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IMPLICIT DEGENERATE EVOLUTION EQUATIONS AND APPLICATIONS*

EMMANUELE DI BENEDETTO† AND R. E. SHOWALTER‡

Abstract. The initial-value problem is studied for evolution equations in Hilbert space of the general form

$$\frac{d}{dt}\mathscr{A}(u)+\mathscr{B}(u)\ni f,$$

where $\mathscr{A}$ and $\mathscr{B}$ are maximal monotone operators. Existence of a solution is proved when $\mathscr{A}$ is a subgradient and either $\mathscr{A}$ is strongly monotone or $\mathscr{B}$ is coercive; existence is established also in the case where $\mathscr{A}$ is strongly monotone and $\mathscr{B}$ is subgradient. Uniqueness is proved when one of $\mathscr{A}$ or $\mathscr{B}$ is continuous self-adjoint and the sum is strictly monotone; examples of nonuniqueness are given. Applications are indicated for various classes of degenerate nonlinear partial differential equations or systems of mixed elliptic-parabolic-pseudo-parabolic types and problems with nonlocal nonlinearity.

1. Introduction. Let $\mathscr{A}$ and $\mathscr{B}$ be maximal monotone operators from a Hilbert space V to its dual V^*. Such operators are in general multi-valued and their basic properties will be recalled below. We shall consider initial-value problems of the form

$$\frac{d}{dt}\mathscr{A}(u)+\mathscr{B}(u)\ni f, \qquad \mathscr{A}u(0)\ni v_0, \tag{1.1}$$

where $f\in L^2(0, T; V^*)$ and $v_0\in V^*$ are given. It is assumed throughout our work that $\mathscr{A}$ is a compact operator from V to V^*. In applications to partial differential equations this assumption limits the order of the operator $\mathscr{A}$ to be strictly lower than that of $\mathscr{B}$. Both operators will be required to satisfy boundedness conditions, and one or the other is assumed to be a subgradient.

The objective of this work is to prove existence of a solution of (1.1) when $\mathscr{A}$ and $\mathscr{B}$ are possibly degenerate. Observe that we must in general assume some condition of coercivity on the pair of operators. To see this, we note that if one of them is identically zero then (1.1) is equivalent to a one-parameter family of "stationary" problems of the form $M(u(t))\ni F(t)$, where M is maximal monotone. But if M is, e.g., a subgradient in a space of finite dimension, it is surjective only if it is coercive. Thus it is appropriate to assume that at least one of $\mathscr{A}$ or $\mathscr{B}$ is coercive. In accord with this remark our work will proceed as follows. First we replace $\mathscr{A}$ by the coercive operator $\mathscr{A}+\varepsilon\mathscr{R}$, where $\varepsilon>0$ and $\mathscr{R}: V\to V^*$ is the Riesz isomorphism determined by the scalar product on V, and we solve the initial-value problem for the "regularized" equation

$$\frac{d}{dt}(\mathscr{A}+\varepsilon\mathscr{R})(u_\varepsilon)+\mathscr{B}(u_\varepsilon)\ni f. \tag{1.2}$$

Here we may take $\varepsilon=1$ with no loss of generality and we make no coercivity assumptions on either $\mathscr{A}$ or $\mathscr{B}$. Next we assume $\mathscr{B}$ is coercive and let $\varepsilon\to 0^+$ in order to recover (1.1) with (possibly) degenerate $\mathscr{A}$. Since $\mathscr{R}$ is of the same order as $\mathscr{B}$ this

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† Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706.

‡ Department of Mathematics RLM 8.100, The University of Texas at Austin, Austin, Texas 78712.

regularization is analogous to the Yoshida approximation. The operator $\mathscr{A}$ is assumed to be a subgradient in the above. Finally, we show the initial-value problem can be solved for (1.2) when $\mathscr{B}$ (but not necessarily $\mathscr{A}$) is a subgradient.

We mention some related work on equations of the form in (1.1). The theory of such implicit evolution equations divides historically into three cases. The first and certainly the easiest is where $\mathscr{B} \circ \mathscr{A}^{-1}$ is Lipschitz or monotone in some space [6], [23]. The second is that one of the operators is (linear) self-adjoint, and this case includes the majority of the applications to problems where singular or degenerate behavior arises due to spatial coefficients or geometry [2], [25]. These situations are described in the book [9] to which we refer for details and a very extensive bibliography. The third case is that wherein both operators are possibly nonlinear. This considerably more difficult case has been investigated by Grange and Mignot [12] and more recently by Barbu [4]. In both of these studies a compactness assumption similar to ours is made. Our boundedness assumptions are more restrictive than those in the papers above, but they assume f is smooth and that both operators are subgradients. By not requiring that $\mathscr{B}$ be a subgradient in (1.1) we obtain a significantly larger class of applications to partial differential equations, especially to systems.

Our work is organized as follows. In § 2 we recall certain information on maximal monotone operators and then state our results on the existence of solutions of the initial-value problems (1.1) and for (1.2). The proofs are given in §§ 3 and 4. Section 5 contains elementary examples of how nonuniqueness occurs, and we show there that uniqueness holds in the situation where one of the operators is self-adjoint. Section 6 is concerned with the structure and construction of maximal monotone operators between Hilbert spaces which characterize certain partial differential equations and associated boundary conditions. These operators are used to present in § 7 a collection of initial-boundary-value problems for partial differential equations which illustrate the applications of our results to the existence theory of such problems.

2. Preliminaries and main results. We begin by reviewing information on maximal monotone operators. Refer to [1], [3], [11] for additional related material and proofs. Then we shall state our existence theorems for the Cauchy problem (1.1).

Let V be a real Hilbert space and A a subset of the product $V \times V$. We regard A as a function from V to 2^V, the set of subsets of V, or as a multi-valued mapping or operator from V into V; thus, $f \in A(u)$ means $[u, f] \in A$. We define the domain $D(A) = \{u \in V: Au \text{ nonempty}\}$, range $R_g(A) = \bigcup\{Au: u \in V\}$ and inverse $A^{-1}(u) = \{v \in V: u \in A(v)\}$ of A as indicated. The operator A is *monotone* if $(f_1 - f_2, u_1 - u_2)_V \geqq 0$ whenever $[u_j, f_j] \in A$ for $j = 1, 2$. This is equivalent to $(I + \lambda A)^{-1}$ being a contraction for every $\lambda > 0$. We call A *maximal monotone* if it is maximal in the sense of inclusion of graphs. Then we have a monotone A maximal monotone if and only if $R_g(I + \lambda A) = V$ for some (hence, all) $\lambda > 0$. If A is maximal monotone we can define its *resolvent* $J_\lambda \equiv (I + \lambda A)^{-1}$, a contraction defined on all V, and its *Yoshida approximation* $A_\lambda = \lambda^{-1}(I - J_\lambda)$, a monotone Lipschitz function defined on all V. For $u \in V$ we have $A_\lambda(u) \in A(J_\lambda(u))$. We denote weak convergence of x_n to x by $x_n \rightharpoonup x$.

LEMMA 2.1. *Let A be maximal monotone, $[x_n, y_n] \in A$ for $n \geqq 1$, $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ and* $\liminf (y_n, x_n)_V \leqq (y, x)_V$. *Then $[x, y] \in A$. If in addition* $\limsup (y_n, x_n)_V \leqq (y, x)_V$, *then $(y_n, x_n)_V \to (y, x)_V$. We observe that A induces on $L^2(0, T; V)$ a maximal monotone operator (denoted also by A) defined by $v \in A(u)$ if and only if $v(t) \in A(u(t))$ for a.e. $t \in [0, T]$.*

A special class of maximal monotone operators arises as follows. If $\varphi: V \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function, we define the *subgradient*

$\partial\varphi \subset V \times V$ by

$$\partial\varphi(x) = \{z \in V : \varphi(y) - \varphi(x) \geqq (z, y - x) \text{ for all } y \in V\}.$$

The operator $\partial\varphi$ is maximal monotone. Furthermore it is useful to consider the convex *conjugate* of φ defined by

$$\varphi^*(z) \equiv \sup\{(z, y)_V - \varphi(y), y \in V\}.$$

The following are equivalent: $z \in \partial\varphi(x)$, $x \in \partial\varphi^*(z)$, and $\varphi(x) + \varphi^*(z) = (x, z)_V$; thus $\partial\varphi^*$ is the inverse of $\partial\varphi$. We mention the following *chain rule* [1]. Let $H^1(0, T; V)$ denote the space of absolutely continuous V-valued functions on $[0, T]$ whose derivatives belong to $L^2(0, T; V)$.

LEMMA 2.2. *If $u \in H^1(0, T; V)$, $v \in L^2(0, T; V)$ and $[u(t); v(t)] \in \partial\varphi$ for a.e. $t \in [0, T]$, then the function $t \to \varphi(u(t))$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\varphi(u(t)) = (w, u'(t)_V), \quad \textit{all } w \in \partial\varphi(u(t)),$$

for a.e. $t \in [0, T]$.

There is a version of a monotone operator from V to its dual space V^* which is equivalent to the above through the Riesz map $\mathcal{R}: V \to V^*$. Thus, $\mathcal{A} \subset V \times V^*$ is monotone if and only if $A \equiv \mathcal{R}^{-1} \circ \mathcal{A}$ is monotone in $V \times V$ and maximal monotone if and only if $R_g(\mathcal{R} + \mathcal{A}) = V^*$ in addition. We shall use these two equivalent notions interchangeably. Our applications to partial differential equations will lead to operators on $V \times V^*$. Also the subgradient is naturally constructed in the $W - W^*$ duality of a Banach (or topological vector) space W. Finally we cite the following chain rule.

LEMMA 2.3. *Let V and W be locally convex spaces with duals V^* and W^*. Let $\Lambda: V \to W$ be continuous and linear with dual $\Lambda^*: W^* \to V^*$. If $\varphi: W \to (-\infty, \infty]$ is proper, convex and lower semicontinuous then so also is $\varphi \circ \Lambda: V \to (-\infty, \infty]$, and if φ is continuous at some point of $R_g(\Lambda)$ we have* [11]

$$\partial(\varphi \circ \Lambda) = \Lambda^* \circ \partial\varphi \circ \Lambda.$$

Our results on the existence of solutions of the Cauchy problem (1.1) are stated as follows.

THEOREM 1. *Let W be a reflexive Banach space and V a Hilbert space which is dense and embedded compactly in W. Denote the injection by $i: V \to W$ and the dual (restriction) operator by i^* $W^* \to V^*$. Assume the following:*

[A_1] *The real-valued φ is proper, convex and lower semicontinuous on W, continuous at some point of V, and $\partial\varphi \circ i: V \to W^*$ is bounded.*

[B_1] *The operator $\mathcal{B}: V \to V^*$ is maximal monotone and bounded. Define $\mathcal{A} \equiv i^* \circ \partial\varphi \circ i$. Then for each given $f \in L^2(0, T; V^*)$ and $[u_0, v_0] \in \mathcal{A}$ there exists a triple $u \in H^1(0, T; V)$, $v \in H^1(0, T; V^*)$, and $w \in L^2(0, T; V^*)$ such that*

$$\frac{d}{dt}(\mathcal{R}u(t) + v(t)) + w(t) = f(t), \tag{2.1a}$$

$$v(t) \in \mathcal{A}(u(t)),\ w(t) \in \mathcal{B}(u(t)), \quad a.e.\ t \in [0, T], \tag{2.1b}$$

$$\mathcal{R}u(0) + v(0) = \mathcal{R}u_0 + v_0. \tag{2.1c}$$

THEOREM 2. *In addition to the above, assume:*

[A_2] *$\partial\varphi \circ i: L^2(0, T; V) \to L^2(0, T; W^*)$ is bounded.*

[B_2] *$\mathcal{B}: L^2(0, T; V) \to L^2(0, T; V^*)$ is bounded and coercive, i.e.,*

$$\lim_{\substack{\|u\|_{L^2(0,T;V)}\to+\infty \\ [u,v]\in\mathcal{B}}} \frac{\int_0^T v(t)(u(t))\,dt}{\|u\|_{L^2(0,T;V)}} = +\infty.$$

Then for each given $f \in L^2(0, T; V^*)$ and $v_0 \in R_g(\mathcal{A})$ there exists a triple $u \in L^2(0, T; V)$, $v \in H^1(0, T; V^*)$, $w \in L^2(0, T; V^*)$ such that

$$\frac{d}{dt} v(t) + w(t) = f(t), \tag{2.2a}$$

$$v(t) \in \mathcal{A}(u(t)), \; w(t) \in \mathcal{B}(u(t)), \quad \textit{a.e. } t \in [0, T], \tag{2.2b}$$

$$v(0) = v_0. \tag{2.2c}$$

Remarks. From Lemma 2.3 it follows that $\mathcal{A} = \partial(\varphi|_V)$ where $\varphi|_V = \varphi \circ i$ is the restriction of φ to V. Since $\mathcal{A}: V \to V^*$ is bounded it follows that $D(\mathcal{A}) = V$; hence,

$$V \subset D(\partial\varphi) \subset \operatorname{dom}(\varphi) \subset W,$$

and φ is continuous on the space V. Also, since $\varphi(0) < \infty$ we may assume with no loss of generality that $\varphi(0) \leqq 0$ and thus $\varphi^*(z) \geqq 0$ for all $z \in V$.

From the compactness of $i^*: W^* \to V^*$ it follows that $\mathcal{A}: V \to V^*$ is compact, i.e., maps bounded sets into relatively compact sets.

Since $\mathcal{B}$ is bounded and maximal monotone we have $D(\mathcal{B}) = V$. It is important for our applications that we have made no assumptions which directly relate $\mathcal{A}$ and $\mathcal{B}$. Specifically, we do not compare $\mathcal{A}(x)$ and $\mathcal{B}(x)$ in angle or in norm.

Finally, we give a variation on Theorem 1 in which only the second operator $\mathcal{B}$ is a subgradient. The compactness assumption on $\mathcal{A}$ is retained.

THEOREM 3. *Let the spaces V and W be given as before. Assume the following:*

[A_3] *The operator $\mathcal{A}: V \to V^*$ is maximal monotone with $R_g(\mathcal{A}) \subset W^*$ and $\mathcal{A}: V \to W^*$ is bounded.*

[B_3] *The real-valued ψ is proper, convex and lower semicontinuous on V and $\mathcal{B} \equiv \partial\psi: V \to V^*$ is bounded.*

Then for given $f \in L^2(0, T; V^*)$ and $[u_0, v_0] \in \mathcal{A}$ there exists a triple $u \in H^1(0, T; V)$, $v \in H^1(0, T; V^*)$ and $w \in L^2(0, T; V^*)$ satisfying (2.1).

3. Proofs of Theorem 1 and Theorem 3. These proofs are very similar; let us consider first Theorem 1. We formulate (2.1) in the space V. Set $A = \mathcal{R}^{-1} \circ \mathcal{A}$, $B = \mathcal{R}^{-1} \circ \mathcal{B}$, etc., and consider the equivalent equation

$$\frac{d}{dt}(u(t) + v(t)) + w(t) = f(t), \tag{3.1a}$$

$$v(t) \in A(u(t)), \qquad w(t) \in B(u(t)), \quad \text{a.e. } t \in [0, T]. \tag{3.1b}$$

Let $\lambda > 0$ and consider the approximation of (3.1) by

$$\frac{d}{dt}(u_\lambda(t) + v_\lambda(t)) + B_\lambda(u_\lambda(t)) = f(t), \tag{3.2a}$$

$$v_\lambda(t) \in A(u_\lambda(t)), \qquad t \in [0, T]. \tag{3.2b}$$

Since $(I + A)^{-1}$ and B_λ are both Lipschitz continuous from V to V, (3.2) has a unique absolutely continuous solution u_λ with $u_\lambda(0) + v_\lambda(0) = u_0 + v_0$. Since $(I + A)^{-1}$ is a function, we have $u_\lambda(0) = u_0$ and $v_\lambda(0) = v_0$.

We derive a priori estimates on u_λ. Take the scalar product in V of (3.2a) with $u_\lambda(t)$ and note

$$(v'_\lambda(t), u_\lambda(t))_V = \frac{d}{dt}\varphi^*(v_\lambda)$$

by Lemma 2.2, where φ^* is the conjugate of $\varphi|_V$ in V. Integrating the resulting identity gives

$$\frac{1}{2}\|u_\lambda(t)\|_V^2 + \varphi^*(v_\lambda(t))$$

$$\leqq \frac{1}{2}\|u_0\|_V^2 + \varphi^*(v_0) + \int_0^t (\|f(s)\|_V + \|B_\lambda(0)\|_V)\|u_\lambda(s)\|_V\, ds, \qquad 0 \leqq t \leqq T.$$

Since $\{B_\lambda(0)\}$ is bounded by the fact that $0 \in D(B)$, $\varphi^* \geqq 0$ and $f \in L^2(0, T; V)$, we have proved the first part of the following lemma.

LEMMA 3.1. *The following are bounded independent of* $\lambda > 0$:

(a) $\|u_\lambda\|_{L^\infty(0,T;V)}$, $\|\mathcal{R}v_\lambda\|_{L^\infty(0,T;W^*)}$, $\|J_\lambda(u_\lambda)\|_{L^\infty(0,T;V)}$, $\|B_\lambda(u_\lambda)\|_{L^\infty(0,T;V)}$,

(b) $\|u'_\lambda\|_{L^2(0,T;V)}$, $\|v'_\lambda\|_{L^2(0,T;V)}$.

Proof. The second and third terms of (a) are bounded because the operators $\mathcal{A}: V \to W^*$ and $J_\lambda \equiv (I + \lambda B)^{-1}: V \to V$ are bounded. Since $B_\lambda(u_\lambda) \in B(J_\lambda(u_\lambda))$ and B is bounded, the last term in (a) is bounded.

To obtain (b) we take the scalar product of (3.2a) by $u'_\lambda(t)$, note that $(v'_\lambda(t), u'_\lambda(t))_V \geqq 0$ by (3.2.b) and the monotonicity of A, and thereby obtain

$$\|u'_\lambda(t)\|_V^2 \leqq (\|f(t)\|_V + \|B_\lambda(u_\lambda(t)\|_V)\|u'_\lambda(t)\|_V,$$

so we bound the first term in (b). The second follows from (3.2.a).

Note that we have $\{\mathcal{R}v_\lambda\}$ bounded in $L^2(0, T; W^*)$ and $\{\mathcal{R}v'_\lambda\}$ bounded in $L^2(0, T; V^*)$. Since W^* is compact in V^* it follows from [17, p. 58] that $\{\mathcal{R}v_\lambda\}$ is (strongly) relatively compact in $L^2(0, T; V^*)$. From this observation and Lemma 3.1 it follows that we may pass to a subsequence, again denoted by u_λ, v_λ, for which we have

(3.3a) $$u_\lambda \rightharpoonup u, \quad B_\lambda(u_\lambda) \rightharpoonup w, \quad u'_\lambda \rightharpoonup u',$$

(3.3b) $$v_\lambda \to v \text{ (strongly)}, \quad v'_\lambda \rightharpoonup v' \quad \text{in } L^2(0, T; V),$$

(3.3c) $$u_\lambda(t) \rightharpoonup u(t) \quad \text{and} \quad v_\lambda(t) \to v(t), \quad \text{all } t \in [0, T].$$

Since $u_\lambda - J_\lambda(u_\lambda) = \lambda B_\lambda(u_\lambda) \to 0$ there follows

(3.3d) $$J_\lambda(u_\lambda) \rightharpoonup u \quad \text{in } L^2(0, T; V).$$

It remains to show that u, v, w satisfy (3.1) and the initial condition. First we use (3.3a) and (3.3b) and Lemma 2.1 to obtain $v \in A(u)$. Next we take the scalar product of (3.2a) with any $x \in V$ and integrate to get

$$(u_\lambda(t) + v_\lambda(t), x)_V + \int_0^t (B_\lambda(u_\lambda(s)), x)_V\, ds = \int_0^t (f(s), x)_V\, ds + (u_0 + v_0, x)_V.$$

Taking the limit as $\lambda \to 0$ gives (since x is arbitrary)

$$u(t) + v(t) + \int_0^t (w - f)\, ds = u_0 + v_0, \qquad 0 \leqq t \leqq T.$$

From this identity we obtain (3.1a) and $u(0)+v(0)=u_0+v_0$; since $v(0)\in A(u(0))$ and $(I+A)^{-1}$ is a function we have $u(0)=u_0$. In order to show $w\in B(u)$, and thereby finish the proof of Theorem 1, it suffices by Lemma 2.1 to show

$$\limsup_{\lambda\to 0}\,(B_\lambda(u_\lambda),J_\lambda(u_\lambda)-u)_{L^2(0,T;V)}\leqq 0.$$

We note further that

$$\begin{aligned}(B_\lambda(u_\lambda),J_\lambda(u_\lambda))&=(B_\lambda(u_\lambda),J_\lambda(u_\lambda)-u_\lambda)+(B_\lambda(u_\lambda),u_\lambda)\\&=-\lambda(B_\lambda(u_\lambda),B_\lambda(u_\lambda))+(B_\lambda(u_\lambda),u_\lambda)\end{aligned}$$

so it suffices to show

$$\limsup_{\lambda\to 0}\,(B_\lambda(u_\lambda),u_\lambda-u)_{L^2(0,T;V)}\leqq 0. \tag{3.4}$$

By (3.2a) it follows (3.4) is equivalent to

$$\liminf_{\lambda\to 0}\,(u'_\lambda+v'_\lambda,u_\lambda-u)_{L^2(0,T;V)}\geqq 0. \tag{3.5}$$

Define $\psi(x)=\frac{1}{2}\|x\|_V^2+\varphi(x)$, $x\in V$ so that $\partial\psi=I+\partial\varphi$. From (3.2b) and Lemma 2.2 we obtain

$$(u'_\lambda(t)+v'_\lambda(t),u_\lambda(t))_V=\frac{d}{dt}\psi^*(u_\lambda(t)+v_\lambda(t)),$$

and integrating yields

$$(u'_\lambda+v'_\lambda,u_\lambda)_{L^2(0,T;V)}=\psi^*(u_\lambda(T)+v_\lambda(T))-\psi^*(u_0+v_0).$$

Similarly we have from (3.1a)

$$(u'+v',u)_{L^2(0,T;V)}=\psi^*(u(T)+v(T))-\psi^*(u_0+v_0).$$

By (3.3c) and weak lower semicontinuity of ψ^* we have

$$\psi^*(u(T)+v(T))\leqq\liminf_{\lambda\to 0}\psi^*(u_\lambda(T)+v_\lambda(T)),$$

and our preceding calculations show that this is equivalent to (3.5).

Remark 3.1. From Lemma 2.1 we find that

$$(B_\lambda(u_\lambda),J_\lambda(u_\lambda))_{L^2(0,T;V)}\to(w,u)_{L^2(0,T;V)}.$$

If we also have B (or $\mathscr{B}$) *strongly monotone* then we can take the limit in the estimate

$$(B_\lambda(u_\lambda)-w,J_\lambda(u_\lambda)-u)_{L^2(0,T;V)}\geqq c\|J_\lambda(u_\lambda)-u\|^2_{L^2(0,T;V)}$$

to conclude that $\{J_\lambda(u_\lambda\}$ and $\{u_\lambda\}$ converge strongly to u in $L^2(0,T;V)$.

Remark 3.2. It is clear that we actually have $v(t)\in A(u(t))$ for every $t\in[0,T]$.

The proof of Theorem 3 closely follows the preceding pattern. That is, formulate (2.1) as the equivalent initial value problem for (3.1) and approximate this by (3.2) with $u_\lambda(0)+v_\lambda(0)=u_0+v_0$ for each $\lambda>0$.

To derive a priori bounds we take the scalar product of (3.2a) with $u'_\lambda(t)$ and integrate to obtain

$$\int_0^T\|u'_\lambda\|_V^2+\psi_\lambda(u_\lambda(T))\leqq\psi_\lambda(u_0)+\int_0^T(f(t),u'_\lambda(t))_V\,dt. \tag{3.6}$$

Here ψ_λ is the Yoshida approximation of ψ. We may assume ψ is nonnegative and the same holds for ψ_λ, so we have the first part of the following.

LEMMA 3.2. *The following are bounded independent of* $\lambda > 0$:

$$\text{(a)} \quad \begin{gathered} \|u_\lambda\|_{L^\infty(0,T;V)}, \quad \|u'_\lambda\|_{L^2(0,T;V)}, \quad \|\mathcal{R}v_\lambda\|_{L^\infty(0,T;W^*)}, \\ \|J_\lambda(u_\lambda)\|_{L^\infty(0,T;V)}, \quad \|B_\lambda(u_\lambda)\|_{L^\infty(0,T;V)}, \end{gathered}$$

$$\text{(b)} \quad \|v_\lambda\|_{L^\infty(0,T;V)}, \qquad \|v'_\lambda\|_{L^2(0,T;V)}.$$

Proof. The bound on the first two terms in (a) follow from (3.6) and the remaining terms in (a) are bounded by $[A_3]$ and $[B_3]$. Next we take the scalar product of (3.2a) with $v'_\lambda(t)$, and obtain (b) as was done in Lemma 3.1.

We may pass to a subsequence satisfying (3.3) and we obtain as in Theorem 1 the triple u, v, w satisfying the equation (3.1a) and initial condition and $v(t) \in Au(t)$, $t \in [0, T]$. It remains to show $w \in B(u)$ and this is equivalent to showing (cf. (3.5))

$$\liminf_{\lambda \to 0} (u'_\lambda + v'_\lambda, u_\lambda)_{L^2(0,T;V)} \geqq (u' + v', u)_{L^2(0,T;V)}. \tag{3.7}$$

Since $u'_\lambda \in L^2(0, T; V)$ we may integrate by parts to compute

$$\begin{aligned} (u'_\lambda + v'_\lambda, u_\lambda)_{L^2(0,T;V)} = \tfrac{1}{2}(\|u_\lambda(T)\|_V^2 - \|u_0\|_V^2) - (v_\lambda, u'_\lambda)_{L^2(0,T;V)} \\ + (v_\lambda(T), u_\lambda(T))_V - (v_0, u_0) \end{aligned} \tag{3.8a}$$

and similarly, since $u' \in L^2(0, T; V)$,

$$\begin{aligned} (u' + v', u)_{L^2(0,T;V)} = \tfrac{1}{2}(\|u(T)\|_V^2 - \|u_0\|_V^2) - (v, u')_{L^2(0,T;V)} \\ + (v(T), u(T))_V - (v_0, u_0)_V. \end{aligned} \tag{3.8b}$$

Finally we observe that (3.7) follows immediately from (3.3) and (3.8).

Remark 3.2. If in addition B is strongly monotone, then $\{u_\lambda\}$ converges strongly to u in $L^2(0, T; V)$.

4. Proof of Theorem 2. Choose $u_0 \in A^{-1}(v_0)$. For each $\lambda > 0$ let $u_\lambda, v_\lambda \in H^1(0, T; V)$, $w_\lambda \in L^2(0, T; V)$ satisfy

$$\lambda u'_\lambda(t) + v'_\lambda(t) + w_\lambda(t) = f(t), \tag{4.1a}$$

$$v_\lambda(t) \in A(u_\lambda(t)), \qquad w_\lambda(t) \in B(u_\lambda(t)), \qquad \text{a.e. } t \in [0, T], \tag{4.1b}$$

$$\lambda u_\lambda(0) + v_\lambda(0) = \lambda u_0 + v_0. \tag{4.1c}$$

The problem (4.1) has a solution by Theorem 1, and our plan is to show that we may take the limit as $\lambda \to 0$ in (4.1) to obtain a solution $u, w \in L^2(0, T; V)$, $v \in H^1(0, T; V)$ of

$$v'(t) + w(t) = f(t) \tag{4.2a}$$

$$v(t) \in A(u(t)), \qquad w(t) \in B(u(t)), \quad \text{a.e. } t \in [0, T], \tag{4.2b}$$

$$v(0) = v_0. \tag{4.2c}$$

With our notation $A = \mathcal{R}^{-1} \circ \mathcal{A}$, etc., (4.2) is equivalent to (2.2).

We proceed to derive a priori estimates. Consider first the initial condition. Since $(\lambda I + A)^{-1}$ is a function it follows from (4.1c) that

$$u_\lambda(0) = u_0, \qquad v_\lambda(0) = v_0, \qquad \lambda > 0. \tag{4.3}$$

LEMMA 4.1. *The following are bounded independent of* $\lambda > 0$:

(a) $\|u_\lambda\|_{L^2(0,T;V)}, \qquad \lambda^{1/2}\|u_\lambda\|_{L^\infty(0,T;V)},$

(b) $\|w_\lambda\|_{L^2(0,T;V)}, \qquad \|\mathcal{R}v_\lambda\|_{L^2(0,T;W^*)}.$

Proof. Take the scalar product of (4.1a) with $u_\lambda(t)$ and integrate to obtain

$$\begin{aligned}&\frac{\lambda}{2}\|u_\lambda(t)\|_V^2+\varphi^*(v_\lambda(t))+\int_0^t (w_\lambda, u_\lambda)_V\\ &\leqq \frac{\lambda}{2}\|u_0\|_V^2+\varphi^*(v_0)+\int_0^t (f, u_\lambda)_V, \qquad 0\leqq t\leqq T.\end{aligned} \tag{4.4}$$

We drop the second (nonnegative) term in (4.4) and note by the monotonicity of B that $(w_\lambda, u_\lambda)_V \geqq (\xi, u_\lambda)_V$ for some $\xi \in B(0)$. Thus (4.4) gives

$$\int_0^T (w_\lambda, u_\lambda)_V \leqq \|f\|_{L^2(0,T;V)}\|u_\lambda\|_{L^2(0,T;V)}+C,$$

and the coercivity of B implies the boundedness of the first term in (a). The second now follows from (4.4) and now part (b) follows from our assumptions [A_1] and [B_1].

LEMMA 4.2. *The following are bounded independent of* $\lambda > 0$:

$$\|v'_\lambda\|_{L^2(0,T;V)}, \qquad \|\lambda u'_\lambda\|_{L^2(0,T;V)}.$$

Proof. Take the scalar product of (4.1a) with $v'_\lambda(t)$. Since $(u'_\lambda(t), v'_\lambda(t))_V \geqq 0$ by the monotonicity of A, we obtain

$$\|v'_\lambda(t)\|_V^2 \leqq (\|f(t)\|_V+\|w_\lambda(t)\|_V)\|v'_\lambda(t)\|_V,$$

from which the first bound is immediate. To obtain the second we take the scalar product of (4.1a) with $u'_\lambda(t)$ and drop the nonnegative term $(u'_\lambda(t), v'_\lambda(t))_V$. This gives

$$\lambda\|u'_\lambda(t)\|_V^2 \leqq (\|f(t)\|_V+\|w_\lambda(t)\|_V)\|u'_\lambda(t)\|_V,$$

and hence the desired bound.

We have now shown that $\{\mathcal{R}v_\lambda\}$ is bounded in $L^2(0, T; W^*)$ and that $\{\mathcal{R}v'_\lambda\}$ is bounded in $L^2(0, T; V^*)$. Since W^* is compact in V^* it follows that $\{\mathcal{R}v_\lambda\}$ is strongly compact in $L^2(0, T; V^*)$. From this observation, Lemma 4.1 and Lemma 4.2 it follows we may pass to a subsequence (which we denote again by $\{u_\lambda\}$, $\{v_\lambda\}$, $\{w_\lambda\}$) for which in $L^2(0, T; V)$ we have

$$u_\lambda \rightharpoonup u, \quad w_\lambda \rightharpoonup w, \quad v_\lambda \rightharpoonup v, \quad v'_\lambda \rightharpoonup v'.$$

Note that $\lambda u_\lambda \to 0$ and it follows that $\lambda u'_\lambda \rightharpoonup 0$ by standard arguments. Furthermore, we may assume $v_\lambda(t) \to v(t)$ in V for all $t \in [0, T]$ by equicontinuity of $\{v_\lambda\}$, and similarly $\lambda u_\lambda(t) \to 0$ in V for all $t \in [0, T]$.

It remains to show that the triple u, v, w obtained above constitutes a solution of (4.2). Let $x \in V$, take the scalar product of x with (4.1a) and integrate to obtain

$$(\lambda u_\lambda(t)+v_\lambda(t), x)_V+\int_0^t (w_\lambda(s), x)_V=\int_0^t (f(s), x)_V\,ds+(\lambda u_0+v_0, x)_V.$$

Since weak convergence in $L^2(0, T; V)$ implies weak convergence in $L^2(0, t; V)$ letting $\lambda \to 0$ gives that

$$(v(t), x)_V+\int_0^t (w(s), x)_V\,ds=\int_0^t (f(s), x)_V\,ds+v_0, \qquad x \in V, \quad t \in [0, T].$$

That is,

$$v(t)+\int_0^t w(s)\,ds=\int_0^t f(s)\,ds+v_0,\quad \text{a.e. } t\in[0,T],$$

and this implies (4.2a) and (4.2c). From Lemma 2.1 there follows $v\in A(u)$ so it remains only to establish $w\in B(u)$. For this it suffices by Lemma 2.1 to show

$$\limsup_{\lambda\to 0}\,(w_\lambda, u_\lambda)_{L^2(0,T;V)}\leqq (w,u)_{L^2(0,T;V)}. \tag{4.5}$$

In order to prove (4.5) we first note by (4.1a) and (4.2a) that it is equivalent to

$$\liminf_{\lambda\to 0}\,(\lambda u'_\lambda+v'_\lambda, u_\lambda)_{L^2(0,T;V)}\geqq (v',u)_{L^2(0,T;V)}. \tag{4.6}$$

Since $u_\lambda(t)\in A^{-1}(v_\lambda(t))=\partial\varphi^*(v_\lambda(t))$ a.e. on $[0,T]$, where φ^* is the conjugate of $\varphi|_V$, we obtain from Lemma 2.2

$$\begin{aligned}(\lambda u'_\lambda+v'_\lambda, u_\lambda)_{L^2(0,T;V)}&=\frac{\lambda}{2}\|u_\lambda(T)\|_V^2+\varphi^*(v_\lambda(T))-\frac{\lambda}{2}\|u_0\|_V^2-\varphi^*(v_0)\\&\geqq\varphi^*(v_\lambda(T))-\frac{\lambda}{2}\|u_0\|_V^2-\varphi^*(v_0).\end{aligned}$$

Similarly we compute

$$(v',u)_{L^2(0,T;V)}=\varphi^*(v(T))-\varphi^*(v_0).$$

Since $\{v_\lambda\}$ are equi-uniformly-continuous we have $v_\lambda(t)\to v(t)$ at every $t\in[0,T]$, so the lower semicontinuity of φ^* gives

$$\liminf_{\lambda\to 0}\,\varphi^*(v_\lambda(T))\geqq\varphi^*(v(T)).$$

In view of the preceding computations this is exactly (4.6).

Remark 4.1. If B is strongly monotone then $\{u_\lambda\}$ converges strongly to u in $L^2(0,T;V)$.

5. Remarks on uniqueness. We first present an example which shows that gross nonuniqueness of solutions of (1.1) can occur, even if both operators are strongly monotone subgradients. Moreover the nonuniqueness occurs in each term of the triple u, v, w, not just in the latter two terms selected, respectively, from $A(u)$ and $B(u)$. Next we shall show that uniqueness does hold for (1.1) when at least one of the operators is continuous, linear and symmetric and the sum of the operators is strictly monotone. Our last example shows that symmetry of the linear operator is essential.

Example 1. Let $V=W=R$, the space of real numbers, and define

$$A(s)=B(s)=s+H(s-1),$$

where

$$H(r)=\begin{cases}1, & r>0,\\ [0,1], & r=0,\\ 0, & r<0\end{cases}$$

denotes the Heaviside function and $f\equiv 0$. Consider the initial-value problem (1.1), which takes the form

$$\begin{aligned}&v'(t)+w(t)=0,\qquad v(0)=2,\\&v(t)-u(t)\in H(u(t)-1),\qquad w(t)-u(t)\in H(u(t)-1).\end{aligned}\tag{5.1}$$

Let g be any maximal monotone graph or continuous function from R to R such that $g(s)=s$ for $s\notin[1,2]$ and $g(s)\subset[1,2]$ for $s\in[1,2]$. Then, if v is a solution of

$$v'(t)+g(v(t))=0, \qquad t\geqq 0, \quad v(0)=2, \tag{5.2}$$

it follows that with $u(t)\equiv A^{-1}(v(t))$ and $w(t)\equiv -v'(t)$ we have a solution of (5.1). This procedure yields an abundance of solutions.

We display some special cases of the above. Pick $c\in[\frac{1}{2},1]$ and define g_c to be the maximal monotone graph such that $g_c(t)=\{c^{-1}\}$, $t\in(1,2)$, and $g_c(t)=\{t\}$, $t\notin[1,2]$. The corresponding solution v_c of (5.2) is given by

$$v_c(t)=2-\frac{t}{c}, \quad 0\leqq t\leqq c, \qquad v_c(t)=e^{c-t}, \quad t\geqq c.$$

With the two functions u_c and w_c given by

$$u_c(t)=1, \qquad w_c(t)=\frac{1}{c} \quad \text{for } 0\leqq t\leqq c,$$

$$u_c(t)=w_c(t)=e^{c-t}, \qquad t\geqq 0,$$

this provides a continuum of solutions of (5.1).

We can give the following elementary sufficient conditions for uniqueness to hold for (1.1) or, equivalently, for (4.2).

THEOREM 4. *Let A and B be monotone operators on a Hilbert space V. Suppose $A+B$ is strictly monotone and that one of A or B is continuous, linear and symmetric. Then for each function $f:[0,T]\to V$ and $v_0\in V$ there is at most one solution u, v, w of (4.2).*

Proof. Suppose A is continuous, linear and symmetric. For $j=1,2$ let u_j, v_j, w_j be a solution of (4.2). Take the scalar product of the difference of (4.2a) with u_1-u_2 to obtain

$$\frac{1}{2}\frac{d}{dt}(A(u_1(t)-u_2(t)),u_1(t)-u_2(t))_V+(w_1(t)-w_2(t),u_1(t)-u_2(t))_V=0.$$

Integrating this identity and using (4.2c) gives

$$\frac{1}{2}(A(u_1(t)-u_2(t)),u_1(t)-u_2(t))_V+\int_0^t(w_1-w_2,u_1-u_2)_V\,ds=0, \qquad 0\leqq t\leqq T,$$

and this implies

$$Au_1(t)=Au_2(t),\ (w_1(t)-w_2(t),u_1(t)-u_2(t))_V=0 \quad \text{a.e. } t\in[0,T].$$

Since $A+B$ is strictly monotone we have $u_1(t)=u_2(t)$; hence $v_1(t)=Au_1(t)=Au_2(t)=v_2(t)$ and, by (4.1a), $w_1(t)=w_2(t)$ a.e. on $[0,T]$.

Suppose now B is continuous, linear and symmetric. Starting with two solutions as above we integrate the corresponding equations (4.2a) to obtain

$$v_j(t)+B(\theta_j(t))=v_0+\int_0^t f, \qquad j=1,2, \tag{5.3}$$

where $\theta_j(t)\equiv\int_0^t u_j$. Taking the difference of (5.3) for $j=1,2$, then the scalar product with $\theta_1'-\theta_2'$ and integrating gives us

$$\int_0^t(v_1-v_2,\theta_1'-\theta_2')_V+\frac{1}{2}(B(\theta_1(t)-\theta_2(t)),\theta_1(t)-\theta_2(t))_V=0. \tag{5.4}$$

Since $v_j(t) \in A(\theta_j'(t))$ a.e., each term is nonnegative. It follows that $B(\theta_1(t)-\theta_2(t))=0$ on $[0, T]$, and thus from (5.4) that

$$(v_1(t)-v_2(t), u_1(t)-u_2(t))_V = 0 \quad \text{a.e. } t \in [0, T],$$

so the desired results follows by strict monotonicity of $A+B$.

Finally we cite an example to show that the symmetry condition cannot be eliminated from Theorem 4.

Example 2. Let $H^1(0, 1)$ be the Sobolev space of those absolutely continuous functions on the interval $(0, 1)$ whose first derivatives belong to $L^2(0, 1)$; set $V = \{v \in H^1(0, 1): v(1) = 0\}$ and note that $V \subset L^2(0, 1) \subset V^*$. Define $\mathscr{A}: V \to V^*$ by $\mathscr{A}(v) = -v'$. Clearly $\mathscr{A}$ is linear and we have

$$\mathscr{A}(v)(v) = -\int_0^1 v'v\, ds = \frac{1}{2}|v(0)|^2 \geqq 0,$$

so $\mathscr{A}$ is monotone. Let β be given by

$$\beta(r) = \begin{cases} \dfrac{r}{2}, & r<0 \text{ or } r>1, \\[2ex] \dfrac{r^2}{2}, & 0 \leqq r \leqq 1, \end{cases}$$

and define $\mathscr{B}: V \to V^*$ by

$$\mathscr{B}(u)(v) = \int_0^1 \beta(u'(s))v'(s)\, ds, \qquad u, v \in V.$$

It is easy to check that $\mathscr{B}$ is a strictly monotone subgradient on V.

Consider the Cauchy problem

$$\frac{d}{dt}\mathscr{A}(u) + \mathscr{B}(u) = 0, \qquad \mathscr{A}u(0) = -1 \tag{5.5}$$

with the above operators. A solution u of (5.5) is a weak solution of the initial-boundary-value problem

$$(-u_x)_t - (\beta(u_x))_x = 0, \qquad 0<x<1, \quad 0<t, \tag{5.6a}$$

$$u_x(0, t) = u(1, t) = 0, \tag{5.6b}$$

$$-u_x(x, 0) = -1, \tag{5.6c}$$

where the subscripts denote partial derivatives. Consider the following two functions:

$$u^{(1)}(x, t) = \begin{cases} \dfrac{x^2+t^2}{2t} - 1, & 0<x<t<1, \\[2ex] x-1, & 0<t<x<1, \end{cases}$$

$$u^{(2)}(x, t) = \begin{cases} \dfrac{t}{2} - 1, & 0<x<\dfrac{t}{2}<\dfrac{1}{2}, \\[2ex] x-1, & 0<\dfrac{t}{2}<x<1, \quad t<1. \end{cases}$$

It is a straightforward computation to check that both $u^{(1)}$ and $u^{(2)}$ are solutions of (5.6), hence, both are solutions of (5.5). Note that the only condition of Theorem 4 not met in this example is the symmetry of $\mathscr{A}$. It shows also that $\mathscr{B}$ being a subgradient is not a satisfactory substitute for $\mathscr{B}$ to be continuous and self-adjoint.

6. Construction of differential operators. We have been discussing evolution equations which contain a pair of nonlinear operators from a Hilbert space V to its dual V^*. In our applications the generalized solutions obtained in our theorems may satisfy natural or variational boundary conditions (e.g., of Neumann type) which are implicit in the functional identity

$$\frac{d}{dt}\mathscr{A}(u(t))+\mathscr{B}(u(t))\ni f(t) \tag{6.1}$$

in V^*. Such boundary conditions are classically recovered by Green's formula so we shall describe an appropriate extension of this formula which requires a minimum of regularity of the generalized solution. The objective is to resolve each term in (6.1) into two parts, a differential operator in distributions over a region Ω, the formal operator, and a constraint on the boundary Γ, the boundary operator. Then we briefly recall basic facts on Sobolev spaces and construct a rather general nonlinear operator $\mathscr{B}$ which will be used in the next section to illustrate theorems in some examples of initial-boundary-value problems.

Assume we are given a linear surjection $\gamma: V\to T$, called a "trace" operator, which is a strict homomorphism onto its range T, called "boundary values" of V. Let V_0 be the kernel of γ and note that the dual operator, $\gamma^*(g)=g\circ\gamma$, is an isomorphism of the dual space T^* onto the annihilator $V_0^\perp$ in V^*. Suppose there is given a continuous seminorm $|\cdot|$ on V for which V_0 is dense in the seminorm space $U\equiv\{V,|\cdot|\}$. Then we naturally identify U^* simultaneously as a subspace of V^* and of V_0^*.

We resolve the operator $\mathscr{A}: V\to 2^{V^*}$ into a formal part in V_0^* and a boundary part in T^*. For each $u\in D[\mathscr{A}]$ set $A_0(u)=\{F|_{V_0}: F\in\mathscr{A}(u)\}$, the set of restrictions to V_0 of functionals in $\mathscr{A}(u)$. Then set $D[\mathscr{A}_0]\equiv\{u\in V: A_0(u)\cap U^*\neq\phi\}$ and define $\mathscr{A}_0: V\to 2^{U^*}$ by $\mathscr{A}_0(u)=A_0(u)\cap U^*$. That is, $\mathscr{A}_0$ is the set of those functionals in $A_0(u)$ which have (unique) continuous extensions in $U^*\subset V^*$. Now let $u\in D[\mathscr{A}_0]$ and $F\in\mathscr{A}(u)$ with $F_0=F|_{V_0}\in U^*$; hence, $F_0\in\mathscr{A}_0(u)$. Then in $V_0^\perp$ we have $F-F_0=\gamma^*(g)$ for a unique $g\in T^*$, so we can define $\partial_{\mathscr{A}}(u)\subset T^*$ to be the set of all such g. Thus, for each $F\in\mathscr{A}(u)$ for which $F_0=F|_{V_0}\in U^*$, there is a unique $g\in T^*$ for which

$$F(v)=F_0(v)+g(\gamma v),\qquad v\in V,$$

and we indicate this by

$$\mathscr{A}(u)=\mathscr{A}_0(u)+\gamma^*(\partial_{\mathscr{A}}(u)),\qquad u\in D[\mathscr{A}_0]. \tag{6.2}$$

In our applications V_0^* is a space of distributions over Ω and T is the space of boundary values of the Sobolev space V, so (6.2) is the abstract *Green's formula* for the operator $\mathscr{A}$.

In many examples the solutions of (6.1) will have the additional regularity properties described below.

LEMMA 6.1. *Let* $v\in H^1(0,T;V^*)$ *with* $v(t)\in\mathscr{A}(u(t))$ *a.e. on* $[0,T]$, *and set* $v_0(t)=v(t)|_{V_0}$ *for each* $t\in[0,T]$. *Let* $v_0(t)\in U^*$ *and define* $g(t)\in T^*$ *by* $v(t)=v_0(t)+\gamma^*(g(t))$ *for* $t\in[0,T]$. *If* $v_0'(t)\in U^*$ *a.e. on* $[0,T]$, *then* $g\in H^1(0,T;T^*)$ *and*

$$v'(t)=v_0'(t)+\gamma^*(g'(t)),\quad \text{a.e. } t\in[0,T].$$

The preceding situation occurs, for example, in the case of linear symmetric $\mathscr{A}$ and in certain other special cases [2,], [9], [17], [25].

Suppose the operator $\mathscr{A}$ is given as above and let a second operator $\mathscr{B}: V \to 2^{V^*}$ be given. Resolve it likewise into two parts:

$$\mathscr{B}(u) = \mathscr{B}_0(u) + \gamma^*(\partial_{\mathscr{B}}(u)), \qquad u \in D[\mathscr{B}_0]. \tag{6.3}$$

Let there be given $f_0 \in L^2(0, T; U^*)$, $g_0 \in L^2(0, T; T^*)$, $v_0 \in R_g[A_0]$ and $g_0 \in T^*$ with $v_0 + \gamma^*(g_0) \in R_g[\mathscr{A}_0]$. Consider a solution of the Cauchy problem

$$\frac{d}{dt}\mathscr{A}(u(t)) + \mathscr{B}(u(t)) \ni f_0(t) + \gamma^*(g_0(t)), \quad \text{a.e. } t \in [0, T],$$

$$\mathscr{A}(u(0)) \ni v_0 + \gamma^*(g_0),$$

that is, a triple u, v, w for which

$$\begin{aligned} &v(t) \in \mathscr{A}(u(t)), \qquad w(t) \in \mathscr{B}(u(t)),\\ &v'(t) + w(t) = f_0(t) + \gamma^*(g_0(t)), \quad \text{a.e. } t \in [0, T],\\ &v(0) = v_0 + \gamma^*(g_0). \end{aligned} \tag{6.4}$$

By restricting the above functionals to V_0 we obtain

$$\begin{aligned} &v_0(t) \in A_0(u(t)), \qquad w_0(t) \in B_0(u(t)),\\ &v_0'(t) + w_0(t) = f_0(t) \quad \text{in } V_0^*, \quad \text{a.e. } t \in [0, T],\\ &v_0(0) = v_0. \end{aligned} \tag{6.5}$$

If Lemma 6.1 applies, then we obtain $w_0(t) \in U^*$ and the identities (6.2) and (6.3) give

$$\begin{aligned} &g_{\mathscr{A}}(t) \in \partial_{\mathscr{A}}(u(t)), \qquad g_{\mathscr{B}}(t) \in \partial_{\mathscr{B}}(u(t)),\\ &g_{\mathscr{A}}'(t) + g_{\mathscr{B}}(t) = g_0(t) \quad \text{in } T^*, \quad \text{a.e. } t \in [0, T],\\ &g_{\mathscr{A}}(0) = g_0. \end{aligned} \tag{6.6}$$

Thus (6.4) implies (6.5) and, in the situation of Lemma 6.1, also (6.6), so we call a solution of (6.4) a *weak solution* of the pair (6.5), (6.6). The first will give a partial differential equation and the second yields variational boundary conditions in our examples.

Let Ω be a bounded open set in R^n which lies locally on one side of its smooth boundary Γ. $H^1(\Omega)$ is the space of functions φ in $L^2(\Omega)$ for which each of the partial derivatives $D_j\varphi = \partial\varphi/\partial x_j$ belongs to $L^2(\Omega)$, $1 \leqq j \leqq n$. Letting D_0 denote the identity on $L^2(\Omega)$, we can express the norm on $H^1(\Omega)$ by

$$\|\varphi\|_{H^1(\Omega)} = \left(\sum_{j=0}^{n} \|D_j\varphi\|^2_{L^2(\Omega)}\right)^{1/2}.$$

We shall let V be a closed subspace of $H^1(\Omega)$ containing $C_0^\infty(\Omega)$ and let $\gamma: V \to L^2(\Gamma)$ be the indicated restriction to V of the trace map [19]. We let T be the range of γ (a subspace of $H^{1/2}(\Gamma)$) and denote the kernel by $V_0 \equiv H_0^1(\Omega)$. Since Γ is smooth there is a unit outward normal vector $n(s) = [n_1(s), \cdots, n_n(s)]$ at each point $s \in \Gamma$. Note that the test functions $C_0^\infty(\Omega)$ are dense in V_0 so the dual V_0^* is the space of (first order) distributions on Ω. We refer to (19) for information on these Sobolev spaces. Specifically, we shall use the trace operator between Sobolev spaces of fractional order.

We shall construct an operator $\mathscr{B}: V \to 2^{V^*}$ which will occur in many of our examples. For each integer k, $-1 \leqq k \leqq n$, let there be given a continuous, convex function $\psi_k; R \to R$ whose subgradient, $\beta_k \equiv \partial\psi_k$, satisfies

$$|w| \leqq C(|s|+1) \quad \text{if } w \in \beta_k(s), \quad s \in R, \quad -1 \leqq k \leqq n, \tag{6.7}$$

where C is some large constant. Then define $\psi: V \to R$ by

$$\psi(u) = \sum_{k=0}^{n} \int_\Omega \psi_k(D_k u(x))\, dx + \int_\Gamma \psi_{-1}(\gamma(u(s)))\, ds, \qquad u \in V.$$

From the estimates (6.7) it follows that ψ is a sum of continuous convex functions so we can compute its subgradient term by term. Recall that the subgradient F of the convex function $v \to \int_\Omega \varphi_k(v)\, dx$ at $w \in L^2(\Omega)$ is determined by $F(x) \in \beta_k(w(x))$, a.e. $x \in \Omega$. Since $D_k : V \to L^2(\Omega)$ is continuous linear, the subgradient of the convex function $v \to \int_\Omega \varphi_k(D_k v)\, dx$ at $u \in V$ is given by $\{D_k^* F: F \in \beta_k(D_k u) \text{ a.e.}\}$. See [11, pp. 26–28] and [1, p. 47] for proofs of these facts. These observations show that the subgradient of ψ is

$$\mathscr{B}(u) = \partial\psi(u) = \sum_{k=0}^{n} D_k^* \beta_k(D_k u) + \gamma^* \beta_{-1}(\gamma u), \qquad u \in V. \tag{6.8}$$

To be precise, we have $F \in \mathscr{B}(u)$ if and only if there exists $f_k \in \beta_k(D_k u)$ in $L^2(\Omega)$, $0 \leqq k \leqq n$, and $f_{-1} \in \beta_{-1}(\gamma u)$ in $L^2(\Gamma)$ for which

$$F(v) = \int_\Omega \sum_{k=0}^{n} f_k(x) D_k v(x)\, dx + \int_\Gamma f_{-1}(s) v(s)\, ds, \qquad v \in V.$$

By restricting the above to $v \in V_0 = H_0^1(\Omega)$ we see the formal part is the distribution

$$F|_{V_0} = -\sum_{k=1}^{n} D_k f_k + f_0 \in V_0^*.$$

We denote this by the equality (of sets)

$$B_0(u) = -\sum_{k=1}^{n} D_k \beta_k(D_k u) + \beta_0(u). \tag{6.9}$$

Let us interpret (6.3) with $U^* = L^2(\Omega)$. First, if $D_k f_k \in U^*$ for $1 \leqq k \leqq n$, then by the classical Green's theorem we have, from above,

$$F(v) - F|_{V_0}(v) = \int_\Gamma \left\{ \sum_{k=1}^{n} f_k(s) n_k(s) + f_{-1}(s) \right\} v(s)\, ds, \qquad v \in V.$$

Thus $u \in D(\mathscr{B})$ and we have shown

$$\sum_{k=1}^{n} f_k n_k + f_{-1} \in \partial_{\mathscr{B}}(u) \quad \text{with } f_k \in \beta_k(D_k u).$$

That is, when the terms are as regular as indicated we have

$$\partial_{\mathscr{B}}(u) = \sum_{k=1}^{n} \beta_k(D_k u) n_k + \beta_{-1}(u). \tag{6.10}$$

Furthermore, $\partial_{\mathscr{B}}(u)$ is defined without these regularity assumptions on the individual terms; it is sufficient to have $F|_{V_0} \in U^*$. Finally, we note that from (6.7) it follows that $\mathscr{B}$ satisfies the assumptions [B_1] of Theorem 1 and [B_3] of Theorem 3. It is also bounded from $L^2(0, T; V)$ to $L^2(0, T; V^*)$ and it will satisfy [B_2] of Theorem 2 if, in addition, there is a pair of numbers K, $c>0$ such that

(6.11)
$\psi_k(s) \geqq c|s|^2 - K$, $s \in R$, $1 \leqq k \leqq n$ and one of the following:
(a) the estimate holds for $k=0$, or
(b) the estimate holds for $k=-1$, or
(c) $v \in V$ and $v = $ constant imply $v \equiv 0$.

From (6.11) we can show that

$$\psi(v) \geqq c_1\|v\|_V^2 - K_1, \qquad v \in V,$$

and this implies the coercivity condition in $[B_2]$.

7. Examples of partial differential equations. We shall describe some examples of initial-boundary-value problems for partial differential equations to illustrate the applications of our results. These examples were chosen merely to suggest a variety of problems that can be resolved by our Theorems, and they are not intended to be best possible in any sense.

(a) *Elliptic-parabolic equations.* For $k=0$ and -1, let $\varphi_k: R \to R$ be convex and continuous with subgradient, $\alpha_k \equiv \partial\varphi_k$, satisfying

$$|w| \leqq C(|s|+1) \quad \text{if } w \in \alpha_k(s), \quad s \in R.$$

Set $W = H^r(\Omega)$, $\frac{1}{2} < r < 1$, $V = H^1(\Omega)$, and note that $V \to W$ is compact and $\gamma: W \to L^2(\Gamma)$ is continuous [19]. Thus we can define by

$$\varphi(v) \equiv \int_\Omega \varphi_0(v(x))\,dx + \int_\Gamma \varphi_{-1}(\gamma(v(s)))\,ds, \qquad v \in W,$$

a continuous and convex function $\varphi: W \to R$ with subgradient

$$\mathscr{A}(u) = \partial\varphi(u) = \alpha_0(u) + \gamma^*(\alpha_{-1}(\gamma u)),$$

bounded from W to W^*. That is, $F \in \mathscr{A}(u)$ if and only if there exist $f_0 \in \alpha_0(u)$ in $L^2(\Omega)$ and $f_{-1} \in \alpha_{-1}(\gamma(u))$ in $L^2(\Gamma)$ for which

$$F(v) = \int_\Omega f_0(x)v(x)\,dx + \int_\Gamma f_{-1}(s)v(s)\,ds, \qquad v \in V, \tag{7.1}$$

so the formal and boundary parts of $\mathscr{A}$ are given, respectively, by

$$A_0(u) = \alpha_0(u), \qquad \partial_{\mathscr{A}}(u) = \alpha_{-1}(\gamma u). \tag{7.2}$$

From Theorem 2 we obtain the existence of a weak solution of the initial-boundary-value problem

$$\begin{aligned}
&\frac{\partial}{\partial t}A_0(u) + B_0(u) \ni f_0 \quad \text{in } L^2(0, T; H^{-1}(\Omega)),\\
&A_0u(0) \ni v_0,\\
&\frac{\partial}{\partial t}\partial_{\mathscr{A}}(u) + \partial_{\mathscr{B}}(u) \ni g_0 \quad \text{in } L^2(0, T; H^{-1/2}(\Gamma)),\\
&\partial_{\mathscr{A}}u(0) \ni b_0.
\end{aligned} \tag{7.3}$$

This is made precise in the form (6.5) and (6.6), where the operators are specified in (6.9), (6.10) and (7.2).

Remarks. By our choice of $V = H^1(\Omega)$, all boundary conditions in (7.3) are of variational type. Dirichlet-type constraints are obtained by taking subspaces of $H^1(\Omega)$.

We require that f_0 and g_0 be square-summable, with values in $H^{-1}(\Omega)$ and $H^{-1/2}(\Gamma)$ respectively, and we assume (6.11) to obtain coercivity of $\mathscr{B}$. The boundedness assumptions on α_k $(k = 0, -1)$ can be relaxed somewhat by using embedding theorems, e.g., of W into $L^p(\Omega)$.

There is no bound on the degeneracy permitted in the operator $\mathscr{A}$; we include even the (uninteresting) elliptic case $\mathscr{A} \equiv 0$. The case of $A_0 = 0$ leads to an evolution on the boundary subject to an elliptic equation in the interior; such problems arise from diffusion in a medium bounded by material of markedly lower diffusivity [25].

The classical porous-media equation and the weak form of the two-phase Stefan free-boundary problem are included in (7.3). In the latter, the enthalpy is given by $\alpha_0(s) = (1 + cH(s))s + LH(s)$, where $L > 0$ is the latent heat of fusion and $H(\cdot)$ is the Heaviside function [14], [16]. Such problems arise in welding, with the nonlinear term $\beta_0(u)$ representing a source of heat due to electrical resistance.

Note that each solution of (5.1) is also a (spatially independent) solution of (7.3), so there is much nonuniqueness in (7.3).

(b) *Pseudoparabolic equations.* Here we set $V = H_0^1(\Omega)$, so $T = \{0\}$ and all boundary conditions are of Dirichlet type. The operator $\mathscr{A}$ is given as above by (7.2); the operator $\mathscr{B}$ is also given as before but we shall only assume (6.7), not (6.11). On the space V we take the (equivalent) scalar product and corresponding Riesz map

$$\mathscr{R}u(v) = \int_\Omega \sum_{k=1}^n D_k u(x) D_k v(x)\, dx, \qquad u, v \in V,$$

so we have $\mathscr{R} = -\Delta_n \equiv -\sum_{k=1}^n D_k^2$. Assume $f_0 \in L^2(0, T; H^{-1}(\Omega))$ and $v_0 \in \alpha_0(u_0)$, $u_0 \in H_0^1(\Omega)$ are given. Then either from Theorem 1 or from Theorem 3 we obtain existence of a solution of the problem

$$
\begin{aligned}
&u \in H^1(0, T; H_0^1(\Omega)), \qquad u(0) = u_0,\\
&v \in H^1(0, T; H^{-1}(\Omega)), \qquad v(0) = v_0,\\
&w \in L^2(0, T; H^{-1}(\Omega)),\\
&\frac{\partial}{\partial t}(v(t) - \Delta_n u(t)) + w(t) = f_0(t),\\
&v(t) \in A_0(u(t)), \qquad w(t) \in B_0(u(t)).
\end{aligned}
\tag{7.4}
$$

The operators A_0 and B_0 are given by (7.2) and (6.9) respectively.

Remarks. The partial differential equation in (7.4) is of the form of a nonlinear parabolic plus the term $(\partial/\partial t)\Delta_n u(x, t)$. Such equations are known to arise in various diffusion problems and are called *pseudoparabolic* [9], [15], [28]. Similar problems with variational boundary conditions can be considered; we obtain weak solutions in the form (6.4). However, since $R_g(A_0 + \mathscr{R}) = H^{-1}(\Omega)$, we cannot use Lemma 6.1, in general, to deduce (6.6). This situation occurs even in the linear case [26].

The operator $-\Delta_n$ in (7.4) can be replaced by the Riesz operator of any equivalent scalar product on $H_0^1(\Omega)$. This trivial observation is useful in introducing elliptic linear operators in its place.

We have not made use of the fact that only one of the operators $\mathscr{A}$, $\mathscr{B}$ need be a subgradient. In particular, we are free to add to one of $\mathscr{A}$ or $\mathscr{B}$ any linear combination of first order derivatives. (See Example (d) below.)

Nonuniqueness of solutions of (7.4) follows from that of solutions of (5.1).

In the preceding examples the nonlinearity arises from the *local* dependence on the solution, e.g., from nonlinear functions of the values of u or ∇u at each point of Ω. We next display examples of *global* nonlinearity arising from the "total energy" or the "total flux" in the system. The following preliminary result will be useful.

LEMMA 7.1. *Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be continuous, bilinear, symmetric and nonnegative real-valued functions on the Hilbert space V. Then for $\alpha, \beta \in R$, the function*

$$\varphi(u) \equiv \tfrac{1}{2}\max\{a(u,u)+\alpha,\, b(u,u)+\beta\}, \qquad u \in V$$

is convex, continuous and its subgradient is given by

$$\partial\varphi(u) = \begin{cases} \{A(u)\} & \text{if } a(u,u)+\alpha > b(u,u)+\beta, \\ \{(\lambda A+(1-\lambda)B)(u),\, 0 \leqq \lambda \leqq 1\} & \text{if } a(u,u)+\alpha = b(u,u)+\beta, \\ \{B(u)\}, & \text{if } a(u,u)+\alpha < b(u,u)+\beta, \end{cases}$$

where $Au(v) = a(u,v)$, $Bu(v) = b(u,v)$, $v \in V$.

Proof. We need only to compute $\partial\varphi(u)$. For the first and last cases we compute the Gateaux derivative $\lim_{t\to 0}\{(\varphi(u+tv)-\varphi(u))/t\}$ to obtain the desired results. Now assume $a(u,u)+\alpha = b(u,u)+\beta$. An easy computation gives

$$t^{-1}(\varphi(u+tv)-\varphi(u)) = \max\left\{a(u,v)+\frac{t}{2}a(v,v),\, b(u,v)+\frac{t}{2}b(v,v)\right\}$$

so we have the equivalence of $f \in \partial\varphi(u)$,

$$f(v) \leqq t^{-1}(\varphi(u+tv)-\varphi(u)), \qquad v \in V, \quad t > 0,$$

and of

$$f(v) \leqq \max\{a(u,v),\, b(u,v)\}, \qquad v \in V.$$

This is equivalent to $f = \lambda Au + (1-\lambda)Bu$ for some λ, $0 \leqq \lambda \leqq 1$.

(c) *Energy-dependent elliptic-parabolic equation.* We shall use Theorem 2 with the operator $\mathcal{B}$ given by (6.8), so we assume (6.7) and (6.11). Choose $V = H^1(\Omega)$ so the space of boundary values is $T = H^{1/2}(\Gamma)$. Define on $W \equiv L^2(\Omega)$ the function

$$\varphi(u) = \frac{1}{2}\max\left\{1, \int_\Omega |u(x)|^2\,dx\right\}, \qquad u \in W.$$

The subgradient $\mathcal{A} = \partial\varphi$ is given by Lemma 7.1 and we have $\mathcal{A} = A_0 = \mathcal{A}_0$, $R_g(\mathcal{A}) = L^2(\Omega)$. Finally, let $v_0 \in L^2(\Omega)$, $f_0 \in L^2(0,T;L^2(\Omega))$, $g_0 \in L^2(0,T;H^{-1/2}(\Gamma))$ be given and define

$$f(t)(v) = \int_\Omega f_0(x,t)v(x)\,dx + g_0(t)(\gamma v), \qquad v \in V.$$

Then we obtain a weak solution of

$$\begin{aligned} &\frac{\partial v}{\partial t} + B_0(u) \ni f_0 && \text{in } L^2(0,T;H^{-1}(\Omega)), \\ (7.5) \qquad &v(x,0) = v_0(x) && \text{in } L^2(\Omega), \\ &\partial_{\mathcal{B}}(u) \ni g_0 && \text{in } L^2(0,T;H^{-1/2}(\Gamma)), \end{aligned}$$

where v is determined by

$$v \in \begin{cases} \{0\}, & \text{if } \int_\Omega |u|^2\,dx < 1, \\ \{\lambda u \colon 0 \leqq \lambda \leqq 1\} & \text{if } \int_\Omega |u|^2\,dx = 1, \\ \{u\}, & \text{if } \int_\Omega |u|^2\,dx > 1. \end{cases}$$

Thus, the type of the equation is either elliptic (with parameter t) or parabolic and depends on the total energy $\int_\Omega |u|^2\,dx$.

(d) *A flux-dependent equation.* Take $V = H_0^1(\Omega)$, $W = L^2(\Omega)$ and $T = \{0\}$. Let the convex function φ_0 and its bounded subgradient $\alpha_0 = \partial\varphi_0$ be given as above in Example (a), and define $\mathscr{A} = \alpha_0$ in $L^2(\Omega)$; cf. (7.2). Denoting the gradient of u by $\vec{\nabla} u$, we define the continuous convex

$$\psi(u) = \frac{1}{2}\max\left\{N, \int_\Omega |\vec{\nabla} u(x)|^2\,dx\right\}, \qquad u \in V.$$

Let $\vec{b} \in R^n$ and define $\mathscr{B} \colon V \to 2^{V^*}$ by

$$\mathscr{B}(u) = \vec{b}\cdot\vec{\nabla} u + \partial\psi(u).$$

Note that $\mathscr{B}$ is maximal monotone, bounded and coercive. Let $v_0 \in R_g(\mathscr{A})$ and $f_0 \in L^2(0, T; H^{-1}(\Omega))$. From Theorem 2 we obtain existence of a solution of the problem

$$u \in L^2(0, T; H_0^1(\Omega)), \quad v \in H^1(0, T; H^{-1}(\Omega)),$$

$$\frac{\partial v}{\partial t} + \vec{b}\cdot\vec{\nabla} u - K\left(\int_\Omega |\vec{\nabla} u|^2\,dx\right)\Delta_n u = f_0, \tag{7.6}$$

$$v(x, t) \in \alpha_0(u(x, t)), \qquad v(x, 0) = v_0(x),$$

where the maximal monotone $K \colon R \to R$ is given by

$$K(s) = \begin{cases} \{0\}, & s < N, \\ [0, 1], & s = N, \\ \{1\}, & s > N. \end{cases}$$

Remarks. In the region where $\int_\Omega |\vec{\nabla} u|^2\,dx < N$ the equation in (7.6) is a *conservation law* of the form

$$\frac{\partial v}{\partial t} + \vec{b}\vec{\nabla} g(v) \ni f_0, \tag{7.7}$$

where the maximal monotone $g \colon R \to R$ is the inverse to α_0. Thus (7.6) suggests a *penalty method* [18] to approximate solutions of (7.7). We shall develop these observations elsewhere.

In order to consider (7.6) in the form (6.1) it is essential that $\mathscr{B}$ is not required to be a subgradient.

(e) *Elliptic-parabolic systems.* Our final example consists of a pair of equations of the type given above in Example (a) that are (nonlinearly) coupled. For $i = 0, 1$ and

$k=0, -1$, let $\varphi_k^i: R \to R$ be convex and continuous with subgradient, $\alpha_k^i \equiv \partial\varphi_k^i$, satisfying

$$(7.8) \qquad |w| \leqq C(|s|+1) \quad \text{for } w \in \alpha_k^i(s),\ s \in R.$$

On the product space $W \equiv H^r(\Omega) \times H^r(\Omega)$, $\frac{1}{2} < r < 1$, we have the continuous trace operator $\gamma([u^1, u^2]) = [\gamma(u^1), \gamma(u^2)]$ which maps W into $L^2(\Gamma) \times L^2(\Gamma)$. Thus we define by

$$\varphi(v) = \sum_{i=1}^{2} \int_\Omega \varphi_0^i(v^i(x))\,dx + \sum_{i=1}^{2} \int_\Gamma \varphi_{-1}^i(\gamma v^i(s))\,ds, \qquad v = [v^1, v^2] \in W,$$

a continuous and convex function whose subgradient is given by

$$\mathscr{A}(u) \equiv \partial\varphi(u) = [\alpha_0^1(u^1) + \gamma^*(\alpha_{-1}^1(\gamma(u^1))),\ \alpha_0^2(u^2) + \gamma^*(\alpha_{-1}^2(\gamma(u^2)))],$$
$$u = [u_1, u_2] \in W.$$

The operator $\mathscr{A}: W \to 2^{W^*}$ is bounded; its formal and boundary parts are given, respectively, by (see (7.2))

$$(7.9) \qquad A_0(u) = [\alpha_0^1(u^1), \alpha_0^2(u^2)], \qquad \partial_{\mathscr{A}}(u) = [\alpha_{-1}^1(\gamma(u^1)), \alpha_{-1}^2(\gamma(u^2))].$$

Hereafter we restrict γ to the product space $V \equiv H^1(\Omega) \times H^1(\Omega)$. Assume we are given a set of continuous and convex functions $\psi_k^i: R \to R$ for $i = 1, 2$, $-1 \leqq k \leqq n$, whose subgradients $\beta_k^i \equiv \partial\psi_k^i$ all satisfy the estimate (6.7). For $i = 1, 2$ we define $\psi^i: H^1(\Omega) \to R$ as in § 6; its subgradient is then given by (see (6.8))

$$\mathscr{B}^i(u^i) = \partial\psi^i(u^i) = \sum_{k=0}^{n} D_k^* \beta_k^i(D_k u^i) + \gamma^* \beta_{-1}^i(\gamma u^i), \qquad u^i \in H^1(\Omega).$$

The formal and boundary parts of $\mathscr{B}^i$ are given by (6.9) and (6.10) for each of $i = 1, 2$. Thus we have two pairs of operators similar to the pair in Example (a). The coupling of the corresponding equations will be attained by a maximal monotone graph $\mu: R \to 2^R$ which is bounded, i.e., (7.8) holds for $w \in \mu(s)$. Then we define a maximal monotone operator M on $R \times R$ by

$$M([s_1, s_2]) = \{[w, -w]: w \in \mu(s_1 - s_2)\}, \qquad [s_1, s_2] \in R \times R.$$

This operator M induces a corresponding operator on $L^2(\Omega) \times L^2(\Omega)$, hence, from V into V^*, which we also denote by M. Finally we define

$$\mathscr{B}([u_1, u_2]) = [\mathscr{B}^1(u_1), \mathscr{B}^2(u_2)] + M(u_1, u_2), \qquad [u_1, u_2] \in V.$$

This $\mathscr{B}$ is the sum of maximal monotone operators, each of which is defined on all of V, so $\mathscr{B}$ is maximal monotone. Similarly $\mathscr{B}$ is bounded, and we note that $\mathscr{B}$ is coercive if both of $\mathscr{B}^1$ and $\mathscr{B}^2$ are coercive.

Assume that we are given the following data:

$$f_0^i \in L^2(0, T; H^{-1}(\Omega)), \qquad g_0^i \in L^2(0, T; H^{-1/2}(\Gamma)), \qquad i = 1, 2,$$
$$[v_0^1, v_0^2] \in R_g(A_0), \qquad (v_{-1}^1, v_{-1}^2] \in R_g(\partial_{\mathscr{A}}).$$

If the functions $\{\beta_k^i: -1 \leqq k \leqq n\}$ satisfy (6.11) for both $i=1$ and $i=2$, then from Theorem 2 it follows that there exists a weak solution of the system

$$\frac{\partial}{\partial t}\alpha_0^1(u^1(x,t)) + B_0^1(u^1(x,t)) + \mu(u^1(x,t)-u^2(x,t)) \ni f_0^1(x,t),$$

$$\frac{\partial}{\partial t}\alpha_0^2(u^2(x,t)) + B_0^2(u^2(x,t)) - \mu(u^1(x,t)-u^2(x,t)) \ni f_0^2(x,t)$$

$$\text{in } L^2(0,T;H^{-1}(\Omega)), \tag{7.10}$$

$$\frac{\partial}{\partial t}\alpha_{-1}^i(\gamma u^i(s,t)) + \partial_{\mathscr{B}^i}(u^i(s,t)) \ni g_0^i, \qquad i=1,2, \quad \text{in } L^2(0,T;H^{-1/2}(\Gamma)),$$

$$\alpha_0^i(u^i(x,0)) \ni v_0^i(x), \qquad i=1,2, \quad \text{in } L^2(\Omega),$$

$$\alpha_{-1}^i(\gamma u^i(s,0)) \ni v_{-1}^i(s), \qquad i=1,2, \quad \text{in } L^2(\Gamma).$$

Remarks. All of the operators in this system are (possibly) multi-valued, so each of the "equations" should be made precise as was done in our preceding examples. See (6.9), (6.10), (7.2) and (7.9) for related computations.

The only requirement on the α_k^i is that they be maximal monotone graphs in R which satisfy the bound (7.8). Thus much degeneracy is possible in the leading operator given by (7.9). Related Stefan-type free-boundary problems can be so considered.

Interesting examples of the coupling term arise in applications to diffusion problems. These include problems with a *semipermeable* membrane, $\mu(s)=s^+$ (where s^+ denotes s if $s>0$ and 0 otherwise), or those with a *threshold* phenomenon $\mu(s)=(s-\varepsilon)^+-(-s-\varepsilon)^+$. The operator M as given above is a subgradient; this is easily verified by showing it is cyclic monotone [1]. However we may add to M nonsymmetric monotone terms, for example, $[-s_2, s_1]$, and thereby obtain systems of the form (6.1) in which $\mathscr{B}$ is not a subgradient.

Systems of equations of *pseudoparabolic* type can be resolved similarly by Theorem 1. For example, we can choose $V=H_0^1(\Omega)\times H_0^1(\Omega)$ with scalar product on each factor as given in Example (b), and obtain existence of a solution of the problem

$$\frac{\partial}{\partial t}(\alpha_0^1(u^1(x,t)) - \Delta_n u^1(x,t)) + B_0^1(u^1(x,t)) + \mu(u^1(x,t)-u^2(x,t)) \ni f_0^1(x,t),$$

$$\frac{\partial}{\partial t}(\alpha_0^2(u^2(x,t)) - \Delta_n u^2(x,t)) + B_0^2(u^2(x,t)) - \mu(u^1(x,t)-u^2(x,t)) \ni f_0^2(x,t) \tag{7.11}$$

$$\text{in } L^2(0,T;H^{-1}(\Omega)),$$

$$u^j \in H^1(0,T;H_0^1(\Omega)), \quad u^j(x,0)=u_j(x), \quad \alpha_0^j(u^j(x,0)) \ni v_j(x)$$

$$j=1,2, \quad \text{in } L^2(\Omega)$$

where the data are given as above with $v_j \in A_0(u_j)$ for $j=1,2$.

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