NOTES AND CORRESPONDENCE

An Improved Bound for the Complex Phase Speed of Baroclinic Instability

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ABSTRACT

An improved bound is obtained for the radius of the semicircle in the complex plane containing the complex phase speed of baroclinically unstable plane wave disturbances. In the limit of long waves, this bound contains a term increasing with β and decreasing with the mean stratification (i.e., decreasing with the baroclinic Rossby radius of deformation). An extension of the bound, valid for finite wavelengths longer than order $(\delta u/\beta)^{1/2}$, where δu is half the range of velocities in the mean shear flow, is also obtained.

1. Introduction

Baroclinic instability is an important mechanism by which fluctuations in the oceans or atmosphere of large enough scale to be in approximate geostrophic balance can spontaneously grow at the expense of a vertically sheared mean flow. Case studies of specific shear flows, solved by analytical or numerical means, have furnished wavenumber ranges in which instability occurs and obtained the dependence of growth rate on wavenumber and physical environmental parameters. A more general approach has been to provide theoretical limits to growth rates of instabilities from consideration of integrated positive-definite properties of geophysical shear flows. One line of this approach has furnished the so-called semicircle bounds on the complex phase speeds of unstable disturbances. Such bounds are of important practical interest because of the limits they place on the growth of baroclinic mesoscale, and longer-scale, eddy variability in the oceans. The semicircle bounds also provide a guide to the degree of modification by the shear of stable modes that lie outside the semicircle (Killworth et al. 1997).

Howard (1961) established a remarkable theorem about plane-wave instabilities in a stratified parallel shear flow. It states that the complex phase speed of an unstable disturbance lies within a semicircle on the complex plane centered on a speed halfway between an upper and a lower bound of the range of velocities that spans the shear flow, and with a radius equal to half the difference between these bounds (Fig. 1). Pedlosky (1963, 1964) extended this theorem to cover quasigeostrophic baroclinic disturbances in rotating, stratified geophysical flows on a β plane. The effect of the earth's curvature is to extend the radius of instability of the semicircle by an amount that depends on the phase speed of a barotropic Rossby wave of the wavenumber under consideration (Fig. 1). In the limit of very long waves (low wavenumber), the barotropic Rossby wave phase speed is unbounded so that the extended semicircle radius is likewise unbounded. This limitation on the complex phase speed (and growth rate) of a baroclinic instability is of very little utility. Cavallini et al. (1988) obtained a bounding semicircle for the complex phase speeds of long-wave disturbances in zonal flows on the sphere (or β plane); it has twice the radius of the Howard semicircle and is centered on the minimum mean flow (Fig. 1).

The stratification of the mean flow appears nowhere in the semicircle radius bounds. This is curious as theoretical case studies of simple shear flows show that instability requires a minimum shear increasing with the β parameter and a measure of the stratification—the baroclinic Rossby radius of deformation—and that growth rate varies with the excess of shear over this critical threshold (Pedlosky 1987). Hence the question arises whether a more severe bound, perhaps depending on stratification, can be placed on the radius of the unstable semicircle. In this paper, we shall establish such a bound. The method follows very closely the development of Howard (1961) and Pedlosky (1964). In section 2 we state the governing equations for baroclinic disturbances in rotating stratified flow and derive the constraint determined by Pedlosky (1964) for the complex phase speed. In section 3 we treat the simpler case of the long-wave limit. The resulting semicircle bound in this case resembles Pedlosky's in form although the barotropic Rossby wave speed dependence of the semicircle bound-

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ary radius is replaced by the long-wave limit of the baroclinic Rossby wave speed. The latter is of course a much smaller bound than the former. Our treatment corrects a flawed derivation of a similar bound by Colin de Verdière (1986). In section 4 the result is extended to shorter wavelengths, but longer than order $(\delta u/\beta)^{1/2}$, where δu is half the range of the velocity profile, at which limit the Pedlosky bound becomes comparable.

2. Quasigeostrophic equations; semicircle theorem

The equation for a small quasigeostrophic pressure disturbance of the form $\mathrm{Re}[\phi(z)e^{ik(x-ct)}]$ on a zonal stratified shear flow $u(z)$ is

$$(u - c)\left[\partial_z\left(\frac{f^2}{N^2}\partial_z\phi\right) - k^2\phi\right] + \left[\beta - \partial_z\left(\frac{f^2}{N^2}\partial_z u\right)\right]\phi = 0, \tag{1}$$

with boundary conditions

$$(u - c)\partial_z\phi - (\partial_z u)\phi \quad \text{at} \quad z = 0, -H \tag{2}$$

(Pedlosky 1987). Here x, z are zonal and vertical coordinates; k is the zonal wavenumber and $c = c_r + ic_i$ is the complex phase speed of the disturbance; $N(z)$ is the buoyancy frequency; $\beta = \partial_y f$ is the rate of variation of Coriolis parameter f: in Eq. (1), however, the latter is considered constant. An alternate form of these equations can be obtained by substituting

$$\eta = \phi/(u - c). \tag{3}$$

Equations (1), (2) then become

$$\partial_z\left[(u - c)^2\frac{f^2}{N^2}\partial_z\eta\right] - k^2(u - c)^2\eta + \beta(u - c)\eta = 0, \tag{4}$$

$$\partial_z\eta = 0 \quad \text{at } z = 0, -H. \tag{5}$$

By multiplying Eq. (4) by η^*, the complex conjugate of η, integrating from $-H$ to 0, and using (5), one obtains

$$\int_{-H}^{0} (u - c)^2\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz = \beta\int_{-H}^{0} (u - c)|\eta|^2\, dz. \tag{6}$$

Taking the imaginary part of this, if $c_i \neq 0$,

$$2\int_{-H}^{0} (u - c_r)\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz = \beta\int_{-H}^{0} |\eta|^2\, dz; \tag{7}$$

and the real part,

$$\int_{-H}^{0} \{(u - c_r)^2 - c_i^2\}\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz = \beta\int_{-H}^{0} (u - c_r)|\eta|^2\, dz. \tag{8}$$

Next, substituting (7) into (8),

$$\int_{-H}^{0} u^2\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz = (c_r^2 + c_i^2)\int_{-H}^{0}\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz + \beta\int_{-H}^{0} u|\eta|^2\, dz. \tag{9}$$

Equation (7) may be rewritten as

$$c_r = \frac{\displaystyle\int_{-H}^{0} u\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz - \frac{\beta}{2}\int_{-H}^{0} |\eta|^2\, dz}{\displaystyle\int_{-H}^{0}\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz}. \tag{10}$$

From this it follows that

$$c_r \leq \max(u(z)) \leq u_u, \tag{11}$$

where u_u is an upper bound on the velocity $u(z)$. If $c_i = 0$, (11) follows from (8), rather than (7), by observing that the left side of the equation is positive. The bound u_u is marked in Fig. 1.

If u_u, u_l are upper and lower bounds on the velocities of the mean shear flow $u(z)$, then

$$0 \geq \int_{-H}^{0} dz(u - u_u)(u - u_l)\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right). \tag{12}$$

By multiplying out the integrand, and using (9) and (7), one sees that

$$0 \geq \left[\left|c - \frac{u_u + u_l}{2}\right|^2 - \left(\frac{u_u - u_l}{2}\right)^2\right] \times \int_{-H}^{0}\left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz + \beta\int_{-H}^{0}\left(u - \frac{u_u + u_l}{2}\right)|\eta|^2\, dz. \tag{13}$$

Because $u \geq u_l$, one deduces that

$$\left|c - \frac{u_u + u_l}{2}\right|^2 \leq \delta u^2 + \beta\delta u L, \tag{14}$$

where $\delta u = (u_u - u_l)/2$ and

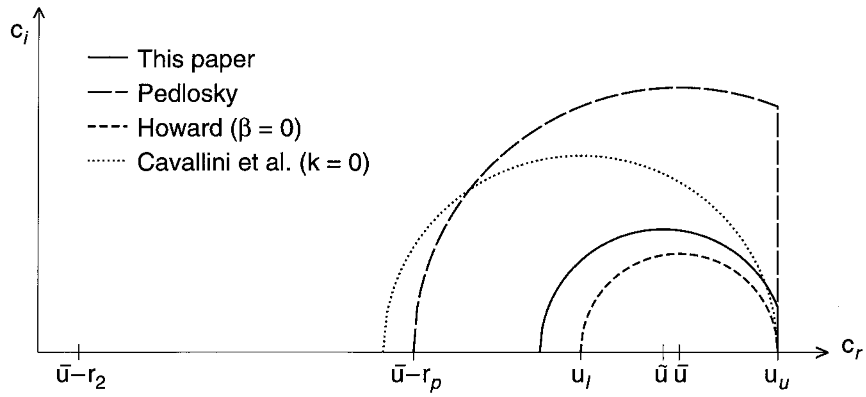

FIG. 1. The Howard (1961) semicircle, encompassing all possible complex phase speeds for parallel shear flow on a flat plane; the Pedlosky (1964) semicircle for disturbances of zonal wavenumber k on the β plane [the example shown is for $k = 0.4(\beta/\delta u)^{1/2}$]; Cavallini et al.'s (1988) semicircle bounding complex phase speeds for long-wave disturbances on zonal flow on the sphere (or β plane); and the bounding semicircle given by this paper (the example shown is for $\lambda_1\delta u/\beta = 100$). On the abscissa the point $\overline{u} - r_2$ is marked (defined in section 4). The distance from this point to the Pedlosky semicircle is $r_2 - r_p$.

$$L = \frac{\displaystyle\int_{-H}^{0} |\eta|^2\, dz}{\displaystyle\int_{-H}^{0} \left(\frac{f^2}{N^2}|\partial_z\eta|^2 + k^2|\eta|^2\right) dz}. \qquad (15)$$

Because the integrand in the denominator of (15) is $\geq k^2|\eta|^2$, it follows that

$$L \leq 1/k^2. \qquad (16)$$

Then it follows from (14) that

$$\left|c - \frac{u_u + u_l}{2}\right|^2 \leq \delta u^2 + \frac{\beta}{k^2}\delta u. \qquad (17)$$

This is the Pedlosky bound for the radius of a semicircle in the complex plane, centered on $(u_u + u_l)/2$, in which c must be contained. It is shown in Fig. 1. [A sector of the semicircle for $c_r > u_u$ may be amputated on account of condition (11). Still, we shall continue to refer to the domain of permitted instability as a "semicircle."] This bound is unsatisfactory in the important long-wave limit as $k \to 0$ (Killworth et al. 1997). In the following sections we shall obtain a stricter bound for L than (16), valid for wavenumbers $k \lesssim (\beta/\delta u)^{1/2}$.

We emphasize that u_u and u_l appearing in inequalities (11)–(14), and (17), may be any upper and lower bounds on $u(z)$, not necessarily the least upper bound and greatest lower bound. This fact will be useful in the sequel.

By defining a vertical average,

$$\overline{\eta} = H^{-1}\int_{-H}^{0} \eta\, dz, \qquad (18)$$

and a deviation from this average,

$$\eta' = \eta - \overline{\eta}, \qquad (19)$$

we may write (15) as

$$L = \frac{|\overline{\eta}|^2 + \overline{|\eta'|^2}}{k^2|\overline{\eta}|^2 + k^2\overline{|\eta'|^2} + H^{-1}\displaystyle\int_{-H}^{0} \frac{f^2}{N^2}|\partial_z\eta'|^2\, dz}. \qquad (20)$$

Colin de Verdière (1986) pointed out that η' may be expanded in eigenfunctions of the following Sturm–Liouville problem:

$$-\partial_z \frac{f^2}{N^2}\partial_z\psi = \lambda\psi, \qquad (21a)$$

$$\partial_z\psi = 0 \quad \text{at } z = 0, -H. \qquad (21b)$$

Denote these eigenfunctions and their eigenvalues by $\psi_n(z)$ and λ_n for $n = 1, 2, \cdots$. It is straightforward to show that the eigenfunctions are orthogonal and can be normalized,

$$H^{-1}\int_{-H}^{0} \psi_m^*\psi_n\, dz = \delta_{mn}, \qquad (22)$$

and have zero vertical average,

$$\overline{\psi_n} = 0. \qquad (23)$$

The eigenvalues are real and strictly positive and can be ordered in magnitude:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots. \qquad (24)$$

Equations (21) pose the classic baroclinic Rossby wave mode problem, obtained by setting $u(z) \equiv 0$ in Eqs. (1) and (2). Hence the eigenvalues λ_n are the inverse squares of the baroclinic radii of deformation for classic Rossby waves (Chelton et al. 1997). [For the special case where N is a constant, the eigenfunctions and eigenvalues are

$$\psi_n = 2^{1/2}\cos(n\pi z/H), \tag{25}$$

$$\lambda_n = n^2\pi^2 f^2/N^2H^2.] \tag{26}$$

Using the eigenfunctions ψ_n as a basis set, η' may be written as an absolutely convergent series,

$$\eta' = \sum_{n=1}^{\infty} a_n\psi_n. \tag{27}$$

Hence

$$\overline{|\eta'|^2} = \sum_{n=1}^{\infty} |a_n|^2, \tag{28}$$

and

$$H^{-1}\int_{-H}^{0}\frac{f^2}{N^2}|\partial_z\eta'|^2\,dz = \sum_{n=1}^{\infty}\lambda_n|a_n|^2 \geq \lambda_1\overline{|\eta'|^2}. \tag{29}$$

The inequality in (29) follows from the ordering of the eigenvalues (24). It must be emphasized that the smallest eigenvalue λ_1 is strictly positive. In Eqs. (21), $\lambda = 0$ if and only if $\psi =$ const. This eigenfunction is eliminated by the requirement (23). It is not needed to construct a representation of η'. The contribution of $\overline{\eta}$ to L will be bounded in a different way.

Substituting (29) into (20), it follows that

$$L \leq \frac{|\overline{\eta}|^2 + \overline{|\eta'|^2}}{k^2|\overline{\eta}|^2 + (k^2 + \lambda_1)\overline{|\eta'|^2}}, \tag{30}$$

or

$$L \leq [k^2 + \lambda_1(1 + |\overline{\eta}|^2/\overline{|\eta'|^2})^{-1}]^{-1}. \tag{31}$$

Colin de Verdière appears to neglect the ratio $|\overline{\eta}|^2/\overline{|\eta'|^2}$ in (31), a step that is not justified.

3. The long-wave limit

In this section we shall obtain a bound on the ratio $|\overline{\eta}|^2/\overline{|\eta'|^2}$ appearing in (31) for the simpler, though important, special case of long-wavelength disturbances, for which we set $k = 0$. In that event, by integrating (1) or (4) from $-H$ to 0, and using (2) or (5), one obtains

$$\beta\int_{-H}^{0}\phi\,dz = \beta\int_{-H}^{0}(u - c)\eta\,dz = 0. \tag{32}$$

Then, substituting (19) and $u(z) = \overline{u} + u'(z)$, into (32), this becomes

$$(c - \overline{u})\overline{\eta} = \overline{u'\eta'}.$$

Taking the absolute values of both sides of this, and applying Schwarz's inequality (Jeffreys and Jeffreys 1956) to the right side,

$$|c - \overline{u}|^2|\overline{\eta}|^2 \leq \overline{u'^2}\,\overline{|\eta'|^2}. \tag{33}$$

The vertical mean $\overline{u}$ must lie between the bounds u_l and u_u, and indeed these bounds may be chosen so that

$$\overline{u} = \frac{u_u + u_l}{2}. \tag{34}$$

Then it follows that

$$|u'| \leq \frac{1}{2}(u_u - u_l) = \delta u. \tag{35}$$

By introducing the parameter

$$\epsilon = (\overline{u'^2})^{1/2}/\delta u \leq 1, \tag{36}$$

(33) may be rewritten as

$$|\overline{\eta}|^2/\overline{|\eta'|^2} \leq \epsilon^2\delta u^2|c - \overline{u}|^{-2}. \tag{37}$$

Using this in (31), with $k = 0$, and then in (14), one obtains

$$|c - \overline{u}|^2 \leq \delta u^2 + \beta\lambda_1^{-1}\delta u\left(\frac{\epsilon^2\delta u^2}{|c - \overline{u}|^2} + 1\right). \tag{38}$$

Multiplication by $|c - \overline{u}|^2$ and factorization of the resulting quadratic gives

$$\left(\frac{|c - \overline{u}|^2}{\delta u^2} - \gamma_+\right)\left(\frac{|c - \overline{u}|^2}{\delta u^2} - \gamma_-\right) \leq 0, \tag{39}$$

where

$$\gamma_\pm = \frac{1}{2}(1 + \hat{\beta}) \pm \frac{1}{2}[(1 + \hat{\beta})^2 + 4\epsilon^2\hat{\beta}]^{1/2}, \tag{40a}$$

$$\hat{\beta} = \beta/\lambda_1\delta u. \tag{40b}$$

Now $\gamma_- < 0$, so the second factor in (39) is positive. Hence

$$|c - \overline{u}|^2 \leq \gamma_+\delta u^2. \tag{41}$$

This is our estimate for the bounding radius of the unstable semicircle for the long-wavelength limit. By writing the radicand in (40a), which is certainly positive, as

$$\{1 + (1 + 2\epsilon^2)\hat{\beta}\}^2 - 4\epsilon^2(1 + \epsilon^2)\hat{\beta}^2$$
$$= \{\hat{\beta} + 1 + 2\epsilon^2\}^2 - 4\epsilon^2(1 + \epsilon^2),$$

one sees that the radical in (40a) is bounded by either of the expressions in braces. Hence the positive root γ_+ is bounded above by

$$\gamma_+ < 1 + (1 + \epsilon^2)\hat{\beta} \quad \text{or} \quad \hat{\beta} + 1 + \epsilon^2, \tag{42}$$

whichever is smaller. Since $\epsilon \leq 1$, from (36), a less strict statement of (42) is

$$\gamma_+ < 1 + 2\hat{\beta} \quad \text{or} \quad \hat{\beta} + 2, \tag{42$'$}$$

whichever is smaller. Because ocean currents are usually concentrated near the surface, so that the ratio ϵ may be substantially smaller than 1, the stricter forms (42) may have considerable advantage over (42)′. Substituting (42) into (41), we see that

$$|c - \overline{u}|^2 \leq \delta u^2 + (1 + \epsilon^2)\beta\lambda_1^{-1}\delta u \quad \text{or}$$
$$(1 + \epsilon^2)\delta u^2 + \beta\lambda_1^{-1}\delta u, \tag{43}$$

whichever is smaller. One should recall that, for the purposes of (43), u_u and u_l may not be the supremum and infimum of the mean flow. These bounds resemble the Pedlosky bound in form, except that the long-wave baroclinic phase speed magnitude $\lambda_1^{-1}\beta$ appears, rather than the barotropic phase speed magnitude β/k^2, apart from factors of $(1 + \epsilon^2)$. For $k \sim 2\pi/(3000\text{ km})$, the barotropic speed is of order 20 m s^{-1}; while for $\lambda_1^{-1} \sim (50\text{ km})^2$, the baroclinic phase speed is about 10 cm s^{-1}. This more than hundredfold difference narrows the bounding radius of the unstable semicircle tremendously (Fig. 1).

4. Moderate wavelengths

In this section we relax the assumption that disturbance wavelengths are long and consider effects of finite, nonzero wavenumber k on the bounding radius of the unstable semicircle. We will obtain a semicircle bound, valid for k less than a finite value, and from which the $k = 0$ bound of section 3 can be recovered as a special case. This treatment thereby clarifies what is meant by the term "long wavelength." For waves longer than a finite threshold, which can be calculated, the new bound is smaller than the Pedlosky bound; for shorter waves, the latter is a better bound. An interesting feature of the derivation is that the Pedlosky bound is itself used to obtain the new bound.

The extension of the results (32) and (33) to nonzero k is somewhat involved. Integrating (4) from $-H$ to 0 and using (5),

$$\int_{-H}^{0} [\beta(u - c)\eta - k^2(u - c)^2\eta]\, dz = 0. \qquad (44)$$

Upon substituting (19), and $u = \overline{u} + u'$, this may be written

$$\left[\beta(\overline{u} - c) - k^2(\overline{u} - c)^2 - k^2\overline{u'^2}\right]\overline{\eta} = -[\beta - 2k^2(\overline{u} - c)]\overline{u'\eta'} + k^2\overline{u'^2\eta'}. \qquad (45)$$

Using Schwarz's inequality, one may obtain

$$|\overline{u'\eta'}| \le \epsilon\delta u\left(\overline{|\eta'|^2}\right)^{1/2}, \qquad (46)$$

where ϵ is defined by (36), and, because $|u'| \le \delta u$,

$$|\overline{u'^2\eta'}| \le \delta u^2\left(\overline{|\eta'|^2}\right)^{1/2}. \qquad (47)$$

(Note that ϵ does not appear in the latter inequality.) Hence, from (45),

$$|\overline{\eta}|/\left(\overline{|\eta'|^2}\right)^{1/2} \le \frac{|\beta/k^2 - 2(\overline{u} - c)|\epsilon\delta u + \delta u^2}{|(\overline{u} - c)\beta/k^2 - (\overline{u} - c)^2 - \epsilon^2\delta u^2|}. \qquad (48)$$

The denominator of (48) may be factored into

$$|c - \overline{u} + r_1|\,|c - \overline{u} + r_2|,$$

where

$$r_{1,2} = \delta u\left[\frac{1}{2\tilde{k}^2} \mp \left(\frac{1}{4\tilde{k}^4} - \epsilon^2\right)^{1/2}\right], \qquad \tilde{k} = k(\delta u/\beta)^{1/2}. \qquad (49a,b)$$

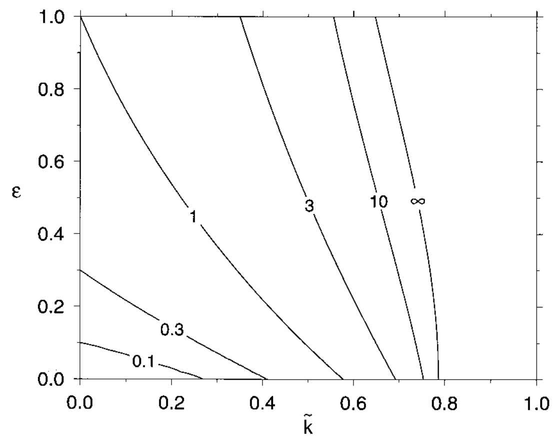

FIG. 2. The parameter M appearing in the bound (54) as a function of $\tilde{k}$ and ϵ. The contour marked ∞ describes $\tilde{k}_0(\epsilon)$, beyond which M is not defined.

The Pedlosky semicircle theorem applies to c. Hence, from (17),

$$|c - \overline{u}| \le r_p \equiv \delta u(1 + \tilde{k}^{-2})^{1/2}. \qquad (50)$$

Consider the conditions under which the quantity

$$r_2 - r_p = \delta u\left[\frac{1}{2\tilde{k}^2} + \left(\frac{1}{4\tilde{k}^4} - \epsilon^2\right)^{1/2} - \left(1 + \frac{1}{\tilde{k}^2}\right)^{1/2}\right] \qquad (51)$$

is strictly positive. This is so as long as

$$\tilde{k} < \tilde{k}_0(\epsilon), \qquad (52)$$

where $\tilde{k}_0(\epsilon)$, a function of ϵ, is shown in Fig. 2. While condition (52) is met, the following inequalities hold:

$$|c - \overline{u} + r_2| \ge r_2 - |c - \overline{u}| \ge r_2 - r_p > 0. \qquad (53)$$

This means that, as long as the scaled wavenumber $\tilde{k}$ is lower than $\tilde{k}_0(\epsilon)$, the complex phase speed c is no nearer $\overline{u} - r_2$ than the span $r_2 - r_p$ (Fig. 1). Then (48) may be bounded by

$$\frac{\overline{\eta}}{\left(\overline{|\eta'|^2}\right)^{1/2}} \le \frac{M\delta u}{|c - \overline{u} + r_1|}, \qquad (54)$$

where

$$M = \frac{(\tilde{k}^2 + \epsilon)\delta u + 2\epsilon r_p\tilde{k}^2}{(r_2 - r_p)\tilde{k}^2}. \qquad (55)$$

The latter is shown in Fig. 2; it is finite and positive as long as restriction (52) is fulfilled. For small $\tilde{k}$, it approaches ϵ.

It remains to introduce the inequality (54) into (31) and (14). Before doing so, we may recall that (14) holds for any choice of upper and lower bounds. In particular, we may choose bounds u_u^+ and u_l^+ so that

$$\frac{u_u^+ + u_l^+}{2} = \overline{u} - r_1 \equiv \tilde{u} \qquad (56)$$

[cf. (34)] and

$$\delta u^{+} = \frac{u_u^{+} - u_l^{+}}{2} = \delta u + r_1. \tag{57}$$

The parameter r_1 is given by the negative sign choice in (49a). It is easy to see that

$$r_1 = \frac{\delta u \epsilon^2}{\dfrac{1}{2\tilde{k}^2} + \left(\dfrac{1}{4\tilde{k}^4} - \epsilon^2\right)^{1/2}} \leq 2\delta u \tilde{k}^2 \epsilon^2. \tag{58}$$

For low $\tilde{k}$, at which the Pedlosky bound (50) becomes large compared to δu, the shift r_1 from $\overline{u}$ to $\tilde{u}$ and the increase from δu to δu^+ are only a small proportion of δu. In general, the restriction on the domain of $\tilde{k}$ given by (52) (see Fig. 2) guarantees that the right side of inequality (58) is $\leq 0.82\ \delta u$. Using u_u^+ and u_l^+ in (14), that inequality becomes, upon substituting from (31) and (54),

$$\begin{aligned} &|c - \tilde{u}|^2 \\ &\leq (\delta u^+)^2 + \beta \delta u^+ \left\{ k^2 + \lambda_1 \left[1 + \frac{M^2 (\delta u^+)^2}{|c - \tilde{u}|^2} \right]^{-1} \right\}^{-1} \\ &\leq (\delta u^+)^2 + \beta \delta u^+ \frac{|c - \tilde{u}|^2 + M^2(\delta u^+)^2}{(\lambda_1 + k^2)|c - \tilde{u}|^2 + k^2 M^2 (\delta u^+)^2}. \end{aligned} \tag{59}$$

Multiplying out the denominator on the right, one obtains an inequality for a quadratic form in $|c - \tilde{u}|^2$. This can be factored into

$$[|c - \tilde{u}|^2 - \tilde{\gamma}_+ (\delta u^+)^2][|c - \tilde{u}|^2 - \tilde{\gamma}_- (\delta u^+)^2] \leq 0, \tag{60}$$

where

$$\begin{aligned} \tilde{\gamma}_{\pm} = &\frac{1}{2}\left(1 + \frac{\dfrac{\beta}{\delta u^+} - k^2 M^2}{\lambda_1 + k^2} \right) \\ &\pm \frac{1}{2}\left[\left(1 + \frac{\dfrac{\beta}{\delta u^+} - k^2 M^2}{\lambda_1 + k^2} \right)^2 + 4M^2 \frac{k^2 + \dfrac{\beta}{\delta u^+}}{\lambda_1 + k^2} \right]^{1/2}. \end{aligned} \tag{61}$$

Now $\tilde{\gamma}_- < 0$ certainly, so the second factor in (60) is positive. Thus the first factor must be nonpositive; that is,

$$|c - \tilde{u}|^2 \leq \tilde{\gamma}_+ (\delta u^+)^2 \equiv \tilde{r}^2. \tag{62}$$

This bound on the radius of the semicircle containing c in the complex plane is the major result of this paper. Some less strict, but perhaps more informative, bounds can be obtained from it. Manipulation of the radicand in (61), which is certainly positive, shows that it can be written as the difference between positive quantities in three distinct ways:

$$\begin{aligned} &\left[1 + \tilde{\beta}(1 + 2M^2) + \frac{k^2 M^2}{\lambda_1 + k^2} \right]^2 - 4\tilde{\beta} M^2 (1 + M^2)\left(\tilde{\beta} + \frac{M^2}{\lambda_1 + k^2} \right) \\ &= \left[\tilde{\beta} - \frac{k^2 M^2}{\lambda_1 + k^2} + 1 + 2M^2 \right]^2 - \frac{4\lambda_1 M^2 (1 + M^2)}{\lambda_1 + k^2} \\ &= \left[\frac{k^2 M^2}{\lambda_1 + k^2} + 1 + \tilde{\beta} + 2\tilde{\beta}\frac{\lambda_1}{k^2} \right]^2 - \frac{4\beta\lambda_1}{k^2}\left(1 + \tilde{\beta} + \frac{\beta\lambda_1}{k^2} \right), \end{aligned} \tag{63a}$$

where

$$\tilde{\beta} = \frac{\beta}{\delta u^+ (\lambda_1 + k^2)}. \tag{63b}$$

Then it follows that the radical in (61) is smaller than any of the bracketed terms in (63). So the positive root of (61) can be bounded by

$$\tilde{\gamma}_+ \leq 1 + \tilde{\beta}(1 + M^2) \quad \text{or}$$
$$\frac{\lambda_1(1 + M^2) + k^2}{\lambda_1 + k^2} + \tilde{\beta} \quad \text{or} \quad 1 + \frac{\beta}{\delta u^+ k^2}, \tag{64}$$

whichever is smallest. The similarity of the first two of (64) to (42) is obvious. Indeed, when $k = 0$ so that $M = \epsilon$, $r_1 = 0$, $\delta u^+ = \delta u$, $\tilde{\beta} = \hat{\beta}$, the first two of (64) and (42) are identical. We see thereby that the long-wave bounds (42) are valid for $k^2 \ll \lambda_1$ as well as condition (52). But the expressions for the bounds (64) show that the former requirement is easily lifted. The sole, essential restriction on the expressions in these bounds is condition (52), that $k < (\beta/\delta u)^{1/2}\tilde{k}_0(\epsilon)$, so that M is defined. As the upper range of k is entered, however, M increases without limit and the first two bounds of (64) become much larger than the third, which does not depend on M. The third bound of (64) is a Pedlosky bound, of the type discussed in remarks following in-

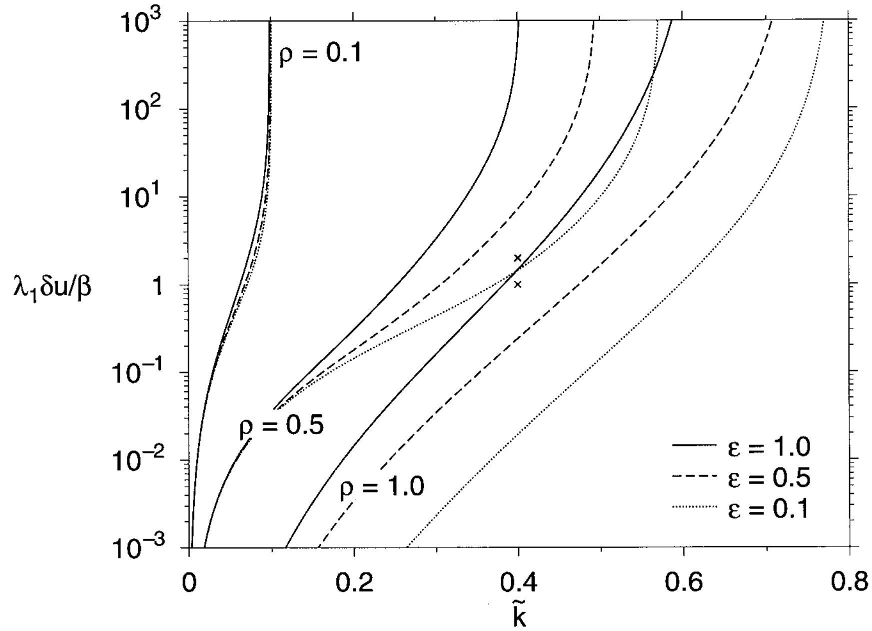

FIG. 3. The ratio ρ of the semicircle bound $\tilde{r}$ of this paper, plus center offset r_1, to the Pedlosky bound r_p. Three ρ contours are shown, 1, 0.5, and 0.1, each for three settings of the ϵ parameter, as a function of $\tilde{k}$ and $\lambda_1 \delta u/\beta$.

equality (17), based on the bounds u_u^+, u_l^+ on velocity. However, there are better Pedlosky bounds available, such as (50) based on u_u, u_l. Yet it is interesting that near the limits of their validity, the bounds (64) furnish a weakened Pedlosky bound.

To gauge the relative utility of the bound given by (61), (62) and the Pedlosky bound (50), we have plotted, in Fig. 3, the ratio $\rho = (\tilde{r} + r_1)/r_p$ as a function of $\tilde{k}$ and $\lambda_1 \delta u/\beta$. When this ratio is less than one, the bound (62) is contained wholly within the Pedlosky bound; when it is greater than one, the Pedlosky bound overlaps the former or is contained within it. (Examples of the bounds in the complex c plane for ratios ρ near 1 are shown in Fig. 4.) For nondimensional wavenumbers $\tilde{k} < 0.07$ and deformation radius such that $\lambda_1 \delta u/\beta > 10$, for example, the bound given by (61), (62) is more than ten times smaller than the Pedlosky bound.

5. Concluding remarks

We have obtained a bound for the radius of a semicircle in the complex phase-speed plane, valid for wavenumbers $k < (\beta/\delta u)^{1/2}\tilde{k}_0(\epsilon)$. The bound is stated formally by inequality (62), with the definition (61) for $\tilde{\gamma}_+$; $\tilde{k}_0(\epsilon)$ is given by the line marked ∞ in Fig. 2, though approximately $\tilde{k}_0 = 0.7$. Formula (61) can be bounded

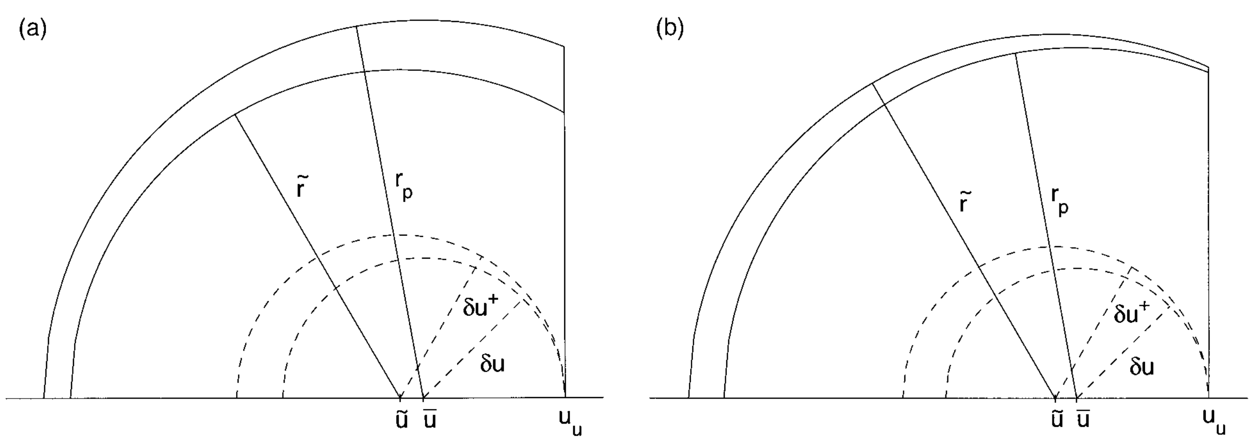

FIG. 4. Two examples comparing the semicircle bound $\tilde{r}$ of this paper, calculated from (61), (62), to the Pedlosky bound r_p. The respective semicircles are centered at $\tilde{u}$ and $\overline{u} = \tilde{u} + r_1$. Also shown are the radii δu^+ and δu to the upper mean velocity bound u_u. The examples are for (a) $\tilde{k} = 0.4$, $\lambda_1 \delta u/\beta = 2$, $\epsilon = 1$; (b) $\tilde{k} = 0.4$, $\lambda_1 \delta u/\beta = 1$, $\epsilon = 1$, and are marked by crosses in Fig. 3.

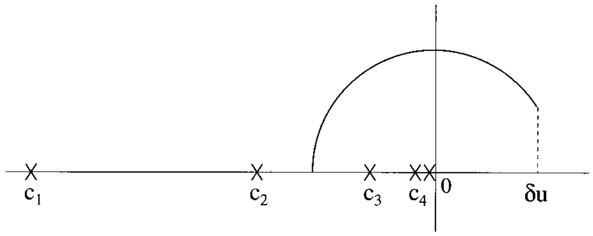

FIG. 5. Schematic showing the zero-shear free-mode phase speeds along the negative real line, with an accumulation point at $c = 0$. All but the gravest modes fall inside the unstable semicircle when perturbed by realistic shear.

by the smallest of three simpler forms, leading to inequality (64). The third of these is merely a weak Pedlosky bound (stronger versions are available). At low wavenumber k and large λ_1 (small baroclinic Rossby deformation radius), the smaller of the other bounds gives a narrower limit to the unstable semicircle radius than even the strong Pedlosky bound. At higher wavenumber and smaller λ_1, and before the validity threshold wavenumber $(\beta/\delta u)^{1/2}\tilde{k}_0$ is reached, the Pedlosky bound gives a narrower limit. The precise transition can be gauged from Fig. 3, as well as the scale of the improvement of the bound: for example, a diminution by more than tenfold of the semicircle bound is obtained for wavenumbers $k(\delta u/\beta)^{1/2} > 0.07$ and $\lambda_1 \delta u/\beta > 10$.

The utility of the kind of semicircle bounds obtained above for the complex phase speed of instabilities is well known. The value of improvements to these bounds is self-evident. Yet there is another use for these semicircle bounds, which has been intimated in recent calculations of modifications of free baroclinic Rossby waves by mean shear (Killworth et al. 1997). The free, zero-shear, Rossby wave modes lie along the negative real line at the points

$$c_n = \frac{-\beta}{\lambda_n + k^2},$$

where the λ_n are eigenvalues of the Sturm–Liouville problem of Eq. (21) above (Fig. 5). How are these modes affected by shear? I offer the following conjecture. Free modes are moved along the negative real axis from their zero-shear position by an amount proportional to the semicircle bound radius and inversely proportional to the magnitude of the zero-shear real phase speed, as long as the latter is larger than the former. This suggests that zero-shear real phase speeds falling well outside the semicircle are susceptible to perturbative calculation of their corrections. (Modes whose zero-shear phase speeds fall close to the edge of the unstable semicircle are difficult to judge.) Modes whose zero-shear phase speeds are within the unstable semicircle have no perturbed counterparts. Instead there may be a finite number of free shear modes, either stable or unstable, supplemented by a continuous spectrum.

Now the countably infinite zero-shear normal modes have an accumulation point at $c = 0$. (Without loss of generality, it is assumed here that $\overline{u} = 0$.) So all but a finite number of these are swallowed up, as it were, by the semicircle, no matter how weak the shear, and replaced by shear modes, possibly unstable, and the continuous spectrum. In fact, a mean velocity profile with a range of order 5 cm s^{-1} will absorb all but the first zero-shear mode or two into the semicircle. It is common to represent the forcing of the ocean by its projection onto the ocean's normal modes, calculated by neglecting mean shear (e.g., Frankignoul et al. 1997). But the modes of the shear-modified system are neither complete nor orthogonal. The implications of this in regard to the physical response of ocean variability to forcing need to be examined. Perhaps the ocean should not be thought of as an analogue of a stretched string or membrane. On the other hand, it may be that the ocean's fundamental, the first baroclinic mode, is sufficiently remote from the semicircle that modifications to it may be handled perturbatively and that, as such, the first mode accounts for a significant proportion of low-frequency ocean variability.

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