OPTIMAL HARVESTING TIME IN FISH FARMING WITH HETEROGENOUS POPULATION

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ABSTRACT

The optimal harvesting time in a fish farm is analyzed in the paper. Fish population is assumed to be heterogeneous with respect to weight, so a distributed parameter dynamic model is considered. Theoretical and numerical results are obtained and compared with the ones concerning homogeneous fish cultures only. The results are applied to the tilapia farming in Mexico for which empirical and market data were obtained. The actual managerial practices turn out to be close to the optimal solution in the model where weight-heterogeneity of the culture is taken into account.

Keywords: aquaculture, heterogeneous population, harvesting time, size-distributed

INTRODUCTION

The production activity in farming concerns with the growth of a livestock since an inmature age to commercial size. This growth is influencing by physiological (metabolism, appetite), exogenous (temperature, stress) and management factors (density, ration, feed frequency). The combination of these elements in the farm constitutes the firm technology.

In particular, industrial fish farming has shown a very significant increase in last decades. Accordingly, many bioeconomic models have been elaborated in order to optimize several relevant management aspects, as the harvesting time, ration size or density, among others (Bjorndal 1988, Cacho 1990, Arnason 1992). These previous models generally assume a homogeneous growth of fish at the same cage. However, variability in sizes inside the same pond or cage is a common phenomenon observed in both extensive and intensive culture. The causes are manifolds, being those related with social behaviour of each species, as hierarchy of competition for food some of the most relevant (Brett 1979, Olla et al. 1997, Irwin et al. 1999, McLean et al. 2000).

The size heterogeneity is an important issue the firms need to regard. The intensive culture is usually conducted with high density levels in order to reach economic efficiency, causing lower dissolved oxygen in cages and stress in fish. This fact carries on usually a decrease in growth and increase in mortality rate that affects mainly to smaller sizes (Via et al. 1998, Bjornsson 1994). As the culture management in farms is influenced by this factor, some practices as optimal harvesting time or ration size can be modified if growth variability is considered.

This paper analyzes the optimal harvesting time in a fish farm where size heterogeneity is included. We make use of a continuous size-structured population model similar to the one proposed by Sinko and Streifer (1967). Although size structured models have been extensively applied to fishery populations (Clark 1985, Horbowy 1996, Basson and Fogarty 1997), the interest of researchers has not been focused on farming. An exception is found in Forsberg (1999), which analyzes the optimal harvesting in an age-distributed model, but assuming linear relationships among variables, as the growth rate. The model we propose includes a logistic growth of fish. We also present a numerical application of the model for culture of tilapia in Yucatán, Mexico, in order to compare the optimal results with the real practices.

OPTIMAL HARVESTING TIME IN A HOMOGENEOUS POPULATION

In this section we revise the theory of the optimal harvesting time so far. More details and references concerning can be found in Bjorndal (1988). So, it is assumed that all juvenile fishes are stocked at the same initial time t = 0 into a cage (pond or farm), all of them with the same weight and with the same growth law. Let the speed of growth at size x be g(x). Then the size of each fish at time t, denoted by x(t), satisfies the equation

$$x(t) = g(x), \quad x(0) = s_{0,}$$

where s_0 is the initial size of the fish. The function g includes all factors affecting growth, such as the body weight, ration size, temperature, etc. For the sake of simplicity, we explicitly incorporate in the growth law only the weight, considering the rest of the factors as exogenous and constant.

Several expressions for g are proposed in the specialized literature, such as logistic, von Bertalanffy or Chapman-Richards function for fish farming, etc. All of them result in an S-shaped curve for the individual weight across time, and the most appropriate ones vary across species and type of culture (Gamito 1998). Generally, we assume that g is a continuously differentiable function defined in the interval [0,w], where w > 0 is the maximal possible weight that a fish can achieve. Correspondingly, we assume that g(0) = 0 and g(w) = 0 and that g is positive in (0,w).

Let N_0 be the number of fish in the cage at the beginning of the culture, t = 0. This number is not maintained all the time due to mortality with a size-dependent rate m(x)≥0. We assume that m is a continuous non-negative function. Thus the evolution of the number of fish, N(t), is given by the differential equation

$$\dot{N}(t) = -m(x)N(t), \quad N(0) = N_{0,}$$

The product B(t) = x(t) N(t) represents the total biomass in the cage at time t. The function p(x) represents the price per kilo of a fish with weight x. It is assumed nonnegative and continuously differentiable.

We consider also the accumulated maintenance costs, C(t), given by

$$C(t) = \int_0^t e^{-r\tau} f(x(\tau))N(\tau)d\tau,$$

where f(x) is the cost of maintaining a fish of size x for one unit of time. This cost may have size-independent component (fixed costs per fish) and size-dependent component (e.g. feed costs per fish, which depends on the size). We assume that f(0)≥0 and f is an increasing continuously differentiable function. The parameter r represents the discount rate.

We consider only one culture cycle, so the rotation problem is not included (although it is easy to incorporate). Thus, the farmer problem is to determine the time when the present value of the revenue obtained by harvesting all the biomass is maximum, that is,

$$\max_t \left\{ \pi(t) = e^{-rt} p(x)B(t) - C(t) \right\}$$

Differentiating in t and using the differential equations for x and N we obtain that the necessary optimality condition becomes

$$p'(x)\, g(x)\, x + p(x)\, g(x) = (r + m(x))\, p(x)\, x + f(x).$$

The expression in the left-hand side represents the marginal revenue from maintaining an additional fish a unit of time growing. The revenue increases due to price appreciation of weight and due to weight increase itself. The expression on the right is the cost of opportunity represented by the discount rate, loss of value due to mortality, and instantaneous maintenance costs.

OPTIMAL HARVESTING TIME IN A HETEROGENEOUS POPULATION

In this section we relax the assumption of identical weight for all individuals, that is, different sizes grow at the same time in the cage. However, we maintain no differences in fish growth across sizes, so that the initial weights determine the posterior heterogeneity in the cage.

Let, as above, N_0 be the initial size of the population, and $n_0(x)$, x in [0,w] be the initial (probability) density of the population. The dynamics of the population is described by the following size-structured model, in which N(t,x) is the number of individuals of size x at time t:

$$N_t(t,x) + (g(x)N(t,x))_x = -m(x)N(t,x), \quad N(0,x) = N_0 n_0(x), \quad 0 < x < w, t > 0, \qquad (1)$$

where m(x)≥q is the mortality rate, g(x) is the growth velocity at size x, as before. The initial (probability) density n_0 is supposed continuous, which is not necessary, but avoids some technicalities connected with the meaning of the solution of equation above and some of the formulas involved.

The above model was proposed by Sinko and Streifer (1967) and Bell and Anderson (1967). Numerical approaches are proposed in Ito et al. (1991) and Angulo and López-Marcos (1999).

The farmer problem is again to optimize the harvesting time in order to obtain the maximum net revenue of the product. Considering the price, p(x), as dependent on size, the value of the biomass in cage is determined by

$$V(t) = \int_0^w p(x)xN(t,x)dx.$$

The accumulated maintenance cost till time t is

$$C(t) = \int_0^t e^{-r\tau} \int_0^w f(x)N(\tau,x)dxd\tau.$$

The farmer maximizes the current value of the net revenue:

$$\max_t \left\{\Pi(t) = e^{-rt}V(t) - C(t)\right\}$$

Differentiating in t, using equation (1) and having in mind that g(w)=0, then the necessary optimality condition $\dot{\Pi}(t) = 0$ becomes

$$\int_0^w \left[\mathrm{p'(x)\,g(x)\,x + p(x)\,g(x) - \ (r + m(x))\,p(x)\,x - f(x)}\right] N(t,x)dx = 0. \qquad (2)$$

One can give similar economic interpretation of the terms in the above expression as in the homogeneous case. Similarly as in the homogeneous case, equation (2) may have more than one solution (see Proposition 1 (i) below).

In order to compare the optimal harvesting times in the homogeneous and in the heterogeneous model, we assume the same growth function g, the same m, r, p and initial population size N_0, and with the initial weight in the homogeneous model equal to the mean weight in the heterogeneous model. That is, the homogeneous model is just a simplification of the heterogeneous one, obtained by assuming that all fish have initially the same weight equal to the mean one.

Namely, we assume that the initial density n_0 is concentrated in the interval [s_0-ε,s_0+ ε] in [0,w] and the mean value is s_0:

$$\int_0^w n_0(s)ds = \int_{s_0-\varepsilon}^{s_0+\varepsilon} n_0(s)ds = 1, \qquad \int_{s_0-\varepsilon}^{s_0+\varepsilon} s n_0(s)ds = s_0. \qquad (3)$$

We present the main result of our paper. Firstly, in addition to the assumptions in section 2, we include that m and p are independent of the size, the functions g and h(x):= p g(x) - f(x) are twice continuously differentiable and strongly concave (that is, the second derivative is strictly negative). Moreover, we assume that h has a positive value for some x in [0,w].

Proposition 1. *There exist numbers $\rho>0$ and $\varepsilon_0>0$ such that for every ε in $(0, \varepsilon_0]$ and non-negative r, m and function f(x) satisfying the above conditions and the inequality $r + m + f(x) \leq \rho$, and for every probability density n0 satisfying (3), equation (2) has exactly two non-negative solutions, larger one equal to the optimal harvesting time t^H. Moreover, the optimal time in the heterogeneous model is bigger than that in the homogeneous model, so $t^H > t^h$.*

We avoid in this version the proof of the proposition. The essence of the proposition is that if the mortality rate, the discount rate, and the maintenance cost are sufficiently small, the homogeneous model tends to underestimate the optimal harvesting time. This finding is supported by the numerical results obtained in the next section for real fish farms. The assumptions for g and f fulfilled for the particular model presented in the next section. We mention, however, that for some (unrealistically) large discount rate, mortality rate, or maintenance costs it may happen that $t^H<t^h$.

A NUMERICAL EXAMPLE

We present in this section a numerical case study of the effect of heterogeneity over the optimal harvesting time. The example was extracted from the real commercial culture of tilapia in Yucatan, Mexico. Table 1 shows the most representative economic and biological data of the culture, obtained from the market and literature. The daily maintenance costs f(x) is defined by fixed and feed costs. Capital depreciation is not initially considered, so only daily labor cost is included in the fixed cost. The feed costs represent over 40% of the total operating cost in a fish farm, and are determined by the conversion rate f_0, which indicates the amount of food necessary to increase the fish weight by one gram. So, function $f(x)=c_0+c_f f_0 g(x)$, where c_f is the feed cost per gram.

Table 1. Data of commercial culture of tilapia in Yucatan, Mexico.

Parameter	Description	Value	Source
m	Daily mortality rate	0.00058	Empirical estimation
p(400)	Price per g. for 400g. fish	$0.002	Local market
p(500)	Price per g. for 500g. fish	$0.003	Local market
cf	Feed cost per g.	$0.0005	Local market
f0	Conversion rate	1.7	Empirical estimation
c0	Fixed cost per day and individual	$0.001107	Local market
N0	Initial number of individuals	2474	Local firm

The growth function g(x) was estimated from empirical data by means of an experiment conducted from April to December 2003. The fish was cultured in four tanks with different fish weights inside each of them. Firstly, we analyzed the growth for the fish with maximum and minimum weight in each tank, having no significant differences in two tanks. We take the most representative one of these two tanks to calibrate the growth model for the tilapia. Assuming four different growth equations (Chapman-Richards, von Bertalanffy, Gompertz and logistic), the logistic equation was selected to replicate the experimental data. Thus, the fish growth model follows the expression $g(x)=ax-bx^2$, where parameters a=0.0182 and b=0.000214 were statistically estimated. The estimated asymptotic weight is w=850 g.

We assume mortality as constant and the initial distribution of fish follows a beta function, that is,

$$N_0(x) = \frac{N_0}{x_1 - x_0} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x - x_0}{x_1 - x_0} \right)^{\alpha-1} \left(1 - \frac{x - x_0}{x_1 - x_0} \right)^{\beta-1}$$

where Γ is the gamma function and N_0 is the total number of the individuals at time t = 0. The culture starts from the minimum (x_0=18.2 g.) and maximum (x_1=47.4 g.) weight found at the initial time in the tank. The mean and variance of the distribution are assumed s_0=20 g. and σ^2=20 respectively, implying figures of ε=17.4, α=3.7445 and β=5.5216. We take also the mean value s_0=30 g. as the initial weight for the homogeneous model.

The optimal harvesting time was calculated for both homogeneous and heterogeneous case, and results for different variances of the initial distribution are shown in Table 2. The market data reflect a fish price oscillating between values of \$2 and 3 per kg. We present in the optimal time calculation these two constant prices and a linear increasing function p(x)=0.00001(x-200), with $0<x\leq w$, which assumes a continuous commercial appreciation for fish bigger than 200 g.

Table 2. Optimal harvesting time for the homogeneous and heterogeneous model for different variances σ^2 of the initial distribution. Three pattern of prices per x g. of fish are included, with $0<x\leq w$.

Fish price	σ^2=0 (homogeneous case)	σ^2=20	σ^2=30	σ^2=40
\$0.002	290	290	291	295
\$0.003	299	299	301	305
$\$10^{-5}$(x-200)	377	377	380	383

The results in Table 2 confirm Proposition 1 of the previous section, that is, the optimal harvesting time is larger if size heterogeneity is considered. This rule is satisfied not only for constant fish prices, but also for a size-dependent price function. However, the gap between optimal times is lower than one week for the highest values of the variance, what does not seem very relevant in the commercial culture. Moreover, the recommended harvesting times around 300 days drive to fish of size 760g., clearly above the real managerial practices of tilapia culture. Farmer tends to take the fish after six or seven months after the stocking, obtaining a fish weight around 400 grams. Other effects influencing the managerial decisions can explain this fact, as the environmental factors (Pascoe et al. 2001). It appears that farmers adopt a precautionary response to the risk inherent to an economic activity related with the environment, what is not captured by the model.

CONCLUSIONS

The optimal harvesting time theory for fish farming developed so far in the literature assumes that the population in a culture cycle has the same weight at any time, so only one representative individual is necessary in the analysis. In this paper, we extend the theory considering heterogeneity of weight in the same culture area. We employ a size-structured population model to describe the evolution of the individuals in a farm and compare the optimal harvesting time with the previous ones obtained in the literature. The results indicate that fish should be maintained longer in the tank or cage if size heterogeneity is taking into account.

The theoretical results were tested numerically with real culture data of tilapia in Yucatan, Mexico. Fish growth and costs data were estimated and two models were designed, one assuming identical homogeneous growth for all individuals in the tank and the other with heterogeneous growth. The numerical figures confirm the theory, but the gap between optimal harvesting times for both homogeneous and heterogeneous growth is not very significant in practical terms. However, this gap is very dependent on the specific growth function, so more relevant consequences over the optimal managerial strategy may be observed in the culture of other species.

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