Much Ado About Weak Measurement

by

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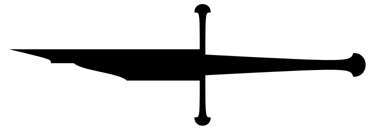

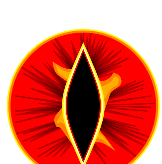

Abstract

written in the style of A Silence of Three Parts from *The Kingkiller Chronicle* by Patrick Rothfuss

I had to write again. My research lay in science, and it was a science of three parts.

The first part made a careful, introductory chapter, begged for by things that were lacking. It summarizes the von Neumann measurement model, derived from the Stern-Gerlach experiment, and it explains how weak values arise from weak interactions between quantum systems and von Neumann probes. If there had been more literature about the error of the weak value approximation and how to bound it, then perhaps I would have read myself into a satisfied sleep and never written this chapter. If there had been talk of operator norms . . . but no, of course there was not. In fact, the focus of the literature was elsewhere, and so I had to provide these things; now the first chapter remains.

In the course of my research, a method came to my attention, which sought to justify the role of the weak value in approximating the effect of a weak interaction. This method, feedback compensation, estimates the effect of the interaction and applies it in reverse to attempt to reset the probe state. Weak values (at least, their real parts) were found to be the optimal estimates of interactions, providing the ideal amount of compensation to reset probes. After explaining how others found this, I added a specific application to the nested Mach-Zehnder interferometer as a proof-of-concept, along with some interpretational caveats. In doing this, I added a second, more tangible chapter to the first. They made a dynamic duo of sorts, a partnership.

The third part was not a common thing to be familiar with. If you worked in quantum cosmology, you would be keenly aware of the need to generalize the Copenhagen interpretation, and might then know about the consistent histories formalism. It comes from the need to determine when probabilities should be assigned to the outcomes of a quantum system, even in the absence of classical measurements.

Consistent histories has been used by others to look at the odd implications of weak values, and a condition relating the consistency of a family of histories to the strangeness of weak values has been derived before, suggesting an incompatibility between non-sharp weak values and consistent families.

My research comes to an end with my extension to this third part (not counting appendices). I deem this appropriate, as it is the most niche science of the three. I mixed weak values and consistent histories while including a weakly coupled probe in the composite system, and accounted for its end state while reasoning about the viable histories of the system. It is the patient, late-hour science of a man who is waiting to graduate.

Key Words: consistent histories, feedback compensation, Mach-Zehnder interferometer, path detection, weak measurement, weak values

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Contents

Chapter 1

The Weak Value Approximation

Abstract

The concept of weak measurements is explained, starting from a baseline of the von Neumann measurement model, based on the Stern-Gerlach experiment. From there, a description of how weak values appear in the von Neumann probe is provided. The chapter is concluded with an investigation of what conditions are sufficient to bound the error of the weak value approximation, conducted with a level of care not found in the literature.

1.1 Von Neumann Measurements

The von Neumann measurement model was first put forth in [1] (which was translated as [2]). However, for a clearer walkthrough of the process, I turned to the work of Pier Mello [3]. Conceptually, the model is based on the Stern-Gerlach experiment. Consider, if you will, taking a particle with spin state $|\psi_{spin}^{(0)}\rangle$ and position state $|\chi(0)\rangle$, which together give us the overall initial system state $|\Psi(0)\rangle = |\psi_{spin}^{(0)}\rangle \otimes |\chi(0)\rangle$. The spin part is the "system" that we seek to measure, while the position will act as the "probe" we will couple to the system.

In Fig. 1.1, we establish the setup of the Stern-Gerlach experiment. We have our particle (which we will take to be spin-$\frac{1}{2}$) traveling along the y-axis. Two magnets establish a non-uniform magnetic field in the path of the particle, and we take Mello's assumptions:

1. $\vec{B} = 0$ outside of the gap between the magnets;
2. B_z is the only component of any significance;
3. $B_z(z) = B'(z)z$ within the gap;
4. translational motion in the y-direction can be simulated with a time-dependent interaction which acts only during the time when the particle is in the gap; and
5. the x degree of freedom can be ignored.

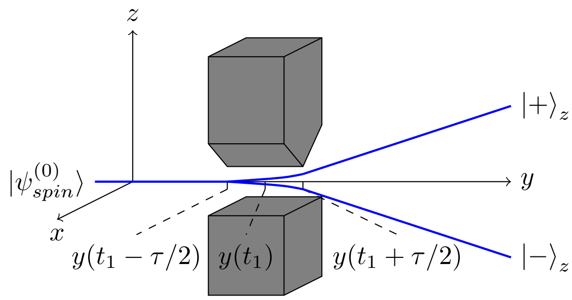


Figure 1.1: Stern-Gerlach Experiment Diagram The spin-$\frac{1}{2}$ particle enters from the left with initial spin state $|\psi^{(0)}_{spin}\rangle$ and travels along the y-axis. It enters the gap between the two magnets (the gray polyhedrons) at $t_1 - \tau/2$, and leaves at time $t_1 + \tau/2$. The non-uniform magnetic field within the gap interacts with the magnetic moment of the particle, causing it to be deflected up if its state is $|+\rangle_z$, or down if its state is $|-\rangle_z$. If the initial state was a superposition of the two spin states, the particle path will be a superposition of these two trajectories.

This gives the model Hamiltonian

$$\begin{aligned}\hat{H}(t) &= \frac{\hat{p}_z^2}{2m} - \vec{\mu}\cdot\vec{B}\theta_{t_1,\tau}(t) \\ &= \frac{\hat{p}_z^2}{2m} - \left[\mu_B B_z'(0)\tau\right]\frac{\theta_{t_1,\tau}(t)}{\tau}\hat{\sigma}_z\hat{z} \\ &= \frac{\hat{p}_z^2}{2m} - \epsilon g(t)\hat{\sigma}_z\hat{z},\end{aligned} \tag{1.1}$$

where $\epsilon = \mu_B B_z'(0)\tau$ and $g(t) = \frac{\theta_{t_1,\tau}(t)}{\tau}$. The function $\theta_{t_1,\tau}$ is an indicator function, which is of unit value when the particle is in the gap:

$$\theta_{t_1,\tau}(t) = \begin{cases} 1, & |t-t_1| < \frac{\tau}{2}; \\ 0, & \text{otherwise.}\end{cases} \tag{1.2}$$

This simulates the interaction, as we desired in assumption 4. Furthermore, note that

$$\int_0^\infty g(t)dt = 1, \tag{1.3}$$

with the strong assumption that $\tau \ll t_1$. As such, we can make the approximation $g(t) \approx \delta(t-t_1)$ and rewrite the Hamiltonian as

$$\begin{aligned}\hat{H}(t) &= \frac{\hat{p}_z^2}{2m} - \epsilon\delta(t-t_1)\hat{\sigma}_z\hat{z} \\ &= \hat{H}_0 + \hat{V}(t),\end{aligned} \tag{1.4}$$

where we have identified $\hat{H}_0 = \frac{\hat{p}_z^2}{2m}$ as the kinetic energy operator for the variable z, and $\hat{V}(t)$ is the interaction itself. By expressing the interaction in the interaction picture and solving the Schrödinger equation, Mello demonstrates that the evolution operator taking the particle to $t > t_1$ is

$$U(t,0) = e^{-i\hat{H}_0(t-t_1)/\hbar}e^{i\epsilon\hat{\sigma}_z\hat{z}/\hbar}e^{-i\hat{H}_0 t_1/\hbar}. \tag{1.5}$$

That particular calculation I shall not replicate, for I have reproduced enough of Mello's work already.

Ultimately, this means that the final state of the overall system is

$$\begin{aligned} |\Psi(t)\rangle &= U(t,0)\,|\Psi(0)\rangle \\ &= e^{-i\hat{H}_0(t-t_1)/\hbar} e^{i\epsilon\hat{\sigma}_z\hat{z}/\hbar} e^{-i\hat{H}_0 t_1/\hbar}\,|\psi^{(0)}_{spin}\rangle\,|\chi(0)\rangle \\ &= e^{-i\hat{H}_0(t-t_1)/\hbar} e^{i\epsilon\hat{\sigma}_z\hat{z}/\hbar} \left[\sum_{\sigma=\pm1} P^{\hat{\sigma}_z}_{\sigma}\right] |\psi^{(0)}_{spin}\rangle\, e^{-i\hat{H}_0 t_1/\hbar}\,|\chi(0)\rangle \\ &= \sum_{\sigma=\pm1} P^{\hat{\sigma}_z}_{\sigma}\,|\psi^{(0)}_{spin}\rangle\, e^{-i\hat{H}_0(t-t_1)/\hbar} e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(t_1)\rangle\,. \end{aligned} \tag{1.6}$$

As such, the spin state component $P^{\hat{\sigma}_z}_{\sigma}\,|\psi^{(0)}_{spin}\rangle$ has become entangled with the position component $e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(t_1)\rangle$.

The operator $\hat{\sigma}_z$ is a Pauli matrix, which Mello terms the "observable for the system proper," and $\hat{z}$ and $\hat{p}_z$ are the position and momentum operators, which are termed the "probe canonical variables." As such, we consider $e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(t_1)\rangle$ to be the probe state, and we can measure the probe state (in either the position or the momentum basis) to gain information about the spin state.

In the physical Stern-Gerlach experiment, we would certainly measure the state of the particle by recording its vertical position upon colliding with a detector screen. However, that involves evolving the position state in time; without time evolution, the two trajectories will never diverge. However, if we examine the momentum of the particle, we need not bother ourselves with how the position evolves in time. Let us suppose that $\hat{H}_0 = 0$, and take our position state to be a Gaussian with standard deviation Δ centered at $z = 0$:

$$|\chi(0)\rangle = \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-z^2/4\Delta^2}\,|z\rangle = \int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-p_z^2/4(\hbar/2\Delta)^2}\,|p_z\rangle\,. \tag{1.7}$$

With $\hat{H}_0 = 0$, we have $U(t,0) = e^{i\epsilon\hat{\sigma}_z\hat{z}/\hbar}$, which means

$$\begin{aligned} |\Psi(t)\rangle &= e^{i\epsilon\hat{\sigma}_z\hat{z}/\hbar}\,|\Psi(0)\rangle \\ &= \sum_{\sigma=\pm1} P^{\hat{\sigma}_z}_{\sigma}\,|\psi^{(0)}_{spin}\rangle\, e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(0)\rangle\,. \end{aligned} \tag{1.8}$$

This gives us the final probe state

$$\begin{aligned} e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(0)\rangle &= e^{i\epsilon\sigma\hat{z}/\hbar} \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-z^2/4\Delta^2}\,|z\rangle \\ &= \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-(z^2/4\Delta^2)+i\epsilon\sigma z/\hbar}\,|z\rangle\,. \end{aligned} \tag{1.9}$$

This makes more sense in the momentum basis, so let us insert the identity:

$$\begin{aligned} e^{i\epsilon\sigma\hat{z}/\hbar}\,|\chi(0)\rangle &= \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-(z^2/4\Delta^2)+i\epsilon\sigma z/\hbar} \int_{-\infty}^{\infty} dp_z\,|p_z\rangle\,\langle p_z|z\rangle \\ &= \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-(z^2/4\Delta^2)+i\epsilon\sigma z/\hbar} \int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt{2\pi\hbar}} e^{-ip_z z/\hbar}\,|p_z\rangle \\ &= \int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}\sqrt{2\pi\hbar}} \exp\left[-\frac{z^2}{4\Delta^2} - \frac{i}{\hbar}\left(p_z - \epsilon\sigma\right) z\right] |p_z\rangle\,. \end{aligned} \tag{1.10}$$

To go further, we will need to complete the square:

$$e^{i\epsilon\sigma\hat{z}/\hbar}|\chi(0)\rangle = \int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dz \frac{1}{\sqrt[4]{2\pi\Delta^2}\sqrt{2\pi\hbar}} \exp\left[-\frac{z^2}{4\Delta^2} - 2\frac{iz}{2\Delta}\frac{\Delta}{\hbar}(p_z - \epsilon\sigma) + \frac{\Delta^2(p_z-\epsilon\sigma)^2}{\hbar^2} - \frac{\Delta^2(p_z-\epsilon\sigma)^2}{\hbar^2}\right]|p_z\rangle$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-(p_z-\epsilon\sigma)^2/4(\hbar/2\Delta)^2} \int_{-\infty}^{\infty}\frac{dz}{2\Delta}\exp\left\{\left[\frac{\Delta(p_z-\epsilon\sigma)}{\hbar} - i\frac{z}{2\Delta}\right]^2\right\}|p_z\rangle\,. \tag{1.11}$$

At this dramatic juncture, we make the change of variables $iz' = -i\frac{z}{2\Delta} + \frac{\Delta(p_z-\epsilon\sigma)}{\hbar}$, which leads us to

$$e^{i\epsilon\sigma\hat{z}/\hbar}|\chi(0)\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-(p_z-\epsilon\sigma)^2/4(\hbar/2\Delta)^2} \int_{\infty-i\Delta(p_z-\epsilon\sigma)/\hbar}^{-\infty-i\Delta(p_z-\epsilon\sigma)/\hbar} (-dz')e^{-z'^2}|p_z\rangle$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-(p_z-\epsilon\sigma)^2/4(\hbar/2\Delta)^2} \int_{-\infty-i\Delta(p_z-\epsilon\sigma)/\hbar}^{\infty-i\Delta(p_z-\epsilon\sigma)/\hbar} dz' e^{-z'^2}|p_z\rangle \tag{1.12}$$
$$= \int_{-\infty}^{\infty} dp_z \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-(p_z-\epsilon\sigma)^2/4(\hbar/2\Delta)^2}|p_z\rangle\,.$$

As we can see, the center of the Gaussian has been shifted by $\epsilon\sigma$, so the momentum wave function entangled with the spin-up state will have an average momentum of ϵ, while the momentum wave function entangled with the spin-down state will have an average momentum of $-\epsilon$.

Let us specify that $|\psi^{(0)}_{spin}\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$ (where $\alpha^2+\beta^2 = 1$). This turns our final state given in Eqn. 1.8 into

$$|\Psi(t)\rangle = \alpha|+\rangle_z e^{i\epsilon\hat{z}/\hbar}|\chi(0)\rangle + \beta|-\rangle_z e^{-i\epsilon\hat{z}/\hbar}|\chi(0)\rangle\,. \tag{1.13}$$

Thus, the probability density function to find the particle in the state $|\sigma\rangle\otimes|p_z\rangle$ is

$$\phi(\sigma,p_z) = \langle\Psi(t)|\left[|\sigma\rangle\langle\sigma|\otimes|p_z\rangle\langle p_z|\right]|\Psi(t)\rangle$$
$$= \begin{cases} \frac{\alpha^2}{\sqrt{2\pi(\hbar/2\Delta)^2}}e^{-(p_z-\epsilon)^2/2(\hbar/2\Delta)^2}, & \sigma = +1; \\ \frac{\beta^2}{\sqrt{2\pi(\hbar/2\Delta)^2}}e^{-(p_z+\epsilon)^2/2(\hbar/2\Delta)^2}, & \sigma = -1. \end{cases} \tag{1.14}$$

Our desire is to use the momentum information (the probe state) to figure out the spin, so we should endeavor to find the conditional probability $\mathscr{P}(\sigma|p_z)$. This function should satisfy $\phi(\sigma,p_z) = \mathscr{P}(\sigma|p_z)\phi_{p_z}(p_z)$, so we can find it by determining the marginal probability density function $\phi_{p_z}(p_z)$, which is

$$\phi_{p_z}(p_z) = \phi(+1,p_z) + \phi(-1,p_z) = \frac{\alpha^2}{\sqrt{2\pi(\hbar/2\Delta)^2}}e^{-(p_z-\epsilon)^2/2(\hbar/2\Delta)^2} + \frac{\beta^2}{\sqrt{2\pi(\hbar/2\Delta)^2}}e^{-(p_z+\epsilon)^2/2(\hbar/2\Delta)^2}. \tag{1.15}$$

This gives us the conditional probability

$$\mathscr{P}(\sigma|p_z) = \frac{\phi(\sigma,p_z)}{\phi_{p_z}(p_z)} = \begin{cases} \frac{\alpha^2 e^{-(p_z-\epsilon)^2/2(\hbar/2\Delta)^2}}{\alpha^2 e^{-(p_z-\epsilon)^2/2(\hbar/2\Delta)^2} + \beta^2 e^{-(p_z+\epsilon)^2/2(\hbar/2\Delta)^2}}, & \sigma = +1; \\ \frac{\beta^2 e^{-(p_z+\epsilon)^2/2(\hbar/2\Delta)^2}}{\alpha^2 e^{-(p_z-\epsilon)^2/2(\hbar/2\Delta)^2} + \beta^2 e^{-(p_z+\epsilon)^2/2(\hbar/2\Delta)^2}}, & \sigma = -1. \end{cases}$$
$$= \begin{cases} \frac{\alpha^2}{\alpha^2+\beta^2 e^{-2p_z\epsilon/(\hbar/2\Delta)^2}}, & \sigma = +1; \\ \frac{\beta^2}{\beta^2+\alpha^2 e^{2p_z\epsilon/(\hbar/2\Delta)^2}}, & \sigma = -1. \end{cases} \tag{1.16}$$

If $\Delta \to \infty$ (which makes for a very wide position Gaussian, but a very sharp distribution in momentum space), then our exponentials $e^{\pm 2p_z\epsilon/(\hbar/2\Delta)^2}$ either explode to infinity or shrink to zero depending on the sign of p_z. In fact, if $p_z = 0$, the exponentials are 1, and we gain no information about our system for any value of Δ (and the result is the same as the $\Delta \approx 0$ case below). Rather, if we assume $p_z > 0$, then

$$\mathscr{P}(\sigma|p_z) \approx \begin{cases} 1, & \sigma = +1; \\ 0, & \sigma = -1, \end{cases} \tag{1.17}$$

whereas if $p_z < 0$, then

$$\mathscr{P}(\sigma|p_z) \approx \begin{cases} 0, & \sigma = +1; \\ 1, & \sigma = -1. \end{cases} \tag{1.18}$$

That is, with the peaks in momentum space highly localized around their new centers at $\pm\epsilon$, the final momentum of the particle is highly correlated with the spin of the particle, so if $p_z > 0$, we know that the particle is spin up, and if $p_z < 0$, we know that the particle is spin down. This is a conventional strong measurement, as the probe gives us perfect information about the system.

If instead $\Delta \approx 0$ (which makes for a very sharp position Gaussian, but a wide distribution in momentum space), then the exponentials in the denominators of Eqn. 1.16 will approach 1, meaning

$$\mathscr{P}(\sigma|p_z) \approx \begin{cases} \alpha^2, & \sigma = +1; \\ \beta^2, & \sigma = -1. \end{cases} \tag{1.19}$$

This is so, because the momentum wave function is so delocalized that it is hard to distinguish a Gaussian centered at $+\epsilon$ from one centered at $-\epsilon$. The momentum of the particle after the interaction is barely correlated with the spin of the particle, so the spin probabilities depend on the initial state, rather than the measured momentum. This makes such a measurement weak, as it provides very little information about the system state.

This setup can be generalized beyond measuring spin, and we may make the probe more abstract than the physical position of the particle. To do so, we define the probe canonical position and momentum as $\hat{q}$ and $\hat{p}$, respectively, and define the inital probe state

$$|\mathscr{Q}\rangle = \int_{-\infty}^{\infty} dq \frac{1}{\sqrt[4]{2\pi\Delta^2}} e^{-q^2/4\Delta^2} |q\rangle = \int_{-\infty}^{\infty} dp \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-p^2/4(\hbar/2\Delta)^2} |p\rangle \,. \tag{1.20}$$

We then define the interaction Hamiltonian to be $\hat{V}(t) = -\epsilon\delta(t - t_1)\hat{A}\hat{q}$, where $\hat{A}$ is the observable for the system proper. The results of our Stern-Gerlach experiment example extend naturally to this more general system.

1.2 Extracting Weak Values

Suppose our system proper begins in state $|\psi(0)\rangle$, and we make a final measurement that discovers the system is in state $|\phi(t_2)\rangle$. We also have an additional degree of freedom in the form of the probe, which begins in state $|\mathscr{Q}\rangle$, as given in Eqn. 1.20. The Hamiltonian for the combined system is

$$\hat{H}(t) = \hat{H}_{sys} - \epsilon\delta(t - t_1)\hat{A}\hat{q}, \tag{1.21}$$

where $\hat{H}_{sys}$ is the Hamiltonian governing the evolution of the system proper, and $\hat{A}$ is an observable for the system proper. As such, for the combined system, we have evolve from $t = 0$ to $t = t_2$ by the evolution operator

$$U(t_2, 0) = e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} e^{-i\hat{H}_{sys}t_1/\hbar}. \tag{1.22}$$

At the time of the final measurement, the projection postulate dictates that our combined system, before renormalization, is in the state

$$|\phi(t_2)\rangle \langle\phi(t_2)|U(t_2, 0)|\psi(0)\rangle |\mathscr{Q}\rangle = |\phi(t_2)\rangle \langle\phi(t_2)|e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} e^{-i\hat{H}_{sys}t_1/\hbar}|\psi(0)\rangle |\mathscr{Q}\rangle . \tag{1.23}$$

To be compact in our notation, let us define $|\psi(t_1)\rangle = e^{-i\hat{H}_{sys}t_1/\hbar}|\psi(0)\rangle$ and $|\phi(t_1)\rangle = e^{-i\hat{H}_{sys}(t_1-t_2)/\hbar}|\phi(t_2)\rangle$, which gives us the state

$$|\phi(t_2)\rangle \langle\phi(t_1)|e^{i\epsilon\hat{A}\hat{q}/\hbar}|\psi(t_1)\rangle |\mathscr{Q}\rangle . \tag{1.24}$$

We then expand the exponential to get

$$|\phi(t_2)\rangle \langle\phi(t_1)| \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j \hat{A}^j|\psi(t_1)\rangle |\mathscr{Q}\rangle = |\phi(t_2)\rangle \langle\phi(t_1)|\psi(t_1)\rangle \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j \frac{\langle\phi(t_1)|\hat{A}^j|\psi(t_1)\rangle}{\langle\phi(t_1)|\psi(t_1)\rangle} |\mathscr{Q}\rangle . \tag{1.25}$$

We compact our notation yet again by defining what Aharonov and Vaidman in [4] call the "weak value" of an observable $\hat{A}$:

$$A_w = \frac{\langle\phi(t_1)|\hat{A}|\psi(t_1)\rangle}{\langle\phi(t_1)|\psi(t_1)\rangle}. \tag{1.26}$$

However, in our current state, we have instead the weak value of $\hat{A}^j$, giving us

$$\begin{aligned} |\phi(t_2)\rangle \langle\phi(t_1)|\psi(t_1)\rangle \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j (\hat{A}^j)_w |\mathscr{Q}\rangle &= |\phi(t_2)\rangle \langle\phi(t_1)|\psi(t_1)\rangle \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{i\epsilon A_w\hat{q}}{\hbar}\right)^j |\mathscr{Q}\rangle \\ &+ |\phi(t_2)\rangle \langle\phi(t_1)|\psi(t_1)\rangle \sum_{j=2}^{\infty} \frac{1}{j!} \left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j \left[(\hat{A}^j)_w - A_w^j\right] |\mathscr{Q}\rangle . \end{aligned} \tag{1.27}$$

Note that two terms of zero value have been removed from the second sum. However, that is not enough, as we want to dispense with the second term of Eqn. 1.27 entirely (see Section 1.3). That would leave us with the final normalized state

$$\frac{\langle\phi(t_1)|\psi(t_1)\rangle}{|\langle\phi(t_1)|\psi(t_1)\rangle|} |\phi(t_2)\rangle e^{i\epsilon A_w\hat{q}/\hbar} |\mathscr{Q}\rangle . \tag{1.28}$$

Note that the operator $e^{i\epsilon A_w\hat{q}/\hbar}$ will shift the momentum state of the probe by ϵA_w, giving us the final probe state

$$\int_{-\infty}^{\infty} dp \frac{1}{\sqrt[4]{2\pi(\hbar/2\Delta)^2}} e^{-(p-\epsilon A_w)^2/4(\hbar/2\Delta)^2} |p\rangle , \tag{1.29}$$

which can subsequently be measured projectively to deduce information about the system proper.

The approximation of Eqn. 1.28 is achieved by manipulation of the coupling constant ϵ and the probe state width Δ. Having a very small Δ means our final probe state will be very wide. For a sufficiently small ϵA_w, we will see a large overlap with the Gaussian momentum state centered at the origin, making them hard to distinguish after a single measurement. By taking the measurement on many identical systems, eventually the distribution of results may have enough data to extract a clear average momentum, thus revealing the weak value of the observable the probe coupled to.

1.3 Error Bounds on Equation 1.27

The literature prefers to focus on the implications of the weak value approximation, rather than the math that supposedly validates the approximation itself. I, however, wish to put this math on center stage, and with greater care, provide a partial validation of the approximation.

In order to get to Eqn. 1.28, we must ensure that the second term of Eqn. 1.27 is small in size. Its norm is

$$\left|\left| |\phi(t_2)\rangle \langle\phi(t_1)|\psi(t_1)\rangle \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j \left[(\hat{A}^j)_w - A_w^j\right]|\mathscr{Q}\rangle \right|\right| \leq |\langle\phi(t_1)|\psi(t_1)\rangle| \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{\epsilon}{\hbar}\right)^j \left|(\hat{A}^j)_w - A_w^j\right| \left|\left||\phi(t_2)\rangle\, \hat{q}^j\, |\mathscr{Q}\rangle\right|\right|. \tag{1.30}$$

The crucial norm of $|\phi(t_2)\rangle\, \hat{q}^j\, |\mathscr{Q}\rangle$ can be worked on to obtain

$$\begin{aligned} \left|\left||\phi(t_2)\rangle\, \hat{q}^j\, |\mathscr{Q}\rangle\right|\right| &= \sqrt{\langle\phi(t_2)|\phi(t_2)\rangle \langle\mathscr{Q}|\hat{q}^{2j}|\mathscr{Q}\rangle} \\ &= \sqrt{\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' \frac{q'^{2j}}{\sqrt{2\pi\Delta^2}} \exp\left(\frac{-q^2 - q'^2}{4\Delta^2}\right)\langle q|q'\rangle} \\ &= \sqrt{\int_{-\infty}^{\infty} dq \frac{q^{2j}}{\sqrt{2\pi\Delta^2}} \exp\left(\frac{-q^2}{2\Delta^2}\right)} \\ &= \sqrt{2\int_{0}^{\infty} dq \frac{q^{2j}}{\sqrt{2\pi\Delta^2}} \exp\left(\frac{-q^2}{2\Delta^2}\right)}. \end{aligned} \tag{1.31}$$

We then make the substitution $x = \frac{q^2}{2\Delta^2}$ to obtain

$$\begin{aligned} \left|\left|\hat{q}^j\, |\mathscr{Q}\rangle\right|\right| &= \sqrt{2\int_{0}^{\infty} dx \frac{\Delta^2(2\Delta^2 x)^{\frac{2j-1}{2}}}{\sqrt{2\pi\Delta^2}} e^{-x}} \\ &= \sqrt{\frac{(\sqrt{2}\Delta)^{2j}}{\sqrt{\pi}} \int_{0}^{\infty} dx x^{(j+\frac{1}{2})-1} e^{-x}}. \end{aligned} \tag{1.32}$$

We desire this form, because the gamma function is defined by $\Gamma(j) = \int_0^{\infty} dx x^{j-1} e^{-x}$ [5], so that integral is $\Gamma\left(j+\frac{1}{2}\right)$. Since $\Gamma\left(j+\frac{1}{2}\right) = \frac{(2j-1)!!}{2^j}\sqrt{\pi}$ [6] (see Appendix A.1), we find

$$\left|\left|\hat{q}^j\, |\mathscr{Q}\rangle\right|\right| = \Delta^j \sqrt{(2j-1)!!}, \tag{1.33}$$

which means

$$\left|\left|\,|\phi(t_2)\rangle\langle\phi(t_1)|\psi(t_1)\rangle\sum_{j=2}^{\infty}\frac{1}{j!}\left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j\left[(\hat{A}^j)_w - A_w^j\right]|\mathscr{Q}\rangle\right|\right| \leq |\langle\phi(t_1)|\psi(t_1)\rangle|\sum_{j=2}^{\infty}\frac{\sqrt{(2j-1)!!}}{j!}\left(\frac{\epsilon\Delta}{\hbar}\right)^j\left|(\hat{A}^j)_w - A_w^j\right|. \tag{1.34}$$

As such, if we wish to make the approximation that takes us to Eqn. 1.28, we need to make the right hand side of Eqn. 1.34 negligibly small, which requires us to bound the sum. We can determine Δ, as we control the preparation of the probe state, and we have control over the coupling strength ϵ, as we control the experimental setup. As such, the only two factors of concern are $\frac{\sqrt{(2j-1)!!}}{j!}$ and $\left|(\hat{A}^j)_w - A_w^j\right|$.

Regarding the former, it can be shown (in Appendix A.2) that, for any $a \geq 1$ there exists a natural number k such that $\frac{\sqrt{(2j-1)!!}}{j!} < a^{-j}$ for all $j \geq k$. This is an exceptional decay, stronger than any exponential growth, so as long as $\left|(\hat{A}^j)_w - A_w^j\right|$ does not grow at a more than exponential rate, the right hand side of Eqn. 1.34 will converge. For now, let it suffice that $\frac{\sqrt{(2j-1)!!}}{j!} \leq 1$ for all $j \in \mathbb{N}$.

Ultimately, we need more than just convergence, which requires us to know how $\left|(\hat{A}^j)_w - A_w^j\right|$ behaves. There are levels of complexity to this, the simplest being if either $|\psi(t_1)\rangle$ or $|\phi(t_1)\rangle$ is an eigenvector of $\hat{A}$. In this case, A_w is simply the associated eigenvalue (unless of course $|\psi(t_1)\rangle$ and $|\phi(t_1)\rangle$ are orthogonal, in which case the probe state is not shifted). The weak value $(\hat{A}^j)_w$ is this eigenvalue to the jth power, so $\left|(\hat{A}^j)_w - A_w^j\right| = 0$, and no approximation is necessary.

Increasing the complexity, let us consider bounded operators. The operator norm is defined by $||\hat{A}||_{op} = \sup_{|||\psi\rangle||\leq 1}||\hat{A}|\psi\rangle||$ ([7] page 167), and thus $||\hat{A}|\psi\rangle|| \leq ||\hat{A}||_{op}|||\psi\rangle||$. This adds the convenience that $||\hat{A}^n|\psi\rangle|| \leq ||A||_{op}||\hat{A}^{n-1}|\psi\rangle|| \leq \cdots \leq ||\hat{A}||_{op}^n|||\psi\rangle||$. As such, we can apply the Cauchy-Schwarz-Bunyakovsky (CSB) inequality to the absolute value of the weak value of $\hat{A}^n$ to find

$$\left|(\hat{A}^n)_w\right| = \left|\frac{\langle\phi(t_1)|\hat{A}^n|\psi(t_1)\rangle}{\langle\phi(t_1)|\psi(t_1)\rangle}\right| \leq \frac{|||\phi(t_1)\rangle||\,||\hat{A}^n|\psi(t_1)\rangle||}{|\langle\phi(t_1)|\psi(t_1)\rangle|} \leq ||\hat{A}||_{op}^n\frac{|||\phi(t_1)\rangle||\,|||\psi(t_1)\rangle||}{|\langle\phi(t_1)|\psi(t_1)\rangle|}. \tag{1.35}$$

As a result,

$$\begin{aligned}\left|(\hat{A}^j)_w - A_w^j\right| &\leq \left|(\hat{A}^j)_w\right| + |A_w|^j \\ &\leq ||\hat{A}||_{op}^j\frac{|||\phi(t_1)\rangle||\,|||\psi(t_1)\rangle||}{|\langle\phi(t_1)|\psi(t_1)\rangle|} + ||\hat{A}||_{op}^j\left(\frac{|||\phi(t_1)\rangle||\,|||\psi(t_1)\rangle||}{|\langle\phi(t_1)|\psi(t_1)\rangle|}\right)^j \\ &= \left(\frac{1}{|\langle\phi(t_1)|\psi(t_1)\rangle|} + \frac{1}{|\langle\phi(t_1)|\psi(t_1)\rangle|^j}\right)||\hat{A}||_{op}^j,\end{aligned} \tag{1.36}$$

and since $|\langle\phi(t_1)|\psi(t_1)\rangle| \leq 1$ we know that $\frac{1}{|\langle\phi(t_1)|\psi(t_1)\rangle|} \leq \frac{1}{|\langle\phi(t_1)|\psi(t_1)\rangle|^j}$ for all $j \in \mathbb{N}$. This allows us to implement the (very loose, but conveniently simple) bound

$$\left|(\hat{A}^j)_w - A_w^j\right| \leq 2\left(\frac{||\hat{A}||_{op}}{|\langle\phi(t_1)|\psi(t_1)\rangle|}\right)^j. \tag{1.37}$$

With this, Eqn. 1.34 becomes

$$\left|\left| \left|\phi(t_2)\right\rangle \left\langle\phi(t_1)|\psi(t_1)\right\rangle \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{i\epsilon\hat{q}}{\hbar}\right)^j \left[(\hat{A}^j)_w - A_w^j\right] \left|\mathscr{Q}\right\rangle \right|\right| \leq 2\sum_{j=2}^{\infty}\left(\frac{\epsilon\Delta}{\hbar}\frac{||\hat{A}||_{op}}{|\left\langle\phi(t_1)|\psi(t_1)\right\rangle|}\right)^j, \tag{1.38}$$

thus, we have a geometric series, which we can shrink as much as we need. Let $r = \frac{\epsilon\Delta}{\hbar}\frac{||\hat{A}||_{op}}{|\langle\phi(t_1)|\psi(t_1)\rangle|} < 1$, and note that

$$\sum_{j=2}^{\infty} r^j = \sum_{j=0}^{\infty} r^j - 1 - r = \frac{1}{1-r} - \frac{1-r^2}{1-r} = \frac{r^2}{1-r}. \tag{1.39}$$

We want to make $2\sum_{j=2}^{\infty} r^j \ll 1$, so this requires $\frac{r^2}{1-r} \ll \frac{1}{2}$, or in other words, $r^2 + \frac{1}{2}r - \frac{1}{2} \ll 0$. This is achieved when $r \ll \frac{1}{2}$, which tells us that we must choose ϵ and Δ such that

$$\epsilon\Delta \ll \frac{|\left\langle\phi(t_1)|\psi(t_1)\right\rangle|}{||\hat{A}||_{op}}\frac{\hbar}{2}. \tag{1.40}$$

Next, we have to deal with unbounded operators, which is a more complicated notion. Here, we have to use more knowledge about the input and output states, $|\psi(t_1)\rangle$ and $|\phi(t_1)\rangle$, which in turn divides into two cases: states with compact support in the state space and states without.

Let us assume without loss of generality that $|\psi(t_1)\rangle$ has compact support, particularly when represented in the eigenvalue spectrum of $\hat{A}$, denoted $\mathrm{sp}(\hat{A})$. Thus,

$$\left|\psi(t_1)\right\rangle = \int_{a\in\mathrm{sp}(\hat{A})} da\ \psi(a)\left|a\right\rangle, \tag{1.41}$$

and let $\mathrm{supp}(\psi(a)) = \{a \in \mathrm{sp}(\hat{A}) : \psi(a) \neq 0\}$ be the support of the wave function $\psi(a)$. It follows that

$$\hat{A}^n\left|\psi(t_1)\right\rangle = \int_{a\in\mathrm{sp}(\hat{A})} da\ \psi(a)a^n\left|a\right\rangle. \tag{1.42}$$

Since $\mathrm{supp}(\psi(a))$ is compact, we can define $a_{max} = \sup\{|a| : a \in \mathrm{supp}(\psi(a))\}$, which will be a finite quantity, and thus

$$\begin{aligned} ||\hat{A}^n\left|\psi(t_1)\right\rangle||^2 &= \left|\left|\int_{a\in\mathrm{sp}(\hat{A})} da\ \psi(a)a^n\left|a\right\rangle\right|\right|^2 \\ &= \int_{a'\in\mathrm{sp}(\hat{A})} da' \int_{a\in\mathrm{sp}(\hat{A})} da\ \psi^*(a')\psi(a)a'^n a^n \left\langle a'|a\right\rangle \\ &= \int_{a\in\mathrm{sp}(\hat{A})} da\ |\psi(a)|^2 a^{2n} \\ &\leq a_{max}^{2n}\int_{a\in\mathrm{sp}(\hat{A})} da\ |\psi(a)|^2 \\ &= a_{max}^{2n}||\left|\psi(t_1)\right\rangle||^2. \end{aligned} \tag{1.43}$$

As such, $||\hat{A}^n|\psi(t_1)\rangle|| \leq a_{max}^n||\,|\psi(t_1)\rangle||$, and so

$$\left|(\hat{A}^j)_w - A_w^j\right| \leq \left(\frac{1}{|\left\langle\phi(t_1)|\psi(t_1)\right\rangle|} + \frac{1}{|\left\langle\phi(t_1)|\psi(t_1)\right\rangle|^j}\right)a_{max}^j. \tag{1.44}$$

This allows us to bound our error effectively by choosing ϵ and Δ such that

$$\epsilon\Delta \ll \frac{|\langle\phi(t_1)|\psi(t_1)\rangle|}{a_{max}}\frac{\hbar}{2}. \tag{1.45}$$

Without compact support, I am unable to conjure such sweeping generalizations. It would be nice if normalization of our states kept the growth of $(\hat{A}^j)_w$ in check, but the right observable can put that in jeopardy. For instance, suppose $|\psi(t_1)\rangle$ is a Gaussian in position space, much like our probe state $|\mathscr{Q}\rangle$:

$$|\psi(t_1)\rangle = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt[4]{2\pi}} e^{-x^2/4} |x\rangle. \tag{1.46}$$

If we operate on this with the observable $e^{\hat{X}^2/4}$, then $e^{\hat{X}^2/4}|\psi(t_1)\rangle$ is no longer normalizable. Now, just because $\left|\left|e^{\hat{X}^2/4}|\psi(t_1)\rangle\right|\right|$ is infinite does not mean that $|\phi(t_1)\rangle$ won't keep $\left(e^{\hat{X}^2/4}\right)_w$ finite, but losing the bound provided by CSB is not promising.

We can add assumptions to try and bound $|(\hat{A}^j)_w|$. The simplest is to assume $|A_w| < \infty$, which is only natural—if it weren't true, the approximation we want to reach in Eqn. 1.28 would be utter nonsense! We may also claim the stronger assumption that $\langle\hat{A}\rangle$ and $\Delta\hat{A}$ are both finite, which is also reasonable. Presumably, the initial state of the system $|\psi(t_0)\rangle$ was set up with sensible statistics, and thus defined expectation value and standard deviation. This in turn tells us (via CSB) that $|\langle\phi(t_1)|\hat{A}|\psi(t_1)\rangle| \leq \langle\psi(t_1)|\hat{A}^2|\psi(t_1)\rangle$, which allows us to bound $|A_w|$. Similarly, if we could bound $\langle\psi(t_1)|\hat{A}^{2j}|\psi(t_1)\rangle$, then we will have a bound for $|(\hat{A}^j)_w|$. We know from Eqn. 1.43 that

$$\langle\psi(t_1)|\hat{A}^{2n}|\psi(t_1)\rangle = \int_{a\in\mathrm{sp}(\hat{A})} da\ |\psi(a)|^2 a^{2n}, \tag{1.47}$$

so we need $|\psi(a)|^2$ to decay faster than any degree of polynomial just to keep the expression finite. We need a more specific function to make the expression bounded in a useful way, but this gives us a fair test bed.

In their work on the matter, Aharonov and Vaidman came up with the condition

$$(2\Delta)^j \frac{\Gamma(j/2)}{(j-2)!}\left|(\hat{A}^j)_w - A_w^j\right| \ll 1. \tag{1.48}$$

Due to notation differences (their use of natural units, and writing the probe state in function notation), my expression was not going to look the same as theirs, but exactly how they came up with their version is still a bit arcane. They do not provide a calculation demonstrating the emergence of this condition, and attempts to work from Eqn. 1.27 to Eqn. 1.48 (or in reverse) do not appear to end up in the right place. Not to mention, their condition is not stated particularly carefully; just making each term of an infinite sum very small is not sufficient to make the sum itself small, especially in the absence of a discussion of the growth of the terms.

Chapter 2

Optimal Estimates of Probe Interactions in the Nested Mach-Zehnder Interferometer

Abstract

The feedback compensation method effectively attempts to correct for the disturbance an auxiliary qubit experiences in a weak interaction with a quantum system. Optimally estimating the required amount of compensation allows the disturbance to be minimized, suggesting that the estimate characterizes the weak interaction to a good approximation. The weak value of the system observable to which the qubit is coupled naturally arises operationally as this optimal estimate (at least, the real part of the weak value does), which dispels concern that weak values only arise as a result of averaging over ensembles. However, by my understanding, it does not dispel the ability to ascribe weak values to interference effects, leaving that avenue of explanation open. I also show that the method continues to hold up in the specific situation of the nested Mach-Zehnder interferometer, demonstrating that weak values do characterize its weak interactions. Still, it is an approximation, and it does not necessarily necessitate the contentious interpretation of a discontinuous trajectory.

2.1 The Feedback Signal

Consider a quantum system with observable $\hat{A}$, which interacts with an auxiliary qubit (which we shall call the 'probe' for forthcoming reasons) via the interaction potential

$$\hat{V}(t) = \epsilon\delta(t - t_1)\hat{A} \otimes \hat{\sigma}_z, \tag{2.1}$$

where ϵ is a parameter controlling the strength of the system-probe coupling, $\hat{\sigma}_z$ is the Pauli operator diagonal in the computational basis $\{|+\rangle_z, |-\rangle_z\}$, and t_1 is the time at which the interaction occurs. The delta function can be thought of as a limit of the function $\frac{\theta_{t_1,\tau}(t)}{\tau}$ as the duration of the interaction τ goes to zero, as was discussed in Section 1.1. Clearly, $V(t)$ must be an energy, but $\hat{\sigma}_z$ is unitless—perhaps it arose from the spin operator $\hat{S}_z$, but the factor of $\frac{\hbar}{2}$ was absorbed into ϵ. As a direct result, $\epsilon\hat{A}$ possesses the dimensions of angular momentum, and thus $\frac{\epsilon\hat{A}}{\tau}$ is the average rate of change in angular momentum during the interaction—a torque. The interaction potential can be seen as exerting a torque on the qubit probe state. The question is, how do we estimate this torque?

The interaction potential leads (via calculations in the interaction picture, as demonstrated for a similar system in [3]) to the unitary operator

$$\hat{U}_{SP} = e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z}. \tag{2.2}$$

If we knew the value of the total change in the qubit's angular momentum—it is proportional to ϵ, so let us call it ϵA—we could apply the unitary operator

$$\hat{U}_Z = e^{i\frac{\epsilon}{\hbar}A\hat{\sigma}_z}, \tag{2.3}$$

which would turn the qubit back to its original state. Rather than presupposing knowledge of A, we can use this compensation scheme to find the change in angular momentum. If we try different values of A and find the correct one, the lack of disturbance in the qubit probe would indicate our success.

We will need information to make our estimations of A. One valuable piece of information is the initial state of the system, represented by the density operator $\hat{\rho}_S$, and we can gain another piece by measuring the system after the probe interaction. The most general measurement is represented by the POVM with operators $\{\hat{E}_S(m)\}_m$, each keyed to an outcome indexed by m. Our estimation of A is a function of these outcomes: $A(m)$. Now, depending on what our measurement outcome is, we apply the operator

$$\hat{U}_Z(m) = e^{i\frac{\epsilon}{\hbar}A(m)\hat{\sigma}_z}, \tag{2.4}$$

which should compensate for the change in angular momentum to a degree dependent upon our choice of estimation $A(m)$. For the best sensitivity to this operation [8], we initially set the probe in the state $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z)$. This gives us the initial expectation value of $\langle\hat{\sigma}_x\rangle\,(\text{in}) = 1$, and any deviation from this at the end (quantified by $1 - \langle\hat{\sigma}_x\rangle\,(\text{out})$) will indicate how well our compensation (based on feedback from the measurement) worked to reset the probe. The feedback compensation scheme is laid out in circuit form in Fig. 2.1.

2.2 Optimal Estimates

Any function A of the outcome m serves as an estimate of the change in angular momentum, so we desire the best function for making this estimate, which would be the function that minimizes $1 - \langle\hat{\sigma}_x\rangle\,(\text{out})$. Over the course of

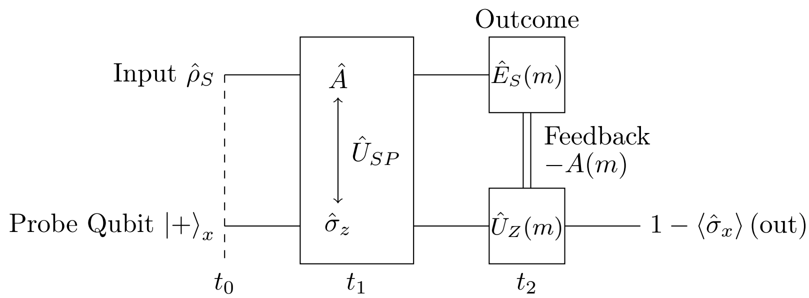


Figure 2.1: Feedback Compensation Setup This is a modification of Fig. 1 from [8]. After the system and probe are initialized at t_0 in the composite state $\hat{\rho}_S \otimes |+\rangle_{x\,x}\langle +|$, they are entangled at t_1 by the unitary operation $\hat{U}_{SP} = e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z}$ before a POVM with operators $\{\hat{E}_S(m)\}_m$ occurs at t_2, the result of which is m. The estimation $A(m)$ is made based on this outcome, and fed back into the probe qubit via the unitary operation $\hat{U}_Z(m) = e^{i\frac{\epsilon}{\hbar}A(m)\hat{\sigma}_z}$. Ultimately, the probe is measured in the Hadamard basis, and the deviation of its expectation value from that of its original state is calculated.

the feedback compensation protocol, the system evolves in the following manner:

$$\hat{\rho}_0 = \hat{\rho}_S \otimes |+\rangle_{x\,x}\langle +|; \tag{2.5}$$

$$\begin{aligned}\hat{\rho}_1 &= \hat{U}_{SP}\hat{\rho}_S \otimes |+\rangle_{x\,x}\langle +|\hat{U}_{SP}^\dagger \\ &= \frac{1}{2}\sum_{j=\pm1}\sum_{k=\pm1} e^{-i\frac{\epsilon}{\hbar}\hat{A}j}\hat{\rho}_S \otimes |j\rangle_{z\,z}\langle k|e^{i\frac{\epsilon}{\hbar}\hat{A}k};\end{aligned} \tag{2.6}$$

$$\begin{aligned}\hat{\rho}_2 &= \sum_m \hat{U}_Z(m)\sqrt{\hat{E}_S(m)}\hat{U}_{SP}\hat{\rho}_S \otimes |+\rangle_{x\,x}\langle +|\hat{U}_{SP}^\dagger\sqrt{\hat{E}_S(m)}\hat{U}_Z(m)^\dagger \\ &= \frac{1}{2}\sum_m\sum_{j=\pm1}\sum_{k=\pm1}\sqrt{\hat{E}_S(m)}e^{-i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]j}\hat{\rho}_S \otimes |j\rangle_{z\,z}\langle k|e^{i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k}\sqrt{\hat{E}_S(m)}.\end{aligned} \tag{2.7}$$

Taking the expectation value of $\hat{\sigma}_x$ at the end nets us

$$\begin{aligned}\langle\hat{\sigma}_x\rangle\,(\text{out}) &= \text{Tr}\{\hat{\rho}_2\hat{\sigma}_x\} \\ &= \frac{1}{2}\sum_m\sum_{j=\pm1}\sum_{k=\pm1}\text{Tr}\left\{\sqrt{\hat{E}_S(m)}e^{-i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]j}\hat{\rho}_S e^{i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k}\sqrt{\hat{E}_S(m)}\right\}{}_z\langle k|\hat{\sigma}_x|j\rangle_z \\ &= \frac{1}{2}\sum_m\sum_{j=\pm1}\sum_{k=\pm1}\text{Tr}\left\{e^{-i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]j}\hat{\rho}_S e^{i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k}\hat{E}_S(m)\right\}\delta_{k,-j} \\ &= \frac{1}{2}\sum_m\sum_{k=\pm1}\text{Tr}\left\{e^{i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k}\hat{\rho}_S e^{i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k}\hat{E}_S(m)\right\}\end{aligned} \tag{2.8}$$

Expanding this to the second order in ϵ brings us

$$\begin{aligned}\langle\hat{\sigma}_x\rangle\,(\text{out}) &\approx \frac{1}{2}\sum_m\sum_{k=\pm1}\text{Tr}\left\{\left[1+i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k-\frac{\epsilon^2}{2\hbar^2}[\hat{A}-A(m)]^2k^2\right]\hat{\rho}_S\left[1+i\frac{\epsilon}{\hbar}[\hat{A}-A(m)]k-\frac{\epsilon^2}{2\hbar^2}[\hat{A}-A(m)]^2k^2\right]\hat{E}_S(m)\right\} \\ &\approx 1-\frac{\epsilon^2}{\hbar^2}\sum_m\text{Tr}\left\{\frac{1}{2}[\hat{A}-A(m)]^2\hat{\rho}_S\hat{E}_S(m)+[\hat{A}-A(m)]\hat{\rho}_S[\hat{A}-A(m)]\hat{E}_S(m)+\frac{1}{2}\hat{\rho}_S[\hat{A}-A(m)]^2\hat{E}_S(m)\right\}.\end{aligned} \tag{2.9}$$

To consolidate terms and push beyond this, we need to expand the terms with $[\hat{A} - A(m)]^2$ in them:

$$\begin{aligned}\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]^2\hat{\rho}_S\hat{E}_S(m) + \hat{\rho}_S[\hat{A} - A(m)]^2\hat{E}_S(m)\right\} &= \sum_m \mathrm{Tr}\left\{[\hat{A}^2 - 2\hat{A}A(m) + A^2(m)]\hat{\rho}_S\hat{E}_S(m)\right.\\ &\qquad \left.+\hat{\rho}_S[\hat{A}^2 - 2\hat{A}A(m) + A^2(m)]\hat{E}_S(m)\right\}\\ &= \sum_m \mathrm{Tr}\left\{\hat{A}^2\hat{\rho}_S\hat{E}_S(m) + \hat{\rho}_S\hat{A}^2\hat{E}_S(m)\right\}\\ &+ 2\sum_m \mathrm{Tr}\left\{\left[-\hat{A}\hat{\rho}_S A(m) - A(m)\hat{\rho}_S\hat{A} + A(m)\hat{\rho}_S A(m)\right]\hat{E}_S(m)\right\}\\ &= 2\mathrm{Tr}\left\{\hat{A}\hat{\rho}_S\hat{A}\right\}\\ &+ 2\sum_m \mathrm{Tr}\left\{\left[-\hat{A}\hat{\rho}_S A(m) - A(m)\hat{\rho}_S\hat{A} + A(m)\hat{\rho}_S A(m)\right]\hat{E}_S(m)\right\}\\ &- 2\sum_m \mathrm{Tr}\left\{\left[\hat{A}\hat{\rho}_S\hat{A} - \hat{A}\hat{\rho}_S A(m) - A(m)\hat{\rho}_S\hat{A} + A(m)\hat{\rho}_S A(m)\right]\hat{E}_S(m)\right\}\\ &+ 2\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\}.\end{aligned} \tag{2.10}$$

As such,

$$\langle\hat{\sigma}_x\rangle\,(\mathrm{out}) \approx 1 - 2\frac{\epsilon^2}{\hbar^2}\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\}. \tag{2.11}$$

We now see that $1 - \langle\hat{\sigma}_x\rangle\,(\mathrm{out})$ is proportional to the quantity $\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\}$, which is a generalization of the square of the statistical deviation of $\hat{A}$ from the estimate [9]. This can be seen as a generalized version of Ozawa's measurement accuracy [8, 10]. It is worth noting that, without measurement and feedback compensation, the place of this statistical deviation is taken by the mean of the square of $\hat{A}$, which is the standard deviation of an observable of zero mean:

$$\begin{aligned}\langle\hat{\sigma}_x\rangle\,(\mathrm{out}) &= \mathrm{Tr}\{\hat{\rho}_1\hat{\sigma}_x\}\\ &= \frac{1}{2}\sum_{j=\pm1}\sum_{k=\pm1}\mathrm{Tr}\left\{e^{-i\frac{\epsilon}{\hbar}\hat{A}j}\hat{\rho}_S e^{i\frac{\epsilon}{\hbar}\hat{A}k}\right\}{}_z\langle k|\hat{\sigma}_x|j\rangle_z\\ &= \frac{1}{2}\sum_{k=\pm1}\mathrm{Tr}\left\{e^{i\frac{\epsilon}{\hbar}\hat{A}k}\hat{\rho}_S e^{i\frac{\epsilon}{\hbar}\hat{A}k}\right\}\\ &\approx 1 - 2\frac{\epsilon^2}{\hbar^2}\mathrm{Tr}\left\{\hat{A}^2\hat{\rho}_S\right\}\\ &= 1 - 2\frac{\epsilon^2}{\hbar^2}\langle\hat{A}^2\rangle\\ &= 1 - 2\frac{\epsilon^2}{\hbar^2}(\Delta\hat{A})^2 \text{ if } \langle\hat{A}\rangle = 0.\end{aligned} \tag{2.12}$$

Returning to $\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\}$, we want to minimize this quantity to reduce the disturbance of the probe state. For this, we will want to expand:

$$\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\} = \langle\hat{A}^2\rangle + \sum_m\left(-A(m)\mathrm{Tr}\left\{\hat{\rho}_S[\hat{E}_S(m)\hat{A} + \hat{A}\hat{E}_S(m)]\right\} + A^2(m)\mathrm{Tr}\left\{\hat{\rho}_S\hat{E}_S(m)\right\}\right). \tag{2.13}$$

It is spatially convenient to define the quantity $A_{opt}(m|\hat{\rho}_S)$ (mimicking the notation of [9]) as

$$A_{opt}(m|\hat{\rho}_S) = \frac{\mathrm{Tr}\left\{\hat{\rho}_S[\hat{E}_S(m)\hat{A} + \hat{A}\hat{E}_S(m)]\right\}}{2\mathrm{Tr}\left\{\hat{\rho}_S\hat{E}_S(m)\right\}}, \tag{2.14}$$

where "opt" stands for "optimal estimate," for reasons to follow immediately. Our expression is now

$$\begin{aligned}\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\} &= \langle\hat{A}^2\rangle + \sum_m \mathrm{Tr}\left\{\hat{\rho}_S\hat{E}_S(m)\right\}\left(-A(m)2A_{opt}(m|\hat{\rho}_S) + A^2(m)\right) \\ &= \langle\hat{A}^2\rangle + \sum_m \mathrm{Tr}\left\{\hat{\rho}_S\hat{E}_S(m)\right\}\left[(A(m) - A_{opt}(m|\hat{\rho}_S))^2 - A^2_{opt}(m|\hat{\rho}_S)\right],\end{aligned} \tag{2.15}$$

which allows us to see that $A(m) = A_{opt}(m|\hat{\rho}_S)$ minimizes the disturbance of the probe state.

When the system is in a pure state $\hat{\rho}_S = |\psi\rangle\langle\psi|$ and the final measurement is projective ($\hat{E}_S(m) = |m\rangle\langle m|$), the optimal estimate is the real component of the weak value associated with the observable [4]:

$$A_{opt}(m|\psi) = \frac{\mathrm{Tr}\left\{|\psi\rangle\langle\psi|\,[|m\rangle\langle m|\,\hat{A} + \hat{A}\,|m\rangle\langle m|]\right\}}{2\mathrm{Tr}\left\{|\psi\rangle\langle\psi|m\rangle\langle m|\right\}} = \frac{1}{2}\left(\frac{\langle m|\hat{A}|\psi\rangle}{\langle m|\psi\rangle} + \frac{\langle\psi|\hat{A}|m\rangle}{\langle\psi|m\rangle}\right) = \frac{1}{2}\left(A_w + A_w^*\right) = \mathrm{Re}\{A_w\}. \tag{2.16}$$

Moreover, if the weak value is real then the disturbance to the probe state when $A(m) = A_{opt}(m|\psi)$ is found to be zero:

$$\begin{aligned}\sum_m \mathrm{Tr}\left\{[\hat{A} - A(m)]\hat{\rho}_S[\hat{A} - A(m)]\hat{E}_S(m)\right\} &= \langle\hat{A}^2\rangle - \sum_m \mathrm{Tr}\left\{\hat{\rho}_S\hat{E}_S(m)\right\}A^2_{opt}(m|\,|\psi\rangle) \\ &= \langle\hat{A}^2\rangle - \sum_m |\langle m|\psi\rangle|^2 \frac{\langle\psi|\hat{A}|m\rangle}{\langle\psi|m\rangle}\frac{\langle m|\hat{A}|\psi\rangle}{\langle m|\psi\rangle} \\ &= \langle\hat{A}^2\rangle - \langle\hat{A}^2\rangle \\ &= 0.\end{aligned} \tag{2.17}$$

We can see that the best estimate of the change in angular momentum due to the weak interaction is proportional to $A_{opt}(m|\hat{\rho}_S)$, and for $A_{opt}(m|\hat{\rho}_S) = A_w \in \mathbb{R}$, the compensation is perfect to second order. A further possible interpretation is that the observable $\hat{A}$ is behaving as though it took the value A_w during the weak interaction, which leads to the contentious conclusion (adopted by [4] and many others in a particle presence context [11, 12, 13, 14]) that $\hat{A}$ actually takes the value A_w, which is not necessarily one of its eigenvalues—nor necessarily even within the bounds of its eigenvalue spectrum!

When an observable $\hat{A}$ is acted on a state $|\psi\rangle$, the outcome of this action is most naturally characterized in the eigenbasis $\{|a\rangle\}_a$:

$$\hat{A}\,|\psi\rangle = \sum_a \hat{A}\,|a\rangle\langle a|\psi\rangle = \sum_a a\,\langle a|\psi\rangle\,|a\rangle\,. \tag{2.18}$$

However, we could just as well characterize it in some other orthonormal basis, such as $\{|m'\rangle\}_{m'}$:

$$\hat{A}\,|\psi\rangle = \sum_{m'} \hat{A}\,|m'\rangle\langle m'|\psi\rangle\,. \tag{2.19}$$

This second way gives one interesting view of a weak value:

$$A_w = \frac{\langle m|\hat{A}|\psi\rangle}{\langle m|\psi\rangle} = \sum_{m'} \langle m|\hat{A}|m'\rangle \frac{\langle m'|\psi\rangle}{\langle m|\psi\rangle}. \tag{2.20}$$

In this way, a weak value captures matrix elements of $\hat{A}$ in the basis $\{|m'\rangle\}_{m'}$ along row m, weighted by their contribution to $|\psi\rangle$ and normalized by the contribution of the postselection $|m\rangle$. Note that the implicit assumption $\langle m|\psi\rangle \neq 0$ ensures that $\langle m|\hat{A}|m\rangle$ is a term of the sum. When the postselection is done in the eigenbasis $\{|a\rangle\}_a$, which diagonalizes $\hat{A}$, we have $\langle a|\hat{A}|a'\rangle = a\delta_{a,a'}$, leading to $A_w = a$.

It would appear that postselection in another basis causes the weak value to incorporate more than is captured by a single eigenvalue, but why this is the case is part of a contentious interpretational topic. Hofmann [8] presents the results of feedback compensation as an additional tool for trying to divine the nature of the weak value. In particular, compensation helps to allay concerns that the weak value arises from a statistical average of outcomes. When compensation is perfect, there is only one outcome for the auxiliary qubit, given a particular postselection (as $\langle\hat{\sigma}_x\rangle = 1$ means $\mathscr{P}_+ = 1$ and $\mathscr{P}_- = 0$).

Additionally, the feedback compensation setting allows for weak values to be operationally defined, independent of one's assumptions about quantum measurement physics. This specific rationale is used by Hofmann to criticize arguments that ascribe the appearance of weak values to interference phenomena. That being said, one need not go farther than Hofmann's assumptions about the nature of the weak interaction (being in the unitary form $e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z}$) to see interference. Consider the weakly coupled system and qubit, with its state grouped by the contributing eigenstates of the observable $\hat{A}$:

$$e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z} |\psi\rangle |+\rangle_x = \sum_a |a\rangle \langle a|\psi\rangle e^{i\frac{\epsilon}{\hbar}a\hat{\sigma}_z} |+\rangle_x . \tag{2.21}$$

We can instead group the states by the eigenstates of an observable $\hat{M}$ to be used in postselection:

$$e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z} |\psi\rangle |+\rangle_x = \sum_{m,a} |m\rangle \langle m|a\rangle \langle a|\psi\rangle e^{-i\frac{\epsilon}{\hbar}a\hat{\sigma}_z} |+\rangle_x . \tag{2.22}$$

We can see at this stage that the factor $\langle m|a\rangle$ bundles together the eigenvectors of $\hat{A}$ by their presence in $|m\rangle$, while the term $\langle a|\psi\rangle$ weights each eigenvector by its amplitude in $|\psi\rangle$. For a weak enough interaction (using the principle of a simplified derivation of weak values [15]; see my examination of its applicability in Section B.2), we can approximate the exponential as linear, giving us

$$
\begin{aligned}
e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z}\left|\psi\right\rangle\left|+\right\rangle_x &\approx \sum_{m,a}\left|m\right\rangle\left\langle m|a\right\rangle\left\langle a|\psi\right\rangle\left(1-i\frac{\epsilon}{\hbar}a\hat{\sigma}_z\right)\left|+\right\rangle_x \\
&= \sum_m \left|m\right\rangle\left[\left\langle m|\psi\right\rangle - i\frac{\epsilon}{\hbar}\left\langle m\right|\left(\sum_a a\left|a\right\rangle\left\langle a\right|\right)\left|\psi\right\rangle\hat{\sigma}_z\right]\left|+\right\rangle_x \\
&= \sum_m \left|m\right\rangle\left[\left\langle m|\psi\right\rangle - i\frac{\epsilon}{\hbar}\left\langle m|\hat{A}|\psi\right\rangle\hat{\sigma}_z\right]\left|+\right\rangle_x \\
&= \sum_m \left|m\right\rangle\left\langle m|\psi\right\rangle\left[1 - i\frac{\epsilon}{\hbar}A_w(m|\psi)\hat{\sigma}_z\right]\left|+\right\rangle_x \\
&\approx \sum_m \left\langle m|\psi\right\rangle\left|m\right\rangle e^{-i\frac{\epsilon}{\hbar}A_w(m|\psi)\hat{\sigma}_z}\left|+\right\rangle_x .
\end{aligned}
\tag{2.23}
$$

In wave mechanics, interference happens when the individual contributions of waves in a superposition are added together. Similarly, the eigenstates of $\hat{A}$ found in superposition in a postselected state $|m\rangle$ all add up to create the ultimate effect of rotating the qubit with $e^{-i\frac{\epsilon}{\hbar}A_w(m|\psi)\hat{\sigma}_z}$, which is also an expression of interference.

2.3 Application: Nested Mach-Zehnder Interferometer

The nested Mach-Zehnder interferometer (setup depicted in Fig. 2.2) is a prime example of the strangeness that accompanies the interpretation of weak values. Using the criterion (espoused in this situation by [11, 12, 13]) that a particle is present where the weak value of its position projector is nonzero (and absent when this weak value is zero), and setting up weak probes (auxiliary qubits) in arms A, B, C, D, and E, one would conclude (see Eqns. C.45–C.50 and C.59–C.64) that a particle (entering from S) that ends up being detected in exit channel F or G was in paths A, B, and C, but not in D or E. This suggests that the particle somehow was present in the inner interferometer (B and C) without entering (D) or leaving (E)—a discontinuous trajectory.

This bizarre assertion is generally shown by calculating the forward-propagating particle state (Eqns. C.29–C.36) and the backward-propagating state of the particular postselection (Eqns. C.37–C.44 for F, C.51–C.58 for G, and C.65–C.72 for H) as though there were no disturbance from the probes. It is generally assumed that the interactions are weak enough to allow this, but it can come across as careless, particularly when ignoring the effects of multiple probes on each other (see Appendix B). By applying the feedback compensation method and minimizing the effect on the auxiliary qubits, I calculated the estimates characterizing the interactions with the weak probes without ignoring all but one of them at a time.

Using five probes, the most direct comparison to the original method would be to define the observable $\hat{\sigma}_{x,D} \otimes \hat{\sigma}_{x,A} \otimes \hat{\sigma}_{x,B} \otimes \hat{\sigma}_{x,C} \otimes \hat{\sigma}_{x,E}$ (where each operator is indexed by the arm of the interferometer its target qubit is meant to interact with) and minimize the quantity $1 - \langle \hat{\sigma}_{x,D} \otimes \hat{\sigma}_{x,A} \otimes \hat{\sigma}_{x,B} \otimes \hat{\sigma}_{x,C} \otimes \hat{\sigma}_{x,E}\rangle$ (out). The gory details are left to Appendix C, but the ultimate result is that the optimal estimates are those expected from the weak

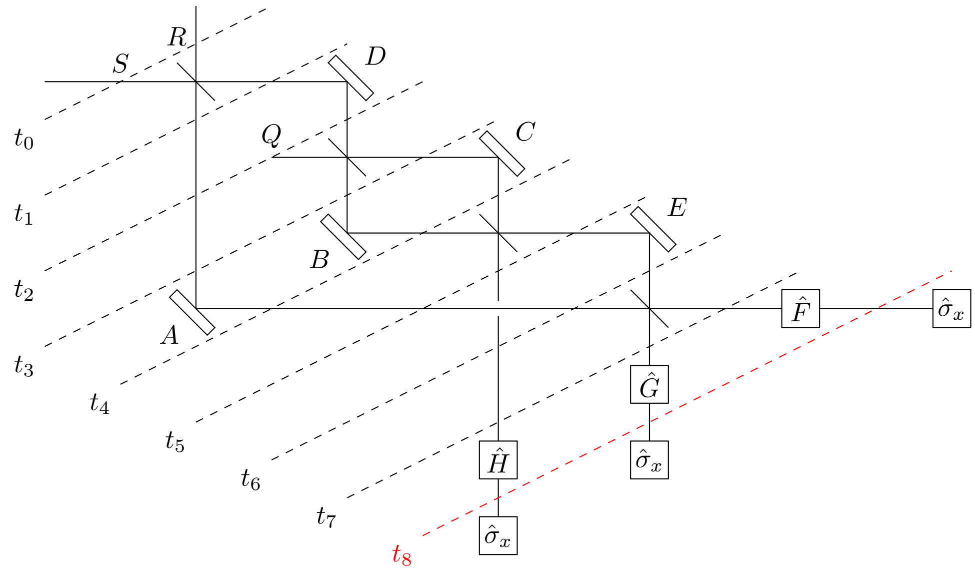


Figure 2.2: Feedback Compensation for the Nested Mach-Zehnder Interferometer A quantum particle enters the interferometer at t_0 through channel S, with all ancillary qubits in state $|+\rangle_x$. Horizontal and vertical solid lines denote paths through the interferometer setup. Slanted solid lines indicate beam splitters, while thin, slanted rectangles indicate mirrors. Slanted dashed lines indicate lines of simultaneity (thus the horizontal and vertical dimensions are not drawn to scale). Note that the time step from t_7 to t_8 is not characterized by unitary evolution. At the boxes labeled $\hat{F}$, $\hat{G}$, and $\hat{H}$, the particle's presence in the associated arm of the interferometer is projectively measured and subject to a unitary feedback operation. Paths are denoted by the mirror they bounce off of (D, A, B, C, E), the final, non-ancillary measurement they are subject to (F, G, H), or a free letter (S, R, Q) if they represent entrance channels. Path labels do not transfer through beam splitters. At the boxes labeled $\hat{\sigma}_x$, the ancillary qubits are measured projectively in the Hadamard basis.

value approach, as given in Table 2.1. In the case of optimal compensation, the disturbance is reduced to zero (see Eqn. C.23), which is what we expected of real weak values in the simplified case (Eqn. 2.17).

Post	D_{opt}	A_{opt}	B_{opt}	C_{opt}	E_{opt}
F	0	1	$-\frac{\beta^2}{2\alpha^2}$	$\frac{\beta^2}{2\alpha^2}$	0
G	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
H	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2.1: Optimal Estimates for 5 Probes Taking these quantities to be the estimates $D(\cdot)$, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $E(\cdot)$ for a given postselection will minimize the disturbance in the qubits captured by Eqn. C.22. These match the weak values derived in Eqns. C.45–C.50, C.59–C.64, and C.73–C.78.

This method also supports an alternative version of the setup proposed by Griffiths [16], who suggested replacing the probes in the arms B and C with a single probe W that measured the projector $\hat{W} = \hat{B} + \hat{C}$—it checks if the particle is in the inner interferometer without determining which arm it is in. When this probe is used, the optimal estimate of the interaction in the inner interferometer vanishes for the postselections F and G, which means the particle presence criterion would suggest that the particle behaved more intuitively, being only in A for its entire trip through the interferometer. It only passes through the inner interferometer on its way to H, which is what the tuning of the interferometer was designed to accomplish. The estimates for the other probes do not change (see

Table 2.2).

Post	D_{opt}	A_{opt}	W_{opt}	E_{opt}
F	0	1	0	0
G	0	1	0	0
H	1	0	1	0

Table 2.2: Optimal Estimates for 4 Probes Taking these quantities to be the estimates $D(\cdot)$, $A(\cdot)$, $W(\cdot)$, $E(\cdot)$ for a given postselection will minimize the disturbance in the qubits captured by Eqn. C.28. These match the weak values derived in Eqns. C.49, C.63, and C.77.

Compared to the state of the five probe system at t_7 (Eqn. C.15), the state of the four probe system (Eqn. C.26) is far simpler, allowing easy calculation of the probabilities for the different postselection outcomes:

$$\mathscr{P}'_F = |\langle F_7|\psi'_7\rangle|^2 = \alpha^4, \tag{2.24}$$

$$\mathscr{P}'_G = |\langle G_7|\psi'_7\rangle|^2 = \alpha^2\beta^2, \tag{2.25}$$

$$\mathscr{P}'_H = |\langle H_7|\psi'_7\rangle|^2 = \beta^2. \tag{2.26}$$

In contrast, the probabilities for the postselection outcomes under the five probe setup (deduced from Table 2.3) are

$$\mathscr{P}_F \approx \alpha^4 + \frac{\beta^4}{2}\frac{\epsilon^2}{\hbar^2}, \tag{2.27}$$

$$\mathscr{P}_G \approx \alpha^2\beta^2 + \frac{\alpha^2\beta^2}{2}\frac{\epsilon^2}{\hbar^2}, \tag{2.28}$$

$$\mathscr{P}_H \approx \beta^2 - \frac{\beta^2}{2}\frac{\epsilon^2}{\hbar^2}. \tag{2.29}$$

It would appear that a little bit of probability has been stolen from H and split between F and G, which suggests the possible interpretation that actually distinguishing the B and C paths leads to diverting particles that would have gone on to H.

Now, one may justifiably wonder at my choice to ensure that $\langle\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\rangle\,(\text{out}) = 1$, as the operator has a larger eigenspace for the eigenvalue 1 than just the state $|+\rangle_{x,D}|+\rangle_{x,A}|+\rangle_{x,B}|+\rangle_{x,C}|+\rangle_{x,E}$. Indeed, if any two or any four of the constituent qubit states are $|-\rangle_x$ instead of $|+\rangle_x$, then the outcome of the measurement is still 1, which calls into question my findings!

To address the issue, I went ahead and calculated the full probability distribution for the measurement of the output channel and the qubit states (Eqns. C.84, C.85, and C.86), and found that my earlier optimal estimates still held true. As can be seen in Table 2.3, applying no compensation leads to only single-qubit flips.

Of course, examination of my calculations will reveal that I did all of my work out to second order in ϵ, as before, and double-qubit flips are all fourth order effects. As such, if significant disturbance must mark passage through the inner interferometer for the particle to reach F or G, then detection of passage through D or E is such an unlikely event, that the disturbance of the particle's passage is best estimated as negligible. If a higher

Post $(\cdot)$	Compensation	$\mathscr{P}_{(\cdot)+++++}$	$\mathscr{P}_{(\cdot)-++++}$	$\mathscr{P}_{(\cdot)+-+++}$	$\mathscr{P}_{(\cdot)++-++}$	$\mathscr{P}_{(\cdot)+++-+}$
F	Optimal	$\alpha^4 + \frac{\beta^4}{2}\frac{\epsilon^2}{\hbar^2}$	0	0	0	0
F	None	$\alpha^4\left(1 - \frac{\epsilon^2}{\hbar^2}\right)$	0	$\alpha^4\frac{\epsilon^2}{\hbar^2}$	$\frac{\beta^4}{4}\frac{\epsilon^2}{\hbar^2}$	$\frac{\beta^4}{4}\frac{\epsilon^2}{\hbar^2}$
G	Optimal	$\alpha^2\beta^2\left(1 + \frac{1}{2}\frac{\epsilon^2}{\hbar^2}\right)$	0	0	0	0
G	None	$\alpha^2\beta^2\left(1 - \frac{3}{2}\frac{\epsilon^2}{\hbar^2}\right)$	0	$\alpha^2\beta^2\frac{\epsilon^2}{\hbar^2}$	$\frac{\alpha^2\beta^2}{2}\frac{\epsilon^2}{\hbar^2}$	$\frac{\alpha^2\beta^2}{2}\frac{\epsilon^2}{\hbar^2}$
H	Optimal	$\beta^2\left(1 - \frac{1}{2}\frac{\epsilon^2}{\hbar^2}\right)$	0	0	0	0
H	None	$\beta^2\left(1 - \frac{5}{2}\frac{\epsilon^2}{\hbar^2}\right)$	$\beta^2\frac{\epsilon^2}{\hbar^2}$	0	$\frac{\beta^2}{2}\frac{\epsilon^2}{\hbar^2}$	$\frac{\beta^2}{2}\frac{\epsilon^2}{\hbar^2}$

Table 2.3: Probabilities With and Without Compensation The probability $\mathscr{P}_{(\cdot)jklmn}$ of finding the system in the state $|(\cdot)_7\rangle\,|j\rangle_{x,D}\,|k\rangle_{x,A}\,|l\rangle_{x,B}\,|m\rangle_{x,C}\,|n\rangle_{x,E}$ with or without optimal compensation (as given in Table 2.1). Probabilities not listed on this table are all zero.

order in ϵ were accepted in the calculations, perhaps corrections to the estimations would be seen that would lead to nonzero compensation being needed in D and E.

To low orders, however, there is no flaw in the use of weak values as optimal estimates characterizing weak interactions in the nested Mach-Zehnder interferometer setup. Indeed, if one takes the probe states entangled with output $|F_7\rangle$ (see the first term of Eqn. C.15) and expands the exponentials to create superpositions of flipped and unflipped qubit states (done in Eqn. C.89), one will see that the resultant state matches the similarly treated pure state with each qubit evolved based on the associated weak value (Eqn. C.90). This suggests, for a weak enough interaction, that weak values are not found as the result of statistical averaging of results—at least, to a good approximation.

Chapter 3

Consistently Weak—Interrogating Weak Measurement with Consistent Histories

Abstract

Consistent histories, a formalism generalizing the Copenhagen interpretation, has been used to examine the eccentricities of weak values, and conditions have been derived showing an incompatibility of strange weak values with consistent families of histories for a quantum system. Herein is briefly laid out the consistent histories formalism, as well as a restatement of prior small results mixing the theory of weak values with consistent histories. Following that, I construct a modest extension, presenting a mixture of weak values and consistent histories that includes a weakly coupled probe in a composite system and accounts for its end state when reasoning about the viable histories of the system.

3.1 Introductory Remarks

Applying consistent histories to the task of understanding weak values has been done before. Kastner [17] demonstrated that strange weak values of projection operators lead to inconsistent families of histories, and Griffiths made mention of this as well on the 10th page of his dive into the nested Mach-Zehnder interferometer [16]. A focus on projectors is reasonable for consistent histories, but it doesn't fully address the entire gamut of weak values—a gamut that includes weak values of more general observables. As such, I intend to press through calculations of consistency with an additional term for the weak probe, and see what obtaining a particular weak value implies.

There are many great resources for understanding consistent histories, including Griffiths himself [18], which cleaves closer to Kastner's use of the approach. Another extensive source that uses notation I prefer was written by Hartle [19] (although I do trade my definition of the class operator $\hat{C}_h$ with that of its adjoint, which I learned from [20, 21]). By far, the presentation closest to how I approach weak values in this work was written by some screwball who couldn't take scientific writing seriously [22].

3.2 Consistent Histories in Brief

The purpose of consistent histories is to generalize the standard Copenhagen interpretation to a state where rules exist to determine when the outcomes of a quantum system can appropriately be assigned probabilities. The Copenhagen interpretation by itself requires the notion of measurements in order to assign probabilities, limiting its application to a universe containing a classical realm of measurement apparatuses and researchers that interact with quantum systems to cause state collapse. In the early universe, before the emergence of the classical domain, the Copenhagen interpretation cannot really be applied, hence the rise of consistent histories out of the needs of quantum cosmologists.

The main object with which a consistent historian works is a *history*, which is a time-ordered set of projectors for a quantum system. Each projector corresponds to a value or set of values for an observable of the system, thus the history specifies what values the system's observables take at each accounted-for time. Generally, one works with *coarse-grained* histories, which specify values of observables for only a subset of important times[1]. Consider the coarse-grained history

$$h = \{\hat{P}_a^{\hat{A}}(t_1), \hat{P}_b^{\hat{B}}(t_2), \hat{P}_c^{\hat{C}}(t_3)\}, \tag{3.1}$$

which indicates that the observable $\hat{A}$ of our system takes value a at time t_1, $\hat{B}$ takes value b at t_2, and $\hat{C}$ takes value c at t_3. The values of observables are not specified between the indicated times, hence its coarseness. This is not the only history of the system, but simply a single branch of possible events among possibly many others, specifying different outcomes for the observables (see Fig. 3.1).

Once we have gathered the full set of histories for a system, we must make sure the set is exhaustive—containing all possible histories the system could experience—and exclusive—containing only distinct histories—before we can begin calculations to assign probabilities to each history. Consider the outcomes of the system depicted in Fig. 3.1, grouped by time of applicability and represented by the associated projector:

$$\Big\{\{\hat{P}_a^{\hat{A}}(t_1), \hat{P}_{a'}^{\hat{A}}(t_1), \hat{P}_{a''}^{\hat{A}}(t_1)\}, \{\hat{P}_b^{\hat{B}}(t_2), \hat{P}_{b'}^{\hat{B}}(t_2)\}, \{\hat{P}_c^{\hat{C}}(t_3), \hat{P}_{c'}^{\hat{C}}(t_3)\}\Big\}. \tag{3.2}$$

For the set of histories for this system to be exclusive and exhaustive, I should be able to select one outcome for each time, say a' at t_1, b' at t_2 and c' at t_3, and have a history in the set which represents it. That is to say, if the history

$$h' = \{\hat{P}_{a'}^{\hat{A}}(t_1), \hat{P}_{b'}^{\hat{B}}(t_2), \hat{P}_{c'}^{\hat{C}}(t_3)\} \tag{3.3}$$

were not in the set, I would not have an exhaustive family.

[1] By contrast, a *fine-grained* history would specify the values of observables at every instant, but such a history would be more exact than we need for most practical purposes, and indeed too exact for the theory to assign a probability to it.

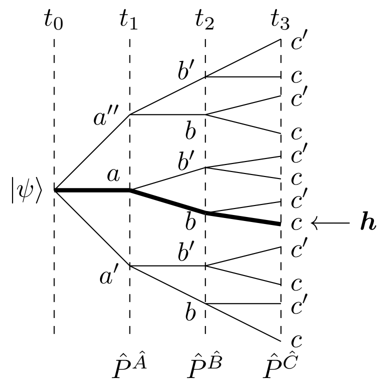


Figure 3.1: Branching Histories Here, we illustrate a set of histories in which an observable $\hat{A}$ (which can have the value $a,\ a'$, or a'') is considered at time t_1, an observable $\hat{B}$ (which can have the value b or b') is considered at time t_2, and an observable $\hat{C}$ (which can have the value c or c') is considered at time t_3. Each path—or branch—represents a history in the set. The dark branch is the particular history h, wherein $\hat{A}$ takes value a at t_1, $\hat{B}$ takes value b at t_2, and $\hat{C}$ takes value c at t_3. Later, we will learn that being able to meaninfully assign probabilities to histories in a set such as this depends on whether or not the discrete branches evince quantum interference. This figure is reproduced from [22] by the reflexive property of permission.

As an exclusive and exhaustive family accounts for all outcomes of an observable at a time, we can sum over all of the available projection operators for that time to attain the identity. For the example given by Fig. 3.1, we have

$$\hat{I} = \hat{P}_a^{\hat{A}} + \hat{P}_{a'}^{\hat{A}} + \hat{P}_{a''}^{\hat{A}}, \tag{3.4}$$

$$\hat{I} = \hat{P}_b^{\hat{B}} + \hat{P}_{b'}^{\hat{B}}, \tag{3.5}$$

$$\hat{I} = \hat{P}_c^{\hat{C}} + \hat{P}_{c'}^{\hat{C}}. \tag{3.6}$$

We can coarse-grain over a collection of outcomes by performing this addition, which creates a new family of histories. For example, consider a family that selects outcomes from the set

$$\left\{\{\hat{P}_a^{\hat{A}}(t_1), \hat{P}_{a'}^{\hat{A}}(t_1) + \hat{P}_{a''}^{\hat{A}}(t_1)\}, \{\hat{P}_b^{\hat{B}}(t_2), \hat{P}_{b'}^{\hat{B}}(t_2)\}, \{\hat{P}_c^{\hat{C}}(t_3), \hat{P}_{c'}^{\hat{C}}(t_3)\}\right\}. \tag{3.7}$$

Histories in this family can specify that $\hat{A}$ takes the value a, or that it takes one of the two values a' or a'', without making a distinction. We could also consider the family selecting from

$$\left\{\{\hat{I}(t_1)\}, \{\hat{P}_b^{\hat{B}}(t_2), \hat{P}_{b'}^{\hat{B}}(t_2)\}, \{\hat{P}_c^{\hat{C}}(t_3), \hat{P}_{c'}^{\hat{C}}(t_3)\}\right\}, \tag{3.8}$$

where histories do not specify the value of $\hat{A}$ at time t_1 at all. This could be represented by removing the time t_1 from the set entirely, which is the sort of coarse-graining already done in the family from Fig. 3.1, where the outcomes from the times between $t_0,\ t_1,\ t_2$, and t_3 are not specified.

There are further representations of histories beyond sets of projections. First, we can express a history as a *class operator*, which is the time-ordered product of the projectors in the history. The class operator for our history

h is

$$\hat{C}_h = \hat{P}_a^{\hat{A}}(t_1)\hat{P}_b^{\hat{B}}(t_2)\hat{P}_c^{\hat{C}}(t_3). \tag{3.9}$$

The class operator can be applied subsequently to the creation of the *branch wave function*, which is the wave function of the system acted on by all of the projectors, achieved by applying the Hermitian adjoint of the class operator. For our history h, the branch wave function is

$$|\psi_h\rangle = \hat{C}_h^{\dagger}\,|\psi\rangle = \hat{P}_c^{\hat{C}}(t_3)\hat{P}_b^{\hat{B}}(t_2)\hat{P}_a^{\hat{A}}(t_1)\,|\psi\rangle\,. \tag{3.10}$$

The times listed are not just labels; consistent histories operates in the Heisenberg picture, so time dependence is centered in the operators, rather than in the wave function. As such, we can also express our history as

$$\begin{aligned} |\psi_h\rangle &= \hat{U}^{\dagger}(t_3,t_0)\hat{P}_c^{\hat{C}}\hat{U}(t_3,t_0)\hat{U}^{\dagger}(t_2,t_0)\hat{P}_b^{\hat{B}}\hat{U}(t_2,t_0)\hat{U}^{\dagger}(t_1,t_0)\hat{P}_a^{\hat{A}}\hat{U}(t_1,t_0)\,|\psi\rangle \\ &= \hat{U}(t_0,t_3)\hat{P}_c^{\hat{C}}\hat{U}(t_3,t_2)\hat{P}_b^{B}\hat{U}(t_2,t_1)\hat{P}_a^{\hat{A}}\hat{U}(t_1,t_0)\,|\psi\rangle\,, \end{aligned} \tag{3.11}$$

in which we can see the state evolving from $|\psi\rangle$ at t_0 until t_1, where it meets the first projector specifying a value of one of the system's observables, then evolving to t_2, where it meets the next projector, then evolving to t_3, where it meets the final projector. The apparent backward evolution $\hat{U}(t_0,t_3)$ at the end is an artifact of using the Heisenberg picture, and the arbitrary global phase it imparts will cancel in our subsequent calculations, with no effect on our results.

The final object to construct is the *decoherence functional*, which contains the information of whether our histories can be assigned probabilities, and what those probabilities are when they exist. For histories h and h' that can be expressed as branch wave functions (ones with pure initial states $|\psi\rangle$), the decoherence functional is simply an inner product:

$$d(h,h') = \langle\psi_{h'}|\psi_h\rangle = \langle\psi|\hat{C}_{h'}\hat{C}_h^{\dagger}|\psi\rangle\,. \tag{3.12}$$

For an exhaustive and exclusive set of histories $\{h\}$, the decoherence functional is normalized:

$$\sum_{h,h'\in\{h\}} d(h,h') = 1. \tag{3.13}$$

There are a few subtly different choices of consistency conditions, but I shall go with the simple "medium decoherence" condition: $d(h,h') = 0$ when $h \neq h'$ [21]. When this is satisfied, the decoherence functional is diagonalized, and we say the family $\{h\}$ is *consistent*. In this situation, the diagonal entries are all that remain, and are still normalized. For our particular example history h, we have

$$d(h,h) = \langle\psi_h|\psi_h\rangle = \langle\psi|\hat{C}_h\hat{C}_h^{\dagger}|\psi\rangle = \langle\psi|\hat{P}_a^{\hat{A}}(t_1)\hat{P}_b^{\hat{B}}(t_2)\hat{P}_c^{\hat{C}}(t_3)\hat{P}_c^{\hat{C}}(t_3)\hat{P}_b^{\hat{B}}(t_2)\hat{P}_a^{\hat{A}}(t_1)|\psi\rangle\,, \tag{3.14}$$

which is the *von Neumann-Lüders rule* for the probability of a particular sequence of measurement outcomes (a the result of measuring $\hat{A}$ at t_1, b the result of measuring $\hat{B}$ at t_2, and c the result of measuring $\hat{C}$ at t_3) given the initial state $|\psi\rangle$ [23, 24] (for a simplified derivation of the two-measurement case from the Born rule, see [22]). However, we have not actually insisted on measurements taking place to get this result from the decoherence functional.

This is where we see the power of the formalism. The von Neumann-Lüders rule may be able to calculate a probability, but it does not in and of itself determine whether it is appropriate to calculate a particular probability for a given system. The decoherence functional, when diagonal, lets us know that probabilities may be assigned, and what the probability $\mathscr{P}(h)$ is for each history h:

$$d(h, h') = \delta_{h,h'} \mathscr{P}(h). \tag{3.15}$$

If this condition is not satisfied, the entire decoherence functional will remain normalized, but the diagonal elements may not, and will not be probabilities. The von Neumann-Lüders rule-like expressions are still calculable, but they do not reflect the outcomes of the system.

As an example, we can examine the double-slit experiment. We start with a quantum system in state $|\psi\rangle$ at time t_0, pass the system through a screen with two thin, parallel slits in it at time t_1, and then have it collide with a final screen at t_2 to measure its final position (see Fig. 3.2). For a particular end position y_f on the final screen, there are two histories we could establish, the first being $h_u = \{\hat{P}^{\hat{Y}}_{\Delta u}(t_1), \hat{P}^{\hat{Y}}_{y_f}(t_2)\}$, which specifies that the system passes through Δu, the region of the upper slit, before reaching the final position. The second history is $h_l = \{\hat{P}^{\hat{Y}}_{\Delta l}(t_1), \hat{P}^{\hat{Y}}_{y_f}(t_2)\}$, which specifies that the system passes through Δl, the region of the lower slit, before reaching the final position. We may attempt to calculate the probability of both with the von Neumann-Lüders rule, obtaining $\mathscr{P}_{h_u} = \langle\psi|\hat{P}^{\hat{Y}}_{\Delta u}(t_1)\hat{P}^{\hat{Y}}_{y_f}(t_2)\hat{P}^{\hat{Y}}_{y_f}(t_2)\hat{P}^{\hat{Y}}_{\Delta u}(t_1)|\psi\rangle$ and $\mathscr{P}_{h_l} = \langle\psi|\hat{P}^{\hat{Y}}_{\Delta l}(t_1)\hat{P}^{\hat{Y}}_{y_f}(t_2)\hat{P}^{\hat{Y}}_{y_f}(t_2)\hat{P}^{\hat{Y}}_{\Delta l}(t_1)|\psi\rangle$. If we wanted to find the total probability of reaching the end position, $\mathscr{P}_{y_f}$, we would be sorely tempted to just add the probabilities of these two histories, but we must be careful to determine if that is appropriate. The sum $\mathscr{P}_{h_u} + \mathscr{P}_{h_l}$ would be the probability density arising from a superposition of single-slit patterns from each of the slits, but we know for a fact that the interference pattern of a double-slit experiment does not arise in this way. Let $h_{y_f} = h_u + h_l = \{\hat{P}^{\hat{Y}}_{\Delta u}(t_1) + \hat{P}^{\hat{Y}}_{\Delta l}(t_1), \hat{P}^{\hat{Y}}_{y_f}(t_2)\}$ be the history specifying that the system passes through the slits at t_1 without specifying either slit in particular. If we calculate $\mathscr{P}_{h_{y_f}}$ via the von Neumann-Lüders rule, we will get a more complicated interference pattern, and see (in Fig. 3.2) that $\mathscr{P}_{h_{y_f}} = \mathscr{P}_{h_u+h_l} \neq \mathscr{P}_{h_u} + \mathscr{P}_{h_l}$. The decoherence functional would have warned us of this, as $d(h_u, h_l) \neq 0$, which tells us that the family consisting of $\{h_u, h_l\}$ is inconsistent. We must coarse grain to the family $\{h_{y_f}\}$ to get consistency and thus be able to assign probabilities to outcomes.

3.3 Weak Values and the Consistency Condition

Say we are given a quantum system in initial state $|\psi(t_0)\rangle$ at time t_0 that is postselected at time t_2 to be in the final state $|\phi(t_2)\rangle$, thus its history includes the projector $\big[\,|\phi(t_2)\rangle\,\langle\phi(t_2)|\,\big](t_2)$. At the intermediate time t_1, we want to know if the system is in some subspace Λ of the eigenspace of the observable $\hat{A}$, so we specify the history further by

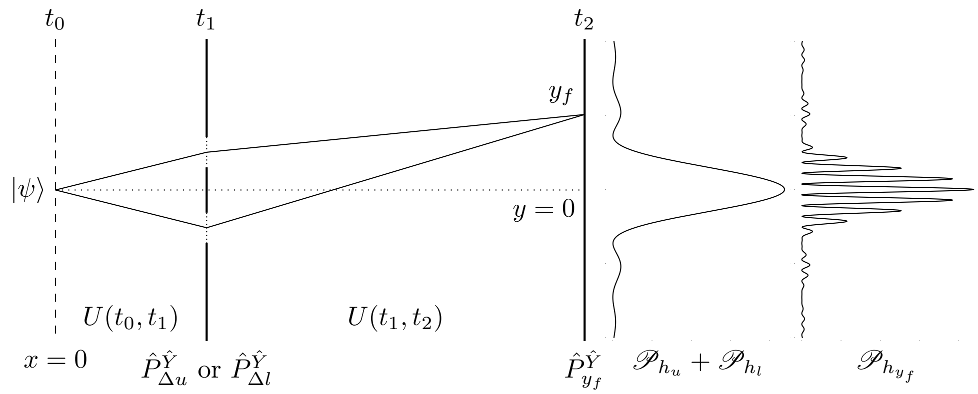


Figure 3.2: Double-Slit Probability Distributions In this figure, the x-axis is horizontal, and the y-axis is vertical. The incident wave function $|\psi\rangle$ passes through two slits at time t_1 and reaches height y_f on the detector screen at time t_2. The intervening moments during which the system undergoes unitary time evolution are marked by the associated propagators ($U(t_0, t_1)$ and $U(t_1, t_2)$). On the right, two probability distributions are depicted: the sum of the probabilities for the individual histories h_u and h_l, which consists of two overlapping single-slit probability distributions, and the probability distribution of the history h_{y_f}, which demonstrates the typical double-slit interference pattern. Clearly, the two are not the same, but the von Neumann-Lüders rule allows us to calculate either. The advantage of the decoherence functional is that it would evaluate the consistency of the families to tell us whether or not it would be appropriate to assign a probability distribution to the associated histories. This figure is modified from Figs. 2.1, 4.2(a), and 4.4(b) of [22] by the reflexive property of permission.

inserting the projector $\hat{P}^{\hat{A}}_{\Lambda}(t_1)$. If $h = \{\hat{P}^{\hat{A}}_{\Lambda}(t_1), \big[\,|\phi(t_2)\rangle\langle\phi(t_2)|\,\big](t_2)\}$, then let $h' = \{\hat{P}^{\hat{A}}_{\Lambda'}(t_1), \big[\,|\phi(t_2)\rangle\langle\phi(t_2)|\,\big](t_2)\}$ be another history in the family, where $\hat{P}^{\hat{A}}_{\Lambda'} = \hat{I} - \hat{P}^{\hat{A}}_{\Lambda}$. Using the decoherence functional to check consistency, we obtain

$$\begin{aligned}
d(h,h') &= \langle\psi(t_0)|\hat{P}^{\hat{A}}_{\Lambda'}(t_1)\big[\,|\phi(t_2)\rangle\langle\phi(t_2)|\,\big](t_2)\hat{P}^{\hat{A}}_{\Lambda}(t_1)|\psi(t_0)\rangle \\
&= \langle\psi(t_0)|\hat{U}^\dagger(t_1,t_0)\hat{P}^{\hat{A}}_{\Lambda'}\hat{U}(t_1,t_0)\hat{U}^\dagger(t_2,t_0)\,|\phi(t_2)\rangle\langle\phi(t_2)|\,\hat{U}(t_2,t_0)\hat{U}^\dagger(t_1,t_0)\hat{P}^{\hat{A}}_{\Lambda}\hat{U}(t_1,t_0)|\psi(t_0)\rangle \\
&= \langle\psi(t_0)|\hat{U}^\dagger(t_1,t_0)\hat{P}^{\hat{A}}_{\Lambda'}\hat{U}(t_1,t_2)\,|\phi(t_2)\rangle\langle\phi(t_2)|\,\hat{U}^\dagger(t_1,t_2)\hat{P}^{\hat{A}}_{\Lambda}\hat{U}(t_1,t_0)|\psi(t_0)\rangle \\
&= \langle\psi(t_1)|\hat{P}^{\hat{A}}_{\Lambda'}\,|\phi(t_1)\rangle\langle\phi(t_1)|\,\hat{P}^{\hat{A}}_{\Lambda}|\psi(t_1)\rangle \\
&= |\,\langle\phi(t_1)|\psi(t_1)\rangle\,|^2\frac{\langle\psi(t_1)|\hat{P}^{\hat{A}}_{\Lambda'}|\phi(t_1)\rangle}{\langle\psi(t_1)|\phi(t_1)\rangle}\frac{\langle\phi(t_1)|\hat{P}^{\hat{A}}_{\Lambda}|\psi(t_1)\rangle}{\langle\phi(t_1)|\psi(t_1)\rangle} \\
&= |\,\langle\phi(t_1)|\psi(t_1)\rangle\,|^2(\hat{P}^{\hat{A}}_{\Lambda'})^*_w(\hat{P}^{\hat{A}}_{\Lambda})_w.
\end{aligned} \tag{3.16}$$

If the family is to be consistent, we must have

$$|\,\langle\phi(t_1)|\psi(t_1)\rangle\,|^2(\hat{P}^{\hat{A}}_{\Lambda'})^*_w(\hat{P}^{\hat{A}}_{\Lambda})_w = 0, \tag{3.17}$$

which is Kastner's result [17], written in the dialect I am familiar with. Assuming $\langle\phi(t_1)|\psi(t_1)\rangle \neq 0$ (otherwise, the postselected outcome would not happen), we can see that either $(\hat{P}^{\hat{A}}_{\Lambda'})^*_w$ or $(\hat{P}^{\hat{A}}_{\Lambda})_w$ must be zero. However, we also know that $\hat{I} = \hat{P}^{\hat{A}}_{\Lambda'} + \hat{P}^{\hat{A}}_{\Lambda}$, so the linearity of weak values tells us that $1 = (\hat{I})_w = (\hat{P}^{\hat{A}}_{\Lambda'})_w + (\hat{P}^{\hat{A}}_{\Lambda})_w$. As such, if either $(\hat{P}^{\hat{A}}_{\Lambda'})_w$ or $(\hat{P}^{\hat{A}}_{\Lambda})_w$ is zero, then the other must be equal to 1. In the case of unsharp or strange weak values for $\hat{P}^{\hat{A}}_{\Lambda}$ (those not landing in its eigenvalue spectrum), we know that neither $(\hat{P}^{\hat{A}}_{\Lambda'})_w$ nor $(\hat{P}^{\hat{A}}_{\Lambda})_w$ is equal to 1 or 0, and the history will be inconsistent.

In the particular case of Vaidman's weak trace criterion for particle presence [11], a particle is defined to be present in a location when the weak value of the projection operator specifying that location is nonzero. Unsharp and strange weak values for particle presence are thus included among the possibilities (not to mention experimentally observed in practice [13]), and so might be seen in the consistent histories approach when building histories with the particle location projectors. This would appear to proscribe the use of consistent histories in examining questions of particle presence in these special situations (or to suggest that the weak trace criterion is off its rocker, depending on which you like more).

Before we write off a union of these two approaches, citing irreconcilable differences, there are two things to consider. First, projectors are a small subset of observables that one can find weak values for, which begs the question, where would the weak values of general observables fit into consistent histories? Second, while the projectors used in consistent histories may need to have sharp weak values to work in a consistent family, the weak values observed in theoretical treatments [4] and experiments are found in the state of the probes. Why should we leave the probes out of our system when we apply the consistent histories formalism?

3.4 Weak Probes and the Consistency Condition

With these questions before me, I set myself to the task of addressing the associated deficiencies, and devised a way to extend Kastner's result. To demonstrate, let us construct a new system, with some simplifications. First, the propagators will be taken to be identities, with the exception of $\hat{U}(t_1, t_0)$, which will incorporate an interaction between the system proper that we are examining and the auxiliary probe that is incorporated into the composite system. This means that, except for the probe-system interaction, the times at which projectors are applied in the histories are just for ordering, and do not result in evolution of the system state.

We begin in the initial composite state $|\psi\rangle |+\rangle_x$, where $|\psi\rangle$ is the system proper, and $|+\rangle_x$ is a qubit probe initially in the spin-up state. The probe evolution will be governed by the unitary operator $e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z}$, which entangles the system and probe by rotating the qubit by some amount that depends on the value of the observable $\hat{A}$. At the final time t_2, we postselect the composite system to be in a state $|\phi, s\rangle = |\phi\rangle |s\rangle_x$, where $s = \pm$. At the intermediate time t_1, we specify that the state of the system proper is in either the subspace Λ or the subspace Λ', which will be represented in the history by either $\hat{P}_\Lambda$ or $\hat{P}_{\Lambda'}$, respectively. This time, Λ and Λ' do not necessarily span the entire state space, so $\hat{P}_\Lambda + \hat{P}_{\Lambda'} \neq \hat{I}$, but they will remain mutually orthogonal. Note also that I have not indicated what observable to associate these projectors with, simply because it need not necessarily be $\hat{A}$. Properly, these projectors should be written in the form $\hat{P} \otimes \hat{I}$ to make clear that they operate on the system and not the probe, but this level of care will be sacrificed for the sake of reducing notational clutter. With these definitions and foibles out of the way, we can now work with the following histories:

$$hs = \{\hat{P}_\Lambda(t_1), \big[\, |\phi, s\rangle \langle \phi, s| \,\big](t_2)\}, \quad h's = \{\hat{P}_{\Lambda'}(t_1), \big[\, |\phi, s\rangle \langle \phi, s| \,\big](t_2)\}, \quad (h{+}h')s = \{\hat{P}_\Lambda(t_1){+}\hat{P}_{\Lambda'}(t_1), \big[\, |\phi, s\rangle \langle \phi, s| \,\big](t_2)\}, \tag{3.18}$$

where $\{h+, h-, h'+, h'-\}$ forms one family, and $\{(h{+}h')+, (h{+}h')-\}$ forms another which is more coarsely grained.

In this particular situation, I will specify a few further properties of our particular intermediate projectors. First and foremost, while $\hat{P}_\Lambda + \hat{P}_{\Lambda'}$ need not be the projector for the entire system state space (also known as the identity), we do want it to be expansive enough that inserting it in the middle of $\langle\phi|\psi\rangle$ has no effect on the inner product. For the purposes of this demonstration, we will accomplish this in the following way:

$$\langle\phi|\hat{P}_\Lambda|\psi\rangle = \langle\phi|\psi\rangle \,, \qquad \langle\phi|\hat{P}_{\Lambda'}|\psi\rangle = 0. \tag{3.19}$$

This specifies that P_Λ does not exclude any states within $|\psi\rangle$ or $|\phi\rangle$ which are necessary parts of their mutual overlap in forming the inner product $\langle\phi|\psi\rangle$. In contrast, $P_{\Lambda'}$ applied to the preselection state or the postselection state makes it orthogonal to the other state, eliminating the inner product. It is worth noting that $P_{\Lambda'}$ does not necessarily remove all overlap between the two states, as there could be component states within both $|\psi\rangle$ and $|\phi\rangle$ which cancel out in the inner product, and thus could be left untouched.

In addition, we want $\hat{P}_\Lambda + \hat{P}_{\Lambda'}$ to have no effect on $\langle\phi|\hat{A}|\psi\rangle$ when inserted before the operator, that way the weak value will go undisturbed:

$$A_w = \frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle} = \frac{\langle\phi| \left[\hat{P}_\Lambda + \hat{P}_{\Lambda'}\right] \hat{A}|\psi\rangle}{\langle\phi| \left[\hat{P}_\Lambda + \hat{P}_{\Lambda'}\right] |\psi\rangle}. \tag{3.20}$$

As a result of these properties, we find that the branch wave function for $(h + h')s$ is

$$\begin{aligned} |\psi_{(h+h')s}\rangle &= |\phi, s\rangle \langle \phi, s| \left[\hat{P}_\Lambda + \hat{P}_{\Lambda'}\right] e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z} |\psi\rangle \,|+\rangle_x \\ &\approx |\phi, s\rangle \, {}_x\langle s| \left(\langle\phi| \left[\hat{P}_\Lambda + \hat{P}_{\Lambda'}\right] |\psi\rangle - i\frac{\epsilon}{\hbar} \langle\phi| \left[\hat{P}_\Lambda + \hat{P}_{\Lambda'}\right] \hat{A}|\psi\rangle \,\hat{\sigma}_z \right) |+\rangle_x \\ &\approx |\phi, s\rangle \langle\phi|\psi\rangle \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}A_w\hat{\sigma}_z} |+\rangle_x, \end{aligned} \tag{3.21}$$

which gives us a consistent family (since $\langle\phi, +|\phi, -\rangle = \langle\phi|\phi\rangle \, {}_x\langle +|-\rangle_x = 0$) in which the probe state is rotated by an amount specified by the weak value of $\hat{A}$.

Alternatively, if we calculate the branch wave functions for hs and $h's$, we get

$$\begin{aligned} |\psi_{hs}\rangle &= |\phi, s\rangle \langle \phi, s| \,\hat{P}_\Lambda e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z} |\psi\rangle \,|+\rangle_x \\ &\approx |\phi, s\rangle \langle\phi|\psi\rangle \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}(\hat{P}_\Lambda\hat{A})_w\hat{\sigma}_z} |+\rangle_x \\ &= |\phi, s\rangle \langle\phi|\psi\rangle \left(\cos\left[\frac{\epsilon}{\hbar}(\hat{P}_\Lambda\hat{A})_w\right] \delta_{s,+} - i\sin\left[\frac{\epsilon}{\hbar}(\hat{P}_\Lambda\hat{A})_w\right] \delta_{s,-} \right), \end{aligned} \tag{3.22}$$

$$\begin{aligned} |\psi_{h's}\rangle &= |\phi, s\rangle \langle \phi, s| \,\hat{P}_{\Lambda'} e^{-i\frac{\epsilon}{\hbar}\hat{A}\otimes\hat{\sigma}_z} |\psi\rangle \,|+\rangle_x \\ &\approx |\phi, s\rangle \left(-i\frac{\epsilon}{\hbar} \langle\phi|\hat{P}_{\Lambda'}\hat{A}|\psi\rangle \,\delta_{s,-} \right). \end{aligned} \tag{3.23}$$

If we want this family to be consistent, then the only fly in the ointment is this off-diagonal term of the decoherence functional:

$$\begin{aligned} d(h-,h'-) = \langle\psi_{h'-}|\psi_{h-}\rangle &= i\frac{\epsilon}{\hbar}\langle\psi|\hat{A}\hat{P}_{\Lambda'}|\phi\rangle\langle\phi|\psi\rangle\left(-i\sin\left[\frac{\epsilon}{\hbar}(\hat{P}_{\Lambda}\hat{A})_w\right]\right) \\ &\approx \frac{\epsilon^2}{\hbar^2}\langle\psi|\hat{A}\hat{P}_{\Lambda'}|\phi\rangle\langle\phi|\hat{P}_{\Lambda}\hat{A}|\psi\rangle\,. \end{aligned} \tag{3.24}$$

Since we are taking the probe coupling ϵ to be very weak, we could gain approximate decoherence by assuming the smallness of ϵ^2 makes this entirely negligible. In turn, this would eliminate the following outcomes:

$$\mathscr{P}_{h'-} = d(h'-,h'-) = \langle\psi_{h'-}|\psi_{h'-}\rangle \approx \frac{\epsilon^2}{\hbar^2}|\langle\phi|\hat{P}_{\Lambda'}\hat{A}|\psi\rangle|^2, \tag{3.25}$$

$$\mathscr{P}_{h-} = d(h-,h-) = \langle\psi_{h-}|\psi_{h-}\rangle \approx \frac{\epsilon^2}{\hbar^2}|\langle\phi|\hat{P}_{\Lambda}\hat{A}|\psi\rangle|^2. \tag{3.26}$$

The only remaining history (of those ending in the postselected state $|\phi\rangle$), would be $h+$, with probability

$$\mathscr{P}_{h+} = d(h+,h+) = \langle\psi_{h+}|\psi_{h+}\rangle \approx |\langle\phi|\psi\rangle|^2\cos^2\left[\frac{\epsilon}{\hbar}(\hat{P}_{\Lambda}\hat{A})_w\right] \approx |\langle\phi|\psi\rangle|^2. \tag{3.27}$$

We can see the Born rule emerging here, with the probability depending only on the initial and final states, as though the weak probe were not there. However, in this history, the weak probe *is* there, and its state has been rotated by $(\hat{P}_{\Lambda}\hat{A})_w$, rather than A_w. Since the probe state is approximately pure after postselection, we can expect to see measurement outcomes of the probe state that reflect this new weak value, rather than the one we anticipate seeing from our further coarse-grained family. As such, it would appear to be imprudent to neglect the ϵ^2 terms, leaving this family inconsistent.

Another resolution to this would be to have a situation in which $(\hat{P}_{\Lambda'}\hat{A})_w = 0$. This would mean that $(\hat{P}_{\Lambda}\hat{A})_w = A_w$, and we can confidently assign zero probability to the system state being in Λ' at the intermediate time. This gives us some idea of the minimum state space the system could occupy during the weak measurement. It must be large enough to encompass all basis states that contribute to both $|\psi\rangle$ and $|\phi\rangle$, except those states that don't ultimately contribute to both $\langle\phi|\psi\rangle$ and $\langle\phi|\hat{A}|\psi\rangle$.

As an example, consider a system in initial state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|2\rangle + |2'\rangle + |3\rangle\right), \tag{3.28}$$

which is postselected in the final state

$$|\phi\rangle = \frac{1}{\sqrt{3}}\left(|2\rangle - |2'\rangle + |3\rangle\right), \tag{3.29}$$

where the numbered states $|n\rangle$ are elements of an orthonormal basis. We will weakly measure the observable $\hat{N}$, the number operator, which acts on the basis states of the system in the following manner: $\hat{N}|n\rangle = n|n\rangle$. The states $|2\rangle$ and $|2'\rangle$ are degenerate in the eigenvalue 2, such that $\hat{N}|2\rangle = 2|2\rangle$ and $\hat{N}|2'\rangle = 2|2'\rangle$. We can see that $\langle\phi|\psi\rangle = \frac{1}{3}$, and $\langle\phi|\hat{N}|\psi\rangle = 1$, so $N_w = 3$. If we are curious to know whether the system proper is in one of the three states $|2\rangle$, $|2'\rangle$, or $|3\rangle$ at the time of our weak measurement, it makes sense to calculate histories which specify these basis states. In terms of their branch wave functions, these histories are

$$\begin{aligned} |\psi_{h_3 s}\rangle &= |\phi, s\rangle \langle\phi, s| \, |3\rangle \langle 3| e^{-i\frac{\epsilon}{\hbar}\hat{N}\otimes\hat{\sigma}_z} |\psi\rangle \, |+\rangle_x \\ &= |\phi, s\rangle \langle\phi|3\rangle \langle 3|\psi\rangle \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}3\hat{\sigma}_z} |+\rangle_x \\ &= |\phi, s\rangle \, \frac{1}{3} \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}3\hat{\sigma}_z} |+\rangle_x, \end{aligned} \tag{3.30}$$

$$\begin{aligned} |\psi_{h_2 s}\rangle &= |\phi, s\rangle \langle\phi, s| \, |2\rangle \langle 2| e^{-i\frac{\epsilon}{\hbar}\hat{N}\otimes\hat{\sigma}_z} |\psi\rangle \, |+\rangle_x \\ &= |\phi, s\rangle \langle\phi|2\rangle \langle 2|\psi\rangle \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}2\hat{\sigma}_z} |+\rangle_x \\ &= |\phi, s\rangle \, \frac{1}{3} \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}2\hat{\sigma}_z} |+\rangle_x, \end{aligned} \tag{3.31}$$

$$\begin{aligned} |\psi_{h_{2'} s}\rangle &= |\phi, s\rangle \langle\phi, s| \, |2'\rangle \langle 2'| e^{-i\frac{\epsilon}{\hbar}\hat{N}\otimes\hat{\sigma}_z} |\psi\rangle \, |+\rangle_x \\ &= |\phi, s\rangle \langle\phi|2'\rangle \langle 2'|\psi\rangle \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}2\hat{\sigma}_z} |+\rangle_x \\ &= |\phi, s\rangle \, \frac{-1}{3} \, {}_x\langle s| e^{-i\frac{\epsilon}{\hbar}2\hat{\sigma}_z} |+\rangle_x. \end{aligned} \tag{3.32}$$

At this stage, we can already see that there is a problem, as none of these branch wave functions are orthogonal (for those sharing the same value of s). However, if we coarse-grain over the last two histories—specifying that the system proper is in the space of the span of $|2\rangle$ and $|2'\rangle$ at the intermediate time, but not specifying which state in particular—we obtain the new history

$$|\psi_{h_{2+2'} s}\rangle = |\phi, s\rangle \langle\phi, s| \left[\, |2\rangle \langle 2| + |2'\rangle \langle 2'| \, \right] e^{-i\frac{\epsilon}{\hbar}\hat{N}\otimes\hat{\sigma}_z} \, |\psi\rangle \, |+\rangle_x = |\psi_{h_2 s}\rangle + |\psi_{h_{2'} s}\rangle = 0. \tag{3.33}$$

Now, the subfamily $\{h_3\pm, h_{2+2'}\pm\}$ is consistent, with only $h_3\pm$ as possible outcomes, both of which leave the probe in the output state with the desired weak value presented. This demonstrates that, though all three basis states contribute to the initial and postselected states, only $|3\rangle$ has a nonzero contribution to both $\langle\phi|\psi\rangle$ and $\langle\phi|\hat{N}|\psi\rangle$, so only it is predicted in the subfamily of histories at the intermediate time.

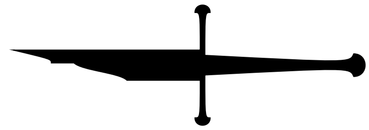
There are a few shortcomings to this approach. First, while I can confidently specify that the system state was not in the subspace projected onto by $\left[\, |2\rangle \langle 2| + |2'\rangle \langle 2'| \, \right]$ at the intermediate time, I cannot technically say that the system state was not in the subspace projected onto by $|2\rangle \langle 2|$ or that projected onto by $|2'\rangle \langle 2'|$, as I cannot calculate those probabilities. It seems intuitive to think so, but it is a level of description that the consistent histories formalism deems inappropriate [16].

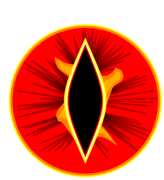
More importantly, this perpetuates a rather annoying habit of weak value theorists in only handling one weak probe at a time. This particular example is arranged as a version of the three-box paradox [16, 18, 25], which is physically realized in the nested Mach-Zehnder interferometer setup of [11, 13, 16]. However, in those experiments and theoretical treatments, the observables are projectors onto the three boxes (which would be $|2\rangle \langle 2|$, $|2'\rangle \langle 2'|$, and $|3\rangle \langle 3|$ in this example), not the number operator I defined above. To properly examine that situation, I would need at least three auxiliary qubits, each one coupled to one of these boxes. This more intensive calculation could provide an interesting perspective on the question of particle presence in the nested interferometer setup.

It is also worth noting that I have only played with projectors in a particular basis. The consistent histories formalism may tell us whether a particular family of histories can have probabilities assigned to it, but when there are multiple consistent families, there is no rule that favors one over another. Two consistent families which share one or more histories will never assign them different probabilities, but the unicity of reality—the idea that there is one unique and true state of a system, or indeed the entire world, at any given time—is not enforced in a reality described by Hilbert space [16]. As such, there could be other consistent families which obtain the same probe state, but predict different intermediate states for the system proper. This can be seen to some degree in the given example, as we could create an even coarser-grained family of histories $\{h_{2+2'+3}\pm\}$ with branch wave functions

$$|\psi_{h_{2+2'+3}s}\rangle = |\phi, s\rangle \langle\phi, s| \left[\, |2\rangle \langle 2| + |2'\rangle \langle 2'| + |3\rangle \langle 3| \,\right] e^{-i\frac{\epsilon}{\hbar}\hat{N}\otimes\hat{\sigma}_z} |\psi\rangle |+\rangle_x = |\psi_{h_2 s}\rangle + |\psi_{h_{2'} s}\rangle + |\psi_{h_3 s}\rangle = |\psi_{h_3 s}\rangle\,. \quad (3.34)$$

This would tell us with certainty that the system proper, when postselected in the state $|\phi\rangle$, is in some state in the span of $|2\rangle$, $|2'\rangle$, and $|3\rangle$. Our earlier description was more specific, but other than the intuitive logic that being certainly in the state $|3\rangle$ means being certainly in the above span (thus favoring the more specific view), there is no mathematical rule that advances one or the other as the true family to use. Other families with stranger combinations of basis states could provide a picture even further removed from the one painted by $\{h_3\pm, h_{2+2}\pm\}$ than that painted by $\{h_{2+2'+3}\pm\}$.





Appendix A

Lemmas for Chapter 1

A.1 Proof of $\Gamma\left(j+\frac{1}{2}\right)=\frac{(2j-1)!!}{2^j}\sqrt{\pi}$

While the claim $\Gamma\left(j+\frac{1}{2}\right)=\frac{(2j-1)!!}{2^j}\sqrt{\pi}$ for $j\in\mathbb{N}$ is provided by [6], the proof is omitted, likely due to its elementary nature. Those familiar with the gamma function know that $\Gamma(x+1)=x\Gamma(x)$ (which is shown by integration by parts), so once the base case, $\Gamma(1+\frac{1}{2})=\frac{\sqrt{\pi}}{2}$, is proved, $\Gamma\left(j+\frac{1}{2}\right)=\frac{(2j-1)!!}{2^j}\sqrt{\pi}$ follows immediately by induction. In particular,

$$\Gamma\left(j+\frac{3}{2}\right)=\left(j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)=\frac{2j+1}{2}\frac{(2j-1)!!}{2^j}\sqrt{\pi}=\frac{(2j+1)!!}{2^{j+1}}\sqrt{\pi}. \tag{A.1}$$

The base case is also simple, requiring a little substitution and some knowledge of Gaussian integrals. We start with

$$\Gamma\left(\frac{3}{2}\right)=\int_0^\infty dx\ \sqrt{x}e^{-x}, \tag{A.2}$$

and make the substitution $z^2=x$, which gives us

$$\begin{aligned}\Gamma\left(\frac{3}{2}\right)&=2\int_0^\infty dz\ z^2e^{-z^2}\\&=\left[-\frac{\partial}{\partial t}\int_{-\infty}^\infty dz\ e^{-tz^2}\right]_{t=1}\\&=\left[-\frac{\partial}{\partial t}\sqrt{\frac{\pi}{t}}\right]_{t=1}\\&=\left[\frac{\sqrt{\pi}}{2}t^{-3/2}\right]_{t=1}\\&=\frac{\sqrt{\pi}}{2}.\end{aligned} \tag{A.3}$$

A.2 Limiting Behavior of $\frac{\sqrt{(2j-1)!!}}{j!}$

To prove that, for any $a\geq 1$ there exists a natural number k such that $\frac{\sqrt{(2j-1)!!}}{j!}<a^{-j}$ for all $j\geq k$, we start with a much simpler fact: for any a, there exists $k\in\mathbb{N}$ such that $k\geq 2e^{16}a^2$. Now, for all $j\geq k$, we know that

$j \geq 2e^{16}a^2$, and we can multiply both sides by j^2 (and play a bit with the right hand side) to get

$$j^3 \geq e^{16}\frac{a^2}{2}(2j)^2. \tag{A.4}$$

If we take the logarithm of both sides, then

$$3\ln j \geq 16 + \ln\frac{a^2}{2} + 2\ln 2j. \tag{A.5}$$

We can then multiply by j to get

$$3j\ln j \geq 16j + j\ln\frac{a^2}{2} + 2j\ln 2j. \tag{A.6}$$

Now we can subtract $42j$ from each side to get

$$3j\ln j - 42j \geq -26j + j\ln\frac{a^2}{2} + 2j\ln 2j. \tag{A.7}$$

Next, we have two more inequalities to work with. The first is a comparison of constants: $3\ln\sqrt{2\pi} > \ln\sqrt{2\pi} + \frac{1}{2}\ln 2$. Second, we know that $\frac{3}{2}\ln j \geq \frac{1}{2}\ln j$ for all $j \geq 1$. As such, we can add these to our expression to get

$$3j\ln j - 42j + 3\ln\sqrt{2\pi} + \frac{3}{2}\ln j \geq -26j + j\ln\frac{a^2}{2} + 2j\ln 2j + \ln\sqrt{2\pi} + \frac{1}{2}\ln 2j. \tag{A.8}$$

Why is this important? Well, the factorial is bounded in the following manner [26]:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \frac{1}{e^{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \frac{1}{e^{12n}}. \tag{A.9}$$

As such, it follows that

$$\begin{aligned} \ln(j!)^3 &> 3\ln\sqrt{2\pi} + 3j\ln j + \frac{3}{2}\ln j - 39j - 3; \\ \ln\left(\frac{a^2}{2}\right)^j (2j)! &< j\ln\frac{a^2}{2} + \ln\sqrt{2\pi} + 2j\ln 2j + \frac{1}{2}\ln 2j - 26j. \end{aligned} \tag{A.10}$$

As such,

$$\begin{aligned} \ln(j!)^3 &> 3\ln\sqrt{2\pi} + 3j\ln j + \frac{3}{2}\ln j - 39j - 3 \\ &\geq 3\ln\sqrt{2\pi} + 3j\ln j + \frac{3}{2}\ln j - 39j - 3j \\ &= 3\ln\sqrt{2\pi} + 3j\ln j + \frac{3}{2}\ln j - 42j \\ &\geq j\ln\frac{a^2}{2} + \ln\sqrt{2\pi} + 2j\ln 2j + \frac{1}{2}\ln 2j - 26j \\ &> \ln\left(\frac{a^2}{2}\right)^j (2j)!. \end{aligned} \tag{A.11}$$

Exponentiating both sides gives us

$$(j!)^3 > \left(\frac{a^2}{2}\right)^j (2j)!, \tag{A.12}$$

and since $j! = \frac{(2j)!!}{2^j}$, we can divide our expression by this to obtain

$$(j!)^2 > a^{2j}(2j-1)!!, \tag{A.13}$$

And then we can take the square root and rearrange to find that

$$\frac{\sqrt{(2j-1)!!}}{j!} < a^{-j}. \tag{A.14}$$

Thus, for any $a \geq 1$, we can select $k \in \mathbb{N}$ such that Eqn. A.14 holds for all $j \geq k$, with the easiest way to find this k being to choose it such that $k \geq 2e^{16}a^2$ (a number well over 17 million).

Of course, less extreme upper bounds on $\frac{\sqrt{(2j-1)!!}}{j!}$ require less extreme lower bounds on k. In particular, assume that there exists $j \in \mathbb{N}$ such that $j! \geq \sqrt{(2j-1)!!}$. It follows that

$$(j+1)! = (j+1)j! \geq (j+1)\sqrt{(2j-1)!!} = \frac{j+1}{\sqrt{2j+1}}\sqrt{(2j+1)!!}, \tag{A.15}$$

and since $(j+1)^2 \geq 2j+1$ for all $j \in \mathbb{N}$, we know $\frac{j+1}{\sqrt{2j+1}} \geq 1$, and thus

$$(j+1)! \geq \sqrt{(2j+1)!!}. \tag{A.16}$$

Since $1! = \sqrt{(2 \cdot 1 - 1)!!}$, we know by induction that $j! \geq \sqrt{(2j-1)!!}$ for all $j \in \mathbb{N}$. Thus, $\frac{\sqrt{(2j-1)!!}}{j!} \leq 1$ for all j, which means we could omit this factor from each term of our sum in Eqn. 1.34 and still have an expression bounding the error in our weak value approximation.

Appendix B

Difficulties with Weak Values Taken in Sequence

B.1 False Starts

While feedback compensation has allowed us to see that weak values are the optimal estimates for weak interactions taken in sequence for the nested Mach-Zehnder interferometer setup (see Chapter 2), it is worth noting that this does not come out of an Eqn. 1.27-like calculation in a clean manner. Let us add a second probe, $|\mathscr{Q}'\rangle$ (with canonical variables q' and p'), and a new interaction to our Hamiltonian from Eqn. 1.21:

$$\hat{H}(t) = \hat{H}_{sys} - \epsilon\delta(t - t_1)\hat{A}\hat{q} - \epsilon\delta(t - t_2)\hat{B}\hat{q}'. \tag{B.1}$$

Evolving to t_3 is accomplished by the operator

$$U(t_3, 0) = e^{-i\hat{H}_{sys}(t_3-t_2)/\hbar} e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} e^{-i\hat{H}_{sys}t_1/\hbar}. \tag{B.2}$$

In turn, the unnormalized state after the final measurement (postselected to be $|\phi(t_3)\rangle$) is

$$|\phi(t_3)\rangle \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle |\mathscr{Q}\rangle |\mathscr{Q}'\rangle . \tag{B.3}$$

The question is, how do we approach the approximation? Work by Vaidman (such as [11, 12]) and colleagues (such as [13]) seems to assume the approximation can be made for one probe while ignoring the presumably negligible effects of the other. This leaves us in what I find to be the rather dissatisfying situation of not actually deriving the approximation directly from a starting point such as Eqn. B.3. That being said, how would we even do this?

It would be a lot simpler if $H_{sys} = 0$, which would simplify Eqn. B.3 to

$$|\phi(t_3)\rangle \langle\phi(t_3)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_0)\rangle |\mathscr{Q}\rangle |\mathscr{Q}'\rangle . \tag{B.4}$$

Before (Eqn. 1.26), I was very careful to define weak values with respect to pre- and postselected states that had been evolved to the same time—the time of the interaction. Here, with $H_{sys} = 0$, these states don't change in time. Without $e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}$, there isn't even a distinction between which interaction, $e^{i\epsilon\hat{B}\hat{q}'/\hbar}$ or $e^{i\epsilon\hat{A}\hat{q}/\hbar}$, happens first.

It is possible to claim

$$|\phi(t_3)\rangle \langle\phi(t_3)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_0)\rangle |\mathscr{Q}\rangle |\mathscr{Q}'\rangle \approx |\phi(t_3)\rangle \langle\phi(t_3)|\psi(t_0)\rangle e^{i\epsilon(\hat{B}\hat{q}'+\hat{A}\hat{q})_w/\hbar} |\mathscr{Q}\rangle |\mathscr{Q}'\rangle, \tag{B.5}$$

and by the linearity of weak values,

$$(\hat{B}\hat{q}' + \hat{A}\hat{q})_w = \frac{\langle\phi(t_3)|(\hat{B}\hat{q}' + \hat{A}\hat{q})|\psi(t_0)\rangle}{\langle\phi(t_3)|\psi(t_0)\rangle} = \frac{\langle\phi(t_3)|\hat{B}|\psi(t_0)\rangle}{\langle\phi(t_3)|\psi(t_0)\rangle}\hat{q}' + \frac{\langle\phi(t_3)|\hat{A}|\psi(t_0)\rangle}{\langle\phi(t_3)|\psi(t_0)\rangle}\hat{q} = B_w\hat{q}' + A_w\hat{q}. \tag{B.6}$$

As such,

$$|\phi(t_3)\rangle \langle\phi(t_3)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_0)\rangle |\mathscr{Q}\rangle |\mathscr{Q}'\rangle \approx |\phi(t_3)\rangle \langle\phi(t_3)|\psi(t_0)\rangle e^{i\epsilon A_w\hat{q}/\hbar} |\mathscr{Q}\rangle e^{i\epsilon B_w\hat{q}'/\hbar} |\mathscr{Q}'\rangle. \tag{B.7}$$

This is tenable, but if $H_{sys} \neq 0$, then we are stuck with Eqn. B.3, and no clear way to handle the core operator on the probe states:

$$\langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle. \tag{B.8}$$

In order to obtain $B_w = \frac{\langle\phi(t_2)|\hat{B}|\psi(t_2)\rangle}{\langle\phi(t_2)|\psi(t_2)\rangle}$ or $A_w = \frac{\langle\phi(t_1)|\hat{A}|\psi(t_1)\rangle}{\langle\phi(t_1)|\psi(t_1)\rangle}$, we somehow need to manufacture $|\psi(t_2)\rangle\langle\phi(t_1)|$ in the middle in place of $e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}$. Exactly how this would be accomplished appears mystifying. The most natural suggestion would be to insert identities: $(I - |\psi(t_2)\rangle\langle\psi(t_2)| + |\psi(t_2)\rangle\langle\psi(t_2)|)$ to the left of $e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}$ and $(I - |\phi(t_1)\rangle\langle\phi(t_1)| + |\phi(t_1)\rangle\langle\phi(t_1)|)$ to the right. Ultimately, this is equivalent to:

$$\begin{aligned}
&\langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar}(I - |\psi(t_2)\rangle\langle\psi(t_2)| + |\psi(t_2)\rangle\langle\psi(t_2)|)e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}(I - |\phi(t_1)\rangle\langle\phi(t_1)| + |\phi(t_1)\rangle\langle\phi(t_1)|)e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle \\
&= \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} |\psi(t_2)\rangle\langle\psi(t_2)|\phi(t_2)\rangle\langle\phi(t_1)| e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle \\
&+ \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar}\left[e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar} - |\psi(t_2)\rangle\langle\psi(t_2)|\phi(t_2)\rangle\langle\phi(t_1)|\right] e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle.
\end{aligned} \tag{B.9}$$

Now, we can take the first term and make the approximation that gives us weak values, but we have the additional task of proving the second term is negligible.

A seemingly simpler task would be to insert similar identities in reverse order. Let $\{|\psi(t_1)\rangle\} \cup \{|\psi^j\rangle\}_{j\in\Lambda}$ and $\{|\phi(t_2)\rangle\} \cup \{|\phi^k\rangle\}_{k\in\Lambda}$ be orthonormal bases (with Λ an indexing set of the required cardinality for the Hilbert space) and use them to make identities:

$$\begin{aligned}
&\langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar}\left(|\phi(t_2)\rangle\langle\phi(t_2)| + \sum_k |\phi^k\rangle\langle\phi^k|\right) e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\left(|\psi(t_1)\rangle\langle\psi(t_1)| + \sum_j |\psi^j\rangle\langle\psi^j|\right) e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle \\
&= \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} |\phi(t_2)\rangle\langle\phi(t_2)|\psi(t_2)\rangle\langle\psi(t_1)| e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle \\
&+ \sum_j \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} |\phi(t_2)\rangle\langle\phi(t_1)|\psi^j\rangle\langle\psi^j| e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle + \sum_k \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} |\phi^k\rangle\langle\phi^k|\psi(t_2)\rangle\langle\psi(t_1)| e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle \\
&+ \sum_k \sum_j \langle\phi(t_2)| e^{i\epsilon\hat{B}\hat{q}'/\hbar} |\phi^k\rangle\langle\phi^k|e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}|\psi^j\rangle\langle\psi^j| e^{i\epsilon\hat{A}\hat{q}/\hbar} |\psi(t_1)\rangle.
\end{aligned} \tag{B.10}$$

There are two problems with this. First, we shouldn't make the weak value approximation using orthogonal pre- and postselections, as their inner product being zero makes the weak value undefined (see the result in Eqn. 3.23). If we choose to make the approximation anyway (neglecting terms of the exponentiated operators' sum forms which are first or higher order in ϵ), orthogonality makes all of the terms in the sums of Eqn. B.10 go away, which appears rather convenient. However, the one term left will have expectation values of observables, rather than the weak values. Based on these difficulties, sequences of weak interactions do not seem to lend themselves to direct approximations in the way single weak interactions do.

B.2 First-Order Approximation of the Exponential

Let us return to Eqn. B.8, and suppose we can approximate $e^{i\epsilon\hat{O}} \approx 1 + i\epsilon\hat{O}$ [15] (which is a big ask, considering the trouble we had to go through in Section 1.3). This would give us

$$\begin{aligned}\langle\phi(t_2)|\, e^{i\epsilon\hat{B}\hat{q}'/\hbar}e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}e^{i\epsilon\hat{A}\hat{q}/\hbar}\,|\psi(t_1)\rangle &\approx \langle\phi(t_2)|\left(1+i\epsilon\hat{B}\hat{q}'/\hbar\right)e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\left(1+i\epsilon\hat{A}\hat{q}/\hbar\right)|\psi(t_1)\rangle \\ &= \langle\phi(t_2)|\, e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\,|\psi(t_1)\rangle + i\epsilon\frac{\hat{q}}{\hbar}\langle\phi(t_2)|\, e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\hat{A}\,|\psi(t_1)\rangle \\ &+ i\epsilon\frac{\hat{q}'}{\hbar}\langle\phi(t_2)|\,\hat{B}e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\,|\psi(t_1)\rangle - \epsilon^2\frac{\hat{q}'\hat{q}}{\hbar^2}\langle\phi(t_2)|\,\hat{B}e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}\hat{A}\,|\psi(t_1)\rangle\,.\end{aligned} \tag{B.11}$$

If we discard the second order term (and allow the system proper's state to evolve so we can get rid of the remaining exponential), we get

$$\begin{aligned}\langle\phi(t_2)|\, e^{i\epsilon\hat{B}\hat{q}'/\hbar}e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}e^{i\epsilon\hat{A}\hat{q}/\hbar}\,|\psi(t_1)\rangle &\approx \langle\phi(t_2)|\psi(t_2)\rangle + i\epsilon\frac{\hat{q}}{\hbar}\langle\phi(t_1)|\hat{A}|\psi(t_1)\rangle + i\epsilon\frac{\hat{q}'}{\hbar}\langle\phi(t_2)|\hat{B}|\psi(t_2)\rangle \\ &= \langle\phi(t_2)|\psi(t_2)\rangle + i\epsilon\frac{\hat{q}}{\hbar}\langle\phi(t_1)|\psi(t_1)\rangle A_w + i\epsilon\frac{\hat{q}'}{\hbar}\langle\phi(t_2)|\psi(t_2)\rangle B_w.\end{aligned} \tag{B.12}$$

This may look problematic at first glance, but we are saved by the fact that $\langle\phi(t_1)|\psi(t_1)\rangle = \langle\phi(t_2)|\psi(t_2)\rangle$, leaving us with

$$\begin{aligned}\langle\phi(t_2)|\, e^{i\epsilon\hat{B}\hat{q}'/\hbar}e^{-i\hat{H}_{sys}(t_2-t_1)/\hbar}e^{i\epsilon\hat{A}\hat{q}/\hbar}\,|\psi(t_1)\rangle &\approx \langle\phi(t_2)|\psi(t_2)\rangle\left(1+i\epsilon\frac{\hat{q}}{\hbar}A_w + i\epsilon\frac{\hat{q}'}{\hbar}B_w\right) \\ &\approx \langle\phi(t_2)|\psi(t_2)\rangle\, e^{i\epsilon(A_w\hat{q}+B_w\hat{q}')/\hbar}.\end{aligned} \tag{B.13}$$

This get us to the desired form, but we should note that we went straight from $e^{i\epsilon\hat{O}} = \sum_{j=0}^{\infty}\frac{(i\epsilon\hat{O})^j}{j!}$ to truncating higher-order terms. In Section 1.3, we actually operated the terms of the sum on states to get numbers (weak values), and demonstrated that those were not growing too quickly to make the terms of the sum small. The claim $e^{i\epsilon\hat{O}} \approx 1 + i\epsilon\hat{O}$ is a much stronger statement if not qualified with assumptions about the states in play, but it can be shown in certain cases, such as for a bounded operator:

$$\left|\left|e^{i\epsilon\hat{O}} - (1+i\epsilon\hat{O})\right|\right|_{op} = \left|\left|\sum_{j=2}^{\infty}\frac{(i\epsilon\hat{O})^j}{j!}\right|\right|_{op} \le \sum_{j=2}^{\infty}\frac{(\epsilon||\hat{O}||_{op})^j}{j!} \le \sum_{j=2}^{\infty}(\epsilon||\hat{O}||_{op})^j = \epsilon^2||\hat{O}||_{op}^2\sum_{j=0}^{\infty}(\epsilon||\hat{O}||_{op})^j. \tag{B.14}$$

This expression is not hard to control, and can be made infinitesimally small with the proper choice of ϵ. For an unbounded operator, however, we need information about the states, and $e^{i\epsilon\hat{O}} \approx 1 + i\epsilon\hat{O}$ is too cavalier of a statement.

If a state being operated on—say $|\psi\rangle$—has compact support, then we can show that $e^{i\epsilon\hat{O}} |\psi\rangle \approx \left(1 + i\epsilon\hat{O}\right) |\psi\rangle$:

$$\left|\left|e^{i\epsilon\hat{O}} |\psi\rangle - (1 + i\epsilon\hat{O}) |\psi\rangle\right|\right| = \left|\left|\sum_{j=2}^{\infty} \frac{(i\epsilon\hat{O})^j}{j!} |\psi\rangle\right|\right| \le \sum_{j=2}^{\infty} \frac{\epsilon^j ||\hat{O}^j |\psi\rangle ||}{j!} \le \sum_{j=2}^{\infty} (\epsilon o_{max})^j || |\psi\rangle || = \epsilon^2 o_{max}^2 \sum_{j=0}^{\infty} (\epsilon o_{max})^j, \tag{B.15}$$

where o_{max} is defined analogously to a_{max} in Section 1.3. This is sufficient to claim that there is an ϵ small enough that we can make the approximation. To apply this to the weak value approximation, we do require that the composite system state has compact support at the time of each weak interaction. This suggests that, while we can appeal to the conditions from Section 1.3, at least a little more care is needed when handling sequential weak measurements.

Appendix C

Calculations for the Nested Mach-Zehnder Interferometer

C.1 Unitary Operators and Evolved States

The first step in calculating the outcomes of the nested Mach-Zehnder interferometer with feedback compensation is to establish the unitary operators that govern time evolution. Based on the model established by Griffiths [16] and utilizing a five probe setup, I devised the following:

$$U_{10} = [\alpha |A_1\rangle + \beta |D_1\rangle] \langle S_0| + [-\beta |A_1\rangle + \alpha |D_1\rangle] \langle R_0| + |Q_1\rangle \langle Q_0| , \tag{C.1}$$

$$U_{21} = |A_2\rangle \langle A_1| + |D_2\rangle \langle D_1| e^{-i\frac{\epsilon}{\hbar}\hat{D}_1 \otimes \hat{\sigma}_{z,D}} + |Q_2\rangle \langle Q_1| , \tag{C.2}$$

$$U_{32} = |A_3\rangle \langle A_2| + \frac{1}{\sqrt{2}}[|B_3\rangle + |C_3\rangle] \langle D_2| + \frac{1}{\sqrt{2}}[|B_3\rangle - |C_3\rangle] \langle Q_2| , \tag{C.3}$$

$$U_{43} = |A_4\rangle \langle A_3| e^{-i\frac{\epsilon}{\hbar}\hat{A}_3 \otimes \hat{\sigma}_{z,A}} + |B_4\rangle \langle B_3| e^{-i\frac{\epsilon}{\hbar}\hat{B}_3 \otimes \hat{\sigma}_{z,B}} + |C_4\rangle \langle C_3| e^{-i\frac{\epsilon}{\hbar}\hat{C}_3 \otimes \hat{\sigma}_{z,C}} , \tag{C.4}$$

$$U_{54} = |A_5\rangle \langle A_4| + \frac{1}{\sqrt{2}}[-|E_5\rangle + |H_5\rangle] \langle B_4| + \frac{1}{\sqrt{2}}[|E_5\rangle + |H_5\rangle] \langle C_4| , \tag{C.5}$$

$$U_{65} = |A_6\rangle \langle A_5| + |E_6\rangle \langle E_5| e^{-i\frac{\epsilon}{\hbar}\hat{E}_5 \otimes \hat{\sigma}_{z,E}} + |H_6\rangle \langle H_5| , \tag{C.6}$$

$$U_{76} = [\alpha |F_7\rangle + \beta |G_7\rangle] \langle A_6| + [\beta |F_7\rangle - \alpha |G_7\rangle] \langle E_6| + |H_7\rangle \langle H_6| . \tag{C.7}$$

The notational convention used is that $U_{tt'}$ evolves a state from time t' to t, and a particle in a particular arm at a particular time is indicated by a state ket with the path label and a subscript to indicate the moment in time, for example: $|B_3\rangle$. The projector associated with a path is indicated similarly, but with a hat instead of a ket enclosure, for example: $\hat{B}_3 = |B_3\rangle \langle B_3|$.

Note that the steps from odd times to even times do not change the path the particle occupies (that is, not the letter, only the number), but they do include transitions across mirrors, where the probe interactions occur. Transitions from even to odd times all include transitions across beam splitters along at least some of the paths.

Applying these operators to a particle entering from path S gives the following states from t_0 to t_7:

$$|\psi_0\rangle = |S_0\rangle\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,, \tag{C.8}$$

$$|\psi_1\rangle = \big[\alpha\,|A_1\rangle + \beta\,|D_1\rangle\big]\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,, \tag{C.9}$$

$$\begin{aligned}|\psi_2\rangle = {} & \alpha\,|A_2\rangle\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \beta\,|D_2\rangle\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,,\end{aligned} \tag{C.10}$$

$$\begin{aligned}|\psi_3\rangle = {} & \alpha\,|A_3\rangle\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{\sqrt{2}}\big[|B_3\rangle + |C_3\rangle\big]e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,,\end{aligned} \tag{C.11}$$

$$\begin{aligned}|\psi_4\rangle = {} & \alpha\,|A_4\rangle\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{\sqrt{2}}\,|B_4\rangle\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{\sqrt{2}}\,|C_4\rangle\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,,\end{aligned} \tag{C.12}$$

$$\begin{aligned}|\psi_5\rangle = {} & \alpha\,|A_5\rangle\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{2}\big[-|E_5\rangle + |H_5\rangle\big]e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{2}\big[|E_5\rangle + |H_5\rangle\big]e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,,\end{aligned} \tag{C.13}$$

$$\begin{aligned}|\psi_6\rangle = {} & \alpha\,|A_6\rangle\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & + \frac{\beta}{2}\,|E_6\rangle\,\Big[-e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E} \\ & \qquad + e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big] \\ & + \frac{\beta}{2}\,|H_6\rangle\,\Big[e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & \qquad + e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\Big]\,,\end{aligned} \tag{C.14}$$

$$\begin{aligned}|\psi_7\rangle = {} & |F_7\rangle\,\Big[\alpha^2\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & \qquad - \frac{\beta^2}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E} \\ & \qquad + \frac{\beta^2}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big] \\ & + |G_7\rangle\,\Big[\alpha\beta\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & \qquad + \frac{\alpha\beta}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E} \\ & \qquad - \frac{\alpha\beta}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big] \\ & + |H_7\rangle\,\Big[\frac{\beta}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E} \\ & \qquad + \frac{\beta}{2}e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\Big]\,.\end{aligned} \tag{C.15}$$

Subsequent to the seventh step, a projective measurement is performed on paths F, G, and H, which leads to non-unitary state collapse. It is after that point that the feedback compensation is applied, depending on which path the particle is found to be in. This requires a transition from the pure state $|\psi_7\rangle$ to the mixed state represented by the density operator

$$\begin{aligned}
\hat{\rho}_8 = {} & |F_7\rangle\langle F_7| \otimes \Big[\alpha^2 e^{i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{-i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}}|+\rangle_{x,E} \\
& - \frac{\beta^2}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{-i\frac{\epsilon}{\hbar}[1-B(F)]\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{-i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}|+\rangle_{x,E} \\
& + \frac{\beta^2}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{-i\frac{\epsilon}{\hbar}[1-C(F)]\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{-i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}|+\rangle_{x,E}\Big] \\
& \times \Big[\alpha^2\, {}_{x,D}\langle+|e^{-i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \\
& - \frac{\beta^2}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(F)]\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}} \\
& + \frac{\beta^2}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(F)]\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}\Big] \\
& + |G_7\rangle\langle G_7| \otimes \Big[\alpha\beta e^{i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{-i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}}|+\rangle_{x,E} \\
& + \frac{\alpha\beta}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{-i\frac{\epsilon}{\hbar}[1-B(G)]\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{-i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}|+\rangle_{x,E} \\
& - \frac{\alpha\beta}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{-i\frac{\epsilon}{\hbar}[1-C(G)]\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{-i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}|+\rangle_{x,E}\Big] \\
& \times \Big[\alpha\beta\, {}_{x,D}\langle+|e^{-i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}} \\
& + \frac{\alpha\beta}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(G)]\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}} \\
& - \frac{\alpha\beta}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(G)]\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}\Big] \\
& + |H_7\rangle\langle H_7| \otimes \Big[\frac{\beta}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{-i\frac{\epsilon}{\hbar}[1-B(H)]\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{i\frac{\epsilon}{\hbar}C(H)\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}|+\rangle_{x,E} \\
& + \frac{\beta}{2} e^{-i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}|+\rangle_{x,D}\, e^{i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}|+\rangle_{x,A}\, e^{i\frac{\epsilon}{\hbar}B(H)\hat{\sigma}_{z,B}}|+\rangle_{x,B}\, e^{-i\frac{\epsilon}{\hbar}[1-C(H)]\hat{\sigma}_{z,C}}|+\rangle_{x,C}\, e^{i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}|+\rangle_{x,E}\Big] \\
& \times \Big[\frac{\beta}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(H)]\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(H)\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}} \\
& + \frac{\beta}{2}\, {}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\; {}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\; {}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(H)\hat{\sigma}_{z,B}}\; {}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(H)]\hat{\sigma}_{z,C}}\; {}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}\Big].
\end{aligned} \tag{C.16}$$

C.2 Quantifying the Disturbance

To find out $1 - \langle\hat{\sigma}_{x,D} \otimes \hat{\sigma}_{x,A} \otimes \hat{\sigma}_{x,B} \otimes \hat{\sigma}_{x,C} \otimes \hat{\sigma}_{x,E}\rangle$ (out), we need to calculate the expectation value as the trace $\mathrm{Tr}\{\hat{I} \otimes \hat{\sigma}_{x,D} \otimes \hat{\sigma}_{x,A} \otimes \hat{\sigma}_{x,B} \otimes \hat{\sigma}_{x,C} \otimes \hat{\sigma}_{x,E}\hat{\rho}_8\}$. There will be a lot of inner products with exponentials and Pauli operators in them, so it helps to perform the calculations in general and apply the forms to the big calculation. The variable *Œ* is the operator estimate used in the feedback compensation ($A(F)$ or $C(G)$, for example), and the inner products it appears in are

$$
\begin{aligned}
{}_x\langle+|e^{-i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z}\hat{\sigma}_x e^{i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z}|+\rangle_x &\approx {}_x\langle+|\left[1-i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z-\frac{\epsilon^2}{2\hbar^2}Œ^2\right]\hat{\sigma}_x\left[1+i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z-\frac{\epsilon^2}{2\hbar^2}Œ^2\right]|+\rangle_x \qquad \text{(C.17)}\\
&\approx 1+i\frac{\epsilon}{\hbar}Œ\,{}_x\langle+|\hat{\sigma}_x\hat{\sigma}_z|+\rangle_x-\frac{\epsilon^2}{2\hbar^2}Œ^2-i\frac{\epsilon}{\hbar}Œ\,{}_x\langle+|\hat{\sigma}_z\hat{\sigma}_x|+\rangle_x\\
&\quad+\frac{\epsilon^2}{\hbar^2}Œ^2\,{}_x\langle+|\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z|+\rangle_x-\frac{\epsilon^2}{2\hbar^2}Œ^2\\
&=1-2\frac{\epsilon^2}{\hbar^2}Œ^2,\\
{}_x\langle+|e^{i\frac{\epsilon}{\hbar}\left[1-Œ\right]\hat{\sigma}_z}\hat{\sigma}_x e^{-i\frac{\epsilon}{\hbar}\left[1-Œ\right]\hat{\sigma}_z}|+\rangle_x &\approx 1-2\frac{\epsilon^2}{\hbar^2}\left[1-Œ\right]^2, \qquad \text{(C.18)}\\
{}_x\langle+|e^{-i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z}\hat{\sigma}_x e^{-i\frac{\epsilon}{\hbar}\left[1-Œ\right]\hat{\sigma}_z}|+\rangle_x &\approx {}_x\langle+|\left[1-i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z-\frac{\epsilon^2}{2\hbar^2}Œ^2\right]\hat{\sigma}_x\left[1-i\frac{\epsilon}{\hbar}\left[1-Œ\right]\hat{\sigma}_z-\frac{\epsilon^2}{2\hbar^2}\left[1-Œ\right]^2\right]|+\rangle_x\\
&\approx 1-i\frac{\epsilon}{\hbar}\left[1-Œ\right]{}_x\langle+|\hat{\sigma}_x\hat{\sigma}_z|+\rangle_x-\frac{\epsilon^2}{2\hbar^2}\left[1-Œ\right]^2-i\frac{\epsilon}{\hbar}Œ\,{}_x\langle+|\hat{\sigma}_z\hat{\sigma}_x|+\rangle_x\\
&\quad-\frac{\epsilon^2}{\hbar^2}\left[1-Œ\right]Œ\,{}_x\langle+|\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z|+\rangle_x-\frac{\epsilon^2}{2\hbar^2}Œ^2\\
&=1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-Œ\right]^2, \qquad \text{(C.19)}\\
{}_x\langle+|e^{i\frac{\epsilon}{\hbar}\left[1-Œ\right]\hat{\sigma}_z}\hat{\sigma}_x e^{i\frac{\epsilon}{\hbar}Œ\hat{\sigma}_z}|+\rangle_x &\approx 1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-Œ\right]^2. \qquad \text{(C.20)}
\end{aligned}
$$

As we can see, the inner product of two probe states is almost 1, less a term on the order of ϵ^2. Two matching states (where either neither probe experienced a weak interaction, or both did) lead to this perturbation from unity being proportional to either $Œ^2$ or $\left[1-Œ\right]^2$, while a mismatched pair of states leads to a perturbation proportional to $\left[\frac{1}{2}-Œ\right]^2$.

The procedure for the requisite calculation is as follows. First, we use Eqns. C.17–C.20 to approximate all of the inner products presented. Second, we distribute all of the products of these approximated inner products and keep only second order terms. In so doing, the multiplication becomes addition, and no cross terms with products of estimates for different probes are present. There are a lot of terms to handle, and while my preparatory work makes the calculation a little less tedious, it doesn't save much space for the two-page monster of an expression that is

$$
\begin{aligned}
&\mathrm{Tr}\{\hat{I}\otimes\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\hat{\rho}_8\}\\
&=\Big[\alpha^2\,{}_{x,D}\langle+|e^{-i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}}\\
&\quad-\frac{\beta^2}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(F)]\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}\\
&\quad+\frac{\beta^2}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(F)]\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}\Big]\\
&\times\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\\
&\times\Big[\alpha^2e^{i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\\
&\quad-\frac{\beta^2}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}[1-B(F)]\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{i\frac{\epsilon}{\hbar}C(F)\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\\
&\quad+\frac{\beta^2}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{i\frac{\epsilon}{\hbar}B(F)\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}[1-C(F)]\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}[1-E(F)]\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big]\\
&+\Big[\alpha\beta\,{}_{x,D}\langle+|e^{-i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}}\\
&\quad+\frac{\alpha\beta}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(G)]\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}\\
&\quad-\frac{\alpha\beta}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(G)]\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}\Big]\\
&\times\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\\
&\times\Big[\alpha\beta e^{i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{-i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\\
&\quad+\frac{\alpha\beta}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}[1-B(G)]\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{i\frac{\epsilon}{\hbar}C(G)\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\\
&\quad-\frac{\alpha\beta}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(G)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(G)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{i\frac{\epsilon}{\hbar}B(G)\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}[1-C(G)]\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{-i\frac{\epsilon}{\hbar}[1-E(G)]\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big]\\
&+\Big[\frac{\beta}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{i\frac{\epsilon}{\hbar}[1-B(H)]\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{-i\frac{\epsilon}{\hbar}C(H)\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}\\
&\quad+\frac{\beta}{2}\,{}_{x,D}\langle+|e^{i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\;{}_{x,A}\langle+|e^{-i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\;{}_{x,B}\langle+|e^{-i\frac{\epsilon}{\hbar}B(H)\hat{\sigma}_{z,B}}\;{}_{x,C}\langle+|e^{i\frac{\epsilon}{\hbar}[1-C(H)]\hat{\sigma}_{z,C}}\;{}_{x,E}\langle+|e^{-i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}\Big]\\
&\times\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\\
&\times\Big[\frac{\beta}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{-i\frac{\epsilon}{\hbar}[1-B(H)]\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{i\frac{\epsilon}{\hbar}C(H)\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\\
&\quad+\frac{\beta}{2}e^{-i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}}\,|+\rangle_{x,D}\,e^{i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}}\,|+\rangle_{x,A}\,e^{i\frac{\epsilon}{\hbar}B(H)\hat{\sigma}_{z,B}}\,|+\rangle_{x,B}\,e^{-i\frac{\epsilon}{\hbar}[1-C(H)]\hat{\sigma}_{z,C}}\,|+\rangle_{x,C}\,e^{i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}}\,|+\rangle_{x,E}\Big]\\
&\approx\alpha^4\left(1-2\frac{\epsilon^2}{\hbar^2}D^2(F)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\big[1-A(F)\big]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(F)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(F)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}E^2(F)\right)\\
&+\alpha^2\beta^2\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-D(F)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-A(F)\right]^2\right)\left[\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(F)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-C(F)\right]^2\right)\right.\\
&\qquad\left.-\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-B(F)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(F)\right)\right]\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-E(F)\right]^2\right)
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
&+\frac{\beta^4}{4}\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-D(F)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}A^2(F)\right)\\
&\quad\times\left[\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-B(F)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(F)\right)+\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(F)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-C(F)\right]^2\right)\right.\\
&\quad\left.-2\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-B(F)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-C(F)\right]^2\right)\right]\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-E(F)\right]^2\right)\\
&+\alpha^2\beta^2\left\{\left(1-2\frac{\epsilon^2}{\hbar^2}D^2(G)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-A(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(G)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(G)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}E^2(G)\right)\right.\\
&\quad+\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-D(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-A(G)\right]^2\right)\left[\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-B(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(G)\right)\right.\\
&\quad\left.-\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(G)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-C(G)\right]^2\right)\right]\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-E(G)\right]^2\right)\\
&\quad+\frac{1}{4}\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-D(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}A^2(G)\right)\\
&\quad\times\left[\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-B(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(G)\right)+\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(G)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-C(G)\right]^2\right)\right.\\
&\quad\left.\left.-2\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-B(G)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-C(G)\right]^2\right)\right]\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-E(G)\right]^2\right)\right\}\\
&+\frac{\beta^2}{4}\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-D(H)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}A^2(H)\right)\\
&\quad\times\left[\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-B(H)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}C^2(H)\right)+2\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-B(H)\right]^2\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[\frac{1}{2}-C(H)\right]^2\right)\right.\\
&\quad\left.+\left(1-2\frac{\epsilon^2}{\hbar^2}B^2(H)\right)\left(1-2\frac{\epsilon^2}{\hbar^2}\left[1-C(H)\right]^2\right)\right]\left(1-2\frac{\epsilon^2}{\hbar^2}E^2(H)\right)\\
&\approx\alpha^4\left(1-2\frac{\epsilon^2}{\hbar^2}D^2(F)-2\frac{\epsilon^2}{\hbar^2}\left[1-A(F)\right]^2-2\frac{\epsilon^2}{\hbar^2}B^2(F)-2\frac{\epsilon^2}{\hbar^2}C^2(F)-2\frac{\epsilon^2}{\hbar^2}E^2(F)\right)\\
&-2\frac{\epsilon^2}{\hbar^2}\alpha^2\beta^2\left(B^2(F)+\left[\frac{1}{2}-C(F)\right]^2-\left[\frac{1}{2}-B(F)\right]^2-C^2(F)\right)\\
&-2\frac{\epsilon^2}{\hbar^2}\frac{\beta^4}{4}\left(\left[1-B(F)\right]^2+C^2(F)+B^2(F)+\left[1-C(F)\right]^2-2\left[\frac{1}{2}-B(F)\right]^2-2\left[\frac{1}{2}-C(F)\right]^2\right)\\
&+\alpha^2\beta^2\left\{\left(1-2\frac{\epsilon^2}{\hbar^2}D^2(G)-2\frac{\epsilon^2}{\hbar^2}\left[1-A(G)\right]^2-2\frac{\epsilon^2}{\hbar^2}B^2(G)-2\frac{\epsilon^2}{\hbar^2}C^2(G)-2\frac{\epsilon^2}{\hbar^2}E^2(G)\right)\right.\\
&\quad-2\frac{\epsilon^2}{\hbar^2}\left(\left[\frac{1}{2}-B(G)\right]^2+C^2(G)-B^2(G)-\left[\frac{1}{2}-C(G)\right]^2\right)\\
&\quad\left.-2\frac{\epsilon^2}{\hbar^2}\frac{1}{4}\left(\left[1-B(G)\right]^2+C^2(G)+B^2(G)+\left[1-C(G)\right]^2-2\left[\frac{1}{2}-B(G)\right]^2-2\left[\frac{1}{2}-C(G)\right]^2\right)\right\}\\
&+\beta^2\left\{1-2\frac{\epsilon^2}{\hbar^2}\left[1-D(H)\right]^2-2\frac{\epsilon^2}{\hbar^2}A^2(H)-2\frac{\epsilon^2}{\hbar^2}E^2(H)\right.\\
&\quad\left.-2\frac{\epsilon^2}{\hbar^2}\frac{1}{4}\left(\left[1-B(H)\right]^2+C^2(H)+2\left[\frac{1}{2}-B(H)\right]^2+2\left[\frac{1}{2}-C(H)\right]^2+B^2(H)+\left[1-C(H)\right]^2\right)\right\}.
\end{aligned}
$$

Moving on, we can now calculate $1-\langle\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\rangle\,(\text{out})$, but let us also rescale it to avoid overuse of the factor $2\frac{\epsilon^2}{\hbar^2}$. To see the best minimization scheme, we must group all of the B and C terms (the real troublemakers). With this intent, I present

$$\begin{aligned}
&\frac{\hbar^2}{2\epsilon^2}\left(1-\mathrm{Tr}\{\hat{I}\otimes\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\hat{\rho}_8\}\right)\\
&\approx \alpha^4\left(D^2(F)+\left[1-A(F)\right]^2+B^2(F)+C^2(F)+E^2(F)\right)+\alpha^2\beta^2\left(B^2(F)+\left[\frac{1}{2}-C(F)\right]^2-\left[\frac{1}{2}-B(F)\right]^2-C^2(F)\right)\\
&+\frac{\beta^4}{4}\left(\left[1-B(F)\right]^2+C^2(F)+B^2(F)+\left[1-C(F)\right]^2-2\left[\frac{1}{2}-B(F)\right]^2-2\left[\frac{1}{2}-C(F)\right]^2\right)\\
&+\alpha^2\beta^2\left\{\left(D^2(G)+\left[1-A(G)\right]^2+B^2(G)+C^2(G)+E^2(G)\right)+\left(\left[\frac{1}{2}-B(G)\right]^2+C^2(G)-B^2(G)-\left[\frac{1}{2}-C(G)\right]^2\right)\right.\\
&\qquad\left.+\frac{1}{4}\left(\left[1-B(G)\right]^2+C^2(G)+B^2(G)+\left[1-C(G)\right]^2-2\left[\frac{1}{2}-B(G)\right]^2-2\left[\frac{1}{2}-C(G)\right]^2\right)\right\}\\
&+\beta^2\left\{\left[1-D(H)\right]^2+A^2(H)+E^2(H)\right.\\
&\qquad\left.+\frac{1}{4}\left(\left[1-B(H)\right]^2+C^2(H)+2\left[\frac{1}{2}-B(H)\right]^2+2\left[\frac{1}{2}-C(H)\right]^2+B^2(H)+\left[1-C(H)\right]^2\right)\right\}\\
&=\alpha^4\left(D^2(F)+\left[1-A(F)\right]^2+B^2(F)+C^2(F)+E^2(F)\right)-\alpha^2\beta^2\left(C(F)-B(F)\right)+\frac{\beta^4}{4}\left(\frac{1}{2}+\frac{1}{2}\right)\\
&+\alpha^2\beta^2\left\{\left(D^2(G)+\left[1-A(G)\right]^2+B^2(G)+C^2(G)+E^2(G)+C(G)-B(G)\right)+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\right)\right\}\\
&+\beta^2\left\{\left[1-D(H)\right]^2+A^2(H)+E^2(H)+\frac{1}{4}\left(\frac{3}{2}-4B(H)+4B^2(H)+\frac{3}{2}-4C(H)+4C^2(H)\right)\right\}\\
&=\alpha^4\left(D^2(F)+\left[1-A(F)\right]^2+E^2(G)+\left[\frac{\beta^2}{2\alpha^2}+B(F)\right]^2-\frac{1}{8}\frac{\beta^4}{\alpha^4}+\left[\frac{\beta^2}{2\alpha^2}-C(F)\right]^2-\frac{1}{8}\frac{\beta^4}{\alpha^4}\right)\\
&+\alpha^2\beta^2\left(D^2(G)+\left[1-A(G)\right]^2+E^2(G)+\left[\frac{1}{2}-B(G)\right]^2-\frac{1}{8}+\left[\frac{1}{2}+C(G)\right]^2-\frac{1}{8}\right)\\
&+\beta^2\left(\left[1-D(H)\right]^2+A^2(H)+E^2(H)+\left[\frac{1}{2}-B(G)\right]^2+\frac{1}{8}+\left[\frac{1}{2}-C(G)\right]^2+\frac{1}{8}\right).
\end{aligned} \tag{C.22}$$

Taking the optimal estimates presented in Table 2.1 minimizes the disturbance in the qubits. The terms left behind in the case of optimal compensation are

$$\alpha^4\left(-\frac{1}{8}\frac{\beta^4}{\alpha^4}-\frac{1}{8}\frac{\beta^4}{\alpha^4}\right)+\alpha^2\beta^2\left(-\frac{1}{8}-\frac{1}{8}\right)+\beta^2\left(\frac{1}{8}+\frac{1}{8}\right)=\frac{1}{4}\left(-\beta^4-\alpha^2\beta^2+\beta^2\right)=0. \tag{C.23}$$

This suggests perfect compensation for the weak interactions.

C.3 An Alternative Proposal

If we instead wish to experiment with the four probe setup (replacing B and C with the W probe that detects presence in the inner interferometer without distinguishing the arms), we make some simple adjustments, starting with the initial state

$$|\psi_0'\rangle = |S_0\rangle\,|+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,W}\,|+\rangle_{x,E}\,. \tag{C.24}$$

We also need a new unitary operator evolving from t_3 to t_4, which captures the new probe interaction. This is given by

$$U'_{43} = |A_4\rangle \langle A_3| \, e^{-i\frac{\epsilon}{\hbar}\hat{A}_3\otimes\hat{\sigma}_{z,A}} + |B_4\rangle \langle B_3| \, e^{-i\frac{\epsilon}{\hbar}\hat{B}_3\otimes\hat{\sigma}_{z,W}} + |C_4\rangle \langle C_3| \, e^{-i\frac{\epsilon}{\hbar}\hat{C}_3\otimes\hat{\sigma}_{z,W}}. \tag{C.25}$$

This leads to a far simpler pure state at t_7, as well as a nicer density operator at t_8:

$$\begin{aligned}
|\psi'_7\rangle = \; & |F_7\rangle \, \alpha^2 \, |+\rangle_{x,D} \, e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}} \, |+\rangle_{x,A} \, |+\rangle_{x,W} \, |+\rangle_{x,E} \\
& + |G_7\rangle \, \alpha\beta \, |+\rangle_{x,D} \, e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}} \, |+\rangle_{x,A} \, |+\rangle_{x,W} \, |+\rangle_{x,E} \\
& + |H_7\rangle \, \beta e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}} \, |+\rangle_{x,D} \, |+\rangle_{x,A} \, e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,W}} \, |+\rangle_{x,W} \, |+\rangle_{x,E} \, , \tag{C.26}
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}'_8 = \; & \alpha^4 \, |F_7\rangle \langle F_7| \otimes e^{i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} \, |+\rangle_{x,D} \, {}_{x,D}\langle +| e^{-i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} \otimes e^{-i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} \, |+\rangle_{x,A} \, {}_{x,A}\langle +| e^{i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} \\
& \quad \otimes e^{i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} \, |+\rangle_{x,W} \, {}_{x,W}\langle +| e^{-i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} \otimes e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \, |+\rangle_{x,E} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \\
& + \alpha^2\beta^2 \, |G_7\rangle \langle G_7| \otimes e^{i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} \, |+\rangle_{x,D} \, {}_{x,D}\langle +| e^{-i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} \otimes e^{-i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} \, |+\rangle_{x,A} \, {}_{x,A}\langle +| e^{i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} \\
& \quad \otimes e^{i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} \, |+\rangle_{x,W} \, {}_{x,W}\langle +| e^{-i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} \otimes e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \, |+\rangle_{x,E} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \\
& + \beta^2 \, |H_7\rangle \langle H_7| \otimes e^{-i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}} \, |+\rangle_{x,D} \, {}_{x,D}\langle +| e^{i\frac{\epsilon}{\hbar}[1-D(F)]\hat{\sigma}_{z,D}} \otimes e^{i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}} \, |+\rangle_{x,A} \, {}_{x,A}\langle +| e^{-i\frac{\epsilon}{\hbar}A(F)\hat{\sigma}_{z,A}} \\
& \quad \otimes e^{-i\frac{\epsilon}{\hbar}[1-W(F)]\hat{\sigma}_{z,W}} \, |+\rangle_{x,W} \, {}_{x,W}\langle +| e^{i\frac{\epsilon}{\hbar}[1-W(F)]\hat{\sigma}_{z,W}} \otimes e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \, |+\rangle_{x,E} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} .
\end{aligned} \tag{C.27}$$

This in turn leads to a far simpler expectation value for the qubit probes:

$$\begin{aligned}
& \mathrm{Tr}\{\hat{I} \otimes \hat{\sigma}_{x,D} \otimes \hat{\sigma}_{x,A} \otimes \hat{\sigma}_{x,W} \otimes \hat{\sigma}_{x,E} \hat{\rho}'_8\} \\
& = \alpha^4 \, {}_{x,D}\langle +| e^{-i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} \hat{\sigma}_{x,D} e^{i\frac{\epsilon}{\hbar}D(F)\hat{\sigma}_{z,D}} |+\rangle_{x,D} \, {}_{x,A}\langle +| e^{i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} \hat{\sigma}_{x,A} e^{-i\frac{\epsilon}{\hbar}[1-A(F)]\hat{\sigma}_{z,A}} |+\rangle_{x,A} \\
& \times \, {}_{x,W}\langle +| e^{-i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} \hat{\sigma}_{x,W} e^{i\frac{\epsilon}{\hbar}W(F)\hat{\sigma}_{z,W}} |+\rangle_{x,W} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} \hat{\sigma}_{x,E} e^{i\frac{\epsilon}{\hbar}E(F)\hat{\sigma}_{z,E}} |+\rangle_{x,E} \\
& + \alpha^2\beta^2 \, {}_{x,D}\langle +| e^{-i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}} \hat{\sigma}_{x,D} e^{i\frac{\epsilon}{\hbar}D(G)\hat{\sigma}_{z,D}} |+\rangle_{x,D} \, {}_{x,A}\langle +| e^{i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}} \hat{\sigma}_{x,A} e^{-i\frac{\epsilon}{\hbar}[1-A(G)]\hat{\sigma}_{z,A}} |+\rangle_{x,A} \\
& \times \, {}_{x,W}\langle +| e^{-i\frac{\epsilon}{\hbar}W(G)\hat{\sigma}_{z,W}} \hat{\sigma}_{x,W} e^{i\frac{\epsilon}{\hbar}W(G)\hat{\sigma}_{z,W}} |+\rangle_{x,W} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}} \hat{\sigma}_{x,E} e^{i\frac{\epsilon}{\hbar}E(G)\hat{\sigma}_{z,E}} |+\rangle_{x,E} \\
& + \beta^2 \, {}_{x,D}\langle +| e^{i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}} \hat{\sigma}_{x,D} e^{-i\frac{\epsilon}{\hbar}[1-D(H)]\hat{\sigma}_{z,D}} |+\rangle_{x,D} \, {}_{x,A}\langle +| e^{-i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}} \hat{\sigma}_{x,A} e^{i\frac{\epsilon}{\hbar}A(H)\hat{\sigma}_{z,A}} |+\rangle_{x,A} \\
& \times \, {}_{x,W}\langle +| e^{i\frac{\epsilon}{\hbar}[1-W(H)]\hat{\sigma}_{z,W}} \hat{\sigma}_{x,W} e^{-i\frac{\epsilon}{\hbar}[1-W(H)]\hat{\sigma}_{z,W}} |+\rangle_{x,W} \, {}_{x,E}\langle +| e^{-i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}} \hat{\sigma}_{x,E} e^{i\frac{\epsilon}{\hbar}E(H)\hat{\sigma}_{z,E}} |+\rangle_{x,E} \\
& \approx \alpha^4 \left(1 - 2\frac{\epsilon^2}{\hbar^2} D^2(F)\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} \big[1 - A(F)\big]^2\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} W^2(F)\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} E^2(F)\right) \\
& + \alpha^2\beta^2 \left(1 - 2\frac{\epsilon^2}{\hbar^2} D^2(G)\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} \big[1 - A(G)\big]^2\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} W^2(G)\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} E^2(G)\right) \\
& + \beta^2 \left(1 - 2\frac{\epsilon^2}{\hbar^2} \big[1 - D(H)\big]^2\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} A^2(H)\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} \big[1 - W(H)\big]^2\right) \left(1 - 2\frac{\epsilon^2}{\hbar^2} E^2(H)\right) \\
& \approx 1 - 2\frac{\epsilon^2}{\hbar^2} \bigg[\alpha^4 \Big(D^2(F) + \big[1 - A(F)\big]^2 + W^2(F) + E^2(F)\Big) + \alpha^2\beta^2 \Big(D^2(G) + \big[1 - A(G)\big]^2 + W^2(G) + E^2(G)\Big) \\
& \qquad + \beta^2 \Big(\big[1 - D(H)\big]^2 + A^2(H) + \big[1 - W(H)\big]^2 + E^2(H)\Big)\bigg] .
\end{aligned} \tag{C.28}$$

The estimates which minimize the disturbance of the probes are given in Table 2.2.

C.4 Calculating Weak Values

Without probe interactions ($\epsilon = 0$), the system state evolves in the following manner:

$$|\psi_0\rangle = |S_0\rangle\,, \tag{C.29}$$

$$|\psi_1\rangle = \alpha\,|A_1\rangle + \beta\,|D_1\rangle\,, \tag{C.30}$$

$$|\psi_2\rangle = \alpha\,|A_2\rangle + \beta\,|D_2\rangle\,, \tag{C.31}$$

$$|\psi_3\rangle = \alpha\,|A_3\rangle + \frac{\beta}{\sqrt{2}}\big[|B_3\rangle + |C_3\rangle\big], \tag{C.32}$$

$$|\psi_4\rangle = \alpha\,|A_4\rangle + \frac{\beta}{\sqrt{2}}\big[|B_4\rangle + |C_4\rangle\big], \tag{C.33}$$

$$|\psi_5\rangle = \alpha\,|A_5\rangle + \beta\,|H_5\rangle\,, \tag{C.34}$$

$$|\psi_6\rangle = \alpha\,|A_6\rangle + \beta\,|H_6\rangle\,, \tag{C.35}$$

$$|\psi_7\rangle = \alpha^2\,|F_7\rangle + \alpha\beta\,|G_7\rangle + \beta\,|H_7\rangle\,. \tag{C.36}$$

If we postselect for a particle that exits through path F, backward evolution of this final state (again, without probes) gives us

$$|\phi_7\rangle = |F_7\rangle\,, \tag{C.37}$$

$$|\phi_6\rangle = \alpha\,|A_6\rangle + \beta\,|E_6\rangle\,, \tag{C.38}$$

$$|\phi_5\rangle = \alpha\,|A_5\rangle + \beta\,|E_5\rangle\,, \tag{C.39}$$

$$|\phi_4\rangle = \alpha\,|A_4\rangle + \frac{\beta}{\sqrt{2}}\big[-|B_4\rangle + |C_4\rangle\big], \tag{C.40}$$

$$|\phi_3\rangle = \alpha\,|A_3\rangle + \frac{\beta}{\sqrt{2}}\big[-|B_3\rangle + |C_3\rangle\big], \tag{C.41}$$

$$|\phi_2\rangle = \alpha\,|A_2\rangle - \beta\,|Q_2\rangle\,, \tag{C.42}$$

$$|\phi_1\rangle = \alpha\,|A_1\rangle - \beta\,|Q_1\rangle\,, \tag{C.43}$$

$$|\phi_0\rangle = \alpha^2\,|S_0\rangle - \alpha\beta\,|R_0\rangle - \beta\,|Q_0\rangle\,. \tag{C.44}$$

In turn, these allow us to derive the weak values for the path projectors given outcome F:

$$D_w = \frac{\langle\phi_1|\hat{D}_1|\psi_1\rangle}{\langle\phi_1|\psi_1\rangle} = 0, \tag{C.45}$$

$$A_w = \frac{\langle\phi_3|\hat{A}_3|\psi_3\rangle}{\langle\phi_3|\psi_3\rangle} = 1, \tag{C.46}$$

$$B_w = \frac{\langle\phi_3|\hat{B}_3|\psi_3\rangle}{\langle\phi_3|\psi_3\rangle} = -\frac{\beta^2}{2\alpha^2}, \tag{C.47}$$

$$C_w = \frac{\langle\phi_3|\hat{C}_3|\psi_3\rangle}{\langle\phi_3|\psi_3\rangle} = \frac{\beta^2}{2\alpha^2}, \tag{C.48}$$

$$W_w = (\hat{B}+\hat{C})_w = B_w + C_w = 0, \tag{C.49}$$

$$E_w = \frac{\langle\phi_5|\hat{E}_5|\psi_5\rangle}{\langle\phi_5|\psi_5\rangle} = 0. \tag{C.50}$$

Similarly, postselecting for a particle that exits through path G gives us the backward-evolved states

$$|\phi_7'\rangle = |G_7\rangle\,, \tag{C.51}$$

$$|\phi_6'\rangle = \beta\,|A_6\rangle - \alpha\,|E_6\rangle\,, \tag{C.52}$$

$$|\phi_5'\rangle = \beta\,|A_5\rangle - \alpha\,|E_5\rangle\,, \tag{C.53}$$

$$|\phi_4'\rangle = \beta\,|A_4\rangle + \frac{\alpha}{\sqrt{2}}\big[|B_4\rangle - |C_4\rangle\big], \tag{C.54}$$

$$|\phi_3'\rangle = \beta\,|A_3\rangle + \frac{\alpha}{\sqrt{2}}\big[|B_3\rangle - |C_3\rangle\big], \tag{C.55}$$

$$|\phi_2'\rangle = \beta\,|A_2\rangle + \alpha\,|Q_2\rangle\,, \tag{C.56}$$

$$|\phi_1'\rangle = \beta\,|A_1\rangle + \alpha\,|Q_1\rangle\,, \tag{C.57}$$

$$|\phi_0'\rangle = \alpha\beta\,|S_0\rangle - \beta^2\,|R_0\rangle + \alpha\,|Q_0\rangle\,, \tag{C.58}$$

and the associated path projector weak values

$$D_w = \frac{\langle\phi_1'|\hat{D}_1|\psi_1\rangle}{\langle\phi_1'|\psi_1\rangle} = 0, \tag{C.59}$$

$$A_w = \frac{\langle\phi_3'|\hat{A}_3|\psi_3\rangle}{\langle\phi_3'|\psi_3\rangle} = 1, \tag{C.60}$$

$$B_w = \frac{\langle\phi_3'|\hat{B}_3|\psi_3\rangle}{\langle\phi_3'|\psi_3\rangle} = \frac{1}{2}, \tag{C.61}$$

$$C_w = \frac{\langle\phi_3'|\hat{C}_3|\psi_3\rangle}{\langle\phi_3'|\psi_3\rangle} = -\frac{1}{2}, \tag{C.62}$$

$$W_w = (\hat{B}+\hat{C})_w = B_w + C_w = 0, \tag{C.63}$$

$$E_w = \frac{\langle\phi_5'|\hat{E}_5|\psi_5\rangle}{\langle\phi_5'|\psi_5\rangle} = 0. \tag{C.64}$$

Finally, for an H postselection, the backward-evolved states are

$$|\phi_7''\rangle = |H_7\rangle\,, \tag{C.65}$$

$$|\phi_6''\rangle = |H_6\rangle\,, \tag{C.66}$$

$$|\phi_5''\rangle = |H_5\rangle\,, \tag{C.67}$$

$$|\phi_4''\rangle = \frac{1}{\sqrt{2}}\big[|B_4\rangle + |C_4\rangle\big], \tag{C.68}$$

$$|\phi_3''\rangle = \frac{1}{\sqrt{2}}\big[|B_3\rangle + |C_3\rangle\big], \tag{C.69}$$

$$|\phi_2''\rangle = |D_2\rangle\,, \tag{C.70}$$

$$|\phi_1''\rangle = |D_1\rangle\,, \tag{C.71}$$

$$|\phi_0''\rangle = \beta\,|S_0\rangle + \alpha\,|R_0\rangle\,, \tag{C.72}$$

and the associated weak values are

$$D_w = \frac{\langle\phi_1''|\hat{D}_1|\psi_1\rangle}{\langle\phi_1''|\psi_1\rangle} = 1, \tag{C.73}$$

$$A_w = \frac{\langle\phi_3''|\hat{A}_3|\psi_3\rangle}{\langle\phi_3''|\psi_3\rangle} = 0, \tag{C.74}$$

$$B_w = \frac{\langle\phi_3''|\hat{B}_3|\psi_3\rangle}{\langle\phi_3''|\psi_3\rangle} = \frac{1}{2}, \tag{C.75}$$

$$C_w = \frac{\langle\phi_3''|\hat{C}_3|\psi_3\rangle}{\langle\phi_3''|\psi_3\rangle} = \frac{1}{2}, \tag{C.76}$$

$$W_w = (\hat{B} + \hat{C})_w = B_w + C_w = 1, \tag{C.77}$$

$$E_w = \frac{\langle\phi_5''|\hat{E}_5|\psi_5\rangle}{\langle\phi_5''|\psi_5\rangle} = 0. \tag{C.78}$$

Comparing these to Tables 2.1 and 2.2 shows us that the weak values are still the best estimates characterizing the weak interactions.

C.5 We Want More: A Probability Distribution

When we want to be sure about the possible outcomes, we need to calculate the probability distribution, which is a process similar to calculating $\mathrm{Tr}\{\hat{I}\otimes\hat{\sigma}_{x,D}\otimes\hat{\sigma}_{x,A}\otimes\hat{\sigma}_{x,B}\otimes\hat{\sigma}_{x,C}\otimes\hat{\sigma}_{x,E}\hat{\rho}_8\}$, except each $\hat{\sigma}_x$ is replaced by a projector onto the Hadamard basis of the qubit. As before, it pays to do some preparatory work:

$$\begin{aligned}
{}_x\langle +|e^{-i\frac{\epsilon}{\hbar}\OE\hat{\sigma}_z}\,|j\rangle_x\,{}_x\langle j|e^{i\frac{\epsilon}{\hbar}\OE\hat{\sigma}_z}\,|+\rangle_x &= {}_x\langle +|\left[\cos\left(\frac{\epsilon}{\hbar}\OE\right)-i\sin\left(\frac{\epsilon}{\hbar}\OE\right)\hat{\sigma}_z\right]|j\rangle_x\,{}_x\langle j|\left[\cos\left(\frac{\epsilon}{\hbar}\OE\right)+i\sin\left(\frac{\epsilon}{\hbar}\OE\right)\hat{\sigma}_z\right]|+\rangle_x \\
&= \cos^2\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{+j}+i\cos\left(\frac{\epsilon}{\hbar}\OE\right)\sin\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{+j}\delta_{-j} \\
&\quad -i\sin\left(\frac{\epsilon}{\hbar}\OE\right)\cos\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{-j}\delta_{+j}+\sin^2\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{-j} \\
&\approx \left(1-\frac{\epsilon^2}{\hbar^2}\OE^2\right)\delta_{+j}+\frac{\epsilon^2}{\hbar^2}\OE^2\delta_{-j} \\
&= \delta_{+j}+(-1)^{\delta_{+j}}\frac{\epsilon^2}{\hbar^2}\OE^2,
\end{aligned} \tag{C.79}$$

$${}_x\langle +|e^{i\frac{\epsilon}{\hbar}\left[1-\OE\right]\hat{\sigma}_z}\,|j\rangle_x\,{}_x\langle j|e^{-i\frac{\epsilon}{\hbar}\left[1-\OE\right]\hat{\sigma}_z}\,|+\rangle_x = \delta_{+j}+(-1)^{\delta_{+j}}\frac{\epsilon^2}{\hbar^2}\left[1-\OE\right]^2, \tag{C.80}$$

$$\begin{aligned}
{}_x\langle +|e^{i\frac{\epsilon}{\hbar}\left[1-\OE\right]\hat{\sigma}_z}\,|j\rangle_x\,{}_x\langle j|e^{i\frac{\epsilon}{\hbar}\OE\hat{\sigma}_z}\,|+\rangle_x &= \cos\left(\frac{\epsilon}{\hbar}\left[1-\OE\right]\right)\cos\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{+j}-\sin\left(\frac{\epsilon}{\hbar}\left[1-\OE\right]\right)\sin\left(\frac{\epsilon}{\hbar}\OE\right)\delta_{-j} \\
&\approx \left(1-\frac{\epsilon^2}{2\hbar^2}\left[1-2\OE+2\OE^2\right]\right)\delta_{+j}-\frac{\epsilon^2}{\hbar^2}\left[1-\OE\right]\OE\delta_{-j} \\
&= \delta_{+j}\left(1-\frac{\epsilon^2}{2\hbar^2}\right)+(-1)^{\delta_{-j}}\frac{\epsilon^2}{\hbar^2}\left[1-\OE\right]\OE.
\end{aligned} \tag{C.81}$$

It also pays to define some new notation to compact expressions. We need the following devices to compact five Kronecker deltas into one messy symbol, and to compact an expression of four delta functions and a varying sign:

$$\Delta_{jklmn} = \delta_{+j}\delta_{+k}\delta_{+l}\delta_{+m}\delta_{+n}, \tag{C.82}$$

$$\Delta^{\pm n}_{jklm} = (-1)^{\delta_{\pm n}}\delta_{+j}\delta_{+k}\delta_{+l}\delta_{+m}. \tag{C.83}$$

Using this notation, our general inner products, and our methodology from the earlier enormous calculation, all outcomes associated with path F are found to have probabilities

$$
\begin{aligned}
\mathscr{P}_{F,j,k,l,m,n} &= \mathrm{Tr}\{|F_7\rangle\langle F_7| \otimes |j\rangle_{x,D\ x,D}\langle j| \otimes |k\rangle_{x,A\ x,A}\langle k| \otimes |l\rangle_{x,B\ x,B}\langle l| \otimes |m\rangle_{x,C\ x,C}\langle m| \otimes |n\rangle_{x,E\ x,E}\langle n|\hat{\rho}_8\} \\
&\approx \alpha^4 \left(\delta_{+j} + (-1)^{\delta_{+j}} \frac{\epsilon^2}{\hbar^2} D^2(F)\right)\left(\delta_{+k} + (-1)^{\delta_{+k}} \frac{\epsilon^2}{\hbar^2}\big[1-A(F)\big]^2\right)\left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(F)\right) \\
&\quad\times \left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(F)\right)\left(\delta_{+n} + (-1)^{\delta_{+n}} \frac{\epsilon^2}{\hbar^2} E^2(F)\right) \\
&\quad+ \alpha^2\beta^2 \left(\delta_{+j}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-j}} \frac{\epsilon^2}{\hbar^2}\big[1-D(F)\big]D(F)\right)\left(\delta_{+k}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-k}} \frac{\epsilon^2}{\hbar^2}\big[1-A(F)\big]A(F)\right) \\
&\quad\times \Bigg[-\left(\delta_{+l}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-l}} \frac{\epsilon^2}{\hbar^2}\big[1-B(F)\big]B(F)\right)\left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(F)\right) \\
&\quad+ \left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(F)\right)\left(\delta_{+m}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-m}} \frac{\epsilon^2}{\hbar^2}\big[1-C(F)\big]C(F)\right)\Bigg] \\
&\quad\times \left(\delta_{+n}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-n}} \frac{\epsilon^2}{\hbar^2}\big[1-E(F)\big]E(F)\right) \\
&\quad+ \frac{\beta^4}{4}\left(\delta_{+j} + (-1)^{\delta_{+j}} \frac{\epsilon^2}{\hbar^2}\big[1-D(F)\big]^2\right)\left(\delta_{+k} + (-1)^{\delta_{+k}} \frac{\epsilon^2}{\hbar^2} A^2(F)\right) \\
&\quad\times \Bigg[\left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2}\big[1-B(F)\big]^2\right)\left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(F)\right) \\
&\quad- 2\left(\delta_{+l}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-l}} \frac{\epsilon^2}{\hbar^2}\big[1-B(F)\big]B(F)\right)\left(\delta_{+m}\left(1-\frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-m}} \frac{\epsilon^2}{\hbar^2}\big[1-C(F)\big]C(F)\right) \\
&\quad+ \left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(F)\right)\left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2}\big[1-C(F)\big]^2\right)\Bigg] \\
&\quad\times \left(\delta_{+n} + (-1)^{\delta_{+n}} \frac{\epsilon^2}{\hbar^2}\big[1-E(F)\big]^2\right) \\
&\approx \alpha^4 \left\{\Delta_{jklmn} + \frac{\epsilon^2}{\hbar^2}\Big[D^2(F)\Delta^{+j}_{klmn} + \big[1-A(F)\big]^2\Delta^{+k}_{jlmn} + B^2(F)\Delta^{+l}_{jkmn} + C^2(F)\Delta^{+m}_{jkln} + E^2(F)\Delta^{+n}_{jklm}\Big]\right\} \\
&\quad+ \alpha^2\beta^2\frac{\epsilon^2}{\hbar^2}\Big[\big[1-B(F)\big]B(F)\Delta^{+l}_{jkmn} + C^2(F)\Delta^{-m}_{jkln} + B^2(F)\Delta^{+l}_{jkmn} + \big[1-C(F)\big]C(F)\Delta^{-m}_{jkln}\Big] \\
&\quad+ \frac{\beta^4}{4}\frac{\epsilon^2}{\hbar^2}\Big[2\Delta_{jklmn} + \big[1-B(F)\big]^2\Delta^{+l}_{jkmn} + C^2(F)\Delta^{+m}_{jkln} + 2\big[1-B(F)\big]B(F)\Delta^{+l}_{jkmn} \\
&\quad+2\big[1-C(F)\big]C(F)\Delta^{+m}_{jkln} + B^2(F)\Delta^{+l}_{jkmn} + \big[1-C(F)\big]^2\Delta^{+m}_{jkln}\Big] \\
&= \alpha^4 \left\{\Delta_{jklmn} + \frac{\epsilon^2}{\hbar^2}\Big[D^2(F)\Delta^{+j}_{klmn} + \big[1-A(F)\big]^2\Delta^{+k}_{jlmn} + B^2(F)\Delta^{+l}_{jkmn} + C^2(F)\Delta^{+m}_{jkln} + E^2(F)\Delta^{+n}_{jklm}\Big]\right\} \\
&\quad+ \alpha^2\beta^2\frac{\epsilon^2}{\hbar^2}\Big[B(F)\Delta^{+l}_{jkmn} + C(F)\Delta^{-m}_{jkln}\Big] + \frac{\beta^4}{4}\frac{\epsilon^2}{\hbar^2}\Big[2\Delta_{jklmn} + \Delta^{+l}_{jkmn} + \Delta^{+m}_{jkln}\Big] \\
&= \left(\alpha^4 + \frac{\beta^4}{2}\frac{\epsilon^2}{\hbar^2}\right)\Delta_{jklmn} + \alpha^4\frac{\epsilon^2}{\hbar^2}\Bigg\{D^2(F)\Delta^{+j}_{klmn} + \big[1-A(F)\big]^2\Delta^{+k}_{jlmn} + E^2(F)\Delta^{+n}_{jklm} \\
&\quad+ \left[\frac{\beta^2}{2\alpha^2} + B(F)\right]^2\Delta^{+l}_{jkmn} + \left[\frac{\beta^2}{2\alpha^2} - C(F)\right]^2\Delta^{+m}_{jkln}\Bigg\}.
\end{aligned}
\tag{C.84}
$$

The probabilities for outcomes associated with path G are

$$
\begin{aligned}
\mathscr{P}_{G,j,k,l,m,n} &= \mathrm{Tr}\{|G_7\rangle\langle G_7| \otimes |j\rangle_{x,D\;x,D}\langle j| \otimes |k\rangle_{x,A\;x,A}\langle k| \otimes |l\rangle_{x,B\;x,B}\langle l| \otimes |m\rangle_{x,C\;x,C}\langle m| \otimes |n\rangle_{x,E\;x,E}\langle n|\hat{\rho}_8\} \\
&\approx \alpha^2\beta^2 \Bigg\{ \left(\delta_{+j} + (-1)^{\delta_{+j}} \frac{\epsilon^2}{\hbar^2} D^2(G)\right) \left(\delta_{+k} + (-1)^{\delta_{+k}} \frac{\epsilon^2}{\hbar^2} \big[1 - A(G)\big]^2\right) \left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(G)\right) \\
&\qquad \times \left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(G)\right) \left(\delta_{+n} + (-1)^{\delta_{+n}} \frac{\epsilon^2}{\hbar^2} E^2(G)\right) \\
&\quad + \left(\delta_{+j}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-j}} \frac{\epsilon^2}{\hbar^2} \big[1 - D(G)\big] D(G)\right) \left(\delta_{+k}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-k}} \frac{\epsilon^2}{\hbar^2} \big[1 - A(G)\big] A(G)\right) \\
&\qquad \times \Bigg[\left(\delta_{+l}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-l}} \frac{\epsilon^2}{\hbar^2} \big[1 - B(G)\big] B(G)\right) \left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(G)\right) \\
&\qquad - \left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(G)\right) \left(\delta_{+m}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-m}} \frac{\epsilon^2}{\hbar^2} \big[1 - C(G)\big] C(G)\right)\Bigg] \\
&\qquad \times \left(\delta_{+n}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-n}} \frac{\epsilon^2}{\hbar^2} \big[1 - E(G)\big] E(G)\right) \\
&\quad + \frac{1}{4}\left(\delta_{+j} + (-1)^{\delta_{+j}} \frac{\epsilon^2}{\hbar^2} \big[1 - D(G)\big]^2\right) \left(\delta_{+k} + (-1)^{\delta_{+k}} \frac{\epsilon^2}{\hbar^2} A^2(G)\right) \\
&\qquad \times \Bigg[\left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} \big[1 - B(G)\big]^2\right) \left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} C^2(G)\right) \\
&\qquad - 2\left(\delta_{+l}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-l}} \frac{\epsilon^2}{\hbar^2} \big[1 - B(G)\big] B(G)\right) \left(\delta_{+m}\left(1 - \frac{\epsilon^2}{2\hbar^2}\right) + (-1)^{\delta_{-m}} \frac{\epsilon^2}{\hbar^2} \big[1 - C(G)\big] C(G)\right) \\
&\qquad + \left(\delta_{+l} + (-1)^{\delta_{+l}} \frac{\epsilon^2}{\hbar^2} B^2(G)\right) \left(\delta_{+m} + (-1)^{\delta_{+m}} \frac{\epsilon^2}{\hbar^2} \big[1 - C(G)\big]^2\right)\Bigg] \\
&\qquad \times \left(\delta_{+n} + (-1)^{\delta_{+n}} \frac{\epsilon^2}{\hbar^2} \big[1 - E(G)\big]^2\right)\Bigg\} \\
&\approx \alpha^2\beta^2 \Bigg\{\Delta_{jklmn} + \frac{\epsilon^2}{\hbar^2}\Big[D^2(G)\Delta^{+j}_{klmn} + \big[1 - A(G)\big]^2\Delta^{+k}_{jlmn} + B^2(G)\Delta^{+l}_{jkmn} + C^2(G)\Delta^{+m}_{jkln} + E^2(G)\Delta^{+n}_{jklm} \\
&\quad - \big[1 - B(G)\big]B(G)\Delta^{+l}_{jkmn} + C^2(G)\Delta^{+m}_{jkln} - B^2(G)\Delta^{+l}_{jkmn} + \big[1 - C(G)\big]C(G)\Delta^{+m}_{jkln} \\
&\quad + \frac{1}{4}\Big(2\Delta_{jklmn} + \big[1 - B(G)\big]^2\Delta^{+l}_{jkmn} + C^2(G)\Delta^{+m}_{jkln} + 2\big[1 - B(G)\big]B(G)\Delta^{+l}_{jkmn} \\
&\qquad + 2\big[1 - C(G)\big]C(G)\Delta^{+m}_{jkln} + B^2(G)\Delta^{+l}_{jkmn} + \big[1 - C(G)\big]^2\Delta^{+m}_{jkln}\Big)\Big]\Bigg\} \\
&= \alpha^2\beta^2 \Bigg\{\left(1 + \frac{\epsilon^2}{2\hbar^2}\right)\Delta_{jklmn} + \frac{\epsilon^2}{\hbar^2}\Bigg[D^2(G)\Delta^{+j}_{klmn} + \big[1 - A(G)\big]^2\Delta^{+k}_{jlmn} + E^2(G)\Delta^{+n}_{jklm} \\
&\quad + \left[\frac{1}{2} - B(G)\right]^2\Delta^{+l}_{jkmn} + \left[\frac{1}{2} + C(G)\right]^2\Delta^{+m}_{jkln}\Bigg]\Bigg\}.
\end{aligned}
\tag{C.85}
$$

Finally, the probabilities for outcomes associated with path H are

$$\begin{aligned}
\mathscr{P}_{H,j,k,l,m,n} &= \mathrm{Tr}\{|H_7\rangle\langle H_7|\otimes|j\rangle_{x,D\,x,D}\langle j|\otimes|k\rangle_{x,A\,x,A}\langle k|\otimes|l\rangle_{x,B\,x,B}\langle l|\otimes|m\rangle_{x,C\,x,C}\langle m|\otimes|n\rangle_{x,E\,x,E}\langle n|\hat{\rho}_8\} \\
&\approx \frac{\beta^2}{4}\left(\delta_{+j}+(-1)^{\delta_{+j}}\frac{\epsilon^2}{\hbar^2}\big[1-D(H)\big]^2\right)\left(\delta_{+k}+(-1)^{\delta_{+k}}\frac{\epsilon^2}{\hbar^2}A^2(H)\right) \\
&\quad\times\Bigg[\left(\delta_{+l}+(-1)^{\delta_{+l}}\frac{\epsilon^2}{\hbar^2}\big[1-B(H)\big]^2\right)\left(\delta_{+m}+(-1)^{\delta_{+m}}\frac{\epsilon^2}{\hbar^2}C^2(H)\right) \\
&\quad+2\left(\delta_{+l}\left(1-\frac{\epsilon^2}{2\hbar^2}\right)+(-1)^{\delta_{-l}}\frac{\epsilon^2}{\hbar^2}\big[1-B(H)\big]B(H)\right)\left(\delta_{+m}\left(1-\frac{\epsilon^2}{2\hbar^2}\right)+(-1)^{\delta_{-m}}\frac{\epsilon^2}{\hbar^2}\big[1-C(H)\big]C(H)\right) \\
&\quad+\left(\delta_{+l}+(-1)^{\delta_{+l}}\frac{\epsilon^2}{\hbar^2}B^2(H)\right)\left(\delta_{+m}+(-1)^{\delta_{+m}}\frac{\epsilon^2}{\hbar^2}\big[1-C(H)\big]^2\right)\Bigg] \\
&\quad\times\left(\delta_{+n}+(-1)^{\delta_{+n}}\frac{\epsilon^2}{\hbar^2}\big[1-E(H)\big]^2\right) \\
&\approx \beta^2\Bigg\{\left(1-\frac{\epsilon^2}{2\hbar^2}\right)\Delta_{jklmn}+\frac{\epsilon^2}{\hbar^2}\Big[\big[1-D(H)\big]^2\Delta^{+j}_{klmn}+A^2(H)\Delta^{+k}_{jlmn}+E^2(H)\Delta^{+n}_{jklm} \\
&\quad+\frac{1}{4}\Big(\big[1-B(H)\big]^2\Delta^{+l}_{jkmn}+C^2(H)\Delta^{+m}_{jkln}-2\big[1-B(H)\big]B(H)\Delta^{+l}_{jkmn} \\
&\quad-2\big[1-C(H)\big]C(H)\Delta^{+m}_{jkln}+B^2(H)\Delta^{+l}_{jkmn}+\big[1-C(H)\big]^2\Delta^{+m}_{jkln}\Big)\Big]\Bigg\} \\
&= \beta^2\Bigg\{\left(1-\frac{\epsilon^2}{2\hbar^2}\right)\Delta_{jklmn}+\frac{\epsilon^2}{\hbar^2}\Bigg[\big[1-D(H)\big]^2\Delta^{+j}_{klmn}+A^2(H)\Delta^{+k}_{jlmn}+E^2(H)\Delta^{+n}_{jklm} \\
&\quad+\left[\frac{1}{2}-B(H)\right]^2\Delta^{+l}_{jkmn}+\left[\frac{1}{2}-C(H)\right]^2\Delta^{+m}_{jkln}\Bigg]\Bigg\}.
\end{aligned} \tag{C.86}$$

These probabilities (both compensated and uncompensated) are stated in Table 2.3, which clearly demonstrates how optimal compensation greatly simplifies these probabilities.

C.6 Probe State Comparison

Finally, it pays to take a look at how similar certain probe states are. For this, we will need to define a little more notation for compactness. Let

$$|o\rangle = |+\rangle_{x,D}\,|+\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,. \tag{C.87}$$

If instead of o, I put the the lowercase version of a letter associated with a probe, then the state represents one with that probe state made down rather than up. For example, if the A probe is in the state $|-\rangle_{x,A}$, we have

$$|a\rangle = |+\rangle_{x,D}\,|-\rangle_{x,A}\,|+\rangle_{x,B}\,|+\rangle_{x,C}\,|+\rangle_{x,E}\,. \tag{C.88}$$

If there is more than one letter in the ket in place of o, then there are multiple inverted probes in the state. This notation was used by Griffiths [16].

From Eqn. C.15, we take the probe states entangled with the outcome $|F_7\rangle$ and calculate the effects of the weak interactions to second order, expressing probe states altered by the weak interactions as superpositions of up and down states. Grouping by probe states using the notation we just established, we find

$$
\begin{aligned}
&\alpha^2 \left|+\right\rangle_{x,D} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}} \left|+\right\rangle_{x,A} \left|+\right\rangle_{x,B} \left|+\right\rangle_{x,C} \left|+\right\rangle_{x,E} \\
&\quad - \frac{\beta^2}{2} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}} \left|+\right\rangle_{x,D} \left|+\right\rangle_{x,A} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}} \left|+\right\rangle_{x,B} \left|+\right\rangle_{x,C} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}} \left|+\right\rangle_{x,E} \\
&\quad + \frac{\beta^2}{2} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}} \left|+\right\rangle_{x,D} \left|+\right\rangle_{x,A} \left|+\right\rangle_{x,B} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}} \left|+\right\rangle_{x,C} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}} \left|+\right\rangle_{x,E} \\
&\approx \alpha^2 \left|+\right\rangle_{x,D} \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}\right) \left|+\right\rangle_{x,A} \left|+\right\rangle_{x,B} \left|+\right\rangle_{x,C} \left|+\right\rangle_{x,E} \\
&\quad + \frac{\beta^2}{2}\left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}\right) \left|+\right\rangle_{x,D} \left|+\right\rangle_{x,A} \left[-\left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}\right) \left|+\right\rangle_{x,B} \left|+\right\rangle_{x,C} \right. \\
&\quad \left. + \left|+\right\rangle_{x,B} \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}\right) \left|+\right\rangle_{x,C}\right] \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}\right) \left|+\right\rangle_{x,E} \\
&= \alpha^2 \left|+\right\rangle_{x,D} \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}\right) \left|+\right\rangle_{x,A} \left|+\right\rangle_{x,B} \left|+\right\rangle_{x,C} \left|+\right\rangle_{x,E} \\
&\quad + \frac{\beta^2}{2}\left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,D}\right) \left|+\right\rangle_{x,D} \left|+\right\rangle_{x,A} i\frac{\epsilon}{\hbar}\left[\left|-\right\rangle_{x,B}\left|+\right\rangle_{x,C} - \left|+\right\rangle_{x,B}\left|-\right\rangle_{x,C}\right] \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,E}\right) \left|+\right\rangle_{x,E} \\
&\approx \alpha^2 \left\{\left(1 - \frac{\epsilon^2}{2\hbar^2}\right)|o\rangle - i\frac{\epsilon}{\hbar}|a\rangle + \frac{\beta^2}{2\alpha^2}\left[i\frac{\epsilon}{\hbar}\left(|b\rangle - |c\rangle\right) + \frac{\epsilon^2}{\hbar^2}\left(|be\rangle - |ce\rangle + |db\rangle - |dc\rangle\right)\right]\right\}.
\end{aligned}
\tag{C.89}
$$

If the weak interactions were characterized by the associated weak values, then we would expect the probes state to have only one term, which would be $\left|+\right\rangle_{x,D}\left|+\right\rangle_{x,A}\left|+\right\rangle_{x,B}\left|+\right\rangle_{x,C}\left|+\right\rangle_{x,E}$ acted upon by the unitary operator $e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}+i\frac{\epsilon}{\hbar}\frac{\beta^2}{2\alpha^2}\hat{\sigma}_{z,B}-i\frac{\epsilon}{\hbar}\frac{\beta^2}{2\alpha^2}\hat{\sigma}_{z,C}}$. We would also want to scale it by $\sqrt{\alpha^4 + \frac{\beta^4}{2}\frac{\epsilon^2}{\hbar^2}}$ to give the outcome $|F_7\rangle$ the proper probability. Expressing and grouping the probe states as in Eqn. C.89 gives us

$$
\begin{aligned}
&\sqrt{\alpha^4 + \frac{\beta^4}{2}\frac{\epsilon^2}{\hbar^2}} \left|+\right\rangle_{x,D} e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}} \left|+\right\rangle_{x,A} e^{+i\frac{\epsilon}{\hbar}\frac{\beta^2}{2\alpha^2}\hat{\sigma}_{z,B}} \left|+\right\rangle_{x,B} e^{-i\frac{\epsilon}{\hbar}\frac{\beta^2}{2\alpha^2}\hat{\sigma}_{z,C}} \left|+\right\rangle_{x,C} \left|+\right\rangle_{x,E} \\
&\approx \alpha^2 \sqrt{1 + \frac{\beta^4}{2\alpha^4}\frac{\epsilon^2}{\hbar^2} + \frac{\beta^8}{16\alpha^8}\frac{\epsilon^4}{\hbar^4}} \left|+\right\rangle_{x,D} \left(1 - \frac{\epsilon^2}{2\hbar^2} - i\frac{\epsilon}{\hbar}\hat{\sigma}_{z,A}\right) \left|+\right\rangle_{x,A} \\
&\quad \times \left(1 - \frac{\beta^4}{4\alpha^4}\frac{\epsilon^2}{2\hbar^2} + i\frac{\beta^2}{2\alpha^2}\frac{\epsilon}{\hbar}\hat{\sigma}_{z,B}\right) \left|+\right\rangle_{x,B} \left(1 - \frac{\beta^4}{4\alpha^4}\frac{\epsilon^2}{2\hbar^2} - i\frac{\beta^2}{2\alpha^2}\frac{\epsilon}{\hbar}\hat{\sigma}_{z,C}\right) \left|+\right\rangle_{x,C} \left|+\right\rangle_{x,E} \\
&= \left(\alpha^2 + \frac{\beta^4}{4\alpha^2}\frac{\epsilon^2}{\hbar^2}\right) \left|+\right\rangle_{x,D} \left[\left(1 - \frac{\epsilon^2}{2\hbar^2}\right)\left|+\right\rangle_{x,A} - i\frac{\epsilon}{\hbar}\left|-\right\rangle_{x,A}\right] \left[\left(1 - \frac{\beta^4}{4\alpha^4}\frac{\epsilon^2}{2\hbar^2}\right)\left|+\right\rangle_{x,B} + i\frac{\beta^2}{2\alpha^2}\frac{\epsilon}{\hbar}\left|-\right\rangle_{x,B}\right] \\
&\quad \times \left[\left(1 - \frac{\beta^4}{4\alpha^4}\frac{\epsilon^2}{2\hbar^2}\right)\left|+\right\rangle_{x,C} - i\frac{\beta^2}{2\alpha^2}\frac{\epsilon}{\hbar}\left|-\right\rangle_{x,C}\right] \left|+\right\rangle_{x,E} \\
&\approx \alpha^2 \left\{\left(1 - \frac{\epsilon^2}{2\hbar^2}\right)|o\rangle - i\frac{\epsilon}{\hbar}|a\rangle + i\frac{\epsilon}{\hbar}\frac{\beta^2}{2\alpha^2}\left(|b\rangle - |c\rangle\right) + \frac{\epsilon^2}{\hbar^2}\left(\frac{\beta^2}{2\alpha^2}\left(|ab\rangle - |ac\rangle\right) + \frac{\beta^4}{4\alpha^4}|bc\rangle\right)\right\}.
\end{aligned}
\tag{C.90}
$$

We can see that the two equations match to first order (a little better, given the match at the $|o\rangle$ term).

As long as we are using Griffiths' notation, it is worth rewriting Eqn. C.15 with it. To do this, we will expand the exponential as

$$e^{-i\frac{\epsilon}{\hbar}\hat{\sigma}_z} = \cos\left(\frac{\epsilon}{\hbar}\right)\hat{I} - i\sin\left(\frac{\epsilon}{\hbar}\right)\hat{\sigma}_x, \tag{C.91}$$

and let $\eta = -i\sin\left(\frac{\epsilon}{\hbar}\right)$ and $\zeta = \cos\left(\frac{\epsilon}{\hbar}\right)$. These are not the same η and ζ Griffiths defined, but they will provide the same $|+\rangle_x \to \zeta |+\rangle_x + \eta |-\rangle_x$ probe interaction that Griffiths used. With this, we obtain

$$
\begin{aligned}
|\psi_7\rangle = |F_7\rangle \Big[& \alpha^2 |+\rangle_{x,D} \left(\zeta |+\rangle_{x,A} + \eta |-\rangle_{x,A}\right) |+\rangle_{x,B} |+\rangle_{x,C} |+\rangle_{x,E} \\
& - \frac{\beta^2}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} \left(\zeta |+\rangle_{x,B} + \eta |-\rangle_{x,B}\right) |+\rangle_{x,C} \left(\zeta |+\rangle_{x,E} + \eta |-\rangle_{x,E}\right) \\
& + \frac{\beta^2}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} |+\rangle_{x,B} \left(\zeta |+\rangle_{x,C} + \eta |-\rangle_{x,C}\right) \left(\zeta |+\rangle_{x,E} + \eta |-\rangle_{x,E}\right) \Big] \\
+ |G_7\rangle \Big[& \alpha\beta |+\rangle_{x,D} \left(\zeta |+\rangle_{x,A} + \eta |-\rangle_{x,A}\right) |+\rangle_{x,B} |+\rangle_{x,C} |+\rangle_{x,E} \\
& + \frac{\alpha\beta}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} \left(\zeta |+\rangle_{x,B} + \eta |-\rangle_{x,B}\right) |+\rangle_{x,C} \left(\zeta |+\rangle_{x,E} + \eta |-\rangle_{x,E}\right) \\
& - \frac{\alpha\beta}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} |+\rangle_{x,B} \left(\zeta |+\rangle_{x,C} + \eta |-\rangle_{x,C}\right) \left(\zeta |+\rangle_{x,E} + \eta |-\rangle_{x,E}\right) \Big] \\
+ |H_7\rangle \Big[& \frac{\beta}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} \left(\zeta |+\rangle_{x,B} + \eta |-\rangle_{x,B}\right) |+\rangle_{x,C} |+\rangle_{x,E} \\
& + \frac{\beta}{2} \left(\zeta |+\rangle_{x,D} + \eta |-\rangle_{x,D}\right) |+\rangle_{x,A} |+\rangle_{x,B} \left(\zeta |+\rangle_{x,C} + \eta |-\rangle_{x,C}\right) |+\rangle_{x,E} \Big] \\
= |F_7\rangle \Big[& \alpha^2 \left(\zeta |o\rangle + \eta |a\rangle\right) - \frac{\beta^2}{2} \left(\zeta^3 |o\rangle + \zeta^2\eta \left(|d\rangle + |b\rangle + |e\rangle\right) + \zeta\eta^2 \left(|db\rangle + |de\rangle + |be\rangle\right) + \eta^3 |dbe\rangle\right) \\
& + \frac{\beta^2}{2} \left(\zeta^3 |o\rangle + \zeta^2\eta \left(|d\rangle + |c\rangle + |e\rangle\right) + \zeta\eta^2 \left(|dc\rangle + |de\rangle + |ce\rangle\right) + \eta^3 |dce\rangle\right) \Big] \\
+ |G_7\rangle \alpha\beta \Big[& \zeta |o\rangle + \eta |a\rangle + \frac{1}{2} \left(\zeta^3 |o\rangle + \zeta^2\eta \left(|d\rangle + |b\rangle + |e\rangle\right) + \zeta\eta^2 \left(|db\rangle + |de\rangle + |be\rangle\right) + \eta^3 |dbe\rangle\right) \\
& - \frac{1}{2} \left(\zeta^3 |o\rangle + \zeta^2\eta \left(|d\rangle + |c\rangle + |e\rangle\right) + \zeta\eta^2 \left(|dc\rangle + |de\rangle + |ce\rangle\right) + \eta^3 |dce\rangle\right) \Big] \\
+ |H_7\rangle & \frac{\beta}{2} \left[\zeta^2 |o\rangle + \zeta\eta \left(|d\rangle + |b\rangle\right) + \eta^2 |db\rangle + \zeta^2 |o\rangle + \zeta\eta \left(|d\rangle + |c\rangle\right) + \eta^2 |dc\rangle\right] \\
= & \left[\zeta\alpha \left(\alpha |F_7\rangle + \beta |G_7\rangle\right) + \zeta^2\beta |H_7\rangle\right] |o\rangle + \zeta\eta\beta |H_7\rangle |d\rangle + \eta\alpha \left[\alpha |F_7\rangle + \beta |G_7\rangle\right] |a\rangle \\
& + \frac{1}{2}\zeta\eta\beta \left[\zeta \left(-\beta |F_7\rangle + \alpha |G_7\rangle\right) + |H_7\rangle\right] |b\rangle + \frac{1}{2}\zeta\eta\beta \left[\zeta \left(\beta |F_7\rangle - \alpha |G_7\rangle\right) + |H_7\rangle\right] |c\rangle \\
& + \frac{1}{2}\eta^2\beta \left[\zeta \left(-\beta |F_7\rangle + \alpha |G_7\rangle\right) + |H_7\rangle\right] |db\rangle + \frac{1}{2}\eta^2\beta \left[\zeta \left(\beta |F_7\rangle - \alpha |G_7\rangle\right) + |H_7\rangle\right] |dc\rangle \\
& + \frac{1}{2}\zeta\eta^2\beta \left[-\beta |F_7\rangle + \alpha |G_7\rangle\right] |be\rangle + \frac{1}{2}\zeta\eta^2\beta \left[\beta |F_7\rangle - \alpha |G_7\rangle\right] |ce\rangle \\
& + \frac{1}{2}\eta^3\beta \left[-\beta |F_7\rangle + \alpha |G_7\rangle\right] |dbe\rangle + \frac{1}{2}\eta^3\beta \left[\beta |F_7\rangle - \alpha |G_7\rangle\right] |dce\rangle .
\end{aligned}
\tag{C.92}
$$

In Griffiths' work, he designated $|\Phi^\kappa\rangle$ as the system state entangled to a particular probe state $|\kappa\rangle$. For example, $|\Phi^{be}\rangle = \frac{1}{2}\zeta\eta^2\beta \left[-\beta |F_7\rangle + \alpha |G_7\rangle\right]$. According to my calculation, $|\Phi^{dbe}\rangle = -|\Phi^{dce}\rangle = \frac{\eta}{\zeta} |\Phi^{be}\rangle$, while Griffiths had $-\frac{\eta}{\zeta} |\Phi^{be}\rangle$ at the end in his Eqn. 33. This isn't the only place I disagreed with him; my Eqn. C.43 does not match Griffiths' Eqn. 5, and the difference is demonstrable—evolving his $|\phi_1\rangle$ forward in time does not lead to $|\phi_7\rangle = |F_7\rangle$. I believe both sign disagreements are simple typos, with no bearing on his findings.

List of Figures

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