this limiting value of the critical load may be attributed to the shear instability of the core.

A form of equation (225) which is useful for design is

$$\lim_{m \to \infty} N_{ocr} = \frac{\pi^2 D_M}{b^2} \frac{1.886}{S \frac{a^2}{b^2}} \quad , \quad \left(S \frac{a^2}{b^2} \geqq 0.215\right) . \tag{226}$$

Equation (226) can be used to compute the critical load for sandwich panels subjected to pure edgewise bending when $\underline{S \frac{a^2}{b^2}} \geqq 0.215$. That this is so is evident from a study of figure 7. In a like manner, study of equation (225) together with figure 6 reveals that the critical load

$$\lim_{m \to \infty} N_{ocr} = \frac{\pi^2 D_M}{b^2} \frac{1.886}{W}, \qquad (W \geqq 0.215) \qquad (227)$$

is applicable for computing the critical load when $\underline{W} \geqq 0.215$.

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IX. APPENDICES

1. Appendix A -- Algebraic Details

Equations (97) and (114) for the elastic energy due to the membrane strains in the upper and lower facing, respectively, may be written in the forms shown in equations (98) and (115) by identical algebraic simplifications. Thus, for the upper facing, let

$$R_{mn} = G_{mn} - \frac{t}{2} C_{mn} \tag{228}$$

$$Q_{mn} = E_{mn} - \frac{t}{2} C_{mn} \tag{229}$$

so that equation (97) may be written as

$$V_{MF} = \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{m^4\pi^4}{a^4} R_{mn}^2 + \frac{n^4\pi^4}{b^4} Q_{mn}^2 + 2\mu \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} R_{mn} Q_{mn} + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (R_{mn} + Q_{mn})^2 \right] \tag{230}$$

$$= \frac{abtE}{8(1-\mu^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\left(\frac{m^2\pi^2}{a^2} R_{mn} + \frac{n^2\pi^2}{b^2} Q_{mn} \right)^2 + \frac{1-\mu}{2} \frac{m^2\pi^2}{a^2} \frac{n^2\pi^2}{b^2} (R_{mn} - Q_{mn})^2 \right] \tag{231}$$

and this latter equation may be seen to be identical with equation (98). A parallel simplification may be effected for the elastic energy due to the membrane strains in the lower facing if it is noted that for the lower facing

$$R'_{mn} = A_{mn}\left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn}\left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn}\frac{t'}{2} + F_{mn}\,c + G_{mn} \tag{232}$$

and

$$Q'_{mn} = A_{mn}\left(\frac{c^3}{6} + \frac{t'c^2}{4}\right) + B_{mn}\left(\frac{c^2}{2} + \frac{t'c}{2}\right) + C_{mn}\frac{t'}{2} + D_{mn}\,c + E_{mn}\,. \tag{233}$$

2. Appendix B -- Integration Formulas

Integration formulas used in this thesis include the following: (i and j are integers)

$$\int_0^b \cos\frac{i\pi y}{b}\cos\frac{j\pi y}{b}\,dy = \frac{b}{2}\,, \qquad \text{for } i = j \tag{234}$$

$$\int_0^b \cos\frac{i\pi y}{b}\cos\frac{j\pi y}{b}\,dy = 0\,, \qquad \text{for } i \neq j \tag{235}$$

$$\int_0^b \sin\frac{i\pi y}{b}\sin\frac{j\pi y}{b}\,dy = \frac{b}{2}\,, \qquad \text{for } i = j \tag{236}$$

$$\int_0^b \sin\frac{i\pi y}{b}\,\sin\frac{j\pi y}{b}\,dy = 0 \quad , \qquad \text{for } i \neq j \tag{237}$$

$$\int_0^b y\,\sin\frac{i\pi y}{b}\,\sin\frac{j\pi y}{b}\,dy = \frac{b^2}{4} \quad , \qquad \text{for } i = j \tag{238}$$

$$\int_0^b y\,\sin\frac{i\pi y}{b}\,\sin\frac{j\pi y}{b}\,dy = 0 \quad , \qquad \text{for } i \neq j \text{ and } i \pm j \text{ even} \tag{239}$$

$$\int_0^b y\,\sin\frac{i\pi y}{b}\,\sin\frac{j\pi y}{b}\,dy = -\frac{4b^2}{\pi^2}\,\frac{ij}{(i^2 - j^2)^2}$$

$$\text{for } i \neq j \text{ and } i \pm j \text{ odd.} \tag{240}$$

3. Appendix C -- Examples of Integrations

Included in this appendix are details of integration of equation (72) and of equation (163). These integrations are typical of many others in this thesis.

Equation (72) may be written as

$$V_{c1} = \frac{E_c}{2} \int_0^a \int_0^b \int_0^c \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} z + B_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\times \sum_{p=1}^{\infty} \sum_{i=1}^{\infty} (A_{pi} z + B_{pi}) \sin \frac{p\pi x}{a} \sin \frac{i\pi y}{b} \, dx\, dy\, dz. \qquad (241)$$

Equation (241) requires the integration of the product of a double infinite series by itself. The integration formulas (236) and (237) show that $\underline{V_{c1}}$ in equation (241) is zero when $\underline{m} \neq \underline{p}$ and/or $\underline{n} \neq \underline{i}$. However, when $\underline{m} = \underline{p}$ and $\underline{n} = \underline{i}$, integration with respect to $\underline{x}$ and $\underline{y}$ in equation (241) gives

$$V_{c1} = \frac{E_c}{2} \frac{ab}{4} \int_0^c \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} z + B_{mn})^2 \, dz. \qquad (242)$$

Integration of equation (242) gives

$$V_{c1} = E_c \frac{ab}{24} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(A_{mn} c + B_{mn})^3 - B_{mn}^3}{A_{mn}} \qquad (243)$$

which is identical with the integrated form given in equation (75).

Equation (163) may be written as (see equation (159))

$$\int_0^a \int_0^b y \left(\frac{\partial w}{\partial x}\right)^2 dx\, dy = \int_0^a \int_0^b y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\times \sum_{p=1}^{\infty} \sum_{i=1}^{\infty} C_{pi} \frac{p\pi}{a} \cos \frac{p\pi x}{a} \sin \frac{i\pi y}{b}\, dx\, dy \;. \qquad (244)$$

Integration of equation (244) gives values other than zero only when $\underline{m} = \underline{p}$. Thus, the integration of equation (244) with respect to $\underline{x}$ gives

$$\int_0^a \int_0^b y \left(\frac{\partial w}{\partial x}\right)^2 dx\, dy = \frac{a}{2} \int_0^b y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{m^2 \pi^2}{a^2} \sin \frac{n\pi y}{b}$$

$$\times \sum_{i=1}^{\infty} C_{mi} \sin \frac{i\pi y}{b}\, dy \;. \qquad (245)$$

Now, when $\underline{i} = \underline{n}$, equation (245) integrates to a value $\underline{I_1}$, where (see integration formula (238))

$$I_1 = \frac{a}{2} \frac{b^2}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^2 \frac{m^2 \pi^2}{a^2} \;. \qquad (246)$$

When $\underline{i} \neq \underline{n}$ and $\underline{i} \pm \underline{n}$ is even, equation (245) integrates to a value $\underline{I_2}$ where (see integration formula (239))

$$I_2 = 0. \qquad (247)$$

When $\underline{i} \neq \underline{n}$ and $\underline{i} \pm \underline{n}$ is odd, equation (245) integrates to a value $\underline{I_3}$ where (see integration formula (240))

$$I_3 = -\frac{a}{2}\,\frac{4b^2}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\,\frac{m^2\pi^2}{a^2}\sum_{i=1}^{\infty} C_{mi}\,\frac{ni}{(n^2 - i^2)^2}\,. \tag{248}$$

The total integral of equation (241) (and hence of equation (163)) is the sum of $\underline{I_1}$, $\underline{I_2}$, and $\underline{I_3}$ given in equations (246), (247), and (248), that is,

$$\int_0^a\!\!\int_0^b y\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy = \frac{ab^2}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}^2\,\frac{m^2\pi^2}{a^2} - \frac{2ab^2}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\,\frac{m^2\pi^2}{a^2}\sum_{i=1}^{\infty} C_{mi}\,\frac{ni}{(n^2 - i^2)^2}$$

$$= \frac{ab^2}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[C_{mn}^2 - \frac{16}{\pi^2}C_{mn}\sum_{i}^{\infty} C_{mi}\,\frac{ni}{(n^2 - i^2)^2}\right] \tag{249}$$

where $\underline{i} \pm \underline{n}$ is odd and $\underline{i} \neq \underline{n}$.

Equation (249) is identical with the integrated form given in equation (163).

4. Appendix D -- Determinant Elements for Case $\underline{a} = 0$, E_c Finite

The elements of the determinant in equation (179) are as follows:

$$H_{1,1} = \frac{E_c}{t'\delta\pi^2}\left(\frac{c^2}{12} + \frac{E_c}{G_{xz}}\frac{a^2}{m^2\pi^2}\right) + \left(-\frac{\theta_{mn}c^3}{12} - \frac{E_c c}{G_{xz}\pi^2}\right)^2$$

$$+ \frac{n^2}{b^2\pi^2}\frac{E_c^2}{G_{xz}^2}\frac{1-\mu}{2}\frac{a^2}{m^2}\frac{c^2}{\pi^2}$$

$$H_{1,2} = \theta_{mn}\left(\frac{c^2 + ct'}{2}\right)\left(-\frac{\theta_{mn}c^3}{12} - \frac{E_c c}{G_{xz}\pi^2}\right)$$

$$H_{1,3} = \theta_{mn}\left(\frac{t'}{2} + c\right)\left(-\frac{\theta_{mn}c^3}{12} - \frac{E_c c}{G_{xz}\pi^2}\right)$$

$$H_{1,4} = \frac{E_c}{t'\delta\pi^2}\frac{G_{yz}}{G_{xz}}\frac{a^2}{m^2}\frac{n^2}{b^2} + \frac{n^2}{b^2}c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\left(-\frac{\theta_{mn}c^3}{12} - \frac{E_c c}{G_{xz}\pi^2}\right)$$

$$+ \frac{n^2}{b^2\pi^2}\frac{E_c}{G_{xz}}\frac{1-\mu}{2}c^2\rho_{mn}$$

$$H_{1,5} = \frac{E_c}{G_{xz}}\frac{c}{\pi^2}\frac{1-\mu}{2}$$

$$H_{1,6} = \left(- \frac{\theta_{mn} c^3}{12} - \frac{E_c c}{G_{xz} \pi^2}\right)$$

$$H_{2,1} = \theta_{mn} \left(\frac{c^2 + ct'}{2}\right)\left(- \frac{\theta_{mn} c^3}{12} - \frac{E_c c}{G_{xz} \pi^2}\right)$$

$$H_{2,2} = \frac{E_c}{t' \delta \pi^2} + \frac{c^2 \theta^2_{mn} (t')^2}{12} + \left[\theta_{mn} \left(\frac{c^2 + ct'}{2}\right)\right]^2 - \frac{N_o}{(t + t')} \frac{m^2}{a^2} \frac{c}{\delta}$$

$$H_{2,3} = \frac{c\theta^2_{mn} (t')^2}{12} + \theta^2_{mn} \left(\frac{t'}{2} + c\right)\left(\frac{c^2 + ct'}{2}\right) - \frac{N_o}{(t + t')} \frac{m^2}{a^2 \delta}$$

$$H_{2,4} = \theta_{mn} \left(\frac{c^2 + ct'}{2}\right) \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)$$

$$H_{2,5} = 0$$

$$H_{2,6} = \theta_{mn} \left(\frac{c^2 + ct'}{2}\right)$$

$$H_{3,1} = \left(- \frac{c^2}{2}\right)\left(- \frac{\theta_{mn} c^3}{12} - \frac{E_c c}{G_{xz} \pi^2}\right)$$

$$H_{3,2} = \frac{E_c}{t'\delta\pi^2\theta_{mn}} + \theta_{mn}\left(\frac{c^2 + ct'}{2}\right)\left(-\frac{c^2}{2}\right)$$

$$H_{3,3} = -\frac{c\theta_{mn}t^3}{3t'} + \theta_{mn}\left(\frac{t'}{2} + c\right)\left(-\frac{c^2}{2}\right) + N_o\left(\frac{t}{t + t'}\right)\frac{m^2}{t'a^2\delta\theta_{mn}}$$

$$H_{3,4} = \frac{n^2}{b^2}\, c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\left(-\frac{c^2}{2}\right)$$

$$H_{3,5} = 0$$

$$H_{3,6} = \left(-\frac{c^2}{2}\right) + \frac{ct^2}{2t'}$$

$$H_{4,1} = \frac{E_c}{t'\delta\pi^2}\,\frac{G_{yz}}{G_{xz}}\,\frac{a^2}{m^2}\,\frac{n^2}{b^2} + \frac{n^2}{b^2}\, c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\left(-\frac{\theta_{mn}c^3}{12} - \frac{E_c\, c}{G_{xz}\pi^2}\right)$$

$$+ \frac{n^2}{b^2\pi^2}\,\frac{E_c}{G_{xz}}\,\frac{1-\mu}{2}\, c^2\rho_{mn}$$

$$H_{4,2} = \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \theta_{mn} \left(\frac{c^2 + ct'}{2}\right)$$

$$H_{4,3} = \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \theta_{mn} \left(\frac{t'}{2} + c\right)$$

$$H_{4,4} = \frac{n^2}{b^2} \frac{G_{yz}}{t' \delta} \rho_{mn} + \left[\frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)\right]^2 + \frac{m^2}{a^2} \frac{n^2}{b^2} \frac{1-\mu}{2} c^2 \rho_{mn}^2$$

$$H_{4,5} = \frac{m^2}{a^2} \frac{1-\mu}{2} c \rho_{mn}$$

$$H_{4,6} = \frac{n^2}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)$$

$$H_{5,1} = t' \frac{c}{\pi^2} \frac{E_c}{G_{xz}}$$

$$H_{5,2} = 0$$

$$H_{5,3} = 0$$

$$H_{5,4} = t'c\theta$$

$$H_{5,5} = \frac{b^2}{n^2} \frac{m^2}{a^2} (t + t')$$

$$H_{5,6} = 0$$

$$H_{6,1} = t' \theta_{mn} (- \frac{c^3}{12})$$

$$H_{6,2} = t' \theta_{mn} (\frac{c^2 + ct'}{2})$$

$$H_{6,3} = \theta_{mn} (\frac{(t')^2 - t^2}{2}) + t'c \theta_{mn}$$

$$H_{6,4} = t'c \theta_{mn}$$

$$H_{6,5} = \frac{b^2}{n^2} \frac{m^2}{a^2} (t + t')$$

$$H_{6,6} = (t + t')$$

5. Appendix E -- Determinant Elements for Case $\alpha = 0$, $E_c = \infty$

The elements of the determinant in equation (180) may be found by taking the limit of the determinant in Appendix D as $E_c \longrightarrow \infty$. This is done by multiplying row 1, row 2, and column 1 by the factor $\frac{1}{E_c}$ and then taking the limit as $E_c \longrightarrow \infty$. Thus, the elements of the determinant in equation (180) are as follows:

$$J_{1,1} = \frac{1}{G_{xz}} \frac{a^2}{m^2 \pi^2} + \frac{t'\delta c^2}{G_{xz}^2 \pi^2} \left(1 + \frac{n^2}{b^2} \frac{a^2}{m^2} \frac{1-\mu}{2}\right)$$

$$J_{1,2} = \frac{c}{2} \left[1 - \frac{ct'\delta}{G_{xz}} \theta_{mn} (c + t')\right]$$

$$J_{1,3} = - \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{n^2}{b^2} - \frac{t'\delta c}{G_{xz}} \left[\theta_{mn} \frac{t'}{2} + \rho_{mn}\, c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \frac{1-\mu}{2}\right)\right]$$

$$J_{1,4} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{n^2}{b^2} - t'\delta \frac{n^2}{b^2} \frac{c^2}{G_{xz}} (1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{mn})$$

$$J_{1,5} = - \frac{t'\delta c}{G_{xz}} \frac{n^2}{b^2} \frac{1+\mu}{2}$$

$$J_{1,6} = - \frac{t'\delta c}{G_{xz}} (\frac{m^2}{a^2} + \frac{n^2}{b^2} \frac{1-\mu}{2})$$

$$J_{2,1} = 0, \qquad J_{2,2} = 1, \qquad J_{2,3} = J_{2,4} = J_{2,5} = J_{2,6} = 0$$

$$J_{3,1} = - \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{n^2}{b^2} - \frac{t'\delta c}{G_{xz}} \left[\theta_{mn} \frac{t'}{2} + \rho_{mn}\, c\, (\frac{m^2}{a^2} + \frac{n^2}{b^2} \frac{1-\mu}{2}) \right]$$

$J_{3,2} = H_{3,2}$ (that is, no change from element $H_{3,2}$ in Appendix D)

$$J_{3,3} = H_{3,3}, \qquad J_{3,4} = H_{3,4}, \qquad J_{3,5} = H_{3,5}, \qquad J_{3,6} = H_{3,6}$$

$$J_{4,1} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{n^2}{b^2} - t'\delta \frac{n^2}{b^2} \frac{c^2}{G_{xz}} (1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{mn})$$

$J_{4,2} = H_{4,2}$, $J_{4,3} = H_{4,3}$, $J_{4,4} = H_{4,4}$,

$J_{4,5} = H_{4,5}$, $J_{4,6} = H_{4,6}$

$$J_{5,1} = - \frac{t'\delta c}{G_{xz}} \frac{n^2}{b^2} \frac{1+\mu}{2}$$

$J_{5,2} = H_{5,2}$, $J_{5,3} = H_{5,3}$, $J_{5,4} = H_{5,4}$,

$J_{5,5} = H_{5,5}$, $J_{5,6} = H_{5,6}$

$$J_{6,1} = - \frac{t'\delta c}{G_{xz}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \frac{1-\mu}{2}\right)$$

$J_{6,2} = H_{6,2}$, $J_{6,3} = H_{6,3}$, $J_{6,4} = H_{6,4}$,

$J_{6,5} = H_{6,5}$, $J_{6,6} = H_{6,6}$

6. Appendix F -- Determinant Elements for Case $a = 2$, $E_c = \infty$

The elements of the determinant in equation (194) may be found by a scheme similar to that used in Appendix E, but it is more convenient to deduce these elements from a study of the determinant in Appendix E together with equations (173) through (178). In any event, the elements of the determinant in equation (194) are listed in order as follows:

All unlisted elements, $K_{p,q}$, have a value of zero.

$$K_{1,1} = \frac{1}{G_{xz}} \frac{a^2}{m^2 \pi^2} + \frac{t'\delta c^2}{G_{xz}^2 \pi^2} \left(1 + \frac{1}{b^2} \frac{a^2}{m^2} \frac{1-\mu}{2}\right)$$

$$K_{1,4} = \frac{c}{2} \left[1 - \frac{ct'\delta}{G_{xz}} \theta_{m1} (c + t')\right]$$

$$K_{1,7} = -\frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{1}{b^2} - \frac{t'\delta c}{G_{xz}} \left[\theta_{m1} \frac{t'}{2} + \rho_{m1} \varsigma \left(\frac{m^2}{a^2} + \frac{1}{b^2} \frac{1-\mu}{2}\right)\right]$$

$$K_{1,10} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{1}{b^2} - t'\delta \frac{1}{b^2} \frac{c^2}{G_{xz}} \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m1}\right)$$

$$K_{1,13} = -\frac{t'\delta c}{G_{xz}} \frac{1}{b^2} \frac{1+\mu}{2}$$

$$K_{1,16} = - \frac{t'\delta c}{G_{xz}} \left(\frac{m^2}{a^2} + \frac{1}{b^2} \frac{1-\mu}{2} \right)$$

$$K_{2,2} = \frac{1}{G_{xz}} \frac{a^2}{m^2 \pi^2} + \frac{t'\delta c^2}{G_{xz}^2 \pi^2} \left(1 + \frac{4}{b^2} \frac{a^2}{m^2} \frac{1-\mu}{2} \right)$$

$$K_{2,5} = \frac{c}{2} \left[1 - \frac{ct'\delta}{G_{xz}} \theta_{m2} (c + t') \right]$$

$$K_{2,8} = - \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{4}{b^2} - \frac{t'\delta c}{G_{xz}} \left[\theta_{m2} \frac{t'}{2} + \rho_{m2}\, c \left(\frac{m^2}{a^2} + \frac{4}{b^2} \frac{1-\mu}{2} \right) \right]$$

$$K_{2,11} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{4}{b^2} - t'\delta \frac{4}{b^2} \frac{c^2}{G_{xz}} \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m2} \right)$$

$$K_{2,14} = - \frac{t'\delta c}{G_{xz}} \frac{4}{b^2} \frac{1+\mu}{2}$$

$$K_{2,17} = - \frac{t'\delta c}{G_{xz}} \left(\frac{m^2}{a^2} + \frac{4}{b^2} \frac{1-\mu}{2} \right)$$

$$K_{3,3} = \frac{1}{G_{xz}} \frac{a^2}{m^2 \pi^2} + \frac{t'\delta c^2}{G_{xz}^2 \pi^2} \left(1 + \frac{9}{b^2} \frac{a^2}{m^2} \frac{1-\mu}{2} \right)$$

$$K_{3,6} = \frac{c}{2}\left[1 - \frac{ct'\delta}{G_{xz}}\,\theta_{m3}\,(c + t')\right]$$

$$K_{3,9} = -\frac{G_{yz}}{G_{xz}}\,\frac{a^2}{m^2}\,\frac{9}{b^2} - \frac{t'\delta c}{G_{xz}}\left[\theta_{m3}\,\frac{t'}{2} + \rho_{m3}\,c\,\left(\frac{m^2}{a^2} + \frac{9}{b^2}\,\frac{1-\mu}{2}\right)\right]$$

$$K_{3,12} = \frac{G_{yz}}{G_{xz}}\,\frac{a^2}{m^2}\,\frac{9}{b^2} - t'\delta\,\frac{9}{b^2}\,\frac{c^2}{G_{xz}}\left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2}\,\rho_{m3}\right)$$

$$K_{3,15} = -\frac{t'\delta c}{G_{xz}}\,\frac{9}{b^2}\,\frac{1+\mu}{2}$$

$$K_{3,18} = -\frac{t'\delta c}{G_{xz}}\left(\frac{m^2}{a^2} + \frac{9}{b^2}\,\frac{1-\mu}{2}\right)$$

$K_{4,4} = 1, \qquad K_{5,5} = 1, \qquad K_{6,6} = 1$

$$K_{7,1} = -\frac{G_{yz}}{G_{xz}}\,\frac{a^2}{m^2}\,\frac{1}{b^2} - \frac{t'\delta c}{G_{xz}}\left[\theta_{m1}\,\frac{t'}{2} + \rho_{m1}\,c\,\left(\frac{m^2}{a^2} + \frac{1}{b^2}\,\frac{1-\mu}{2}\right)\right]$$

$$K_{7,4} = \frac{c\delta\pi^2}{12}\,\theta_{m1}^2\,(t')^3 + t'\delta\pi^2\left[\theta_{m1}\left(\frac{c^2 + ct'}{2}\right)\right]\left[\theta_{m1}\,\frac{t'}{2} + \frac{m^2}{a^2}\,\rho_{m1}\,c\right]$$

$$K_{7,5} = -\frac{16}{9} N_o \, a \, \frac{t'}{t + t'} \, \frac{m^2}{a^2}$$

$$K_{7,7} = \frac{\pi^2}{b^2} G_{yz} \, \rho_{m1} + \delta\pi^2 \theta^2_{m1} \left[\frac{t^3}{3} + \frac{(t')^3}{12} \right] + t' \delta\pi^2 \left\{ \left[\theta_{m1} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m1} \, c \right]^2 + \frac{m^2}{a^2} \frac{1}{b^2} c^2 \frac{1-\mu}{2} \rho^2_{m1} \right\}$$

$$K_{7,8} = -\frac{16}{9} \frac{N_o \, a}{c} \frac{m^2}{a^2}$$

$$K_{7,10} = -\frac{\pi^2}{b^2} G_{yz} \, \rho_{m1} + t' \delta\pi^2 \left[\frac{1}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \left(\theta_{m1} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m1} \, c\right) - \frac{m^2}{a^2} \frac{1}{b^2} \frac{1-\mu}{2} c^2 \rho^2_{m1} \right]$$

$$K_{7,13} = -\frac{t^2}{2} \delta\pi^2 \frac{1}{b^2} \theta_{m1} + t' \delta\pi^2 \left[\frac{1}{b^2} \left(\theta_{m1} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m1} \, c\right) - \frac{m^2}{a^2} \frac{1}{b^2} \frac{1-\mu}{2} c \, \rho_{m1} \right]$$

$$K_{7,16} = -\frac{t^2}{2}\delta\pi^2 \frac{m^2}{a^2}\theta_{m1} + t'\delta\pi^2 \left[\frac{m^2}{a^2}\left(\theta_{m1}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m1}\, c\right) + \frac{m^2}{a^2}\frac{1}{b^2}\frac{1-\mu}{2} c\, \rho_{m1}\right]$$

$$K_{8,2} = -\frac{G_{yz}}{G_{xz}}\frac{a^2}{m^2}\frac{4}{b^2} - \frac{t'\delta c}{G_{xz}}\left[\theta_{m2}\frac{t'}{2} + \rho_{m2}\, c\left(\frac{m^2}{a^2} + \frac{4}{b^2}\frac{1-\mu}{2}\right)\right]$$

$$K_{8,4} = -\frac{16}{9}\frac{N_o a t'}{t+t'}\frac{m^2}{a^2}$$

$$K_{8,5} = \frac{c\delta\pi^2}{12}\theta^2_{m2}(t')^3 + t'\delta\pi^2\theta_{m2}\left(\frac{c^2+ct'}{2}\right)\left(\theta_{m2}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m2}\, c\right)$$

$$K_{8,6} = -\frac{48}{25}N_o a\frac{t'}{t+t'}\frac{m^2}{a^2}$$

$$K_{8,7} = -\frac{16}{9}\frac{N_o a}{c}\frac{m^2}{a^2}$$

$$K_{8,8} = \frac{4\pi^2}{b^2} G_{yz}\, \rho_{m2} + \delta\pi^2 \theta^2_{m2} \left[\frac{t^3}{3} + \frac{(t')^3}{12} \right] + t'\,\delta\pi^2 \left[\left(\theta_{m2} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m2}\, c\right)^2 + \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c^2 \rho^2_{m2} \right]$$

$$K_{8,9} = -\frac{48}{25} \frac{N_o a}{c} \frac{m^2}{a^2}$$

$$K_{8,11} = -\frac{4\pi^2}{b^2} G_{yz}\, \rho_{m2} + t'\,\delta\pi^2 \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \left(\theta_{m2} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m2}\, c\right) - \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c^2 \rho^2_{m2} \right]$$

$$K_{8,14} = -\frac{t^2}{2} \delta\pi^2 \frac{4}{b^2} \theta_{m2} + t'\delta\pi^2 \left[\frac{4}{b^2} \left(\theta_{m2} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m2}\, c\right) - \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c\, \rho_{m2} \right]$$

$$K_{8,17} = -\frac{t^2}{2}\delta\pi^2\frac{m^2}{a^2}\theta_{m2} + t'\delta\pi^2\left[\frac{m^2}{a^2}\left(\theta_{m2}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m2}\,c\right)\right.$$

$$\left. + \frac{m^2}{a^2}\frac{4}{b^2}\frac{1-\mu}{2}\,c\,\rho_{m2}\right]$$

$$K_{9,3} = -\frac{G_{yz}}{G_{xz}}\frac{a^2}{m^2}\frac{9}{b^2} - \frac{t'\delta c}{G_{xz}}\left[\theta_{m3}\frac{t'}{2} + \rho_{m3}\,c\left(\frac{m^2}{a^2} + \frac{9}{b^2}\frac{1-\mu}{2}\right)\right]$$

$$K_{9,5} = -\frac{48}{25}N_o\alpha\frac{t'}{t+t'}\frac{m^2}{a^2}$$

$$K_{9,6} = \frac{c\delta\pi^2}{12}\theta_{m3}^2(t')^3 + t'\delta\pi^2\theta_{m3}\left(\frac{c^2+ct'}{2}\right)\left(\theta_{m3}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m3}\,c\right)$$

$$K_{9,8} = -\frac{48}{25}\frac{N_o\alpha}{c}\frac{m^2}{a^2}$$

$$K_{9,9} = \frac{9\pi^2}{b^2} G_{yz}\, \rho_{m3} + \delta\pi^2 \theta_{m3}^2 \left[\frac{t^3}{3} + \frac{(t')^3}{12}\right] + t'\,\delta\pi^2 \left[\left(\theta_{m3} \frac{t'}{2}\right.\right.$$

$$\left.\left. + \frac{m^2}{a^2} \rho_{m3}\, c\right)^2 + \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c^2 \rho_{m3}^2\right]$$

$$K_{9,12} = -\frac{9\pi^2}{b^2} G_{yz}\, \rho_{m3} + t'\delta\pi^2 \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \left(\theta_{m3} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m3}\, c\right)\right.$$

$$\left. - \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c^2 \rho_{m3}^2\right]$$

$$K_{9,15} = -\frac{t^2}{2} \delta\pi^2 \frac{9}{b^2} \theta_{m3} + t'\delta\pi^2 \left[\frac{9}{b^2} \left(\theta_{m3} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m3}\, c\right)\right.$$

$$\left. - \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c\, \rho_{m3}\right]$$

$$K_{9,18} = -\frac{t^2}{2} \delta\pi^2 \frac{m^2}{a^2} \theta_{m3} + t'\delta\pi^2 \left[\frac{m^2}{a^2} \left(\theta_{m3} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m3}\, c\right)\right.$$

$$\left. + \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c\, \rho_{m3}\right]$$

$$K_{10,1} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{1}{b^2} - t'\delta \frac{1}{b^2} \frac{c^2}{G_{xz}} (1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m1})$$

$$K_{10,4} = t'\delta\pi^2 \left[\theta_{m1} (\frac{c^2 + ct'}{2})\right] \left[\frac{1}{b^2} c (1 - \frac{G_{yz}}{G_{xz}})\right]$$

$$K_{10,7} = -\frac{\pi^2}{b^2} G_{yz} \rho_{m1} + t'\delta\pi^2 \left\{ \left[\theta_{m1} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m1} c\right] \left[\frac{1}{b^2} c (1 - \frac{G_{yz}}{G_{xz}})\right] - \frac{m^2}{a^2} \frac{1}{b^2} \frac{1-\mu}{2} c^2 \rho_{m1}^2 \right\}$$

$$K_{10,10} = \frac{\pi^2}{b^2} G_{yz} \rho_{m1} + t'\delta\pi^2 \left\{ \left[\frac{1}{b^2} c (1 - \frac{G_{yz}}{G_{xz}})\right]^2 + \frac{m^2}{a^2} \frac{1}{b^2} \frac{1-\mu}{2} c^2 \rho_{m1}^2 \right\}$$

$$K_{10,13} = t'\delta\pi^2 \left\{ \frac{1}{b^2} \left[\frac{1}{b^2} c (1 - \frac{G_{yz}}{G_{xz}})\right] + \frac{m^2}{a^2} \frac{1}{b^2} \frac{1-\mu}{2} c \rho_{m1} \right\}$$

$$K_{10,16} = t' \delta \pi^2 \frac{m^2}{a^2} \frac{1}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m1}\right)$$

$$K_{11,2} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{4}{b^2} - t'\delta \frac{4}{b^2} \frac{c^2}{G_{xz}} \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m2}\right)$$

$$K_{11,5} = t'\delta\pi^2 \left[\theta_{m2} \left(\frac{c^2 + ct'}{2}\right)\right] \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)\right]$$

$$K_{11,8} = - \frac{4\pi^2}{b^2} G_{yz} \rho_{m2} + t'\delta\pi^2 \left\{ \left[\theta_{m2} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m2} c\right] \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)\right] - \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c^2 \rho_{m2}^2 \right\}$$

$$K_{11,11} = \frac{4\pi^2}{b^2} G_{yz} \rho_{m2} + t'\delta\pi^2 \left\{ \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right)\right]^2 + \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c^2 \rho_{m2}^2 \right\}$$

$$K_{11,14} = t'\delta\pi^2 \left\{ \frac{4}{b^2} \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right] + \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c \rho_{m2} \right\}$$

$$K_{11,17} = t'\delta\pi^2 \frac{m^2}{a^2} \frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m2}\right)$$

$$K_{12,3} = \frac{G_{yz}}{G_{xz}} \frac{a^2}{m^2} \frac{9}{b^2} - t'\delta \frac{9}{b^2} \frac{c^2}{G_{xz}} \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m3}\right)$$

$$K_{12,6} = t'\delta\pi^2 \left[\theta_{m3} \left(\frac{c^2 + ct'}{2}\right) \right] \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right]$$

$$K_{12,9} = - \frac{9\pi^2}{b^2} G_{yz} \rho_{m3} + t'\delta\pi^2 \left\{ \left[\theta_{m3} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m3} c \right] \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right] - \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c^2 \rho_{m3}^2 \right\}$$

$$K_{12,12} = \frac{9\pi^2}{b^2} G_{yz} \rho_{m3} + t' \delta \pi^2 \left\{ \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right]^2 + \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c^2 \rho^2_{m3} \right\}$$

$$K_{12,15} = t' \delta \pi^2 \left\{ \frac{9}{b^2} \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right] + \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c \rho_{m3} \right\}$$

$$K_{12,18} = t' \delta \pi^2 \frac{m^2}{a^2} \frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}} - \frac{1-\mu}{2} \rho_{m3}\right)$$

$$K_{13,1} = - \frac{t' \delta c}{G_{xz}} \frac{1}{b^2} \frac{1+\mu}{2}$$

$$K_{13,4} = t' \delta \pi^2 \left[\theta_{m1} \left(\frac{c^2 + ct'}{2} \right) \right] \frac{1}{b^2}$$

$$K_{13,7} = -\frac{t^2\delta\pi^2}{2}\,\theta_{m1}\,\frac{1}{b^2} + t'\delta\pi^2\left\{\left[\theta_{m1}\,\frac{t'}{2} + \frac{m^2}{a^2}\,\rho_{m1}\,c\right]\frac{1}{b^2} - \frac{m^2}{a^2}\,\frac{1}{b^2}\,\frac{1-\mu}{2}\,c\,\rho_{m1}\right\}$$

$$K_{13,10} = t'\delta\pi^2\left\{\left[\frac{1}{b^2}\,c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\right]\frac{1}{b^2} + \frac{m^2}{a^2}\,\frac{1}{b^2}\,\frac{1-\mu}{2}\,c\,\rho_{m1}\right\}$$

$$K_{13,13} = \delta\pi^2\,\frac{1}{b^2}\left(\frac{1}{b^2} + \frac{m^2}{a^2}\,\frac{1-\mu}{2}\right)(t + t')$$

$$K_{13,16} = \delta\pi^2\,\frac{m^2}{a^2}\,\frac{1}{b^2}\,\frac{1+\mu}{2}\,(t + t')$$

$$K_{14,2} = -\frac{t'\delta c}{G_{xz}}\,\frac{4}{b^2}\,\frac{1+\mu}{2}$$

$$K_{14,5} = t'\delta\pi^2\,\theta_{m2}\left(\frac{c^2 + ct'}{2}\right)\frac{4}{b^2}$$

$$K_{14,8} = -\frac{t^2 \delta \pi^2}{2} \theta_{m2} \frac{4}{b^2} + t' \delta \pi^2 \left[\left(\theta_{m2} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m2} c \right) \frac{4}{b^2} - \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c \rho_{m2} \right]$$

$$K_{14,11} = t' \delta \pi^2 \left\{ \left[\frac{4}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \right] \frac{4}{b^2} + \frac{m^2}{a^2} \frac{4}{b^2} \frac{1-\mu}{2} c \rho_{m2} \right\}$$

$$K_{14,14} = \delta \pi^2 \frac{4}{b^2} \left(\frac{4}{b^2} + \frac{m^2}{a^2} \frac{1-\mu}{2} \right) (t + t')$$

$$K_{14,17} = \delta \pi^2 \frac{m^2}{a^2} \frac{4}{b^2} \frac{1+\mu}{2} (t + t')$$

$$K_{15,3} = -\frac{t' \delta c}{G_{xz}} \frac{9}{b^2} \frac{1+\mu}{2}$$

$$K_{15,6} = t' \delta \pi^2 \theta_{m3} \left(\frac{c^2 + ct'}{2} \right) \frac{9}{b^2}$$

$$K_{15,9} = -\frac{t^2\delta\pi^2}{2}\theta_{m3}\frac{9}{b^2} + t'\delta\pi^2\left[(\theta_{m3}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m3}c)\frac{9}{b^2}\right.$$

$$\left. - \frac{m^2}{a^2}\frac{9}{b^2}\frac{1-\mu}{2}c\,\rho_{m3}\right]$$

$$K_{15,12} = t'\delta\pi^2\left\{\left[\frac{9}{b^2}c\,(1 - \frac{G_{yz}}{G_{xz}})\right]\frac{9}{b^2} + \frac{m^2}{a^2}\frac{9}{b^2}\frac{1-\mu}{2}c\,\rho_{m3}\right\}$$

$$K_{15,15} = \delta\pi^2\frac{9}{b^2}(\frac{9}{b^2} + \frac{m^2}{a^2}\frac{1-\mu}{2})(t + t')$$

$$K_{15,18} = \delta\pi^2\frac{m^2}{a^2}\frac{9}{b^2}\frac{1+\mu}{2}(t + t')$$

$$K_{16,1} = -\frac{t'\delta c}{G_{xz}}(\frac{m^2}{a^2} + \frac{1}{b^2}\frac{1-\mu}{2})$$

$$K_{16,4} = t'\delta\pi^2\theta_{m1}(\frac{c^2 + ct'}{2})\frac{m^2}{a^2}$$

$$K_{16,7} = -\frac{t^2\delta\pi^2}{2}\,\theta_{m1}\,\frac{m^2}{a^2} + t'\delta\pi^2\left[\left(\theta_{m1}\,\frac{t'}{2} + \frac{m^2}{a^2}\,\rho_{m1}\,c\right)\frac{m^2}{a^2}\right.$$

$$\left. + \frac{m^2}{a^2}\,\frac{1}{b^2}\,\frac{1-\mu}{2}\,c\,\rho_{m1}\right]$$

$$K_{16,10} = t'\delta\pi^2\left[\frac{1}{b^2}\,c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\frac{m^2}{a^2} - \frac{m^2}{a^2}\,\frac{n^2}{b^2}\,\frac{1-\mu}{2}\,c\,\rho_{m1}\right]$$

$$K_{16,13} = \delta\pi^2\,\frac{m^2}{a^2}\,\frac{1}{b^2}\,\frac{1+\mu}{2}\,(t + t')$$

$$K_{16,16} = \delta\pi^2\,\frac{m^2}{a^2}\left(\frac{m^2}{a^2} + \frac{1}{b^2}\,\frac{1-\mu}{2}\right)(t + t')$$

$$K_{17,2} = -\frac{t'\delta c}{G_{xz}}\left(\frac{m^2}{a^2} + \frac{4}{b^2}\,\frac{1-\mu}{2}\right)$$

$$K_{17,5} = t'\delta\pi^2\theta_{m2}\left(\frac{c^2 + ct'}{2}\right)\frac{m^2}{a^2}$$

$$K_{17,8} = -\frac{t^2}{2}\delta\pi^2\theta_{m2}\frac{m^2}{a^2} + t'\delta\pi^2\left\{\left[\theta_{m2}\frac{t'}{2} + \frac{m^2}{a^2}\rho_{m2}c\right]\frac{m^2}{a^2} + \frac{m^2}{a^2}\frac{4}{b^2}\frac{1-\mu}{2}c\,\rho_{m2}\right\}$$

$$K_{17,11} = t'\delta\pi^2\left[\frac{4}{b^2}c\left(1 - \frac{G_{yz}}{G_{xz}}\right)\frac{m^2}{a^2} - \frac{m^2}{a^2}\frac{4}{b^2}\frac{1-\mu}{2}c\,\rho_{m2}\right]$$

$$K_{17,14} = \delta\pi^2\frac{m^2}{a^2}\frac{4}{b^2}\frac{1+\mu}{2}(t + t')$$

$$K_{17,17} = \delta\pi^2\frac{m^2}{a^2}\left(\frac{m^2}{a^2} + \frac{4}{b^2}\frac{1-\mu}{2}\right)(t + t')$$

$$K_{18,3} = -\frac{t'\delta c}{G_{xz}}\left(\frac{m^2}{a^2} + \frac{9}{b^2}\frac{1-\mu}{2}\right)$$

$$K_{18,6} = t'\delta\pi^2\theta_{m3}\left(\frac{c^2 + ct'}{2}\right)\frac{m^2}{a^2}$$

$$K_{18,9} = - \frac{t'^2 \delta \pi^2}{2} \theta_{m3} \frac{m^2}{a^2} + t' \delta \pi^2 \left[\left(\theta_{m3} \frac{t'}{2} + \frac{m^2}{a^2} \rho_{m3} c \right) \frac{m^2}{a^2} \right.$$

$$\left. + \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c \rho_{m3} \right]$$

$$K_{18,12} = t' \delta \pi^2 \left[\frac{9}{b^2} c \left(1 - \frac{G_{yz}}{G_{xz}}\right) \frac{m^2}{a^2} - \frac{m^2}{a^2} \frac{9}{b^2} \frac{1-\mu}{2} c \rho_{m3} \right]$$

$$K_{18,15} = \delta \pi^2 \frac{m^2}{a^2} \frac{9}{b^2} \frac{1+\mu}{2} (t + t')$$

$$K_{18,18} = \delta \pi^2 \frac{m^2}{a^2} \left(\frac{m^2}{a^2} + \frac{9}{b^2} \frac{1-\mu}{2} \right) (t + t')$$

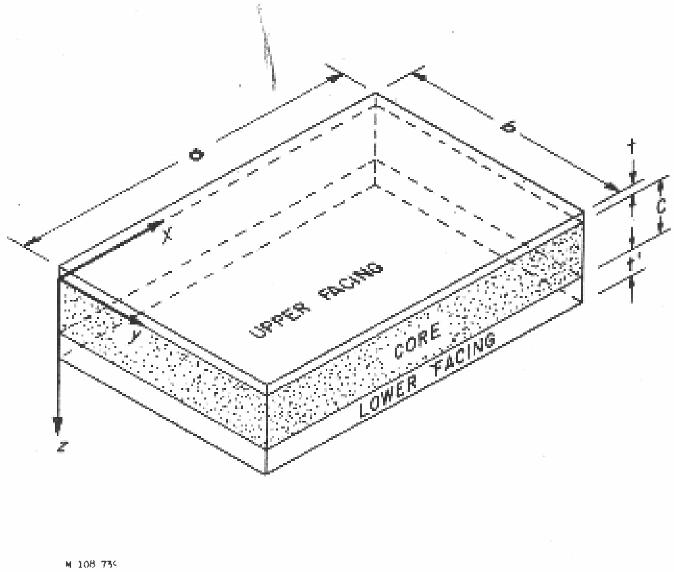


Figure 1. --Isometric drawing of a sandwich panel.

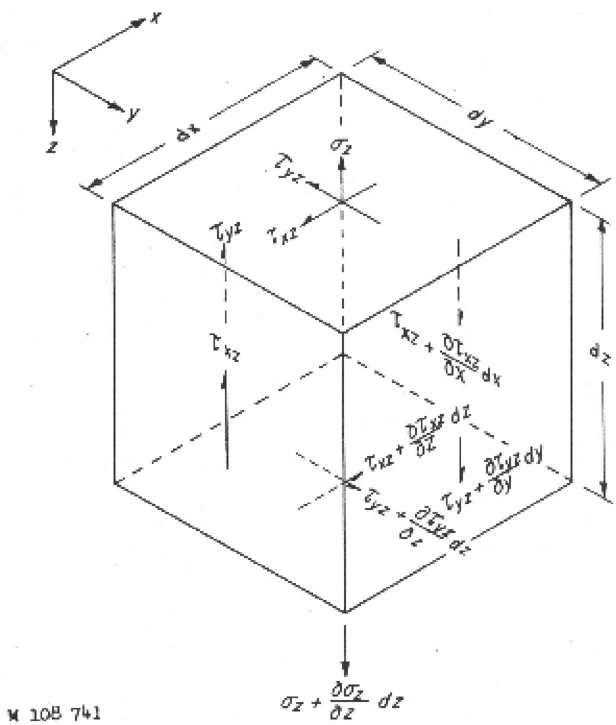


Figure 2. --Differential element of the core.

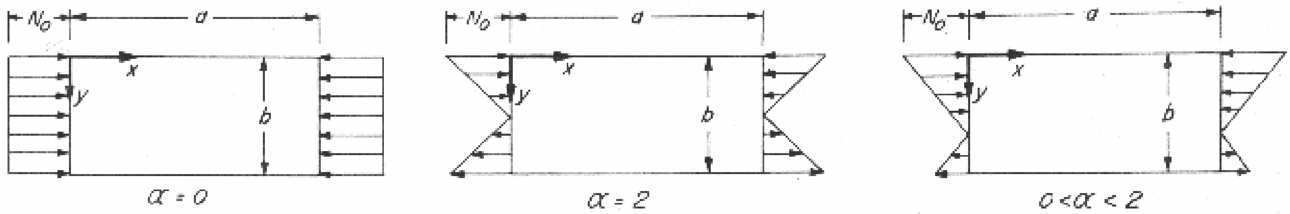
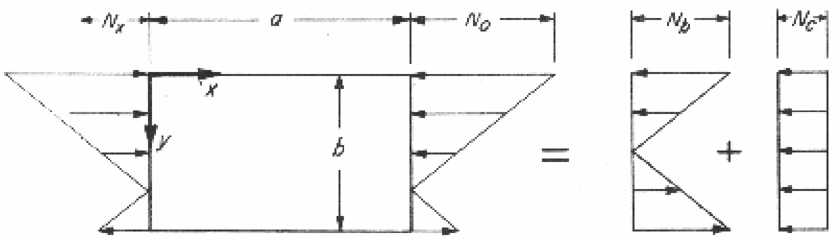


Figure 3. --Top view of sandwich panel showing different combinations of edgewise bending and compression as defined by α. N_o is the load in pounds per inch of sandwich.

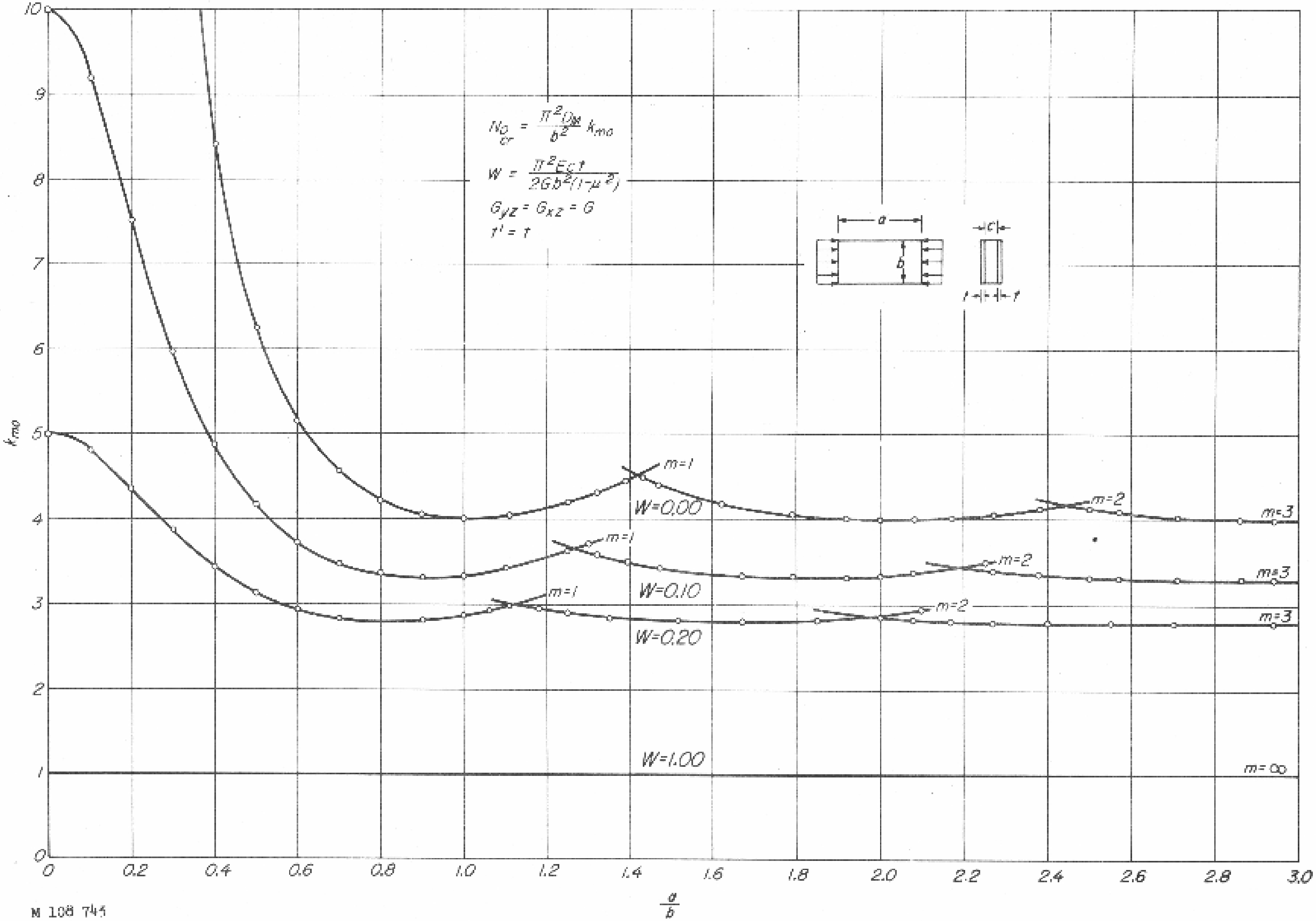


$$\alpha = \frac{2N_b}{N_b + N_c} = \frac{2N_b}{N_o}$$

$$N_o = N_b + N_c$$

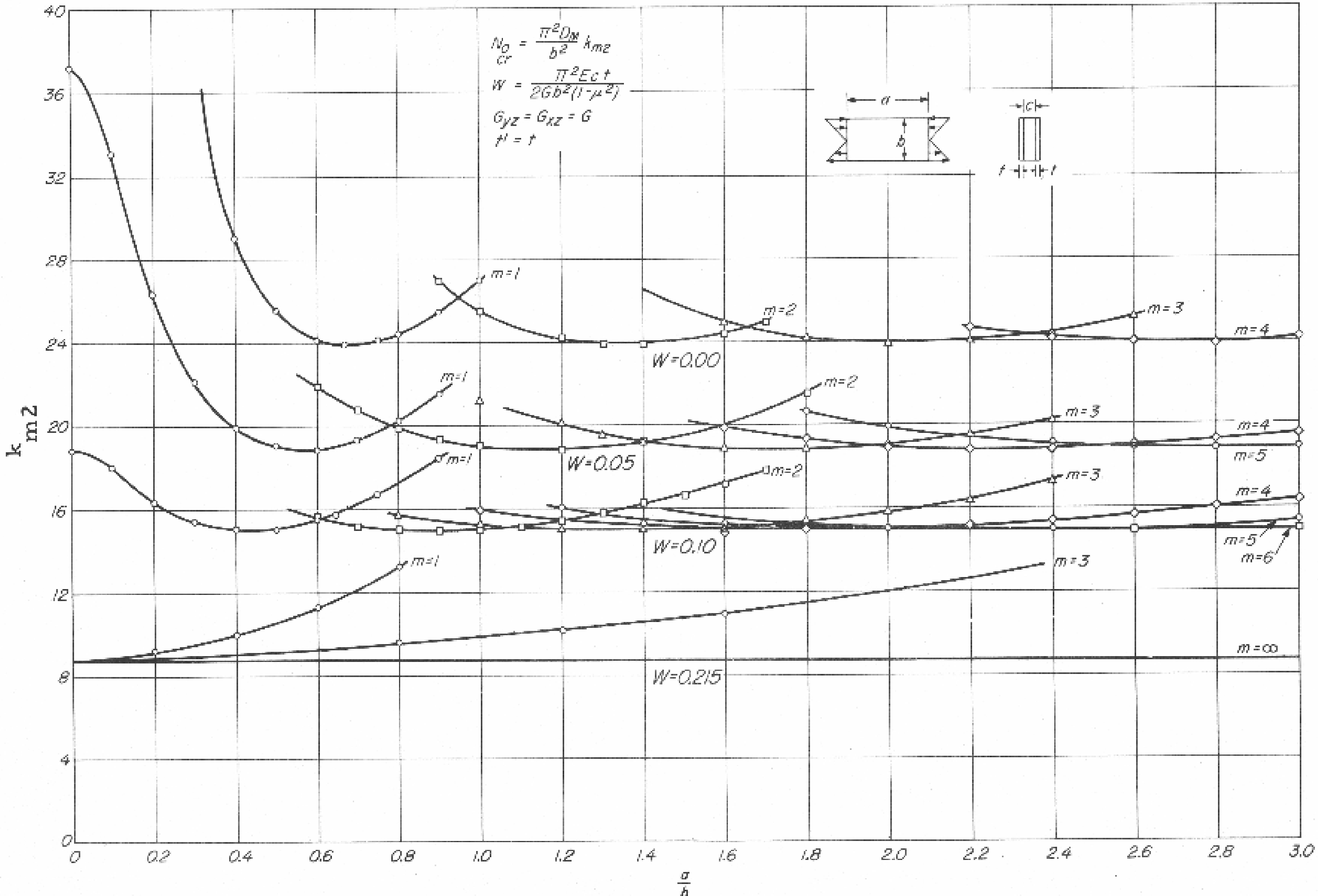
M 109 506

Figure 4. --Pictorialized definition of α, N_o, N_b, N_c, and N_x.



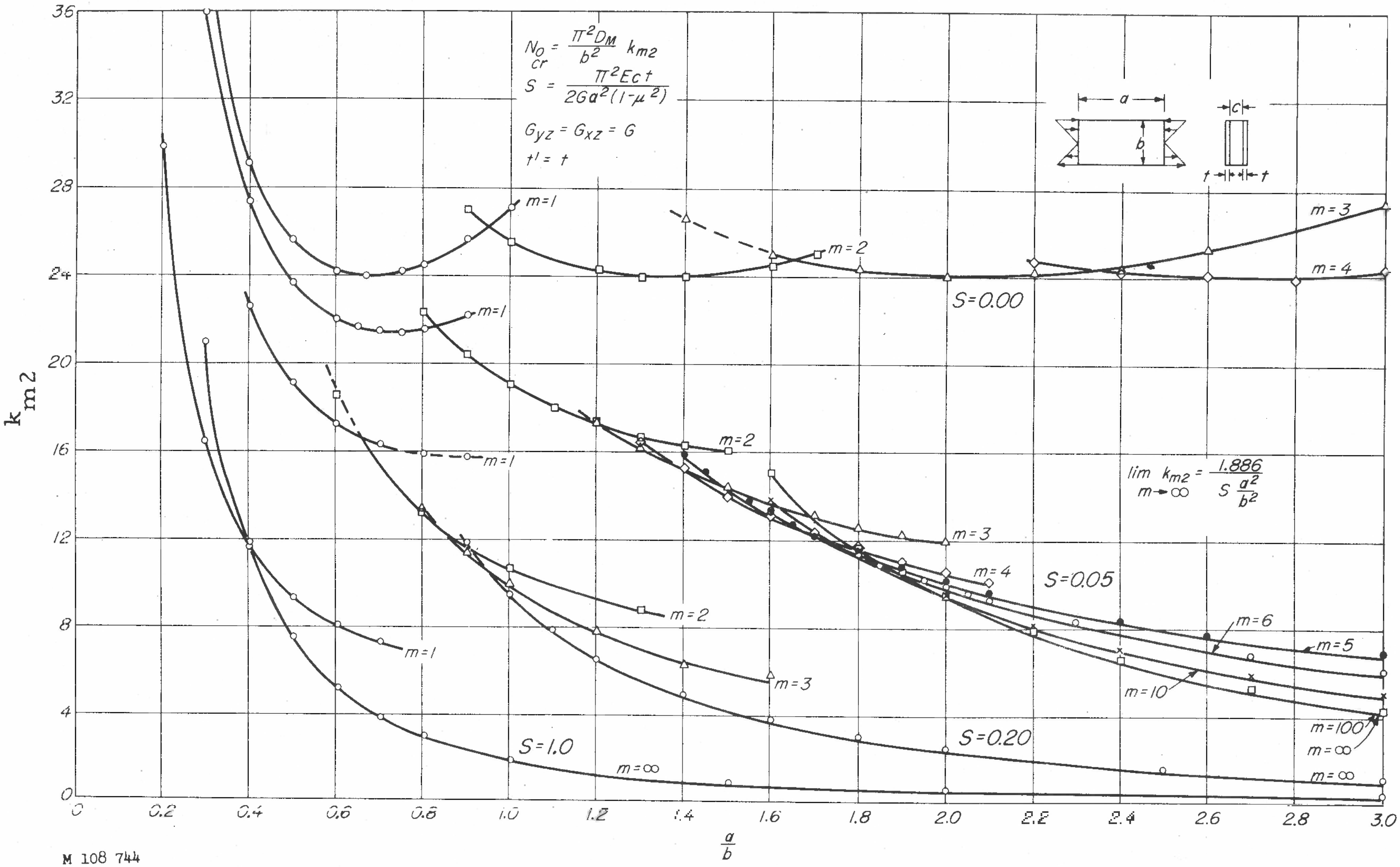
M 108 741

Figure 5. --Critical load factor k_{m0} versus $\frac{a}{b}$ for α = 0. (Pure edgewise compression.) Families of curves designated by parameter **W**.



M 108 742

Figure 6. --Critical load factor k_{m2} versus $\frac{a}{b}$ for $\underline{a} = 2$. (Pure edgewise bending.) Families of curves designated by parameter $\underline{W}$.



M 108 744

Figure 7. --Critical load factor k_{m2} versus $\frac{a}{b}$ for $\underline{a}$ = 2. (Pure edgewise bending.) Families of curves designated by parameter $\underline{S}$.