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Formulation of the undertow using linear wave theory

G. Guannel[1,a)] and H. T. Özkan-Haller[2]
[1]School of Civil and Construction Engineering, Oregon State University, Corvallis, Oregon 97331, USA
[2]College of Earth, Ocean, and Atmospheric Sciences, Oregon State University, Corvallis, Oregon 97331, USA

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The undertow is one of the most important mechanisms for sediment transport in nearshore regions. As such, its formulation has been an active subject of research for at least the past 40 years. Still, much debate persists on the exact nature of the forcing and theoretical expression of this current. Here, assuming linear wave theory and keeping most terms in the wave momentum equations, a solution to the undertow in the surf zone is derived, and it is shown that it is unique. It is also shown that, unless they are erroneous, most solutions presented in the literature are identical, albeit simplified versions of the solution presented herein. Finally, it is demonstrated that errors in past derivations of the undertow profile stem from inconsistencies between (1) the treatment of advective terms in the momentum equations and the wave action equation, (2) the expression of the mean current equation and the surface shear stress, and (3) the omission of bottom shear stress in the momentum equation. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4872160]

I. INTRODUCTION

The undertow is a wave-induced current, generated to compensate for the shoreward mass flux of the waves.[1,2] Near the bed, it interacts with wave motion to dictate the amount of sediment put in suspension.[3–6] In the water column, it moves sediment offshore, counteracting the suspended flux due to waves.[7–9] Hence, this current is crucial in determining the amount and direction of sediment movement in nearshore regions. It is therefore important to accurately determine the vertical profile of the undertow as well as its value in the proximity of the bed.

While 2D or 3D circulation models (e.g., the Regional Ocean Modeling System,[10] the Princeton Ocean Model,[11] etc. . .) are currently receiving more attention within the coastal engineering community, steady state, wave-averaged undertow models are still widely used in the modeling of coastal hydrodynamics[12] and sediment transport.[13] These models, which are relatively easy and cheap to run, have achieved good results in reproducing observed undertow profiles U in both laboratory and field settings.[11,14–17] In an Eulerian reference frame, models for the undertow are generally of the form[2,14]

$$\frac{\partial \bar{\tau}}{\partial z} = \frac{\partial}{\partial z}\left(\rho \nu_t \frac{\partial U(x,z)}{\partial z}\right) = F_U, \tag{1}$$

where $\bar{\tau}$ is the turbulent shear stress in the water column, ν_t is the turbulent eddy viscosity, F_U is a forcing function, with x pointing shoreward, z pointing upwards, and origin at the still water level.

To obtain an expression for the undertow $U(x, z)$, Eq. (1) must be integrated twice in the vertical. Three boundary conditions are available to solve this equation:

1. a shear stress at the top of the domain (referred to as surface shear stress herein), which has been defined as the wave trough level,[18,19] or the mean water level,[2,20]

[a)]Author to whom correspondence should be addressed. Electronic mail: greg.guannel@gmail.com

2. a shear stress at the bed,[21] and
3. the continuity equation, which dictates that the depth-averaged undertow is compensated by the shoreward volume flux due to waves and rollers.

However, only two boundary conditions are necessary to solve Eq. (1). Thus, a wide range of solutions have been presented in the literature using different sets of boundary conditions. This leads to the impression that the undertow problem is still theoretically unresolved. However, based on physical grounds, and as is shown herein, there should only be a unique solution to Eq. (1) that is consistent with the boundary conditions, assuming that the eddy viscosity ν_t is known.

To integrate Eq. (1) once, the forcing F_U needs to be computed, which most models assume to be uniform with depth. Some models justify this depth-uniformity based on experimental evidence,[18,22–24] while others rely on linear long-wave theory.[17,18,21,25] Once the depth-uniformity has been justified, methods of computation of F_U differ. Indeed, to infer the value of F_U, some models rely on outputs from wave models,[18,21,26,27] while others invoke surface or bottom shear stress boundary conditions,[25,28,29] or the continuity equation.[28,30]

After the first integration is completed, an expression for the vertical gradient of the mean shear stress is obtained, which needs to be integrated once again to obtain $U(x, z)$. Here again, methods differ. Some models use a bottom shear-stress boundary condition,[14,26,27] while others use a surface shear stress.[18,19,28] Among the models that use the surface shear stress as a boundary condition, there is a wide variation in the choice of formulation: Garcez-Faria *et al.*[17] used the expression of Stive and Wind;[18] Apotsos *et al.*[28] used the expression of Stive and de Vriend;[20] Rakha[31] used the expression of Deigaard and Fredsøe;[19] Tajima and Madsen[25] derived a new expression. Although these surface shear stress expressions all yielded good estimates of calibrated undertow profiles, some of them[18,25] have erroneous physical implications.[32,33] Moreover, some models that use a bottom boundary condition yield different results than models that use a surface shear stress boundary condition.[17,34]

Considering that, in principle, there should be only one solution to Eq. (1), the wide variety of solutions and theoretical assumptions presented in the literature might seem puzzling and calls for a unifying framework to solve this problem.

In this paper, a consistent solution to the undertow problem using linear wave theory is presented, and it is shown that this solution is valid in all relative water depths of the waves. Discrepancies in the existing formulations of the undertow that have been developed are highlighted, and their differences are explained and reconciled when possible.

The paper is organized as follows. In Sec. II, we will show that the forcing of the undertow is constant over depth for linear water waves. We will also derive an expression for surface shear stress, which turns out to be identical to the expression suggested by Deigaard and Fredsøe,[19] but is valid for a wider range of conditions. The derivations are conducted by using linear wave theory in its most general form, i.e., they are not simplified by assuming shallow water waves. Because recent publications have recognized the role played by mean current advective terms and bed shear stress in the forcing of the undertow,[17,28,35] these terms are retained in the derivation but a weak vertical variation of the undertow is assumed when treating these terms. In Sec. III, we will discuss the findings presented in Sec. II, and illustrate some of the theoretical concepts presented in this paper. We conclude the paper in Sec. IV.

II. GENERAL FORMULATION OF THE UNDERTOW PROFILE

To derive a theoretical formulation of the undertow profile, we will first develop an expression for its forcing, then we will discuss the choice of boundary conditions necessary to integrate that forcing in the water column and, finally, we will present the solution of that integration.

The problem is set up in a 2DV (x, z) Eulerian reference system, where x is pointing shoreward, and z is pointing upwards, with an origin at the still water level (SWL). The free water surface elevation, $z = \eta$, and the still water depth, $z = -d$, are both referenced from the SWL. The total water depth is defined by $h = d + \bar{\eta}$, where $z = \bar{\eta}$ is the mean water level (MWL), and where the overbar represents wave averaging of any function $\psi(x, z, t)$: $\overline{\psi\,(x, z, t)} = \frac{1}{T}\int\limits_{t}^{t+T} \psi\,(x, z, t)\,dt$, with T the wave period.

The horizontal and vertical velocities of water particles are described by $u(x, z, t)$ and$w\,(x, z, t)$, respectively. They are decomposed into their mean, wave and turbulent components as $v\,(x, z, t) = V\,(x, z) + \tilde{v}\,(x, z, t) + v'\,(x, z, t)$, where v represents either the horizontal or vertical velocity. These quantities satisfy $\bar{v} = V,\ \overline{\tilde{v}} = 0,$ and $\overline{v'} = 0$, and it is assumed that turbulent and wave velocities are uncorrelated: $\overline{\tilde{u}u'} = 0$. It is also assumed that the turbulence is nearly isotropic and that the turbulent normal stresses are small (Stive and Wind,[18] Svendsen,[36] Chaps. 11 and 12), which means that $\overline{u'^2} \approx 0$ and $\overline{w'^2} \approx 0$. Next, we show that, assuming that mean velocities $V = (U, W)$ are confined between the MWL ($z = \bar{\eta}$) and the bed ($z = -d$), a mean horizontal current, the undertow, develops to balance the shoreward mass flux above the MWL due to waves.

The total mass flux (or momentum) M^T reads

$$M^T = \overline{\int_{-d}^{\eta} \rho u dz} = \int_{-d}^{\bar{\eta}} \rho U dz + \overline{\int_{\bar{\eta}}^{\eta} \rho \tilde{u} dz} + \overline{\int_{\bar{\eta}}^{\eta} \rho u' dz}, \tag{2}$$

which is rewritten as

$$M^T = M^m + M^w + M^t, \tag{3}$$

where M^m is the mass flux (momentum) due to mean current, and is, per the definition presented above, confined to a region between bed ($z = -d$) and MWL ($z = \bar{\eta}$). M^w is the net mass flux (momentum) due to waves, which is confined to a region between MWL and free surface ($z = \eta$). Finally, M^t is the mass flux due to turbulent motion, and is also confined to a region between MWL and free surface. This term is assumed to be zero since it is beyond the scope of this paper to model turbulence. However, some of the effects of M^t will be heuristically captured in the roller and turbulent eddy viscosity terms that will be introduced later on.

To relate the mean current momentum to wave momentum, the continuity equation is first integrated from bed to free surface elevation, and averaged over a wave period. The continuity equation reads

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

After invoking the kinematic free surface and bottom boundary conditions, which, for a sloping bottom, are[37]

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - w = 0 \text{ at } z = \eta, \tag{5}$$

$$u\frac{\partial d}{\partial x} + w = 0 \text{ at } z = -d, \tag{6}$$

another expression for the total momentum M^T is obtained:

$$\rho\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial M^T}{\partial x} = 0. \tag{7}$$

Under steady state conditions, this last equation becomes $\partial_x M^T = 0$, where ∂_α is shorthand for a partial derivation with respect to a variable α. Because the shoreline is a fixed boundary that can be considered impermeable, Eq. (7) is rewritten as $M^T = \text{Constant} = 0$, which, following Eq. (3), yields $M^m + M^w = 0$.

This last equation indicates that, under 2DV conditions, the net shoreward mass flux due to the waves must be returned in its entirety. In other words, the shoreward mass flux due to the waves M^w generates an offshore mean current U, the undertow:[36]

$$M^w = -M^m = -\int_{-d}^{\bar{\eta}} \rho U dz = -\rho h U_r, \tag{8}$$

where U_r is the depth averaged value of the undertow (defined between MWL and bed).

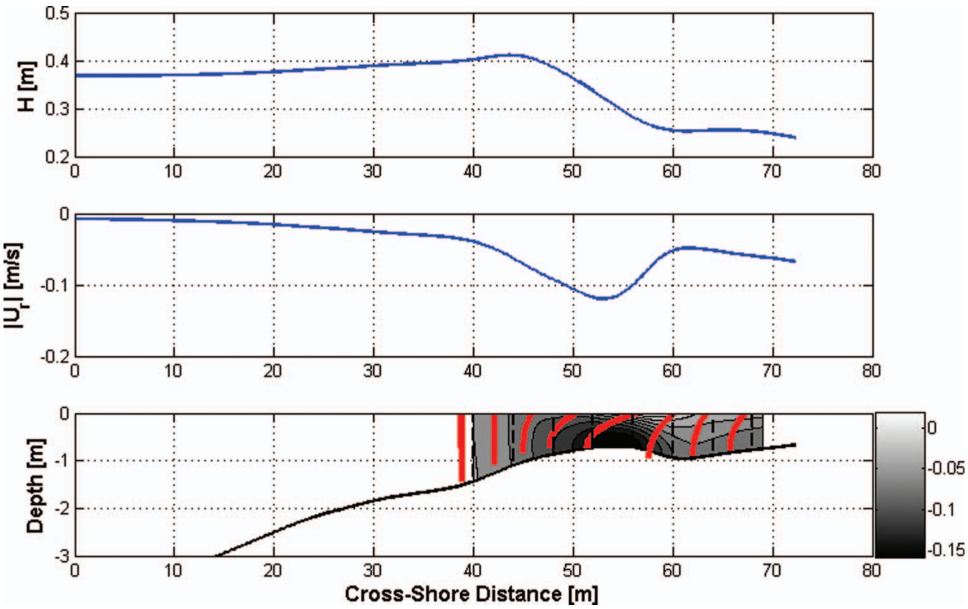

FIG. 1. Wave and undertow profiles computed using an offshore wave height of 0.6 m and wave period of 4 s. (Top panel) Wave height profile. (Middle panel) Depth-averaged undertow U_r. (Bottom panel) Contour of undertow velocity values on top of the bathymetry profile. Profiles of undertow in the surf zone are plotted every 4 m (thick vertical lines, indicated by red). Thinner dashed vertical lines represent the origin associated with each undertow profile.

To visualize the cross-shore variation of the depth averaged undertow, we modeled the evolution of a 0.6 m wave height, with a peak period of 4 s, over a barred beach profile similar to the one presented in Scott *et al.*[38] As shown in Fig. 1, U_r gains strength as the waves shoal and break over the bar. Interestingly, the depth averaged mean current reaches an absolute maximum a few meters shoreward of the breakpoint, near the trough of the bar. In the surf zone, U_r is stronger than offshore of the bar, even though the waves are smaller. Figure 1 also shows the vertical profiles of undertow at discrete cross-shore locations. Those profiles, which are discussed in Sec. III, were obtained by solving the undertow equations presented herein and with an empirical eddy viscosity formulation[39] $\nu_t \approx 0.01h\sqrt{gh}$.

A. Forcing of the undertow

In this section, we derive a solution for the forcing of the undertow in the surf zone. We first derive a general solution that can be solved using any wave theory. Next, we simplify the expression of the forcing using linear wave theory.

1. General solution

The governing equations for inviscid unidirectional flows between $z = \eta(x, t)$ and $z = -d$ can be written as a function of the pressure $p(x, z, t)$ by

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{9}$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{10}$$

To derive an equation for the forcing of the mean current, we decompose the velocities into mean and fluctuating parts, and wave-average the horizontal momentum equation [Eq. (9)], neglecting horizontal turbulent mixing terms $\overline{u'^2}$ (see above):

$$\frac{\partial U}{\partial t} + \frac{\partial \overline{\tilde{u}^2}}{\partial x} + \frac{\partial U^2}{\partial x} + \frac{\partial \overline{\tilde{u}\tilde{w}}}{\partial z} + \frac{\partial UW}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial \overline{u'w'}}{\partial z}. \tag{11}$$

The vertical gradient of mean turbulent shear stress ($\bar{\tau}(z) = -\partial_x \overline{u'w'}$) is expressed by the vertical mixing of horizontal mean momentum $\bar{\tau}(z) = -\partial_z(\nu_t \partial_z U)$, where ν_t is an eddy viscosity coefficient representative of the turbulence level in the water column. The wave-averaged pressure is also rewritten by wave-averaging the vertical momentum equation [Eq. (10)], and neglecting the mixing term $\overline{w'^2}$ (see above):

$$\bar{p} = -\rho\overline{\tilde{w}^2} - \rho W^2 + \rho g(\bar{\eta} - z). \tag{12}$$

Contrary to previous work cited herein, the mean vertical velocity W is retained in the derivation. Finally, following Garcez-Faria *et al.*,[17] the expression of the wave shear stress $\partial_z \overline{\tilde{u}\tilde{w}}$ is simplified by using the decomposition presented by Rivero and Arcilla:[40]

$$\frac{\partial \overline{\tilde{u}\tilde{w}}}{\partial z} = \overline{\tilde{\omega}\tilde{w}} - \frac{1}{2}\frac{\partial}{\partial x}\left(\overline{\tilde{u}^2} - \overline{\tilde{w}^2}\right), \tag{13}$$

where $\tilde{\omega}$ represents the wave-induced vorticity. Equation (11) thus becomes

$$\rho\frac{\partial U}{\partial t} + \underbrace{\frac{1}{2}\frac{\partial}{\partial x}\rho\left(\overline{\tilde{u}^2} - \overline{\tilde{w}^2}\right) + \rho\overline{\tilde{\omega}\tilde{w}}}_{F} + \underbrace{\rho g\frac{\partial \bar{\eta}}{\partial x}}_{P} + \underbrace{\rho\frac{\partial U^2}{\partial x} - \rho\frac{\partial W^2}{\partial x} + \rho\frac{\partial UW}{\partial z}}_{G} = \frac{\partial \bar{\tau}}{\partial z}, \tag{14}$$

where F is a force due to action of waves only, P is a pressure force induced by gradients in MWL, and G is a force induced by advection of mean currents. This expression is exact and can be solved with any wave theory.

2. Solution using linear wave theory

We use linear wave theory to solve Eq. (14), but retain horizontal and vertical mean velocities. We assume that the energy density of a wave field is represented by $E_w = 1/8\rho g H^2$, where $H = \sqrt{8\overline{\eta^2}}$. Wave period T is related to wavelength L and associated wavenumber $k = 2\pi/L$ by the linear dispersion relationship, which reads in the presence of a depth-uniform current U_o:$\sigma = U_o k + \sigma_r$, where $\sigma_r^2 = gk\tanh(kh)$ is the relative frequency of the wave. The wave energy density travels at the wave group velocity C_{ga}:

$$C_{ga} = nC + U_o = C_g + U_o = \frac{1}{2}\left(\frac{2kh}{\sinh(2kh)} + 1\right)C + U_o, \tag{15}$$

where the relative celerity C of the wave is

$$C = \frac{\sqrt{gk\tanh kh}}{k} = \frac{2\pi/T - U_o k}{k}. \tag{16}$$

We simplify the wave stress terms in Eq. (14) to obtain (see Appendix A for details)

$$\rho\frac{1}{2}\frac{\partial}{\partial x}\left(\overline{\tilde{u}^2} - \overline{\tilde{w}^2}\right) = \frac{\partial}{\partial x}\left(\frac{S_{xx} - E_w/2}{h}\right), \tag{17}$$

where S_{xx} represents the radiation stress due to waves only—rollers are not included yet. The vorticity term in Eq. (14) disappears under the assumptions of linear wave theory.

Equation (17) shows that, in Eq. (14), the combination of the wave velocity advective term $\tilde{u}^2$, the dynamic pressure term $\tilde{w}^2$ and the wave-induced shear stress $\overline{\tilde{u}\tilde{w}}$ generates a depth uniform forcing term for the undertow. This result is in line with previous experimental observations,[18,22,23] but is valid only if one uses linear wave theory: using a weakly nonlinear wave theory, Zou *et al.*[41] expressed the wave-induced vertical shear stress $\overline{\tilde{u}\tilde{w}}$ as a function of the bed shear stress, which causes $\partial_z \overline{\tilde{u}\tilde{w}}$ to become depth-varying.

Finally, second-order terms in W are assumed to be negligible (see Appendix B), and Eq. (14) is rewritten as

$$\rho\frac{\partial U}{\partial t}+\frac{\partial}{\partial x}\left(\frac{S_{xx}-E_w/2}{2h}\right)+\rho g\frac{\partial\bar{\eta}}{\partial x}+\rho\frac{\partial U^2}{\partial x}+\rho\frac{\partial UW}{\partial z}=\frac{\partial\bar{\tau}}{\partial z}. \tag{18}$$

Equation (18), which reduces to the one derived by Newberger and Allen[11] in shallow water (see their Eq. (B3)), shows that the undertow is forced by two depth uniform forces per unit volume, F and P, and a depth-varying force per unit volume G:

$$F_U=\underbrace{\frac{\partial}{\partial x}\left(\frac{S_{xx}-E_w/2}{2h}\right)}_{F(x)}+\underbrace{\rho g\frac{\partial\bar{\eta}}{\partial x}}_{P(x)}+\underbrace{\left[\rho\frac{\partial U^2}{\partial x}+\rho\frac{\partial UW}{\partial z}\right]}_{G(x,z)}. \tag{19}$$

The depth uniform force F, first term on the right-hand side (RHS), is generated by the wave momentum flux; we will see in Sec. II B that it can be expressed as a function of the total radiation stress gradient, as Newberger and Allen[11] also showed. The second part of the forcing P, the middle term on the RHS, is a depth uniform pressure gradient due to wave setup/setdown ($\partial_x\bar{\eta}$). The third part of the forcing G, the last term between brackets on the RHS, is a depth-varying force caused by the horizontal and vertical advection of the mean current, $U(x, z)$.

B. Boundary conditions

To solve the wave-averaged mean current equation, Eq. (18), two boundary conditions are necessary. The first condition is the depth-averaged continuity, Eq. (8). The second condition is a boundary condition, which can either be a mean bottom shear stress condition τ_b,[17,21,29] or a mean shear stress at the MWL or at the trough level.[2,15,18,30] In this section, we first present expressions for bottom and surface shear stress that have been proposed in the literature. Next, we use the equations presented above to show that the expression of surface shear stress originally derived by Deigaard and Fredsøe[19] can be arrived at by relaxing assumptions of shallow water linear waves.

1. Existing expressions for mean bottom and surface shear stress, $\overline{\tau_b}$ and $\overline{\tau_s}$

The mean bottom shear stress, $\overline{\tau_b}=\rho\nu_t\partial_z U_{|z_o}$, where z_o represents the bed level, is usually expressed empirically by using a friction factor f_{wc},[42] such as in

$$\overline{\tau_b}=\frac{2}{\pi}\rho f_{wc}\tilde{u}_b U_\delta, \tag{20}$$

where $\tilde{u}_b$ represents the wave orbital velocity at the bed, and U_δ is the mean velocity right above the wave bottom boundary layer (WBBL). In practice, $U_\delta=U_{z=-d}$, for models which assume a slip boundary condition.

Similarly, various expressions for the surface shear stress have been suggested. One of the first expressions of the surface shear stress was developed by Dally:[2]

$$\overline{\tau_s}=-\frac{1}{2}\frac{\partial E}{\partial x}. \tag{21}$$

Dally[2] derived this expression by integrating the wave-averaged horizontal momentum equation [Eq. (14)] between a level z in the water column, ($z<\bar{\eta}$), and the free water surface η. He neglected wave stress terms between MWL and free surface and assumed vertical velocity was negligible. He also assumed that all stress terms between MWL and free surface can be averaged at the MWL. Later, Stive and Wind,[18] following Svendsen[43] (see also Svendsen[36], p. 613), derived an identical expression for the surface shear stress at the trough level. They integrated the momentum equation between trough level and free surface, but also assumed shallow water conditions and a locally horizontal flat bottom. More recently, Tajima and Madsen[25] re-evaluated the same integral that Dally[2] calculated, but assumed that linear wave theory holds from trough to crest, and that the mean velocity is constant in that region. They arrived at an expression that differs from Eq. (21), because it includes wave and mean current variables (for details, see Ref. 25). These three expressions of the surface shear stress, which are based on different assumptions, generate a surface shear stress in regions where no breaking occurs. This outcome is physically unrealistic because, if a shear stress existed in these regions, the work done by this internal force would have to dissipate the wave energy flux. However, in the absence of wave breaking no dissipation exists, so no surface shear stress can be generated.[32,33]

Another expression for $\overline{\tau_s}$ was heuristically developed by Deigaard,[44] following the work of Deigaard and Fredsøe,[19] by equating the work done by the shear stress at the trough level to wave and roller dissipation:

$$\overline{\tau_s} = \frac{D_w}{C} - \frac{\partial 2E_r}{\partial x}, \tag{22}$$

where D_w is wave dissipation, E_r is the average roller kinetic energy,[21] and C is the wave celerity. Stive and de Vriend[20] arrived at the same expression of the surface shear stress by depth-integrating the linearized momentum equation from bed to the mean water level. They assumed linear shallow water wave theory to express wave-induced radiation stress and neglected bottom shear stress in the depth-integrated, depth-averaged total momentum equation (see Sec. II B 2). Newberger and Allen[11] also arrived at the same expression by assuming linear shallow water wave theory, as well as depth uniform horizontal current, and integrating the full horizontal momentum equations, nonlinear terms included, from the mean water level to the free surface. They expressed the mean surface shear stress by expanding M^w at the MWL using a Taylor expansion of the horizontal momentum equation [Eq. (9)]. They added the roller contribution *ad hoc*.

Finally, Stive and de Vriend,[20] following the work of Nairn *et al.*[45], modified Eq. (22) to read

$$\overline{\tau_s} = \frac{D_w}{\mathrm{C}} - \frac{1}{\mathrm{C}}\frac{\partial 2E_r C}{\partial x}. \tag{23}$$

Compared to Eq. (22), this expression assumes that $\partial_x C \ll 1$, which is consistent with the assumption of a flat bottom. The extra term in $\overline{\tau_s}$ in Eq. (23), represents the exchange of mass (momentum) between wave and roller during the roller growth and decay phases (Deigaard;[44] see also the Appendix of Ref. 20). Deigaard[44] excluded that term in his derivation of the surface shear stress.

All expressions of surface shear stress presented so far in this section have been used to model the undertow. For example, Garcez-Faria *et al.*[17] and Spielmann *et al.*[34] referred to Stive and Wind,[18] Deigaard[44] and Rakha[31] used Eq. (22), and Apotsos *et al.*[28] used Eq. (23). Although the difference between Eqs. (22) and (23) can be considered to be relatively minor,[31] these expressions are not identical, and they are based on different assumptions.

For illustration, we compare in Fig. 2 the different formulations of surface shear stress using the outputs of the numerical experiment presented in Fig. 1. Only the expressions of Deigaard and

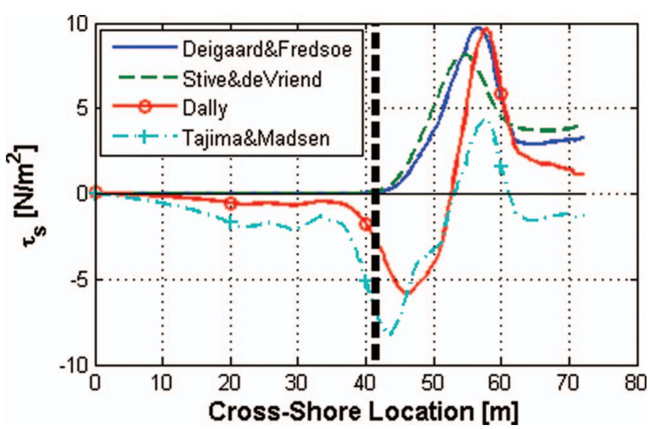

FIG. 2. Comparison between the various formulations of shear stress at the MWL using the wave data presented in Fig. 1. Formulations of Deigaard[44] and Stive and de Vriend[20] correctly predict a relatively small, if not zero, shear stress offshore of the breaking point (indicated by thick vertical dashed line), a region where wave and roller dissipation should also be relatively small, if not zero.

Fredsøe[19] and Stive and de Vriend[20] generate zero shear stress offshore of the surf zone, where dissipation does not occur. Furthermore, it appears that differences between these two formulations are small.

2. Derivation of surface shear stress

All the expressions of surface shear stress reviewed above were derived assuming shallow water waves, negligible bottom shear stress and, with the exception of Newberger and Allen,[11] by neglecting the nonlinear advection terms. However, as Guannel[46] showed, undertow profiles are often observed in intermediate waters depths (i.e., $kh > \pi/10$, according to linear wave theory). In this section, we close this theoretical gap and demonstrate the generality of the surface shear stress expression shown above [Eq. (22)] by evaluating the time evolution of the total momentum equation [Eq. (3)], advection terms included. We do not make any assumptions about the relative water depth and do not neglect bottom shear stress in the momentum equations [Eqs. (9) and (10)]. We assume that waves are described by linear wave theory, and that higher-order terms involving horizontal currents are depth-uniform. This assumption is necessary in order to obtain an equation for the wave momentum evolution, $\partial_t M^w$. In the remainder of this paper, the tilde over wave velocities are dropped ($\tilde{u} = u$), as well as the overbar over mean shear stress ($\bar{\tau} = \tau$).

To obtain an equation for the surface shear stress, we first obtain an evolution equation for M^T by depth-integrating the horizontal momentum equation from the bed to the free water surface. Second, we obtain an evolution equation for M^m by depth-integrating the wave-averaged mean momentum equation [Eq. (18)], from the bed to the mean water surface. In the process, the shear stress at the MWL, τ_s, explicitly appears. Third, we obtain an expression for the evolution of M^w by re-arranging the various terms of the wave action equation, following the work of Smith.[47] Finally, we use the time derivative of the depth-integrated momentum equation to equate the evolution equations of total, mean and wave momentum, $\partial_t M^T = \partial_t M^M + \partial_t M^w$, and obtain an expression for τ_s in the absence of rollers. The roller term is included in the momentum equations following Svendsen[14] to obtain the same equation for the surface shear stress shown in Sec. II B 1, Eq. (22).

First, an expression for the evolution of total momentum, $\partial_t M^T$, is obtained by wave-averaging the depth-integrated horizontal momentum equation, Eq. (9), from $z = -d$ to $z = \eta$ (Svendsen,[36] p.544), and by neglecting wind stress and atmospheric pressure:

$$\frac{\partial M^T}{\partial t} + \frac{\partial S_{xx}}{\partial x} + \rho g h \frac{\partial \bar{\eta}}{\partial x} + \rho \frac{\partial h U_r^2}{\partial x} + 2\frac{\partial U_r M^w}{\partial x} + \tau_b = 0. \tag{24}$$

Although Svendsen[36] neglected mean vertical currents in his derivation, this assumption is not necessary to obtain Eq. (24). In the depth integration of the horizontal momentum equation [Eq. (9)], the nonlinear term containing vertical current [third term on the RHS of Eq. (9)] is combined with the first and second terms on the RHS of the same equation. Next, after applying Leibniz rule to the integral, terms containing the total vertical velocity vanish because of the surface and bed boundary conditions, Eqs. (5) and (6) [for details, see Svendsen,[36] Eqs. (11.4.6) and (11.4.7)]. Furthermore, terms containing pressure at the free surface and at the bottom appear in the depth integration of the horizontal momentum equation.

We obtain the expression for the vertical variation of pressure from the depth integration of the vertical momentum equation, Eq. (10). Vertical velocity appears in the expression of the pressure at the free surface, but atmospheric pressure is neglected. Vertical velocity also appears in the expression of the pressure at the bed via a term containing the cross-shore gradient of a shear stress term (see Svendsen,[36] p. 539). This term is commonly assumed small since a non-zero contribution from this term (in a wave averaged formulation) would result in a physically counter-intuitive scenario where the pressure at the bottom of a column of water is less than the weight of the water column. For a detailed discussion of this argument, the reader is referred to Svendsen,[36] Sec. 11 D. As a result of these considerations, Eq. (24) is consistent with the assumption that first-order mean vertical currents are not neglected. (We show in Appendix B that second order mean vertical currents can be neglected.)

To obtain an evolution equation for the mean current momentum, $\partial t M^m$, the wave-averaged mean momentum equation, Eq. (18), is integrated between the bed $-d$ and MWL $\bar{\eta}$, and the Leibniz rule is invoked to obtain:

$$\frac{\partial}{\partial t}\int_{-d}^{\bar{\eta}} \rho U dz + h\frac{\partial}{\partial x}\left(\frac{S_{xx}-E/2}{2h}\right) + \rho g h\frac{\partial \bar{\eta}}{\partial x} + \frac{\partial}{\partial x}\int_{-d}^{\bar{\eta}} \rho U^2 dz + \rho U^2_{|-d}\frac{\partial d}{\partial x}$$

$$-\rho\left(UW\right)_{|-d} - \rho U^2_{\bar{\eta}}\frac{\partial \bar{\eta}}{\partial x} + \rho\left(UW\right)_{|\bar{\eta}} + \tau_b - \tau_s = 0. \qquad (25)$$

To simplify this expression, we wave-average the kinematic boundary conditions, Eqs. (5) and (6), and obtain (see Refs. 47 and 48, and also Appendix A of Ref. 11)

$$\frac{\partial \bar{\eta}}{\partial t} + U\frac{\partial \bar{\eta}}{\partial x} - W + \frac{1}{\rho}\frac{\partial M^w}{\partial x} = 0 \text{ at } z = \bar{\eta}, \qquad (26)$$

$$U\frac{\partial d}{\partial x} + W = 0 \text{ at } z = -d, \qquad (27)$$

where $M^w = E_w/C$ (Ref. 49).

Combining Eqs. (25)–(27), and assuming depth uniform currents in the second order current terms, an equation for the mean momentum equation appears as

$$\frac{\partial M^m}{\partial t} + h\frac{\partial}{\partial x}\left(\frac{S_{xx}-E/2}{2h}\right) + \rho g h\frac{\partial \bar{\eta}}{\partial x} + \frac{\partial h U_r^2}{\partial x} + U_r\left(\frac{\partial M^w}{\partial x} + \rho\frac{\partial \bar{\eta}}{\partial t}\right) + \tau_b - \tau_s = 0. \qquad (28)$$

Finally, the evolution equation of net wave momentum $\partial_t M^w$ is derived from the wave action equation,[50–52] which is expressed as

$$\frac{\partial}{\partial t}\frac{E_w}{\sigma_r} + \frac{\partial}{\partial x}\frac{E_w\left(C_g + U_r\right)}{\sigma_r} = -\frac{D_w}{\sigma_r}. \qquad (29)$$

In this equation, the dissipation term D_w represents the dissipation of wave energy, neglecting effects of bottom friction. Following the work of Smith[47] (Sec. II C) and after a few manipulations summarized in Appendix C, an evolution equation for the wave momentum appears

$$\frac{\partial M^w}{\partial t} + \frac{\partial S_{xx}}{\partial x} - h\frac{\partial}{\partial x}\left(\frac{S_{xx}-E/2}{2h}\right) + \frac{D_w}{C} + \frac{\partial U_r M^w}{\partial x} + M^w\frac{\partial U_r}{\partial x} = 0. \qquad (30)$$

This equation was first derived by Longuet-Higgins in 1973 for waves in the absence of mean currents. It shows that, under steady state conditions, the radiation stress gradient (2nd term on the LHS) can be expressed as a function of (1) a portion of the depth uniform forcing of the undertow F [3rd term on the LHS, see Eq. (19)], (2) wave dissipation (4th term on the LHS), which only exists when waves are breaking, and (3) wave-current interaction terms (5th and 6th terms on the LHS).

Next, the evolution equations for total, mean and wave momentum are combined to generate an expression for the surface shear stress τ_s:

$$\frac{\partial M^T}{\partial t} - \frac{\partial M^w}{\partial t} - \frac{\partial M^m}{\partial t} = \rho U_r\frac{\partial \bar{\eta}}{\partial t} + \frac{D_w}{C} - \tau_s = 0, \qquad (31)$$

which, for steady state, becomes

$$\tau_s = D_w/C, \qquad (32)$$

where C is the relative speed of the wave with respect to the current U_r, given by Eq. (33).

This expression, which was derived at the MWL and not at the trough level, shows that the dissipation of wave energy exerts a stress at the MWL; the same result could not be found at the trough level following the method described herein. This expression is also identical to the one derived by Newberger and Allen,[11] Deigaard and Fredsøe,[19] or Stive and de Vriend,[20] except that here, contrary to those references, the shallow water assumption has been relaxed. Under the assumption of depth-uniform second-order horizontal current, all nonlinear terms vanish exactly, and thus have no influence on the surface stress. This finding, which differs from Tajima and Madsen,[25] is consistent with physical reasoning:[19,33] the work of an internal force, i.e., $\tau_s C$, only generates dissipation.

Wave dissipation is characterized by the formation of a roller, which absorbs some of the excess momentum of the waves. Hence, to complete Eq. (32), it is necessary to account for the roller momentum flux, referred to as M^r. In the absence of a formal derivation for roller momentum evolution, we used the heuristic expressions for roller momentum and roller radiation stress derived by Svendsen.[14] The total momentum equation thus becomes[21,36]

$$\frac{\partial M^T}{\partial t}+\frac{\partial S_{xx}}{\partial x}+\frac{\partial R_{xx}}{\partial x}+\rho g h\frac{\partial\bar{\eta}}{\partial x}+\rho\frac{\partial hU_r^2}{\partial x}+2\frac{\partial U_r M^w}{\partial x}+\tau_b=0, \tag{33}$$

where $R_{xx}=2E_r$. Furthermore, following Svendsen,[14] the depth-averaged return current U_r becomes

$$\rho h U_r=-M_t^w=-\left(M^w+M^r\right)=-\left(E/C+2E_r/C\right), \tag{34}$$

where $M^r=2E_r/C$ is the momentum of the roller, and E_r is the roller energy. With this definition of U_r, we once again evaluate the time derivative of Eq. (3), but written as $\partial_t M^T=\partial_t M^m+\partial_t M_T^w$. Equation (22) for the surface shear stress thus appears again: $\tau_s=D_w/C-\partial_x 2E_r$.

In this section, we formally derived an expression for the surface shear stress similar to the one reported by Deigaard.[44] The shallow water wave approximation was relaxed, and bottom shear stress and mean currents were included. From the approach presented herein, the mean surface shear stress is expressed at the MWL, similarly to Stive and de Vriend[20] and Newberger and Allen.[11]

C. Solutions

As mentioned earlier, the solution to the mean momentum equation and for the profile of undertow in the water column, Eq. (18), were obtained by integrating it twice in z, and by applying either one of the boundary conditions presented in Sec. II B 1. This solution requires knowledge of the wave variables as well as eddy viscosity in the fluid. In previous publications, researchers compared the merits of applying a surface versus a bottom shear stress condition, in addition to using mass conservation.[17,34,53] Others presented a formulation of the undertow based on the difference between surface and bottom shear stresses[25] (i.e., $\partial_z(\tau(z))=f(\tau_s-\tau_b)$]. In this section, we examine the various possible solutions to Eq. (18), and show that they are all equivalent. Specifically, we show that there is no difference in a solution computed using surface shear stress or bottom shear stress as a boundary condition. We also show that the solution is similar whether one takes the difference between surface and bottom shear stress or uses the depth uniform forces F and P presented in Eq. (18) to express the vertical variation of undertow.

First, we integrate Eq. (18) once in z, and use the bottom shear stress boundary condition:

$$\tau(z)=(F+P)(z+d)+\rho\frac{\partial}{\partial x}\int_{-d}^{z}U^2dz+\rho UW+\tau_b, \tag{35}$$

where the wave-averaged kinematic bottom boundary condition, Eq. (27), was invoked. Alternatively, when a mean shear stress at the MWL is used, the expression of the undertow reads

$$\tau(z)=(F+P)(z-\bar{\eta})-\rho\frac{\partial}{\partial x}\int_{z}^{\bar{\eta}}U^2dz+\rho UW-\rho U_{|\bar{\eta}}\frac{\partial M^w}{\partial x}+\tau_s, \tag{36}$$

where the wave-averaged free surface kinematic boundary condition, Eq. (26), was invoked.

To show that these two expressions are similar, both equations are subtracted to obtain

$$hF+hP+\rho\frac{\partial}{\partial x}\int_{-d}^{\bar{\eta}}U^2dz+\rho U_{|\bar{\eta}}\frac{\partial M^w}{\partial x}=\tau_s-\tau_b, \tag{37}$$

which is the steady-state mean momentum equation for depth-varying mean currents, Eq. (25). After we express $(F+P)$ from this equation, and insert it in Eq. (36), a third expression for the solution to Eq. (18) appears:

$$\tau(z) = \frac{\tau_s - \tau_b}{h}(z+d) + \tau_b - \left(\rho\frac{\partial}{\partial x}\int_{-d}^{\bar{\eta}} U^2 dz + \rho U_{|\bar{\eta}}\frac{\partial M^w}{\partial x}\right)\frac{z+d}{h} + \rho\frac{\partial}{\partial x}\int_{-d}^{z} U^2 dz + \rho U W. \tag{38}$$

Thus, the solution to the mean current equation, Eq. (18), is the same whether one imposes a surface or a bottom shear stress boundary condition. Equations (35) and (36) yield the same expression for the mean shear stress $\tau(x, z)$ as was argued by Svendsen.[54] Additionally, the comparison of Eqs. (35) and (36), and (38) shows that the solution is also the same if one expresses it using the depth-uniform forcing $F + P$ [Eq. (35) and (36)] or a difference of surface and bottom shear stresses $\tau_s - \tau_b$ [Eq. (38)].

We once again assume that the vertical variation of second order nonlinear terms is negligible, which means that terms on the RHS of Eqs. (35) and (36), and (38) are depth uniform. Under this assumption, these equations become (see Appendix D for details)

$$\rho\nu_t\frac{\partial U}{\partial z} = \left[h\frac{\partial}{\partial x}\left(\frac{S_{xx} - E/2}{2h}\right) + \rho g\frac{\partial\bar{\eta}}{\partial x} + \rho U_r\frac{\partial U_r}{\partial x}\right](z+d) + \tau_b, \tag{39}$$

and

$$\rho\nu_t\frac{\partial U}{\partial z} = \left[h\frac{\partial}{\partial x}\left(\frac{S_{xx} - E/2}{2h}\right) + \rho g\frac{\partial\bar{\eta}}{\partial x} + \rho U_r\frac{\partial U_r}{\partial x}\right](z-\bar{\eta}) + \tau_s, \tag{40}$$

and

$$\rho\nu_t\frac{\partial U}{\partial z} = \frac{\tau_s - \tau_b}{h}(z+d) + \tau_b. \tag{41}$$

The nonlinear term in Eq. (39) or (40) is half the term in Garcez-Faria *et al.*[17] because the derivation presented above included mean vertical velocity. All nonlinear terms vanished from Eq. (41).

III. DISCUSSION

We have expressed the various forcing mechanisms of the undertow, assuming linear wave theory. We also demonstrated that, although one can create three formulations of the undertow depending on the boundary conditions, these formulations are interchangeable if a consistent framework is employed, and they all lead to the same solution. Hence, theoretically, it is possible to reproduce observed profiles of the undertow $U(x, z)$ for a given wave field if the wave momentum flux is reproduced accurately,[55] and the value of the eddy viscosity ν_t is known. We generated profiles of undertow using the three different solutions presented in Sec. II C, using the dataset presented in Fig. 1; the eddy viscosity in the water column was approximated as[39] $\nu_t \approx 0.01h\sqrt{gh}$. We found that the undertow in the surf zone is convex, and strongest shoreward of the breakpoint (Fig. 1). Furthermore, the difference between the three solutions was of the order of mm/s, with the strongest deviations observed when a bottom boundary condition was used. This is not too surprising as the estimate of the bottom shear stress is approximate.

To investigate the forcing of the undertow, the mean momentum equation [Eq. (28); see also, e.g., Dingemans *et al.*[32]] is rewritten as

$$\rho h\frac{\partial U_r}{\partial t} = -\underbrace{\rho hg\frac{\partial\bar{\eta}}{\partial x}}_{I} - \underbrace{\left[h\frac{\partial}{\partial x}\left(\frac{S_{xx} - E/2}{2h}\right) - \overline{\tau_s}\right]}_{II} - \underbrace{\left[\rho\frac{\partial hU_r^2}{\partial x} + U_r\frac{\partial M^w}{\partial x}\right]}_{III} - \underbrace{\overline{\tau_b}}_{IV}, \tag{42}$$

where each term was multiplied by ρh. The first process (term *I* on the RHS, denoted by $d\eta$) is the pressure gradient induced by wave setup/setdown. The second process (term *II* on the RHS) is a wave-induced force. It is composed of a portion of the depth uniform force in Eq. (18), denoted by hF, and a stress term which is related to the shoreward directed surface shear stress, denoted by τ_s; this latter term is only present when wave dissipation occurs. The third process is an advective and wave-current interaction force (term *III* on the RHS, denoted by *Adv)*, and is also a mass source/sink at the surface caused by the change in wave momentum, as pointed out by Smith[47] and Newberger

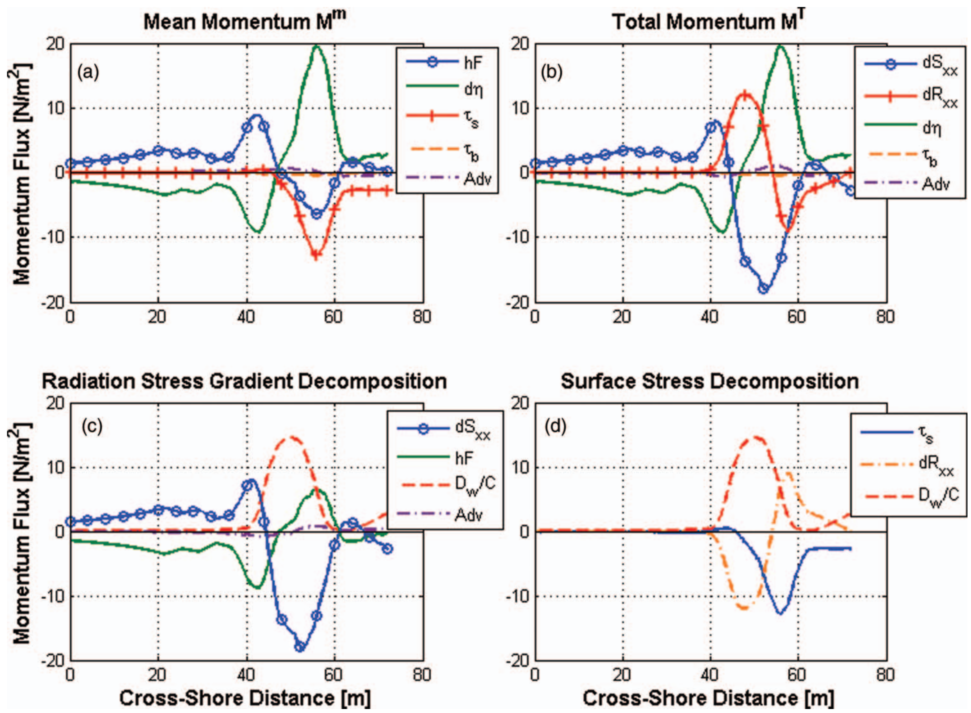

FIG. 3. Top subplots: balance of terms in the mean momentum equation [Eq. (33) or (42); subplot a], the total momentum equation [Eq. (28) or (43); subplot b]. Bottom subplots: decomposition of radiation stress gradient according to Eq. (30) (subplot c), and decomposition of the surface shear stress according to Eq. (22) (subplot d). Symbols used in legend are explained in the text.

and Allen.[11] The fourth process is caused by the bottom shear stress (term *IV* on the RHS, denoted by τ_b). Written this way, it becomes clear how the expression for the forcing of the undertow [Eqs. (39)–(41)] validates the conceptual explanation presented by Dyhr-Nielsen and Sørensen,[1] who state that the mean current is forced by those four processes.

We use the aforementioned dataset presented in Fig. 1 to explore the relative importance of those terms (Fig. 3(a); note that those terms were multiplied by the water depth h before plotting). Except in the inner surf zone (cross-shore distance $X > 60$ m) the depth-uniform force F, along with the surface shear stress term τ_s balance gradients in MWL $d\eta$. In the inner surf zone, the depth-uniform force F works together with gradients in MWL $d\eta$ to oppose the surface shear stress τ_s. Bed shear stress and advective terms play a relatively minor role, and have approximately the same strength. These observations indicate that, depending on the location of the breaker, the body force is either balanced by gradients in MWL or by the surface shear stress. These observations also highlight the important role played by the surface shear stress in generating the undertow. Guannel[46] conducted the same analysis for different flume and field datasets and found similar properties of the forcing of the undertow. It is worth mentioning that the relative role of the surface shear stress varies between datasets.

The role of the pressure gradient $d\eta$ in generating the undertow is further examined by exploring the relative importance of the terms in the total momentum equation, which is rewritten here for convenience as

$$\frac{\partial M^T}{\partial t} + \underbrace{\rho g h \frac{\partial \bar{\eta}}{\partial x}}_{I} + \underbrace{\frac{\partial S_{xx}}{\partial x}}_{II} + \underbrace{\frac{\partial R_{xx}}{\partial x}}_{III} + \underbrace{\rho \frac{\partial h U_r^2}{\partial x} + 2 \frac{\partial U_r M^w}{\partial x}}_{IV} + \underbrace{\tau_b}_{V} = 0. \tag{43}$$

Under steady state conditions, the pressure gradient generated by the MWL (term *I*, denoted by $d\eta$) is balanced by the combined action of gradients of radiation stress due to waves and roller (terms *II* and *III*, respectively, denoted by dS_{xx} and dR_{xx}, respectively), the action of advective terms (term

IV, denoted by *Adv*) and the bed shear stress (term *V*, denoted by τ_b). Offshore of the breakpoint (Fig. 3(b), cross-shore position $X < 40$ m), the gradient in MWL is principally balanced by the wave radiation stress gradient. As the roller develops during breaking and advances in the surf zone, the balance between dS_{xx} and dR_{xx} is more complex, as they continually change sign. Hence, the role of the roller in balancing $d\eta$ – which balances the body force F along most of the profile – is unclear. Interestingly, the roller also pushes the location of the setdown shoreward, since $d\eta$ changes sign shoreward of the location where dS_{xx} does. Finally, as observed previously, the bed shear stress and advective terms play a relatively weak role, and have the same relative strength. Consequently, it is possible that, in most models where advective terms are ignored but bed shear stress is calibrated to match observations of MWL,[28] the bottom friction coefficient captures both bed friction and advective term effects.

The analysis of the total and mean momentum balances showed that, along with the pressure gradient induced by changes in MWL, total radiation stress gradients and surface shear stress play a major role in determining the value of mean return current U_r. As mentioned earlier, the surface stress is non-zero when waves start to dissipate their energy. Dissipation of wave energy results in a non-zero body force term, but also in a dissipation term.

To gain some insight into the behavior of the radiation stress gradient dS_{xx} [Eq. (30), 2nd term on the LHS], the profiles of the different terms in Eq. (30) are presented in Fig. 3(c). The radiation stress gradient is balanced principally by the body force hF [Eq. (30), 3rd term of the LHS] and the wave dissipation term [Eq. (30), 4th term on the LHS]. It is interesting to note that the wave dissipation term alters this balance and, inside the surf zone, works together with the body force hF to balance the radiation stress gradient. Advective terms play a relatively minor role in the decomposition of the radiation stress gradient, but, in all cases, strengthen it in the surf zone. Finally, decomposition of surface shear stress [Eq. (22)] in Fig. 3(d) shows that roller growth ($dR_{xx} < 0$) counteracts the effects of wave dissipation, and reduces the strength of the surface shear stress. When the roller decays ($dR_{xx} > 0$), it acts in conjunction with wave dissipation to push water shoreward at the MWL. In all cases, the effect of the roller is to push the location of maximum surface shear stress shoreward and sustain the surface shear stress when wave dissipation weakens. Consequently, as mentioned previously, the roller term plays an important role in generating the surface shear stress, which, together with the body force, balances the pressure gradient induced by changes in MWL.

IV. CONCLUSION

We have shown for the first time that, under the confines of linear wave theory, nearly all of the theoretical formulations of undertow that have been published can be reconciled. Throughout the derivations, we relied on the evolution equations of total momentum [Eq. (24)], mean momentum [Eq. (28)], and wave momentum [Eq. (29)], which are all linked by the equation of mass conservation [Eq. (3)]. Various formulations of the undertow based on linear wave theory are presented and the expression for surface shear stress originally presented by Deigaard and Fredsøe[19] was re-derived by relaxing some of their assumptions. The derivations are valid for all relative water depths, and included effects of bottom shear stress as well as mean currents. Second order mean current terms were assumed to be negligible. Further, we found that the forcing of the mean shear stress ($\tau = \nu_t \partial_z U$) is depth-uniform, and the mean shear stress is linear. The forcing can be expressed as a function of gradients in wave velocity and MWL, or by taking the difference between surface and bottom shear stresses.

Errors between solutions occur if one is not consistent in the methodology used to solve the three evolution equations. Specifically, three common types of inconsistencies were identified as follows:

1. Inclusion of advective terms in the depth-averaged cross-shore momentum equation [Eq. (24)] and in the mean current equation [Eq. (18)], but not in the wave action equation, Eq. (29).
2. Omission of the wave stress $\overline{\tilde{u}\tilde{w}}$ in the mean current equation [Eq. (18)], but inclusion of surface shear stress terms τ_s [Eq. (22)] in the solution.
3. Omission of bottom shear stress in wave-averaged total momentum equation.

Now that the theoretical foundation for the solution to the undertow problem assuming linear wave theory has been established, the next step is to evaluate the performance of the theoretical model at reproducing observed profiles of undertow. The skill of this model will be dependent on the performance of the wave and roller model.[55] Most importantly, the model performance will be a function of the formulation of the eddy viscosity.[35,46] The investigation of the performance of these different formulations will be the subject of a subsequent publication.

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APPENDIX A: SIMPLIFICATION OF WAVE STRESS TERMS

Consistent with linear wave theory, it is assumed that the flow is irrotational, which means that $\tilde{\omega} = 0$. The vertical derivative of $\overline{\tilde{u}\tilde{w}}$ is decomposed as[40]

$$\frac{\partial}{\partial z}\left(\overline{\tilde{u}\tilde{w}}\right) = -\frac{1}{2}\frac{\partial}{\partial x}\left(\overline{\tilde{u}^2} - \overline{\tilde{w}^2}\right). \tag{A1}$$

Dropping the tildes over wave velocity ($\tilde{u} = u$), this expression is combined with the other wave velocity terms in Eq. (14) to obtain[17]

$$\frac{\partial}{\partial x}\left(\overline{u^2} - \overline{w^2}\right) + \frac{\partial \overline{uw}}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x}\left(\overline{u^2} - \overline{w^2}\right). \tag{A2}$$

This latter equation is further simplified by using the linear wave theory expressions of u and w to obtain a depth uniform term[2,40,56]

$$\rho\left(\overline{u^2} - \overline{w^2}\right) = \frac{2kE_w}{\rho \sinh 2kh}. \tag{A3}$$

The wave radiation stress is expressed as[56,57]

$$\begin{aligned} S_{xx} &= \overline{\int_{-h}^{\eta} \rho\tilde{u}^2 + p dz} - \frac{1}{2}\rho g\left(h + \bar{\eta}\right)^2 \\ &= E_w\left(2C_g/C - 1/2\right) \\ &= \frac{2khE_w}{\sinh 2kh} + \frac{E_w}{2}, \end{aligned} \tag{A4}$$

so, Eq. (A3) becomes

$$\rho\left(\overline{u^2} - \overline{w^2}\right) = \frac{S_{xx} - E/2}{h}. \tag{A5}$$

APPENDIX B: NON-DIMENSIONAL ANALYSIS FOR MEAN VERTICAL VELOCITY

To estimate the relative order of mean vertical velocity with respect to mean horizontal velocity, horizontal and vertical velocities are first decomposed into mean, wave and turbulent components:

$$u = U + \tilde{u} + u', \tag{B1}$$

$$w = W + \tilde{w} + w'. \tag{B2}$$

This decomposition is applied to the continuity equation, Eq. (4), and we obtain after wave-averaging:

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0. \tag{B3}$$

Next, mean velocities are assumed to vary over a horizontal length scale $X = L_U$, which is much longer than the wave length scale $L_w = 2\pi/k$, a vertical length scale equal to the total water depth $Z = d$, and a time scale T_U which is much larger than the wave timescale $T_w = 2\pi/\sigma$. Non-dimensional velocities U^* and W^* are defined as $U^* = T_U U/L_U$ and $W^* = T_U W/L_U$, and non-dimensional length scales are defined as $x^* = x/X$ and $z^* = z/Z$. Hence the wave-averaged continuity equation becomes

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = \frac{L_u}{T_u}\frac{1}{L_u}\frac{\partial U^*}{\partial x^*} + \frac{L_U}{T_U}\frac{1}{h}\frac{\partial W^*}{\partial z^*}, \tag{B4}$$

which yields $W \sim \varepsilon U$, with $\varepsilon = h/L$. Consequently, in the wave-averaged horizontal momentum equation, Eq. (14), we have

$$\frac{\partial U^2}{\partial x} + \frac{\partial W^2}{\partial z} = \frac{L_U^2}{T_U^2}\frac{1}{L_u}\frac{\partial U^{*2}}{\partial x^*} + \frac{L_U^2}{T_U^2}\frac{1}{L_U}\frac{\partial W^{*2}}{\partial z^*}, \tag{B5}$$

which means that $\partial_x W^2 \sim \varepsilon^2 \partial_x U^2$. We now decide that, although we will keep terms of order ε in the remainder of this paper, we will neglect terms of order ε^2. Consequently, terms involving W^2 are neglected, but terms involving W are kept in the wave-averaged horizontal momentum equation.

APPENDIX C: EVOLUTION EQUATION FOR M^w

The wave action equation reads[36,58]

$$\frac{\partial}{\partial t}\frac{E_w}{\sigma_r} + \frac{\partial}{\partial x}\frac{E_w\left(C_g + U_r\right)}{\sigma_r} = -\frac{D}{\sigma_r}, \tag{C1}$$

where $A = E_w/\sigma_r$ is the wave action, and $\sigma_r^2 = (2\pi/T - U_r k)^2 = gk \tanh kh$. Expanding this equation by recognizing that the net wave momentum M^w is $M^w = Ak = kE_w/\sigma_r = E_w/C$ gives

$$\frac{\partial M^w}{\partial t} + \frac{\partial}{\partial x}M^w\left(C_g + U_r\right) - A\left(\frac{\partial k}{\partial t} + \left(C_g + U_r\right)\frac{\partial k}{\partial x}\right) = -\frac{kD}{\sigma_r}. \tag{C2}$$

To simplify this expression, the kinematical conservation equation is invoked (see Ref. 59, p. 23)

$$\frac{\partial k}{\partial t} + \nabla\left(\sigma_r + kU_r\right) = \frac{\partial k}{\partial t} + \left(C_g + U_r\right)\frac{\partial k}{\partial x} + k\frac{\partial U_r}{\partial x} + \frac{\partial \sigma_r}{\partial h}\frac{\partial h}{\partial x} = 0, \tag{C3}$$

where we used $\partial_k \sigma_r = C_g$. Hence, Eq. (C2) becomes

$$\frac{\partial M^w}{\partial t} + \frac{\partial}{\partial x}M^w\left(C_g + U_r\right) = -\frac{kD}{\sigma_r} - A\left(k\frac{\partial U_r}{\partial x} + \frac{\partial \sigma_r}{\partial h}\frac{\partial h}{\partial x}\right). \tag{C4}$$

Equation (C4) is further simplified by expressing $A\partial_h\sigma_r$ using the expression for C_g and trigonometric identities:

$$\frac{E_w}{\sigma_r}\frac{\partial \sigma_r}{\partial h} = \frac{E_w}{\sigma_r}\frac{\sigma_r k}{\sinh 2kh} = \frac{2n-1}{2kh}kE, \tag{C5}$$

where $n = C_g/C$. Consequently, Eq. (C4) becomes (see Ref. 47, Eq. (2.27))

$$\frac{\partial M^w}{\partial t} + \frac{\partial}{\partial x}M^w\left(C_g + U_r\right) = -\frac{kD}{\sigma_r} - M^w\frac{\partial U_r}{\partial x} - M^w\frac{E\,(n-1/2)}{h}\frac{\partial h}{\partial x}. \tag{C6}$$

Recognizing that $S_{xx} = 2M^w C_g - E/2 = E\,(2n - 1/2)$, Eq. (C6) is rewritten as

$$\frac{\partial M^w}{\partial t} + \frac{\partial S_{xx}}{\partial x} - \frac{\partial}{\partial x}\frac{E\,(2n-1)}{2} - h\frac{E\,(2n-1)}{2}\frac{\partial}{\partial x}\left(\frac{1}{h}\right) + \frac{\partial M^w U_r}{\partial x} + M^w\frac{\partial U_r}{\partial x} = -\frac{D}{C}. \tag{C7}$$

Finally, the wave action equation becomes

$$\frac{\partial M^w}{\partial t} + \frac{\partial S_{xx}}{\partial x} - h\frac{\partial}{\partial x}\left(\frac{S_{xx} - E/2}{2h}\right) + \frac{\partial M^w U_r}{\partial x} + M^w\frac{\partial U_r}{\partial x} = -\frac{D}{C}. \tag{C8}$$

APPENDIX D: DERIVATION OF UNDERTOW FORCING FOR DEPTH-UNIFORM HORIZONTAL CURRENT

In Sec. II A, we derived an expression for the solution of the wave-averaged momentum equation, Eq. (18), using a bottom shear stress boundary condition

$$\tau(z) = (F + P)(z + d) + \rho \frac{\partial}{\partial x} \int_{-d}^{z} U^2 dz - \rho U^2 \frac{\partial z}{\partial x} + \rho U W + \tau_b. \quad \text{(D1)}$$

We can simplify the nonlinear terms in this expression assuming a weak vertical variation of the mean horizontal current, i.e., $U(x, z) = U_r(x, z) + \varepsilon U_1(x, z)$, with $\varepsilon \ll 1$:

$$\begin{aligned} \rho \frac{\partial}{\partial x} \int_{-d}^{z} U^2 dz + \rho U W &= \int_{-d}^{z} \rho \frac{\partial U_r^2}{\partial x} + \rho \frac{\partial U_r W}{\partial z} dz \\ &= \rho \frac{\partial U_r^2}{\partial x}(z + d) + \rho (W(z) - W(-d)), \\ &= \rho U_r \frac{\partial U_r}{\partial x}(z + d) - \rho U_r^2 \frac{\partial z}{\partial x}, \end{aligned} \quad \text{(D2)}$$

where $\partial_x z$ was neglected and the integral of the wave-averaged continuity equation between z and $-d$ was used: $(z + d)\,\partial_x U_r + W(z) - W(-d) = 0$. Hence Eq. (D1) becomes

$$\rho \nu_t \frac{\partial U}{\partial z} = \left[\frac{\partial}{\partial x} \left(\frac{S_{xx} - E/2}{2h} \right) + \rho g \frac{\partial \bar{\eta}}{\partial x} + \rho U_r \frac{\partial U_r}{\partial x} \right] (z + d) + \tau_b. \quad \text{(D3)}$$

Similarly, the expression for the solution of the wave-averaged momentum equation, Eq. (18), using a shear stress at the MWL as boundary condition was

$$\rho \nu_t \frac{\partial U}{\partial z} = (F + P)(z - \bar{\eta}) - \rho \frac{\partial}{\partial x} \int_{z}^{\bar{\eta}} U^2 dz - \rho U^2 \frac{\partial z}{\partial x} + \rho U W - \rho U_{|\bar{\eta}} \frac{\partial M^w}{\partial x} + \tau_s, \quad \text{(D4)}$$

and after simplifying the nonlinear terms in this equation, assuming a weak vertical variation of the mean horizontal current, and invoking the wave-averaged continuity equation between z and MWL $\bar{\eta}$, we obtain

$$\rho \frac{\partial}{\partial x} \int_{z}^{\bar{\eta}} U^2 dz - \rho U^2 \frac{\partial z}{\partial x} + \rho U W - \rho U_{|\bar{\eta}} \frac{\partial M^w}{\partial x} = \rho U_r \frac{\partial U_r}{\partial x} (\bar{\eta} - z), \quad \text{(D5)}$$

and we obtain in Eq. (D4)

$$\rho \nu_t \frac{\partial U}{\partial x} = \left[\frac{\partial}{\partial x} \left(\frac{S_{xx} - E/2}{2h} \right) + \rho g \frac{\partial \bar{\eta}}{\partial x} + \rho U_r \frac{\partial U_r}{\partial x} \right] (z - \bar{\eta}) + \tau_s. \quad \text{(D6)}$$

Subtracting Eq. (D6) from Eq. (D3) yields the steady state mean momentum equation

$$h \frac{\partial}{\partial x} \left(\frac{S_{xx} - E/2}{2h} \right) + \rho g h \frac{\partial \bar{\eta}}{\partial x} + \rho h U_r \frac{\partial U_r}{\partial x} + \tau_b - \tau_s = 0. \quad \text{(D7)}$$

Finally, if $F + P$ is expressed from Eq. (D3), Eq. (D7) becomes

$$\rho \nu_t \frac{\partial U}{\partial x} = \frac{\tau_s - \tau_b}{h} (z + d) + \tau_b, \quad \text{(D8)}$$

where all explicit nonlinear terms have now vanished.

Equation (D7) is identical to the steady state momentum equation derived in Sec. II B 2, Eq. (28), because

$$\frac{\partial U_r^2 h}{\partial x} + U_r \frac{\partial M^w}{\partial x} = 2hU_r \frac{\partial U_r}{\partial x} - U_r h \frac{\partial U_r}{\partial x}$$
$$= hU_r \frac{\partial U_r}{\partial x}. \tag{D9}$$

These equalities are exact because $hU_r = -M^w$. But, in the surf zone, $hU_r = -M^r + M^w$. So, for Eq. (D9) to hold, M^w is re-defined as $M^w = M^w + M^r$(Ref. 14). But if we do so, then we can no longer arrive at the wave action equation, Eq. (29), since it was derived in Appendix C by taking $M^w = E_w/C$.

Consequently, for all equation to be consistent, the roller needs to be taken into account in the evolution of wave action. This problem illustrates once again the need to develop a rigorous equation for the wave roller evolution that can be incorporated in the momentum equation.

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