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Title THE APPLICATION OF DYNAMIC PROGRAMMING TO THE DESCRIPTION AND OPTIMAL CONTROL OF DYNAMIC PROCESSES

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The demand for improved controllers creates the need for synthesis techniques which emphasize optimality. Optimality automatically infers stability. This thesis presents one such technique which is not limited to linear systems. The theory of dynamic programming is applied to the synthesis of optimal controllers with emphasis on missile flight control systems. It is shown how the theory copes with time dependence, nonlinearities and random aspects which prevail in the missile field.

The first chapter introduces the basic concept and terminology of dynamic programming through its application to deterministic and stochastic versions of a first order control process. The second chapter presents a general formulation of processes. In the third chapter the description and optimization of trajectories is

discussed. Under suitable assumptions concerning time dependence the problems can be explicitly formulated in two dimensions. This permits their solution on existing digital computers. The final chapter treats homing guidance. A simple method of implementing control so that a missile seeks a low drag ballistic trajectory terminating at intercept is deduced.

These examples demonstrate the usefulness of dynamic programming as a tool for synthesizing and analyzing advanced flight control systems. With this theory synthesis of sophisticated controllers can be attacked with mathematical rigor rather than through trial and error.

THE APPLICATION OF DYNAMIC PROGRAMMING TO THE DESCRIPTION AND OPTIMAL CONTROL OF DYNAMIC PROCESSES

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THE APPLICATION OF DYNAMIC PROGRAMMING TO THE DESCRIPTION AND OPTIMAL CONTROL OF DYNAMIC PROCESSES

INTRODUCTION

Rapid advancement in technology carries with it an engineeering demand for improved controllers. Little improvement can be expected if stability continues to be emphasized by employing linear synthesis techniques. Rather, techniques are required which emphasize optimality. The mathematical theory of dynamic programming satisfies this requirement.

It is the purpose of this report to apply the theory of dynamic programming to the synthesis of optimal controllers, with special emphasis to missile flight control systems. The intent is to show that this theory copes with the time dependence, nonlinearities, and random aspects which characterize the missile field.

The name "Dynamic Programming" was coined by Bellman [3] to describe the mathematical theory of multi-stage decision processes. These processes are dynamic since they evolve with time. They have a programming context since their evaluation can be controlled by a sequence of decisions. All feedback control processes are of this type. For example, the function of a guidance computer is to carry out a sequence of decisions which minimize a terminal miss distance. The determination of an optimal sequence of decisions presents a problem which can be profitably discussed in terms of dynamic programming.

The material presented is divided into four chapters.

In the first chapter the basic concepts and terminology of dynamic programming are introduced. The theory is used to determine optimal control for a simple first order system subject to input limiting. The theory leads quite naturally to a mode of control commonly referred to as bistable control. The problem is solved by two formulations: the discrete and the continuous. The discrete formulation gives insight into the role the digital computer can play as a tool for synthesis. The continuous formulation caters to common prejudice toward analytic solution. Following this, a stochastic version of the same system is considered in order to indicate the broad scope of the theory.

The second chapter presents an abstract formulation of both descriptive and control processes. Their distinction lies in the absence or presence of control decisions. The discussion of descriptive processes leads to the theory of invariant imbedding and the discussion of control processes, to the theory of dynamic programming.

In the discussion the functional equations which describe general processes are deduced. These equations embody all the functional equations used throughout the text.

Chapter III treats the solution of trajectory problems. First a descriptive process which yields to analytic solution is considered in detail. The problem is to determine the range from an arbitrary state to impact arising from the ballistic flight of a missile subject only to a parallel gravitational field. This problem is solved by two methods. The first method treats the problem in terms of three state variables with time as the independent variable. The second method treats it in terms of two state variables with altitude as the independent variable.

The second method of solution provides a model for the treatment of a trajectory control problem. It is shown how to determine the optimal launch angle and angle of attack program to maximize the range of a missile, subject to prescribed terminal conditions on altitude and velocity. Under suitable assumptions concerning time dependence the problem can be solved by a discrete formulation which employs two state variables. Thus the method is readily handled by present day digital computers. In the formulation presented, as many constraints as are necessary to realistically describe the process are welcomed. Furthermore, nonlinear aerodynamic characteristics cause no trouble.

In the final chapter the subject of terminal guidance is approached in terms of dynamic programming. This leads to a very simple maximum effort type control policy. The sign of the control force is determined by the sign of extrapolated miss distance. As a consequence, control can be implemented so that a missile will seek a low drag ballistic trajectory terminating at intercept. It is necessary to incorporate in the extrapolated miss distance only knowledge of the gravitational force acting on the missile and knowledge of the missile's axial accelerations. The chief advantage of this type of navigation is that it minimizes needless maneuvers and eliminates error early in flight.

At present the class of problems that can be handled by dynamic programming is limited primarily by available computer memory size. As computer technology increases computer capacity, so will the usefulness of dynamic programming be increased.

CHAPTER I

CONTROL OF A SIMPLE DYNAMIC PROCESS

1.0 Introduction

In this section the theory of dynamic programming will be introduced by means of its application to a simple control process. By relating basic concepts and terminology to a particular problem it is hoped that these concepts will be more readily understood.

Consider as an example a process whose scalar output $y(t)$ satisfies the following differential equation:

$$\frac{dy}{dt} + ay = r(t) + f(t) \tag{1-1}$$

$$y(0) = C.$$

Here $f(t)$ is a forcing function through which control is exerted, $r(t)$ is a random function of time such as noise, a is a constant, and C is the initial condition. It is assumed that the system equation is time independent. To make the process realistic the solution will be subject to the restraint

$$|f(t)| \leq L \tag{1-2}$$

where L is a constant.

The problem is to choose f(t) so as to minimize the deviation of y(t) from zero over some time interval (0, T). Once f(t) is known it will be shown that a controller can be implemented. The system is illustrated in Figure 1.

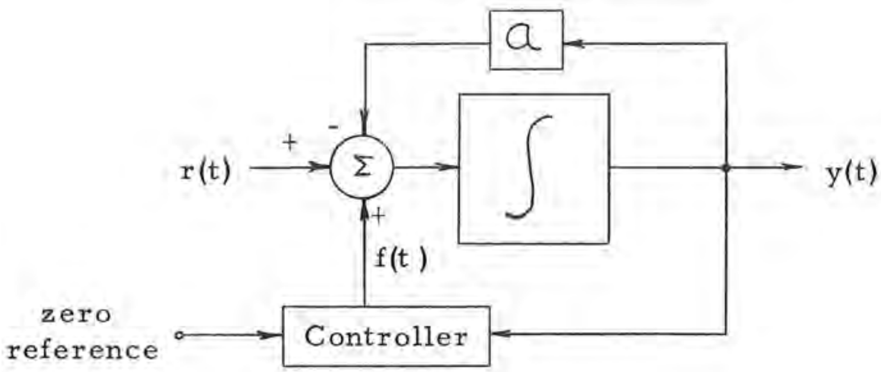

Figure 1. A Simple Control Process.

Three special cases of this problem can be considered. The first case is the deterministic problem for which r(t) = 0, the second case is the stochastic process for which the statistical nature of r(t) is known, and the third case is an adaptive process where the statistical properties of r(t) are unknown. The first two cases will be considered here. The adaptive case is discussed in the literature [10] .

The dynamic programming technique can be considered in two forms, the discrete and the continuous. The discrete formulation is emphasized here because of its adaptability to solution by digital computers. Although the problem considered here is simple and yields to analytic solution, many problems will demand use of a computer. Both discrete and continuous techniques will be presented for comparison.

1.1. The Deterministic Process

For the deterministic problem, the system is described by

$$\frac{dy}{dt} + ay = f(t) \tag{1-3}$$

with the constraint

$$|f(t)| \leq L. \tag{1-4}$$

In order to determine how well the system performs it is necessary to choose some measure of error, $J(f)$. The measure chosen here will be the integrated absolute value of error defined by

$$J(f) = \int_0^T |y(t)|\,dt, \qquad f = f(t). \tag{1-5}$$

Many different measures of error might have been chosen. The prime concern is to choose $f(t)$ to minimize the value of $J(f)$.

1.1.1 The Discrete Formulation

The system may now be expressed in discrete form. It is assumed that the controller's knowledge of the variable $y(t)$ arrives via discrete sampling of the continuous process. Further assume that sampling is uniform and occurs every Δ seconds until $N\Delta = T$, where N is an integer. The derivative of $y(t)$ at time $k\Delta$ may be approximated by

$$\frac{dy}{dt} \cong (y_{k+1} - y_k)/\Delta , \qquad (1\text{-}6)$$

and the system equation may be put in the difference equation form

$$y_{k+1} = y_k + (f_k - ay_k)\Delta \qquad (1\text{-}7)$$

$$y_0 = C$$

$$|f_k| \leq L .$$

Here $k = 0, 1, 2, \cdots, N-1$; C is the initial value of the variable y; and f_k is the value of f at the k^{th} sampling instant. In discrete form equation (1-5) becomes

$$J_N(f_k) = \sum_{k=0}^{N-1} |y_k| \Delta. \qquad (1\text{-}8)$$

The terminology introduced now is consistent with Bellman's [3] and is used throughout.

$J_N(f_k)$ is called the criterion function since it defines the process goal. A "policy" is any rule for making decisions which yields an allowable sequence

$$\{f_k\} = (f_0, f_1, \cdots, f_{N-1}), \qquad |f_k| \leq L. \tag{1-9}$$

It is very basic and important to any programming procedure that if an optimal policy is employed, the value of $J_N(f_k)$ can be an explicit function only of C and N. This leads to the definition of the auxiliary functional

$$F_N(C) = \underset{\{f_k\}}{\text{Min}} \; J_N(f_k), \qquad k = 0, 1, \cdots, N\ , \tag{1-10}$$

where the minimization is taken over all permissible sequences $\{f_k\}$.

To determine the optimal policy, a technique basic to dynamic programming is employed. The original problem is imbedded in a more general problem with an arbitrary number of stages n and an arbitrary initial condition y. Accordingly, $F_n(y)$ is defined as

$$F_n(y) = \underset{\{f_k\}}{\text{Min}} \; J_n(f_k), \qquad k = 0, 1, \cdots, n\ . \tag{1-11}$$

$F_n(y)$ is interpreted as the n-stage return or the n-stage cost arising from an optimal policy and an initial state y.

The problem is now contained in the following equations:

$$F_n(y) = \operatorname{Min} \sum_{k=0}^{n-1} |y_k| \Delta, \qquad n = 1, 2, \cdots, N, \tag{1-12}$$

$$y_{k+1} = y_k + (f_k - a y_k)\Delta, \qquad y_0 = y, \quad |f_k| \le L. \tag{1-13}$$

To arrive at a solution first consider a one-stage process. For all values of the initial condition y,

$$F_1(y) = |y| \Delta. \tag{1-14}$$

This is true since the result of an initial decision f is not apparent until the beginning of the second stage.

To treat the $(n+1)$-stage process assume knowledge of the n-stage return and reason as follows. Regardless of the initial choice of f_0, the remaining decisions must be optimal with respect to the new state y_i arising from the initial condition. The n-stage return is $F_n(y[\Delta])$, which has been assumed known for any state $y(\Delta)$. However, f_0 is not chosen arbitrarily but is chosen to minimize the $(n+1)$-stage return. The mathematical statement of this argument is

$$F_{n+1}(y) = \operatorname*{Min}_{|f| \le L} [|y| \Delta + F_n(y[\Delta])], \qquad n = 1, 2, 3, \cdots, N. \tag{1-15}$$

Here $|y|\Delta$ is the incremental return arising from the first stage and $F_n(y[\Delta])$ is the return arising from the remaining n stages.

From the system equation (1-13) $y(\Delta)$ is,

$$y(\Delta) = y + (f_0 - ay)\Delta . \tag{1-16}$$

Thus:

$$F_{n+1}(y) = \min_{|f| \leq L} [|y|\Delta + F_n(y + [f_0 - ay]\Delta)] , \quad n = 1, 2, 3, \cdots, N , \tag{1-17}$$

and

$$F_1(y) = |y|\Delta . \tag{1-18}$$

Equations (1-17) and (1-18) provide an iterative computation for $F_N(y)$ and yield the sequence of optimal decisions. Note that the bracketed expression is the return for an arbitrary initial decision f_0 and an optimal policy thereafter. The minimization with respect to f_0 guarantees that the $(n+1)$-stage return is optimal.

To carry out the computations begin with the 2-stage process:

$$F_2(y) = \min [|y|\Delta + F_1(y + [f - ay]\Delta)] , \tag{1-19}$$

$$= |y|\Delta + \min F_1(y + [f - ay]\Delta). \tag{1-20}$$

But it has already been determined that $F_1(y) = |y|\Delta$; hence,

$$F_2(y) = |y|\Delta + \min_{|f| \leq L} |[y + (f - ay)\Delta]|\Delta. \tag{1-21}$$

The minimizing value of f satisfies

$$f = -L\,\mathrm{sgn}\,y, \qquad |y|(1-a\Delta)-L\Delta \geq 0 \tag{1-22}$$

or

$$f = -y(1-a\Delta)/\Delta, \qquad |y|(1-a\Delta)-L \leq 0\,. \tag{1-23}$$

It follows that

$$F_2(y) = |y|(1+[1-a\Delta]\,\Delta-L\Delta^2, \quad |y|(1-a\Delta)-L\Delta \geq 0 \quad (1\text{-}24)$$

or

$$F_2(y) = |y|\,\Delta, \qquad |y|(1-a\Delta)-L\Delta \leq 0. \quad (1\text{-}24)$$

Now the 3-stage return may be computed. Depending upon the value of $|y|$, three possibilities exist.

$$F_3(y) = |y|(1+[1-a\Delta]+[1-a\Delta]^2)\Delta-L\Delta^2(2+[1-a\Delta]),$$

$$|y|(1-a\Delta)-L\Delta > |y|(1-a\Delta)^2-L\Delta(1+[1-a\Delta]) \geq 0\,;$$

$$F_3(y) = |y|(1+[1-a\Delta])\Delta-L\Delta^2, \tag{1-25}$$

$$|y|(1-a\Delta)-L\Delta \geq 0 \geq |y|(1-a\Delta)^2-L\Delta(1+[1-a\Delta]);$$

$$F_3(y) = |y|\,\Delta,$$

$$0 \geq |y|(1-a\Delta)-L\Delta > |y|(1-a\Delta)^2-L\Delta(1+[1-a\Delta])\,.$$

The solution for the n-stage process can be expressed by a recurrence relation. Begin by postulating that

$$F_n(y) = (|y|R_k - LS_k)\Delta , \qquad k \leq n , \tag{1-26}$$

where k is determined from the inequalities

$$|y|U_1 - L\Delta V_1 > \cdots > |y|U_{k-1} - L\Delta V_{k-1} \geq 0 \geq |y|U_k - L\Delta V_k > \cdots ,$$

and R_i, S_i, V_i, and U_i are independent of y for all i. It has been shown that the assumption is true for $n = 1, 2,$ and 3. It will be shown that it is true for the $(n+1)$-stage process; hence, the result will follow by induction.

First determine recurrence relations for U_n and V_n. Assume the inequalities are valid and write

$$\begin{aligned} |y|U_m - L\Delta V_m &> |y + (-Lsgny - ay)\Delta|U_m - L\Delta V_m \\ &= ||y|(1-a\Delta) - L\Delta|U_m - L\Delta V_m \\ &= |y|(1-a\Delta)U_m - L\Delta U_m - L\Delta V_m . \end{aligned} \tag{1-28}$$

Thus

$$|y|U_m - L\Delta V_m > |y|U_{m+1} - L\Delta V_{m+1} , \tag{1-29}$$

where

$$U_{m+1} = (1-a\Delta)U_m \tag{1-30}$$

$$V_{m+1} = V_m + U_m . \tag{1-31}$$

The inequality is certainly true for $m = 1$ with

$$U_1 = 1 - a\Delta, \qquad V_1 = 1 .$$

By induction the inequality holds for all m.

Using these recurrence relations U_k and V_k can be constructed for $k = 2, 3, \cdots, n$. Of special interest is the value of k for which

$$|y| U_{k-1} - L\Delta V_{k-1} \geq 0 \geq |y| U_k - L\Delta V_k. \tag{1-32}$$

This value of k depends upon the magnitude of y.

To determine recurrence relations for R_k and S_k write

$$\begin{aligned} F_{n+1}(y) &= \underset{|f| \leq L}{\mathrm{Min}} \; [\,|y|\,\Delta + F_n(y + [f - ay]\;\Delta)] \\ &= \underset{|f| \leq L}{\mathrm{Min}} \; [\,|f|\,\Delta + (\,|y + [f - ay]\;\Delta\,|R_k - LS_k)\Delta] \end{aligned} \tag{1-33}$$

Carrying out this minimization over f gives

$$F_{n+1}(y) = |y|\,(1 + [1 - a\Delta]\, R_k)\Delta - L(\Delta R_k + S_k)\Delta, \tag{1-34}$$

for $\qquad |y|\,(1 - a\Delta) - L\Delta \geq 0$;

$$F_{n+1}(y) = |y|\,\Delta - LS_k\Delta\,, \tag{1-35}$$

for $\qquad |y|\,(1 - a\Delta) - L\Delta \leq 0$.

Equation (1-35) corresponds to the special case $k = 1$ if $R_1 = 1$, $S_1 = 0$. Hence, equation (1-35) can be used as a starting point for calculating R_m and S_m.

The recurrence relations for R_m and S_m are derived simply by induction. These are

$$\begin{aligned} R_{m+1} &= 1 + (1-a\Delta)R_m \\ &= V_{m+1} , \end{aligned} \tag{1-36}$$

$$\begin{aligned} S_{m+1} &= S_m + R_m \Delta \\ &= \Delta \sum_{i=1}^{m} V_i \end{aligned} \tag{1-37}$$

It is easy to verify that equations (1-34) through (1-37) yield the proper solutions for the 1, 2, and 3-stage process. By induction the solution can be extended to the n-stage process.

In Appendix I this will be demonstrated with a numerical example.

It is interesting to note that the optimal policy depends only on y and not upon R_k or S_k. In fact, at any stage the problem is just that of the 2-stage process whose solution was given by equations (1-22) and (1-23), or

$$f = -L \operatorname{sgn} y, \quad |y|(1-ay) - L\Delta \geq 0 \; ; \tag{1-38}$$

$$f = -y(1-a\Delta)/\Delta, \; |y|(1-ay) - L\Delta \leq 0 \quad . \tag{1-39}$$

Since the optimal policy at each stage is independent of previous policies, it is fairly straightforward to implement this policy. An implementation is shown in Figure 2.

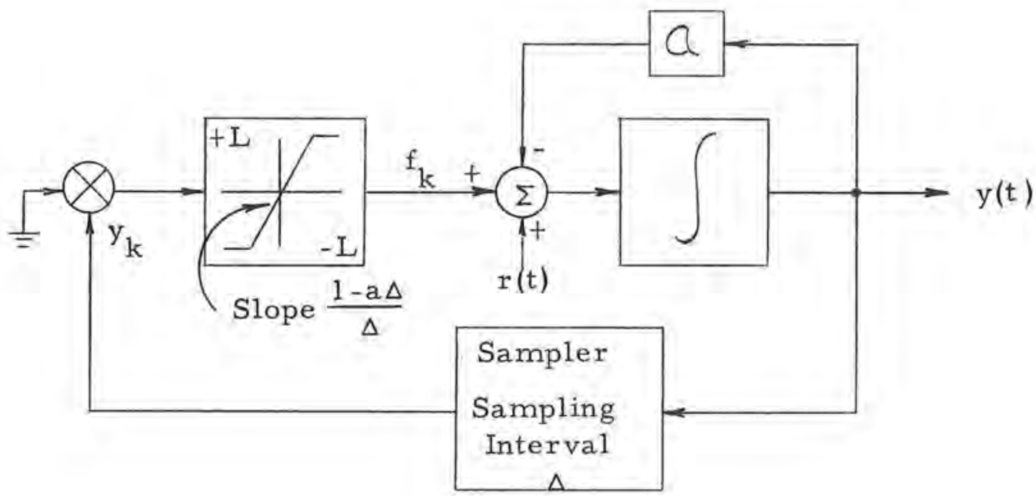

Figure 2. Optimal Control of the Deterministic Process, $r(t) = 0$.

1.1.2 The Continuous Formulation

If the sample interval in the discrete case approaches zero, the continuous case is approached. In Figure 2 the slope of the limiter characteristic approaches infinity and the result is an idealized bistable controller.

The continuous formulation can be demonstrated by proceeding in a manner parallel to that of the discrete formulation. Instead of the discrete criterion function, consider its continuous counterpart,

$$J(f) = \int_0^T |y(t)|\, dt, \qquad f = f(t). \tag{1-40}$$

The minimum value of $J(f)$ is a function only of the initial state y and the duration of the process T. To emphasize this fact the auxiliary functional is defined as

$$F(y,t) = \underset{|f|(t) \leq L}{\text{Min}} \int_0^T |y(t)|\, dt\ , \tag{1-41}$$

where the minimizing $f(t)$ is specified over the whole interval $(0, T)$. Thus, $F(y, T)$ is the return over the interval $(0, T)$ if the system starts in the state y and if an optimal policy is employed throughout.

Equation (1-41) can be rewritten as follows:

$$F(y,T) = \underset{|f(t)| \leq L}{\text{Min}} \left[\int_0^\Delta |y(t)|\, dt + \int_\Delta^T |y(t)|\, dt\right]. \tag{1-42}$$

The choice of $f(t)$ over the interval $(0, \Delta)$ transforms the initial state y into a new state $y(\Delta)$. The problem at this point is the same as the original problem. Starting in the new state $y(\Delta)$ at time Δ an optimal policy is to be determined over the interval

(Δ, T). Since the system equation is time independent this is equivalent to starting in the state $y(\Delta)$ at time zero and determining the optimal policy over the interval $(0, T-\Delta)$.

If an optimal policy is employed over the remaining interval, then by definition of the return function,

$$F(y[\Delta], T-\Delta) = \int_{\Delta}^{T} |y(t)| dt \tag{1-43}$$

It follows that

$$F(y, T) = \underset{f(t)}{\text{Min}} \left[\int_{0}^{\Delta} |y(t)| dt + F(y[\Delta], T-\Delta) \right], \tag{1-44}$$

where the minimization is taken only over the interval $(0, \Delta)$.

Let Δ be a very small increment of time, then to a first order approximation,

$$F(y, T) = \underset{|f| \leq L}{\text{Min}} \left[|y| \Delta + F(y + [f - ay] \Delta, T-\Delta) \right] \tag{1-45}$$

where $y(\Delta)$ has been defined from the system equation. On the assumption that $F(y, T)$ has continuous first partial derivatives, $F(y+[f-ay] \Delta, T-\Delta)$ can be expanded in a Taylor Series about the state (y, T). Thus,

$$F(y, T) = \underset{|f| \leq L}{\text{Min}} \left[|y| \Delta + F(y, T) + (f-ay)\Delta F_y - \Delta F_T + 0(\Delta) \right], \tag{1-46}$$

where $F_y = \partial F/\partial y$,

and $F_T = \partial F/\partial T$,

and $0(\Delta)$ contains higher order terms in Δ. Since $F(y,T)$ is not a function of f it is unaffected by the minimization and can be subtracted from both sides of equation (1-46). Dividing through by Δ and taking the limit as Δ approaches zero gives

$$\min_{|f| \leq L} \left[|y| + (f-ay)F_y - F_T \right] = 0. \tag{1-47}$$

Here the bracketed expression is being minimized and the result set equal to zero. The solution of this nonlinear partial differential equation (1-47) must satisfy the boundary condition

$$F(y,0) = 0 .$$

Thus, a property of the optimal policy may immediately be deduced. It is obvious that the minimizing f satisfies

$$f = -L \operatorname{sgn} F_y, \qquad F_y \neq 0. \tag{1-48}$$

If $F_y = 0$ then f is arbitrary. Thus, an alternative form for equation (1-47) is

$$|y| - ayF_y - L|F_y| - F_T = 0 , \tag{1-49}$$
$$F(y,0) = 0 .$$

The solution of this equation yields $F(y,T)$ and the zero crossings of F_y.

Now proceed as follows. Consider the regions in which the signs of y and F_y are invariant. In each region solve for $F(y,T)$. The complete solution is obtained by matching the conditions at regional boundaries. For high order linear systems this procedure is very difficult to carry out manually, but the present example causes no difficulty.

Consider the case $y > 0$ and $F_y > 0$. The partial differential equation (1-49) takes the form

$$y - (L+ay)F_y - F_T = 0 , \qquad (1\text{-}50)$$

$$F(y,0) = 0 .$$

The solution to (1-50) can be obtained by the method of characteristics [6]. First convert the equation to its associated system of ordinary equations:

$$dt = dy \,|\, (L+ay)| = dF/y, \qquad y > 0, \quad F_y > 0. \qquad (1\text{-}51)$$

The first two terms yield an equation whose general solution is

$$T - \ln (L+ay)/a = C_1 . \qquad (1\text{-}52)$$

The second two terms yield

$$F + (L/a^2)\ln(L+ay) - y/a = C_2. \tag{1-53}$$

These two independent integrals represent two families of integral surfaces of (1-47). For fixed values of C_1 and C_2 the two surfaces intersect to give a characteristic curve in 3-space.

If arbitrary functional dependence is permitted then

$$C_2 = G(C_1) \tag{1-54}$$

and the locus of the intersections generates a surface in 3-space which is an integral surface of equation (1-50). It follows then that the general solution is

$$F + (L/a^2)\ln(L+ay) - y/a = G(T - \ln[L+ay]/a). \tag{1-55}$$

The functional form of G can be determined by employing the boundary condition $F(y, 0) = 0$ in equation (1-55). This yields

$$G(-\ln[L+ay]/a) = (L/a^2)\ln(L+ay) - y/a. \tag{1-56}$$

If $x = -\ln(L+ay)/a$,

$$G(x) = -(L/a)x + L/a^2 - \epsilon^{-ax}/a^2. \tag{1-57}$$

It follows that

$$F(y, T) = y(1-\epsilon^{-aT})/a + (L/a^2)(1-aT-\epsilon^{-aT}). \tag{1-58}$$

The region in which this solution is valid is defined by $y > 0$ and $F_y > 0$, where

$$F_y = (1-\epsilon^{-aT})/a\,. \tag{1-59}$$

Note that $F_y > 0$ for all $T > 0$ and by definition of $F(y, T)$, $F_T \geq 0$. Since

$$F_T = y\epsilon^{-aT} - (L/a)(1-\epsilon^{-aT}) \tag{1-60}$$

it is deduced that $0 \leq T \leq T_0$, where

$$T_0 = \ln(1+ay/L)/a\,. \tag{1-61}$$

The case where $y < 0$ and $F_y < 0$ can be treated in the same manner. For this case equation (1-47) has the form

$$\begin{aligned} -y + (L-ay)F_y - F_T &= 0\,, \\ F(y,0) &= 0\,. \end{aligned} \tag{1-62}$$

The solution to equation (1-62) is

$$F(y,T) = -y(1-\epsilon^{-aT})/a + (L/a^2)(1-aT-\epsilon^{-aT}), \tag{1-63}$$

and

$$F_y = -(1-\epsilon^{-aT})/a\,, \tag{1-64}$$

$$F_T = -y\epsilon^{-aT} - (L/a)(1-\epsilon^{-aT})\,. \tag{1-65}$$

To satisfy $F_T > 0$ it is required that $0 \leq T \leq T_0$, where

$$T_0 = \ln(1 - ay/L)/a\,. \tag{1-66}$$

The cases $y > 0$, $F_y < 0$ and $y < 0$, $F_y > 0$ do not have valid solutions since the sign constraints on y and F_y cannot be satisfied simultaneously. This must be since the two cases considered exhaust all non-zero values of y and F_y.

If the two sets of valid solutions are observed, it is seen that the complete solution satisfies

$$F(y,T) = |y|(1-\epsilon^{-aT})/a - (L/a^2)(\epsilon^{-aT} + aT - 1) \tag{1-67}$$
$$0 \leq T \leq T_0\,,$$

where

$$T_0 = \ln(1 + a|y|/L)/a. \tag{1-68}$$

The optimal policy is

$$\begin{aligned} f &= -L \operatorname{sgn} F_y \\ &= -L \operatorname{sgn}[(1-\epsilon^{-aT})\operatorname{sgn}(y/a)] \end{aligned}$$

for $0 \leq T \leq T_0$. This simplifies to

$$f = -L \operatorname{sgn} y, \qquad t \neq 0, \quad y \neq 0, \tag{1-69}$$

and if $T = 0$ or $y = 0$, f may be defined equal to zero.

There remains the case $T > T_0$. Since $F_T = 0$ by necessity the solution is simply

$$F(y, T) = F(y, T_0), \quad T \geq T_0$$

and the same optimal policy is in force.

Note that the solution (1-67) has the form of equation (1-26). Thus

$$R_k \Delta \approx (1-\epsilon^{-aT})/a\,, \tag{1-70}$$

and

$$LS_k \approx (L/a^2)(\epsilon^{-aT}+aT-1). \tag{1-71}$$

The two solutions are compared numerically in Appendix I.

1.2 A Stochastic Version

Consider again the system described by

$$\frac{dy}{dt} + ay = r(t) + f(t) \tag{1-72}$$

with $$y(0) = 0,$$

and $$|f(t)| \leq L,$$

where $r(t)$ is a stationary random disturbance. It is assumed that $r(t)$ has a known distribution function $P(r)$ which is illustrated by the smooth curve in Figure 3. The value of $P(r_0)$ is the probability that r lies in the interval $-\infty < r \leq r_0$.

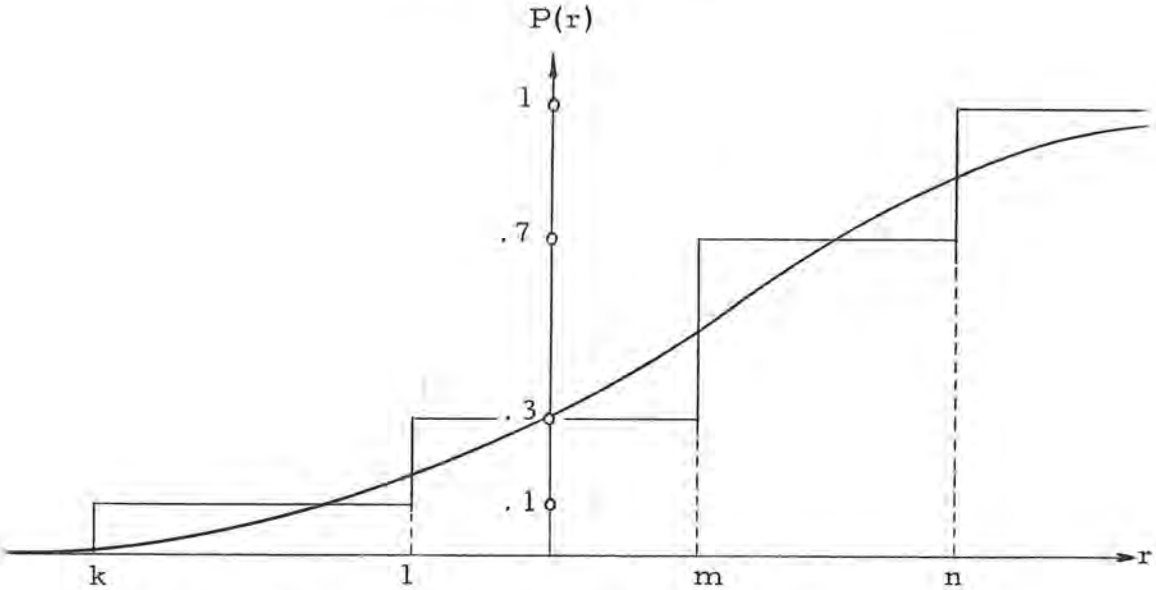

Figure 3. Distribution Function for $r(t)$.

Suppose that at some time t the system state is y and a decision f is to be made. Because of the presence of $r(t)$ the controller does not know exactly which new state $y(\Delta)$ arises after a time interval Δ following the decision. To render the problem formulation tractable assume that after the decision is made and before the next stage, $y(\Delta)$ is known to the controller. Assume also that system performance is measured in terms of some average value of the criterion function. This will be called the expected return.

If $P(r)$ is a smooth function it can be expressed as

$$dP(r) = p(r)dr ,$$

where $p(r)$ is the probability density function. Thus, $p(r)$ acts as a weighting function and the integration is over all r.

If the criterion function is

$$J(f) = \int_0^T |y(t)|\,dt, \qquad (1\text{-}73)$$

equation (1-17) can be modified to be

$$F(y, T) = \underset{|f| \leq L}{\text{Min}} \left[|y| \Delta + \int_{-\infty}^{+\infty} F(y+[f-ay]\Delta + r\Delta,\ T-\Delta)p(r)dr \right. \qquad (1\text{-}74)$$

where $F(y, T)$ is the expected return over the interval $(0, T)$, starting in the state y and using an optimal policy. The first term of the bracketed expression is the incremental return in the small interval $(0, \Delta)$. The integrand is the expected return over the remaining interval (Δ, T) if at time Δ the disturbance has the value r. Thus, the integral is over r.

To carry out the computation of equation (1-73) revert to the discrete formulation of the problem and write

$$F_n(y) = \underset{|f| \leq L}{\text{Min}} \left[|y| \Delta + \int_{-\infty}^{+\infty} F_{n-1}(y+[f-ay]\Delta + r\Delta)p(r)dr, \right. \qquad (1\text{-}75)$$

$$F_1(y) = |y|\Delta.$$

To perform the integration assume that $r(t)$ can take on only discrete values. Thus, the distribution function might be represented by the step form of Figure 3, in which the possible values of $r(t)$ are k, l, m, or n. Equation (1-75) now takes on the form

$$F_n(y) = \underset{|f| \leq L}{\mathrm{Min}} \left[|y| \Delta + 0.1 F_{n-1}(z+k\Delta) + 0.2 F_{n-1}(z+\ell\Delta) + 0.4 F_{n-1}(z+m\Delta) + 0.3 F_{n-1}(z+n\Delta) \right], \tag{1-76}$$

$$F_1(y) = |y| \Delta ,$$

where $\quad z = y + (f-ay)\Delta .$

The iterative computation is of the type which generally can be carried out with ease by a digital computer. The more discrete values of $r(t)$ that are chosen the better the continuous distribution will be approximated, but the more computation the computer must perform. However, these computations are simple and since the values of $F_{n-1}(y)$ are in high speed storage, they consume little time.

Because of the similarity between (1-76) and the equations for both the discrete and the continuous cases, it is natural to suspect that the optimal policy is again a maximum effort type satisfying

$$f = -L\,\mathrm{sgn}[y(1-a\Delta)+\hat{r}\Delta] , \quad |y(1-a\Delta)+\hat{r}\Delta| - L\Delta > 0 , \tag{1-77}$$

$$f = -[y(1-a\Delta)+\hat{r}\Delta] / \Delta, \qquad |y(1-a\Delta)+\hat{r}\Delta| - L\Delta \leq 0 ,$$

where $\hat{r}$ is the expected value of r defined by

$$\hat{r} = \int_{-\infty}^{\infty} r p(r) dr \; . \tag{1-78}$$

In the limit as Δ approaches 0

$$f = -L \operatorname{sgn} y, \qquad |y| > 0 \; ; \tag{1-79}$$
$$f = -\hat{r}, \qquad |y| = 0 \; .$$

It has been implicitly implied that $|\hat{r}| < L$.

If the assumptions made are valid, the controller illustrated in Figure 2 with $\Delta = 0$ can be employed here. When $y = 0$ feedback automatically forces f to assume a time average value of $-\hat{r}$.

1.3 Discussion

The method of treating the preceding control process can be generalized in the following terms:

a. There exists a physical process which can be described at a particular stage (or sampling instant) by a number of state variables y_k which comprise a state vector y.

b. At each stage the controller must specify the value of one or more control variables f_k which comprise a control vector f.

c. The effect of a decision f is to transform the state vector y to a new state vector $y(\Delta)$ at the beginning of the next stage.

d. The decision is to be optimal in the sense that it is a member of a sequence of decisions which optimizes a given control criterion $J_N(f)$.

The use of one control criterion has been demonstrated in this section. For more difficult problems the choice of criterion function will generally influence the relative ease with which a solution is obtained.

At this point it seems appropriate to ask why this method of solution might be attractive. Some of the reasons are apparent already. The prime reasons are :

a. Given an N-stage process with Y state variables, the iterative technique reduces the single (NY)-dimensional problem to a sequence of N,Y-dimensional problems. This has the computational advantage of requiring less computer storage capacity.

b. In solving a particular problem it is imbedded in a more general problem; thus, it is easier to determine the important structural features of the solution to the general problem.

c. Problems which escape solution by any classical approach because of their non-analytic structure generally yield to solution by the functional equation technique, unless they are too unwieldy from a computational viewpoint.

d. If the optimal policy is not readily obtained, the functional equation generally provides a means for converging upon the optimal policy by means of successive approximations.

e. A control function is specified as a function of the process variables, which are measurable. Thus, inherent in the solution are the specifications for the controller.

f. In synthesizing controllers the use of the functional equation shifts the emphasis away from stability and toward optimality. Optimality automatically implies stability. Feedback is implicit in the generation of optimal policies.

g. In general, the presence of process constraints decreases computational time because they narrow the permissible values of the control variables.

CHAPTER II

AN ABSTRACT FORMULATION OF DYNAMIC PROCESSES

2.0 Introduction

In this chapter abstract processes will be considered, and generalized functional equations will be deduced. The notion of causality is emphasized here since it is this notion which forms the basis for the derivation of functional equations.

To elaborate, consider the treatment of the deterministic system in Chapter I. The assertion that the optimal value of the criterion function depends only on the initial state and the process duration is an assertion of causality. It cannot be denied that the treatment of the stochastic system had the same basis. By using statistical regularity it was possible to remove the random variable from the formulation and to replace it by its known properties.

In the generalizations made, no distinction between linear and non-linear processes are made since the formalism incorporates both.

The purpose or goal of a physical process will generally provide the foundation for its description. With a physical process is associated a large number of variables whose values influence the goal of the process. These variables are quantities which can be measured or assigned numerical values. For example, missile velocity is a variable whose interaction with a number of other significant variables influences miss distance.

On the basis of observation and foresight all of the process variables which would seem to be significant are specified. These variables are divided into two subsets -- state variables and parameters. The criterion for this division is the following: state variables interact only with state variables, parameters interact with parameters and state variables. As an example, a change in angle of attack produces change in a missile's velocity vector but does not deflect its control surfaces or change its weight. On the other hand, a control surface deflection changes its angle of attack and the velocity vector. It also changes the weight of the vehicle if the actuator fluid is spewed into the atmosphere. Thus, angle of attack, velocity, and the variables dependent upon these quantities are state variables. Control surface deflection and weight are parameters. A detailed discussion of the distinction between state variables and parameters is presented in [3] .

The choice of state variables defines the system. A vector whose components are the state variables is the "state vector". Henceforth, a system state will mean a specific set of values of the components of the state vector.

Once the state variables are selected the remaining variables are assigned to a parameter subset. Parameters may be constant, variable, random, or adjustable. For example, control surface area is constant, local atmospheric density is variable, atmospheric distortion of the input signal to a radar tracking system is random, and the angular deflection of a control surface is adjustable.

The adjustable parameters are perhaps the most interesting since the act of changing their values changes the interdependence of the state variable.

Among the state variables some are chosen and designated the system outputs. The outputs are closely associated with the goal of the process. System inputs are chosen and they also may be associated with the goal. Their properties are measured but they in no way are affected by the remaining state variables or the parameters.

A parameter common to all dynamic processes is time. Since it plays a unique role in the description of processes the dependence of the process variables on time is explicitly listed. It will be seen that time manifests itself as the number of stages remaining in a process.

The existence or non-existence of variable parameters serves to classify processes into two types. These are respectively control processes and descriptive processes. In the work to follow, the functional equation technique is used to handle both types. This technique applied to descriptive processes yields the theory of invariant imbedding; applied to control processes it yields the theory of dynamic programming. The designation of state variables and parameters is carried out in the following chapter where both a descriptive and a control process will be discussed.

2.1 Descriptive Processes

The abstract formulation will begin with a descriptive process. Let a system be characterized by a state vector y. At discrete times a transform $T(y)$ is applied, yielding a sequence of states

$$\begin{aligned} y_0 &= \text{initial condition} \\ y_1 &= T(y_0) \\ &\vdots \\ y_N &= T(y_{N-1}) \,. \end{aligned}$$

The nature of the transformation $T(y)$ depends upon the parameters of the process and interaction of the state variables.

It is apparent that the values of y_N depend only upon the initial state y and the number of transformations N. This is expressed by writing $y_N = F_n(y)$. Notice that the problem may be started in the state y and N transformations performed, or can be started in the state y_1 and $N-1$ transformations performed, or it can be started in any state y_k and $(N-k)$ transformations performed. In any case the resulting state is y_N.

The causal relationship yields the basic functional equation

$$\begin{aligned} F_n(y) &= F_{n-1}(T(y)), \qquad n = 1, 2, \cdots, N, \\ F_0(y) &= y \ , \end{aligned} \tag{2-1}$$

which permits a recursive computation of y_N. The verbal transliteration of equation (2-1) is the following: if the system is in the state y and n transformations are performed, the resultant state is $y_n = F_n(y)$. After the first transformation the new state is $T(y)$. Starting with this new state and performing $(n-1)$ transformations the resultant state is also y_n.

This can be demonstrated by employing equation (2-1) recursively to yield

$$F_n(y) = F_{n-1}(T(y)) = F_{n-2}(T(T(y))) = \cdots = F_0(\underbrace{T(\cdots T}_{n}(y)\cdots). \tag{2-2}$$

But $F_0(y) = y$; hence

$$F_0(\underbrace{T(\cdots T}_{n}(y)\cdots)) = T(\underbrace{T\cdots T}_{n}(y)\cdots). \tag{2-3}$$

It follows that

$$F_{n-k}(\underbrace{T\cdots T(y)\cdots}_{k}) = T(\underbrace{T\cdots T(y)\cdots}_{n}) = y_n, \tag{2-4}$$

which is true by definition of the process.

The following diagram (Figure 4) demonstrates causality if the process is described by a single state variable y. Suppose that the initial condition y_0 at time $t = 0$ causes $y(t)$ to assume the continuum of values in Figure 4.

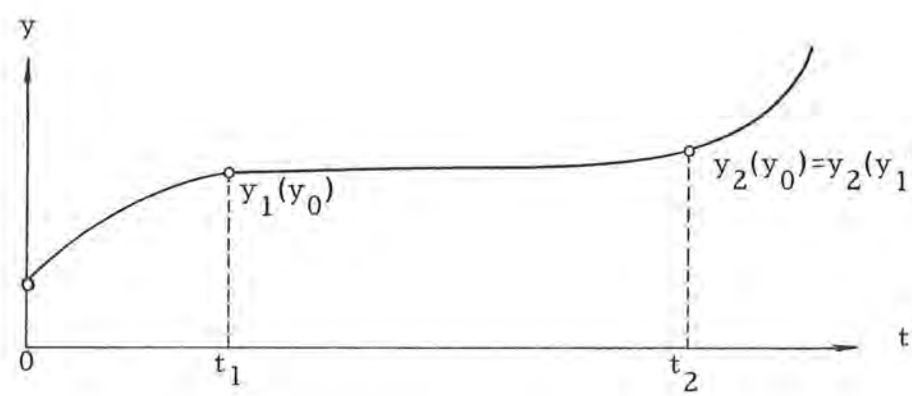

Figure 4. Diagram Illustrating Causality.

The value of y at t_1 can be expressed by $y_1(y_0)$ and the value of y at t_2 can be expressed by $y_2(y_0)$. In doing so it is acknowledged that the initial condition and the time are sufficient to specify y. Now observe that if the system starts with the condition $y_1(y_0)$ at time t_1, the value of y at t_2 is $y_2(y_1(y_0))$ and this value must equal $y_2(y_0)$.

In summary, the original problem was to determine the resultant system state arising from a family of N transformations applied to a particular state vector. To solve this problem it was imbedded in a more general problem in which the initial state and the number of transformations are arbitrary. In this general problem a single transformation maps the vector into itself. Thus this space is an invariant of the generalized process -- hence the name, invariant imbedding.

2.2 Control Processes

In a control process adjustable parameters are given whose values must be chosen to optimize the performance of a suitably defined system with respect to a given criterion of optimality. This criterion is intimately associated with the goal of the process and is expressed by the criterion function, a pre-assigned function of the state variables and perhaps the cost of control. Its value is a measure of system performance. If the value of the criterion function is determined primarily by the terminal state the process will be called a terminal control process, and if its value is determined by the whole sequence of states the process will be called a general control process.

Any rule for making allowable control decisions; that is, any rule for adjusting parameters is termed a "policy". An optimal policy optimizes (maximizes or minimizes) the criterion function. The functional equation governing the system is obtained by applying Bellman's Principle of Optimality [2]:

> An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Assume that only the terminal state of the system is of interest. The criterion function is some function of this terminal state, say $R(y_N)$, and a policy consists of a selection of N consecutive transformations $T_1, T_2, \cdots, T_N$. If the initial state of the system is y these transformations yield successively the states

$$
\begin{aligned}
y_0 &= \text{initial condition} \\
y_1 &= T_1(y_0)\,, \\
y_2 &= T_2(y_1)\,, \\
&\;\;\vdots \\
y_N &= T_N(y_{n-1})
\end{aligned}
\tag{2-5}
$$

It is assumed that the transformations are to be chosen to maximize $R(y_N)$.

Observe that if an optimal policy is employed the value of $R(y_N)$ is a function only of the initial state y and the number of transformations N. With this in mind a basic auxiliary function is defined as

$$F_n(y) = \underset{(T_1, T_2, \cdots, T_n)}{\text{Max}} [R(y_n)] \tag{2-6}$$

where the maximum is taken over all possible sequences of allowable transformations and $n = 1, 2, \cdots, N$. $F_n(y)$ is defined as the n-stage return starting with an initial state y and using an optimal policy. In the literature $F_n(y)$ is also referred to as the n-stage yield or cost.

Now the principle of optimality is employed to derive the functional equation. Let f be a vector whose components are the adjustable parameters, and let $T(y, f)$ be the transformation of y corresponding to a particular choice of f. If the initial state of the system is y, then following some initial decision f the new system state is $T(y, f)$. By definition the return or yield from the following $(n-1)$ stages is $F_{n-1}(T(y, f))$. If the n-stage return is to be a maximum, f must be chosen so that

$$F_n(y) = \operatorname*{Max}_{f} F_{n-1}(T(y, f)), \qquad n = 2, 3, \cdots, N, \quad (2\text{-}7)$$

where

$$F_1(y) = \operatorname*{Max}_{f} R(T(y, f)) .$$

Of course the designer does not have complete freedom in the choice of f because all adjustable parameters have physical constraints. Strictly speaking the constraint $f \subset S$, where S is a closed space containing all allowable vectors f, should always be written.

In a manner similar to that employed for descriptive processes the problem has been imbedded within a family of more general problems. Originally it was desired to determine the optimal control policy for a system starting with a particular initial state and subject to a particular number of transformations. Instead of solving the isolated problem a more general problem is solved in which the initial state and number of transformations are arbitrary. Thus, properties of the optimal policies associated with the members of a family of similar processes are deduced.

Suppose now that the criterion function depends not only on the terminal state of the system but also upon intermediate states. Thus, each control decision results in a contribution to the overall return. Let the contribution of a single state be $r(y, f)$; that is, the contribution depends only upon the state and the control decision.

If the system starts in the state y and an initial decision f is made the single stage return is $r(y, f)$ and the new system state is $T(y, f)$. By definition the optimal return over the remaining $(n-1)$ stages is $F_{n-1}(T(y, f))$; hence, the total return is $r(y, f) + F_{n-1}(T(y))$. If the n-stage return is to be a maximum, f must be chosen so that

$$F_n(y) = \underset{f \subset S}{\mathrm{Max}}[r(y,f) + F_{n-1}(T[y,f])] \;, \quad n = 2, 3, \cdots N,$$
$$F_1(y) = \underset{f \subset S}{\mathrm{Max}}[r(y,f) + R(T[y,f])] \;. \tag{2-8}$$

If it is assumed that the transformations and the criterion function are time dependent this can be denoted by the subscript k in $T_k(y, f)$ and $r_k(y, f)$. Thus k must fix the time with respect to the initiation of the process if the iterative computation is to be valid. Accordingly, the total process duration is fixed at some value equal to $N\Delta$. Then the computation proceeds backwards by defining $F_n(y)$ to be the return over the last n stages of the process starting in the state y and using an optimal policy. Then,

$$F_n(y) = \underset{f \subset S}{\mathrm{Max}}[r_{N-n}(y,f) + F_{n-1}(T_{N-n}[y,f])] \;, \quad n = 2, 3, \cdots, N,$$

and (2-9)

$$F_1(y) = \underset{f \subset S}{\mathrm{Max}}[r_{N-1}(y,f) + R(T_{N-1}[y,f])]$$

gives $F_n(y)$ as computed in terms of the return realizable in the future. The subscripts in equation (2-9) refer to the time at which the functions are applicable. The process is completed at time equal to N.

If the control process is stochastic rather than deterministic the transformation arising from a decision f is not known. Rather, the initial vector y is transformed into a stochastic vector z with an associated distribution function $G_f(y, z)$, dependent upon u and the decision f. It is assumed that z is known after the decision is made and before the next stage. It is also agreed to evaluate a policy in terms of some average value of the criterion function. This will be called the "expected return".

$F_n(y)$ is defined to be the expected n-stage return starting with an initial state y and using an optimal policy. If an initial decision f is made the expected return is $\int_{-\infty}^{+\infty}[r(y,f)+F_{n-1}(z)]\, g_f(y,z)dz$, where $g_f(y, z)dz = dG_f(y, z)$. Each possible return is weighted by its occurrence probability and averaged over all z. It follows that

$$F_n(y) = \underset{f \subset S}{\text{Max}} \int_{-\infty}^{+\infty}[r(y, f)+F_{n-1}(z)]\, g_f(y, z)dz, \quad n=2, 3, \cdots, N, \tag{2-10}$$

$$F_1(y) = \underset{f \subset S}{\text{Max}}\, r(y, f).$$

In the remaining text problems of the type discussed here will be presented.

For further details on this abstract point of view see Chapter III of [1] and [2] .

CHAPTER III

SOLUTION OF TRAJECTORY PROBLEMS

3.0 Introduction

Basic problems in the flight mechanics of missiles are of two types. It is desired either to describe the trajectory arising from determining causes or to specify a trajectory which optimizes some criterion function. The first problem gives rise to descriptive processes, the second gives rise to control processes.

The first problem to be considered here is a descriptive process which permits analytic solution under simplifying assumptions. Following this a non-linear control process requiring a discrete formulation and machine solution will be considered. The descriptive process leads to a method for reducing the dimensionality of the control process from three to two. In the discussion of these problems, a trajectory is confined to a single vertical plane.

3.1 Description of Uncontrolled Ballistic Trajectories

In this chapter the problem of determining the range covered on a stationary, flat earth during the ballistic flight of a missle will be investigated. To permit an analytic solution the atmosphere will be disregarded and it will be assumed that the missile is subjected only to a constant parallel, vertical gravitational field. The geometry is illustrated in Figure 5.

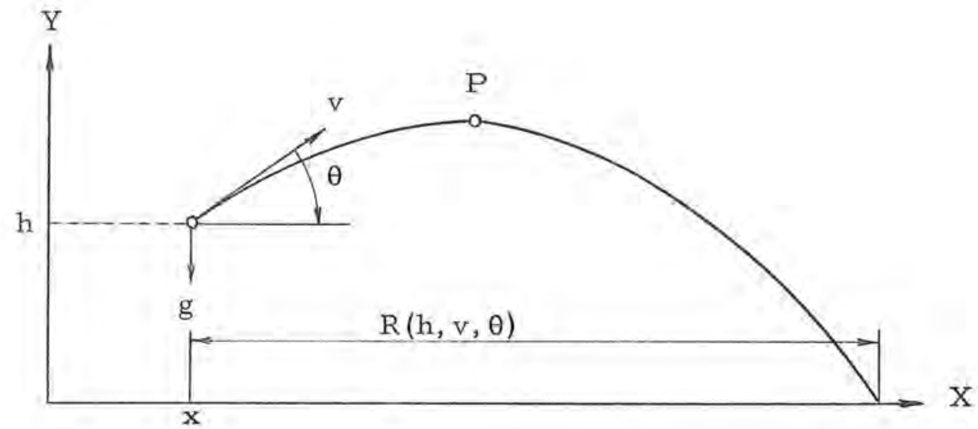

Figure 5. Geometry Describing a Simplified Trajectory.

The goal of this process is to exhibit range as a result of determining causes. These are the components of the initial state vector. To formulate the problem, consider all variables which significantly affect the goal:

x = horizontal coordinate

h = altitude

v = missile velocity

θ = angle between the velocity vector and the horizontal

T = remaining flight time

g = gravitational constant

t = time .

The range is independent of the initial value of x, since x serves only to reference the range to some origin. If it is assumed that the process duration T is sufficient to permit missile impact then the range is also independent of T. Time itself plays no significant role since there are no time dependent parameters.

This leaves only h, v, θ, and g for consideration. Among these only g can induce unilateral changes in those remaining. It follows that the state variables are h, v, and θ, and the sole parameter is g.

The interaction between the state variables is given by the following differential equations:

$$\frac{dh}{dt} = v \sin\theta,$$

$$\frac{dv}{dt} = -g \sin\theta , \tag{3-1}$$

$$\frac{d\theta}{dt} = -(g/v)\cos\theta .$$

Now to proceed to determine the range $R(h, v, \theta)$ from an arbitrary state (h, v, θ) to impact. If the system starts in this arbitrary state and a small increment of time Δ goes by, an increment of range of approximately $\Delta v \cos\theta$ is realized. The state variables undergo the following transformations:

$$\begin{aligned} h(\Delta) &= h + \Delta v \sin\theta \ , \\ v(\Delta) &= v - \Delta g \sin\theta \ , \\ \theta(\Delta) &= \theta - \Delta(g/v)\cos\theta \ . \end{aligned} \tag{3-2}$$

This notation implies that $(h[\Delta] \, , \, v[\Delta] \, , \, \theta[\Delta])$ is the state following (h, v, θ) after a small time lapse of Δ. Causality then yields:

$$\begin{aligned} R(h,v,\theta) = \Delta v\cos\theta + R(h+\Delta v\sin\theta,\ v - \Delta g\sin\theta, \\ \theta - \Delta(g/v)\cos\theta). \end{aligned} \tag{3-3}$$

Expanding R in a Taylor Series about (h, v, θ),

$$\begin{aligned} R = \Delta v\cos\theta + R + \Delta v\sin\theta\, R_h - \Delta g\sin\theta\, R_v \\ - \Delta(g/v)\cos\theta\, R_\theta + 0(\Delta) \ , \end{aligned} \tag{3-4}$$

where $\quad R_h = \partial R/\partial h,$

$\quad R_v = \partial R/\partial v,$

$\quad R_\theta = \partial R/\partial \theta,$

and $0(\Delta)$ contains higher order terms in Δ.

Now subtract R from both sides of the equation, divide by Δ and take the limit as Δ approaches zero. Since

$$\lim_{\Delta\to 0} 0(\Delta)/\Delta = 0 \tag{3-5}$$

the result is the following linear, first order, partial differential equation:

$$v \cos\theta = g \sin\theta R_v + (g/v) \cos\theta R_\theta - v \sin\theta R_h. \qquad (3\text{-}6)$$

To solve this equation the method of characteristics discussed in [4] will be employed. First convert equation (3-6) to its associated system of ordinary differential equations:

$$\frac{dv}{g \sin\theta} = \frac{v d\theta}{g \cos\theta} = \frac{-dh}{v \sin\theta} = \frac{dR}{v \cos\theta} \quad . \qquad (3\text{-}7)$$

The first and third terms yield an equation whose general solution is:

$$h + v^2/2g = C_1. \qquad (3\text{-}8)$$

The first and second term yield:

$$v \cos\theta = C_2, \qquad (3\text{-}9)$$

and the second and fourth terms yield:

$$R - (v^2/g) \cos\theta \sin\theta = C_3. \qquad (3\text{-}10)$$

The three independent integrals of (3-7) given by equations (3-8), (3-9), and (3-10) represent three families of integral surfaces of (3-6). For fixed values of C_1, C_2, and C_3 the three surfaces intersect to give a characteristic curve in 4-space. If arbitrary functional dependence is permitted,

$$C_3 = F(C_1, C_2). \tag{3-11}$$

The locus of the intersections generate a surface in 4-space which is an integral surface of equation (3-6). It follows that the general solution of (3-6) is

$$R = v^2/g \cos\theta \sin\theta + F(H + v^2/2g, \quad v\cos\theta). \tag{3-12}$$

The form of the functional F can be determined by introducing the boundary condition

$$R(0, v, \theta) = 0, \tag{3-13}$$

for $0 \geq \theta \geq -\pi/2$. Substituting this condition into the general solution gives

$$F(v^2/2g, \; v\cos\theta) = -(v^2/g)\cos\theta \sin\theta . \tag{3-14}$$

If $y = v^2/2g$ and $z = v\cos\theta$,

$$F(y, z) = (z/g)\sqrt{2gy - z^2} , \tag{3-15}$$

since θ is negative. It follows that

$$F(h + v^2/2g, \; v\cos\theta) = (v/g)\cos\theta \sqrt{2gh + v^2 \sin\theta} , \tag{3-16}$$

and the range equation becomes

$$R(h, v, \theta) = (v^2/g) \cos\theta \sin\theta + (v/g) \cos\theta \sqrt{2gh + v^2 \sin\theta} \quad (3\text{-}17)$$

where $\pi/2 \geq \theta \geq -\pi/2$.

This same problem will now be solved by another method which will be exploited later to reduce dimensionality. If it is assumed that $\theta = 0$ only at the highest point P of the trajectory, this fact can be used to solve the problem in two steps. First the range and change in altitude to the point P can be determined as a function of the initial state (h, v, θ), where $\pi/2 \geq \theta \geq 0$.

Because of the restriction on the shape of the trajectory, time can be eliminated from the equations of motion; and h can be treated as the independent variable. Thus equations(3-1) become:

$$\begin{aligned} \frac{dv}{dh} &= -gv \\ \frac{d\theta}{dh} &= -(g/v^2) \tan\theta \end{aligned} \quad (3\text{-}18)$$

Since the right side of these equations is independent of h, the range and the change in altitude to point P depends only upon v and θ.

Let $D(v, \theta)$ be the range to point P if the system starts in the state (h, v, θ). If an increment of altitude Δ is experienced, the system generates an increment of range $\Delta/\tan\theta$. The state variables v and θ undergo the following transformations:

$$v(\Delta) = v - \Delta(g/v) \ ,$$
$$\theta(\Delta) = \theta - \Delta(g/v^2 \tan\theta) \ . \tag{3-19}$$

It follows that

$$D(v,\theta) = \Delta/\tan\theta + D[v-\Delta(g/v),\ \theta-\Delta(g/v^2\tan\theta)] \ . \tag{3-20}$$

The corresponding partial differential equation is

$$(g/v)D_v + (g/v^2\tan\theta)D_\theta = 1/\tan\theta \ , \tag{3-21}$$

and the equivalent system of ordinary differential equations is

$$v\,dv = v^2\tan\theta\,d\theta = g\tan\theta\,dD \ . \tag{3-22}$$

Proceeding as before the general solution is

$$D = (v^2/g)\cos\theta\sin\theta + F(v\cos\theta) \ , \tag{3-23}$$

where F is an arbitrary functional.

F is determined by introducing the boundary condition $D(v, 0) = 0$. Substituting this condition in the general solution gives $F(v\cos\theta) = 0$. The solution satisfying the boundary condition is then

$$D(v,\theta) = (v^2/g)\cos\theta\sin\theta \tag{3-24}$$

Now define $A(v, 0)$ to be the change in altitude from the initial state to P. The causal relationship is now

$$A(v, \theta) = \Delta + A(v - \Delta g/v, \ \Delta g/v^2 \tan\theta); \tag{3-25}$$

hence,

$$(g/v)A_v + (g/v^2 \tan\theta)A_\theta = 1 \ . \tag{3-26}$$

The equivalent system of ordinary differential equations is

$$v\,dv = v^2 \tan\theta\, d\theta = g\,dA, \tag{3-27}$$

and the general solution is

$$A = v^2/2g + F(v\cos\theta) \ . \tag{3-28}$$

The boundary condition applicable here is $A(v, 0) = 0$. This yields

$$F(v) = -v^2/2g \ . \tag{3-29}$$

It follows that

$$F(v\cos\theta) = -(v\cos\theta)^2/2g \ , \tag{3-30}$$

and the solution is

$$A(v, \theta) = (v\sin\theta)^2/2g \ . \tag{3-31}$$

Finally, define $V(v, \theta)$ to be the velocity at P if the system starts in the state (v, θ). Since there are no horizontal forces, it is apparent that

$$V(v, \theta) = v \cos \theta \, . \tag{3-32}$$

The second step in solving this problem is to determine the range from the point P to impact. Let this range be $E(h, v, 0)$. It is easily shown that

$$E(h, v, 0) = (v/g)\sqrt{2gh} \, . \tag{3-33}$$

The range can now be computed from an arbitrary initial state (h, v, θ) where $\theta \leq 0 \leq \pi/2$. Let the system state at P be (h_p, v_p, θ_p); then, as a function of the initial state

$$h_p = h + A(v, \theta) = h + (v \sin \theta)^2/2g,$$

$$v_p = V(v, \theta) = v \cos \theta \, , \tag{3-34}$$

$$\theta_p = 0.$$

It follows that

$$R(h, v, \theta) = D(v, \theta) + E(h_p, v_p, \theta_p) \, . \tag{3-35}$$

On performing the indicated substitutions, equation (3-17) is reconstructed.

It has been learned that this equation holds for $\pi/2 \geq 0 \geq -\pi/2$. An expression of this type is useful for guiding a missile to a prescribed range.

The important observation to make concerning this method of solution is that the trajectory has been divided up into two sections, in each of which altitude varies monotonically. This property is used to treat the control process which follows. Although the assumption of monotonically varying altitude has simplified this problem, it will be pointed out in the next section that this assumption limits the application of the control policy that will be developed. The problem has been solved in terms of the important variables (h, v, θ); hence, the need for performing integrations with respect to time has been bypassed.

3.2 Control of One Class of Ballistic Trajectories

In this section the problem of how to control ballistic trajectories of the type discussed in section 3.1 will be considered.

The class of trajectories to be considered contains those trajectories in which altitude varies monotonically on both sides of the highest point of the trajectory. Many possible trajectories having local maximum altitudes are omitted from consideration. If a missile must re-enter the atmosphere it might be desirable for it to skip off of the atmosphere one or more times before beginning a monotonic descent. Such a problem could not be solved with the following approach.

The specific problem to be considered is the following:

> What missile launch angle and permissible angle of attack program yield a maximum horizontal range subject to terminal restrictions on altitude and velocity?

The launch angle is just the angle at which the missile is launched with respect to the local horizontal. The angle of attack is the angle between the missile's velocity vector and its body axis. It is by commanding an angle of attack that a missile is maneuvered. An angle of attack gives rise to a lift or maneuver force normal to the missile's body axis and an axial or drag force.

It is assumed that angle of attack is an adjustable parameter rather than a state variable. [1] As in the preceding problem it is assumed that the earth is flat and has a constant vertical gravitational field.

To formulate the problem, all variables which significantly affect range must be introduced. These are:

h altitude

v missile velocity

θ angle between velocity vector and horizontal

T remaining flight time

α angle of attack

g gravitational constant

f missile thrust, $f = f(t)$

w missile weight, $w = w(t)$

t time.

---

[1] This is equivalent to assuming a perfect autopilot.

Both f and w are dependent upon altitude and velocity, but it is assumed that this dependence is not significant. In addition, constraints will be introduced later.

It is apparent that α, g, f, w, and t, are parameters since changes in any of these quantities induce unilateral changes in the remaining variables. Thus, h, v, θ, and T are state variables.

There are several quantities, such as atmospheric density and dynamic pressure, to which names and symbols can be assigned; however, these quantities all have a static dependence upon the preceding variables. Their effect is to give rise to missile forces which can be accounted for by the following quantities:

$$\begin{aligned} A &= A(h, v, |\alpha|) , \\ N &= N(h, v, |\alpha|) \end{aligned} \tag{3-36}$$

where A = axial drag force and N = lift force.

The forces acting on the missile are illustrated in Figure 6.

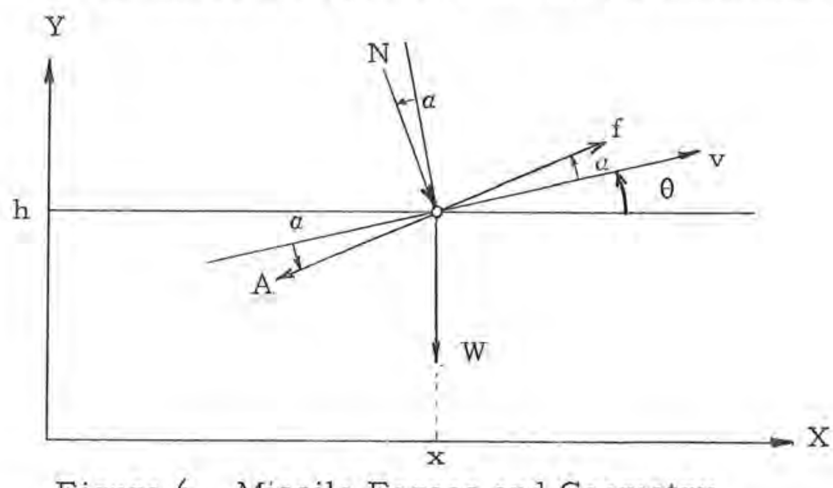

Figure 6. Missile Forces and Geometry

The interaction of the state variables is expressed by the following equations of motion:

$$
\begin{aligned}
\frac{dh}{dt} &= v \sin\theta \, , \\
\frac{dv}{dt} &= (g/w)[(f-A)\cos\alpha - N\sin\alpha - w\sin\theta] \, , \\
\frac{d\theta}{dt} &= (g/wv)[(f-A)\sin\alpha + N\cos\alpha - w\cos\theta] \, , \\
\frac{dT}{dt} &= -1 \, .
\end{aligned}
\tag{3-37}
$$

The constraints which must be observed are:

$$
\begin{aligned}
N(h, v, |\alpha|) &\le C_1 , \quad \text{a structural constraint} \, , \\
|\alpha| &\le C_2 \, , \quad \text{a control constraint} \, , \\
h(t+T) &= h_T \, , \\
v(t+T) &= v_T \, , \\
h(0) &= 0 \, , \\
v(0) &= 0 \, .
\end{aligned}
\tag{3-38}
$$

The initial and terminal values of θ will not be specified but will depend upon the optimal control policy. The total process duration $(t+T) = T_0$ is unknown and for the present can be thought of as a parameter.

The goal of the process is simply to maximize the change in x, the range, over the duration of the process subject to the system constraints. A measure of the goal is the following criterion functional:

$$J(\alpha) = \int_0^{T_0} v \cos\theta \, dt, \quad \alpha = \alpha(t) . \tag{3-39}$$

Because of the time dependence and non-linearities in the equations of motion, any attempt to obtain an analytic solution must be abandoned. Instead, a discrete formulation of the process can be used. A discrete formulation prepares the problem for solution by a digital computer and it clears the way for a mathematically rigorous treatment of the problem. As Bellman points out [2] both continuous and discrete formulations are approximations to the actual physical process; hence, one is concerned only with the value of either mathematical model rather than with the similarities between models.

The computational problem is now considered. There are four state variables, h, v, θ, and T. The abstract discussion of time dependent control processes in Chapter II indicates that T can be delegated to the parameter set. To do so the total process duration must be fixed at some value T_0 and the computation carried out backward in time from the termination of the process.

The computations must be repeated for each value of T_0. Such a procedure still leaves three state variables.

Suppose $R_n(h, v, \theta)$ is defined to be the maximum range over the last n stages of the process if the system starts in the state (h, v, θ). At the n^{th} stage $R_n(h, v, \theta)$ is computed in terms of $R_{n-1}(h, v, \theta)$. Both functions should be stored in table form in high speed storage. To carry out the computations, a range of variation is assigned to each state variable and only discrete values are chosen within each range. If the number of values $h, v,$ and θ are respectively P, Q, and R, a computer storage capacity of $2PQR$ is required. With the IBM 7094 computer this figure cannot exceed 60,000. It is apparent that a coarse grid is necessary and precision is lost.

There are probably several ways to overcome the storage problem. Three methods come to mind. The problem can be solved with a coarse grid and the solution used to narrow the range of variation for each state variable. The solution can then be repeated with a finer grid and the new bounds of variation. Another method discussed in [5] employs successive approximations to reduce the dimensionality of the problem from three to one. A third very interesting method involves expansion in terms of a set of orthogonal functions and is discussed in [6]. All of these methods overcome the dimensionality problem at the expense of the increased computer time.

To avoid these approximation methods the dimensionality of the formulation must be reduced. This can be done under certain assumptions. It frequently happens that the missile is uncontrolled or program controlled during the thrust phase of flight. This is done primarily to reduce induced drag and to thus maximize velocity at burnout. If this is the case, the thrust phase is a descriptive process which offers no computational problem. It is straightforward to compute in a step-wise manner the system state at the end of thrust arising from prescribed initial conditions. Running through a set of initial launch angles gives a corresponding set of terminal conditions. These conditions serve as initial conditions for the remaining flight interval.

It is assumed that the system equations are time dependent only during the thrust phase and that no control decisions have to be made during this interval. Let the range achieved at the end of thrust be $C(\theta_L)$ if the launch angle is θ_L, and let the resulting system state be (h_0, v_0, θ_0). It is now possible to consider the simpler problem of maximizing the range if the system starts in the state (h_0, v_0, θ_0) and terminates in the state (h_T, v_T, θ) where θ is unspecified.

To solve this problem the procedure described in the first section of this chapter can be used. First, the trajectory is divided into two sections, in each of which altitude varies monotonically. The fact that $\theta = 0$ at P, the point of maximum altitude is used to carry out this division.

Referring to Figure 7 assume that the maximum altitude h_p is specified (two possibilities are illustrated). Suppose that $D_p(h_0, v_0, \theta_0)$ is the maximum possible range if the system starts in the state $(h_p, v, 0)$ where v is the only variable with a free boundary condition.

Because of the intimate relationship between range and velocity, a trajectory which maximizes range subject to boundary conditions must at the same time maximize terminal velocity. In other words, minimization of drag along a path satisfying boundary conditions is implicit in the optimization procedure. This means that a maximum velocity v_p can be associated with each h_p. Both v_p and D_p can be computed.

Suppose further that $E(h_p, v_p, 0)$ is the maximum possible range if the system starts in the state $(h_p, v_p, 0)$ and terminates in the state (h_p, v_p, θ), where θ is unspecified. Then, the maximum overall range is

$$R(0, 0, \theta_L) = C(\theta_L) + \max_{h_p} D_p(h_0, v_0, \theta_0) + E(h_p, v_p, 0) , \qquad (3\text{-}39)$$

where the maximum is taken over all reasonable values of h_p,

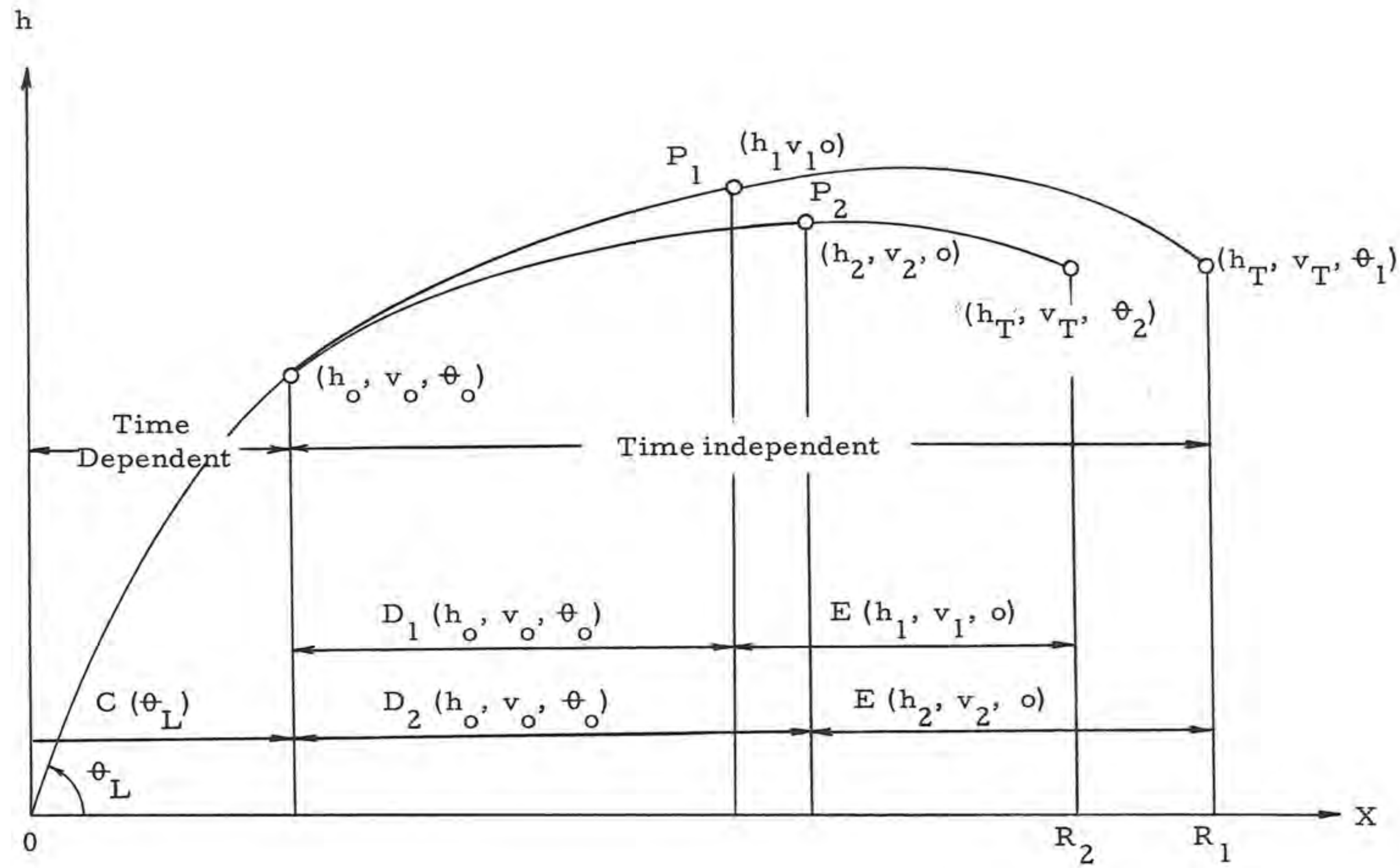

Figure 7. Representation of a Class of Trajectories.

and $R(0, 0, \theta_L)$ is the maximum range from the initial state $(0, 0, \theta_L)$ to the terminal state (h_T, v_T, θ). The optimal launch angle is determined by maximizing $R(0, 0, \theta_L)$ over all values of θ_L.

The functional equation needed to carry out the computations can now be derived. The time dependence has been eliminated from the system equations by considering control only during the power-off portion of flight. Dimensionality is reduced by considering h as the independent variable. The equations of motion given in (3-37) become:

$$\frac{dv}{dh} = f(h, v, \theta, \alpha)$$
$$\frac{d\theta}{dh} = g(h, v, \theta, \alpha) , \tag{3-40}$$

where the functional forms $F(h, v, \theta, \alpha)$ and $g(h, v, \theta, \alpha)$ have been adopted for simplicity.

First consider the problem of maximizing range from maximum altitude h_p to the terminal state (h_T, v_T, θ) where θ is unspecified. For this problem the criterion function expressed in (3-38) has the equivalent form

$$J(\alpha) = \int_{h_p}^{h_T} \cot\theta \, dh. \tag{3-41}$$

Since equations (3-40) are altitude dependent, the problem must be solved in the same manner as a time dependent problem. That is, the terminal altitude is fixed and the computation proceeds backward.

Continuing the problem with a discrete formulation, let $h_p = N\Delta$ and $h_T = K\Delta$, where $N > K$. If $N = K$ the problem is simplified but much the same. Now,

$$J(a_{K+1}, a_{K+2}, \cdots, a_N) = -\sum_{n=1}^{N-K} \cot \theta_{N-n} \Delta, \tag{3-42}$$

where the notation implies that J is a function of the sequence of decisions $(a_{K+1}, a_{K+2}, \cdots, a_N)$. Equations (3-40) become

$$\begin{aligned} v_{n+1} &= v_n + \Delta f(h_n, v_n, \theta_n, a_n) , \\ \theta_{n+1} &= \theta_n + \Delta g(h_n, v_n, \theta_n, a_n) , \end{aligned} \tag{3-43}$$

and the constraints become

$$\begin{aligned} N(h_n, v_n, |a_n|) &\leq C_1 , \\ |a_n| &\leq C_2 , \\ h_k &= h_T , \\ h_N &= h_p , \\ \theta_N &= 0 , \\ v_k &= v_T , \\ 0 \geq \theta &\geq -\pi/2 . \end{aligned}$$

The important observation is now made that <u>if an optimal policy is employed</u> the value of J is a function only of the initial state $(h_p, v, 0)$. To solve this problem it must be imbedded in a more general problem in which the initial state at some intermediate altitude is arbitrary. This state is denoted by (h, v, θ).

Define the basic auxiliary functional as

$$E_n(v, \theta) = \underset{(a_{K+1}, a_{K+2}, \cdots, a_{K+n})}{\text{Max}} \left[\sum_{i=1}^{n} \cot \theta_r \Delta \right], \qquad (3\text{-}44)$$

where $E_n(v, \theta)$ is the range over the last n stages if at the altitude $(K+n)\Delta$ the system is in the state (v, θ) and if an optimal sequence of decisions $(a_{K+1}, a_{K+2}, \cdots, a_{K+n})$ is made.

Assume that at some discrete altitude denoted by $(K+n)\Delta$ the system state (v, θ) and the decision a is made. After an incremental change Δ in altitude the system is in a new state $[v(\Delta), \theta(\Delta)]$. If an optimal sequence of decisions is made over the remaining $(n-1)$ stages, the range over these stages is $E_{n-1}[v(\Delta), \theta(\Delta)]$ by definition. If a is not chosen indiscriminately, but chosen to maximize range over the last n stages then

$$E_n(v, \theta) = \underset{a \subseteq S}{\text{Max}} \left[-\Delta \cot \theta + E_{n-1}(v[\Delta], \theta[\Delta]) \right], \qquad (3\text{-}45)$$

where
$$v(\Delta) = v + \Delta f(K+n)\Delta, v, \theta, a),$$
$$\theta(\Delta) = \theta + \Delta g(K+n)\Delta, v, \theta, a),$$

and S is the set of all permissible α which satisfy the constraints

$$N[(K+n)\Delta,\ v,\ |\alpha|] \leq C_1\ ,$$
$$|\alpha| \leq C_2\ .$$

In addition only negative θ satisfying $0 \geq \theta \geq -\pi/2$ are permitted.

Note that equation (3-45) is a mathematical statement of the principle of optimality. By solving this equation iteratively the problem is solved; however, a starting point must be established. This is provided by the single stage process. Since the terminal velocity is constrained, only those states (v, θ) for which it is possible to satisfy this terminal constraint after an incremental change Δ in altitude can be considered. It follows that

$$E_1(v, \theta) = -\Delta \cot \theta\ , \tag{3-46}$$

where v and θ satisfy

$$v_T = v + \Delta f([K+1]\Delta,\ v,\ \theta,\ \alpha)\ . \tag{3-47}$$

Once $E_1(v, \theta)$ and the associated decision $\alpha_1(v, \theta)$ have been solved for over the permissible range of v and θ, equation (3-45) is employed to solve for $E_2(v, \theta)$ and $\alpha_2(v, \theta)$, $E_3(v, \theta)$ and $\alpha_3(v, \theta)$, and so on. At each stage the velocity range can be extended if the added velocities, after an incremental change in altitude, are transformed into values contained in the velocity range of the preceding stage.

At any stage computations have to be made for small θ. Since $\cot\theta$ behaves badly in a region about $\theta = 0$, this linear approximation for an increment of range is inadequate. A much better approximation can be obtained with

$$E_n(v,\theta) = \max_{\alpha \subset S} \Big[-\Big(\sqrt{\theta^2 + 2g\Delta[N-w]/wv^2} - |\theta| \Big) wv^2 / g[N-w] + E_{n-1}(v[\Delta], \theta[\Delta]) \Big], \tag{3-48}$$

for small θ. This incremental range approximation is derived in Appendix II.

At any stage n, the value of $E_n(v, 0)$ is of special interest. This is the optimal range from maximum altitude to the terminal state corresponding to the initial state $((K+N)\Delta, v, 0)$. The accumulated information can be arranged as in Table 1.

Table 1. Record of $E_n(v, 0)$ for Specified Terminal Altitude and Terminal Velocity.

	Maximum Altitude $h_p = (K+n)$			
n	1	2		r
v_1	E_{11}	E_{12}	. . . .	E_{1r}
v_2	E_{21}	E_{22}	. . .	E_{2r}
	⋮			⋮
v_s	E_{s1}	E_{s2}	. . .	E_{sr}

To maximize the range from the state (h_0, v_0, θ_0) to maximum altitude, define

$$J(a_{n-U}, a_{n-U+1}, \cdots, a_{N-1}) = \sum_{n=U}^{N-1} \Delta \cot \theta_n , \qquad (3\text{-}49)$$

where $h_p = N\Delta$, $h_0 = U\Delta$, and $N > U$. Here J is the value of the range from the initial altitude $U\Delta$ to maximum altitude $N\Delta$ if the sequence of decisions $(a_{N-U}, a_{N-U+1}, \cdots, a_{N-1})$ is made.

For this case the auxiliary functional is

$$D_n(v, \theta) = \underset{(a_{N-n}, a_{N-n+1}, \cdots, a_{N-1})}{\text{Max}} \sum_{r-n}^{N-1} \Delta \cot \theta_r , \qquad (3\text{-}50)$$

where $D_n(v, \theta)$ is the maximum range over the last n stages if at the altitude $(N-n)\Delta$ the system is in the state (v, θ) and if an optimal sequence of decisions is made. Employing the principle of optimality,

$$D_n(v, \theta) = \underset{a \subset S}{\text{Max}} \; [\Delta \cot \theta + D_{n-1}(v[\Delta], \; \theta[\Delta])], \qquad (3\text{-}51)$$

where $\quad v(\Delta) = v + \Delta f[(N-n\Delta), v, \theta, a]$,

and $\quad \theta(\Delta) = \theta + \Delta g[(N-n)\Delta, \; v, \theta, a]$.

Here θ satisfies $\pi/2 \geq \theta \geq 0$. For small θ the approximation derived in Appendix II can again be used.

The single stage process is defined by

$$D_1(v, \theta) = (wv^2/g[N-w] (\sqrt{\theta^2 + 2g\Delta[N-w]/wv^2} - |\theta|), \quad (3\text{-}52)$$

where only those values of v and θ which satisfy

$$0 = \theta + \Delta g([N-1]\,\Delta,\ v,\ \theta,\ a)\ , \quad (3\text{-}53)$$

and $\theta > 0$ can be considered. It is reasonable to limit the range of θ to small values for the first stage.

It is likely that more than one set of initial states (h_0, v_0, θ_0) will be of interest. Such a set can be associated with each fixed control program employed during the thrust phase. Let $h_0 = U\Delta$ be the lowest altitude contained in the sets of interest and let y_i refer to the set (h_i, v_i, θ_i). Then the iterative computation of (3-51) is carried out until $n = N-U$. At any iteration if a set y_i can be associated, $D(y_i) = D(h_i, v_i, \theta_i)$ is recorded. The computations are repeated for several values of h_p and the following table of pertinent information is compiled (Table 2):

Table 2. Record of $D_n(v, \theta)$ for Specified Initial State y_i.

	Maximum Altitude $h_p = (U+n)\Delta$			
	1	2		r
y_1	D_{11}	D_{12}	. . .	D_{1r}
y_2	D_{21}	D_{22}	. . .	D_{2r}
⋮	⋮			⋮
y_n	D_{n1}	. . .		D_{nr}

Following the pattern of the descriptive process the velocity at maximum altitude arising from an initial state y_i and an optimal control policy can be determined. This velocity is a maximum for the prescribed boundary conditions.

Accordingly let $V_n(v, \theta)$ be the velocity after n stages if at the altitude $(N-n)\Delta$ the system is in the state (v, θ) and if an optimal control policy is employed. Then,

$$V_n(v, \theta) = \underset{a \subset S}{\text{Max}} \; V_{n-1} \; (v[\Delta] \, , \; \theta[\Delta]). \tag{3-54}$$

The single stage process is

$$V_1(v, \theta) = v + \Delta f\,[(N-1)\,\Delta, \; v, \; \theta, \; a] \; , \tag{3-55}$$

where $$0 = \theta + \Delta g[(N-1)\Delta, \; v, \; \theta, \; a] \; , \tag{3-56}$$

and $\theta > 0$.

The iterative computation of (3-54) is carried out until $n = K$; then the computations are repeated for a range of h_p. The results of these computations can be tabulated as follows:

Table 3. Record of $V_n(v, 0)$ for Specified Initial y_i .

	Maximum Altitude $h_p = (V+n)$		
n	1	2	
y_1	V_{11}	V_{12} . . .	V_{1r}
y_2	V_{21}	V_{22} . . .	V_{2r}
⋮	⋮		⋮
y_n	V_{n1}	V_{n2} . . .	V_{nr}

An important aspect of the maximization procedure is that the presence of constraints simplifies the computations. The more constraints, the smaller the allowable choice of α and the more rapid the numerical search. Usually an optimal initial decision α in an n-stage process is similar to that for an (n+1)-stage process, given the same initial state. Hence, the computer need only search the neighborhood of the previously determined policy. Computational aspects of dynamic programming are discussed in [6].

Sufficient information is now available to solve the problem. Given an initial state y_i, the maximum range corresponding to each value of n in Table 2 is determined. In corresponding positions in Table 3 the resulting velocities are determined. Thus, the state y_i is transformed to $(h_p, v_p, 0)$. For each pair of values (h_p, v_p), Table 1 gives the range to the terminal state (h_T, v_T, θ).

If $R(0, 0, \theta_L)$ is defined to be the total maximum range, then

$$R(0, 0, \theta_L) = C(\theta_L) + \max_{h_p}[D_p(h_i, v_i, \theta_i) + E(h_p, v_p, 0)] , \quad (3\text{-}56)$$

and the optimal value of θ_L maximizes $R(0, 0, \theta_L)$.

The remaining problem is to implement the optimal control policy. One of two philosophies can be followed: feedback can be exploited or a particular sequence of decisions can be programmed.

It is important to remember that the mathematical model employed to describe the process is not perfect. After executing a decision the resulting state is not exactly the predicted state. Thus, if a predetermined sequence of decisions based only upon the initial state and incremental changes in altitude is carried out, error accumulates. As a consequence, the terminal conditions are not satisfied simultaneously. If precision is not of prime intent, this is the simplest procedure for implementation. Since complete knowledge of $\alpha(h, v, 0)$ is available, the original control process can be converted to a purely descriptive process and the policy can be predetermined as a function of the initial conditions. An initial state (h_0, v_0, θ_0) is specified and the initial decision $\alpha(h_0, v_0, \theta_0)$ is looked up. Using the discrete form of equation (3-40) the next state is computed. The optimal decision for this state is looked up and the succeeding state computed. These computations are continued until the process is completed. The resulting sequence of decisions can be programmed.

Thus, a purely descriptive (or open loop) process is constructed which has no ability to detect error. It can only be hoped that the chosen mathematical model is realistic so that the accumulated error is small.

If precision is important, feedback must be exploited. This is accomplished by executing an optimal decision based upon the actual existing state of the system. Thus, provisions must be made for implementing the function $\alpha(h, v, \theta)$ for any state (h, v, θ) the system might find itself in. Implementing control policies is a complex problem in itself. Conceptually, the simplest method is to store $\alpha(h, v, \theta)$ in a grid for immediate use. With advances in micro-miniature circuitry and the utilization of logical design techniques, implementing optimal control of many variable systems may become feasible in the near future.

CHAPTER IV

AN APPROACH TO TERMINAL NAVIGATION

4.0 Introduction

To satisfy the performance requirements of a homing or command-homing missile a means of navigation is required which efficiently utilizes the acceleration capability of a missile. To do this it is necessary to eliminate needless missile maneuvers, counteract error early in flight, and minimize the time required to counteract error.

Present guidance techniques favor proportional navigation, where the system's commanded acceleration is proportional to the rate of rotation of the tracking angle [9]. Some of the deficiencies in proportional navigation are listed:

a. Disturbances such as reference error, launch error and target evasion lead to corrective accelerations which are sustained over the whole flight interval. For constant target evasion the corrective acceleration increases to a maximum at the end of flight. This maximum is more than twice the magnitude of the evasive acceleration of the target. Since the acceleration capability of a missile is usually a decreasing function of flight time, it will frequently happen that insufficient acceleration capability exists during the terminal maneuver and a large miss occurs.

b. Inherent dynamic characteristics of the system, such as missile velocity variation, are interpreted by the controller as an equivalent target evasion. As a consequence the missile maneuvers needlessly, it experiences excessive induced drag and the probability of a large miss is increased.

c. The gravitational force acting on the missile is generally overcome by an acceleration command bias. Three undesirable effects arise from this practice: the bias gives rise to induced drag; it decreases the effective upward acceleration capability of the missile; and it induces a secondary bias by causing asymetrical clipping of noise.

d. The system is highly sensitive to drift in the measurement of the tracking angle. Drift can lead to miss distances which are greater than those arising from target evasion.

e. A missile utilizing proportional navigation is denied targets which are prey for missiles with the same configuration but with a more intelligent controller.

It is the intention to present a system which in principle overcomes or minimizes the preceding deficiencies.

Since system details are of secondary importance in this treatment of guidance they are delegated to Appendix III. There the generalized navigation equation used in the following text is derived.

Homing guidance is one more example of a multi-stage decision process. At each instant the controller must determine, from available knowledge of the system state, the optimal command to minimize the terminal miss. Conveniently, the mathematical model used to describe the process yields a closed form solution for the optimal policy.

For a comparison of the approach here to that of final value control theory the reader is referred to [7] and [8].

4.1 A General Formulation and Solution

Using the formulation of dynamic programming consider the problem of minimizing the terminal value of $|y|$ for a system described by the differential equation

$$G\ddot{y} = Gc(t) + e(t) - L(t)f(t), \tag{4-1}$$

where $|f(t)| \leq 1$. This equation is derived in Appendix III. It describes the relative target-missile ordinate y when the inherent characteristics of the system give rise to a known time function $c(t)$ and an unknown time function $e(t)$. The quantity $L(t)$ is related to the corrective acceleration capability of the missile and $f(t)$ is the fractional command of this capability. The linear operator G describes the time response of the missile's autopilot. For a perfect autopilot $G = 1$ and for an autopilot with a first order lag $G = 1 + \mu d/dt$, where μ is the time constant.

The derivation of equation (4-1) is based upon linearizing assumptions which are valid only if the missile launch station has reasonable knowledge of the optimal launch angle. This implies that it has an estimate of the total flight time. Let this estimate be T. The problem then is to choose $f(t)$, subject to its constraint, so that $|y(T)|$ is minimized.

First assume that the controller is unable to deduce any properties of $e(t)$. The controller is designed on the assumption that $e(t) = 0$ and the perturbing effects of non-zero $e(t)$ are evaluated later. Thus the system is described by

$$G\ddot{y} = Gc(t) - L(t)f(t), \quad |f(t)| \leq 1. \tag{4-2}$$

Observe that the system state is completely specified at an arbitrary time s by the values of $y(s), \dot{y}(s), \cdots, y^{(n)}(s)$ and the duration of the process $(T-s)$. Here it is assumed that the highest order derivative of y, determined by G, is $(n+1)$. The n state variables and the time imply on $(n+1)$-dimensional problem if the problem is handled as in Chapter III. Fortunately, the number of varibales can be reduced to two if equation (4-2) is linearized.

Define the linear operator

$$H = G^{d^2} / dt^2,$$

and the function

$$h(t) = G\,c(t) .$$

Equation (4-2) can now be expressed in the following operational form:

$$Hy = h(t) - L(t)f(t), \quad |f(t)| \leq 1.$$

If the values of the state variables are singled out at time s the solution of this equation is expressed as follows:

$$y(t) = g(t,s) + \int_s^t K(t,\tau)[h(\tau) - L(\tau)f(\tau)]\,d\tau, \qquad (4\text{-}3)$$

$$t > s.$$

The function $g(t,s)$ is the solution of $Hy = 0$ which satisfies the initial conditions at time s. The integral expresses the solution in terms of the known impulse response $K(t, \tau)$ and the forcing function $h(t) - L(t)f(t)$.

The aim here is to minimize $|y(t)|$ starting the process in some known state at time $t = 0$. Since the miss distance is a function of the control policy $f(t)$ employed over the flight interval $(0, T)$, the criterion functional to be minimized is

$$J(f) = |y(T)|, \qquad f = f(t).$$

To solve this problem it is imbedded in a more general problem in which $J(t)$ is to be minimized starting at an arbitrary time s.

If t is replaced by T in equation (4-3),

$$J(f) = \left| M - \int_s^T K(T, \tau) L(\tau) f(\tau) d\tau \right|, \qquad |f(t)| \leq 1,$$

where

$$M = g(T, s) + \int_s^T K(T, \tau) h(\tau) d\tau .$$

The variable M is the miss distance which would result if there was no correction. It is completely specified by the system state at time s. Thus, the n state variable can be replaced by the single state variable M since both contain identical information as far as $|y(T)|$ is concerned.

Accordingly, the auxiliary function is defined to be

$$F(M, s) = \min_{|f(t)| \leq 1} J(f) .$$

It is required that $F(M, s)$ satisfy the boundary condition

$$F(M, T) = |M| .$$

The functional equation for the process can now be derived. If the system is in the state M at time s and a decision f is made, after a small increment of time Δ, M is transformed into a new state $M(\Delta)$ satisfying

$$M(\Delta) = M - \int_{s}^{s+\Delta} K(T, \tau)L(\tau)f(\tau)d\tau$$

$$\approx M - \Delta K(T, s)L(s)f.$$

By definition, the miss distance arising from this state and an optimal policy is $F[M-\Delta K(T, s)L(s)f, \ s+\Delta]$. If the initial decision f is to be optimal it is required that

$$F(M, s) = \min_{|f| \leq 1} F[M-\Delta K(T, s)L(s)f, \ s+\Delta] .$$

Expanding the right side in a Taylor Series,

$$F(M,s) = \min_{|f| \leq 1} [F(M,s) - \Delta K(T,s)L(s)fF_M + \Delta F_s + 0(\Delta)] ,$$

where $F_M = \partial F/\partial M$, $F_s = \partial F/\partial s$, and $0(\Delta)$ contains higher order terms in Δ. Subtracting $F(M, s)$ from both sides, dividing through by Δ and letting Δ approach zero gives

$$\min_{|f| \leq 1} [F_s - K(T,s)L(s)fF_M] = 0, \tag{4-4}$$

with $F(M, T) = |M|$.

For systems of interest $K(T, s)$ is a positive function of s in the interval $(0, T)$. The function $L(s)$ is certainly positive. It is apparent then that the minimum of the bracketed expression occurs if

$$f = \operatorname{sgn} F_M, \qquad F_M \neq 0 .$$

Equation (4-4) can now be written as

$$F_s - K(T,s)L(s)|F_M| = 0, \qquad F_M \neq 0, \tag{4-5}$$

or alternatively,

$$F_s - K(T,s)L(s)F_M = 0, \qquad F_M > 0 ,$$

$$F_s + K(T,s)L(s)F_M = 0, \qquad F_M < 0 .$$

Before solving these equations it is possible to deduce some properties of the expected solution. If at some time s the uncontrolled miss $|M|$ is sufficiently large, maximum correction acceleration exerted over the remaining flight interval is insufficient to reduce the terminal miss to zero. However, the terminal miss will be a minimum and the optimal control policy is unique. On the other hand, if at some time s the uncontrolled miss is sufficiently small there should be many optimal control policies which will guarantee a zero terminal miss. That is, the missile can exert corrective acceleration early in the flight, late in flight, or in some other manner. The solution to equation (4-4) should have these properties.

Following the solution of equation (4-4) the control criterion will be modified to yield directly a unique control policy which is satisfactory.

For the case $F_M < 0$, equation (4-5) becomes

$$F_s - K(T, s)L(s)F_m = 0, \tag{4-6}$$

and its associated system of ordinary differential equations is

$$ds = -dM/K(T, s)L(s) = dF/0.$$

Two independent solutions of this system are:

$$F = C_1$$

$$M + \int_{v(s)}^{u(s)} K(T, \tau)L(\tau)d\tau = C_2$$

where either $v(s) = s$, $u(s) = C$ or $u(s) = s$, $v(s) = C$. The constant C has to be determined. The general solution is then

$$F = Q[M + \int_{v(s)}^{u(s)} K(T, \tau)L(\tau)d\tau] ,$$

where Q is an undetermined functional.

Q is determined by introducing the boundary condition $F(M, T) = |M|$. This can be satisfied only if $v(T) = u(T) = T$ and $Q(x) = |x|$. It follows that $C = T$ and

$$F(M, s) = |M + \int_{s}^{T} K(T, \tau)L(\tau)d\tau| ,$$

or

$$F(M, s) = |M - \int_{s}^{T} K(T, \tau)L(\tau)d\tau| . \tag{4-7}$$

Since the solution is applicable only if $F_M > 0$ it is required that

$$F_M = \mathrm{sgn}\,(M \pm \int_{s}^{T} K(T, \tau)L(\tau)d\tau) = +1 .$$

Now determine F_s .

$$F_s = \mp [K(T, s)L(s)]\,\mathrm{sgn}\,(M \neq \int_{s}^{T} K(T, \tau)L(\tau)d\tau)$$

$$= \mp \; K(T, s)L(s).$$

If the solution is to satisfy (4-6) the form of (4-7) must be chosen. It follows that

$$F_s = +K(T, s)L(s) ,$$

and

$$M > \int_s^T K(T, \tau)L(\tau)d\tau .$$

Now consider the case $F_M < 0$ and $F_s + K(T, s)L(s)F_M = 0$. In a similar manner it is deduced that

$$F(M, s) = \left| M + \int_s^T K(T, \tau)L(\tau)d\tau \right| ,$$

where

$$F_M = -1 ,$$

$$F_s = -K(T, s)L(s) ,$$

and

$$M < -\int_s^T K(T, \tau)L(\tau)d\tau .$$

There remains to consider $F_M = 0$. From equation (4-4) it is noted that the choice of f is arbitrary. By definition,

$$F(M, s) = \underset{|f(t)| \leq 1}{\mathrm{Min}} \left| M - \int_s^T K(T, \tau)L(\tau)f(\tau)d\tau \right| .$$

If $F_M = 0$ then $F(M, s)$ is independent of M. This is possible only if

$$M = \int_{s}^{T} K(T,\tau)L(\tau)f(\tau)d\tau , \tag{4-8}$$

in which case $F(M,S) = 0$. Any policy $f(\tau)$ which satisfies equation (4-8) will do.

If M satisfies equation (4-8) it can be said that

$$|M| \leq \int_{s}^{T} K(T,\tau)L(\tau)d\tau .$$

The controller need not worry about satisfying equation (4-8) since an arbitrary policy will eventually lead the system to a state satisfying

$$|M| = \int_{s_0}^{T} K(T,\tau)L(\tau)d\tau, \ S_0 > s .$$

At this time the policy becomes determinate ; that is,

$$f = \operatorname{sgn} M ,$$

and equation (4-8) is automatically satisfied.

The complete solution is then

$$F(M,s) = |M| - \int_{s}^{T} K(T,\tau)L(\tau)d\tau, \quad |M| > \int_{s}^{T} K(T,\tau)L(\tau)d\tau,$$

$$F(M,s) = 0 \qquad , \quad |M| \leq \int_{s}^{T} K(T,\tau)L(\tau)d\tau .$$

The optimal control policy which yields this solution is:

$$f = \operatorname{sgn} M, \qquad |M| > \int_s^T K(T,\tau)L(\tau)d\tau ,$$

$$f \text{ arbitrary} \qquad |M| \leq \int_s^T K(T,\tau)L(\tau)d\tau .$$

It was stated earlier that one requirement for efficient control is to counteract error early in flight. Since f is arbitrary for a certain range of M, in this range f can be chosen to satisfy this performance requirement. The resulting policy is:

$$f = \operatorname{sgn} M, \qquad |M| > 0 ,$$

$$f = 0, \qquad M = 0 .$$

Though uniqueness has not been proved, one could hardly expect to discover a simpler one than the one presented.

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APPENDICES

APPENDIX I

A NUMERICAL EXAMPLE

In Chapter I forcing function control of a first order process was discussed. In the discrete formulation the N^{th} stage return was defined as

$$F_N(y) = \underset{\{f_k\}}{\text{Min}} \sum_{k=0}^{N-1} |y_k| \Delta ,$$

and the optimal control policy was deduced:

$$f = -L \operatorname{sgn} y, \qquad |y|(1-a\Delta) \geq L\Delta,$$

$$f = -y(1-a\Delta)/\Delta, \qquad |y|(1-a\Delta) \leq L\Delta,$$

The solution for $F_N(y)$ is the following:

$$F_N(y) = |y| R_k \Delta - LS_k \Delta ,$$

where $\quad R_{m+1} = 1 + (1-a\Delta)R_m, \qquad R_1 = 1,$

and $\quad S_{m+1} = S_m + \Delta R_m , \qquad S_1 = 0.$

The subscript k is determined from

$$|y| U_{k-1} - LV_{k-1} \geq 0 \geq |y| U_k - LV_k ,$$

where $$U_{m+1} = (1-a\Delta)U_m, \qquad U_1 = 1-a\Delta,$$

and $$V_{m+1} = V_m + \Delta U_m, \qquad V_1 = 1.$$

In the continuous formulation the return at time = T was defined as

$$F(y, T) = |y|(1-\epsilon^{-aT})/a-(\epsilon^{-aT}+aT-1)(L/a^2),$$

where $0 \leq T \leq T_0$ and

$$T_0 = (1/a) \ln (1+|y|a/L).$$

If $T > T_0$ then

$$F(y, T) = F(y, T_0)$$

A numerical solution will be carried out for both cases. Consider process durations of $N > K$ and $T > T_0$ to guarantee sufficient time to drive the error to zero. For convenience choose $\Delta = 0.1$, $a = 1$, and $L = 1$. In the following table the coefficients of $F_N(y)$ will be generated recursively.

Table I-1. Coefficients Required for Computations of $F_N(y)$ and $F(y, T)$: $b = 1$, $L = 1$, $\Delta = 0.1$.

T	$K=T/\Delta$	U_k	$V_k=R_k$	S_k	$\lvert y_k \rvert$	$\lvert y_T \rvert$	R(T)	S(T)
0.1	1	.900	1.00	0	0.11	0.11	0.95	0.05
0.2	2	.810	1.90	0.10	0.23	0.22	1.81	0.19
0.3	3	.729	2.71	0.29	0.37	0.35	2.59	0.41
0.4	4	.656	3.44	0.56	0.52	0.49	3.30	0.70
0.5	5	.591	4.10	0.91	0.69	0.65	3.93	1.07
0.6	6	.531	4.69	1.31	0.88	0.82	4.51	1.49
0.7	7	.478	5.22	1.78	1.09	1.01	5.03	1.97
0.8	8	.431	5.70	2.31	1.32	1.23	5.51	2.49
0.9	9	.387	6.13	2.87	1.58	1.46	5.93	3.07
1.0	10	.349	6.51	3.49	1.87	1.72	6.32	3.68
1.1	11	.314	6.86	4.14	2.19	2.00	6.67	4.33
1.2	12	.282	7.18	4.82	2.54	2.32	6.99	5.01
1.3	13	.254	7.46	5.54	2.93	2.67	7.27	5.73
1.4	14	.229	7.71	6.29	3.37	3.06	7.53	6.47
1.5	15	.206	7.94	7.06	3.86	3.48	7.77	7.23

In Table I-1 the value of $|y_k|$ satisfies $|y_k| U_k - LV_k = 0$. To compute $F_N(y)$, where $|y_{k-1}| < |y| \leq y_k$; then

$$F_N(y) = |y| R_k - LS_k \Delta.$$

For comparative purposes R(T) and S(T) have been evaluated where these quantities satisfy

$$F(y, T) = |y| R(T)\Delta - LS(T)\Delta;$$

hence,

$$R(t) = (1-\epsilon^{-aT})/a\Delta,$$

$$S(t) = (\epsilon^{-aT} + aT - 1)/a^2\Delta.$$

It follows then that

$$R(T) \approx R_k ,$$

and

$$S(T) \approx S_k .$$

The quantity $|y_T| = \epsilon^T - 1$ is that value of $|y|$ which satisfies

$$T_0 = (1/a)\log(1+a/L|y|),$$

where $$T_0 = K\Delta .$$

Thus, $$|y_k| \approx |y_T| .$$

APPENDIX II

APPROXIMATIONS TO IMPROVE COMPUTATIONAL PRECISION

In Chapter III it is necessary to compute incremental changes δ in range arising from incremental changes Δ in altitude. If θ (the angle between the velocity vector and the horizontal) is large, an adequate approximation is

$$\delta = \Delta \cot \theta .$$

However, as θ approaches zero δ increases without limit; hence, the approximation is invalid. In this appendix is derived another expression for δ which is applicable if θ is small.

From equations (3-37)

$$\frac{d\theta}{dh} = (g/wv^2 \sin\theta)[(f-a)\sin\alpha + N\cos\alpha] - g\cot\theta/v^2; \qquad \text{(II-1)}$$

and if θ and α are small,

$$\frac{d\theta}{dh} = (g/v^2)(N-w)/w\theta . \qquad \text{(II-2)}$$

Assume that over an incremental change in altitude the velocity remains constant. Integrating equation (II-2) from a small angle to another small angle θ_1, where $\theta > \theta_1$, gives

$$\theta_1^2 = \theta^2 + (2g/wv^2)(N-w)h. \tag{II-3}$$

Here h is a small change in altitude which has the sign of θ. The angle θ_1 also has the same sign as θ.

If equation (II-3) is substituted into the following equation;

$$\int_0^\delta dx = \int_0^h \cot\theta_1 hd \approx \int_0^h (1/\theta_1)dh, \tag{II-4}$$

and the integration is carried out, the result is

$$\delta = \pm(wv^2/g[N-w])(\sqrt{\theta^2 + 2gh[N-w]/wv^2} - |\theta|). \tag{II-5}$$

Here $\quad \operatorname{sgn}\delta = \operatorname{sgn}\theta$

and $\quad h = \Delta \operatorname{sgn}\delta$.

APPENDIX III

DERIVATION OF THE NAVIGATION EQUATION FOR A HOMING MISSILE SYSTEM

In this appendix the linearized navigation equation employed in Chapter IV will be derived. To minimize complexity the target and the missile will be constrained to move in a single vertical plane. Motions in this plane are referred to the initial line of sight. The following variables will be used :

V_m	missile velocity
V_T	target velocity
R	missile-target range
σ	tracking angle
r	angle between missile velocity vector and reference
θ	angle between target velocity vector and reference
ϕ	angle between the reference and the horizontal
g	gravitational constant
t	time

The geometry is illustrated in Figure III-1.

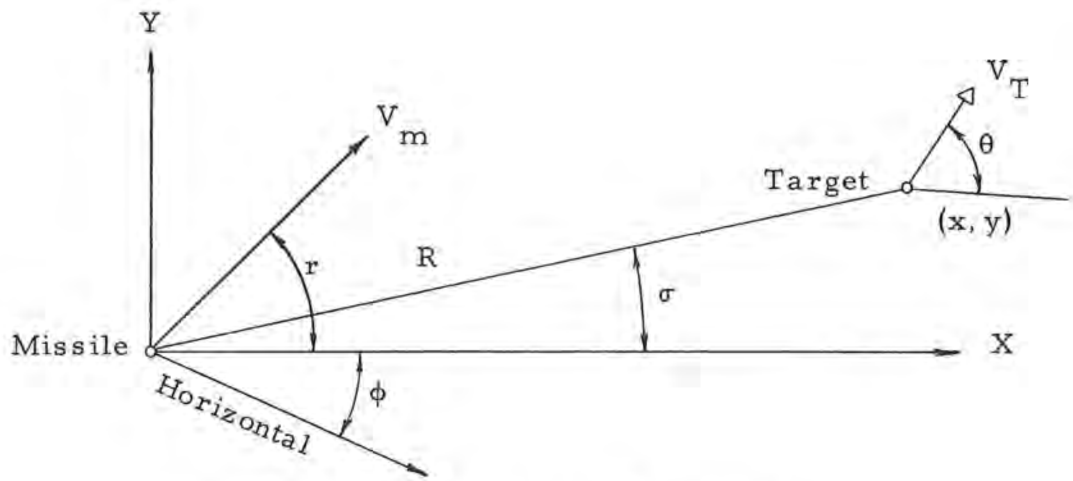

Figure III-1. Missile-target Geometry.

It is convenient to construct an orthogonal coordinate system whose axes translate in the plane under consideration. The origin remains coincident with the missile's center of gravity and the X-axis remains parallel to the reference.

If (x, y) are the coordinates of the target in this translating system, then

$$R^2 = x^2 + y^2 .$$

The miss distance will be defined to be that value of y when $x = 0$.

To linearize the navigation equation refer the missile and target trajectories to straight line trajectories selected to minimize the angular deviations which occur during the angular deviations which occur during flight. These reference trajectories are illustrated in

Figure III-2. The angles made with the reference by the missile and target reference trajectories at the initiation of the process are respectively r_0 and θ_0.

Now define the deviation angle

$$\gamma = r - r_0 ,$$

$$\Lambda = \theta - \theta_0 .$$

Throughout flight it is assumed that γ, Λ, and also σ, are small.

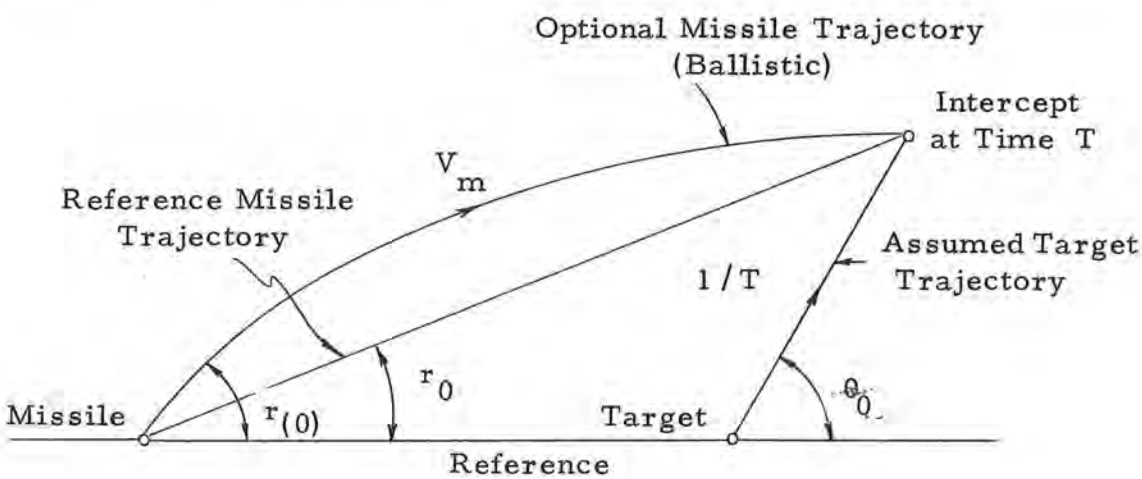

Figure III-2. Reference Trajectories.

From Figure III-1,

$$\dot{x} = v_T \cos\theta - v_m \cos r, \tag{III-1}$$

$$\dot{y} = v_T \sin\theta - v_m \sin r .$$

Replacing θ and r by their preceding definitions and employing small angle approximations gives :

$$\dot{x} = v_T \cos\theta_0 - v_m \cos r_0 - \Lambda V_T \sin\theta_0 + \gamma v_m \sin r_0, \qquad \text{(III-2)}$$

$$\dot{y} = v_T \sin\theta_0 - v_m \sin r_0 + \Lambda v_T \cos\theta_0 - \gamma v_m \cos r_0.$$

The first equation of (III-2) is useful for obtaining an estimate of the total flight time. If the initial range is x_0, an approximation for the flight time is that value of T for which

$$x_0 + \int_0^T (v_T \cos\theta_0 - v_m \cos r_0)dt = 0 . \qquad \text{(III-3)}$$

A more exact method is to compute the missile's flight time over ballistic trajectories to a discrete set of space points. This information and associated launch angles can be stored for immediate use. If the target's motion is predicted it is possible to specify the flight time and the missile launch angle.

Now differentiate the second equation of (III-2) with respect to time. This gives,

$$\ddot{y} = -\dot{\gamma} v_m \cos r_0 - \gamma \dot{v}_m \cos r_0 \qquad \text{(III-4)}$$

$$+ \frac{d}{dt}(v_T \sin\theta_0 - v_m \sin r_0 + \Lambda v_T \cos\theta_0)$$

Some of the terms in this equation may be negligible. For example, if the missile and target deviate slightly from the reference trajectories the terms $\gamma\dot{v}_m \cos r_0$ and $\Lambda v_T \cos\theta_0$ are small. If the target velocity is constant the term $\dot{v}_T \sin\theta_0$ can be neglected. Those terms which are significant may not be predictable; however, a reasonable knowledge of v_m and r_0 is assumed.

The missile's trajectory is controlled by exerting forces on the missile which induce changes in the deviation angle γ. Let the maximum acceleration arising from these control forces be $N(t)$, and let $f(t)$ denote the fractional command of this maximum capability. Then, if a_m is the induced acceleration and G the differential operator expressing the missile's response to the forces, it follows that

$$Ga_m = N(t)f(t), \qquad |f(t)| \leq 1. \tag{III-5}$$

If the control forces are exerted normal to the missile's velocity vector

$$a_m = v_m\dot{\gamma} - g\cos(r+\phi), \tag{III-6}$$

where $g\cos(r+\phi)$ is the acceleration normal to the velocity vector arising from the earth's gravitational force. If the control forces are exerted normal to the reference then

$$a_m = v_m\dot{\gamma}\cos r + \dot{v}_m \sin r - g\cos\phi. \tag{III-7}$$

Solve for $v_m\dot{\gamma}$ in equation (III-6) and substitute the result into (III-4). Then

$$\ddot{y} = m(t) - a_m \cos r_0 , \tag{III-8}$$

where $m(t) = g\cos(r+\phi)\cos r_0 - \gamma\dot{v}_m \cos r_0$

$$+ \frac{d}{dt}(v_T \sin\theta_0 - v_m \sin r_0 + \Lambda v_T \cos\theta_0).$$

Performing the operation G on both sides of equation (III-8) and introducing equation (III-5) gives

$$G\ddot{y} = Gm(t) - N(t)f(t)\cos r_0 . \tag{III-9}$$

Now if equation (III-7) is solved for $v_m\dot{\gamma}$ and the same operation applied, the result is

$$G\ddot{y} = Gn(t) - N(t)f(t), \tag{III-10}$$

where $n(t) = -g\cos\phi + \dot{v}_m \sin r_0 - \gamma\dot{v}_m \cos r_0$ (III-11)

$$+ \frac{d}{dt}(v_T \sin\theta_0 - v_m \sin r_0 + \Lambda v_T \cos\theta_0).$$

To deduce $n(t)$ we assume that $r \approx r_0$.

The occurrence of γ in both $m(t)$ and $n(t)$ is regarded as a perturbation rather than a means of control. Its influence here is overshadowed by its direct influence in $v_m\dot{\gamma}$.

The general navigation equation which incorporates both methods of control is the following:

$$G\ddot{y} = Gc(t) + e(t) - L(t)f(t) . \tag{III-12}$$

The terms $c(t)$ and $L(t)$ are determined by the control mode and $e(t)$ is a perturbing function arising from the unpredictable components of $m(t)$ or $n(t)$.