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An Inertial Subrange in Microstructure Spectra

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An inertial subrange was found in spectra calculated from vertical profiles of temperature gradient recorded in the bottom boundary layer of the Oregon shelf. Spectra were calculated for 53-cm vertical segments. An ensemble average of those spectra that were fully resolved and had high Cox number was compared to the universal form. Good agreement was found with the Batchelor form. The high wave number end of the inertial range was resolved. A relationship between the Kolmogorov constant for temperature, β, and the Batchelor constant, q, was established, $\beta q^{-2/3} = 0.172$ (± 0.012). If $\beta = 0.5$, as determined from atmospheric data, then $q = 4.95$ ($4.28 < q < 6.65$) and the transition from the inertial to the viscous-convective range occurs at a wave number $k = 0.035\, k_K$ ($0.021 < k/k_K < 0.043$) where k_K is the Kolmogorov wave number.

INTRODUCTION

Kolmogorov [1941] proposed that the small-scale properties of turbulent fluids are universal when scaled by factors depending only on ϵ, the dissipation rate for kinetic energy, and ν, the kinematic viscosity, if the Reynolds number is sufficiently large. In this equilibrium range of wave numbers, the eddies are statistically independent of the mechanism which generates the turbulence. If the Reynolds number is large enough, the equilibrium range will include an inertial subrange with length scales large enough that viscosity is not important. Neither production nor dissipation of turbulent energy occurs in the eddies in this subrange.

Obukhoff [1949] and *Corrsin* [1951] independently assumed that Kolmogorov's hypotheses apply to the distribution of temperature in a turbulent fluid if the temperature variations are so small that buoyancy effects are not important. They each deduced the existence of an inertial-convective range where temperature variance is neither produced nor dissipated. In this range, statistical properties, such as the temperature spectrum, depend only on ϵ and χ, the dissipation rate of temperature variance.

Batchelor [1959] determined the form of the temperature spectrum for higher wave numbers where viscous and diffusive effects are important. For fluid with large Prandtl number ($\nu \gg D$, where D is the thermal diffusivity), such as water, the Batchelor spectrum contains two subranges, the viscous-convective, where viscosity is effective but thermal diffusivity is not, and the viscous-diffusive, where both are important. The spectra conventionally shown are computed from the temperature gradient (Figure 1).

A number of observations of temperature or temperature-gradient spectra have been made in the ocean [e.g., *Grant et al.*, 1968; *Gregg*, 1976, 1977; *Nasmyth*, 1970; *Elliot and Oakey*, 1976; *Dillon and Caldwell*, 1980*a*]. These observations have, in general, two purposes: to determine whether turbulence theory applies to the ocean and to gain understanding of small-scale flow dynamics. Comparison with the predicted universal form has been unfavorable in several cases [*Nasmyth*, 1970; *Elliot and Oakey*, 1976]. *Dillon and Caldwell* [1980*a*] found good agreement with the Batchelor spectrum in the surface mixed layer when the Cox number, $\langle (dT/dz)^2 \rangle / \langle dT/dz \rangle^2$ is greater than 2500. They suggest that the broadening of the spectrum at lower Cox number is caused by contamination by fine structure.

The inertial subrange had not been identified in vertical temperature microstructure spectra until a recent paper in which one case in the surface layer was described [*Gregg and Sanford*, 1980]. *Batchelor* [1959] suggests that the Batchelor form may be found even if the Reynolds number is not high enough for an inertial range to exist. One reason that the inertial range is rarely seen in vertical microstructure may be that the turbulence is seldom stationary over large enough vertical scales [*Dillon and Caldwell*, 1980*a*]. Fine structure and internal waves may also contaminate the spectrum [*Gregg*, 1977].

An attempt to determine whether the inertial subrange ever exists in the ocean must take these difficulties into account. Observations should be made in a region of small stratification so that the vertical component of the turbulence is not suppressed. High Cox numbers and a lack of fine structure will prevent contamination of the spectrum. The bottom layer on the Oregon shelf satisfies these conditions [*Caldwell*, 1976, 1978; *Newberger and Caldwell*, 1981], as did the surface layer in the Sargasso Sea when *Gregg and Sanford* [1980] made their observations. The spectral cutoff (defined to be the wave number at which the temperature-gradient spectrum falls to 10% of its maximum) often occurs at very high wave number so that part of the inertial range, if it exists, will be reached in spectra from regions of small vertical extent.

DATA

Sixteen vertical profiles (Figure 2) of temperature and temperature gradient were made in a 2.5-h period in 85 m of water on the Oregon shelf (45°N) on October 22, 1977. The profiles were made with one freely falling microstructure instrument, and during the sampling period no changes were made affecting the descent rate of the instrument. This instrument responds quickly to changes in the vertical velocity of the water [*Dillon and Caldwell*, 1980*b*] so that the speed of the instrument through the water (9.1 cm s^{-1}) did not change during the sampling period.

Temperature-gradient spectra, corrected for thermistor response [*Dillon and Caldwell*, 1980*a*], were calculated for each 53 cm segment (512 points) of the bottom layer for each of these profiles. Of these segments, 77 were chosen for this analysis. Three factors determined which segments were chosen. First, the Cox number for each chosen segment was large (>2000) [*Dillon and Caldwell*, 1980*a*]. Second, the cutoff wave

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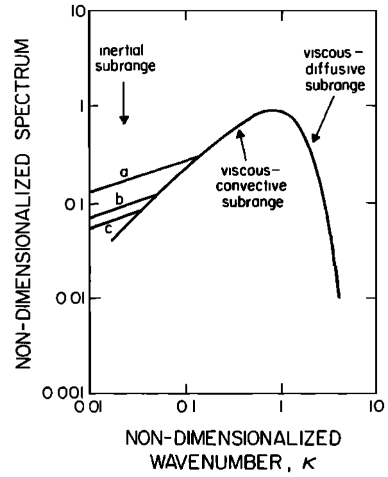

Fig. 1. The theoretical scaled temperature-gradient spectrum for line a, $\beta\alpha^{4/3} = 0.576$ ($\beta = 0.5$, $q = 2.0$); line b, $\beta\alpha^{4/3} = 0.318$ ($\beta = 0.5$, $q = 4.95$); line c, $\beta\alpha^{4/3} = 0.248$ ($\beta = 0.31$, $q = 2(3)^{1/2}$).

number (the wave number at which the spectrum falls to 10% of its maximum value) lay within the range of resolved wave numbers. Finally, the spectral estimates were required to be clearly above instrumental noise.

Taylor's hypothesis is required to express vertical microstructure spectra in terms of wave number. *Lumley* [1965] has shown that violations of this hypothesis caused by large turbulent intensities can cause distortion of the high wave number spectrum. The distortion depends on the relative magnitude of the typical turbulent velocity fluctuation, u', and the advection velocity, here the instrument descent rate, 9.1 cm s^{-1}. The mean current, measured by a nearby current meter, varied from 10 to 19 cm s^{-1} during this experiment. By using the usual estimate for the friction velocity, u_* ($u_* = \bar{U}/30$), u_* is at most 0.7 cm s^{-1}. In the bottom layer, u' and u_* are comparable, and Lumley's analysis indicates that with u' of this magnitude, distortion owing to this effect is negligible.

THEORETICAL BACKGROUND

The one-dimensional Batchelor spectrum for the temperature gradient can be written as [*Gibson and Schwarz,* 1963]

$$S(k) = \pi^{1/2}q^{1/2}\chi k_B^{-1}D^{-1}f[(2q)^{1/2}kk_B^{-1}] \qquad (1)$$

where q is a universal constant, k_B the Batchelor wave number $[k_B = (\epsilon\nu^{-1}D^{-2})^{1/4}]$, ϵ the dissipation rate for kinetic energy, ν the kinematic viscosity, D the thermal diffusivity, and χ the dissipation rate for temperature variance satisfying

$$\chi = 6D\int_0^\infty S(k)\,dk = 6D\left\{\frac{\overline{dT}}{dz}\right\}^2 \qquad (2)$$

The universal function f is given by

$$f(\zeta) = (2\pi)^{-1/2}\zeta\left[\exp(-\tfrac{1}{2}\zeta^2) - \zeta\int_\zeta^\infty \exp(-\tfrac{1}{2}y^2)\,dy\right] \qquad (3)$$

The transition from the inertial to the viscous-convective range occurs at wave number $k_* = C_*Pr^{-1/2}k_B = C_*k_K$ where k_K is the Kolmogorov wave number $[k_K = (\epsilon\nu^{-3})^{1/4}]$, Pr is the Prandtl number ($Pr = \nu/D$), and C_* is a universal constant. For wave numbers in the inertial range

$$S(k) = \beta\chi\epsilon^{-1/3}k^{1/3} \qquad (4)$$

where β is also universal. The three constants, β, q, and C_* are related by the requirement that the spectrum be continuous at k_*, so that there are two independent constants that must be determined to describe the predicted spectrum.

Caldwell et al. [1980, 1981] have shown that for vertical temperature gradient spectra the cutoff wave number is related to the Batchelor wave number. For the Batchelor spectrum, the relationship depends only on the universal constant q. Numerical evaluation of (3) shows that the universal function f falls to 10% of its maximum when the argument ζ is equal to 2.225. This implies that the cutoff occurs at $k = \alpha k_B$ where $\alpha = 2.225(2q)^{-1/2}$. This relationship can be used to define a scaling wave number $k_s = \alpha k_K = \alpha Pr^{-1/2}k_B$.

The spectrum is then nondimensionalized by multiplying (1) and (4) by $(k_s\nu\chi^{-1})$:

$$S(k)\{k_s\nu\chi^{-1}\} = \alpha(\pi qPr)^{1/2}f[(2q)^{1/2}Pr^{-1/2}\alpha\kappa] \qquad \kappa_* \le \kappa \qquad (5)$$

$$S(k)\{k_s\nu\chi^{-1}\} = \alpha^{4/3}\beta\kappa^{1/3} \qquad \kappa_0 < \kappa \le \kappa_* \qquad (6)$$

where $\kappa = k/k_s$ is the nondimensional wave number, $\kappa_* = k_*/k_s$ and κ_0 is the low wave number end of the inertial range. In this form (Figure 2) the spectrum above κ_* is independent of all parameters except the Prandtl number ($\alpha q^{1/2} = 1.573$), but values in the inertial range depend on both q and β.

There have been various attempts to evaluate the universal constants β, q, and C_*. The Kolmogorov constant β is best known. *Williams and Paulson* [1977] found $\beta \simeq 0.5$ from atmospheric measurements. *Paquin and Pond* [1971] found $\beta \simeq 0.4$, also in the atmosphere. *Grant et al.* [1968] found $\beta = 0.31$ in the ocean, while *Gibson and Schwarz* [1963] found $\beta = 0.35 \pm 0.05$ (standard deviation) in the laboratory.

Grant et al. [1968] estimated $q = 3.9 \pm 1.5$ and $C_* = 0.024 \pm 0.008$ (standard error estimates), *Gibson* [1968] suggested on theoretical grounds that $3^{1/2} < q < 2(3)^{1/2}$. *Gibson et al.* [1970] found C_* to be between 0.03 and 0.04.

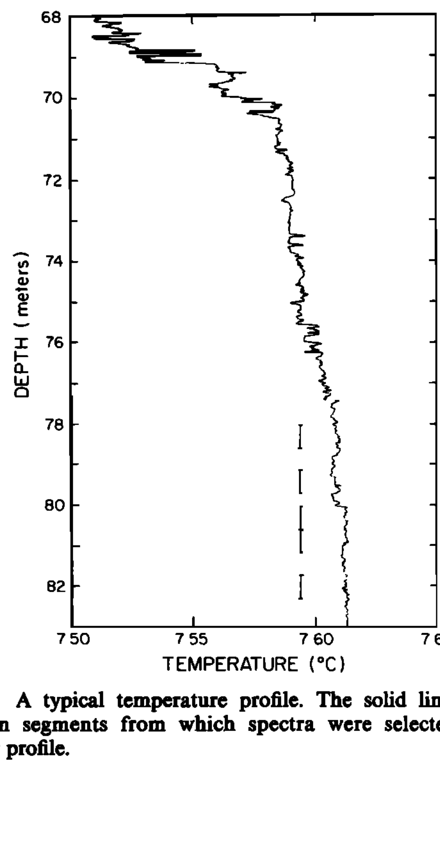

Fig. 2. A typical temperature profile. The solid lines indicate the 53-cm segments from which spectra were selected for this particular profile.

SCALED SPECTRUM

If the universal scaling is appropriate to the spectra from the bottom layer, the nondimensionalized spectra should collapse to the form of Figure 1. Although there is considerable scatter at the low wave number end where each point represents a single spectral estimate (Figure 3), good agreement with the universal form is revealed when the spectra are ensemble-averaged (Figure 4). Because of the statistical basis of the Batchelor theory, the scatter in Figure 3 is expected. The three lowest wave numbers do appear to lie in the inertial subrange. The values of the averaged spectrum at the inertial-range points together with (6) imply that $\beta\alpha^{4/3} = 0.318$ (standard deviation of the mean 0.020). This is equivalent to $\beta q^{-2/3} = 0.172$ (standard deviation of the mean 0.012). Limits can be placed on these estimates by considering the largest and smallest values of $\beta\alpha^{4/3}$ such that (6) lies within the standard error of the mean for each of the inertial-range points. Thus $0.256 < \beta\alpha^{4/3} < 0.347$ and $0.140 < \beta q^{-2/3} < 0.190$. The wave number for transition from the inertial range can be determined from the intersection of (5) and (6) with $\beta\alpha^{4/3} = 0.318$ and occurs at $\kappa_* = 0.049$ ($0.035 < \kappa_* < 0.057$). An independent estimate of ϵ, from velocity spectra for example, would be required to evaluate the constants separately.

Atmospheric data are more extensive and probably more accurate than oceanic measurements. In the inertial range, the difference in viscosity between air and water is not important. Therefore we assume $\beta = 0.5$ [*Williams and Paulson*, 1977] to determine q and C_*. We find that $q = 4.95$ ($6.65 > q > 4.28$) and $C_* = 0.035$ ($0.021 < C_* < 0.043$). If in fact $\beta = 0.31$ [*Grant et al.*, 1968], the lowest estimate of β, $q = 2.42$ and $C_* = 0.047$. Experimental evidence tends to support the lower value of C_*.

CONCLUSIONS

The inertial subrange is observed in temperature-gradient spectra from microstructure data recorded in the bottom layer on the Oregon shelf. Good agreement is found for higher wave numbers between the ensemble-averaged spectrum and the Batchelor form. A relationship between the universal constants β and q, $\beta q^{-2/3} = 0.172$, is established. If $\beta = 0.5$, as indicated by atmospheric measurements, $q = 4.95$ and $C_* = 0.035$. These observations were made under conditions nearly ideal for the existence of fully developed turbulence. In this case the inertial range exists, but such favorable conditions may be uncommon in the ocean.

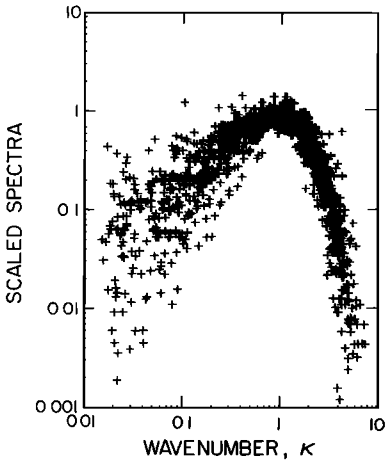

Fig. 3. Composite of the 77 nondimensionalized spectra.

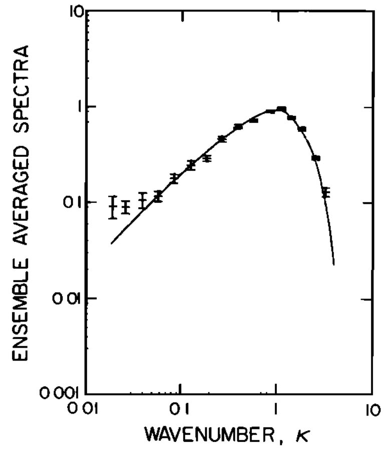

Fig. 4. The ensemble-averaged spectrum (average of spectra that are shown in Figure 3). The bars represent the standard error of the mean; the solid line is the Batchelor spectrum.

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