Orienting the Cube in Minkowski Space & Orienting Myself in Mathematics

By
Jo O'Harrow

A THESIS

submitted to

Oregon State University

Honors College

in partial fulfillment of
the requirements for the
degree of

Honors Baccalaureate of Science in Mathematics
(Honors Associate)

Presented May 31, 2022
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Abstract approved:

Tevian Dray

This is a two-part thesis. The first part is a generalization of vector calculus tools to Minkowski Space, a non-Euclidean 3-dimensional geometry that has a distance function that is not positive definite. We orient a cube in Minkowski Space using the generalized Stokes' Theorem to relate a divergence integral to a flux integral, generalized to the language of differential forms, in Minkowski Space. Furthermore, we compute the curvature of the hyperboloid in Minkowski Space, and of the Poincare disk model for hyperbolic geometry intrinsically using differential forms. This computation suggests an immediate application for our orientation of the cube to compute the curvature extrinsically using the shape operator. The second part of this thesis discusses traditional approaches to philosophy of mathematics, the emerging-in-popularity project of humanist mathematics, and finally how feminist theory and in particular feminist philosophy of science might inform a new philosophy of mathematics or critically expand the humanist project.

Key Words: Minkowski Space, Hyperbolic Geometry, Philosophy of Mathematics

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Toni Doolen, Dean, Oregon State University Honors College

I understand that my project will become part of the permanent collection of Oregon State University Honors College. My signature below authorizes release of my project to any reader upon request.

Jo O'Harrow, Author

Orienting the Cube in Minkowski Space & Orienting Myself in Mathematics

Jo O'Harrow

June 3, 2022

Contents

1 Introduction

This is a two part honors thesis written by Jo O'Harrow, with the tremendous support of Dr. Tevian Dray, at Oregon State University. My work in this thesis builds on both my training as a geometer, as well as a feminist philosopher. In many ways these two areas have informed each other; a piece of that will be presented through this thesis. The first part of this thesis will be more mathematically technical.

The mathematically-oriented chapter of this thesis includes an introduction to Minkowski Space, an argument to orient the cube in Minkowski Space, and finally a discussion and computation of curvature on the Hyperboloid in Minkowski Space. A result of this work is that the scope of applicability for existing vector calculus machinery, such that any engineer would possess, will be expanded into Minkowski Space. Our computation of curvature on the Hyperboloid is done using differential forms, however, using vector calculus machinery one can expect the same result and as such, this is a provided example for the applications of our orientation proof.

The second major chapter of this thesis discusses the potential for feminist philosophy of science lensing in qualitative inquiry for projects geared toward the overarching goal of humanizing mathematics. This lensing overlaps in significant ways with humanist philosophy of mathematics, but also enhances the project in novel ways that are discussed in the chapter. We end with a discussion of research questions which apply a feminist philosophy of science lens to inquire on the subjectivity and humanity of mathematics.

2 Orienting the Cube in Minkowski Space

"There's a very subtle sign involved"

– Tevian Dray (out of context)

2.1 Roadmap

In this paper, we will begin with a discussion of the history of Minkowski Space and hyperbolic geometry. Then, we will more technically introduce Minkowski Space and the tools needed to generalize vector calculus machinery to this setting. Next, we will present an argument to orient the cube in Euclidean space using the Euclidean divergence theorem. Then, we will generalize that argument by first extending the cross product, Hodge Dual operator, and divergence to Minkowski Space and, using those tools, orient the cube in Minkowski Space.

After our orientation of the cube, we move to a computation of curvature on the hyperboloid using differential forms. In completing those computations, we begin with the hyperboloid itself, which is embedded within Minkowski Space. We will provide a computation of the Gaussian curvature of the upper half of the Hyperboloid in Minkowski Space using differential forms. The form-based computation of curvature, which we complete for two isometries of the hyperboloid, ultimately provides an "answer-sheet" to verify a computation of curvature using our orientation, and we will give it thorough attention to verify its correctness.

Next will be a discussion of the stereographic projection from the Hyperboloid that results in the Poincaré Disk. This discussion will set us up with the tools necessary to investigate curvature in the Poincaré Disk. Because the Poincaré Disk models hyperbolic geometry, we expect its curvature to be a negative constant, and equal to the curvature seen in the hyperboloid. This result will be confirmed in the section on curvature in the Poincaré Disk.

An intrinsic geometric property is something that is preserved under geometric equivalence (defined through the concept of an isometry). The Theorema Egregium, proven by Gauss, tells us that curvature is one such intrinsic geometric property [22]. We will end this section with a brief discussion of why these models are equivalent. Because of that equivalence these two computations are ultimately verifying each other's correctness.

Finally, we will briefly discuss potential applications of this work, including a follow-up computation of curvature for the Hyperboloid embedded in Minkowski Space using the shape operator.

To put our goals into a list, we aim to accomplish the following:

1. Introduce Minkowski Space, the Divergence and General Stokes' Theorem, and the Hyperboloid.

2. Orient "the cube" in Euclidean Space using the divergence theorem

3. Extend the cube orientation argument from Euclidean Space to orient "the cube" in Minkowski Space. This will effectively extend the scope of applicability for various tools from vector calculus, including the shape operator.

4. Compute Curvature on the Hyperboloid using differential forms. Complete this computation using two isometries of the hyperboloid to double-check that we have the correct result.

5. Use our orientated cube to compute curvature on the Hyperboloid using vector calculus. This provides one example of a vector calculus computation in Minkowski Space for which the oriented cube can be used for.

Our last goal will not be accomplished in this thesis, but instead will motivate our discussion of future work.

2.2 Introduction

2.2.1 History and Context

Minkowski 3-Space ($\mathbb{M}^3$) is intuitively the points in $\mathbb{R}^3$ with a non-Euclidean distance function. It is common in relativity to use Euclidean space, and the topology and smooth structure induced by the Euclidean metric, while using a possibly quite different metric. This is the case for Minkowski Space. Thus, the open sets in Minkowski 3-Space are the same open sets from $\mathbb{R}^3$ that we know and love. The unique line element for Minkowski Space allows for a non-Euclidean model used by physicists to describe special relativity, in which time and space are measured using the same units. Furthermore, the metric for Minkowski Space enables the existence of "lightlike", zero magnitude but non-zero, vectors.

"The cube", illustrated in figure (1) for both Euclidean Space and Minkowski Space, is a cube at a given point that has arbitrary length Δ. Our orientation of the cube will involve orienting each face of the cube so that the overall orientation is consistent with a generalized Divergence Theorem in Minkowski Space. Because we are using the same differential structure as for Euclidean Space, this orientation for the boundary of our box will match the orientation for the box in Euclidean Space in the language of differential forms. That said, when mapping differential forms to normal vectors, we will end up with a different result than the orientation for the Euclidean case mapped to normal vectors.

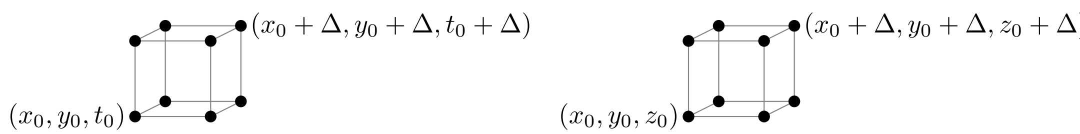

Figure 1: The Box in Euclidean space (right) and Minkowski space (left)

The ability to measure time and space with the same units, and the existence of zero magnitude vectors, are two properties of Minkowski Space that are fundamental to the theory of special relativity. In special relativity, notions of distance to a point are relative to the observer, but can be related to another observer's distance to that point through a rotation. Minkowski 4-space is

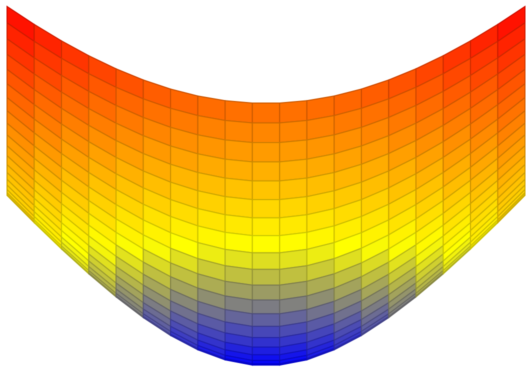

Figure 2: The Hyperboloid

the mathematical structure underlying special relativity, and Minkowski 3-space can be thought of as a “toy model” for this more general case. In considering Minkowski Space as a model for our universe, the upper Hyperboloid, seen in figure (2) represents the points that a traveler could reach in a constant unit of time (and when $\rho = 1$, one year) into the future [8, 25].

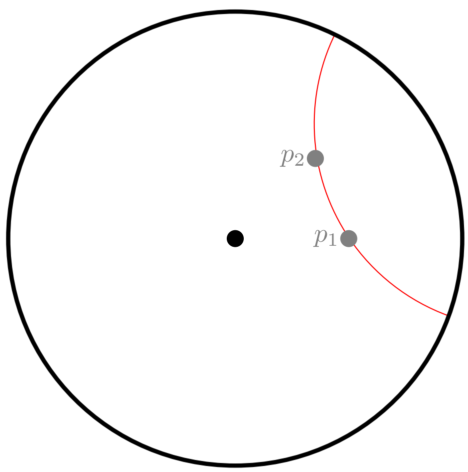

Figure 3: Poincaré Disk

The Hyperboloid in Minkowski 3-space is a model for hyperbolic geometry. The Poincaré Disk, seen in figure (3), is an equivalent model for hyperbolic geometry. The Poincaré Disk is the unit disk in $\mathbb{R}^2$, without boundary, with a different, non-Euclidean, metric.

Hyperbolic geometry is a neutral geometry, meaning it satisfies the School Mathematics Study

Group's postulates for Euclidean Geometry numbers one through fifteen [25]. Hyperbolic geometry's violation of the Euclidean parallel postulate makes it necessarily a non-Euclidean geometry: in hyperbolic geometry, for each point not on a given line, there are infinitely many lines parallel to that line which go through the given point. While hyperbolic geometry is often associated to the study of relativity, it has its beginnings earlier, becomming pertinent to the study of special relativity in physics in the early 1900's. Hyperbolic geometry has constant and negative Gaussian curvature at each point [25]. We will compute this explicitly for an arbitrary point on the Hyperboloid, and for an arbitrary point in the Poincaré Disk.

2.3 Background

2.3.1 Minkowski Space

Here we rigorously introduce three-dimensional Minkowski Space ($\mathbb{M}^3$). We will use the following definition adapted from Lee [18]:

Definition 2.1 (Pseudo-Riemannian Manifold)**.** A Pseudo-Riemannian Manifold is a pair (M, g) where M is a smooth manifold and g is a pseudo-Riemannian metric on M.

Let M be $\mathbb{R}^3$ with the metric topology and smooth structure consistent with the Euclidean distance function. Because our underlying space is $\mathbb{R}^3$, this gives us an inner product for vectors in Minkowski space. Here, we adopt the convention of relativists and relax the definition of an inner product to require, in the place of positive definiteness, that our map is non-degenerate. Three-dimensional Minkowski space is a pseudo-Riemannian manifold (M, g), with g an inner product defined as follows [8, 18]:

$$g\left(a_1\,\hat{x} + a_2\,\hat{y} + a_3\,\hat{t}, b_1\,\hat{x} + b_2\,\hat{y} + b_3\,\hat{t}\right)$$
$$= a_1 b_1 + a_2 b_2 - a_3 b_3 \;\; \forall p \in \mathbb{R}^3.$$

We will equivalently denote g via:

$$g\left(a_1\,\hat{x} + a_2\,\hat{y} + a_3\,\hat{t}, b_1\,\hat{x} + b_2\,\hat{y} + b_3\,\hat{t}\right)$$
$$= (a_1\hat{x} + a_2\hat{y} + a_3\hat{t}) \cdot (b_1\hat{x} + b_2\hat{y} + b_3\hat{t})$$

We are using the fact that $T_p(\mathbb{R}^3)$ has a known orthonormal basis $\{\hat{x}, \hat{y}, \hat{z}\}$ and we are relabeling $\hat{z}$ as $\hat{t}$ due to the surprising fact that

$$g\left(\hat{t}, \hat{t}\right) = -1 \;\; \forall p \in \mathbb{R}^3.$$

Because we are using the smooth structure induced from the Euclidean norm, the derivatives for functions will be the same as they are in the Euclidean case. Thus, we can differentiate functions in Minkowski space by treating them as functions in Euclidean space. On the other hand, tools that are defined relative to the cross product or dot product in Euclidean space must be generalized according to the Minkowski metric, because they are computed according to the inner product on the tangent space at each point of our manifold.

A Riemannian metric is not the same thing as the "metric" used in topology for metric spaces. A Riemannian metric is a smooth, symmetric, covariant 2-tensor field on M, that is, a map from $T_pM \times T_pM \to \mathbb{R}$ that is positive definite at each point. Similarly, a pseudo-Riemannian metric is a smooth, symmetric, 2-tensor field on M that is non-degenerate at each point.

In Minkowski space, we are working with an indefinite distance function, so-named because of the possibility that squared distance is negative. Because of the close association to relativity, we consider negative distances to be time-like and positive distances to be space-like. Distances of zero are considered light-like [8].

Now, let

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dt\,\hat{t}$$

be the vector differential. In the algebraic context, our vector differential is the following 1-1 tensor,

$$d\vec{r} = dx \otimes \hat{x} + dy \otimes \hat{y} + dt \otimes \hat{t}.$$

In this context, we can consider our vector differential to be a map:

$$dx \otimes \hat{x} + dy \otimes \hat{y} + dt \otimes \hat{t} : T_p\mathbb{M}^3 \times Alt^1(T_p\mathbb{M}^3) \to \mathbb{R}$$

$$(b_1\hat{x} + b_2\hat{y} + b_3\hat{t}, a_1dx + a_2dy + a_3dt) \mapsto (a_1b_1 + a_2b_2 + a_3b_3).$$

We can also, and from here on will, opt to express the vector differential through the language of differential form-valued vectors as:

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dt\,\hat{t}$$

In considering our vector differential as a differential form-valued vector, the inner product of $d\vec{r}$ with other differential form-valued vectors in the tangent space is well defined via:

$$\begin{aligned}
&g(\sigma_x\,\hat{x} + \sigma_y\,\hat{y} + \sigma_t\,\hat{t}, \alpha_x\,\hat{x} + \alpha_y\,\hat{y} + \alpha_t\,\hat{t}) \\
&= \sigma_x\alpha_x\,g(\hat{x},\hat{x}) + \sigma_y\alpha_y\,g(\hat{y},\hat{y}) + \sigma_t\alpha_t\,g(\hat{t},\hat{t}) \\
&= \sigma_x\alpha_x + \sigma_y\alpha_y - \sigma_t\alpha_t,
\end{aligned}$$

where the σ_i, α_i coefficients are differential forms. Now we have a dot product well defined for our vector differential, and we are prepared to compute the line element for $\mathbb{M}^3$ as:

$$\begin{aligned}
ds^2 &= d\vec{r} \cdot d\vec{r} \\
&= dx^2 + dy^2 - dt^2.
\end{aligned}$$

We can classify geometries based on the signature of their metric (in this case, a pseudo-Riemannian metric). Due to the single negative sign in our line element, Minkowski space is classified as having signature -1. Signature profoundly impacts the Hodge Dual operator, and for our case, we have that:

$$**(1) = -1.$$

The Hyperboloid lives in Minkowski space, and is defined as the set

$$H = \{(x, y, t) \mid x^2 + y^2 - t^2 = -\rho^2\}$$

where $\rho \neq 0 \in \mathbb{R}$ is a constant. Alternatively, we could define the Hyperboloid to be the set of roots for the polynomial $x^2 + y^2 - z^2 + \rho^2$ [8]. We characterize the Hyperboloid as an analog to the sphere in Minkowski space. Remarkably, despite appearances, every point on the Hyperboloid is the same distance away from the origin in Minkowski Space.

Despite the fact that distance in Minkowski Space is indefinite, when we restrict ourselves to the "upper" half of the Hyperboloid ($t > 0$), we obtain a positive definite notion of distance between points. Given two points, p_1, p_2 on the Hyperboloid, the line between two points on the Hyperboloid is the intersection of the plane between each point and the origin in Euclidean 3-space, and the Hyperboloid. The distance between the two points is the arc length of the segment of that line between those two points using the restriction of the Minkowski line element.

This conversation about line elements is relevant to our eventual computation of curvature on both the Hyperboloid and the Poincaré Disk. Because both the Hyperboloid and Poincaré Disk are two-dimensional surfaces, we may parametrize them with two variables. In so doing, we rewrite our line element for Minkowski Space in terms of our new variables, and in both cases we will be left with a positive definite notion of distance on our surface. What's more, once we are working with a positive definite distance function, our line element gives us an orthonormal basis for our surface that we can use to compute curvature. Before completing those computations, however, we will begin with an orientation of the cube in Minkowski Space.

2.3.2 Vector Calculus in Minkowski Space

To explain what we mean by "vector calculus tools", we must first distinguish between intrinsic and extrinsic properties of a manifold. An intrinsic geometric property is a property of a manifold that is independent of an embedding for that manifold. On the other hand, an extrinsic property is relative to an embedding. Surprisingly, there are intrinsic properties, including Gaussian curvature, that can be computed using extrinsic tools [22]. The cube is embedded in Minkowski Space, and the underlying manifold for Minkowski Space is $\mathbb{R}^3$.

Because of this, we can generalize several important extrinsic tools from Euclidean vector calculus to the Minkowski case. The main tools we aim to generalize are the dot product, the cross product, and normal vectors for a surface.

The dot product is immediately generalized using our inner product. The cross product and normal vectors are a bit trickier: first, we will first compute a generalized cross product, then, we will orient each face of the cube so that the orientations are consistent with a generalized Divergence Theorem in Minkowski Space. Because of the close relationship between orientation and normal vectors, this will go hand-in-hand with determining the normal vectors for each face on the cube. This determines the normal vectors for an arbitrary surface by essentially telling us which Euclidean normal vectors correspond to "flipped" normal vectors in Minkowski Space.

2.3.3 The General Form of Stokes' Theorem

The Divergence Theorem is a famous theorem from vector calculus. In short, this theorem states that if you have a compact subset $X \subseteq \mathbb{R}^3$ and a given vector field $\vec{F}$, then the surface integral of the flux of $\vec{F}$ out of X and the volume integral of the divergence of $\vec{F}$ must be equal. Similar theorems from calculus, including the Fundamental Theorem of Calculus, relate boundary conditions to a higher dimensional subset between the boundary. It turns out that both the Fundamental Theorem of Calculus and the Divergence theorem are actually specific cases of a more general theorem, called the General form of Stokes' Theorem, which is articulated in the language of differential geometry. We present the General Form of Stokes' Theorem below. Notably, the Euclidean divergence theorem can be derived from the general form of Stokes' Theorem. Furthermore, the upcoming computation to orient the box in Minkowski Space will rely on an application of the General form of Stokes' Theorem in a context very similar to the Euclidean divergence theorem, but instead for Minkowski Space.

Theorem 2.1 (General form of Stokes' Theorem)**.** *Let X be a compact, oriented k-dimensional manifold with boundary. Then ∂X is a $(k-1)$-dimensional manifold with boundary orientation. If ω is a smooth $k-1$ form on X, then [18]*

$$\int_{\partial X} \omega = \int_X d\omega.$$

Notably, this will not be sufficient for our box argument because our box is not a smooth manifold; the box has corners. That said, the General form of Stokes' Theorem can still be carefully applied to our argument. As with most integral theorems, it is sufficient for the boundary to be piecewise smooth. For the scope of this thesis, we will brush this technicality under the rug.

2.4 Orienting the Box in Euclidean Space

This section presents an example of orienting the cube in Euclidean space, when we have a well-established notion of the Hodge Dual operator, cross product, and dot product. For specifics on how these act in Euclidean space, see [9]. We will not introduce these tools rigorously in this section, but provide the computational structure and argument that motivates our next and more thorough argument.

To begin, let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with components given by $\vec{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$. Then there is a natural correspondence (see the tangent-cotangent isomorphism in [18]) between $\vec{F}$ and a 1-form $F = F_x dx + F_y dy + F_z dz$. We define:

$$divF = *d*F.$$

Then,

$$
\begin{aligned}
*d*F &= *(d(*F)) \text{ by associativity of } * \\
&= *(d(*(F_x dx + F_y dy + F_z dz))) \\
&= *(d(F_x * dx + F_y * dy + F_z * dz)) \text{ by linearity of } * \\
&= *(d(F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy)) \\
&= *(d(F_x dy \wedge dz) + d(F_y dz \wedge dx) + d(F_z dx \wedge dy)) \\
&= * \left(\frac{\partial F_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_y}{\partial y} dy \wedge dz \wedge dx + \frac{\partial F_z}{\partial z} dz \wedge dx \wedge dy \right) \\
&= * \left(\frac{\partial F_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_y}{\partial y} dx \wedge dy \wedge dz + \frac{\partial F_z}{\partial z} dx \wedge dy \wedge dz \right) \\
&= \frac{\partial F_x}{\partial x} * (dx \wedge dy \wedge dz) + \frac{\partial F_y}{\partial y} * (dx \wedge dy \wedge dz) + \frac{\partial F_z}{\partial z} * (dx \wedge dy \wedge dz) \\
&= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
&= \vec{\nabla} \cdot \vec{F}
\end{aligned}
$$

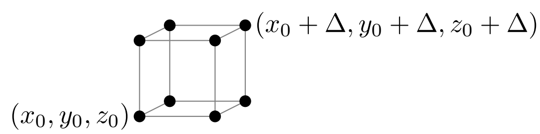

Figure 4: The Euclidean Box

Now, let X be the box pictured in figure (4). Let $dV = dx \wedge dy \wedge dz$. In Euclidean space, because of the signature, the Hodge Dual operator (see [9]) acts such that:

$$(*d*F)dV = d*F. \tag{1}$$

According to the generalized Stokes' Theorem we have

$$\int_X d{*}F = \int_{\partial X} {*}F. \tag{2}$$

In Euclidean space, the vector differential is

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}.$$

Then, using the fact that $F = \vec{F} \cdot d\vec{r}$ and $d\vec{S} = {*}d\vec{r}$, equation (2) is equivalent to:

$$\int_X \text{div}\vec{F}\, dV = \int_{\partial X} \vec{F} \cdot d\vec{S}.$$

Following the convention in ([11]), we define $d\vec{S}$ for each face individually. In the language of vector calculus, we are restricting $d\vec{S}$ to each face. Starting with the top face, we have $dz = 0$ and thus,

$$\begin{aligned} d\vec{r}_1 &= dx\hat{x} \\ d\vec{r}_2 &= dy\hat{y} \\ d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= (dx \wedge dy)\, \hat{z}. \end{aligned}$$

For the right face, $dx = 0$ and thus,

$$\begin{aligned} d\vec{r}_1 &= dy\hat{y} \\ d\vec{r}_2 &= dz\hat{z} \\ d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= (dy \wedge dz)\, \hat{x} \end{aligned}$$

Finally, on the back face $dy = 0$,

$$
\begin{aligned}
d\vec{r}_1 &= dz\hat{z} \\
d\vec{r}_2 &= dx\hat{x} \\
d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\
&= (dz \wedge dx)\,\hat{y}.
\end{aligned}
$$

We define $d\vec{S}$ on the bottom to be -1 times $d\vec{S}$ on the top, so

$$d\vec{S} = -(dx \wedge dy)\,\hat{z}.$$

Similarly, on the left face

$$d\vec{S} = -(dy \wedge dz)\,\hat{x}.$$

Finally, on the front face

$$d\vec{S} = -(dz \wedge dx)\,\hat{y}.$$

A correct computation of the flux integral relies on, and in fact verifies, a correct orientation of the cube. We proceed, breaking the flux integral up into each face:

$$
\begin{aligned}
\int_{\partial X} \vec{F} \cdot d\vec{S} = &\int_{\text{top face}} \vec{F} \cdot d\vec{S} \\
&+ \int_{\text{back face}} \vec{F} \cdot d\vec{S} \\
&+ \int_{\text{right face}} \vec{F} \cdot d\vec{S} \\
&+ \int_{\text{bottom face}} \vec{F} \cdot d\vec{S} \\
&+ \int_{\text{left face}} \vec{F} \cdot d\vec{S} \\
&+ \int_{\text{front face}} \vec{F} \cdot d\vec{S}.
\end{aligned}
$$

Starting with the top face,

$$
\begin{aligned}
\int_{\text{top face}} \vec{F} \cdot d\vec{S} &= \int_{y_0}^{y_0+\Delta} \int_{x_0}^{x_0+\Delta} \vec{F} \cdot d\vec{S} \\
&= \int_{y_0}^{y_0+\Delta} \int_{x_0}^{x_0+\Delta} F_z(x, y, z_0 + \Delta)\, dx \wedge dy.
\end{aligned}
$$

Similarly, the flux integral for the bottom face is given by

$$
\begin{aligned}
\int_{\text{bottom face}} \vec{F} \cdot d\vec{S} &= \int_{y_0}^{y_0+\Delta} \int_{x_0}^{x_0+\Delta} \vec{F} \cdot d\vec{S} \\
&= \int_{y_0}^{y_0+\Delta} \int_{x_0}^{x_0+\Delta} -F_z(x, y, z_0)\, dx \wedge dy.
\end{aligned}
$$

For the right face,

$$
\begin{aligned}
\int_{\text{right face}} \vec{F} \cdot d\vec{S} &= \int_{z_0}^{z_0+\Delta} \int_{y_0}^{y_0+\Delta} \vec{F} \cdot d\vec{S} \\
&= \int_{z_0}^{z_0+\Delta} \int_{y_0}^{y_0+\Delta} F_x(x_0 + \Delta, y, z)\, dy \wedge dz.
\end{aligned}
$$

On the left face,

$$\begin{aligned}\int_{\text{right face}} \vec{F}\cdot d\vec{S} &= \int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta} \vec{F}\cdot d\vec{S}\\ &= \int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta} -F_x(x_0,y,z)\,dy\wedge dz.\end{aligned}$$

For the back face,

$$\begin{aligned}\int_{\text{back face}} \vec{F}\cdot d\vec{S} &= \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} \vec{F}\cdot d\vec{S}\\ &= \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} F_y(x,y_0+\Delta,z)\,dz\wedge dx.\end{aligned}$$

For the front face,

$$\begin{aligned}\int_{\text{front face}} \vec{F}\cdot d\vec{S} &= \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} \vec{F}\cdot d\vec{S}\\ &= \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} F_y(x,y_0,z)\,dz\wedge dx.\end{aligned}$$

Combining the integrals with matching bounds, we have that:

$$\begin{aligned}\int_{\partial X} \vec{F}\cdot d\vec{S} = &\int_{y_0}^{y_0+\Delta}\int_{x_0}^{x_0+\Delta} \left(F_z(x,y,z_0+\Delta)-F_z(x,y,z_0)\right)\,dx\wedge dy\\ &+\int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} \left(F_y(x,y_0+\Delta,z)-F_y(x,y_0,z)\right)\,dz\wedge dx\\ &+\int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta} \left(F_x(x_0+\Delta,y,z)-F_x(x_0,y,z)\right)\,dy\wedge dz.\end{aligned} \tag{3}$$

Integration of differential forms is defined in terms of the usual multiple integral integration since we are in the Euclidean case. Consequently, we can identify $dx\wedge dy$ with $dxdy$ and so on. Then, equation (3) is equivalent to:

$$\begin{aligned}\int_{\partial X} \vec{F}\cdot d\vec{S} = &\int_{y_0}^{y_0+\Delta}\int_{x_0}^{x_0+\Delta} \left(F_z(x,y,z_0+\Delta) - F_z(x,y,z_0)\right)\, dxdy \\ &+ \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} \left(F_y(x,y_0+\Delta,z) - F_y(x,y_0,z)\right)\, dzdx \\ &+ \int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta} \left(F_x(x_0+\Delta,y,z) - F_x(x_0,y,z)\right)\, dydz.\end{aligned} \tag{4}$$

By the Fundamental Theorem of Calculus, we can rewrite equation (4) as

$$\begin{aligned}\int_{\partial X} \vec{F}\cdot d\vec{S} = &\int_{y_0}^{y_0+\Delta}\int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta} \frac{\partial F_z}{\partial x} dxdydz \\ &+ \int_{x_0}^{x_0+\Delta}\int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta} \frac{\partial F_y}{\partial y} dydzdx \\ &+ \int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta}\int_{x_0}^{x_0+\Delta} \frac{\partial F_x}{\partial x} dxdydz. \\ = &\int_{z_0}^{z_0+\Delta}\int_{y_0}^{y_0+\Delta}\int_{x_0}^{x_0+\Delta} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dxdydz.\end{aligned}$$

Up to the identification of dV with $dxdydz$, our flux integrand is equal to the divergence integrand. Thus, we have confirmed our orientation of the cube in Euclidean space. The normal vectors are the (unit-length) outward pointing vectors perpendicular to each face, and are given by the cross products of tangent vectors as computed in this section. In the next section, we will ellaborate on our computations as well as generalize the tools needed to complete this argument in Minkowski Space.

2.5 Orienting the Box in Minkowski Space

We can carefully use the coordinate patches for the cube as a submanifold with boundary of $\mathbb{M}^3$ to correctly orient each face. That said, we can also save ourselves the headache of working in that setting by using the Divergence theorem to carefully pick orientations, through trial and error, or each face on our cube so that the flux and divergence integrals of an arbitrary vector field defined over the cube are consistent. This means that our orientation will hold up to a flipping of normal vectors; note that we would not get a stronger result were we to use the other approach because we would still have to make a sign choice. Thus, our orientation of the cube is based on the Generalized Stokes' Theorem.

2.5.1 The correspondance between vectors and one forms in $\mathbb{M}^3$

Let $\vec{F} : \mathbb{M}^3 \to \mathbb{R}^3$ where

$$\vec{F} = F_x\,\hat{x} + F_y\,\hat{y} + F_t\,\hat{t}$$
$$F_x : \mathbb{M}^3 \to \mathbb{R}$$
$$F_y : \mathbb{M}^3 \to \mathbb{R}$$
$$F_t : \mathbb{M}^3 \to \mathbb{R}$$

be a smooth vector field in Minkowski Space. There is a module isomorphism, φ, between 1-forms and vector fields (called the tangent-cotangent ismorphism [18]) satisfying at each point $p \in \mathbb{M}^3$ [9]:

$$\begin{aligned}
\varphi : T_pM &\to T_p^*M \\
\varphi(\vec{F}(p)) &= \varphi(F_x(p)\,\hat{x} + F_y(p)\,\hat{y} + F_t(p)\,\hat{t}) \\
&= d\vec{r} \cdot \vec{F} \\
&= (dx\,\hat{x} + dy\,\hat{y} + dt\,\hat{t}) \cdot (\vec{F}(p)) \\
&= (dx\,\hat{x} + dy\,\hat{y} + dt\,\hat{t}) \cdot (F_x(p)\,\hat{x} + F_y(p)\,\hat{y} + F_t(p)\,\hat{t}) \\
&= F_x(p)\,dx + F_y(p)\,dy - F_t(p)\,dt.
\end{aligned}$$

2.5.2 Hodge Dual in $\mathbb{M}^3$

In this section, $*$ refers to the Hodge dual operator. We first review how a pseudo-Riemannian metric extends to an inner product for arbitrary k-forms (with $1 \leq k \leq n$). Then we will define the Hodge Dual operator, and finally we will compute the Hodge Dual operator for Minkowski Space.

Let g be a pseudo-Riemannian metric on an n-manifold M. We would like to extend g to be an inner product on $Alt^k(T_pM)$ for $1 \leq k \leq n$. We will follow the construction outlined in [9]. First, note that we can use the tangent-cotangent isomorphism φ to extend g to an inner product on T_p^*M (equivalently $Alt^1(T_pM)$). This inner product is defined by

$$g(\alpha, \beta) = g\left(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\right),$$

Where φ is the isomorphism introduced in the previous section, and $\alpha, \beta \in T_p^*M$. Now, let α, β be k-forms with $k \leq n$ on T_pM for $p \in M$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for T_pM. Let $\{\sigma_1, \cdots, \sigma_n\}$ be the corresponding dual basis for T_p^*M. Then the set

$$\{\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$$

is a basis for $Alt^k(T_pM)$. If we require that this basis is orthonormal relative to g and that g is an inner product, this completely defines g. In practice, using our basis we can express α and β as:

$$\begin{aligned}\alpha &= \sum_{|1 \leq i_1 < \ldots < i_k \leq n} (f_{\alpha i_1} \sigma_{i_1}) \wedge \cdots \wedge \sigma_{i_k} \\ &= \alpha_1 \wedge \ldots \wedge \alpha_k \\ \beta &= \sum_{|1 \leq i_1 < \ldots < i_k \leq n} (f_{\beta i_1} \sigma_{i_1}) \wedge \cdots \wedge \sigma_{i_k} \\ &= \beta_1 \wedge \ldots \wedge \beta_k\end{aligned}$$

where $f_{\alpha j}$ and $f_{\beta j}$ are 0-forms for each j. Then $g(\alpha, \beta)$ satisfies:

$$g(\alpha,\beta) = \left| \begin{bmatrix} g(\alpha_1,\beta_1) & \cdots & g(\alpha_k,\beta_1) \\ \vdots & \ddots & \vdots \\ g(\alpha_1,\beta_k) & \cdots & g(\alpha_k,\beta_k) \end{bmatrix} \right| .$$

Now, let (h, U) be a smooth chart for M so that $h : U \subset M \to \mathbb{R}^n$, and let α, β be smooth $k-$forms defined on U. For an n–manifold and $0 \leq k \leq n$, The Hodge dual operator is the unique map which takes k-forms on U to $(n-k)$-forms on U and satisfies:

$$\alpha \wedge *\beta = g(\alpha, \beta)\omega$$

where ω is the pullback of the volume form in $\mathbb{R}^n$ under h. This property completely defines our operator. For convenience, although the Hodge Dual operator is defined relative to each k, we colloquially refer to any of them as "the Hodge Dual operator". In relativity, where it is common to use Euclidean Space as the manifold, we can loosely forget the pullback relative to our coordinate charts and treat ω literally as the volume form.

To complete our calculation for divergence and the cross product in Minkowski space, we first need to know where $*$ maps forms in Minkowski Space. Our generalized definitions for the cross product, as well as divergence, use the Hodge Dual operator. From above, for $p \in \mathbb{M}^3$, we know that $*$ must satisfy:

$$* : Alt^k(T_p\mathbb{M}^3) \to Alt^{3-k}(T_p\mathbb{M}^3), k \in \{0, 1, 2, 3\}$$
$$\alpha, \beta \in \Lambda^k(T_p\mathbb{M}^3) \mapsto *\alpha \text{ s.t. } \alpha \wedge *\beta = g(\alpha, \beta)\omega \text{ where } \omega = dx \wedge dy \wedge dt$$

where g is the inner product on 1-forms induced by the metric tensor as described above. Note that $\{dx, dy, dt\}$ is an orthonormal basis for $T_p\mathbb{M}^3$, $p \in \mathbb{M}^3$. Now, relabel these elements arbitrarily as $\{\sigma_1, \sigma_2, \sigma_3\}$. Because $*\sigma_1$ is a 2−form, we know that

$$*\sigma_1 = a_1\, \sigma_1 \wedge \sigma_2 + a_2\, \sigma_2 \wedge \sigma_3 + a_3\, \sigma_1 \wedge \sigma_3,$$

where a_1, a_2, a_3 are 0-forms. Now we use the defining property of the Hodge Dual operator to conclude that

$$\begin{aligned} 0 &= g(\sigma_2, \sigma_1)\omega \\ &= \sigma_2 \wedge *\sigma_1 \\ &= \sigma_2 \wedge (a_1\, \sigma_1 \wedge \sigma_2 + a_2\, \sigma_2 \wedge \sigma_3 + a_3\, \sigma_1 \wedge \sigma_3) \\ &= a_3\, \sigma_2 \wedge \sigma_1 \wedge \sigma_3, \end{aligned}$$

and therefore $a_3 = 0$. Repeating our logic from above,

$$\begin{aligned} 0 &= g(\sigma_2, \sigma_1)\omega \\ &= \sigma_3 \wedge *\sigma_1 \\ &= \sigma_3 \wedge (a_1\, \sigma_1 \wedge \sigma_2 + a_2\, \sigma_2 \wedge \sigma_3 + a_3\, \sigma_1 \wedge \sigma_3) \\ &= a_1\, \sigma_3 \wedge \sigma_1 \wedge \sigma_2, \end{aligned}$$

and thus $a_1 = 0$. This justifies the claim that

$$\sigma_1 = a_2\, \sigma_2 \wedge \sigma_3.$$

We have used properties of the wedge product implicitly in our justification. The wedge product is defined rigorously in [18]. Since our relabelling was arbitrary, we have shown that $*\sigma_i$ does not include any non-zero σ_i terms for each $1 \leq i \leq 3$. Thus, proceeding in a simplified calculation to determine $*dx$,

$$\begin{aligned} dx \wedge *dx &= g(dx, dx)\omega \\ &= dx \wedge dy \wedge dt \end{aligned}$$

so that

$$*dx = dy \wedge dt. \tag{5}$$

Similarly, we compute $*dy$:

$$\begin{aligned} dy \wedge *dy &= g(dy, dy)\omega \\ &= dx \wedge dy \wedge dt \\ &= -dy \wedge dx \wedge dt \end{aligned}$$

which implies that

$$*dy = -dx \wedge dt. \tag{6}$$

Finally,

$$\begin{aligned} dt \wedge *dt &= g(dt, dt)\omega \\ &= -dx \wedge dy \wedge dt \\ &= -dt \wedge dx \wedge dy \end{aligned}$$

which proves that

$$*dt = -dx \wedge dy. \tag{7}$$

We now use those computations to see how $*$ acts on two forms. We apply $*$ to both sides of equation (5), and find that

$$* * dx = *(dy \wedge dt).$$

Now, because of the signature of our metric, for any k form α we must have $* * \alpha = -\alpha$ [9]. Thus, we find that the above equation simplifies to:

$$-dx = *(dy \wedge dt).$$

We now compute $*(dx \wedge dt)$. We begin by applying the Hodge Dual operator to both sides of equation (6):

$$* * dy = - * (dx \wedge dt).$$

We must have $* * dy = -dy$, and thus

$$-dy = - * (dx \wedge dt).$$

For convenience, we rewrite this as

$$dy = *(dx \wedge dt).$$

Finally, we procede in a computation of $*(dx \wedge dy)$. We apply $*$ to both sides of equation (7) to find that

$$* * dt = - * (dx \wedge dy).$$

As before, we have $* * dt = -dt$ and consequently

$$-dt = - * (dx \wedge dy)$$

which is equivalent to

$$dt = *(dx \wedge dy).$$

Finally, by convention we define $*1 = \omega$. On the other hand, $*\omega = -1$ for the same reason, given in [9], that $* * \alpha = -\alpha$. Ultimately, the minus sign comes from the signature of our metric.

2.5.3 Divergence in $\mathbb{M}^3$

Let $\vec{F}$ be a vector in $\mathbb{M}^3$. That is,

$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_t\hat{t}.$$

Because we start our computation with $\vec{F}$, we will consider $\vec{F}$ to be our fundamental object, however it is worth noting we would arrive at a different result if we began with a fundamental 1-form, say

$$G = G_x dx + G_y dy + G_t dt,$$

due to the differences in sign on coefficients through our isomorphism, φ. Recall from the previous section that

$$\varphi(\vec{F}) = F_x dx + F_y dy - F_t dt.$$

We will again use the definition that

$$Div(\vec{F}) = *d*F.$$

Then,

$$\begin{aligned}
*d*F &= *d*(F_x dx + F_y dy - F_t dt) \\
&= *d(F_x *dx + F_y *dy - F_t *dt) \\
&= *d(F_x(dy \wedge dt) + F_y(-dx \wedge dt) - F_t(-dx \wedge dy)) \\
&= *(d(F_x) \wedge dy \wedge dt - d(F_y) \wedge dx \wedge dt + d(F_t) \wedge dx \wedge dy) \\
&= * \left(\frac{\partial F_x}{\partial x} dx \wedge dy \wedge dt - \frac{\partial F_y}{\partial y} dy \wedge dx \wedge dt + \frac{\partial F_t}{\partial t} dt \wedge dx \wedge dy \right) \\
&= * \left(\frac{\partial F_x}{\partial x} dx \wedge dy \wedge dt + \frac{\partial F_y}{\partial y} dx \wedge dy \wedge dt + \frac{\partial F_t}{\partial t} dx \wedge dy \wedge dt \right) \\
&= - \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_t}{\partial t} \right)
\end{aligned}$$

where the minus sign is due to the fact, discussed in Section 2.6.2, that $*(dx \wedge dy \wedge dt) = -1$.

2.5.4 Cross Product in $\mathbb{M}^3$

We define the generalized cross product via:

$$\vec{a} \times \vec{b} = \varphi^{-1} \left(* \left(\varphi(\vec{a}) \wedge \varphi(\vec{b}) \right) \right).$$

Then, if

$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_t\hat{t}$$
$$\vec{b} = b_x\hat{x} + b_y\hat{y} + b_t\hat{t},$$

we have

$$\begin{aligned}
\vec{a} \times \vec{b} &= \varphi^{-1}\left(*\left(\varphi(a_x\hat{x} + a_y\hat{y} + a_t\hat{t}) \wedge \varphi(b_x\hat{x} + b_y\hat{y} + b_t\hat{t})\right)\right) \\
&= \varphi^{-1}\left(*\left((a_x dx + a_y dy - a_t dt) \wedge (b_x dx + b_y dy - b_t dt)\right)\right) \\
&= \varphi^{-1}(*((a_x b_y - a_y b_x)dx \wedge dy + (b_y a_t - a_y b_t)dy \wedge dt + (a_t b_x - a_x b_t)dx \wedge dt)) \\
&= \varphi^{-1}((a_x b_y - a_y b_x)dt - (b_y a_t - a_y b_t)dx + (a_t b_x - a_x b_t)dy) \\
&= (a_y b_t - a_t b_y)\hat{x} + (a_t b_x - a_x b_t)\hat{y} + (a_y b_x - a_x b_y)\hat{t}.
\end{aligned}$$

2.5.5 Orienting the Cube

Consider box illustrated in figure (5), which was introduced earlier under different labels for our Euclidean argument, where $x_0, y_0, t_0, \Delta \in \mathbb{R}$. Let this box be the submanifold $X \subset \mathbb{M}^3$.

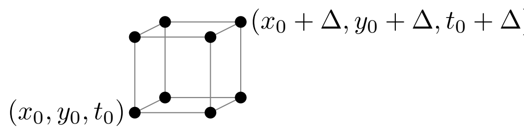

Figure 5: Minkowski Box

Let $\vec{F}$ be a smooth vector field on $\mathbb{M}^3$,

$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_t\hat{t}.$$

We can extend the Hodge Dual operator $*$ to the vector differential by treating the vector differential as a vector-valued one form. In that case, we then can define a general surface element, $d\vec{S}$ as

$$
\begin{aligned}
d\vec{S} &= *d\vec{r} \\
&= *(dx\,\hat{x} + dy\,\hat{y} + dt\,\hat{t}) \\
&= (dy \wedge dt)\hat{x} + (-dx \wedge dt)\hat{y} + (-dx \wedge dy)\hat{t}.
\end{aligned}
$$

Now we consider a 2-form which is the Hodge dual of the corresponding 1-form F :

$$
\begin{aligned}
*F &= *(d\vec{r} \cdot \vec{F}) \\
&= *(F_x dx + F_y dy - F_t dt) \\
&= F_x(dy \wedge dt) + F_y(-dx \wedge dt) - F_t(-dx \wedge dy) \\
&= \vec{F} \cdot *d\vec{r} \\
&= \vec{F} \cdot d\vec{S}.
\end{aligned}
$$

Because $\vec{F}$ is a smooth vector field over X, F is a smooth 1-form over X. Furthermore, because F is a smooth 1-form over the cube, $*F$ is a smooth 2-form over the cube. Thus, we can use Stokes' Theorem to assert that

$$\int_X d(*F) = \int_{\partial X} *F \tag{8}$$

Recall from before that

$$\mathrm{div}(\vec{F}) = *(d*F).$$

If ω is the volume form, then

$$\mathrm{div}(\vec{F})\,\omega = -d(*F). \tag{9}$$

In that case, using equation (8) and equation (9), we have

$$\int_X -\text{div}(\vec{F})dV = \int_{\partial X} \vec{F} \cdot d\vec{S}. \tag{10}$$

We will use this theorem to orient the cube, beginning with computing the integral $\int_{\partial X} \vec{F} \cdot d\vec{S}$ for an arbitrary vector field $\vec{F}$. We will break up $\int_{\partial X} \vec{F} \cdot d\vec{S}$ into integrals over each face of X, so that:

$$\begin{aligned}\int_{\partial X} \vec{F} \cdot d\vec{S} = & \int_{\text{front face}} \vec{F} \cdot d\vec{S} \\ & + \int_{\text{back face}} \vec{F} \cdot d\vec{S} \\ & + \int_{\text{left face}} \vec{F} \cdot d\vec{S} \\ & + \int_{\text{right face}} \vec{F} \cdot d\vec{S} \\ & + \int_{\text{top face}} \vec{F} \cdot d\vec{S} \\ & + \int_{\text{bottom face}} \vec{F} \cdot d\vec{S}.\end{aligned}$$

Starting with the top face, we have $dt = 0$. Using the vector differential for Minkowski Space,

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dt\hat{t},$$

and restricting the vector differential to the top face, we have

$$d\vec{r} = dx\hat{x} + dy\hat{y}.$$

Since $t = \text{const}$ on our face, a family of curves may be parametrized by either x or y. Because x, y vary independently on our surface, the family that is parametrized by x and a family that is parametrized by y will be distinct. Thus,

$$d\vec{r}_1 = dy\hat{y};\ dx = 0$$
$$d\vec{r}_2 = dx\hat{x};\ dy = 0.$$

The definition of a general surface element from ([11]) agrees with the more general definition provided above over our surface. We will use the definition for the surface element that is restricted to a surface for our computation, preceding so that:

$$\begin{aligned} d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= dy\hat{y} \times dx\hat{x} \\ &= dxdy\hat{t}. \end{aligned}$$

Having computed the surface element for the top face, we express our first integral as:

$$\begin{aligned} \int_{topface} \omega &= \int_{topface} -F_t dxdy \\ &= \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} -F_t(x, y, t_0 + \Delta) dxdy \end{aligned}$$

We will define $d\vec{S}$ for the bottom face to be the same as for the top face, times minus one. Thus,

$$\begin{aligned} \int_{bottomface} \omega &= \int_{bottomface} F_t dxdy \\ &= \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} F_t(x, y, t_0) dxdy. \end{aligned}$$

Now we focus on the right face. Restricting the vector differential on this face, where $x = \text{const}$, and again using the argument for distinct families of curves based on independent parametrization, we have:

$$d\vec{r}_1 = dt\hat{t};\ dx, dy = 0,$$

$$d\vec{r}_2 = dy\hat{y};\ dx, dt = 0,$$

which gives us a surface element such that

$$\begin{aligned} d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= dt\hat{t} \times dy\hat{y} \\ &= -dtdy\hat{x}. \end{aligned}$$

Thus, the integral for flux over the right face is

$$\begin{aligned} \int_{rightface} \omega &= \int_{rightface} -F_x dtdy \\ &= \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} -F_x(x_0+\Delta, y, t)dtdy. \end{aligned}$$

Using our previous reasoning,

$$\begin{aligned} \int_{leftface} \omega &= \int_{leftface} -F_x(-dtdy) \\ &= \int_{leftface} F_x dtdy \\ &= \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} F_x(x_0, y, t)dtdy. \end{aligned}$$

Finally we consider the back face! Then

$$d\vec{r}_1 = dx\hat{x};\ dy, dt = 0,$$

$$d\vec{r}_2 = dt\hat{t};\ dy, dx = 0,$$

which implies that we have the following general surface element:

$$\begin{aligned} d\vec{S} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= dx\hat{x} \times dt\hat{t} \\ &= -dxdt\hat{y}. \end{aligned}$$

Using the above surface element on the back face, we have

$$\begin{aligned} \int_{\text{back face}} \vec{F} \cdot d\vec{S} &= \int_{\text{back face}} -F_y dxdt \\ &= \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} -F_y(x, y_0+\Delta, t) dxdt. \end{aligned}$$

Alternatively, for the top face,

$$\begin{aligned} \int_{\text{top face}} \omega &= \int_{\text{top face}} F_y dxdt \\ &= \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} F_y(x, y_0, t) dxdt. \end{aligned}$$

Putting all of these expressions together, we have

$$\begin{aligned} \int_{\partial X} \vec{F} \cdot d\vec{S} &= \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} F_t(x, y, t_0) dxdy \\ &- \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} F_t(x, y, t_0+\Delta) dxdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} F_x(x_0, y, t) dtdy \\ &- \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} F_x(x_0+\Delta, y, t) dtdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} F_y(x, y_0, t) dxdt \\ &- \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} F_y(x, y_0+\Delta, t) dxdt. \end{aligned} \tag{11}$$

When we combine integrals that agree on their bounds, (11) can be rewritten as

$$\begin{aligned}\int_{\partial X} \vec{F} \cdot d\vec{S} = &\int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} F_t(x,y,t_0) - F_t(x,y,t_0+\Delta) dxdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} F_x(x_0,y,t) - F_x(x_0+\Delta,y,t) dtdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} F_y(x,y_0,t) - F_y(x,y_0+\Delta,t) dxdt.\end{aligned} \tag{12}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} F_t(x,y,t_0) - F_t(x,y,t_0+\Delta) &= \int_{t_0}^{t_0+\Delta} \frac{\partial F_t(x,y,t)}{\partial t} \text{ for fixed } x,y \\ F_x(x_0,y,t) - F_x(x_0+\Delta,y,t) &= \int_{x_0}^{x_0+\Delta} \frac{F_x(x,y,t)}{\partial x} \text{ for fixed } t,y \\ F_y(x,y_0,t) - F_y(x,y_0+\Delta,t) &= \int_{y_0}^{y_0+\Delta} \frac{\partial F_y(x,y,t)}{\partial y} \text{ for fixed } x,t. \end{aligned} \tag{13}$$

Thus, (12) becomes:

$$\begin{aligned}\int_{\partial X} \vec{F} \cdot d\vec{S} = &\int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} \int_{t_0}^{t_0+\Delta} \frac{\partial F_t(x,y,t)}{\partial t} dtdxdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{y_0}^{y_0+\Delta} \int_{x_0}^{x_0+\Delta} \frac{\partial F_x(x,y,t)}{\partial x} dxdtdy \\ &+ \int_{t_0}^{t_0+\Delta} \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} \frac{\partial F_y(x,y,t)}{\partial y} dydxdt.\end{aligned} \tag{14}$$

Because we are integrating over a box, we may switch our order of integration so that (14) becomes:

$$\begin{aligned}
&\int_{\partial X} \vec{F} \cdot d\vec{S} \\
&= \int_{x_0}^{x_0+\Delta} \int_{y_0}^{y_0+\Delta} \int_{t_0}^{t_0+\Delta} \left(\frac{\partial F_t(x,y,t)}{\partial t} + \frac{\partial F_x(x,y,t)}{\partial x} + \frac{\partial F_y(x,y,t)}{\partial y} \right) dV \\
&= \int_X \left(\frac{\partial F_t}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) dV \\
&= -\int_X div(\vec{F}) dV.
\end{aligned} \tag{15}$$

We can see that our integrands from (10) agree up to a sign, which means we have correctly chosen orientations for our cube for the flux integral computation according to the equality of integrands implied by Stokes theorem. Our chosen orientation thus gives us the following normal vectors for the top, right, and back faces, displayed in figure (6):

$$\hat{n}_{top} = \hat{t}$$

$$\hat{n}_{right} = -\hat{x}$$

$$\hat{n}_{back} = -\hat{y}.$$

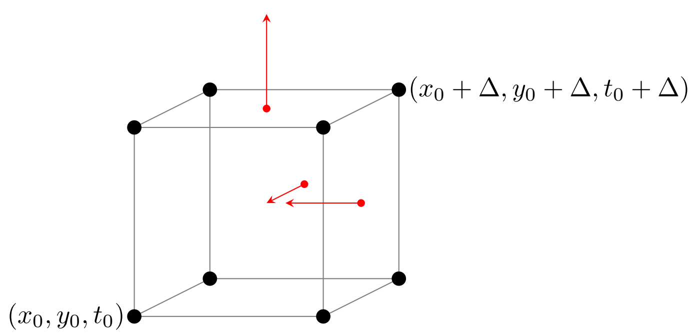

Figure 6: Oriented Minkowski Box

2.6 A Differential Forms Computation for Curvature on the Hyperboloid

2.6.1 Curvature of the Hyperboloid

Defining Curvature

The curvature we consider in this paper is a specific type of curvature: Gaussian Curvature. Gaussian curvature is an intrinsic property of a two-dimensional surface. That is, regardless of the space such a surface is embedded in, its Gaussian curvature will remain unchanged. In fact, Gaussian curvature is the only surface curvature with this property [9].

We assume familiarity of k-forms, and begin here with a discussion of connection forms. The connection forms are 1-forms which uniquely define the exterior derivative operator on vectors. We are working with the Levi-Civita connection forms, which exist and are determined uniquely by assuming they are metric compatible and torsion free [9].

Let M be an n-dimensional Riemannian or pseudo-Riemannian manifold, and let $\{\sigma_i\}$ be an orthonormal basis for T_p^*M for $p \in M$ dual to a basis $\{\hat{e}_i\}$ for T_pM. A connection is a choice of 1-forms, $\{\omega_{ij} \mid 1 \leq i, j \leq n\}$, which once selected characterize an exterior derivative on vectors via:

$$d(\hat{e}_j) = \sum_{i=1}^{n} \omega_{ji}\, \hat{e}_i.$$

A connection, $\{\omega_{ij} \mid 1 \leq i, j \leq n\}$ is torsion free if it satisfies

$$d\sigma_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \sigma_j\,, \tag{16}$$

and metric compatible if it satisfies

$$\omega_{ij} = -\omega_{ji} \tag{17}$$

for each ω_{ij}.

Definition 2.2 (The Levi-Civita Connection)**.** The Levi-Civita connection is the unique connection that is torsion free and metric compatible.

From here on, we assume we are working with the Levi-Civita connection. The definition of a

Levi-Civita connection immediately implies that $\omega_{ij} = 0$ when i = j because of metric compatibility.

To define curvature as an intrinsic property of our surface, we begin by defining it extrinsically, and through substitution arrive at an intrinsic definition. If we include a third basis element, the vector normal to our tangent plane at a given point which very notably depends on where we have embedded our surface, we are able to define Gaussian curvature extrinsically as:

$$\omega_{13} \wedge \omega_{32} = -K\sigma_1 \wedge \sigma_2 \tag{18}$$

We introduce here something called the curvature 2-forms. In 3-dimensional space these have the form

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}, \tag{19}$$

and in fact for this case,

$$\Omega_{ij} = 0$$

for each i, j. From that we conclude

$$-d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}. \tag{20}$$

For ω_{12}, by equation (20), we have

$$\begin{aligned} -d\omega_{12} &= \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22} + \omega_{13} \wedge \omega_{32} \\ &= \omega_{13} \wedge \omega_{32}, \end{aligned} \tag{21}$$

because $\omega_{11},\ \omega_{22} = 0$. Similarly, for ω_{21}, by equation (20) and equation (21), we have

$$\begin{aligned} -d\omega_{21} &= \omega_{21} \wedge \omega_{11} + \omega_{22} \wedge \omega_{21} + \omega_{23} \wedge \omega_{31} \\ &= \omega_{32} \wedge \omega_{13} \\ &= d\omega_{12}. \end{aligned} \tag{22}$$

Combining equation (18) and (20), we arrive at an intrinsic formulation for curvature. In particular, we have

$$-d\omega_{12} = -K\,\sigma_1 \wedge \sigma_2, \tag{23}$$

or equivalently

$$d\omega_{12} = K\,\sigma_1 \wedge \sigma_2, \tag{24}$$

where K is the Gaussian curvature of our surface. Equation (24) and equation (22) tell us that

$$d\omega_{21} = K\,\sigma_2 \wedge \sigma_1. \tag{25}$$

For the remainder of our computations, we restrict our attention to the Hyperboloid and Poincare disk as surfaces independent of an embedding. Both of these are 2-dimensional manifolds, and thus we can assume that we have an orthonormal dual basis $\{\sigma_1, \sigma_2\}$. Having done so, there are ω_{12}, ω_{21} that uniquely characterize the connection forms because of metric compatibility. In that case, substituting equation (16) with our dual basis elements, we find that

$$\begin{aligned} d\sigma_1 &= -(\omega_{11} \wedge \sigma_1 + \omega_{12} \wedge \sigma_2) \\ &= -\omega_{12} \wedge \sigma_2\,, \end{aligned} \tag{26}$$

and similarly,

$$d\sigma_2 = -\omega_{21} \wedge \sigma_1. \tag{27}$$

2.6.2 Computing Curvature of the Hyperboloid

Now we begin our computation of curvature for the hyperboloid. We will restrict to the upper bowl, and then use the patch given in [8]:

$$\vec{x} = (\rho \sinh(\beta)\cos(\phi),\ \rho\sinh(\beta)\sin(\phi),\ \rho\cosh(\beta))$$

We may interpret this patch as a description of the t coordinate in terms of ρ and β, since for each fixed value of $t > \rho$ we have $x^2 + y^2 = t^2 - \rho^2$, which is a circle with squared radius $t^2 - \rho^2$ parametrized by ϕ. Using this patch, we have:

$$
\begin{aligned}
x &= \rho \sinh(\beta)\cos(\phi) \\
y &= \rho \sinh(\beta)\sin(\phi) \\
t &= \rho \cosh(\beta).
\end{aligned}
$$

Applying the differential operator d on each side, we find that:

$$
\begin{aligned}
dx &= \rho(-\sinh(\beta)\sin(\phi)d\phi + \cos(\phi)\cosh(\beta)d\beta) \\
dy &= \rho(\cos(\phi)\sinh(\beta)d\phi + \sin(\phi)\cosh(\beta)d\beta) \\
dt &= \rho \sinh(\beta)d\beta
\end{aligned}
$$

Using the line element for Minkowski space given in [8], we substitue for dx, dy, and dt to find:

$$
\begin{aligned}
(ds)^2 &= d\vec{r} \cdot d\vec{r} \\
&= dx^2 + dy^2 - dt^2 \\
&= \rho^2(-\sinh(\beta)\sin(\phi)d\phi + \cos(\phi)\cosh(\beta)d\beta)^2 + \rho^2(\cos(\phi)\sinh(\beta)d\phi + \sin(\phi)\cosh(\beta)d\beta)^2 - (\rho\sinh(\beta)d\beta)^2 \\
&= (\rho\sinh(\beta)d\phi)^2 + (\rho d\beta)^2.
\end{aligned}
\tag{28}
$$

As described in [9], our line element gives us an orthonormal basis for 1-forms using the coordinates from our patch, ϕ and β, where:

$$
\begin{aligned}
\sigma_1 &= \rho d\beta \\
\sigma_2 &= \rho \sinh(\beta)d\phi.
\end{aligned}
$$

These are orthonormal in the following sense: they correspond to an orthonormal basis in the tangent space of our surface at each point. The "dot product" induced by our metric ensures that $\sigma_1 \cdot \sigma_2 = 0$ and each σ_i acts on a corresponding normal vector in the tangent space by sending it to one. From here, we compute $d\sigma_2$ and find that:

$$d\sigma_2 = \rho \cosh(\beta) d\beta \wedge d\phi.$$

Next, we use equation (27) and substitute such that:

$$\begin{aligned} d\sigma_2 &= -\omega_{21} \wedge \sigma_1 \\ \rho \cosh(\beta) d\beta \wedge d\phi &= -\omega_{21} \wedge \rho d\beta \end{aligned} \tag{29}$$

From equation (29), we see that

$$\omega_{21} = \cosh(\beta) d\phi. \tag{30}$$

Next, we compute $d\omega_{21}$ and find that

$$d\omega_{21} = \sinh(\beta) d\beta \wedge d\phi$$

Finally, using equation (25), we find that

$$\sinh(\beta) d\beta \wedge d\phi = K\rho^2 \sinh(\beta) d\phi \wedge d\beta. \tag{31}$$

From equation (31), we can see that

$$K = \frac{-1}{\rho^2}.$$

Thus, we have found the Gaussian curvature of the hyperboloid to be negative and constant.

2.6.3 Stereographic Projection

The stereographic projection of the Hyperboloid to the open disk D_ρ with radius ρ functions as follows: let p_0 be the point where $x = y = 0$ on the lower bowl. For each point on the upper bowl, draw a line from that point to p_0. Where this line intersects D_ρ is where this point is mapped to under the stereographic projection.

Because we are working in the disk, we can think about position in terms of two coordinates: ϕ and R. Note that the angle out from the x-axis, ϕ, is preserved under our mapping. Then it is left to determine R; using similar triangles, we find that

$$\frac{R}{\rho} = \frac{\rho \sinh(\beta)}{\rho + \rho \cosh(\beta)},$$

so that

$$R = \frac{\rho \sinh(\beta)}{1 + \cosh(\beta)}. \tag{32}$$

Then a point in the Poincaré Disk has coordinates:

$$(R\cos(\phi),\ R\sin(\phi),\ 0)$$

Assuming the coordinates for the Poincaré Disk, we would like to rewrite the line element for the hyperboloid in terms of R and ϕ. As a spoiler for the reader who would like to skip the detailed computation, we will find that the line element in the Poincaré Disk is given by:

$$(ds)^2 = \left(\frac{2\rho^2 R}{\rho^2 - R^2} d\phi\right)^2 + \left(\frac{2\rho^2}{\rho^2 - R^2} dR\right)^2.$$

2.6.4 Computation of Line Element

Using [8] as a guide, we will prove that

$$(ds)^2 = \left(\frac{2\rho^2 R}{\rho^2 - R^2} d\phi \right)^2 + \left(\frac{2\rho^2}{\rho^2 - R^2} dR \right)^2 \tag{33}$$

is equivalent in the Poincaré Disk to the line element from equation (28) in the Hyperboloid. Because ϕ maps to ϕ, we begin with a verification that

$$\frac{2\rho^2 R}{\rho^2 - R^2} = \rho \sinh(\beta),$$

or equivalently,

$$\frac{2\rho R}{\rho^2 - R^2} = \sinh(\beta).$$

We proceed:

$$\begin{aligned}
&\frac{2\rho R}{\rho^2 - R^2} \\
&= \frac{2\rho\left(\dfrac{\rho\sinh(\beta)}{1+\cosh(\beta)}\right)}{\rho^2 - \dfrac{\rho^2\sinh^2(\beta)}{1+2\cosh(\beta)+\cosh^2(\beta)}} \\
&= \frac{2\sinh(\beta)}{(1+\cosh(\beta))\left(1-\dfrac{\sinh^2(\beta)}{1+2\cosh(\beta)+\cosh^2(\beta)}\right)} \\
&= \frac{2\sinh(\beta)}{1-\dfrac{\sinh^2(\beta)}{1+2\cosh(\beta)+\cosh^2(\beta)}+\cosh(\beta)-\dfrac{\cosh(\beta)\sinh^2(\beta)}{1+2\cosh(\beta)+\cosh^2(\beta)}} \\
&= \frac{2\sinh(\beta)}{\dfrac{1+2\cosh(\beta)+\cosh^2(\beta)-\sinh^2(\beta)+\cosh(\beta)+2\cosh^2(\beta)+\cosh^3(\beta)-\cosh(\beta)\sinh^2(\beta)}{1+\cosh(\beta)+\cosh^2(\beta)}} \\
&= \frac{2\sinh(\beta)(1+2\cosh(\beta)+\cosh^2(\beta)}{2+2\cosh(\beta)+\cosh(\beta)+2\cosh^2(\beta)+\cosh(\beta)(\cosh^2(\beta)-\sinh^2(\beta))} \\
&= \frac{2\sinh(\beta)(1+2\cosh(\beta)+\cosh^2(\beta)}{2(1+2\cosh(\beta)+\cosh^2(\beta)} \\
&= \sinh(\beta).
\end{aligned}$$

To finish our verification that equation (28) and equation (33) equivalent in their respective models, we would like to show that

$$\frac{2\rho^2}{\rho^2 - R^2}dR = \rho d\beta,$$

or equivalently:

$$\frac{2\rho}{\rho^2 - R^2} dR = d\beta.$$

Since we no longer have a direct mapping of β to R, we will need to do a bit more work here to solve for dR in terms of $d\beta$. We find that

$$\begin{aligned} d(R) &= d\left(\frac{\rho\sinh(\beta)}{1+\cosh(\beta)}\right) \\ &= \left(\frac{\rho\cosh(\beta)}{1+\cosh(\beta)} - \frac{\rho\sinh^2(\beta)}{(1+\cosh(\beta))^2}\right) d\beta. \end{aligned}$$

From here, we substitute for dR with the above calculation:

$$\begin{aligned}
&\frac{2\rho}{\rho^2 - R^2} dR \\
&= \frac{2\rho}{\rho^2 - R^2}\left(\frac{\rho\cosh(\beta)}{1+\cosh(\beta)} - \frac{\rho\sinh^2(\beta)}{(1+\cosh(\beta))^2}\right) \\
&= \frac{2\rho}{\rho^2 - \dfrac{\rho^2\sinh^2(\beta)}{(1+\cosh(\beta))^2}}\left(\frac{\rho\cosh(\beta) + \rho\cosh^2(\beta) - \rho\sinh^2(\beta)}{(1+\cosh(\beta))^2}\right) \\
&= \frac{2(\cosh(\beta)+1)}{\left(1 - \dfrac{\sinh^2(\beta)}{(1+\cosh(\beta))^2}\right)(1+\cosh(\beta))^2} \\
&= \frac{2(\cosh(\beta)+1)}{(1+\cosh(\beta))^2 - \sinh^2(\beta)} \\
&= \frac{2(\cosh(\beta)+1)}{1 + 2\cosh(\beta) + \cosh^2(\beta) - \sinh^2(\beta)} \\
&= \frac{2(\cosh(\beta)+1}{2(\cosh(\beta)+1)} \\
&= 1.
\end{aligned}$$

This computation completes our proof that the line elements in [8] are correct for the Poincaré Disk.

2.6.5 The Poincaré Disk

We now provide a general characterization of the Poincaré Disk. The points in this model are the Euclidean points in the interior of the unit disk,

$$\{(x, y) \mid x^2 + y^2 < 1\}.$$

Since we are assuming the neutral geometry postulates, the lines in this model are all defined by two distinct points.

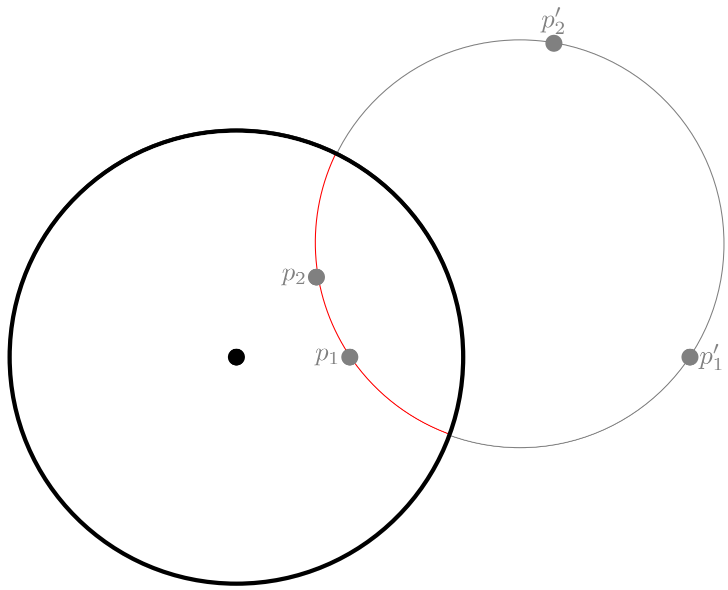

Figure 7: Hyperbolic line between p_1 and p_2 (red)

The point $(0, 0)$ is the center of the Poincaré Disk, and the disk has radius 1. Given two points p_1, p_2, to find a line containing them first let p_1' be the inversion of p_1. This is the point colinear (in the Euclidean sense) to $(0, 0)$ and p_1 which satisfies

$$||p_1|| \cdot ||p_1'|| = 1,$$

using the Euclidean norm. Now, let C be the circle that goes through p_1', p_1, and p_2. Notably, this circle will also contain the inversion of p_2, so this circle is invariant upon relabeling of our points. The hyperbolic line through p_1, p_2 in the Poincaré Disk is the intersection of C with the open unit disk. Furthermore, we equip the Poincaré Disk with a non-Euclidean metric [25]. This is illustrated for a particular pair of points p_1, p_2 in figure (7).

2.6.6 Computation of Curvature in the Poincaré Disk

In this section, we will compute the curvature in the Poincaré Disk using the intrinsic definition of Gaussian curvature from equation (24). We begin with our dual basis, which we selected using the line element to be:

$$\sigma_1 = \frac{2\rho^2}{\rho^2 - R^2} dR$$

$$\sigma_2 = \frac{2\rho^2 R}{\rho^2 - R^2} d\phi.$$

From here, we solve for $d\sigma_1$ and find that

$$\begin{aligned} d\sigma_2 &= \left(\frac{2\rho^4 - 2\rho^2 R^2 + 4\rho^2 R^2}{(\rho^2 - R^2)^2} \right) dR \wedge d\phi \\ &= -\left(\frac{2\rho^4 + 2\rho^2 R^2}{(\rho^2 - R^2)^2} \right) d\phi \wedge dR. \end{aligned}$$

Now we equation (27) to conclude that

$$\begin{aligned} \omega_{21} &= \left(\frac{\rho^2 + R^2}{\rho^2 - R^2} \right) d\phi \\ &= \left(\frac{\rho^2}{\rho^2 - R^2} + \frac{R^2}{\rho^2 - R^2} \right) d\phi. \end{aligned}$$

Next we compute $d\omega_{21}$, which we find to be:

$$\left(\frac{2\rho^2 R}{(\rho^2 - R^2)^2} + \frac{2R}{\rho^2 - R^2} + \frac{2R^3}{(\rho^2 - R^2)^2}\right) dR \wedge d\phi \tag{34}$$

$$= \frac{4\rho^2 R}{(\rho^2 - R^2)^2} dR \wedge d\phi \tag{35}$$

$$= -\frac{4\rho^2 R}{(\rho^2 - R^2)^2} d\phi \wedge dR. \tag{36}$$

The final piece of the puzzle is $\sigma_2 \wedge \sigma_1$, for which we have:

$$\sigma_2 \wedge \sigma_1 = \frac{4\rho^4 R}{(\rho^2 - R^2)^2} d\phi \wedge dR.$$

Now, using equation (25), we find that

$$K = \frac{-1}{\rho^2}.$$

Thus, we have finally verified that Gaussian curvature is equal on the Poincaré Disk and on the hyperbola.

2.7 Conclusion

We began by discussing the Euclidean box argument that motivates our orientation of the cube in Minkowski Space. After that, we introduced and generalized various tools to the Minkowski 3-space setting, including divergence, the module isomorphism induced by the vector differential, and the cross product. Finally, we used Stokes' theorem to correctly pick an orientation for each face of the cube. More specifically, our orientation of the cube relied on the assumption that each face was orientable, and furthermore, that the integrands would be equal if a correct orientation was selected. The latter assumption is grounded in the work we did to generalize the Euclidean divergence theorem to our context using Stokes' Theorem.

In our computations of curvature in each model, we verified the equivalence of the Poincaré Disk and the hyperboloid as models of Hyperbolic Geometry. This result is further validated by the fact that stereographic projection from the hyperboloid to the Poincaré Disk is an isometry; in determining the line element for the Poincaré Disk according to our mapping under stereographic projection, we guarantee that distance between points is preserved under mapping. This notion of an isometry loosely follows the definition outlined more rigorously in [22].

The existence of any such isometry implies that the two models will have the same intrinsic geometric properties [22]. Considering this in conjunction with Gauss' Theorema Egregium, it comes as no surprise that our separate computations for curvature yield an identical result. That said, we have independently found the curvature for each model to be negative, constant, and equal to $\frac{-1}{\rho^2}$.

2.8 Future Work

The most immediate project following this work is to apply the orientation of the cube to compute the Gaussian curvature of the Hyperboloid extrinsically. This can be easily verified using our forms-based computation of curvature in the preceding section. Following that, I would love to expand the accessibility of the results in this paper so that someone with a solid understanding of vector calculus and the shape operator could seriously use the oriented cube to do their own computations.

3 Orienting Myself in Mathematics

"The real political task in a society such as ours is to criticize the workings of institutions that appear to be both neutral and independent, to criticize and attack them in such a manner that the political violence that has always exercised itself obscurely through them will be unmasked, so that one can fight against them."

– Michel Foucault

3.1 Introduction

I would like to begin by saying, first and foremost, I am not anti-math. In my tenure as a student of mathematics, I have gone through periods of general interest, active dread, and extreme fixation toward my studies. Ultimately, however, I can't imagine a future where I ever stop doing mathematics because deep down, I always love it. Furthermore, I recognize that mathematics as we think of it today has admittedly been used to cause a lot of harm, but it has also been used to reduce human suffering, produce life-saving interventions, and bring joy to casual audiences who learn that *math is actually pretty fun*.

While I love mathematics, I am also committed to improving the culture surrounding it. This includes improving access to mathematics, as well as improving the experience of doing mathematics once access is granted, for *everyone* and especially those currently marginalized in the field. Furthermore, improving the culture of mathematics also means expanding what counts as "valid" mathematics and who is considered a mathematical knower. The consequences of doing so are twofold; first, the experiences of mathematicians would be improved. This is important because folks who love to do mathematicsshouldn't have to decide between doing mathematics and feeling comfortable in their profession. Furthermore, in improving access and culture, our field could have higher retention of brilliant individuals who carry crucial perspectives. For a field that often claims to be fundamental to the other sciences, mathematics is not represented by the necessary stakeholders. One project aimed at improving the culture of mathematics is "humanizing mathematics". I have been drawn toward this project in particular, and I see it as an important goal with which to orient our social work in mathematics. That said, the philosophy of humanist mathematics, while an important step forward, is an insufficient theoretical framework from which to achieve this goal.

In this section, I will consider how a feminist philosophy of science lens could be used within philosophy of mathematics, and how it can contribute to the project of humanizing mathematics. In particular, a feminist philosophy of science lens both challenges humanist philosophy of mathematics and reinforces many of the critiques made by such an approach toward platonic and formalist philosophies of mathematics. In addition, feminist philosophy of science can deepen and support existing contributions toward the social and cultural construction of mathematics made by humanist philosophers of mathematics. My scope of expertise, both on philosophy of mathematics and feminist philosophy is insufficient to detail the extent of these possibilities or anticipate a cohesive union of these disparate philosophies. That said, I aim to introduce feminist philosophy of science in relation to humanist philosophy of mathematics, and argue that it is worth spending the time to develop this line of thought more thoroughly.

Aside: On the Title of this Chapter

The language of "Orienting Myself in Mathematics" can be interpreted at several levels. Most literally, my author positioning will serve as a way to contextualize myself and my experiences in relation to mathematics. On a more abstract level, this qualitative inquiry is my attempt to reconcile my understanding of the world and my love for mathematics, by adopting a philosophy of mathematics that is consistent with my own beliefs and values.

3.2 Audience

This writing will be the most accessible to feminist philosophers of science and philosophers of mathematics. That said, in the spirit of feminist qualitative inquiry [12, 24], I aim for this writing to be accessible to any mathematician and, more broadly, to anyone affected by mathematics. Thus, if my parents are able to read and understand this, I will be overjoyed.

3.3 Author Positioning

To orient myself in mathematics, I will begin, according to the feminist research methodologies introduced by Linda Tuhiwai Smith and Venus Evans-Williams [24, 12], by positioning myself in this field. In addition, similarily to how Venus Evans-Williams uses autoethnography to inform her research methodologies via her lived experiences, I will use autoethnographic writing to explain how my positionality has led me to my current conceptualizing of philosophy of mathematics.

I am a senior mathematics student at Oregon State University; I am queer, white, and possess the dual diagnoses of ADHD and OCD. I grew up in a rural unincorporated community in Western Oregon where I did not have access to as much as a grocery store. I was discouraged from taking advanced math courses on more than one occasion by my public school teachers, but several were tremendously supportive of my development as a scientist.

My father did not attend college and worked mostly blue-collar jobs for the earlier part of my life. My mom was college educated, and as a high school guidance counselor she was able to help me apply for scholarships and internships prior to college. I have been lucky in that I could work only summers in high school and was able to attain a full-ride scholarship, and unlucky in that I continued to work through most of college to fund my housing, food, and healthcare.

Before I was eligible for food stamps, I experienced extreme food insecurity. This was also the time I began the core classes for the mathematics major. I did not experience racism in my mathematics classes, and I rarely experienced direct homophobia. On the other hand, I was stalked by two of my classmates, and harrassed by many others. Most of my professors believed in my abilities as a mathematician and actively challenged me to apply myself, albeit sometimes too much so. As I worked through my degree, I lost two family members. My mother was diagnosed with breast cancer, underwent treatment, and is now in remission. As I finish this thesis, my sibling is undergoing treatment for ALS Leukemia.

All of this is the liminal space I have walked between experiencing oppression, bad luck, good luck, and white privilege. Some of my fortune is due to institutional structures, and some is due to sheer luck, and similarly for my misfortunes. These experiences immediately caused me to consider how my educational experience failed to support me fully when I struggled, and at the same time elevated me unfairly over more marginalized students. My mathematics education was largely based on the assumption that equality could be achieved by measuring every student with the same metrics and treating each student the same. That is not the case, because student needs are vastly different and student performance is largely affected by things such as experiencing food insecurity, experiencing sexual harrassment, and working possibly several jobs, or on the other hand not working and having parents with technical degrees. Teachers could provide more flexibility for working students, and department scholarships could consider factors like hours worked per week in conjunction with GPA and unpaid research experience.

My experiences have affected my access to the institution that produces mathematicians and the resources to do mathematics, but furthermore my experiences have affected my relationship with mathematics itself. Thus, while feminist philosophy of science could certainly be used to investigate mathematics at an institutional level, I am more interested at the moment in how feminist philosophy of science could be used within philosophy of mathematics.

3.4 Philosophy of Mathematics

Broadly, the philosophy of mathematics is the study of philosophical questions about mathematics from the field of philosophy. This is closely related, but distinct, from foundations of mathematics research done from the field of mathematics [16]. In the philosophy of mathematics, there are various theories for how mathematics should be conceptualized, and accurately explaining and connecting all of these to this writing is far beyond my scope of knowledge and expertise. A curious reader should look into [16, 5]. One of the most influential philosophies of mathematics is platonism, through which "mathematics objects are real, objective, and mind independent abstract objects existing outside of space and time" [16, 5]. This includes the conception that mathematics is discovered by mathematicians [3].

Another hugely influential philosophy of mathematics is formalism, which "regards mathematics as the study of formal deductive systems, and mathematical truth is just provability in the system" [3]. The reduction of mathematics to logic has been challenged substantially by mathematicians, even those who see logical argument as an ultimate goal in the project of mathematics, who nonetheless recognize a certain appeal to mathematics that is beyond logic. For example, Paul Lockhart's "A Mathematician's Lament" famously situates mathematics as art and a part of human culture. Paul Lockhart goes so far as to call mathematics the "purest of arts" and presents mathematical objects as ideas which, much like a child's toy, are moved around and played with by various individuals [20]. In addition to demonstrating the popular belief that mathematics is culturally and emotionally valuable, this simple example is significant to my critiques against maintstream philosophies of mathematics. Toys are played with but not (significantly) physically changed depending on who holds them. The analogy between mathematical entities and child's toys implies a static quality about mathematical entities–a representation of them as "out there", that is ultimately challenged by a feminist philosophy of science approach to mathematical knowing.

Paul Lockhart's appeal to the beauty of mathematics, and the wide circulation and adoptation of ideas from his essay, demonstrate that the the work of many mathematicians is clearly motivated by than a desire for logic and truth. As further evidence of this, consider the case of proof automation;

many mathematicians resist the automation of proofs, in this case seeing some value beyond mere truth generated computationally. That said, we also work, and are trained, to produce correct proofs and mathematical statements. This has certainly been the goal of my advisor and I in the body of mathematics included for this thesis!

I am not equipped to comment on the objectivity of mathematics in any definitive way. On one hand, I recognize that we construct axiomatic systems that allow for some notion of truth after a potentially subjective proof experience. On the other, the mathematics we do cumulatively as a field is flawed logically and sometimes lacking in "rigor". At least for now, how truth comes to be accepted is a human endeavor, and thus it seems intractible for us to succeed in having a logically perfect, axiomatic, body of work. In my head, there is a mathematics "out there", but ultimately this mathematics doesn't seem to exist. The mathematics we actually interact with is a living, breathing, body of human knowers and human relationships. In some sense, mathematics is only fallible as much as we entertain the idea of an uncorrupted and perfect mathematics beyond our touch; mathematical truth is what we construct it to be.

Both humanist philosophy of mathematics and feminist philosophy of science position mathematics as the truth, definitions, and conventions currently accepted within the field. That said, humanist philosophy of mathematics measures accuracy relative to an infallible mathematical system. Let's assume the existence of such a system and temporarily wear the hat of a formalist. When we compare our system to what is conceptualized as mathematics by either a humanist philosophy of mathematics or a feminist philosophy of science lensing toward mathematics, we see that "mathematics" is fallible. Concepts like folklore in mathematics, summarized on Wikipedia as "an unpublished result with no clear originator, but which is well-circulated and believed to be true among the specialists" [1], challenge our understanding of mathematics as consistent with . Another example of the fallibility of mathematics is the case where mathematicians get it wrong. This summer, one of the research teams at my "Research Experience for Undergraduates" site spent a large part of their time correcting a proof for a result that had gone, incorrectly, through peer reviews and into print, and which was to serve as the basis of their future work. Because both humanist philosophy of mathematics and feminist philosophy of science admit an understanding of mathematics as socially constructed, this fallibility is accepted as a part of mathematics. It's worth noting now that this isn't necessarily a positive artifact of the social construction of mathematics in either philosophy, but rather it just "is".

3.4.1 Humanizing Mathematics

Reuben Hersh is a research mathematician who is largely credited with developing a humanist philosophy of mathematics. The humanist philosophy of mathematics seeks to reconcile the activities of mathematicians with our understanding of mathematics. This philosophy is conflated in [5] with a socioconstructivist philosophy of mathematics. Humanist philosophy of mathematics establishes mathematics as a part of human culture that is constructed by humans. In this philosophy, "mathematical knowledge isn't infallible, and there are different versions of proof and rigor" [3]. In line with platonism, the humanist philosophy of mathematics sees mathematical objects after construction as independent of their creator and mathematical statements as describing mathematical objects [5]. This notion will be challenged by the concept of situated knowledge from feminist philosophy of science.

Ultimately, my training as a feminist philosopher, and my own experiences in mathematics, lead me to agree that mathematics is indeed a part of human culture. That said, feminist philosophy of science challenges the notion of mathematical objects as separate, abstract objects, and instead situates them within social context, relationships, and a variety of other dynamics that knowledge is conceived as situated under this framework. One consequence of adopting a feminist philosophy of science lensing toward mathematics is the relationality and culture of mathematicians, as advanced by humanist philosophy of mathematics, is emphasized further because, through this lens, culture and relationships partially constitute mathematical entities. Furthermore, feminist theory provides further space to ponder the social construction of mathematics through the theory of feminist postmodernism. Finally, a feminist philosophy of science framework move towards a non-essentialist conception of humans, human culture, and the role of mathematics therein.

3.5 Feminist Philosophy of Science

My original vision for this inquiry was largely inspired by Piper H's thesis [14]. I wanted to discuss the lives of each cited mathematician, and I wanted to include context for my life as this work was written, in an effort to radically contextualize the mathematics that I do. This developed into a set of case studies challenging the overall objectivity of mathematics, and the institution that trains mathematicians. Ultimately, working out the theoretical lensing for that research was a rich inquiry in and of itself and has become the focus of my qualitative inquiry. I chose to use a feminist philosophy of science lens because of my background in feminist philosophy; I took a variety of women, gender, and sexuality courses as well as feminist philosophy as an undergraduate, and I have continued to recreationally read, blog about, and critique various feminist philosophical pieces recreationally.

The primary lens through which I will conduct my analysis is feminist philosophy of science. Various expert scientists have contributed to active feminist inquiry on the epistemology and ontology of their own field, including Banu Subramaniam, Robin Wall Kimmerer, and Karen Barad [23, 17, 4]. It is my goal to contribute an argument for developing a feminist philosophy of mathematics, using a feminist philosophy of science framework, in my own field.

Here I will provide a brief overview of the elements from feminist philosophy of science that are particularly relevant to my project, and which have shaped my areas of study. The material for this overview is prepared based on [2], as well as my own scholarship broadly across feminist philosophy, which will be cited throughout.

Background: Situating Feminist Philosophy of Science

Western philosophies have been significant in their assumption of humanism and essentialism. This has been challenged vigorously by feminist theorists. In my domain of experience, I have encountered such challenges from feminist philosophers of disability like Susan Wendell (see [26]) who push back against the naturalization of disability and reveal how disability is significantly constructed by the organization and structure of society.

In addition, feminist scholars of color have significantly pushed back against the essentialist, global, category of “women” assumed under traditional western feminist philosophers. This includes Kimberly Crenshaw’s genius intersectional feminist framework [7], which articulates how unique types of oppression create different and unique experiences of womanhood and patriarchy for women in distinct social groups. Furthermore, in “Feminism Without Borders” Chandra Talpade Mohanty articulates a strong critique against a universal “woman” or a universal experience of patriarchy [21]. In fact, as Mohanty shows, such conceptions are essentialist and ultimately naturalize gendered oppression. Furthermore, she carefully argues that implicit in the universal constitution of womanhood by many western feminists is an affluent, white, American woman.

Finally, in “Full Surrogacy Now”, Sophie Lewis brilliantly moves feminist reproductive justice toward questioning family, kinship, and birth [19]. She highlights how Western ontologies have ultimately limited conceptions of pregnancy and birth to what is understood as essential to birth, and suggests that we should radically reconsider what is seen as fundamental to childbirth, including birth mortality, and pain during pregnancy and childbirth.

All of these contributions have been in line with feminist philosophical stances that reject humanism and the naturalization of gendered oppression. In parallel to these counters to an essential human, an essential woman, or an essential birth experience, feminist philsophy of science has challenged the objectivity promised by western science and ultimately of essential knowledge. In traditional western epistemologies, knowledge is held separately from the subject of knowledge, and furthermore, knowledge is fundamentally true.

Assumptions implicit under Western epistemology have been effectively exposed by Indigenous scholars. For example, Whitt, Roberts, Norman, and Grieves describe two “prevailing convictions in western philosophy and the science which it sustains.” These convictions are the claims that ”knowledge of nature is ultimately distinct, and separable from, nature” and “what is known are true propositions about reality.” Furthermore, these scholars outline an eye-opening alternative to what they define as Western representational knowledge systems, revealing that the very conception of knowledge as separated from the subject of knowledge is contingent on a choice of ontology [27].

Feminist Philosophy of Science

Similar to Indigenous feminist philosophers, feminist philosophers of science have grounded into critiques of knowledge as representational, informed largely by the theory of knowledge as situated. Under the framework of situated knowledge, knowledge is held in the relationships of the knower, possibly to the subject of knowledge, to themselves, or to other knowers. Unlike other philosophies of mathematics, including mainstream humanist philosophy of mathematics, in a feminist philosophy of science lens knowledge doesn't exist "beyond" knowers but instead is situated within and *among* knowers and thinkers.

Situated knowledge has been informed significantly by feminist standpoint theory, feminist postmodernism, and feminist empiricism. Each theory has challenged and evolved the other in its critiques, and modern feminist philosophy of science is informed by this evolution and convergence of theories. Situated knowledge stresses several relationships to knowledge that are significant in the context of mathematics. Of the eight major constituents of situated knowledge highlighted by Elizabeth Anderson in ([2]), the following are immediately relevant to advancing or critiquing claims made by a humanist philosophy of mathematics:

1) Embodiment "People experience the world by using their bodies, which have different constitutions and are differently located in space and time."

3) Emotions, attitudes, interests, and values "People often represent objects in relation to their emotions, attitudes and interests, which differ from how others represent these objects."

5) Know-how "People have different skills, which may also be a source of different propositional knowledge."

6) Cognitive Styles "People have different styles of investigation and representation."

7) Background beliefs and worldviews "People form different beliefs about an object, in virtue of different background beliefs."

8) Relations to other inquirers "People may stand in different epistemic relations to other inquirers-for example, as informants, assistands, students–which affects their access to information and their ability to convey their beliefs to others." Epistemic authority and epistemic injustice is ellaborated on in great detail in [2].

Because situated knowledge is ultimately informed by our three distinct philosophies, I will detail each of those philosophies in the next section. Rather than present these philosophies as competing with each other, I will explain each and how it can enhance the other.

Feminist Standpoint Theory

Feminist standpoint theory is grounded on the assumption that women, holding subjugated identities, have a clearer and thus more objective understanding of gendered oppression than cis-gender men. This assumption is grounded in a critical catchum that this understanding is only granted in conjunction with critical awareness gained through political struggle and theoretical indoctrination in feminism. In this sense, the grounds of standpoint expertise are restricted to women who largely identify as feminists and who have struggled for liberation in the face of their oppression [2].

Feminist standpoint theory implies that any such women are uniquely qualified to detect and eliminate bias in matters where gender is implicated. As standpoint theory relates to feminist philosophy of science, this theory suggests that women are uniquely equipped to detect gender bias in science. Thus, an ultimate implication of this theory is that objectivity of science is improved by increasing the participation of women [2].

Feminist standpoint theory is not necessarily inconsistent with feminist empiricism or postmodernism, but responds to the question of political objectivity differently. For example, a feminist standpoint theorist might reject the scientific method, in a manner inconsistent with most feminist empiricists, but they may also adopt and fully utilize both philosophies.

Feminist standpoint theory has been criticized on several grounds. From mainstream philosophies of science which present scientists as objective, this theory is charged with incorrectly portraying scientists as biased. From a feminist perspective, this theory is critiqued for over-universalizing the category of "woman". As Mohanty discusses, behind such universal classification is the assumption of a specific, usually white and affluent, woman [21]. Ultimately, this theory provides a framework that suggests expertise to a group of subjugated knowers, and I suggest we take it "with a grain of salt" and adapt it in local contexts as much as it makes sense to.

Feminist standpoint theory is a relevant theory to my project in that it enhances my validity as a feminist theorist. That said, I will not consider for the scope of this project the ways in which gender is implicated further in our field or in the structures of our logical systems. I am not dismissing the possibility that our very proof structures and definitions are in some ways gendered and in fact find that likely; see, for example, [15]. Instead, I am acknowledging that this is beyond the scope of my resources at the time to consider.

Postmodernism

Postmodernism is a branch of philosophy concerned with the rejection of and skepticism toward essentializing or universalizing claims of existence, reason, subject and self dichotomy, and science [15]. This theory, informed significantly by postmodernist philosophers including Michael Foucault, "stresses the locality, partiality, contingency, instability, uncertainty, ambiguity, and essential contestability of any particular view of the world and the good" [2].

Before situating postmodernism further into a feminist philosophy of science lens, I will discuss a couple of key applications of postmodernist philosophies. From a feminist philosophy of disability lens, postmodernism provides grounds to argue toward a social model of disability, where disability is not produced by essential classifications but instead through environmental conditions and as such, it can be considered (at least in many cases) as "socially constructed" [26]. A concrete example is a wheelchair user who cannot access a room on the second floor due to a lack of ramps in the building. Instead of placing disability on behalf of the individual, we can recognize that the omission of a ramp has resulted in their lack of access, thus partially constructing their disability.

Another application that we have already seen is Mohanty's anti-universalizing appeals for categories like "woman" [21]. The experience and customs associated to women and womanhood differ by culture, historical moment, and presentation on behalf of the individual. Thus, there can be no essential "woman". Shifting from gender toward sex, which is often portrayed as rooted in the biological instead of social, a postmodernist lens encourages us to take not even those categories for experience as final. As Sophie Lewis demonstrates in her groundbreaking writing, there cannot even be an essential female experience.

So what does postmodernism imply for feminist philosophy of science? Postmodernism immediately critiques our preceding two theories as being too broad or too rigid. Furthermore, postmodernism asserts that "there can be no complete, unified theory of the world that captures the whole truth... The selection of any one theory is a choice that cannot be justified by appeal to the 'objective' truth or reality" [2]. As Michel Foucault suggests, truth itself is constructed within various epistemes, each of which is in turn created by a comibation of social and historical factors [13]. One such episteme Foucault identifies is the episteme of western science; thus, postmodernism provides a lens by which to question the very structures of science, discovery, and experimentation as being socially constructed and thus having room for gender bias.

From a more specific context, Sandra Harding discusses challenges presented to "pure mathematics" that are in line with such postmodernist skepticism. For example, the field of formal semantics is critiqued as individuating objects in a highly gendered manner by Merrill and Jaakko Hintikka. She further discusses how historical moments and social images have created logic and routes to proof in mathematics [15].

Finally, Sandra Harding crucially notes that "no conceptual system can provide the justificatory grounds for itself." As such, even pure mathematics must be derived from human intentions and values. Thus, just as a geometry is axiomatically constructed, the logical system from which axiomatic construction is permitted is produced relationally and ultimately socially constructed.

Postmodernism contributes to a feminist philosphy of mathematics in a final significant way; in valuing and recognizing a plurality and constant evolution of perspective, we can better recognize the potential of individuals with unique beliefs, perspectives, or worldviews as knowers of mathematics.

Empiricism

Empiricism is broadly the philosophy that "experience provides the sole or primary justification for knowledge" [2]. Within feminist empiricist philosophy of science, there are two main philosophies: community-based social knowledge and values-as-evidence. Both philosophies agree that "values, as with any number of descriptive beliefs, are often implicit and assumed in scientific research, operating as auxiliary or background assumptions" [6]. Notably, in addition to the discussion of values informing practice, several empiricists have argued for an empiricist philosophy of mathematics. For Quine, such a philosophy assumes that "the natural sciences are the ultimate arbiters concerning mathematical existence and mathematical truth" [16]. On the other hand, Tevian Dray, who works at the intersection of theoretical physics and mathematical physics, has claimed that mathematics is only ever a crude approximation for the natural world, and is constructed axiomatically by mathematicians [10]. Tevian's philosophy on this matter more accurately summarizes my own relationship to mathematics and logic, and notably still leaves space to interpret a constructivist approach to mathematics and the imbuing of values in such construction.

The community-based social knowledge approach argues that objectivity arises from democracatic decision-making and diverse representation in science. On the other hand, the values-as-evidence approach conceptualizes that "values are beliefs informed by the evidence of experience", and furthermore, that objectivity is increased by interrogating and testing values in the same way that other propositions are tested in science. Sharyn Clough provides a more thorough overview of the above material, and discusses how feminist values hold up to interrogation and can inform scienctific processes, in her essay "Using Values as Evidence When There's Evidence For Values: A Pragmatist Approach" [6].

Objectivity

Contrary to representations in the media, feminist philosophy of science is explicit and nearly universal in its goal to improve objectivity in science [2, 6]. Objectivity in research is defined by Sharyn Clough, an empiricist, as research that "captures as much of the available evidence as possible, obscures or discounts as little of the available evidence as possible, is based on as representative a sampling of the relevant evidence as possible, and explains as much as possible of the variation in the evidence at issue" [6]. That said, there are a variety of appraoches within feminist philosophy of science toward objectivity, but they are fitted to the lens of each philosopher and do not necessarily imply an abstract, "pure" notion of objectivity [2]. In her summary of feminist philosophy of science, Elizabeth Anderson presents the following critiques posed by feminist philosophers of science toward western science's claims toward objectivity:

Subject/object dichotomy "what is really ('objectively") real exists independently of knowers."

Aperspectivity "'objective' knowledge is ascertained through 'the view from nowhere,' a view that transcends or abstracts from our particular locations."

Detachment "knowers have an 'objective' stance toward what is known when they are emotionally detached from it."

Value-neutrality "knowers have an 'objective' stance toward what is known when they adopt an evaluatively neutral attitude toward it."

Control "'objective' knowledge of an object (the way it "really" is) is attained by controlling it, especially by experimental manipulation, and observing the regularities it manifests under control."

External guidance "'objective' knowledge consists of representations whose content is dictated by the way things really are, not by the knower."

Many of these critiques are especially relevant in the context of mathematics. For example, platonism and, possibly to a lesser extent, humanist philosophy of mathematics both product subject/object dichotomy as portraying mathematical entities as separate from mathematicians. Furthermore, aperspectivity is assumed by many mathematicians when they claim that mathematics is a universal language. This hides the subtleties of mathematics as situated in an individual's culture and perhaps misses the unique and possibly quite different interpretations at play when two mathematicians work together.

Although doing mathematics is regularly portrayed as neutral or "pure", this is contrary to my experience in mathematics. Mathematicians often fondly remark on proofs ("cute" is a favorite descriptor in our department) and relate to favorite structures warmly and affectionately. Mathematicians are described as passionate and eccentric. Notably, the emotions it is appropriate to feel toward mathematical entities are policed in what I feel is a highly gendered manner. Furthermore, excess passion is part of a larger class of neurodivergence which is, in the words of a past advisor, "fetishized" in our field to the point of exploitation of mathematicians. That said, mathematicians place strong emphasis on certain emotions and the ability of emotions to connect us closely to the objects we work with; such connection fosters what many professors refer to as "mathematical intuition."

Emotional connection to mathematical entities is actually in line with what Elizabeth Anderson highlights as a method to *improve* objectiviy in science. According to Anderson, feminist theory also suggests methodologies for more objective scientific research [2]. This part of the lensing is crucial, because the overall goal of feminist philosophers of science is not to attack science, but to improve science. So far, we have recognized some of the shortcomings of our philosophies of mathematics and the limitations of their promises for objectivity. We now consider Anderson's producers, as implied by feminist theory, for more objective research:

Feminist/nonsexist Research Methods Feminist and nonsexist research methods encapsulates the application of research methods which avoid gender bias. Various feminist scholars have articulated ways in which the structure of inquiry, or the design of individual experiments, are biased in gendered ways.

Emotional Engagement Rather than avoiding and attempting to remain neutral toward the subjects of study, feminist philosophers of science argue that we should engage at an emotional level with them. By doing so, we can more accurately predict and understand them. In the words of Anderson, "loving attention toward the object enhances perception of". Furthermore, because knowledge is situated within the knower, emotions are an important tool for inquiry, and inform our intuition for the truth.

Reflexivity Reflexivity means that "inquirers place themselves on the same causal plane as the object of knowledge." Reflexivity requires that an inquirer makes their social position explicit in their inquiry. This convention was adopted above when I discussed my own social positioning.

Democratic discussion Feminist theorists posit that objectivity is improved through critical and cooperative discussion. Further, democratic discussion must include a representative group, and further, that the participants in knowledge production are not subject to illegitimately garnered epistemic authority or denial of such authority on the basis of social power. This is also referred to as "equality of intellectual authority."

3.6 Philosophical Takeaways

I have struggled to place my understanding of embodied and more generally situated mathematical knowledge within the main philosophies of mathematics. Feminist philosophy of science provides a crucial lens for understanding the deeply human, and furthermore situated, aspect of the work that we do as mathematicians. Furthermore, this lens critiques what we assume to be the source of our objectivity, and inspires a new line of inquiry that encourages collaboration, full and fair representation, and open critique. Finally, the relational aspect of situated knowledge underscores the human and cultural aspect of mathematics that philosophers including Reuben Hersh have argued for. In my experience as a mathematician, mathematics is not abstractly "out there" but deeply contingent on the context it is done in; feminist philosophy of science, and more broadly feminist theory are a promising philosophy of mathematics in that they admit for such contextualization.

3.6.1 Future Philosophizing

My writing so far has largely advanced the argument that this framework shows potential for advancing and critically challenging humanist philosophy of mathematics. Because of my lack of deep expertise on either feminist theory or philosophy of mathematics, future work would most immediately be including the expertise and perspectives of philosophers from these two domains. Furthermore, if this theory truly holds water, I envision a feminist philosophy of mathematics, which would require much more work and research to develop fully and thoroughly. Many major elements of feminist theory were left out of this analysis, and there is a lot of room for future exploration and development of a feminist philosophy of mathematics. Furthermore, an immediate corollary to this writing, and the original inspiration for working out this lensing, would be to use this framework to inquire on the subjectivity of mathematics through four major lines of inquiry that I will discuss in the next section.

Further projects, which I have not had time to consider, are what shortcomings there are for feminist philosophy of science's "situated knowledge" in describing mathematical knowledge. For example, does this place knowledge mostly at the level of the individual, instead of among and potentially beyond mathematicians? Furthermore, I think it is worthwhile to examine how gender is implicated in the construction of specific mathematical entities and, the development of or tendency to select certain proof methods.

3.7 Application of Feminist Philosophy of Science Lensing: A Proposal for Qualitative Inquiry in Humanizing Mathematics

The preceding philosophical work was largely inspired to serve as the lensing for a body of work that remains unfinished. Accordingly, I will discuss the most immediately pressing questions on which I would like to apply a feminist philosophy of mathematics lensing. The following are the main questions I seek to investigate:

1. What existing critiques are there in the literature for the title “pure math”?

2. What ontological assumptions are baked into the distinction between “pure” and “applied” mathematics?

3. Where is there space for subjectivity or human-ness in “pure” mathematics?

For most of this proposal, I would like to elaborate more on the third question. I would like to research the subjectivity of “pure” mathematics. Because I have been trained as a mathematician in a rigid axiomatic system, I want to look at how even this system has space for humanity and subjectivity. There is a common notion that “pure mathematics” is truly value neutral, ends-oriented, and standardized because it relies on proofs of mathematical statements. In my experience, this is very much not the case.

Even if there is an underlying very rigid logical structure to things, I think the myth of objectivity in our field causes a loss of creativity and perpetuates a scarcity model in who has the ability to do math. What’s more, treating mathematics as separate from the people who do it causes bad people to continue to be rewarded and recognized in our field. I would like to critique the process of mathematics as objective by interviewing others, share my personal experiences at the intersection of mathematics and being marginalized, and, in addition, provide personal context for every person cited in my project to radicalize the citation process in math.

I would like to analyze four key areas that I see as relevant to launching a critique against objectivity of "pure mathematics": process vs. outcome, ethics of mathematicians in their lives beyond math, relationship to mathematical structures, and institutional structures influence on the doing of mathematics.

I originally planned to incorporate interviews with other mathematicians to analyze both process vs. outcome, and the relationship to mathematical structures. That said, due to the resources and time needed to obtain an IRB approval, I have written a proposal for work that would not require IRB approval. Instead, per my colleague Kay Ohsiek's inspirational example, I would like to incorporate autoethnography to discuss these components and detail my own experiences. While I worry that I would like to be taken less seriously if I only document my own experiences, I do have a cumulative three years of research experience in the domain of "pure mathematics", and the testimonies of many other professional mathematicians that I can cite in my work to back up my analytical claims.

3.7.1 Methodologies and Methods

Amplification For this method, I would like to reach out to a philosophy professor who is a white woman working at the intersection of western scientific ontologies and Indigenous ontologies. She has mentioned that she uses "Amplification" as a methodology to recognize that she is uplifting the voices of others who contributed the original theories she is pulling from. I would like to ask her for good texts and resources to learn about this methodology. I think this is really relevant to what I want to do, since especially women of color in mathematics have been advancing critiques that have informed my existing stance and motivation to do this kind of research work. Essentially, other folks have been doing this, and I want to lift up their voices.

Autoethnography This narrative methodology is informed by the methodology used and adopted by Venus Evans-Winters in her book, and by a desire to bring to light the barriers hidden both systematically and explicitly by "Math".

3.7.2 The four areas

Process vs. Outcome I have been frustrated with a fixation on outcome instead of process, because oftentimes even if an effort fails to prove the result we believe is true, the effort either suggests a more effective proof strategy or is itself a very cool, potentially useful but also potentially just fun and interesting, idea to explore. Research into this component would largely consist of my own experiences in my department, as well as my experiences as a researcher. I would like to use autoethnographic methods to include these narratives, and explain how these have informed my existing critiques in this domain.

Ethics This component describes the tendencancy in our field to see mathematics as separate from the person who produced it. This is self-reinforcing with myths of mathematical genius, without whom we would never have certain contributions, and has famously led to the tolerance of individuals famous for their mathematics who also are Nazis (Teichmuller, or someone who developed the tools in my own field, Friedrich Hirzebruch). Another famous example is Serge Lang, who was an AIDs denier and whose textbooks are still used in our curriculum even though there are a wealth of other texts about the same content that we could be using.

In addition to tolerating atrocious individuals and saving space for them simply due to a scarcity model of mathematics contributions, we leave out narratives of mathematicians who contributed immensely to important social movements. One example is Eugene Lawler, who attended anti-war protests and was even bailed out of jail by famous complexity theorist Richard Karp after protesting the Vietnam War. Another example is Alexander Grothendieck, often hailed as the founder of algebraic geometry, who in addition to drawing cooky categorical diagrams founded an anticaptialist and pro environment legion of French mathematicians and vocally and through his work, protested wars the U.S. was involved with going so far as to move into regions that were being actively bombed while doing his work as a sign of protest.

Because of myths that mathematics is supposed to be all-consuming and that a mathematician has no time to contribute to social movements, the hiding of these narratives is very detrimental to our community and the training of future mathematicians, as well as causing people who are socially-minded to question their place in mathematics and ultimately sometimes leave mathematics because of a lack of attention on these exceptional narratives.

Research in this component would involve simply looking into and documenting the lives of every mathematician cited in my work, including my mentor Tevian Dray and myself. This would be largely biographical and involve a lot of digging through archives to learn about the people who created the mathematics that I use.

Relationships to Mathematical Structures This component is inspired by a class I took in differential geometry, and the relationship I developed to a theoretical construct called differentiable manifolds. I love manifolds, and honestly that's not an understatement. In addition, I have certain images, feelings, and physical embodied knowledge about manifolds. Intuition is very important in mathematics as it guides us to using tools for our proofs; I am curious about how our relationships to structures affects our intuition and how that all comes together to create an environment for the mathematics we do within ourselves. This relationality with completely imaginary things is bizarre, but from conversations with friends everyone does this and in fact, there seem to be differences in how we relate to constructs. For this component of research I would like to pick a few basic structures to have in common that all mathematicians at the graduate or upper-division undergraduate level would be acquainted with, and discuss my own visualizations and feelings toward these structures. Furthermore, I would like to include my artwork, song lyrics, doodles, proof scratch work, and any other personal "scratch work" that helped me arrive at the material for the mathematical side of my thesis. This also overlaps with the Process vs. Outcome component.

This section is informed somewhat by Indigenous concepts of relationality to the environment [27], as well as conversations with friends about their relationship to math, and the overarching and pleasantly surprising emphasis on intuition and embodied knowledge that is found in higher level mathematics.

Institutional Structures This is the final component I want to investigate, and it ultimately would look at institutional aids and barriers, and how they affect the ability to sit down and focus and do math. Some things I have envisioned are having to work through college or graduate school in a different field, as well as facing systemic racism or gender bias. Their are also structures in place to help marginalized people in our department, including a club for women and non-binary mathematicians. For this component, I ultimately want to hear about everyone's experiences getting to the blackboard (or whiteboard, paper, whatever medium they are using to do math), and in addition how structures might affect focus or emotions and how those play out on doing math.

Currently the Association for Women in Mathematics at OSU is running a winter blog series of free-form art centering the experiences of marginalized people. With the club's permission, I would like to include some of these contributions in my research. Another source for testimonies on this part would be from unofficial group(s) of mathematician activists at our university including graduate students, undergraduates, and sympathetic faculty, working toward various activist projects in our department. Activists have written and sent letters to the department on a variety of subjects, including the honors society for undergraduates and the qualifying exam requirement for graduate students. With the permission of testimony contributors and the permission of any wider collectives, I would like to include some of their work in my research analyses. This would be hard to navigate ethically, as I have responsibility to the community who has done this work. My friend Tali Ilkovitch and I have talked about the ethical impacts of doing archival work on behalf of a community working toward systemic change, and I think this is going to be an ongoing conversation if I pursue this work.

The other part of this research would be autobiographical, and include personal barriers I experienced in mathematics, and in my life for which there was inadequate support from my higher ed institution that ultimately impacted my ability to learn and produce mathematics. This would essentially be a dialogue of what was going on while I did this work. Much of this autoethnographic writing would overlap with the author positioning presented earlier in this chapter.

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