TRAVELLING WAVE ANALYSIS OF NONLINEAR MAXWELL MODELS WITH APPLICATIONS TO NONLINEAR OPTICS

by

Noah Langlie

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Vrushali A. Bokil

In 1877 John Kerr described an experiment that demonstrated a quadratic change in refractive index in a plate glass placed in a strong external electric field. This results in a nonlinear relationship between the average electric polarization within the materials and the intensity of the applied electric field. This opened the door for a new area of electromagnetic material science by incorporating nonlinearity into the basic Maxwell system, which in general describes a linear relationship between the electric and magnetic fields. Since then multiple other nonlinear effects have been found in materials that need to be incorporated into Maxwell's equations to accurately model the dynamical evolution of the polarization driven by the electric field.

In this thesis, we explore a model of one linear and two nonlinear effects that are incorporated into the Maxwell's Equations via the macroscopic polarization. This will include a single linear Lorentz dispersion, the nonlinear instantaneous electronic Kerr response as well as the non-instantaneous Raman vibrational response. We will consider one spatial dimension and investigate electromagnetic (EM) wave propagation in these nonlinear materials. To do so, we will include these effects in our constitutive equations for the relationship between the electric field and displacement and reduce our system of partial differential equations (PDEs) into a system of nonlinear ordinary differential equations (ODEs) by assuming traveling wave solutions. Using linear stability analysis from dynamical systems theory allows us to predict behavioral changes in the electric and magnetic fields for an EM traveling wave passing through a material. We will consider the stability of steady states through an eigenvalue analysis of the linearized ODE system and consider the character of arising bifurcations. We have proved that varying the response time parameter of the Lorentz and Raman effects produces a degenerate Hopf bifurcation, and the varying the velocity of our traveling wave solution results in a pitchfork bifurcation. We will also look for changes in behavior arising from a Leapfrog time discretization of the ODE system of the Maxwell Lorentz-Kerr model relative to those in the continuum case, with the predicted stability being preserved in the discrete case under certain conditions.

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Symbol	Parameter Definition	Units
B	Magnetic Induction	T
D	Electric Displacement	C/m^2
E	Electric Field	V/m
H	Magnetic Field	$T*m/H$
β	$\epsilon_s - \epsilon_\infty$	Unitless
Γ	Damping constant in the Lorentz Model	$1/s$
Γ_v	Damping constant in the Raman Model	$1/s$
ϵ_s	Permittivity in the limit of zero frequency	Unitless
ϵ_∞	Permittivity in the limit of infinite frequency	Unitless
ϵ_0	Permittivity of free space	F/m
θ	Relative strength of the Raman and Kerr effects	Unitless
μ_0	Permeability of free space	H/m
Ω	Spatial Boundary	m
ω_0	Resonance Frequency of the Lorentz Oscillator	$1/s$
ω_v	Resonance Frequency of the Raman effect	$1/s$
a	Nonlinear Coupling constant	m^2/V^2

Symbol	Variable Definition	Units
c	Speed of light in a vacuum	m/s
k_0	Difference in the initial conditions of D, and E.	v/m
v	Velocity of the wavefront	m/s
α	Functions of (E, Y) redefined throughout	Unitless
ϕ	Magnitude of the Lorentz Effect	V/m
B	Magnetic Induction	T
D	Electric Displacement	C/m^2
E	Electric Field	V/m
H	Magnetic Field	$T * m/H$
t	Time Variable	s
Q	Magnitude of the Raman effect	V^2/m^2
ξ	traveling wave coordinate	m
X	$Q'(\xi)$	V^2/m^3
Y	$E'(\xi)$	V/m^2
z	One dimensional spatial variable	m

1 INTRODUCTION

In this thesis, we investigate the behavior of travelling wave solutions of nonlinear Maxwell models modeling electromagnetic (EM) wave propagation in a nonlinear optical medium. We consider Maxwell's Equations in one spatial dimension and constitutive equations that model instantaneous and non instantaneous linear and third order responses in the electric polarization. These include the non-instantaneous linear Lorentz response, the nonlinear, instantaneous electronic Kerr response, and the nonlinear, non-instantaneous Raman vibrational response. The nonlinear instantaneous Kerr response models the Kerr effect, in which the refractive index of the medium varies quadratically with the electric field intensity.

After presenting the nonlinear Maxwell model, we use dynamical systems theory to perform a bifurcation analysis to determine both local and global properties of the systems of ordinary differential equations (ODEs) that result under the assumptions of a travelling wave solution. This will allow us to make conclusions about the effects that different responses have under the variations of parameter values, such as the velocity of the travelling wave and the damping constant of the Lorentz model, resulting in a variety of different types of global behavior.

1.1 Maxwell's Equations

Maxwell's Equations are a system of partial differential equations (PDEs) that model the evolution of the EM fields in a material. They are, for a nonconductive, charge free environment, in differential form, Faraday's Law, the Maxwell-Ampere Law, and Gauss's Laws, respectively given by [4],

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{1.1.1a}$$

$$\nabla \times H = \frac{\partial D}{\partial t}, \tag{1.1.1b}$$

$$\nabla \cdot D = 0, \text{ and} \tag{1.1.1c}$$

$$\nabla \cdot B = 0, \tag{1.1.1d}$$

where the B represents the magnetic induction field, H the magnetic field, E the electric field and D the electric displacement field. Maxwell's Equations are incomplete, and must be completed by adding *Constitutive Laws* that encode the response of the material to the EM field. In a linear medium we have

$$D = \epsilon E \tag{1.1.2a}$$

$$B = \mu H \tag{1.1.2b}$$

Where ϵ is the permitivity, and μ is the permeability in the material. Therefore, if we assume these conditions, we can reduce the Maxwell System to depend only on the magnetic induction and electric fields:

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{1.1.3a}$$

$$\nabla \times B = \epsilon\mu\frac{\partial E}{\partial t}, \tag{1.1.3b}$$

$$\nabla \cdot E = 0, \tag{1.1.3c}$$

$$\nabla \cdot B = 0. \tag{1.1.3d}$$

We can show that this set of equations is equivalent to the electromagnetic wave equation, which is known to have traveling wave solutions. By taking the curl of Faraday's Law and the Maxwell-Ampere Law, yielding

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}(\nabla \times B) = -\epsilon\mu \frac{\partial^2 E}{\partial t^2}, \tag{1.1.4a}$$

$$\nabla \times (\nabla \times B) = \epsilon\mu \frac{\partial}{\partial t}(\nabla \times E) = -\epsilon\mu \frac{\partial^2 B}{\partial t^2}. \tag{1.1.4b}$$

We can then use the vector identity

$$\nabla \times (\nabla \times V) = \nabla(\nabla \cdot V) - \nabla \cdot (\nabla V). \tag{1.1.5}$$

Therefore, by Gauss's Laws,

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla \cdot (\nabla E) = -\nabla \cdot (\nabla E), \tag{1.1.6a}$$

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla \cdot (\nabla B) = -\nabla \cdot (\nabla B). \tag{1.1.6b}$$

Combining this with our previous result leads us to conclude that

$$\nabla^2 E = -\mu\epsilon \frac{\partial^2 E}{\partial t^2}, \tag{1.1.7a}$$

$$\nabla^2 B = -\mu\epsilon \frac{\partial^2 B}{\partial t^2}, \tag{1.1.7b}$$

which is the second order wave equation. Since we assumed a linear relationship in the medium, the same equations also apply to the D field and the H field. Note that this result is independent of the number of spatial dimensions, a fact which we will use to justify our traveling wave assumption in a single spatial dimension. When we consider Maxwell's Laws in one spatial dimension, we need to decide what we mean by curl, and divergence. The latter trivially reduces, but curl is an operation that must be computed in a minimum of three dimensions. For this thesis we will define the one dimensional curl as

$$\nabla \times V = \frac{\partial V}{\partial z} \tag{1.1.8}$$

With this definition for our 'scalar cross product,' we can define Faraday's Law and the Maxwell-Ampere Law in a charge free, nonconductive, uniform material, with one spatial dimension, as

$$\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}, \tag{1.1.9a}$$

$$\frac{\partial H}{\partial z} = \frac{\partial D}{\partial t}. \tag{1.1.9b}$$

A common assumption in nonlinear optics is a linear relationship between the magnetic induction and the magnetic field, with the permeability being that of free space which is a common assumption in nonlinear optics [11]:

$$B = \mu_0 H. \tag{1.1.10}$$

We will also make the assumptions necessary for the form of Maxwell's equations specified above. This leaves the constitutive equation between the electric displacement and the electric field for us to introduce nonlinearities to account for several physical effects. This will come in the form of polarization P, where

$$D = \epsilon_0 (E + P) \tag{1.1.11}$$

where polarization represents the average amount of the material that forms a dipole moment in the presence of a strong electric field.

1.1.1 The Lorentz Model for Polarization

The simplest correction to the instantaneous linear model is to add the retarded linear Lorentz effect. This models an oscillator in the medium with a specific resonance frequency and response time that depends on the responsiveness of the medium. Frequently in this field, multiple independent oscillator are used to better approximate data with various response times and resonating frequencies. However for simplicity we will only consider a single Lorentz oscillator in this report.

1.1.2 The Kerr Effect

In 1877, John Kerr performed an experiment demonstrating a quadratic relation between a strong electric field intensity and the refractive index of a material [12]. To do this, he used Nicols and a generated electric field to measure the nonlinear relationship between the polarization and the field. In the case of lower energy or intensity, this effect is negligible due to its quadratic nature. However, for optical pulses at higher intensity we see this effect more apparently with near instantaneous response time relative to the retarded effects. As such in this report we will consider a model of the Kerr Effect (also known as the Quadratic Electro-Optical Effect) as cubic term in the polarization.

1.1.3 The Raman Effect

The Raman Effect is a very famous and important effect in various fields of science. It is used in chemistry to identify compounds through Raman Spectroscopy. It is a vibrational response to an EM wave of interacting photons which result in virtual energy states that absorb and emit photons of specific energies. This can be used to identify chemical compounds through precise use of monochromatic lasers to create a narrow frequency of light and detectors with a fine tolerance to determine the frequency emitted. The Raman Effect is also known as Raman scattering and is a fundamental phenomenon in particle physics described as the inelastic scattering of two electrons interacting via a virtual photon. This effect is important because the polarization of the material is dependent on the magnitude of the Raman Effect.This operates in smaller time scales than the Kerr Effect, owing to delayed emission of photons when the molecules are in excited states. As such an oscillator will be needed to model the response time.

1.2 Traveling Waves

We will note that for the following sections we will use subscript notation for partial derivative for compactness of the form

$$\frac{\partial f}{\partial x} = f_x. \tag{1.2.1}$$

We look for traveling wave solutions which reduce our system of PDEs into ODEs. Since we are dealing with EM waves a common and reasonable assumption is to consider traveling wave solutions with a velocity that is constant in the material. As such, we will assume a one dimensional traveling wave solution in all field variables. This means that for all different functions dependent on z, our one dimensional spatial variable, and t, our time variable, we can take v, the velocity of the wavefront, and make the substitution $\xi = z - vt$ where ξ is our wave variable ξ. This results in our two dimensional functions dependent on both space and time becoming dependent on a single variable. Thus, for all functions f, we have

$$f(z,t) = f(z - vt) = f(\xi). \tag{1.2.2}$$

This has further implications for the differential equations since, by the chain rule,

$$f' = \frac{df}{d\xi} = f_t \frac{\partial t}{\partial \xi} = f_z \frac{\partial z}{\partial \xi}.$$

Therefore,

$$f' = -\frac{1}{v} f_t = f_z. \tag{1.2.3}$$

We will show in the next chapter that the traveling wave assumption reduces our system of PDEs into a system of ODEs. Using the theory of dynamical systems we will analyze the systems of ODEs for local and global properties.

1.3 Constitutive Laws for Lorentz, Kerr and Raman Effects

We now consider our Maxwell system under the linear relationship (1.1.10) to remove the H field with result

$$\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}, \tag{1.3.1a}$$

$$\frac{1}{\mu_0}\frac{\partial B}{\partial z} = \frac{\partial D}{\partial t}. \tag{1.3.1b}$$

Therefore, if we assume a traveling solution of velocity v, get the derivatives in terms of time and equate the two. We get the relationship between the electric field and displacement of

$$D_t = -\frac{1}{v\mu_0}E_z. \tag{1.3.2}$$

Assuming a traveling wave solution, as outlined in the next section, we can apply (1.2.3) to get both derivatives in terms of the same variable. Thus,

$$v^2\mu_0 D_t = E_t. \tag{1.3.3}$$

Using this relationship in later derivations, we will be able eliminate D or E since we will have a second relationship between D and E via the polarization relation. In a linear dielectric medium there is a linear relationship between the E-field and the D-field. This follows if we consider a field P that represents the amount of polarization in the medium, such that

$$D = \epsilon_0 P. \tag{1.3.4}$$

In linear dielectric medium we define the constant of proportionality as the electric susceptiblity χ_e, meaning that

$$P = \chi_e E. \tag{1.3.5}$$

As we take the limit as the frequency of the wave goes to infinity we see we can define the corresponding permitivity as

$$\epsilon_\infty = \chi_e \epsilon_0, \tag{1.3.6}$$

and then substitute (1.3.5) into (1.3.4),

$$D = \epsilon_\infty E, \tag{1.3.7}$$

which displays that the electric field is proportional to the electric displacement in a linear medium. In a nonlinear material we have

$$D = \epsilon_0 (P^{\text{Linear}} + P^{\text{Nonlinear}}). \tag{1.3.8}$$

Furthermore, we will consider both instantaneous and residual effects [3] meaning that,

$$P^{\text{Linear}} = \epsilon_\infty E + \phi, \tag{1.3.9a}$$

$$P^{\text{Nonlinear}} = a(1-\theta)E|E|^2 + a\theta Q E, \tag{1.3.9b}$$

ϕ is the lorentz oscillator, Q is the Raman oscillator, a is the third order coupling constant, θ is the relative strength of instantaneous electronic Kerr and residual Raman effects. The first term in each definition represents the instananeous response, while the second contains the residual responses. In this thesis we will consider a single Lorentz pole for the linear residual effect, the Kerr effect for the nonlinear instantaneous response, and the Raman Molecular Vibratational Response for the nonlinear residual response. By ϕ and Q we will denote functions that correspond to the Lorentz Pole and Raman effect, respectively. We will define Γ to be the reciprocal of the Lorentz response time, and Γ_v to be the reciprocal of the Raman response time. As such we have Lorentz and Raman oscillators that model the evolution of ϕ and Q as

$$\phi_{tt} + \Gamma\phi_t + {\omega_0}^2\phi = \beta{\omega_0}^2 E, \tag{1.3.10a}$$

$$Q_{tt} + \Gamma_v Q_t + {\omega_v}^2 Q = {\omega_v}^2 E^2. \tag{1.3.10b}$$

In the above ω_0 is the resonance frequency of the Lorentz oscillator, ω_v is the corresponding resonance frequency of the Raman oscillator, and β is the difference between the permitivities at zero and infinite frequencies. This models the delay in response in both the Raman and Lorentz poles as they take effect by coupling these oscillators to our equation for electric displacement. Since our Lorentz model for ϕ is linear, ϕ will be proportional to the polarization it induces. The Raman effect will have an additional term E term to take into account nonlinearities in its equation with similar coefficients to the Kerr effect. Therefore, our model, given these assumptions, is the following system of PDEs.

$$E_z = -B_t, \tag{1.3.11a}$$

$$B_z = \mu_0 D_t \tag{1.3.11b}$$

$$\phi_{tt} + \Gamma\phi_t + {\omega_0}^2\phi = \beta{\omega_0}^2 E, \tag{1.3.11c}$$

$$Q_{tt} + \Gamma_v Q_t + {\omega_v}^2 Q = {\omega_v}^2 E^2, \tag{1.3.11d}$$

$$D = \epsilon_0(\epsilon_\infty E + \phi + a(1-\theta)E^3 + a\theta Q E). \tag{1.3.11e}$$

1.4 Energy Identities

In [2] it was found that an energy analysis can be done on this model with the defininitywith the conclusion that for $\theta \in [0, \frac{3}{4}]$, and an assumption of periodic boundary conditions on the spatial region Ω for all fields it follows that the rate of change of the energy in the system ε_θ can be computed as

$$\frac{d}{dt}\varepsilon_\theta = -\int_\Omega \left(\frac{\epsilon_0 \Gamma}{\beta {\omega_0}^2}(\phi_t)^2 + \frac{a\theta\epsilon_0\Gamma_v}{2{\omega_v}^2}(Q_t)^2 \right) dz. \tag{1.4.1}$$

where

$$\begin{aligned} \varepsilon_\theta(t) := \int_\Omega \Bigg(&\mu_0 H^2 + \epsilon_0\epsilon_\infty E^2 + \frac{\epsilon_0}{\epsilon_s - \epsilon_\infty}\phi^2 + \frac{\epsilon_0}{\omega_0^2}\phi_t^2 + \frac{a\theta\epsilon_0}{2\omega_v^2}Q_t^2 \\ &+ \frac{a\theta\epsilon_0}{2}(Q + E^2)^2 + \frac{a\epsilon_0}{2}(3 - 4\theta)E^4 \Bigg) dz \end{aligned} \tag{1.4.2}$$

For physical reasons, all parameters must be non-negative, for reason of which we expect that the energy in the system will be conserved or decay over time. In the limit as the dampening terms of the Lorentz and Raman terms go to zero (i.e., $\Gamma = \Gamma_v = 0$), the energy is conserved, which is consistent with energy loss being dependent on the retarding effects. Outside of the limit, the rate of change of our energy will be negative and dependent on the magnitude of these terms. In order to preserve this identity, will assume that θ falls within $[0, \frac{3}{4}]$, which includes the case where the Raman effect is neglected ($\theta = 0$). Due to the phenomenological nature of these models, we can also slightly modify these equations in order to get results that more closely matches our data. One such method is to introduce multiple Lorentz poles into the constitutive laws in which each pole has a different frequency and driving constant. However, for this thesis we will consider only one pole.

1.5 Nonlinear Schrödinger Equation

An alternative model that is frequently used to model an optical Kerr medium is the nonlinear Schrödinger equation [14]

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + f(u) = 0, \tag{1.5.1}$$

where $f(u)$ is the part of equation containing nonlinearities we wish to model. For the Kerr effect, a model for f is

$$f(u) = a|u|^2 u. \tag{1.5.2}$$

Frequently this model is used for its simplicity, and its effectiveness at an approximation to the Maxwell system. However, this equation is still an approximation to the Maxwell's equations used in this thesis, and as such may not preserve all the behavior of the Maxwell models.

2 QUALITATIVE BEHAVIOR OF THE MAXWELL-LORENTZ SYSTEM WITH KERR EFFECT

2.1 Bifurcation Analysis of Maxwell's Equations

In this section we will reduce our model into a system of ODEs to preform a bifurcation analysis. This entails finding the equilibrium points, computing the Jacobian of the system at these points, and computing the associated eigenvalues. This will tell us if the points are stable, unstable, or something more complex. This will undoubtedly be in terms of the parameters, and by examining how changes in the parameters change the behavior of our system, we can infer the physical implications of the parameters. We will divide the analysis into sub-cases each neglecting certain parameters in order to elucidate the effect of each term of the model.

2.1.1 Undamped Lorentz Model with Kerr Effect

The simplest case we will consider is to neglect the Raman effect as well as the dampening in the Lorentz Pole ($\theta = 0, \Gamma = 0$). This reduces (1.3.11) into three equations: our Maxwell Relation, the constitutive equation for the electric polarization and the Lorentz oscillator:

$$E_t = v^2 \mu_0 D_t, \tag{2.1.1a}$$

$$D = \epsilon_0(\epsilon_\infty E + \phi + aE^3), \tag{2.1.1b}$$

$$\phi_{tt} + {\omega_0}^2 \phi = \beta {\omega_0}^2 E. \tag{2.1.1c}$$

Assuming a traveling wave solution on functions E, D, and ϕ, i.e., assuming

$$f(z,t) = f(z - vt) = f(\xi) \text{ with } \xi := z - vt,$$

we can reduce our system of PDEs into a planar system of two ODEs. Since we are assuming that $E(z,t) = E(\xi)$ we have, by the chain rule,

$$\frac{\partial E}{\partial z} = \frac{\partial E}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi} := Y. \tag{2.1.2}$$

Rearranging this equation yields

$$Y = -\frac{1}{v}E_t. \tag{2.1.3}$$

Repeating this process gives

$$\frac{dY}{d\xi} = \frac{1}{v^2}E_{tt}. \tag{2.1.4}$$

The next step is to solve for E_{tt} in terms of E and Y. If we differentiate (2.1.1a) we get the relationship

$$E_{tt} = \mu_0 v^2 D_{tt}. \tag{2.1.5}$$

We then substitute this into (2.1.4), yielding

$$\frac{dY}{d\xi} = \mu_0 D_{tt}. \tag{2.1.6}$$

To get a second equation involving D_{tt} in terms of E and Y we take a time derivative of (2.1.1b) which, by product rule, produces

$$D_t = \epsilon_0(\epsilon_\infty E_t + \phi_t + 3aE^2 E_t) = \epsilon_0 \phi_t + \epsilon_0(\epsilon_\infty + 3aE^2)E_t. \tag{2.1.7}$$

Substuting in (2.1.1a) to change the E_t term to a D_t term and grouping together D_t terms yields the equation

$$D_t = \frac{\epsilon_0 \phi_t}{1 - \epsilon_0 \mu_0 v^2(\epsilon_\infty + 3aE^2)}. \tag{2.1.8}$$

By taking a time derivative and applying the quotient rule, this equations becomes

$$D_{tt} = \epsilon_0 \frac{(\phi_{tt})(1-\epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2)) - (\phi_t)(6a\epsilon_0\mu_0 v^2(EE_t))}{(1-\epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2}. \tag{2.1.9}$$

We now need to consider how factor out the ϕ terms. Beginning by integrating (2.1.1a) and assuming that the initial conditions of D and E are identical, it follows that

$$E = \mu_0 v^2 D. \tag{2.1.10}$$

Substituting in our equation (2.1.1b),

$$E = \epsilon_0\mu_0 v^2(\epsilon_\infty E + \phi + aE^3).$$

Solved for ϕ,

$$\phi = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2\right)E. \tag{2.1.11}$$

This equation also gives us enough information to find ϕ_{tt} in terms of E since, if we take (2.1.1c) and get ϕ_{tt} alone, we have

$$\phi_{tt} = {\omega_0}^2(\beta E - \phi).$$

If we then substitute in (2.1.11),

$$\phi_{tt} = {\omega_0}^2\left(\beta - \frac{1}{\epsilon_0\mu_0 v^2} + \epsilon_\infty + aE^2\right)E. \tag{2.1.12}$$

Now we differentiate (2.1.11) to get

$$\phi_t = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aE^2\right)E_t. \tag{2.1.13}$$

Therefore if we take (2.1.9) and substitute in (2.1.3) to remove E_t, (2.1.13) to remove ϕ_t and (2.1.12) to remove ϕ_{tt}, we can remove the undesired functions:

$$\begin{aligned} D_{tt} = \epsilon_0 \Bigg(& \frac{({\omega_0}^2(\beta - \frac{1}{\epsilon_0\mu_0 v^2} + \epsilon_\infty + aE^2)E)(1-\epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))}{(1-\epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2} \\ & - \frac{((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty + 3avEY)E)(6a\epsilon_0\mu_0 v^2(-vEY))}{(1-\epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2}\Bigg). \end{aligned} \tag{2.1.14}$$

Simplified,

$$D_{tt} = \frac{1}{\mu_0}\frac{6aE^2Y - {\omega_0}^2(\frac{c^2}{v^2} - \epsilon_s)E - aE^3}{c^2 - \epsilon_\infty c^2 - 3av^2E^2}. \tag{2.1.15}$$

Therefore, from (2.1.6) and the above result, we finally have the equation

$$\frac{dY}{d\xi} = \frac{6aE^2Y - {\omega_0}^2(\frac{c^2}{v^2} - \epsilon_s)E - aE^3}{c^2 - \epsilon_\infty c^2 - 3av^2E^2}. \tag{2.1.16}$$

We will then make the change of variables as follows[10],

$$z = \frac{c}{\omega_0\sqrt{\epsilon_\infty}}\tilde{z}, \tag{2.1.17a}$$

$$t = \frac{1}{\omega_0}\tilde{t}, \tag{2.1.17b}$$

$$\xi = \frac{c}{\omega_0\sqrt{\epsilon_\infty}}\tilde{\xi}, \tag{2.1.17c}$$

$$v = \frac{c}{\sqrt{\epsilon_\infty}}\tilde{v}, \tag{2.1.17d}$$

$$E = \sqrt{\frac{\epsilon_\infty}{3a}}\tilde{E}, \tag{2.1.17e}$$

$$Y = \frac{\epsilon_\infty\omega_0}{c\sqrt{3a}}\tilde{Y}. \tag{2.1.17f}$$

Note that omit the tilde for simplicity. This yields the following system of ODEs.

$$\frac{dE}{d\xi} = Y, \tag{2.1.18a}$$

$$\frac{dY}{d\xi} = \frac{2v^2EY^2 - (v^{-2} - \epsilon_s/\epsilon_\infty)E + E^3/3}{1 - v^2 - v^2E^2}. \tag{2.1.18b}$$

We will let (E_∞, Y_∞) denote the equilibria points, which are defined by

$$\left.\frac{dE}{d\xi}\right|_{(E_\infty, Y_\infty)} = 0, \tag{2.1.19a}$$

$$\left.\frac{dY}{d\xi}\right|_{(E_\infty, Y_\infty)} = 0. \tag{2.1.19b}$$

It follows from the first equation that $Y_\infty = 0$. Therefore, for the second,

$$0 = -(v^{-2} - \epsilon_s/\epsilon_\infty)E_\infty + {E_\infty}^3/3.$$

This polynomial has three solutions each representing a different equilibrium of the system with positions

$$(E_\infty, Y_\infty) = (0,0) \text{ and } (E_\infty, Y_\infty) = \left(\pm\sqrt{3\left(\frac{1-v^2}{v^2} - \frac{\beta}{\epsilon_\infty}\right)}, 0 \right). \tag{2.1.20}$$

In [11] it was found that the zero equilibrium of this system is nonhyperbolic, and for $v \in \left(\sqrt{\frac{2\epsilon_\infty}{3\epsilon_s - \epsilon_\infty}}, \sqrt{\frac{\epsilon_\infty}{\epsilon_s}}\right)$ it was found that the nonzero equilibria are saddle nodes connected by a heteroclinic connection call kink-antikink solutions. Thus, this model exhibits traveling wave solutions in the form of periodic orbits and heteroclinic cycles. See Figure 2.5. If we then consider velocity outside this range, the three equilibria coalesce into a single saddle at the origin. This is consistant with our energy analysis since at steady state either centers or saddle will occur. If we then consider velocity outside this range, the three equilibria coalesce into a single saddle at the origin. This is consistent with our energy analysis since at steady state either orbits or saddles will occur.

2.1.2 Dampened Lorentz Model with Kerr Effect

We now expand our scope to consider nonzero dampening in the Lorentz pole. This means that only the Raman effect is neglected ($\theta = 0$), leaving us with the system of PDEs:

$$E_t = v^2 \mu_0 D_t, \tag{2.1.21a}$$

$$D = \epsilon_0(\epsilon_\infty E + \phi + aE^3), \tag{2.1.21b}$$

$$\phi_{tt} + \Gamma\phi_t + {\omega_0}^2\phi = \beta{\omega_0}^2 E. \tag{2.1.21c}$$

Assuming a traveling wave solution on functions E, D, and ϕ, that is., assuming

$$f(z,t) = f(z - vt) = f(\xi) \text{ with } \xi := z - vt,$$

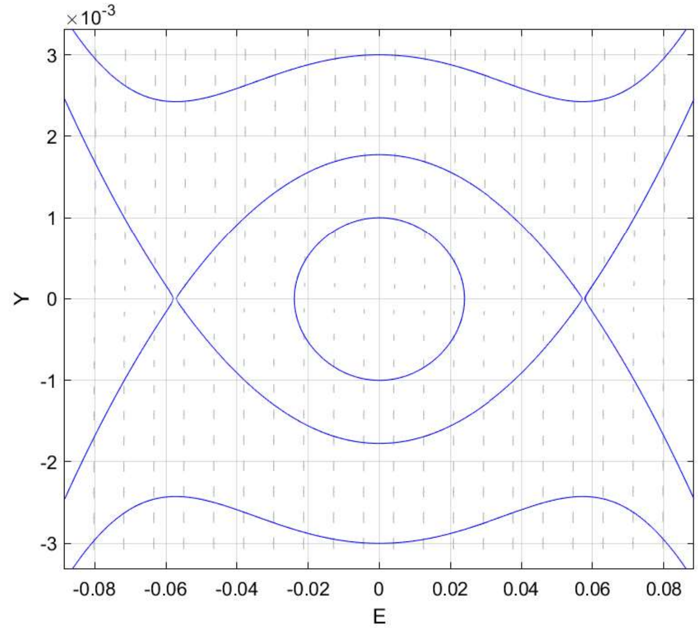

FIGURE 2.1: Phase Portrait of Undamped Lorentz-Kerr System showing solution curves generated in PPlane with $v = .6545, \epsilon = 7/3$

we can reduce our system of PDEs into a planar system of two ODEs as follows. Since we are assuming that $E(z,t) = E(\xi)$ we have, by the chain rule,

$$\frac{\partial E}{\partial z} = \frac{\partial E}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi} := Y. \tag{2.1.22}$$

Rearranging this equation yields

$$Y = -\frac{1}{v}E_t. \tag{2.1.23}$$

Repeating this process gives

$$\frac{\partial Y}{\partial z} = \frac{\partial Y}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial Y}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial Y}{\partial \xi}. \tag{2.1.24}$$

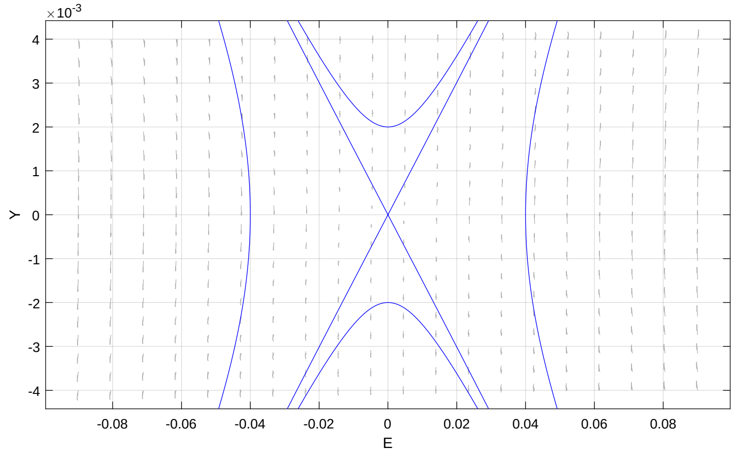

FIGURE 2.2: Phase Portrait of Undamped Lorentz-Kerr System showing solution curves generated in PPlane with $v = .6545, \epsilon = 2.4$

Therefore,

$$\frac{dY}{d\xi} = \left(\frac{-1}{v}\right) Y_t = \frac{1}{v^2} E_{tt}. \tag{2.1.25}$$

We now need to solve for E_{tt} in terms of E and Y. If we differentiate (2.1.21a) and then substitute into (2.1.25) we have

$$\frac{dY}{d\xi} = \mu_0 D_{tt}. \tag{2.1.26}$$

To get a second equation involving D_{tt} in terms of E and Y we take a time derivative of (2.1.21b), which produces

$$D_t = \epsilon_0(\epsilon_\infty E_t + \phi_t + 3aE^2 E_t) = \epsilon_0 \phi_t + \epsilon_0(\epsilon_\infty + 3aE^2)E_t.$$

Substuting in (2.1.21a) to change the E_t term to a D_t term and grouping together D_t terms yields the equation

$$D_t = \frac{\epsilon_0 \phi_t}{1 - \epsilon_0 \mu_0 v^2(\epsilon_\infty + 3aE^2)}. \tag{2.1.27}$$

The time derivative of this equation is, by quotient rule,

$$D_{tt} = \epsilon_0 \frac{(\phi_{tt})(1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2)) + (\phi_t)(6a\epsilon_0\mu_0 v^2(EE_t))}{(1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2}. \tag{2.1.28}$$

We now need to remove the ϕ terms. Beginning by integrating (2.1.21a),

$$E - E_0 = \mu_0 v^2 (D - D_0), \tag{2.1.29}$$

and then assuming appropriate conditions for D and E, it follows that

$$E = \mu_0 v^2 D. \tag{2.1.30}$$

Substituting in (2.1.21b) gives

$$E = \epsilon_0\mu_0 v^2(\epsilon_\infty E + \phi + aE^3).$$

Solved for ϕ,

$$\phi = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2\right) E. \tag{2.1.31}$$

The time derivative of this equation is

$$\phi_t = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aE^2\right) E_t. \tag{2.1.32}$$

From (2.1.21c) we can solve ϕ_{tt}:

$$\phi_{tt} = {\omega_0}^2(\beta E - \phi) - \Gamma\phi_t.$$

Hence, by (2.1.31) and (2.1.32),

$$\phi_{tt} = {\omega_0}^2 \left(\beta - \frac{1}{\epsilon_0\mu_0 v^2} + \epsilon_\infty + aE^2\right) E - \Gamma \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aE^2\right) E_t. \tag{2.1.33}$$

Now, plugging (2.1.32), and (2.1.33) into (2.1.28) produces

$$D_{tt} = \epsilon_0 \frac{({\omega_0}^2(-\frac{1}{\epsilon_0\mu_0 v^2} + \epsilon_s + aE^2)E - \Gamma(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aE^2)E_t)(1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))}{(1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2} + \frac{((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aE^2)E_t)(6a\epsilon_0\mu_0 v^2(EE_t))}{(1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3aE^2))^2}. \tag{2.1.34}$$

Simplified,

$$D_{tt} = \epsilon_0 (c^2 \frac{6av^2EY^2 - {\omega_0}^2(\frac{c^2}{v^2} - \epsilon_s)E + a\omega_0^2 E^3}{c^2 - \epsilon_\infty v^2 - 3av^2E^2} + \Gamma c^2 \frac{Y}{v}).$$

And, by (2.1.25),

$$\frac{dY}{d\xi} = \frac{6av^2EY^2 - {\omega_0}^2[(\frac{c^2}{v^2} - \epsilon_s)E - aE^3]}{c^2 - \epsilon_\infty v^2 - 3av^2E^2} + \Gamma \frac{Y}{v}. \tag{2.1.35}$$

For which if we then repeat the change of variables:

$$\xi = \frac{c}{\omega_0 \sqrt{\epsilon_\infty}} \tilde{\xi}, \tag{2.1.36a}$$

$$\Gamma = \frac{\omega_0}{\sqrt{3a}} \tilde{\Gamma}, \tag{2.1.36b}$$

$$v = \frac{c}{\sqrt{\epsilon_\infty}} \tilde{v}, \tag{2.1.36c}$$

$$Y = \frac{\sqrt{\epsilon_\infty}\omega_0}{c\sqrt{3a}} \tilde{Y}, \tag{2.1.36d}$$

$$E = \sqrt{\frac{\epsilon_\infty}{3a}} \tilde{E}, \tag{2.1.36e}$$

with the result that, after dropping tildes, we get the nondimensional system.

$$\frac{dE}{d\xi} = Y, \tag{2.1.37a}$$

$$\frac{dY}{d\xi} = \frac{2v^2EY^2 - (v^{-2} - \epsilon_s/\epsilon_\infty)E + E^3/3}{1 - v^2 - v^2E^2} + \Gamma \frac{Y}{v}. \tag{2.1.37b}$$

Notice the odd, but consistent implication that,

$$\frac{dY}{d\xi}_{\Gamma \neq 0} = \frac{dY}{d\xi}_{\Gamma = 0} + \Gamma \frac{Y}{v}. \tag{2.1.38}$$

Again, we will omit the tilde for simplicity. This permits a bifurcation analysis of the parameter Γ, beginning by determining if the equilibria of the system move from the undamped case and, if so, how. To find the nullclines we will make the above substitution and solve directly. Clearly, the E nullcline remains solely along the $Y = 0$ axis; it is the Y nullclines that concern us. Solving,

$$0 = \left.\frac{dY}{d\xi}\right|_{\Gamma\neq 0, E_\infty, Y_\infty=0} = \left.\frac{dY}{d\xi}\right|_{\Gamma=0, E_\infty, Y_\infty=0}$$
$$= \frac{-(v^{-2} - \epsilon_s/\epsilon_\infty)E_\infty + {E_\infty}^3/3}{1 - v^2 - v^2 {E_\infty}^2}.$$

Therefore,

$$0 = (v^{-2} - \epsilon_s/\epsilon_\infty)E_\infty - {E_\infty}^3/3.$$

This equation is the same as the undamped case, so we are left with the same equilibria points. We now move to a linearization and stability analysis of the zero equilibrium. To begin we find the terms in the Jacobian matrix computed at the origin:

$$\left.\frac{\partial}{\partial E}\frac{dE}{d\xi}\right|_{E=Y=0} = 1, \tag{2.1.39a}$$
$$\left.\frac{\partial}{\partial Y}\frac{dE}{d\xi}\right|_{E=Y=0} = 0, \tag{2.1.39b}$$
$$\left.\frac{\partial}{\partial E}\frac{dY}{d\xi}\right|_{E=Y=0} = \frac{\frac{1}{v^2} - \frac{\epsilon_s}{\epsilon_\infty}}{1 - v^2}, \tag{2.1.39c}$$
$$\left.\frac{\partial}{\partial Y}\frac{dY}{d\xi}\right|_{E=Y=0} = \Gamma\frac{1}{v}. \tag{2.1.39d}$$

For the purposes of bifurcation analysis we will label the third term δ and the last term γ, which is merely Γ scaled.

$$J_\Gamma(0,0) = \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix}$$

Therefore the characteristic equation of the Jacobin for eigenvalue λ is:

$$(\gamma - \lambda)(0 - \lambda) - \delta = \lambda^2 - \gamma\lambda - \delta = 0.$$

Computation of eigenvalues reveals

$$\lambda = \frac{\gamma \pm \sqrt{\gamma^2 + 4\delta}}{2},$$

or, equivalently,

$$\lambda = \frac{\Gamma\frac{1}{v} \pm \sqrt{\left(\Gamma\frac{1}{v}\right)^2 + 4\frac{\frac{1}{v^2} - \frac{\epsilon_s}{\epsilon_\infty}}{1 - v^2}}}{2}.$$

We also wish to consider the nonzero equilibria with appropriate Jacobian derivatives. Computation reveals

$$\frac{\partial}{\partial E}\frac{dE}{d\xi}\bigg|_{E=\pm\sqrt{3(\frac{1-v^2}{v^2} - \frac{\beta}{\epsilon_\infty})}, Y=0} = 1, \tag{2.1.40a}$$

$$\frac{\partial}{\partial Y}\frac{dE}{d\xi}\bigg|_{E=\pm\sqrt{3(\frac{1-v^2}{v^2} - \frac{\beta}{\epsilon_\infty})}, Y=0} = 0, \tag{2.1.40b}$$

$$\frac{\partial}{\partial E}\frac{dY}{d\xi}\bigg|_{E=\pm\sqrt{3(\frac{1-v^2}{v^2} - \frac{\beta}{\epsilon_\infty})}, Y=0} = \frac{2(v^{-2} - \epsilon_s/\epsilon_\infty)}{1 - v^2 - 3v^2(v^{-2} - \epsilon_s/\epsilon_\infty)}, \tag{2.1.40c}$$

$$\frac{\partial}{\partial Y}\frac{dY}{d\xi}\bigg|_{E=\pm\sqrt{3(\frac{1-v^2}{v^2} - \frac{\beta}{\epsilon_\infty})}, Y=0} = \frac{\Gamma}{v}. \tag{2.1.40d}$$

For the purposes of bifurcation analysis we will again label the third term be δ and the last term γ, the latter of which merely is Γ scaled.

$$J_\Gamma(0,0) = \begin{pmatrix} 0 & 1 \\ \gamma & \delta \end{pmatrix}$$

Therefore the characteristic equation of the Jacobin for eigenvalue λ is:

$$(\gamma - \lambda)(0 - \lambda) - \delta = \lambda^2 - \gamma\lambda - \delta = 0$$

and computation reveals the eigenvalues to be:

$$\lambda = \frac{\gamma \pm \sqrt{\gamma^2 + 4\delta}}{2}.$$

Reinputting our original variables yields the eigenvalues

$$\lambda = \frac{\frac{\Gamma}{v} \pm \sqrt{\left(\frac{\Gamma}{v}\right)^2 + 4\frac{2(v^{-2} - \epsilon_s/\epsilon_\infty)}{1 - v^2 - 3v^2(v^{-2} - \epsilon_s/\epsilon_\infty)}}}{2}.$$

These eigenvalues allow us to do a Bifurcation Analysis on all relevant parameters: $(\Gamma, v, \epsilon_s, \epsilon_\infty)$ more easily effort if we group together their products to give the new set of parameters Γ, v, ϵ where

$$\epsilon = \frac{\epsilon_s}{\epsilon_\infty}.$$

In summary, with this new trio of parameters the zero equilibrium eigenvalues can we rewritten as

$$\lambda = \frac{\frac{\Gamma}{v} \pm \sqrt{(\frac{\Gamma}{v})^2 + 4\frac{v^{-2} - \epsilon}{1 - v^2}}}{2},$$

and nonzero equilibria eigenvalues as

$$\lambda = \frac{\frac{\Gamma}{v} \pm \sqrt{(\frac{\Gamma}{v})^2 + 8\frac{(v^{-2} - \epsilon)}{((3\epsilon - 1)v^2 - 2)}}}{2}.$$

With respect to physical meaning and scaling factors we have limited the ranges of these parameters to $\Gamma \in [0, \infty), v \in (0, \sqrt{\epsilon_\infty})$, and $\epsilon \in (1, \infty)$. Note that ϵ_∞ could potentially take any positive value. We now consider regions of various behavior for our parameters. Note that the nonzero equilibria only have a real position if we assume both $\epsilon - v^{-2} > 0$ and $v \neq 1$.

Zero Equilibrium

Case 1 $\Gamma = 0$

For this case our eigenvalues reduce to:

$$\lambda = \pm\sqrt{\frac{v^{-2} - \epsilon}{1 - v^2}}$$

Subcases

($v^{-2} < \epsilon$ and $v < 1$) or ($v^{-2} > \epsilon$ and $v > 1$)

For this case we have a pair of real eigenvalues one positive and one negative resulting in a saddle node.

($v^{-2} \geq \epsilon$ and $v < 1$) or ($v^{-2} \leq \epsilon$ and $v < 1$)

For this case we have a pair of eigenvalues with zero real parts resulting in a non-hyperbolic equilibrium.

Case 2 $0 < \Gamma < 2v\sqrt{\frac{\epsilon - v^{-2}}{1-v^2}}$ (Assuming bound is real)

Since we assumed the upper bound is real we only have two subcases, both which have the same behavior.

Subcases

($v^{-2} < \epsilon$) and ($1 > v$)

($v^{-2} > \epsilon$) and ($1 < v$)

This results in a pair of eigenvalues we positive real parts and nonzero imaginary parts, indicating an unstable spiral.

Case 3 $\Gamma > 2v\sqrt{\frac{v^{-2} - \epsilon}{1-v^2}}$ (Assuming bound is real) Since we assumed the lower bound is real we only have two subcases.

Subcases

($v^{-2} < \epsilon$) and ($1 > v$)

($v^{-2} > \epsilon$) and ($1 < v$)

Both eigenvalues are purely real and positive resulting in an unstable source. **Case 4** $\Gamma > 0$ (Assuming previous bound is imaginary) **Subcases**

$v^{-2} = \epsilon$

Results in one purely real positve eigenvalues and one zero eigenvalues, yielding a non-

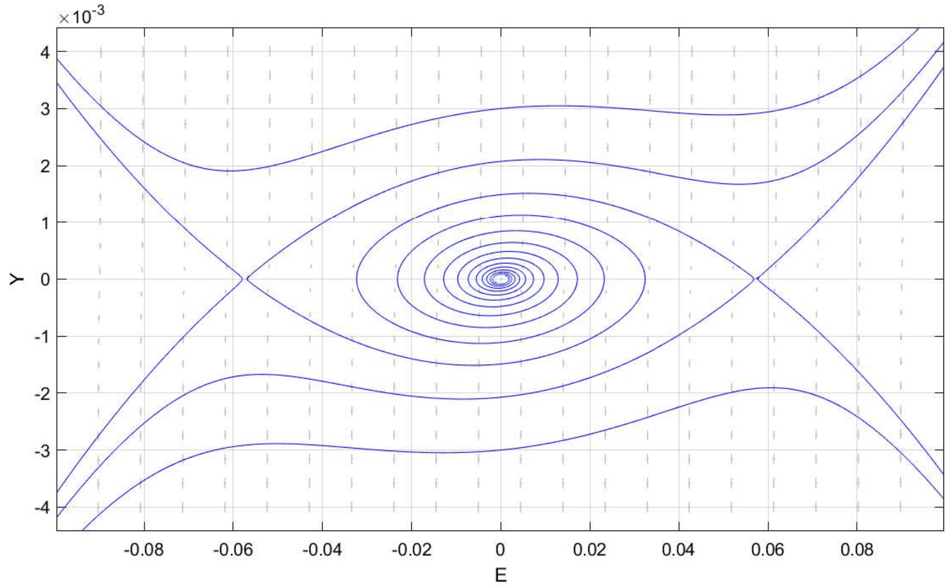

FIGURE 2.3: Phase Portrait of Case 2 of the ODEs system generated from the Lorentz-Kerr model with damping generated in PPlane with $v = .6545, \epsilon = 7/3$, and $\Gamma = .05$

hyperbolic equilibrium.

$v^{-2} < \epsilon$

Yields two purely real eigenvalues, one positive and one negative resulting in a saddle node.

Nonzero Equilibrium

Note that for this section we will assume $v^{-2} - \epsilon > 0$ which is required for the nonzero equilbiria to exist.

Case 1 $\Gamma = 0$

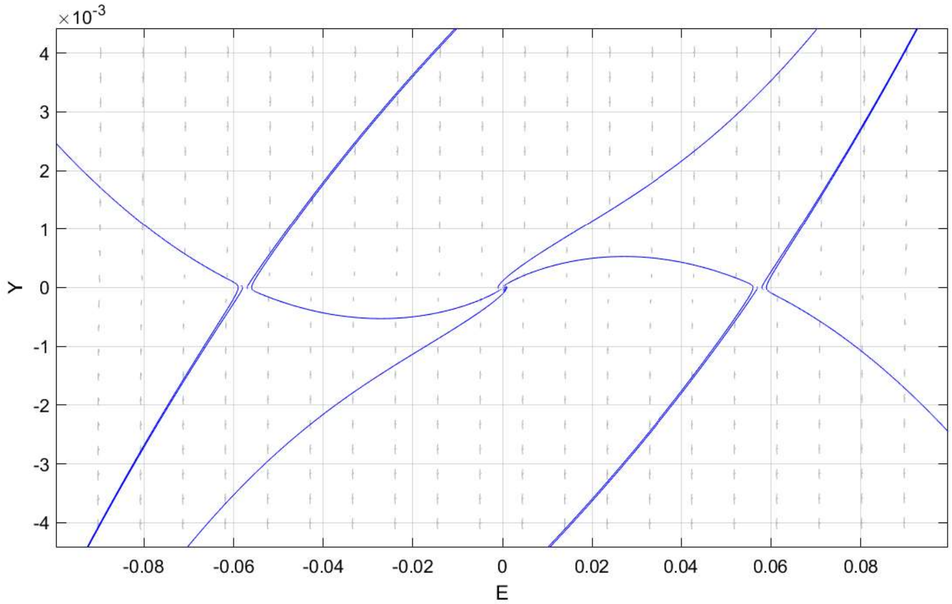

FIGURE 2.4: Phase Portrait of Case 3 of the ODEs system generated from the Lorentz-Kerr model with dampening generated in PPlane with $v = .6545, \epsilon = 2.4$, and $\Gamma = .1$

For this case our eigenvalues reduce to:

$$\lambda = \pm 2\sqrt{\frac{2(v^{-2} - \epsilon)}{((3\epsilon - 1)v^2 - 2)}}$$

Subcases

$v^{-2} > \epsilon$

This implies we have a pair of purely real eigenvalues, one positive and one negative resulting in a saddle node.

Case 2 $0 < \Gamma < 2\sqrt{\frac{2(v^{-2}-\epsilon)}{((3\epsilon-1)v^2-2)}}$

Since we assumed the upperbound was real we only have one subcase.

Subcases

$v^{-2} < \epsilon$

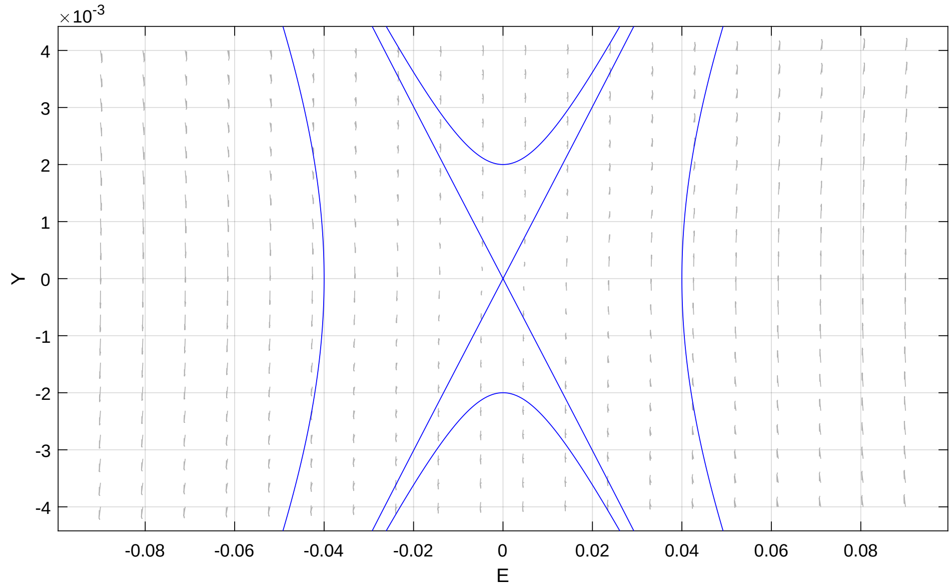

FIGURE 2.5: Phase Portrait of Case 4 of the ODEs system generated from the Lorentz-Kerr model with damping generated in PPlane with $v = .6545$, $\epsilon = 2.4$, and $\Gamma = .05$

It follows that have two purely real eigenvalues one negative and one positive resulting in a saddle node.

Case 3 $\Gamma > 2\sqrt{\frac{2(v^{-2}-\epsilon)}{((3\epsilon-1)v^2-2)}}$

Since we assumed the lowerbound was real we only have one subcase.

Subcases

$v^{-2} > \epsilon$

It follows that have two purely real eigenvalues one negative and one positive resulting in a saddle node.

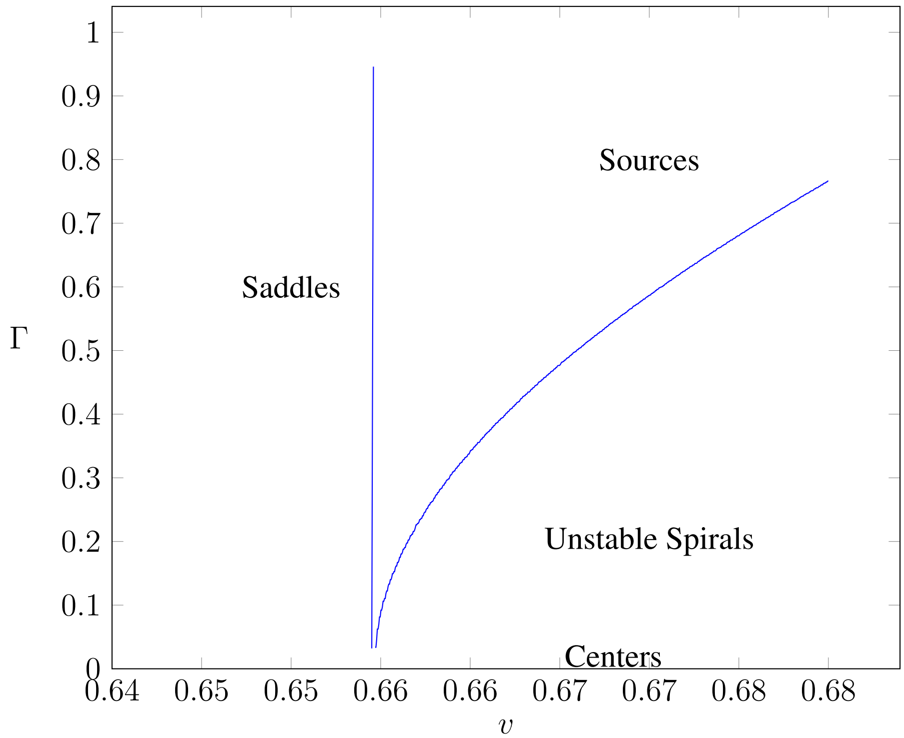

FIGURE 2.6: Bifurcation Diagram of Lorentz-Kerr model ODE system about Zero Equilibrium with $\epsilon = 7/3$. The behavior of the point changes as the lines are crossed, with centers along $\Gamma = 0$ and connection of the curves when $\epsilon = v^{-2}$

Based on this behavior it is reasonable to conjecture that a Hopf Bifurcaiton exists in the dampening parameter. To prove the existance of the Hopf Bifurcation, we will consider the linearized system corresponding to our above Jacobian.

$$\frac{dY}{d\xi} = Y, \tag{2.1.41a}$$

$$\frac{dY}{d\xi} = \delta E + \gamma Y. \tag{2.1.41b}$$

Theorem 1 (Hopf Bifurcation Theorem)**.** *[13] Consider a nonlinear planar system of*

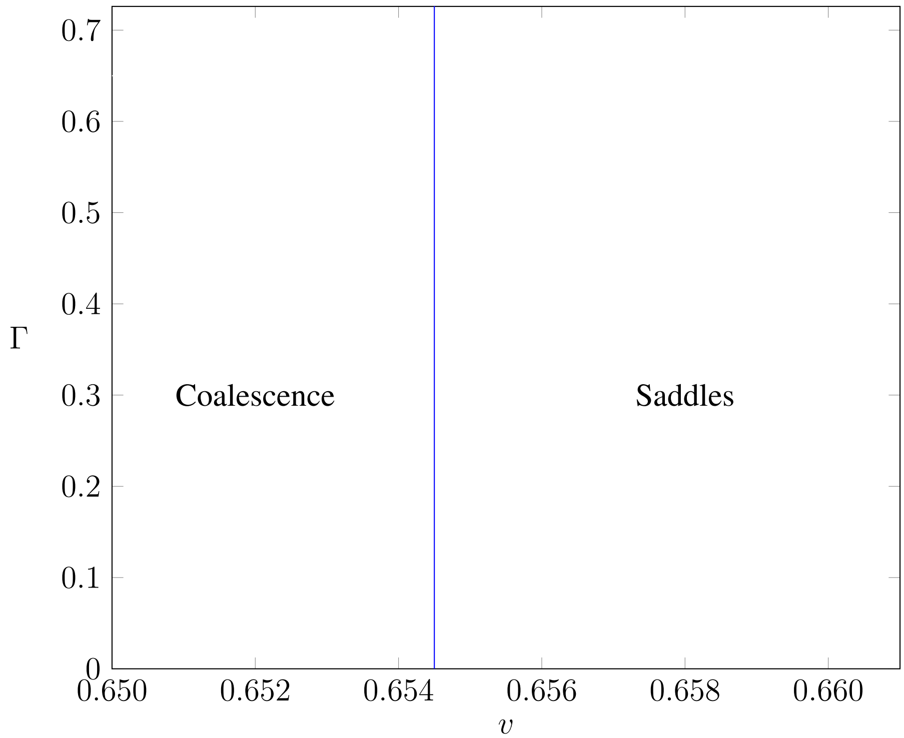

FIGURE 2.7: Bifurcation Diagram of Lorentz-Kerr model ODE system about Nonzero Equilibrium with $\epsilon = 7/3$. The behavior of the point changes as the lines are crossed. We note that to the left of the line $\epsilon = v^{-2}$ the nonzero equilibria merge with the zero equilbrium at the origin and do not move.

autonomous ODEs in the form

$$\frac{d\mathbf{X}}{dt} = J(\Gamma)(\mathbf{X}) + \mathbf{F}^{\text{nonlinear}}(\mathbf{X};\Gamma)$$

where X is a vector of unknown field variables and Γ is a parameter. Assume there is an equilibrium point at the origin. Then, a Hopf-Bifurcation occurs at the origin with a Hopf Bifurcation Point of $\Gamma = \Gamma_0$ if the following conditions hold.

1. *The nonlinear terms in $\mathbf{F}_{\text{nonlinear}}$ have continuous third order partial derivatives.*

2. *The Jacobian of the system at the origin* $J(\Gamma)$ *exists for small values of* $\Gamma - \Gamma_0$.

3. $A(\Gamma)$ *has complex eigenvalues dependent on* Γ *of the form* $\alpha(\Gamma) \pm i\beta(\Gamma)$, *with* $\alpha(\Gamma_0) = 0$ *and* $\beta(\Gamma_0) = \omega \neq 0$, *i.e. the equilibrium is hyperbolic for small* Γ *and non-hyperbolic for* $\Gamma = \Gamma_0$.

4. *(***Transversality condition:***) The eigenvalues of the Jacobian cross the imaginary axis with nonzero speed* d, *where*

$$d = \frac{\partial}{\partial \Gamma}\alpha(\Gamma)\bigg|_{\Gamma=\Gamma_0},$$

ω being the real part of the eigenvalues dependent on Γ.

Furthermore, we can classify the Hopf-Bifurcation as follows.

(Genericity condition:) *Assume we have a pair of ODEs, with* $\mathbf{F}^{\text{nonlinear}} = (f, g)^T$. *Define* a *dependent variable on the third order derivatives of* $\mathbf{F}^{\text{nonlinear}}$ *as*

$$\begin{aligned} a = &\frac{1}{16}(f_{EEE} + f_{EYY} + g_{YYY} + g_{EEY}) \\ &+ \frac{1}{16\omega(0)}(f_{EY}(f_{EE} + f_{YY}) - g_{EY}(g_{EE} + g_{YY}) - f_{EE}g_{EE} + f_{YY}g_{YY}) \end{aligned}$$

, with all derivative terms being computed at the origin.

Then, for $\Gamma > \Gamma_0$, *if* $ad > 0$ *(and for* $\Gamma < \Gamma_0$ *if* $ad < 0$*), the system of ODEs has a non-degenerate Hopf Bifurcation. In the case where* $ad < 0$ *the Hopf bifurcation is called* **super-critical**. *In the case that* $ad > 0$, *the Hopf bifurcation is called;* **sub-critical**. *If* $a = 0$ *then the bifurcation is called a* **degenerate Hopf**. *Note that if we can rule out limit cycles almost everywhere in our region, the sub-classification is degenerate Hopf.*

We shall apply this theorem to the zero equilibrium with parameter Γ with the concession that Γ can be negative. Since our system of ODEs is infinitely differentiable for all partial derivatives, it follows that it has third-order continuous partial derivatives. Since the only term in our linear system that depends on Γ is proportionally dependent, Γ it follows that the Jacobian is valid close to zero. The eigenvalues we computed earlier clearly show that, when $\Gamma = 0$, our eigenvalues are purely imaginary and for small values are complex. This leaves the Transversality Condtion which will require a derivative to prove, in particularly the derivative of the real part of our eigenvalues in terms of the parameter as computed at zero:

$$\frac{d}{d\Gamma}\gamma = \frac{1}{v} \neq 0$$

Since this derivative is nonzero, we conclude that the Transversality Condition holds, meaning that indeed a Hopf Bifurcation exists at $\Gamma = 0$ at the origin. Computation of a results in 0 because none of the derivatives that match the order of the system survive the ones that do not. We therefore can conclude that this is a Degenerate Hopf Bifurcation. To prepare to extend to higher dimensional systems, we will now introduce another method to come to the conclusion of degeneracy.

Theorem 2 (Bendixon-Dulac Criterion)**.** *[6] Let $f(E,Y)$, $g(E,Y)$ and ϕ be functions C^1 in a simply connected domain $D \in \mathbb{R}^2$ such that $T = \frac{\partial(f\phi)}{\partial E} + \frac{\partial(g\phi)}{\partial Y}$ does not change sign in D and vanishes at most on a set of measure zero. Then the system*

$$E' = f(E,Y), \tag{2.1.42a}$$

$$Y' = g(E,Y), \tag{2.1.42b}$$

does not have periodic orbits in D. Note that if ϕ exists then it is called a Dulac function.

Consider the above system and conjecture

$$\phi(E,Y) = (1 - v^2 - v^2E^2)^{2/v^2} \tag{2.1.43}$$

is a Dulac function. We then compute T,

$$T = \frac{\partial(f\phi)}{\partial E} + \frac{\partial(g\phi)}{\partial Y} = Y\frac{\partial \phi}{\partial E} + \phi\frac{\partial g}{\partial Y} =$$
$$-4v^2EY(1-v^2-v^2E^2)^{2/v^2-1} + (1-v^2-v^2E^2)^{2/v^2}\left(\frac{4v^2EY}{1-v^2-v^2E^2} + \frac{\Gamma}{v\omega_0}\right)$$
$$= \frac{\Gamma}{v}(1-v^2-v^2E^2)^{2/v^2}. \tag{2.1.44}$$

Note that this quantity is not defined for large E, as the function breaks down at a discontinuity in the system and is therefore a reasonable boundary. Therefore we conclude ϕ is indeed a Dulac function. The system, therefore does not have periodic orbits for nonzero Γ, and is zero when $\Gamma = 0$ as anticipated from our plots of the region. We again conclude that this Hopf Bifurcation is degenerate.

In future sections we will want to investigate behavior of the system for an arbitrary number of dimensions in

Theorem 3 (Bendixon Criterion in $\mathbb{R}^n$)**.** *[5] We will define our system of ODEs as:*

$$\frac{dx}{dt} = f(x,t),\ x(t) \in \mathbb{R}^n, \forall t \in [0,T]$$

A simple closed rectifiable curve invariant to the definition above cannot exist if one of the following conditions is satisfied on $\mathbb{R}^{\ltimes}$:

1. $\sup\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}(|\frac{\partial f_q}{\partial x_r}| + |\frac{\partial f_q}{\partial x_s}|) : 1 \leq r < s \leq n\} < 0$,,
2. $\sup\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}(|\frac{\partial f_r}{\partial x_q}| + |\frac{\partial f_s}{\partial x_q}|) : 1 \leq r < s \leq n\} < 0$,
3. $\lambda_1 + \lambda_2 < 0$,
4. $\inf\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}(|\frac{\partial f_q}{\partial x_r}| + |\frac{\partial f_q}{\partial x_s}|) : 1 \leq r < s \leq n\} > 0$,

5. $\inf\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}(|\frac{\partial f_r}{\partial x_q}| + |\frac{\partial f_s}{\partial x_q}|) : 1 \leq r < s \leq n\} > 0$,

6. $\lambda_{n-1} + \lambda_n > 0$.

To prove that the closed curve is the correct behavior of the system in the nonhyperbolic case $G = 0$ we will introduce a new theorem to prove existence of limit cycles.

Definition 2.1.1. Hamiltonian System:[7] Let D be any open subset of $\mathbb{R}^{2n}$ and let $\mathcal{H} \in C^2(D)$ where $\mathcal{H} = \mathcal{H}(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$. A system of the form

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \mathbf{y}}$$

$$\dot{\mathbf{y}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}$$

is called a Hamiltonian system with n degrees of freedom on D.

We know that the Hamiltonian of this system exists and can be expressed as

$$\mathcal{H} = \frac{1}{2}\mathcal{A}_t^2 + \frac{2}{3}\mathcal{A}_t^3 + \frac{1}{2}\mathcal{A}_z^2 + \frac{1}{2\beta_0}\phi_t^2 - \frac{1}{2\beta_0}\phi^2$$

where $\mathcal{A}_z = H$, $\beta_0 = \frac{\beta}{\epsilon_\infty}$.

Theorem 4. *[7] Any nondegenerate critcical point of an analytic Hamiltonian system(*) is either a saddle or a center; furthermore (x_0, y_0) is a saddle for (*) iff it is a saddle for the Hamiltonian function $\mathcal{H}(x, y)$ and strict local maximum or minimum of the function $\mathcal{H}(x, y)$ is a center for (*).*

In order to prove that the origin in the nonhyperbolic case is a center, we will need to get our Hamiltonian in terms of E, Y. It follows by the Ampere-Maxwell Law and our previous assumptions that,

$$\mathcal{A}_{zz} = H_z = D_t,$$

$$\mathcal{A}_{tz} = H_t = D_z,$$

Therefore,

$$\mathcal{A}_t = D \text{ and } \mathcal{A}_z = \int D_z dt.$$

Under the assumptions made in the previous derivation,

$$D = \frac{E}{\mu_0 v^2},$$

$$\int D_z dt = \int \frac{E_z}{\mu_0 v^2} dt = \int \frac{Y}{\mu_0 v^2} dt = \int -\frac{E_t}{\mu_0 v^3} dt = -\frac{E}{\mu_0 v^3}$$

$$\phi_t = (\frac{c^2}{v^2} - \epsilon_\infty + \frac{3aEY}{v})$$

$$\phi = (\frac{c^2}{v^2} - \epsilon_\infty - 3aE^2)E.$$

It therefore follows that without non-dimensionalization,

$$\mathcal{H}(E,Y) = \frac{1}{2}(-\frac{E}{\mu_0 v^2})^2 + \frac{2}{3}(-\frac{E}{\mu_0 v^2})^3 + \frac{1}{2}(\frac{E}{\mu_0 v^3})^2 + \frac{1}{2\beta_0}(\frac{c^2}{v^2} - \epsilon_\infty + \frac{3aEY}{v})^2 - \frac{1}{2\beta_0}((\frac{c^2}{v^2} - \epsilon_\infty - 3aE^2)E)^2 \tag{2.1.45}$$

Therefore,

$$\frac{\partial \mathcal{H}}{\partial E} = (\frac{E}{\mu_0^2 v^4}) + 2(-\frac{E^2}{\mu_0^3 v^6}) + (\frac{E}{\mu_0^2 v^6}) + \frac{1}{\beta_0}((\frac{c^2}{v^2} - \epsilon_\infty + \frac{3aEY}{v})\frac{3aY}{v} - (\frac{c^2}{v^2} - \epsilon_\infty - 3aE^2)E(\frac{c^2}{v^2} - \epsilon_\infty - 9aE^2)) \tag{2.1.46}$$

$$\frac{\partial \mathcal{H}}{\partial Y} = \frac{1}{\beta_0}(\frac{c^2}{v^2} - \epsilon_\infty + \frac{3aEY}{v})(\frac{3aE}{v})) \tag{2.1.47}$$

It follows that at a critical point

$$\frac{\partial \mathcal{H}}{\partial E} = \frac{\partial \mathcal{H}}{\partial Y} = 0.$$

Trivially one such solution is $(E, Y) = (0, 0)$, meaning that the nonhyperbolic equilibria is a critical point of the Hamiltonian making it a center or a saddle dependent on the sign of higher order derivatives. It follows that,

$$\begin{aligned}
\frac{\partial^2 \mathcal{H}}{(\partial E)^2} &= \frac{1}{\mu_0^2 v^4} - \frac{4E}{\mu_0^3 v^6} + \frac{1}{\mu_0^2 v^6} + \frac{1}{\beta_0}((\frac{c^2}{v^2} - \epsilon_\infty + \frac{3aY}{v})\frac{3aY}{v} \\
&\quad + (6aE^2)(\frac{c^2}{v^2} - \epsilon_\infty - 9aE^2) - (\frac{c^2}{v^2} - \epsilon_\infty - 3aE^2)(\frac{c^2}{v^2} - \epsilon_\infty - 9aE^2) \\
&\quad + (\frac{c^2}{v^2} - \epsilon_\infty - 3aE^2)(18aE^2)), &\text{(2.1.48a)} \\
\frac{\partial^2 \mathcal{H}}{\partial E \partial Y} &= \frac{1}{\beta}(\frac{c^2}{v^2} - \epsilon_\infty + \frac{6aEY}{v})(\frac{3a}{v}), &\text{(2.1.48b)} \\
\frac{\partial^2 \mathcal{H}}{(\partial Y)^2} &= \frac{1}{\beta_0}(\frac{9a^2E^2}{v^2}). &\text{(2.1.48c)}
\end{aligned}$$

It therefore follows that for the bounds we put on our parameters,

$$\mathcal{H}_{EE}(0,0)\mathcal{H}_{YY}(0,0) - \mathcal{H}_{EY}(0,0) = \frac{1}{\beta_0}(\frac{c^2}{v^2} - \epsilon_\infty)(\frac{3a}{v}) \neq 0$$

Therefore this critical point is either a minimum or maximum making the equilibrium a center in the nonhyperbolic case.

Definition 2.1.2. [9]**Kamke Function:** Conisder a function $F \in D \subset \mathbb{R}^n$, it follows that such a function is a Kamke Function if for every $x, y \in D$, if $x \leq y$ with $x_i = y_i$ pointwise for some i, then $F_i(x) \leq F_i(y)$ where $F = (F_1, F_2, ..., F_n)$.

Theorem 5. *[9] Consider our nonlinear, autonomous, smooth planar system of ordinary differential equations, which we denote as,*

$$X' = F(X)$$

and let F be a Kamke function. Let there exist functions $\psi, \nu : [0,T] \to \mathbb{R}^n$ such that $\psi(t) \leq \nu(t), 0 \leq t \leq T, \psi(0) \leq \psi(T), \nu(0) \geq \nu(T)$ and,

$$D_{-}\psi(t) \leq F(\psi(t)), D^{-}\nu(t) \geq F(\nu(t)), 0 \leq t \leq T.$$

Then there exists a periodic solution x(t) such that $\psi(t) \leq x(t) \leq \nu(t), 0 \leq t \leq T$. Note that D_{-} and D^{-} are the upper and lower Dini Differential Operators.

Definition 2.1.3. Dini Differential Operators Let a, b be real numbers such that $a < b$ and f a function such that $f([a,b]^n) \to \mathbb{R}^n$. Then the four Dini Differential Operators are

$$D_{-}f(x) = \liminf_{|h| \to 0^{-}} \frac{f(x+h) - f(x)}{h}, \tag{2.1.49a}$$

$$D^{-}f(x) = \liminf_{|h| \to 0^{-}} \frac{f(x+h) - f(x)}{h}, \tag{2.1.49b}$$

$$D_{+}f(x) = \limsup_{|h| \to 0^{-}} \frac{f(x+h) - f(x)}{h}, \tag{2.1.49c}$$

$$D^{+}f(x) = \limsup_{|h| \to 0^{-}} \frac{f(x+h) - f(x)}{h}. \tag{2.1.49d}$$

whe note that if f is differentiable then all of these operators are equivalent to the regular definition of differentiation.

We consider the system in question:

$$\frac{dE}{d\xi} = Y, \tag{2.1.50a}$$

$$\frac{dY}{d\xi} = \frac{2v^2EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2E^2}, \tag{2.1.50b}$$

where $v^{-2} - \epsilon < 0$. We call the righthand side F, and we will define the following vectors

$$P_0 = \begin{pmatrix} E \\ Y \end{pmatrix}, P_1 = \begin{pmatrix} E \\ Y - \delta \end{pmatrix}, P_2 = \begin{pmatrix} E - \delta \\ Y \end{pmatrix}$$

It follows that F is Kamke if,

$$F_1(P_0) \geq F_1(P_1)$$

$$F_2(P_0) \geq F_2(P_2)$$

for all $\delta \geq 0$ such that $E - \delta \in D$. This system is equivalent to,

$$Y \geq Y - \delta$$

$$\frac{2v^2 EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2 E^2} \geq \frac{2v^2(E-\delta)Y^2 - (v^{-2} - \epsilon)(E-\delta) + (E-\delta)^3/3}{1 - v^2 - v^2(E-\delta)^2}$$

The first equation is trivially true, for second equation we will consider cases. Note that if either E or δ are zero, the second inequality holds trivially. Now assume $E, E-\delta > 0$, then

$$\begin{aligned} \frac{2v^2(E-\delta)Y^2 - (v^{-2} - \epsilon)(E-\delta) + (E-\delta)^3/3}{1 - v^2 - v^2(E-\delta)^2} &\leq \frac{2v^2(E)Y^2 - (v^{-2} - \epsilon)(E) + (E)^3/3}{1 - v^2 - v^2(E-\delta)^2} \\ &\leq \frac{2v^2 EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2 E^2} \end{aligned}$$

Therefore, the second equation holds. Assume $E, E - \delta < 0$ then,

$$\begin{aligned} -\frac{2v^2(E-\delta)Y^2 - (v^{-2} - \epsilon)(E-\delta) + (E-\delta)^3/3}{1 - v^2 - v^2(E-\delta)^2} &\geq -\frac{2v^2(E)Y^2 - (v^{-2} - \epsilon)(E) + (E)^3/3}{1 - v^2 - v^2(E-\delta)^2} \\ &\geq -\frac{2v^2 EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2 E^2} \end{aligned}$$

Which implies our second equation. Finally, consider the case where $E > 0$ and $E-\delta < 0$ then,

$$\begin{aligned} &\frac{2v^2(E-\delta)Y^2 - (v^{-2} - \epsilon)(E-\delta) + (E-\delta)^3/3}{1 - v^2 - v^2(E-\delta)^2} \\ &\qquad \leq \frac{2v^2(E)Y^2 - (v^{-2} - \epsilon)(E) + (E)^3/3}{1 - v^2 - v^2(E-\delta)^2} \\ &\qquad \leq \frac{2v^2 EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2 E^2} \end{aligned}$$

Therefore, our second equation holds, implying that F is Kamke across its domain.

Now that we know that F is Kamke, we can using this formula to prove the origin is a center. Let,

$$\psi(\xi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\nu(\xi) = r_0 \begin{pmatrix} cos(\xi) \\ sin(\xi) \end{pmatrix}$$

where $T = 2\pi$. Clearly, $|\psi(t)| \leq |\nu(t)|$,$\psi(0) = \psi(T)$ and $\nu(0) = \nu(T)$. Notice that since we are proving that periodic solutions exist between two circles, we will take the absolute value of everything rather than convert to radial coordinates. It also follows that,

$$D_{-}\psi_1 = 0$$

$$D_{-}\psi_2 = 0$$

$$D^{-}\nu_1 = -r_0 sin(t)$$

$$D^{-}\nu_2 = r_0 cos(t)$$

$$F_1(\psi(\xi)) = r_0 sin(t)$$

$$F_2(\psi(\xi)) = 0$$

$$F_1(\nu(\xi)) = r_0 sin(t)$$

$$F_2(\nu(\xi)) = \frac{2v^2 r_{max}^3 cos(t) sin^2(t) + |v^{-2} - \epsilon| r_{max} cos(t) + r_{max}^3 cos^3(t)/3}{1 - v^2 - v^2 r_{max}^2 cos^2(t)}$$

It therefore follows that trivially,

$$|D_{-}\psi_1| = |F_1(\psi)|$$

$$|D^{-}\nu_1| = |F_1(\nu)|$$

$$|D_{-}\psi_2| = |F_1(\psi)|$$

Nontrivially,

$$|D^{-}(\nu_2)| = |r_0 cos(\xi)| < |\frac{r_0 cos(\xi)}{1 - v^2 - v^2 r_0^2 cos^2(\xi)}| < |\frac{r_0 cos(\xi)}{1 - v^2 - v^2 r_0^2 cos^2(\xi)}|(r_0^2(min(2v^2, 1/3)|$$

$$< |\frac{r_0 cos(\xi)}{1 - v^2 - v^2 r_0^2 cos^2(\xi)}||2v^2 r_0^2 sin^2(\xi) + |v^{-2} - \epsilon| + r_0^2 cos^2(\xi)/3| = |F_2(\nu(\xi))|$$

With the restriction that $r_0 > min(1/(2v^2), \sqrt{3})$. Therefore, for each corresponding value of r_0 in our restricted region, there exists a periodic solution $X(t)$ such that,

$$|\psi(\xi)| \leq |X(\xi)| \leq |\nu(\xi)|$$

Therefore, the origin is a center in the nonhyperbolic case.

We now consider the bifurcation occuring at the origin. We see three equilibria coalesce into one, which suggests some form of Pitchfork Bifurcation.

Theorem 6 (Pitchfork Bifurcation Theorem)**.** *[8] Consider a nonlinear planar system of autonomous ODEs in the form*

$$\frac{\mathbf{dX}}{dt} = \mathbf{F},$$

where $\mathbf{X}$ *is a vector of unknown field variables and* v *is a parameter. Assume there is an equilibrium point at the origin. Then a Pitchfork-Bifurcation occurs at the origin with* $v = v_0$ *being the Pitchfork Bifurcation Point if the following conditions hold.*

1. *The all terms in* $\mathbf{F}$ *have continuous third order partial derivatives.*

2. *The Jacobian of the system at the origin, $J(v)$, exists for small values of $v - v_0$.*

3. *Either one equilibria exists before $v = v_0$ and three exist after, or three exist before and one exists after.*

4. *The the function is odd:*

$$-\mathbf{F}(\mathbf{X}; v) = \mathbf{F}(-\mathbf{X}; v) \tag{2.1.51}$$

5. *At the bifurcation point the following hold,*

   $\frac{\partial \mathbf{F}}{\partial v}(0,0)_{v=v_0} = 0$

   $\frac{\partial^2 \mathbf{F}}{\partial \mathbf{X} \partial v}(0,0)_{v=v_0} > 0$

   $\frac{\partial^3 \mathbf{F}}{\partial^3 \mathbf{X}}(0,0)_{v=v_0} < 0$

We will look at our system at the point where $v = \epsilon^{-2}$. It follows that we have assumed the system is sufficiently smooth, and already know that the Jacobian exists for values close to the bifurcation. We also know from our analysis that condition 3 is satisfied. We now look at the other conditions, starting with 4. It follows that,

$$-\frac{dE}{d\xi}(E, Y) = -Y = \frac{dE}{d\xi}(-E, -Y)$$

$$\begin{aligned} -\frac{dY}{d\xi}(E, Y) &= -\frac{2v^2 E Y^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2 E^2} \\ &= \frac{2v^2(-E)(-Y)^2 - (v^{-2} - \epsilon)(-E) + (-E)^3/3}{1 - v^2 - v^2(-E)^2} = \frac{dY}{d\xi}(-E, -Y) \end{aligned}$$

Therefore condition 4 holds. Finally we consider condition 5 which will require computation of many different partial derivatives. It follows that since the first equation does not depend on v or E and only depends on Y in the first order,

$$\frac{\partial}{\partial v}\frac{dE}{d\xi} = \frac{\partial^2}{\partial \mathbf{X} \partial v}\frac{dE}{d\xi} = \frac{\partial^3}{\partial^3 \mathbf{X}}\frac{dE}{d\xi} = 0$$

Regaurdless of the other other equation, this result indicates that the fifth condition will not be satisfied. We therefore conclude that this particular bifurcation has many of the characteristics of the pitchfork, but does not fit the qualifications of a pitchfork bifurcation.

3 QUALITATIVE BEHAVIOR OF MAXWELL-LORENTZ MODEL WITH BOTH KERR AND RAMAN EFFECTS

We now wish to include the Raman effect in our analysis and neglect none of the parameters in questions. This means we have five equations to consider: the Maxwell Laws, the constitutive equation for D, and the Raman and Lorentz oscillators:

$$E_z = -B_t, \tag{3.0.1a}$$

$$B_z = \mu_0 D_t, \tag{3.0.1b}$$

$$D = \epsilon_0(\epsilon_\infty E + \phi + a(1-\theta)E^3 + a\theta QE), \tag{3.0.1c}$$

$$\phi_{tt} + \Gamma\phi_t + {\omega_0}^2\phi = \beta{\omega_0}^2 E, \tag{3.0.1d}$$

$$Q_{tt} + \Gamma_v Q_t + {\omega_v}^2 Q = {\omega_v}^2 E^2. \tag{3.0.1e}$$

Assuming a traveling wave solution on functions B, E, D, Q, and ϕ, that is, assuming

$$f(z,t) = f(z - vt) = f(\xi) \text{ with } \xi := z - vt,$$

we can reduce our system of PDEs into a planar system of four ODEs as follows: We start by using the rearrangment in the introduction to get

$$E_t = v^2\mu_0 D_t, \tag{3.0.2}$$

assuming that $E(z,t) = E(\xi)$ the chain rule yields

$$\frac{\partial E}{\partial z} = \frac{\partial E}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi} := Y, \tag{3.0.3a}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial Q}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial Q}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial Q}{\partial \xi} := X. \tag{3.0.3b}$$

Rearranging these equations,

$$Y = -\frac{1}{v}E_t, \tag{3.0.4a}$$

$$X = -\frac{1}{v}Q_t. \tag{3.0.4b}$$

Repeating this process gives

$$\frac{dY}{d\xi} = \frac{1}{v^2}E_{tt}, \tag{3.0.5a}$$

$$\frac{dX}{d\xi} = \frac{1}{v^2}Q_{tt}. \tag{3.0.5b}$$

Beginning with (3.0.5b) and substituting (3.0.1e) and (3.0.4b), we arrive with X' in terms of E, Q and X:

$$\frac{dX}{d\xi} = \left(\left(\frac{\omega_v}{v}\right)^2 (E^2 - Q) + \frac{\Gamma_v}{v}X\right). \tag{3.0.6}$$

Differentiating (3.0.2) results in

$$\frac{dY}{d\xi} = \mu_0 D_{tt}. \tag{3.0.7}$$

To get a second equation involving D_{tt} in terms of E and Y we take a time derivative of (3.0.1c), which produces

$$D_t = \epsilon_0(\epsilon_\infty E_t + \phi_t + 3a(1-\theta)E^2 E_t + a\theta(Q_t E + QE_t)).$$

Substuting in (3.0.2) and grouping together D_t terms yields the equation

$$D_t = \frac{\epsilon_0(\phi_t + a\theta Q_t E)}{1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3a(1-\theta)E^2 + a\theta Q)}. \tag{3.0.8}$$

This result is consistent with the cases where we set $\theta = 0$. Since the denominator is composed entirely of the functions we seek in the final result, we shall call it α turning (3.0.8) into,

$$D_t = \frac{\epsilon_0(\phi_t + a\theta Q_t E)}{\alpha}. \tag{3.0.9}$$

Note that

$$\alpha_t = -\epsilon_0\mu_0 v^2 a(6(1-\theta)EE_t + \theta Q_t) = \epsilon_0\mu_0 v^3 a(6(1-\theta)EY + \theta X), \tag{3.0.10}$$

that is, α_t is also comprised of the variables we are considering. If we take a time derivative of (3.0.9) it follows, by quotient rule, that

$$D_{tt} = \epsilon_0 \frac{(\phi_{tt} + a\theta(Q_{tt}E + Q_tE_t))\alpha - (\phi_t + a\theta Q_t E)\alpha_t}{\alpha^2}. \tag{3.0.11}$$

Some of these terms are undesired and will need to be removed. To begin, we remove the ϕ terms via integration of (3.0.2) and the assumption that the initial conditions of D and E are identical. It follows that

$$E = \mu_0 v^2 D. \tag{3.0.12}$$

Substituting in our equation (3.0.1c),

$$E = \epsilon_0\mu_0 v^2(\epsilon_\infty E + \phi + a(1-\theta)E^3 + a\theta QE).$$

Solved for ϕ,

$$\phi = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - a(1-\theta)E^2 - a\theta Q\right)E. \tag{3.0.13}$$

Differentiating with respect to t, we have

$$\phi_t = \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3a(1-\theta)E^2\right)E_t - a\theta(Q_tE + QE_t). \tag{3.0.14}$$

Making use of (3.0.1d) we can solve for ϕ_{tt}:

$$\phi_{tt} = {\omega_0}^2(\beta E - \phi) - \Gamma\phi_t.$$

Substituting this into this equation (3.0.13) and (3.0.14),

$$\begin{aligned}\phi_{tt} = {\omega_0}^2 &\left(\beta E - \left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - a(1-\theta)E^2 - a\theta Q\right)E\right) \\ &-\Gamma\left(\left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3a\theta E^2\right)E_t - a\theta(Q_tE + QE_t)\right).\end{aligned} \tag{3.0.15}$$

Now, considering the term Q_{tt} it follows by rearrangement of (3.0.1e) that

$$Q_{tt} = {\omega_v}^2(E^2 - Q) - \Gamma_v Q_t. \tag{3.0.16}$$

Note, further, that by (3.0.5b) and (3.0.16),

$$\frac{dX}{d\xi} = \frac{1}{v^2}({\omega_v}^2(E^2 - Q) + v\Gamma_v X). \tag{3.0.17}$$

Therefore using our definitions for Y and X we can solve for Y' in terms of our variables by solving the following system.

$$\frac{dY}{d\xi} = \mu_0\epsilon_0 \frac{(\phi_{tt} + a\theta(Q_{tt}E + v^2XY))\alpha - (\phi_t - av\theta E)\alpha_t}{\alpha^2} \tag{3.0.18a}$$

$$Q_{tt} = {\omega_v}^2(E^2 - Q) + v\Gamma_v X, \tag{3.0.18b}$$

$$\phi_{tt} = {\omega_0}^2(\beta E - (\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - a(1-\theta)E^2 - a\theta Q)E)$$
$$+ v\Gamma((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3a\theta E^2)Y + a\theta(XE + QY)), \tag{3.0.18c}$$

$$\phi_t = -v((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3a(1-\theta)E^2)Y + a\theta(XE + QY)), \tag{3.0.18d}$$

$$\phi = (\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - a(1-\theta)E^2 - a\theta Q)E, \tag{3.0.18e}$$

$$\alpha_t = \epsilon_0\mu_0 v^3 a(6(1-\theta)EY + \theta X), \tag{3.0.18f}$$

$$\alpha = 1 - \epsilon_0\mu_0 v^2(\epsilon_\infty + 3a(1-\theta)E^2 + a\theta Q). \tag{3.0.18g}$$

Therefore we have the system comprised of (3.0.18a), (3.0.17) and (3.0.3a)(3.0.3b) as follows:

$$\frac{dE}{d\xi} = Y, \tag{3.0.19a}$$

$$\frac{dQ}{d\xi} = X, \tag{3.0.19b}$$

$$\frac{dY}{d\xi} = \mu_0\epsilon_0 \frac{(\phi_{tt} + a\theta(Q_{tt}E + v^2XY))\alpha - (\phi_t - av\theta E)\alpha_t}{\alpha^2}, \tag{3.0.19c}$$

$$\frac{dX}{d\xi} = \frac{1}{v^2}({\omega_v}^2(E^2 - Q) + v\Gamma_v X). \tag{3.0.19d}$$

with the equations for $\alpha, \alpha_t, Q_{tt}, \phi_{tt},$ and ϕ_t as stated above. We now wish to find the equilibria. The first two equations trivially imply that for all equilibria,

$$Y_\infty = 0 \text{ and } X_\infty = 0,$$

which, put into the fourth, implies,

$$E_\infty^2 = Q_\infty.$$

Therefore, if we input these results into our other equation imply that at equilibria,

$$\alpha = 1 - \frac{v^2}{c^2}(\epsilon_\infty + a(3-2\theta)E_\infty^2), \tag{3.0.20a}$$

$$\alpha_t = 0, \tag{3.0.20b}$$

$$Q_{tt} = 0, \tag{3.0.20c}$$

$$\phi_{tt} = \omega_0^2(\beta E_\infty - (\frac{c^2}{v^2} - \epsilon_\infty - a(1-2\theta)E_\infty^2)E_\infty, \tag{3.0.20d}$$

$$\phi_t = 0, \tag{3.0.20e}$$

$$\phi = (\frac{c^2}{v^2} - \epsilon_\infty - a(1-2\theta)E_\infty^2)E_\infty. \tag{3.0.20f}$$

Therefore,

$$0 = c^2 \frac{(\omega_0^2(\beta E_\infty - (\frac{c^2}{v^2} - \epsilon_\infty - a(1-2\theta)E_\infty^2)E_\infty)(1 - \frac{v^2}{c^2}(\epsilon_\infty + a(3-2\theta)E_\infty^2))}{(1 - \frac{v^2}{c^2}(\epsilon_\infty + a(3-2\theta)E_\infty^2))^2}.$$

Simplified,

$$E_\infty(\beta - (\frac{c^2}{v^2} - \epsilon_\infty - a(1-2\theta)E_\infty^2)) = 0.$$

Therefore we have the zero equilibrium,

$$(E_\infty, Y_\infty, Q_\infty, X_\infty) = (0,0,0,0),$$

and also a pair of nonzero equilibria,

$$(E_\infty, Y_\infty, Q_\infty, X_\infty) = (-\sqrt{\frac{1}{(1-2\theta)a}}\sqrt{\frac{c^2}{v^2} - \epsilon_s}, 0, \frac{1}{a(1-2\theta)}(\frac{c^2}{v^2} - \epsilon_s), 0),$$

$$(E_\infty, Y_\infty, Q_\infty, X_\infty) = (\sqrt{\frac{1}{(1-2\theta)a}}\sqrt{\frac{c^2}{v^2} - \epsilon_s}, 0, \frac{1}{a(1-2\theta)}(\frac{c^2}{v^2} - \epsilon_s), 0),$$

which are equivalent in E and Y to the previous case where $\theta = 0$. We will now evaluate the Jacobian at the origin by computing the following nontrivial partial derivatives.

$$\frac{\partial}{\partial Y}\frac{dE}{d\xi} = 1. \tag{3.0.21a}$$

$$\frac{\partial}{\partial E}\frac{dY}{d\xi} = {\omega_0}^2 \frac{\beta - \frac{c^2}{v^2} + \epsilon_\infty}{(c^2)(1 - \frac{v^2}{c^2}\epsilon_\infty)}. \tag{3.0.21b}$$

$$\frac{\partial}{\partial Y}\frac{dY}{d\xi} = \frac{\Gamma}{v}. \tag{3.0.21c}$$

$$\frac{\partial}{\partial X}\frac{dQ}{d\xi} = 1. \tag{3.0.21d}$$

$$\frac{\partial}{\partial Q}\frac{dX}{d\xi} = -\frac{{\omega_v}^2}{v^2}. \tag{3.0.21e}$$

$$\frac{\partial}{\partial X}\frac{dX}{d\xi} = \frac{\Gamma_v}{v}. \tag{3.0.21f}$$

$$\tag{3.0.21g}$$

Therefore our Jacobian is

$$J_{\Gamma,\Gamma_v}(0,0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \psi & \frac{\Gamma}{v} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{{\omega_v}^2}{v^2} & \frac{\Gamma_v}{v} \end{pmatrix},$$

where,

$$\psi = {\omega_0}^2 \frac{\beta - \frac{c^2}{v^2} + \epsilon_\infty}{(c^2)(1 - \frac{v^2}{c^2}\epsilon_\infty)}$$

which corresponds to the characteristic equation

$$0 = \left(\lambda\left(\frac{\Gamma}{v} - \lambda\right) - \psi\right)\left(\lambda\left(\frac{\Gamma_v}{v} - \lambda\right) - \frac{{\omega_v}^2}{v^2}\right).$$

Therefore one of the following expressions must be zero.

$$0 = \left(\lambda\left(\frac{\Gamma}{v} - \lambda\right) - \psi\right),$$

$$0 = \left(\lambda\left(\frac{\Gamma_v}{v} - \lambda\right) - \frac{{\omega_v}^2}{v^2}\right).$$

Yielding corresponding solutions

$$\lambda = \frac{\frac{\Gamma}{v} \pm \sqrt{(\frac{\Gamma}{v})^2 - 4\psi}}{2}$$

and

$$\lambda = \frac{\frac{\Gamma_v}{v} \pm \sqrt{(\frac{\Gamma_v}{v})^2 - 4\frac{{\omega_v}^2}{v^2}}}{2}.$$

These eigenvalues demonstrate that there is a change in stability occuring at the the origin in both the damping constants of the Lorentz and Raman oscillators. With the possibility of a full hopf bifurcation occuring if we set the two constants to be proportional to each other by a positive constant. To determine if we have a bifurcation at the origin where, for some $m > 0$ $m\Gamma = \Gamma_v$, we reinvoke the Hopf Bifurcation theorem. Since our system is infinitely differentiable on our region of interest, it follows that it has continuous third order partial derivatives in all four variables. The Jacobian computed above indeed exists for all values of Γ and Γ_v and therefore exists for values close to the origin. Moreover, our eigenvalues are indeed of the form we need and last we check that the Transversality condition hold with the first pair of eigenvalues:

$$d = \frac{\partial}{\partial\Gamma}\alpha(\Gamma)_{\Gamma=0} = \frac{\partial}{\partial\Gamma}\frac{\Gamma}{v}_{\Gamma=0} = \frac{1}{v}$$

Since the velocity is always positive,the condition holds. This implies that we have at least a simple Hopf at the origin. To determine if limit cycles exists, we invoke the sixth condition of the Bendixon Criterion:

$$\lambda_4 + \lambda_3 = m\frac{\Gamma}{v} > 0$$

As this condition holds, no limit cycles exist and we conclude that a degerate Hopf Bifurcation exist at the origin. Any approximation we make of the system should preserve this behavior.

For a complete bifurcation analysis we proceed to consider the cases on the regions of our parameters. Due to physical restrictions, we know that $\Gamma, \Gamma_v \in [0, \infty)$, $v \in (0, c)$, $\omega_v, \omega_0 \in (0, \infty)$, $\epsilon_s \in [\epsilon_\infty, \infty)$, and $\epsilon_\infty \in (0, \epsilon_s]$, with c being the only physical constant in our eigenvalues.

Case 1

$$\Gamma = \Gamma_v = 0$$

In this case energy in the system is conserved, but the qualitative behavior could still change. **Subcase 1**

$$\frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} \geq 0$$

It this case all eigenvalues have zero real parts resulting in a nonhyperbolic equilibrium with likely center behavior. **Subcase 2**

$$\frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} < 0$$

In this case we have a pair of nonzero eigenvalues with zero real part, and a pair of purely real nonzero eigenvalues with opposite sign. As such we have a nonhyperbolic equilibrium with likely saddle node behavior.

Case 2

$$\Gamma\Gamma_v \neq 0$$

This is the most general case with energy lost in the system, behavior now depends on all eigenvalues.

Subcase 1

$$\frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} < 0$$

This cases yields at least two real eigenvalues of opposite sign resulting in a saddle node.

Subcase 2

$$(\frac{\Gamma}{2v})^2 > \frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} \geq 0 \text{ and } \Gamma_v > 2\omega_v$$

This yields 4 positive real eigenvalues resulting in an unstable source.

Subcase 3

$$\frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} \geq (\frac{\Gamma}{2v})^2 \text{ and } \Gamma_v < 2\omega_v$$

This results in four real eigenvalues, three of which are positve and one is negative resulting in a saddle.

Subcase 4

$$\frac{\epsilon_s - \frac{c^2}{v^2}}{1 - \frac{v^2}{c^2}\epsilon_\infty} \geq (\frac{\Gamma}{2v})^2$$

This yields four eigenvalues, at least two of which are complex all of which have positive real parts resulting in an unstable spiral.

We will now consider the nonhyperbolic equilbria, starting with the the case where $\Gamma = \Gamma_v = 0$ and $\frac{\epsilon_s - \frac{c^2}{v^2}}{1-\frac{v^2}{c^2}\epsilon_\infty} \geq (\frac{\Gamma}{2v})^2$ at the origin. By [10] we know that this system is Hamiltonian with corresponding function,

$$\mathcal{H} = \frac{1}{2}(1+Q)\mathcal{A}_t^2 + \frac{2}{3}\mathcal{A}_t^3 + \frac{1}{2}\mathcal{A}_z^2 + \frac{1}{2\beta_0}\phi_t^2 - \frac{1}{2\beta_0}\phi^2 + \frac{1}{4b}Q_t^2 + \frac{1}{4b}\omega_v^2 Q^2$$

where $\mathcal{A}_z = H$, $b = \frac{\omega_v^2}{\omega_0^2}\frac{\theta}{3(1-\theta)}$.

It follows by the Maxwell-Ampere Law and our previous assumptions that,

$$\mathcal{A}_{tz} = H_t = D_z$$

$$\mathcal{A}_{zz} = H_z = D_t$$

3.1 A Three Equation Approximation to the Four Equation System

As an attempt at an approximation, consider a very rough linearization of the Raman effect. For this derivation, will introduce a new variable u such that

$$u = \phi + a\theta E Q. \tag{3.1.1}$$

Therefore by (3.0.1d) and (3.0.1e) we can create an oscillator equation for u with the right hand side containing E, Y, Y', Q and Q_t as follows:

$$u_{tt} + \Gamma u_t + {\omega_0}^2 u = \beta {\omega_0}^2 E + a\theta E({\omega_v}^2 E^2 + (\Gamma - \Gamma_v)Q_t + ({\omega_0}^2 - {\omega_v}^2)Q) + a\theta(E_t Q + 2E_t Q_t + E_{tt} Q)$$

Since we wish for this to depend solely on E, E_t, E_{tt}, Q, we will make the approximation

$$Q_{tt} \approx 0.$$

This assumes some form of linearity in the Raman effect globally so the best application is in smaller intervals of time and space. Therefore, by (3.0.1e),

$$Q_t = \frac{{\omega_v}^2}{\Gamma_v}(E^2 - Q)$$

It follows that

$$\begin{aligned} u_{tt} + \Gamma u_t + {\omega_0}^2 u = \beta {\omega_0}^2 E + a\theta E[{\omega_v}^2 E^2 + (\Gamma - \Gamma_v)\frac{{\omega_v}^2}{\Gamma_v}(E^2 - Q) + ({\omega_0}^2 - {\omega_v}^2)Q] \\ + a\theta[E_t Q + 2\frac{{\omega_v}^2}{\Gamma_v}E_t(E^2 - Q) + E_{tt}Q] := h(E, E_t, E_{tt}, Q) \end{aligned} \tag{3.1.2}$$

Hence for the new system we have the four equations

$$B_t = -E_z, \tag{3.1.3a}$$

$$D_t = -\frac{1}{\mu_0}B_z, \tag{3.1.3b}$$

$$D = \epsilon_0(\epsilon_\infty E + u + a(1-\theta)E^3), \tag{3.1.3c}$$

$$u_{tt} + \Gamma u_t + {\omega_0}^2 u = h(E, E_t, E_{tt}, Q). \tag{3.1.3d}$$

Assuming a traveling wave solution on functions B, E, D, Q, and u, that is, assuming

$$f(z,t) = f(z - vt) = f(\xi) \text{ with } \xi := z - vt,$$

we can reduce our system of PDEs into a planar system of three ODEs as follows. Since we are assuming that $E(z,t) = E(\xi)$ we have, by the chain rule,

$$\frac{\partial E}{\partial z} = \frac{\partial E}{\partial t}\frac{\partial t}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial E}{\partial \xi} := Y \tag{3.1.4}$$

Rearranging this equation,

$$Y = -\frac{1}{v}E_t. \tag{3.1.5}$$

Repeating this process yields

$$\frac{dY}{d\xi} = \frac{1}{v^2} E_{tt}. \tag{3.1.6}$$

Notice that by (3.1.1), (3.0.1e) reduces to

$$Q_t = \frac{{\omega_v}^2}{\Gamma_v}(E^2 - Q). \tag{3.1.7}$$

This implies by (3.1.4) that

$$\frac{dQ}{d\xi} = -\frac{1}{v} Q_t = -\frac{1}{v} \frac{{\omega_v}^2}{\Gamma_v}(E^2 - Q). \tag{3.1.8}$$

We now need to solve for E_{tt} in terms of E, Y and Q. If we take (3.1.3a) and (3.1.3b) and use (3.1.4) to get everything in terms of time, it follows that

$$B_t = \frac{1}{v} E_t, \tag{3.1.9a}$$

$$D_t = \frac{1}{\mu_0} \frac{1}{v} B_t. \tag{3.1.9b}$$

Therefore if we equate these two equations to remove B_t, we arrive at the equation,

$$E_t = \mu_0 v^2 D_t, \tag{3.1.10}$$

which, after substituting (3.1.6) and differentiating yields

$$\frac{dY}{d\xi} = \mu_0 D_{tt}. \tag{3.1.11}$$

To get a second equation involving D_{tt} in terms of E and Y we take a time derivate of (3.1.3c), which produces

$$D_t = \epsilon_0(\epsilon_\infty E_t + u_t + 3a(1 - \theta)E^2 E_t).$$

Substuting in (3.1.10) and grouping together D_t terms yields the equation

$$D_t = \frac{\epsilon_0 u_t}{1 - \epsilon_0 \mu_0 v^2 (\epsilon_\infty + 3a(1 - \theta)E^2)}. \tag{3.1.12}$$

Since the denominator is comprised entirely of functions we will be working with we shall call it α turning (3.1.12) into

$$D_t = \frac{\epsilon_0 u_t}{\alpha}. \tag{3.1.13}$$

Note that,

$$\alpha_t = -6a\frac{v^2}{c^2}(1-\theta)EE_t, \tag{3.1.14}$$

which implies that α_t also contains our desired functions. If we differentiate (3.1.13) then, by the quotient rule, we have

$$D_{tt} = \epsilon_0 \frac{u_{tt}\alpha - u_t\alpha_t}{\alpha^2}. \tag{3.1.15}$$

We now need to remove the u terms. We begin by integrating (3.1.10) and assuming that the initial condtions of D and E are identical. It follows that

$$E = \mu_0 v^2 D \tag{3.1.16}$$

Substituting in our equation (3.1.3c),

$$E = \epsilon_0\mu_0 v^2(\epsilon_\infty E + u + aE^3);$$

solved for u

$$u = ((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2)E). \tag{3.1.17}$$

Taking a time derivative produces

$$u_t = (\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t). \tag{3.1.18}$$

Using (3.1.3d) to solve for u_{tt} results in

$$u_{tt} = h - {\omega_0}^2 u - \Gamma u_t,$$

from which, if we then substitute in (3.1.17) and (3.1.18)

$$u_{tt} = \left(h - {\omega_0}^2\left(\left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2\right)E\right) - \Gamma\left(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t\right)\right) \tag{3.1.19}$$

Therefore plugging (3.1.17),(3.1.18), (3.1.19) and our definition of Y' into (3.1.15) produces,

$$\begin{aligned} D_{tt} = \epsilon_0 &\frac{(h - {\omega_0}^2((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2)E) - \Gamma(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t))\alpha}{\alpha^2} \\ &- \frac{(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t)\alpha_t}{\alpha^2} \end{aligned} \tag{3.1.20}$$

Therefore, by (3.1.6),

$$\begin{aligned} \frac{dY}{d\xi} = \mu_0\epsilon_0 &\frac{(h - {\omega_0}^2((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2)E) - \Gamma(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t))\alpha}{\alpha^2} \\ &- \frac{(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t)\alpha_t}{\alpha^2} \end{aligned} \tag{3.1.21}$$

This, however, is incomplete. There is an E_{tt} term inside the h function. As such we will factor it out by introducing a function g such that

$$g(Q, E, E_t) = h(Q, E, E_t, E_{tt}) - a\theta E_{tt} Q = h - a\theta v^2 \frac{dY}{d\xi} Q. \tag{3.1.22}$$

Making use of g, we arrive at a solution for $\frac{dY}{d\xi}$, namely

$$\begin{aligned} \frac{dY}{d\xi} = \mu_0\epsilon_0 &\Bigg(\frac{(g - {\omega_0}^2((\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - aE^2)E) - \Gamma(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t))\alpha}{\alpha^2} \\ &- \frac{(\frac{1}{\epsilon_0\mu_0 v^2} - \epsilon_\infty - 3aEE_t)\alpha_t}{\alpha^2}\Bigg)\left(\frac{\alpha}{\alpha + a\theta v^2\mu_0\epsilon_0 Q}\right) \end{aligned} \tag{3.1.23}$$

We now wish to observe the equilibrium and bifurcation structure of the system and compare to the four-equation case in which the Q_{tt} is not neglected. If the behavior is sufficiently similar we will consider this an acceptable approximation and draw our phase portrait. We begin by finding the equilibria of the system. We will denote these points by

$(Q_{\infty,i}, E_{\infty,i}, Y_{\infty,i})$. Trivially we know that at equilibrium

$$Y_\infty = 0 \text{ and } {E_\infty}^2 = Q_\infty$$

Now looking at (3.1.23) to find E_∞ and assuming that $k_0 = 0$, we get

$$0 = E_\infty \left(\beta {\omega_0}^2 + a\theta({\omega_v}^2 {E_\infty}^2 + ({\omega_0}^2 - {\omega_v}^2) Q_\infty) - \frac{{\omega_0}^2 c^2}{v^2} + {\omega_0}^2 \epsilon_\infty + a(1-\theta){\omega_0}^2 {E_\infty}^2 \right)$$

We therefore have the zero equilibrium,

$$(Q_{\infty,1}, E_{\infty,1}, Y_{\infty,1}) = (0,0,0)$$

and nonzero equilibria,

$$(Q_{\infty,2}, E_{\infty,2}, Y_{\infty,2}) = (\frac{1}{a}(\frac{c^2}{v^2} - \beta - \epsilon_\infty), \sqrt{\frac{1}{a}(\frac{c^2}{v^2} - \beta - \epsilon_\infty)}, 0)$$

$$(Q_{\infty,3}, E_{\infty,3}, Y_{\infty,3}) = (\frac{1}{a}(\frac{c^2}{v^2} - \beta - \epsilon_\infty), -\sqrt{\frac{1}{a}(\frac{c^2}{v^2} - \beta - \epsilon_\infty)}, 0)$$

These equilibria are identical to the equilibria negating the Raman effect. We now move on to computation of the jacobian at the zero equilibria. The appropriate partial derivatives are computed to be,

$$\frac{\partial}{\partial Q}\frac{dQ}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Q}(-v\frac{{\omega_v}^2}{\Gamma_v}(E^2-Q))\bigg|_{E=Y=Q=0} = v\frac{{\omega_v}^2}{\Gamma_v} \tag{3.1.24a}$$

$$\frac{\partial}{\partial E}\frac{dQ}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial E}(-v\frac{{\omega_v}^2}{\Gamma_v}(E^2-Q))\bigg|_{E=Y=Q=0} = 0 \tag{3.1.24b}$$

$$\frac{\partial}{\partial Y}\frac{dQ}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Y}(-v\frac{{\omega_v}^2}{\Gamma_v}(E^2-Q))\bigg|_{E=Y=Q=0} = 0 \tag{3.1.24c}$$

$$\frac{\partial}{\partial Q}\frac{dE}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Q}Y\bigg|_{E=Y=Q=0} = 0 \tag{3.1.24d}$$

$$\frac{\partial}{\partial E}\frac{dE}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial E}Y\bigg|_{E=Y=Q=0} = 0 \tag{3.1.24e}$$

$$\frac{\partial}{\partial Y}\frac{dE}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Y}Y\bigg|_{E=Y=Q=0} = 1 \tag{3.1.24f}$$

$$\frac{\partial}{\partial Q}\frac{dY}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Q}\mu_0\epsilon_0\frac{w\alpha+u_t\alpha_t}{\alpha(\alpha-a\theta v^2\mu_0\epsilon_0 Q)}\bigg|_{E=Y=Q=0} = 0 \tag{3.1.24g}$$

$$\frac{\partial}{\partial E}\frac{dY}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial E}\mu_0\epsilon_0\frac{w\alpha+u_t\alpha_t}{\alpha(\alpha-a\theta v^2\mu_0\epsilon_0 Q)}\bigg|_{E=Y=Q=0} = \psi \tag{3.1.24h}$$

$$\frac{\partial}{\partial Y}\frac{dY}{d\xi}\bigg|_{E=Y=Q=0} = \frac{\partial}{\partial Y}\mu_0\epsilon_0\frac{w\alpha+u_t\alpha_t}{\alpha(\alpha-a\theta v^2\mu_0\epsilon_0 Q)}\bigg|_{E=Y=Q=0} = \delta\Gamma \tag{3.1.24i}$$

Where,

$$\delta = \frac{\frac{c^2}{v^2}-\epsilon_\infty}{v(1-\frac{v^2}{c^2}\epsilon_\infty)} > 0 \text{ and } \psi = {\omega_0}^2\frac{\beta-\frac{c^2}{v^2}-\epsilon_\infty}{1-\frac{v^2}{c^2}\epsilon_\infty}$$

Therefore we have the corresponding jacobian,

$$J_{\Gamma,\Gamma_v}(0,0,0) = \begin{pmatrix} v\frac{{\omega_v}^2}{\Gamma_v} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \psi & \delta\Gamma \end{pmatrix}$$

With characteristic equation,

$$0 = (v\frac{{\omega_v}^2}{\Gamma_v}-\lambda)((\delta\Gamma-\lambda)(-\lambda)-\psi)$$

Therefore we have eigenvalues,

$$\lambda_1 = v\frac{{\omega_v}^2}{\Gamma_v},$$

$$\lambda_2 = \frac{\delta\Gamma - \sqrt{(\delta\Gamma)^2 + 4\psi}}{2},$$

$$\lambda_3 = \frac{\delta\Gamma + \sqrt{(\delta\Gamma)^2 + 4\psi}}{2}.$$

This preserves our Hopf bifurcation for constant Γ, however we no longer have the possibility of compex eigenvalues dependent on Γ_v. This approximation also assumes that the damping constant of the Raman effect is nonzero, and is ill behaved for sufficiently small Γ_v with optimal behavior when it is on the order of ${\omega_v}^2$.

4 A NUMERICAL METHOD FOR THE DAMPED MAXWELL-LORENTZ MODEL WITH KERR EFFECT

4.1 The Euler Time discretization

In this chapter we consider numerical methods for the Maxwell models discussed in the previous chapter. We will discretize the ODEs of the damped Maxwell-Lorentz -Kerr model with the Euler method, a finite difference method for ODEs [2]. We will repeat our bifurcation analysis on the discretized systems, find the fixed points, and their stability and show consistency of the discrete system with the continuous case for some nonzero step sizes. We will utilize a uniform grid with step size $\Delta\xi > 0$. We denote discrete ξ values as $\xi_n = n\Delta\xi$, for $n \in \mathbb{N}$. We also denote the approximate value of the fields E and Y at point ξ_n as E^n and Y^n respectively, i.e.

$$E(\xi_n) \approx E^n,\ Y(\xi_n) \approx Y^n.$$

The Euler method is a first order finite difference scheme and makes the forward approximation

$$\left.\frac{dE}{d\xi}\right|_{\xi_n} \approx \frac{E^{n+1} - E^n}{\Delta\xi}. \tag{4.1.1}$$

Note that this scheme is known to be first order accurate in $\Delta\xi$. We will discretize the system of ODEs found in section 2.1.2, where $\Gamma \neq 0$ and $\theta = 0$. Using the transformations in that section we obtain the system of ODEs,

$$\frac{dE}{d\xi} = Y, \tag{4.1.2a}$$

$$\frac{dY}{d\xi} = \frac{2v^2EY^2 - (v^{-2} - \epsilon)E + E^3/3}{1 - v^2 - v^2E^2} + \Gamma\frac{Y}{v}. \tag{4.1.2b}$$

For simplicity in this section we will write

$$\frac{dE}{d\xi} = f(E, Y), \tag{4.1.3a}$$

$$\frac{dY}{d\xi} = g(E, Y). \tag{4.1.3b}$$

We will approximate E, Y on the discrete grid consisting of the points $\xi_0, \xi_1, \xi_2, \ldots \xi_n \ldots$, along with their derivatives. Our discrete scheme can therefore be written as the following set of discrete equations

$$D_{\Delta\xi}E^n =: \frac{E^{n+1} - E^n}{\Delta\xi} = f\Big(E^n, Y^n\Big), \tag{4.1.4a}$$

$$\tilde{D}_{\Delta\xi}Y^n := \frac{Y^{n+1} - Y^n}{\Delta\xi} = g\Big(E^n, Y^n\Big). \tag{4.1.4b}$$

Thus, under discretization, our system of ODEs becomes the algebraic system of difference equations,

$$E^{n+1} = E^n + (\Delta\xi)Y^n, \tag{4.1.5a}$$

$$Y^{n+1} = Y^n + (\Delta\xi)\left(\frac{2(v^2)(E^n)(Y^n)^2 - (v^{-2} - \epsilon)E^n + (E^n)^3/3}{1 - v^2 - v^2(E^n)^2} + \frac{\Gamma}{v}Y^n\right) \tag{4.1.5b}$$

Thus, our system is now written in the form of a two dimensional first order discrete dynamical system

$$E^{n+1} = F(E^n, Y^n), \tag{4.1.6a}$$

$$Y^{n+1} = G(E^n, Y^n). \tag{4.1.6b}$$

4.2 Analysis of Discrete Dynamical Systems

To begin our study of this system, we first define analogous quantities for discrete dynamical systems to those defined in the dynamical systems theory used previously

Definition 4.2.1. (**Fixed Point**) The point (E_∞, Y_∞) is called a fixed point of the system if it is a constant solution to the system of discrete equations, i.e. for which

$$E_\infty = F(E_\infty, Y_\infty), \tag{4.2.1a}$$

$$Y_\infty = G(E_\infty, Y_\infty). \tag{4.2.1b}$$

Definition 4.2.2. (**Amplification Matrix**) We define the amplification matrix or Jacobian computed at the fixed point (E_∞, Y_∞) to be the matrix

$$J(E_\infty, Y_\infty) = \begin{pmatrix} F_E(E_\infty, Y_\infty) & F_Y(E_\infty, Y_\infty) \\ G_E(E_\infty, Y_\infty) & G_Y(E_\infty, Y_\infty) \end{pmatrix}. \tag{4.2.2}$$

We note that if ρ is the spectral radius (largest absolute value of all the eigenvalues), then ρ will determine the stability of the fixed points.

We first consider the position of fixed points which are the discrete analog of the equilibria in the continuous case. Ideally, these points would be identical in both cases. If (E_∞, Y_∞) is a fixed point, then we have that

$$E_\infty = E_\infty + (\Delta\xi)Y_\infty, \tag{4.2.3a}$$

$$Y_\infty = Y_\infty + (\Delta\xi)\left(\frac{2(v^2)(E_\infty)(Y_\infty)^2 - (v^{-2} - \epsilon)E_\infty + ({E_\infty}^3)/3}{1 - v^2 - v^2(E_\infty)^2} + \frac{\Gamma}{v}(Y_\infty)\right). \tag{4.2.3b}$$

Our first equation implies that since $\Delta\xi > 0$, $Y_\infty = 0$. Substituting this into our second equation yields,

$$0 = (\Delta\xi)\frac{-(v^{-2} - \epsilon)E_\infty + ({E_\infty}^3)/3}{1 - v^2 - v^2(E_\infty)^2},$$

Which yields three fixed points the zero,

$$(E_\infty, Y_\infty) = (0, 0)$$

and the nonzero fixed points

$$(E_\infty, Y_\infty) = (\pm\sqrt{3(\epsilon - v^{-2})}, 0).$$

These are identical to the continuous case which is indeed ideal. We now need to figure out if the properties of stability still hold at these points in the discrete case. This, however, requires a new definition of stability. We first present some theorems on the stability of the difference equations as dependent on the eigenvalues of J.

Theorem 7. *[1] The fixed point* (E_∞, Y_∞)*, which is a constant solution of the discrete linearized system of difference equations* (4.2.3a)*, is asymptotically stable iff the eigenvalues of* J *are within the unit disk of the complex plane, if and only if*

$$|\mathrm{Tr}(J)| < 1 + \det(J) < 2,$$

where in the Jury conditions above, $\mathrm{Tr}(\mathrm{J})$ *is the Trace of the Jacobian matrix at the fixed point and* $\det(J)$ *is the determinant of the Jacobian at he fixed point.*

Theorem 8. *[1] The fixed point* (E_∞, Y_∞) *of the linearized system* (4.2.3a) *is stable iff the eigenvalues of* J *are within the unit disk of the complex plane where those on the unit circle are semisimple.*

We start by computing the various components of the Amplification Matrix at the zero equilibrium. We have

$$F_E(E_\infty, Y_\infty) = 1, \tag{4.2.4a}$$

$$F_Y(E_\infty, Y_\infty) = \Delta\xi, \tag{4.2.4b}$$

$$G_E(E_\infty, Y_\infty) = \Delta\xi\left(\frac{2v^2(1 - v^2 + v^2E_\infty^2)Y_\infty^2 - (v^{-2} - \epsilon)(1 - v^2)}{(1 - v^2 - v^2(E_\infty^2))^2})\right),$$

$$+ \Delta\xi\left(\frac{v^2(\epsilon - 1 - \frac{5}{3}E_\infty^2)E_\infty^2}{(1 - v^2 - v^2(E_\infty^2))^2})\right), \tag{4.2.4c}$$

$$G_Y(E_\infty, Y_\infty) = 1 - \frac{\Delta\xi\Gamma}{v} + (\Delta\xi)\left(\frac{4v^2Y_\infty E_\infty}{1 - v^2 - v^2{E_\infty}^2}\right). \tag{4.2.4d}$$

Notice that the characteristic equation of the Jacobian is

$$\lambda^2 - (F_E + G_Y)\lambda + (F_EG_Y - G_EF_Y) = 0.$$

Consider the zero fixed point. In this case, the Jacobian is given by

$$J(E_\infty = 0, Y_\infty = 0) = \begin{pmatrix} 1 & \Delta\xi \\ \Delta\xi(\epsilon - \frac{1}{v^2}) & 1 + \Delta\xi\frac{\Gamma}{v} \end{pmatrix}. \tag{4.2.5}$$

To determine the stability of the zero fixed point, we determine the Jury condition to be

$$\Delta\xi^2(\epsilon - \frac{1}{v^2}) < 0.$$

Since $\Delta\xi > 0$, this implies a condition on v and ϵ. Thus, if $\epsilon < v^{-2}$, then the Jury conditions are satisfied and the zero fixed point is asympototically stable.

5 CONCLUSION

In this report, we have considered inclusion of the nonlinear factors in Maxwell's equations. In particular, we have considered models for the Kerr and Raman effects in nonlinear optics. Under the assumption of a traveling wave solution in the corresponding nonlinear materials, we have analyzed the system of ordinary differential equations that arise. Degenerate Hopf bifurcations and Pitchfork bifurcations are shown to arise in a bifurcation analysis in which different parameters in the nonlinear model are varied. Our analysis was for the case of a one spatial dimensional system. We have also considered discretization of the ODE systems that result from Maxwell's equations under the assumption of a travelling wave solution. In the future we will consider nonlinear Maxwell models for Kerr, Lorentz and Raman effects in two and three spatial dimensions and also consider PDE based discretizations.

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