Equivariant Homology of Representation Spheres for the Nonabelian Group of Order 21

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Thesis directed by Dr. Agnès Beaudry

We present two methods for computing Mackey functor-valued Bredon homology, one using an explicit equivariant cell structure and another using an isotropy separation sequence. For the non-abelian group of order 21, we identify a representation sphere whose associated homology computation does not simplify to previously known computations regarding finite cyclic groups. The explicit cell structure for this representation sphere is given, and the homology is computed in full using an isotropy separation sequence.

Dedication

To my beloved 婆婆, who was wise enough to laugh at all of it.

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Chapter 1

Introduction

1.1 History of the problem

Given the success of cohomology in the field of algebraic topology, it makes sense that we would want an analogue for spaces that have a group action. One such equivariant cohomology theory was introduced by Bredon [Bre67] and Illman [Ill75], and this theory is now called Bredon cohomology. Like nonequivariant cohomology, Bredon cohomology is “stable” in the sense that suspension by spheres S^n with a trivial group action yields the usual shift in degree. In 1981, Lewis, May, and McClure [LMM81] extended Bredon cohomology so that suspension by representation spheres of a group G would also produce an appropriate shift in degree. This extended cohomology theory is known as $RO(G)$-graded cohomology, though some sources may use the term “Bredon homology” to describe the $RO(G)$-graded case as well. A more detailed history of the developments mentioned above can be found in the introduction of the book by Costenoble and Waner [CW16].

$RO(G)$-graded cohomology has many advantages over its predecessors, and was used in the landmark result resolving the Kervaire invariant problem [HHR16]. Despite its power, one significant drawback is that it is not easy to compute, and as a result, the bank of known computations is small. For example, results about the cohomology of C_{2^n} were used in [HHR16], and the authors had to compute those results therein. At a talk in 2017, May [May17] mentioned that the $RO(G)$-graded homology of a point had not yet been computed for any non-abelian groups. Kriz and Lu [KL20] followed up by performing those computations for the permutation group Σ_3. Since then, the bank of computations has grown to include finite abelian groups $(C_2)^n$ [HK17] and C_{p^n} [HHR17], dihedral groups D_{2p} [KL20, Zou18] and D_8 [Zhu18], symmetric group Σ_4 [Zhu18], and quaternion group Q_8 [Lu21]. The goal of this thesis is to provide the equivariant homology of a point for the nonabelian group of order 21.

Remark: During the writing of this thesis, work by Angeltveit [Ang22] computing the equivariant cohomology for semidirect products $C_p \rtimes C_q$ was made public. Though Angeltveit's work does subsume the final result of this thesis, we will establish the independence of the results in this thesis and discuss differences in methods used in Section 1.3.

1.2 Document structure and summary of results

Since $RO(G)$-graded homology involves knowledge about the group G and its real representation spheres, Chapter 2 gives classical background on real representation theory and the ring $RO(G)$.

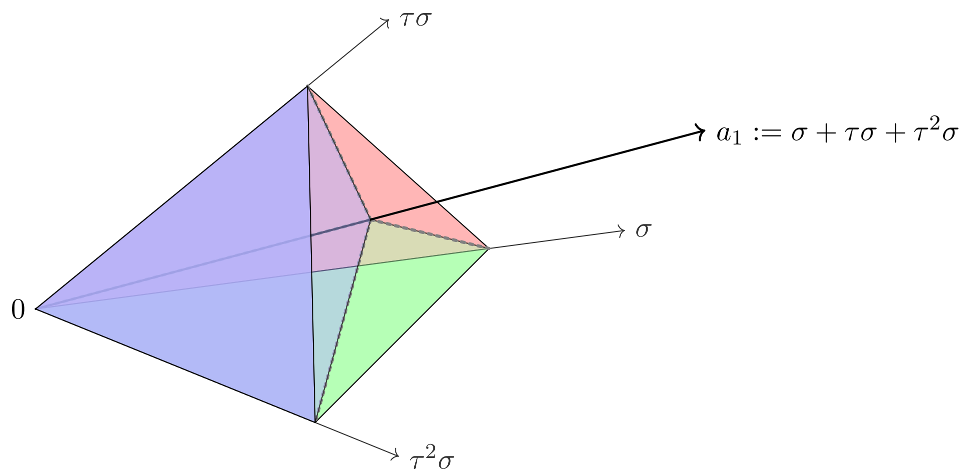
Chapter 3 introduces Bredon homology. The presentation here is partly inspired by Hill's Handbook chapter [Hil20] and gives background on G-spaces, G-CW complexes, and representation spheres. Since equivariant homology uses the language of Mackey functors and coefficient systems instead of abelian groups, we define those here as well. A definition of Bredon homology using coefficient systems, drawn from [Wil75], is provided. This definition is then shown to be consistent with the Mackey functor-valued homology discussed in Section 3.3 of [HHR16]. We provide a detailed description of this equivalence in the proof of lemma 3.4.1. This lemma, shown below, gives an explicit chain complex for computing Bredon homology.

Lemma 3.4.1. The Bredon homology of X evaluated at G/K using constant $\underline{\mathbb{Z}}$ coefficients, denoted $H\underline{\mathbb{Z}}_\bullet(X)(G/K)$ is the homology of the chain complex

$$\cdots \longrightarrow \mathbb{Z}\{T_n\}^K \xrightarrow{e_n} \mathbb{Z}\{T_{n-1}\}^K \longrightarrow \cdots$$

where the differential e_n is the map of the same name described in Section 3.3, extended linearly to free abelian groups.

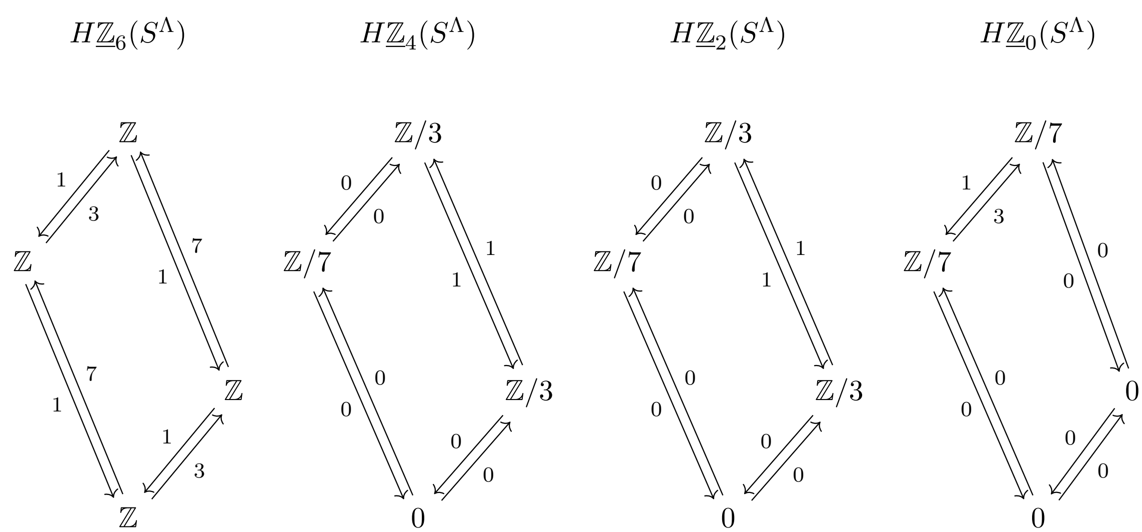
Then Chapters 4 and 5 apply the information in the previous two chapters to a specific group of interest G_{21}, the non-abelian group of order 21. We discuss the group structure and representation theory of this group, as well its group homology. Since we will be interested in certain representation spheres of G_{21}, we also give a G-CW stucture for a degree-2 representation λ and establish the existence of a subcomplex Y inside the degree-6 representation Λ. This subcomplex Y will play a key role in the computations of Chapter 7. The explicit G-CW structure for Λ is reserved for Chapter 5 because its development is more involved. While creating the cell structure for S^{Λ}, we develop a process for converting certain CW-structures into G-CW structures. The method subdivides cells to create proper G-CW cells, like in the picture below.



The method of subdivision applies to more general settings as well, so several examples of how it can be used in the context of different group representations are also provided.

Chapters 6 and 7 cover the computation of the homology of the 6-dimensional representation sphere S^{Λ}. Chapter 6 is concerned with parts of the computation that can be reduced to previously known computations. Using the fact that $H\underline{\mathbb{Z}}_n(X)(G/H) \cong H\underline{\mathbb{Z}}_n(i_H^* X)(H/H)$ from Lemma 3.4.3, we restrict our attention from G_{21} to its proper subgroups, which are finite cyclic groups. Detailed computations for each proper subgroup are provided, and the outcome is consistent with known results about finite cyclic groups. Next, Chapter 7 presents the rest of the computation relating to G_{21} itself. Drawing from Section 17.2.4 of [Hil20], we use an isotropy separation sequence to isolate the contributions of cells in S^Λ that have trivial isotropy from those that have nontrivial isotropy. The contributions of free cells of S^Λ are found using the key observation that $H\underline{\mathbb{Z}}_n^G(EG_+ \wedge S^\Lambda) \cong H_{n-6}(G_{21}, \mathbb{Z})$. The contributions of non-free cells are computed using the subcomplex Y discussed in Chapter 4. The results of Chapters 6 and 7 combine to produce the homology of S^Λ, which we state here.

Theorem. *Let G_{21} be the non-abelian group of order 21 and Λ the irreducible degree-6 representation of G_{21}. Then the Mackey functor-valued integer-graded Bredon homology of S^Λ is*



and $H\underline{\mathbb{Z}}_n(S^\Lambda) = 0$ for any $n \neq 0, 2, 4, 6$.

Remark: Parts of the Mackey functor corresponding to the subgroups which are conjugate to C_3 have been suppressed for succinctness.

1.3 Comparisons with other work

During the production of this thesis, both Angeltveit [Ang22] and Liu [Liu21] released works that are related to the topics of this thesis. We now compare each of these works with the contents of this document.

The most significant overlap between this thesis and the Angeltveit document is the computation of $H\underline{\mathbb{Z}}_n(S^V)$ when V is an irreducible real representation of the non-abelian group of order 21. Angeltveit's method uses the algebraic structure of cohomological Mackey functors under restriction to Sylow subgroups. Cohomological Mackey functors are ones that satisfy an additional condition regarding the composition of its transfer and restriction maps. The Mackey functors that form the output of Bredon homology are indeed cohomological Mackey functors, so Angeltveit's method can be used to recover the equivariant homology of a point for a group G by investigating the representations of G under restriction to its Sylow subgroups. Theorem 3.1 of [Ang22] is the precise statement of this idea. Since the Bredon homology of a point for finite cyclic groups is known and they are precisely the Sylow subgroups of semidirect product $C_p \rtimes C_q$, one application of this method is to recover the homology of a point for $C_q \rtimes C_q$. Example 6.6 in [Ang22] is one such computation, and it is here that the results of this thesis overlap the most clearly.

This thesis discusses two methods for computing the homology of a point for the nonabelian group of order 21, and both of them are different from Angeltveit's technique. In Section 5.1 of this document, we present an explicit G-CW cell structure for the 6-dimensional representation sphere S^{Λ}. In contrast, Angeltveit uses a C_q-CW structure that is compatible with the group action by C_p but is not a cell structure for the action by G. This C_q-CW structure is sufficient because Angeltveit is pinpointing the action of C_p on $H\underline{\mathbb{Z}}_n(S^V)(G/C_q)$, and Remark 6.3 in [Ang22] points out that it would also be possible to compute the desired homology by using an explicit G-CW cell structure like the one given in this thesis.

The second method of computation mentioned in this thesis is isotropy separation, which is used in Chapter 7 to compute $H\underline{\mathbb{Z}}^G_n(S^{\Lambda})$. While this method is not used in the abovementioned work of Angeltveit, it is used in related work by Liu that discusses the homotopy of G-spectra [Liu21]. In this work, Liu examines $E\mathscr{F}_+$ and $\widetilde{E\mathscr{F}}$ to find the homotopy of the G-spectrum X. The spaces EG_+ and $\widetilde{EG}$ used in the computations of Chapter 7 of this thesis are particular examples of $E\mathscr{F}_+$ and $\widetilde{E\mathscr{F}}$ when the family $\mathscr{F}$ consists of only the identity element of G. Liu uses this splitting of spectra to compute an explicit example involving the G-spectrum $X = H\underline{\mathbb{Z}}$ and dihedral group D_{2p} in Section 6 and gives a description of how this splitting works for semi-direct products in Section

7 without computing an explicit example.

In summary, Liu's work gives a method for computing the homology of a point for the non-abelian group of order 21 via isotropy separation without performing said computation, and this thesis uses a closely-related method in Chapter 7. Angeltveit's work produces the homology groups of Chapter 7 via a different method that examines the structure of Mackey functors. Any of the above overlap is unintentional and the work of this thesis was done independently. Finally, the explicit G-CW structure and method of subdivision in Chapter 5 is not in either of the abovementioned works.

Chapter 2

Some Representation Theory

In this chapter we summarize some useful facts from representation theory. Ultimately we will be interested in the real representations of finite groups, so this section will cover the relevant facts for that goal. The bulk of this material is drawn from Serre's classic text on the topic [Ser77]. Throughout this section, G will denote a finite group.

2.1 Representations over $\mathbb{C}$

Definition 2.1.1. A **representation** of a finite group G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a vector space. A representation is **real** or **complex** if the field of coefficients for V is $\mathbb{R}$ or $\mathbb{C}$, respectively. The **degree** of a representation is the dimension of vector space V.

Remark: We may sometimes take the alternative but equivalent perspective that a representation is a vector space V together with a linear G action. For this reason, we will also refer to a representation as simply V.

Definition 2.1.2. A representation $\rho : G \to \mathrm{GL}(V)$ is **irreducible** if V is not zero and no proper subspace of V is stable under G.

Example 2.1.3. Let $G = C_n$ be the finite cyclic group of order n, and choose a generator γ. Then $\lambda_n : G \to \mathbb{C}^*$ given by $\gamma \mapsto \exp(i2\pi/n)$ is a degree-1 complex representation of G. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can form $\lambda'_n : G \to GL(\mathbb{R}^2)$, which is a degree-2 real representation of G.

Definition 2.1.4. Let H be a subgroup of G and let $\rho : H \to \mathrm{GL}(V)$ be a representation of H. Let $n = [G : H]$. Take $g_1, \cdots, g_n$ to be a complete set of representatives for all cosets of H. Let $W = \bigoplus_i g_i V$, i.e. n copies of V indexed by the coset representatives. Let G act on W via

$$g \cdot \sum g_i v_i = \sum g_j \rho(h_i) v_i, \text{ where } g g_i = g_j h_i. \tag{2.1}$$

Then the **induced representation** of G is

$$\mathrm{Ind}_H^G(\psi) : G \to \mathrm{GL}(W)$$

where each element $g \in G$ is an automorphism of W by the group action defined in statement (2.1).

Definition 2.1.5. Let V be a finite-dimensional vector space over a field F and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. The **character** of ρ is the function $\chi_\rho : G \to F$ given by

$$\chi_\rho(g) = \mathrm{Tr}\big(\rho(g)\big).$$

In other words, the character is the composition of the representation ρ with the trace function $\mathrm{GL}(V) \to F$.

We now summarize some facts about characters below.

- If two representations have the same character, then they are isomorphic as representations.
- For complex characters of a group G, group elements that share a conjugacy class will have the same image under χ. In other words, complex characters are class functions.
- The character of a direct sum of representations is the sum of the characters of the summands.
- The character of a tensor product of representations is the product of the characters of the factors.

We can organize the characters of all irreducible representations of a group into a character table. Since the number of irreducible characters over $\mathbb{C}$ is equal to the number of conjugacy classes of G ([Ser77] Sec 2.5 Thm 7) and each character is constant on conjugacy classes, the character table is organized such that each column represents one irreducible character, and each row is one conjugacy class of G, often designated by a single representative of that class.

Example 2.1.6. Let λ_3 be the degree-1 complex representation of C_3 given by mapping the generator $\tau \mapsto \exp(i2\pi/3)$. Let λ_3' be the representation mapping $\tau \mapsto \exp(i4\pi/3)$. These two representations, together with the trivial representation, are the three irreducible complex representations of C_3. They are listed in the character table below. In the first column, χ_0 is the character for the trivial representation while χ_1 and χ_2 are characters for λ_3 and λ_3', respectively. Since $C_3 = \langle\tau\rangle$ is abelian, every element is its own conjugacy class.

	χ_0	χ_1	χ_2
e	1	1	1
τ	1	$e^{i2\pi/3}$	$e^{i4\pi/3}$
τ^2	1	$e^{i4\pi/3}$	$e^{i2\pi/3}$

Figure 2.1: Characters of irreducible complex characters of C_3

Definition 2.1.7. Let $R(G)^+$ be the monoid of characters of G, where the binary operation is the usual function addition in $\mathbb{C}$. Observe that the characters of $R(G)^+$ can also be multiplied, with the character of the trivial representation serving as the multiplicative identity. Then **ring of virtual characters**, denoted $R(G)$, is the ring formed by taking the group completion of $R(G)^+$ under the addition operation.

Remark: Every element of $R(G)$ can be formed from irreducible characters. More explicitly, let $\chi_1, \ldots, \chi_h$ be the distinct irreducible complex characters of a finite group G. Then every character of G is a linear combination of the irreducible characters with coefficients from $\mathbb{Z}_{\geq 0}$.

2.2 Representations over $\mathbb{R}$

Let us now consider representations over fields other than $\mathbb{C}$. We stated earlier that for complex representations of a finite group G, the number of conjugacy classes of G is equal to the number of irreducible representations over $\mathbb{C}$. For real representations, there is an analogous statement, but the notion of conjugacy class must be modified. We summarize the appropriate modification below, and then discuss the specific case when our field of interest is $\mathbb{R}$.

Let m be an integer such that $|G|$ divides m. Let K be a field of characteristic zero and let L be the field obtained by adjoining the m^{th} roots of unity to K. Then for each automorphism φ in the Galois group $\mathrm{Gal}(L/K)$, there exists a unique $t \in (\mathbb{Z}/m\mathbb{Z})^*$ such that for $z \in L$,

$$\text{if } z^m = 1, \text{ then } \varphi(z) = z^t. \tag{2.2}$$

Definition 2.2.1. Let Γ_K be the subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$ consisting of the elements t described statement (2.2). Two elements $g', g \in G$ are **Γ_K-conjugate** if there exists $t \in \Gamma_K$ such that g' and g^t are conjugate by an element of G.

In the same way that conjugacy is an equivalence relation, Γ_K-conjugacy is also an equivalence relation on G, and we can use it to partition G accordingly. These Γ_K classes will be the appropriate analogue (over $\mathbb{R}$) to replace conjugacy classes (over $\mathbb{C}$).

Definition 2.2.2. Let **the ring of virtual K characters**, denoted $R_K(G)$, be the ring generated by characters of representations of G over a field K of characteristic zero.

Lemma 2.2.3 (Serre 12.1 Prop 32)**.** *Let χ_i be the characters of the distinct irreducible representations of G over K. Then*

(1) The χ_i form a basis of $R_K(G)$.

(2) The χ_i are mutually orthogonal with respect to the bilinear form

$$\langle \chi_1, \chi_2 \rangle = (1/|G|) \sum_{s \in G} \chi_1(s^{-1})\, \chi_2(s).$$

Lemma 2.2.4 (Serre 12.4 Cor 1)**.** *In order that a class function f on G with values in a field K to belong to $K \otimes R_K(G)$, it is necessary and sufficient that it be constant on the Γ_K-classes of G.*

Lemma 2.2.5 (Serre 12.4 Cor 2)**.** *Let χ_i be the characters of the distinct irreducible representations of G over K. Then the χ_i form a basis for the space of functions on G which are constant on Γ_K-classes, and their number is equal to the number of Γ_K-classes.*

In summary, if K is a field of characteristic zero and L/K is the field extension formed by adjoining m^{th} roots of unity (with $|G|$ dividing m), then representations over L can be "collapsed" to form representations over the ground field K. The general principle relies on the fact that some representations over the ground field K split when a K-vector space is extended to an L-vector space. The above-described method then reassembles the pieces to produce K-representations.

Now we apply this method to the specific case where our field of interest is $\mathbb{R}$. Again let m be an integer such that $|G|$ divides m, and adjoin the m^{th} roots of unity to $\mathbb{R}$ to obtain the field $\mathbb{C}$. We know $\mathbb{C}$ is a degree-2 Galois extension over $\mathbb{R}$ and so the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ must be isomorphic to $\mathbb{Z}/2$. In fact, $\text{Gal}(\mathbb{C}/\mathbb{R})$ must be generated by complex conjugation, since conjugation is an automorphism of $\mathbb{C}$ that fixes $\mathbb{R}$. Then the elements of $(\mathbb{Z}/m\mathbb{Z})^*$ that satisfy statement (2.2) are $\Gamma_{\mathbb{R}} = \{1, -1\}$. Two elements $g', g \in G$ are $\Gamma_{\mathbb{R}}$-conjugate if and only if g' and g^{-1} are conjugate by an element of G. In other words, the $\Gamma_{\mathbb{R}}$-classes of G can be formed from the conjugacy classes of G by taking the union of the classes of g and g^{-1} for all $g \in G$.

Example 2.2.6. Let us examine the specific case where G is a finite cyclic group. In this case, we can view the elements of G as roots of unity on the unit circle in $\mathbb{C}$. Then any root of unity that is purely real is a $\Gamma_{\mathbb{R}}$-class with only a single element, while any root of unity with nonzero imaginary component, together with its inverse, is a $\Gamma_{\mathbb{R}}$-class with two elements. Thus $G = C_n$ has $(n+1)/2$ irreducible real representations if n is odd and $(n+2)/2$ such representations if n is even.

By Lemma (2.2.5), each $\Gamma_{\mathbb{R}}$ class will correspond to one irreducible real representation of G. For $G = C_n$, the correspondence is listed below.

- The $\Gamma_{\mathbb{R}}$ class of the identity corresponds to the degree-1 trivial representation.
- If n is even, then we have a second purely real root of unity, namely $e^{\pi i} = -1$. The $\Gamma_{\mathbb{R}}$ class

$\{-1\}$ corresponds to the degree-1 sign representation given by $g \mapsto \text{sign}(g)$ where $\text{sign}(g)$ is the sign of the permutation representation of $g \in G$.

- The remaining $\Gamma_{\mathbb{R}}$ classes each consist of two elements, namely $\exp(\pm i2\pi k/n)$ for $k = 1, \ldots, \lfloor (n-1)/2 \rfloor$. The corresponding representation for each class is the degree-2 real representation φ_k that rotates $\mathbb{R}^2$ by an angle of $2\pi k/n$ in the counterclockwise direction. Explicitly, if γ is our chosen generator of C_n, then

$$\gamma \mapsto \begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix}.$$

The representations listed above generate a table of characters of real representations of C_n, shown below in figures (2.2) and (2.3). Each column represents an irreducible real character and each row is labelled with a representative from a $\Gamma_{\mathbb{R}}$-class of $C_n = \langle \gamma \rangle$.

Let $j, k = 1, \ldots, \frac{n-1}{2}$

	χ_0	χ_k
1	1	2
γ	1	$2\cos\left(\frac{2\pi k}{n}\right)$
γ^2	1	$2\cos\left(\frac{4\pi k}{n}\right)$
$\vdots$	$\vdots$	$\vdots$
γ^j	1	$2\cos\left(\frac{2\pi jk}{n}\right)$

Figure 2.2: Real characters of C_n, n odd

Let $j, k = 1, \ldots, \frac{n-2}{2}$.

	χ_0	χ_k	$\chi_{n/2}$
1	1	2	1
γ	1	$2\cos\left(\frac{2\pi k}{n}\right)$	-1
γ^2	1	$2\cos\left(\frac{4\pi k}{n}\right)$	1
$\vdots$	$\vdots$	$\vdots$	
γ^j	1	$2\cos\left(\frac{2\pi jk}{n}\right)$	$(-1)^j$
$\gamma^{n/2}$	1	$2\cos\left(\frac{2\pi(n/2)k}{n}\right)$	$(-1)^{n/2}$

Figure 2.3: Real characters of C_n, n even

We have now established a method for finding all irreducible real characters from the set of irreducible complex characters. We know two representations are isomorphic if and only if they have the same character. This allows us to choose between two equivalent viewpoints - characters or isomorphism classes of representations. Since the latter viewpoint is more useful for the topological applications to come, we will describe the change in perspective below and henceforth only use the new perspective.

2.3 The ring $RO(G)$

The integer-graded homology theory introduced by Bredon [Bre67] and Illman [Ill75] was "stable" in the sense that suspension by spheres with trivial action would generate a shift in degree analogous to the suspension isomorphism for nonequivariant homology. But suspension by only trivial spheres is highly restrictive, so Lewis, May, and McClure [LMM81] extended that homology theory in a way that gives a suspension isomorphism for representation spheres of G as well. This extension means that Bredon homology was no longer graded on the integers only, but on a larger ring of representations of G. For this reason, we are interested in $RO(G)$, the ring of orthogonal representations. For now we establish the definition of $RO(G)$.

Definition 2.3.1. Let V be a vector space with an inner product. A representation $\rho : G \to GL(V)$ is **orthogonal** if the image of ρ is contained in $O(V)$, the orthogonal group within $GL(V)$.
Note: For $V = \mathbb{R}$ with the standard inner product and a finite group G, a representation is orthogonal when the determinant of $\rho(g)$ is ± 1 for all $g \in G$.

Definition 2.3.2. Let $RO(G)^+$ be the monoid of real, orthogonal representations of G, where the binary operation is the direct sum of vector spaces and the identity element is the trivial representation. Observe that the representations in $RO(G)^+$ also have a tensor product. Then the **ring of real, orthogonal representations**, denoted $RO(G)$, is the ring formed by taking the group completion of $RO(G)^+$ under the addition operation.

Remark: A typical element of $RO(G)$ is a "virtual" representation of G written as a $\mathbb{Z}$-linear combination of irreducible representations of G. These representations are "virtual" because while the sum of representations produces an honest representation, the difference of representations is purely formal.

Lemma 2.3.3. *If G is a finite group, then every real representation of G is isomorphic to an orthogonal representation.*

Proof. We'll use the "Weyl unitary trick" to define a new inner product. Let $\rho : G \to V$ be a representation of G and let $\langle -, - \rangle$ be an inner product on V. Define a new inner product $(-, -)$ such that for $v, w \in V$,

$$(v, w) := \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle.$$

Note that the sum has no convergence issues because G is finite. The representation (V, ρ) is orthogonal with respect to the new inner product since for any $h \in G$,

$$\big(\rho(h)v, \rho(h)w\big) = \sum_{g \in G} \langle \rho(gh)v, \rho(gh)w \rangle = \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle = (v, w).$$

Let T be the change-of-basis transformation from some orthonormal basis with respect to $\langle -, - \rangle$ to a new orthonormal basis with respect to $(-, -)$. Then T is also an isomorphism of representations that maps (V, ρ) to an orthogonal representation. □

Lemma 2.3.4. *If G is a finite group, then $R_{\mathbb{R}}(G) \cong RO(G)$ as rings.*

Proof. The isomorphism $RO(G) \to R_{\mathbb{R}}(G)$ takes each representation to its character. □

Since we will only be discussing finite groups, we can see from Lemma (2.3.4) that we may conflate the isomorphic rings $R_{\mathbb{R}}(g)$ and $RO(G)$. Our interest in representations of G is strongly geometric, in the sense that we will want to think about spheres and disks formed from vector spaces, flipping and turning under the action of G. For this reason, it behooves us to lean into the geometric perspective of real vector spaces in $RO(G)$ rather than the algebraic perspective of characters in $R_{\mathbb{R}}(G)$. So henceforth we will deal only with the ring $RO(G)$.

Example 2.3.5. Let $G = C_3$. The irreducible real representations of G are the trivial representation and λ_3, the degree-2 representation given by rotating the real plane by $2\pi/3$. Then a typical element of $RO(C_3)$ is denoted $a + b\lambda_3$ for $a, b \in \mathbb{Z}$.

While the irreducible real characters of G form a basis for $RO(G)$, the precise relations for the ring $RO(G)$ can be seen by examining the character table for the irreducible representations of G.

Example 2.3.6. Let $RO(C_3)$ be the ring described in Example 2.3.5 and let λ_3 be the representation described in Example 2.1.6 with the identification $\mathbb{C} \cong \mathbb{R}^2$. Let χ_0 be the character for the trivial representation and χ_1 be the character for λ_3. Then the character table below presents the two irreducible real representations of $C_3 = \langle \tau \rangle$. The labels on the left side of the table are representatives of the $\Gamma_{\mathbb{R}}$-classes of C_3. Computing the pairwise products of these characters gives

	χ_0	χ_1
e	1	2
τ	1	-1

Figure 2.4: Irreducible real characters of C_3

the following results.

$$\chi_0 \cdot \chi_0 = \chi_0$$

$$\chi_0 \cdot \chi_1 = \chi_1$$

$$\chi_1 \cdot \chi_1 = \chi_1 + 2\,\chi_0$$

Since χ_0 serves as the unit of $RO(C_3)$, the first two lines are unremarkable. Thus $RO(C_3) \cong \mathbb{Z}[\lambda_3]/({\lambda_3}^2 - \lambda_3 - 2)$. ★

Chapter 3

Background on Mackey Functors and Bredon homology

In this chapter, we will give a definition of Bredon homology that uses an explicit chain complex. Before doing so, we will also discuss Mackey functors. This is because while non-equivariant singular homology is presented as a series of abelian groups, Bredon homology is presented as a series of Mackey functors. These definitions will form the relevant background to the computations in Chapters 6 and 7 that produce the final result.

Throughout this chapter, G will be a finite group.

3.1 Background on G-spaces

We begin by listing some conventions. The G-spaces discussed in this document will be based G-spaces. A space with a disjoint basepoint, denoted $+$, will be written as X_+. The basepoint of all other spaces will be denoted by $*$. Maps between G-spaces are assumed to be equivariant, i.e. for a map of G-spaces $f : X \to Y$ and any $g \in G$, we must have $g \cdot f(x) = f(g \cdot x)$. A G-homotopy between maps $f, g : X \to Y$ is a (non-equivariant) homotopy $F : X \times I$ with the additional data of a G-action where G acts trivially on I and diagonally on $X \times I$.

For any point x in a G-space X, the stabilizer of x is some subgroup H and we say that x has orbit type G/H. This notation is sensible because the orbit of x is isomorphic (as G-sets) to the cosets G/H. Then rather than referring to single points in X, we will refer to the appropriate equivariant analogue, which is the whole orbit of x. We choose one of the points in the orbit to represent the whole orbit, so we write $G/H \times x$, or $G/H_+ \wedge x_+$ when we have need for a basepoint. This notation extends to more than just points, and rather than a single point x, we can also have disks, spheres, and more.

Having now established our conventions, we will discuss some specific G-spaces.

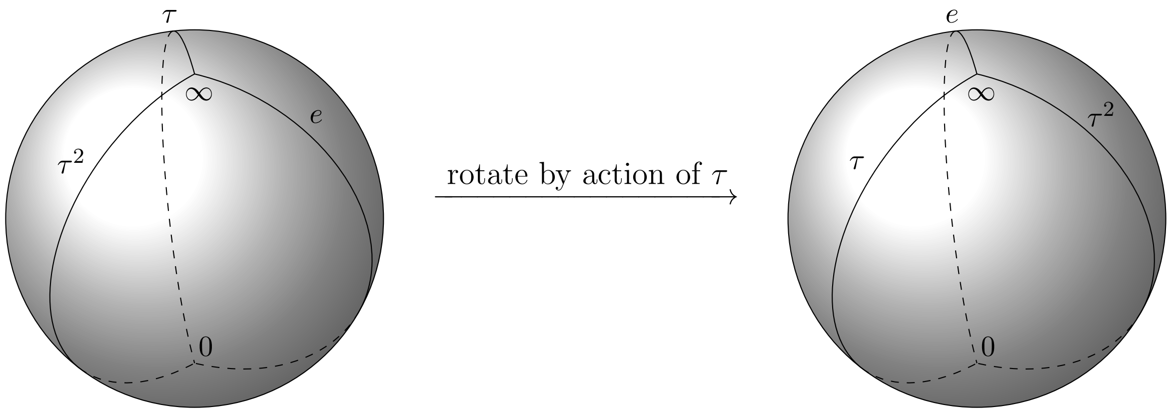
Definition 3.1.1. A G-space X is a **G-CW-complex** if it is a CW-complex with a cellular group action of G. The **n-cells** of X are orbits of n-disks (as opposed to single n-disks in the nonequivariant case) of the form $G/H_+ \wedge D^n_+$ where H is a subgroup of G. The **n-skeleton** of X is defined iteratively: the 0-skeleton $X^{(0)}$ is a finite G-set, and the n-skeleton is formed from the $(n-1)$-skeleton by attaching cells $G/H_+ \wedge D^n_+$ along its boundary $G/H_+ \wedge S^{n-1}_+$.

Definition 3.1.2. A **representation sphere** for a representation $\rho : G \to \mathrm{GL}(V)$ is the G-space formed by taking the one-point compactification of V.

Remark: Notice that when a real representation V undergoes one-point compactification, rays from the origin become semicircles connecting the poles of S^V. For this reason we will use the same notation to denote a ray from the origin in V, the unit vector contained in that ray, and the semicircle formed from that ray. In addition, we will use the convention of naming cells of G-CW structures by listing 1-cells inside square brackets. More precisely, for linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^m$, let $[v_1, \ldots, v_n]$ denote the n-dimensional subset of $\mathbb{R}^{nm}$ consisting of non-negative linear combinations of $v_1, \ldots, v_n$. The following Example 3.1.3 will demonstrate these conventions.

Example 3.1.3. Let $G = C_3$ and let λ be the 2-dimensional real representation where the generator $\tau \in G$ rotates $\mathbb{R}^2$ by an angle of $2\pi/3$ counterclockwise. Then its representation sphere S^λ is S^2 with rotation by 1/3 as shown in the figure below. This G-CW structure of S^λ consists of

- two fixed 0-cells: $G/G_+ \wedge 0_+$ and $G/G_+ \wedge \infty_+$
- a single 1-cell: $G/e_+ \wedge [e]_+$
- a single 2-cell: $G/e_+ \wedge [e, \tau]_+$, where $[e, \tau]$ is the 2-dimensional panel between the 1-cells $[e]$ and $[\tau]$

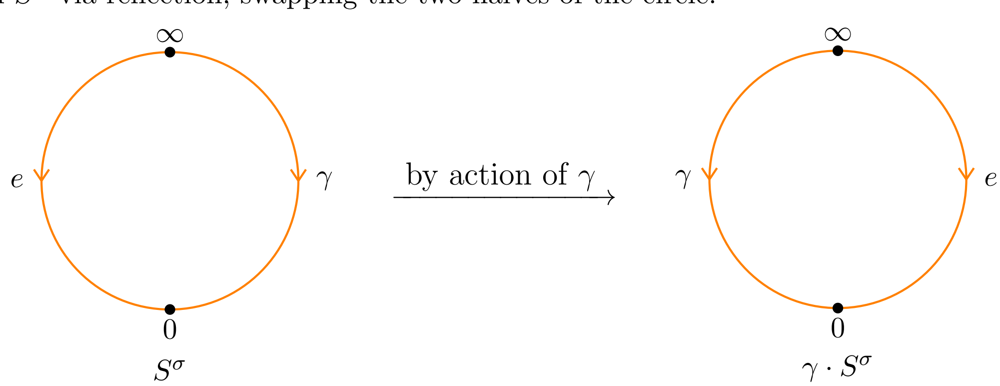


★

Remark: The G-CW structure in Example 3.1.3 can be generalized for any finite cyclic group C_k. Let λ be the 2-dimensional real representation of C_k where the generator $\tau \in C_k$ rotates $\mathbb{R}^2$ by an angle of $2\pi/k$. Then $\mathbb{R}^2$can be divided into k radial sectors, and the resulting representation sphere S^λ will have a cell structure looks like a "beach ball" with k panels. The action of the group will rotate the beach ball. Again there will be two fixed 0-cells, a single 1-cell with orbit type C_k/e, and a single 2-cell with orbit type C_k/e.

Now we discuss how representation spheres are affected by combining representations via direct sum. Let V, W be two representations of G. Then the representation sphere for $V \oplus W$ is G-homotopy equivalent to the smash product $S^V \wedge S^W$. If S^V and S^W have G-CW structures, it is sometimes straightforward to obtain a cell structure on $S^V \wedge S^W$. The following is an example is one such case.

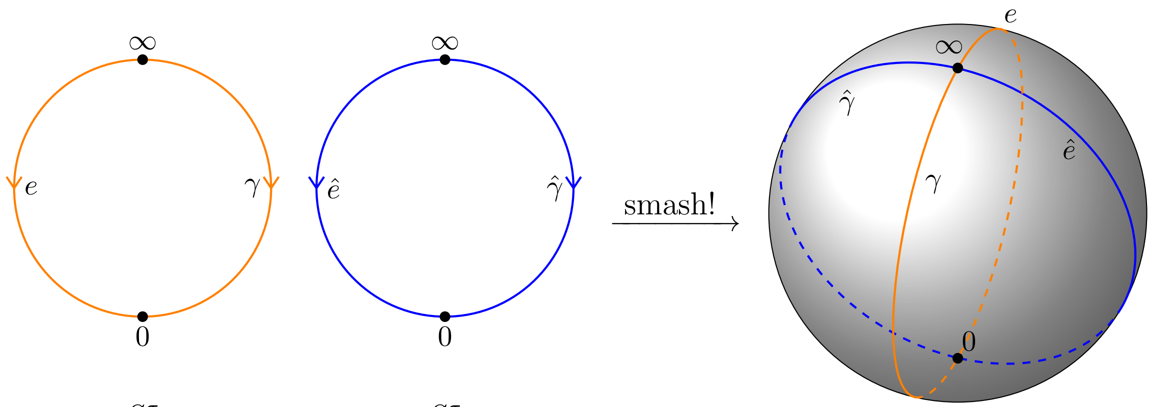
Example 3.1.4. Let $G = C_2$ and let σ be the sign representation. Then the generator γ of C_2 acts on S^σ via reflection, swapping the two halves of the circle.



One possible G-CW structure of S^{σ} consists of

- one fixed 0-cell plus a basepoint at ∞: $G/G_+ \wedge 0_+$
- a single 1-cell: $G/e_+ \wedge [e]_+$

Then the representation $2\sigma = \sigma \oplus \sigma$ has representation sphere $S^{2\sigma}$, drawn below.



S^{σ} S^{σ} $S^{2\sigma}$

One possible G-CW structure of $S^{2\sigma}$ consists of

- one fixed 0-cell plus a basepoint at ∞: $G/G_+ \wedge 0_+$
- two 1-cells: $G/e_+ \wedge [e]_+$, one drawn in orange and the other in blue
- two 2-cells: $G/e_+ \wedge [e, \hat{\gamma}]_+$ and $G/e_+ \wedge [e, \hat{e}]_+$ ★

The previous example was tractable because the spheres had sufficiently low dimension that we could rely on drawings and geometric intuition. But in general, when given G-CW structures on representation spheres, it may not be an easy task to produce a G-CW structure on the smash product of representation spheres. There may be convenient results in very particular cases. To see method for producing cell structures on representation spheres that capitalizes on the tidy structure of C_{p^n}, see the discussion of representation spheres in the introduction of [HHR17].

3.2 Background on Mackey functors

In this section, we give background on Mackey functors. The reason for this is that while (nonequivariant) singular homology produces a series of abelian groups, the equivariant case uses multiple abelian groups to keep track of infomation for both the acting group and also all of its subgroups. For this reason, the Bredon homology we compute later will produce a *diagram* of abelian groups rather than just a single abelian group.

The following definition of a Mackey functor is drawn from the one given in [Dre73].

Definition 3.2.1. A **Mackey functor** $\underline{M}$ is a pair of functors $\underline{M} = (\underline{M}_*, \underline{M}^*)$ from the category of finite G-sets to the category of abelian groups with the following properties:

(a) $\underline{M}_*$ is covariant and $\underline{M}^*$ is contravariant.

(b) $\underline{M}_*$ and $\underline{M}^*$ take disjoint unions of G-sets to direct sums of abelian groups.

(c) $\underline{M}_*$ and $\underline{M}^*$ agree on objects. In other words, for any finite G-set A, $\underline{M}_*(A) = \underline{M}^*(A)$.

(d) $\underline{M}$ takes any pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\delta]{} & Y \end{array}$$

to the commutative square

$$\begin{array}{ccc} \underline{M}(A) & \xrightarrow{\alpha_*} & \underline{M}(B) \\ {\scriptstyle\gamma^*}\uparrow & & \uparrow{\scriptstyle\beta^*} \\ \underline{M}(X) & \xrightarrow[\delta_*]{} & \underline{M}(Y) \end{array}$$

where the covariant maps are produced by $\underline{M}_*$ and the contravariant ones by $\underline{M}^*$.

The covariant maps produced by $\underline{M}$ are called **transfer maps** while the contravariant ones are **restriction maps**.

Remark: In this document, Mackey functors will be underlined. When a Mackey functor $\underline{M}$ is evaluated on a G-set B, we will write $\underline{M}(B)$. If the G-set happens to be a single orbit G/H, then we may write $\underline{M}(G/H)$. In some specific places where the notation is particular cumbersome, we may also write $\underline{M}^H$ to mean evaluation of $\underline{M}$ at G/H, and this peculiarity will be mentioned when it arises.

Because Mackey functors satisfy property (b), and G-sets are disjoint unions of orbits of the form G/H for various subgroups H, we may define Mackey functors by discussing their behavior on single orbits G/H rather than more general G-sets. For this reason, when describing a Mackey functor, it will suffice to describe the behavior of the functor on the orbit category, a subcategory of the category of finite G-sets.

Definition 3.2.2. Let G be a finite group. The **orbit category** $\mathcal{O}_G$ is the category whose objects are the G-sets G/H for each subgroup H in G, and whose morphisms are equivariant maps between these G-sets.

Remark: For subgroups K, H of G, there exists a map from G/K to G/H in the orbit category if and only if K is subconjugate to H. To see this, suppose that there exists an equivariant map $f : G/K \to G/H$ such that $f(eK) = gH$. Then for any $k \in K$, we also have

$$gH = f(eK) = f(kK) = kf(K) = kgH$$

Since for any k we have $gH = kgH$, we know $g^{-1}Kg \subseteq H$. For the converse, suppose $g^{-1}Kg \subseteq H$ and take $aK \in G/K$. Then the map given by $aK \mapsto agH$ is indeed an equivariant map $G/K \to G/H$.

The orbit category describes every possible orbit type that can arise from a G-action for a particular group, as well as every restriction and transfer that can arise between G-sets consisting of a single orbit. Larger G-sets consisting of multiple orbits can be formed via disjoint union. Since restricting a Mackey functor to G-sets of the form G/H is sufficient, it is common to visualize a Mackey functor as a structure that resembles a subgroup lattice. This structure is called a Lewis diagram, after those seen in [Lew88]. In the examples below, we will draw Mackey functors as Lewis diagrams.

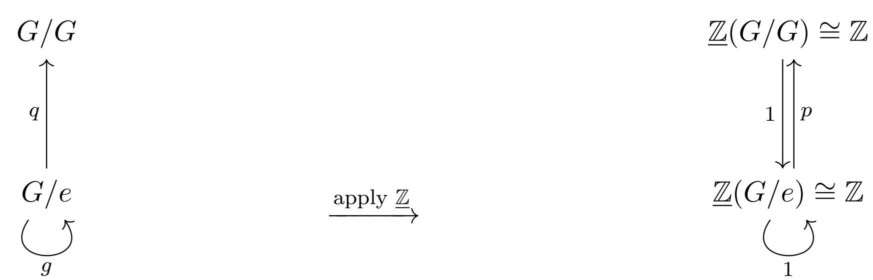
Definition 3.2.3. The **constant Mackey functor**, denoted $\underline{\mathbb{Z}}$ is given by

$$\underline{\mathbb{Z}}(G/H) := \hom^G(G/H, \mathbb{Z}) \cong \mathbb{Z}.$$

For a given map of G-sets $\alpha : G/H \to G/K$, the restriction $\operatorname{res} : \underline{\mathbb{Z}}(G/H) \to \underline{\mathbb{Z}}(G/K)$ is given by precomposition with α while the transfer $\operatorname{tr} : \underline{\mathbb{Z}}(G/K) \to \underline{\mathbb{Z}}(G/H)$ is given by the fibers of each coset of K. Thus for single orbits G/K and G/H, the restriction and transfer are the identity and multiplication by $|K : H|$, as shown below.

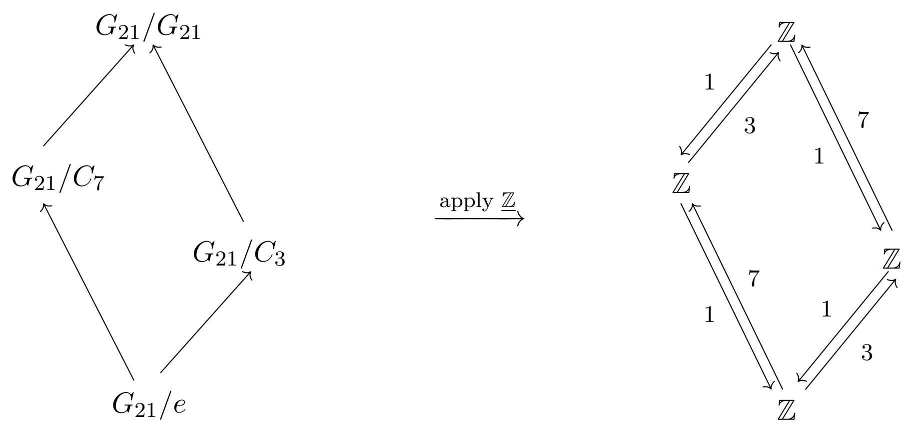
$$\begin{array}{c} \hom^G(G/K, \mathbb{Z}) \cong \mathbb{Z} \\ {\scriptstyle 1}\downarrow \;\; \uparrow{\scriptstyle |K:H|} \\ \hom^G(G/H, \mathbb{Z}) \cong \mathbb{Z} \end{array}$$

Example 3.2.4. Let G be the cyclic group of order p. The orbit category for G is shown on the left. The map q is the quotient map, and every element of G gives an endomorphism on G/e via left multiplication. For simplicity, only one such map (labelled g) is shown. The Mackey functor $\underline{\mathbb{Z}}$ applied to the orbit category is shown on the right. The restriction and transfer arising from q are the identity and multiplication by p, respectively. Note that both the restriction and the transfer arising from left multiplication by $g \in G$ are the identity, so there is only one map from $\underline{\mathbb{Z}}(G/e)$ to itself.



Orbit category of C_p Constant Mackey functor $\underline{\mathbb{Z}}$ for C_p

Example 3.2.5. Let G_{21} be $C_7 \rtimes C_3$, the nonabelian group of order 21. (See Chapter 4 for a closer look at this group). A partial orbit category for G_{21} is shown on the left, and the constant Mackey functor $\underline{\mathbb{Z}}$ for this group is shown on the right.



Definition 3.2.6. Let B be a finite G-set. The **permutation Mackey functor**, denoted $\underline{\mathbb{Z}}\{B\}$, is given by

$$\underline{\mathbb{Z}}\{B\}(G/H) := \hom^G(G/H, \mathbb{Z}\{B\}).$$

The restriction maps are given by precomposition, while the transfer maps are given by summing over the fibers.

Remark: Notice that if B is the one-point G-set G/G, then $\underline{\mathbb{Z}}\{G/G\} \cong \underline{\mathbb{Z}}$. Thus the constant Mackey functor is a particular case of the more general permutation Mackey functor.

Lastly, we record two definitions and a lemma that will be useful later.

Definition 3.2.7. A (covariant) **coefficient system** $\underline{M}$ for a group G is a functor from the orbit category $\mathcal{O}_G$ to the category of abelian groups.

Remark: A Mackey functor can be thought of as two coefficient systems, one covariant and one contravariant, satisfying some additional conditions. Because of this close relationship between the two, we will use underlined symbols to denote both Mackey functors and coefficient systems, and we will be more specific if there is ambiguity.

Definition 3.2.8. A Mackey functor $\underline{M}$ is **cohomological** if evaluating $\underline{M}$ on the map of G-sets $G/K \xrightarrow{f} G/H$ produces restriction and transfer maps f^* and f_* that satisfy

$$f_* \circ f^* = |K : H|$$

In other words, the composition is multiplication by the index of K in H.

Lemma 3.2.9. *Any subfunctor of a cohomological Mackey functor is cohomological. Any quotient of a cohomological Mackey functor is cohomological in the sense that given the following short exact sequence of Mackey functors*

$$\begin{array}{ccccc} \underline{M} & & \underline{M}' & & \underline{M}'/\underline{M} \\ A & \hookrightarrow & A' & \longrightarrow & A'/A \\ \text{res}\downarrow\uparrow\text{tr} & & \text{res}\downarrow\uparrow\text{tr} & & \text{res}'\downarrow\uparrow\text{tr}' \\ B & \hookrightarrow & B' & \longrightarrow & B'/B \end{array}$$

where $\underline{M}'$ is cohomological, we know $\underline{M}'/\underline{M}$ is also cohomological.

Proof. The morphisms of a subfunctor are unchanged except that the domains are restricted. For quotients, the composition $\text{tr}' \circ \text{res}'$ is determined by the effect of $\text{tr} \circ \text{res}$ on any choice of coset representative, so the composition $\text{tr}' \circ \text{res}'$ is also multiplication by the index. □

3.3 Background on Bredon homology

In Chapter 1, we mentioned that Bredon homology is an equivariant homology theory that has many of the desirable qualities of a homology theory, but is difficult to compute. For one, the dimension axiom only guarantees that the *integer-graded* part of the equivariant homology of a point is zero, while the homology is often nontrivial on non-integer parts of the larger ring of (virtual) representations $RO(G)$.

Despite the fact that $RO(G)$-graded homology is difficult to compute, it does have properties that ease the diffiulties. Since RO(G)-graded homology has a suspension isomorphism for representation spheres, we know

$$H_V^G(G/H_+) \cong H_{V\oplus W}^G(S^W \wedge G/H_+).$$

This means we can find the homology of a point by investigating the homology of representation spheres instead. This approach will make use of the G-CW cell structures of representation spheres discussed earlier. This perspective has the advantage of closely mirroring the perspective of nonequivariant cellular homology. Drawing from [Wil73] and [Wil75], we will convert G-CW structures into chain complexes of coefficient systems. We will then show that when the coefficient system can be extended to a Mackey functor, this definition is consistent with the Mackey functor version seen in [HHR16].

There are other compatible definitions of Bredon homology as well. To see a presentation using G-spectra, see [HHR16]. To see a presentation in the language of categories that is ultimately equivalent to the cellular perspective we are about to present, see [May96].

Now let us establish some notation and definitions that will be used to build a chain complex for computing Bredon homology. Let X be a G-CW complex and let its n-skeleton be denoted X^n. Since the n-cells of X are n-disks of the form $G/H_+ \wedge D_+^n$ (see Definition 3.1.1), the quotient X^n/X^{n-1} is a wedge of n-spheres. Let T_n be a discrete G-set that indexes the n-cells of X such that $X^n/X^{n-1} \cong (T_n)_+ \wedge S^n$.

Definition 3.3.1. The **cellular n-chains coefficient system** of a G-CW complex X of finite type, denoted $\underline{C}_n^{\text{cell}}(X)$, is a (contravariant) coefficient system whose value at G/K is given by

$$\underline{C}_n^{\text{cell}}(X)(G/K) := H_n\left((X^n)^K, (X^{n-1})^K, \mathbb{Z}\right).$$

Since $X^n/X^{n-1} \cong (T_n)_+ \wedge S^n$, ultimately $\underline{C}_n^{\text{cell}}(X)$ evaluated at G/K is the free abelian group on $(T_n)_+^K$. Given a map $G/K \to G/L$ in the orbit category, the corresponding map is the one described in the remark following definition 3.2.2

The pairs $\left((X^{n-2})^K, (X^{n-1})^K\right)$ and $\left((X^{n-1})^K, (X^n)^K\right)$ have their respective long exact sequences in homology. The two long exact sequences can be patched together at $H_{n-1}\left((X^{n-1})^K\right)$, as shown in the diagram below.

$$\begin{array}{ccccc}
H_{n-1}\left((X^{n-2})^K\right) & & & & \\
& \searrow & & & \\
H_n\left((X^n)^K, (X^{n-1})^K\right) & \xrightarrow{\delta_*} & H_{n-1}\left((X^{n-1})^K\right) & \longrightarrow & H_{n-1}\left((X^n)^K\right) \\
& & & \searrow^{j_*} & \\
& & & & H_{n-1}\left((X^{n-1})^K, (X^{n-2})^K\right)
\end{array}$$

Let $d_n : \underline{C}_n^{\text{cell}}(X)(G/K) \to \underline{C}_{n-1}^{\text{cell}}(X)(G/K)$ be the composition $j_* \circ \delta_*$, and consider the relative homology groups that form the domain and codomain of this map. We have the following equivalences:

$$\begin{aligned}
H_n\left((X^n)^K, (X^{n-1})^K\right) &\cong \widetilde{H}_n\left((T_n)_+^K \wedge S^n;\, \mathbb{Z}\right) \\
&\cong \widetilde{H}_0\left((T_n)_+^K;\, \mathbb{Z}\right) \\
&\cong \mathbb{Z}\left[T_n^K\right] \\
&\cong \mathbb{Z}\left\{\text{Fin}^G(G/K, T_n)\right\},
\end{aligned}$$

where the last line denotes the free abelian group on the set $\text{Fin}^G(G/K, T_n)$, and $\text{Fin}^G(G/K, T_n)$ denotes the morphisms from G/K to T_n in the category of finite G-sets. Notice that as the input of $\text{Fin}^G(-, T_n)$ varies over $\mathcal{O}_G$, the maps d_n form a natural transformation between the following two coefficient systems.

$$d_n : \mathbb{Z}\left\{\text{Fin}^G(-, T_n)\right\} \to \mathbb{Z}\left\{\text{Fin}^G(-, T_{n-1})\right\}$$

In particular, when the input is T_n itself, we have

$$d_n : \mathbb{Z}\left\{\text{Fin}^G(T_n, T_n)\right\} \to \mathbb{Z}\left\{\text{Fin}^G(T_n, T_{n-1})\right\}$$

Let the image of the identity map on T_n be denoted e_n. Since e_n is an element of the free abelian group $\mathbb{Z}\left\{\text{Fin}^G(T_n, T_{n-1})\right\}$, it is of the form $e_n = \sum a_f f$ where the maps f are in $\text{Fin}^G(T_n, T_{n-1})$. Define $\underline{N}(e_n)$ to mean $\underline{N}(e_n) := \sum a_f \underline{N}(f)$. With these definitions in hand, we are ready to describe the chain complex that ultimately computes Bredon homology.

Definition 3.3.2. Let $\underline{N}$ be a covariant coefficient system. Then the **cellular chain complex** $\underline{C}^G_\bullet(X)$ for a G-space X with coefficients in $\underline{N}$ is the chain complex

$$\cdots \longrightarrow \underline{N}(T_n) \xrightarrow{\underline{N}(e_n)} \underline{N}(T_{n-1}) \longrightarrow \cdots$$

Remark: In the next three definitions, the superscripts are used to indicate that $H^K_\bullet(X;\underline{N})$ the result of evaluating a coefficient system at G/K. It is equivalent to the clunkier notation $H_\bullet(X;\underline{N})(G/K)$.

Definition 3.3.3. The **Bredon homology of** X with coefficients in $\underline{N}$, denoted $H^G_\bullet(X;\underline{N})$ is the homology of the cellular chain complex $\underline{C}^G_\bullet(X)$

This completes the definition of Bredon homology when evaluated at G/G. We now extend this definition to the case of evaluation at G/K for a subgroup K, and then provide the maps that link the groups together into a coefficient system.

Definition 3.3.4. Let K be a subgroup of G. The **Bredon homology of** X evaluated at G/K, denoted $H^K_\bullet(X;\underline{N})$, is the homology of the chain complex

$$\cdots \longrightarrow \underline{N}(G/K \times T_n) \xrightarrow{\underline{N}(\mathrm{id}\times e_n)} \underline{N}(G/K \times T_{n-1}) \longrightarrow \cdots$$

where the differential $\underline{N}(\mathrm{id} \times e_n)$ is given by

$$\underline{N}(\mathrm{id} \times e_n) := \sum a_f \underline{N}(\mathrm{id}_{G/K} \times f)$$

Given a map of G-sets $q : G/L \to G/K$, we also have the map of spaces $q \times \mathrm{id}_X : G/L \times X \to G/K \times X$, which induces a map on homology $H^L_\bullet(X;\underline{N}) \to H^K_\bullet(X;\underline{N})$. The groups and maps assemble to form the coefficient system $H\underline{N}_\bullet(X)$.

Definition 3.3.5. The **coefficient system-valued Bredon homology of** X with coefficients in $\underline{N}$, denoted $H\underline{N}_\bullet(X)$, is a coefficient system that takes G/K to

$$H\underline{N}_\bullet(X)(G/K) := H^K_\bullet(X;\underline{N})$$

and takes the map $G/L \to G/K$ to the map on homology induced by $q \times \mathrm{id}_X : G/L \times X \to G/K \times X$.

This completes the definition of integer-graded, coefficient system-valued Bredon homology. As mentioned in the introduction, there is value in extending the grading from $\mathbb{Z}$ to the ring of virtual representations $RO(G)$, and we can find the relevant theorem that describes when it is possible to do so as Theorem 5.2 in [May96].

Theorem 3.3.6 (Lewis–May–McClure [May96]). *Let G be a compact Lie group and let M and N be contravariant and covariant coefficient systems. The ordinary cohomology theory $\widetilde{H}^*_G(-;M)$ extends to an $RO(G)$-graded cohomology theory if and only if M extends to a Mackey functor. The ordinary homology theory $\widetilde{H}^G_*(-;N)$ extends to an $RO(G)$-graded homology theory if and only if N extends to a coMackey functor.*

The constant $\underline{\mathbb{Z}}$ coefficients used in the remainder of this document do indeed extend to a Mackey functor, which means that suspension by representation spheres is sensible. Thus we can compute the $RO(G)$-graded homology of a point by investigating the homology of representation spheres instead. This is the main focus of the rest of this document.

3.4 Bredon homology with constant $\underline{\mathbb{Z}}$ coefficients

For constant $\underline{\mathbb{Z}}$ coefficients, the above chain complex has an alternate presentation. In this section we'll show that when $\underline{N} = \underline{\mathbb{Z}}$, the groups $\underline{N}(G/K \times T_n)$ in the chain complex for computing Bredon homology becomes $\mathbb{Z}\{T_n\}^K$, the K-fixed points of the free abelian group on T_n. We'll also compute an example.

Lemma 3.4.1. *The Bredon homology of X evaluated at G/K using constant $\underline{\mathbb{Z}}$ coefficients, denoted $H\underline{\mathbb{Z}}_\bullet(X)(G/K)$ is the homology of the chain complex*

$$\cdots \longrightarrow \mathbb{Z}\{T_n\}^K \xrightarrow{e_n} \mathbb{Z}\{T_{n-1}\}^K \longrightarrow \cdots$$

where the differential e_n is the map of the same name described in Section 3.3, extended linearly to free abelian groups.

Proof. Beginning with the definition of Bredon homology in Definition 3.3.4, we have the following isomorphisms.

$$\begin{aligned}
\underline{C}_n(X;\mathbb{Z})(G/K) &:= \underline{\mathbb{Z}}(G/K \times T_n) \\
&\cong \hom^G(G/K \times T_n, \mathbb{Z}) \\
&\cong \hom^G(G/K, \hom(T_n, \mathbb{Z})) && (3.1)\\
&\cong \hom^G(G/K, \mathbb{Z}\{T_n\}) && (3.2)\\
&\cong \mathbb{Z}\{T_n\}^K && (3.3)
\end{aligned}$$

The isomorphism in line 3.1 is given by assigning $\varphi(s,t) \in \hom^G(G/K \times T_n, \mathbb{Z})$ to $\widetilde{\varphi} : G/K \to \hom(T_n, \mathbb{Z})$ such that for $gK \in G/K$, we have $\widetilde{\varphi}(gK)(t) = \varphi(gK, t)$. Line 3.2 holds because a (nonequivariant) set map $\sigma : T_n \to \mathbb{Z}$ gives an element of $\mathbb{Z}\{T_n\}$ via $\sum_{t \in T_n} \sigma(t) \cdot t$. Lastly, the isomorphism in line 3.3 is given by assigning $\psi : G/K \to \mathbb{Z}\{T_n\}$ to $\psi(eK)$, the image of the identity coset, which is indeed a K-fixed point in $\mathbb{Z}\{T_n\}$. □

Remark: The form of Bredon homology given in Lemma 3.4.1 is consistent with the definition of Bredon homology found in [HHR16]. This can be seen most clearly in Section 3.3, where the authors discuss the special case of the constant $\underline{\mathbb{Z}}$ Mackey functor.

Remark: Bredon homology produces cohomological Mackey functors, and in the case of $\mathbb{Z}$ coefficients, we can see that this is true because of Lemma 3.2.9. The Mackey functors of the chain complex are given by $\mathbb{Z}$ so they are cohomological. Then passing to homology involves taking subgroups and quotients so the output Mackey functor is also cohomological.

We use Lemma 3.4.1 to compute an example of Bredon homology using constant $\underline{\mathbb{Z}}$ coefficients.

Example 3.4.2. Let $G = C_3$ and let X be the representation sphere S^λ with G-CW structure given in Example 3.1.3. In this cell structure, the G-sets T_n are empty except in dimensions 0, 1, and 2, and they are

$$T_0 \cong G/G_+ \cong \{0, \infty\} \qquad T_1 \cong G/e \cong \{e, \tau, \tau^2\} \qquad T_2 \cong G/e \cong \{e, \tau, \tau^2\}$$

Apply Lemma 3.4.1 to obtain a chain complex. Since the subgroups of G are only itself and the trivial subgroup, its orbit category has two "levels", which will be apparent in the chain complex below. Note that the bottom level is the chain complex for the non-equivariant sphere S^2 while the top level is the C_3-fixed points of the bottom level. The diagonal map Δ is given by $e+\tau+\tau^2 \to e+\tau+\tau^2$, while the fold map ∇ takes generators e, τ, τ^2 to $e+\tau+\tau^2$. The augmentation map ε maps each generator to 1.

$$\begin{array}{ccccccc}
n=2 & & n=1 & & n=0 & & \\
\mathbb{Z}[e+\tau+\tau^2] & \xrightarrow{\tau - e} & \mathbb{Z}[e+\tau+\tau^2] & \xrightarrow{\infty - 0} & \mathbb{Z}[0,\infty] & \xrightarrow{\varepsilon} & \mathbb{Z} \\
\Delta \downarrow \uparrow \nabla & & \Delta \downarrow \uparrow \nabla & & 1 \downarrow \uparrow 1 & & \\
\mathbb{Z}[e,\tau,\tau^2] & \xrightarrow{\tau - e} & \mathbb{Z}[e,\tau,\tau^2] & \xrightarrow{\infty - 0} & \mathbb{Z}[0,\infty] & \xrightarrow{\varepsilon} & \mathbb{Z}
\end{array}$$

Then we pass to homology to produce the final result $H\underline{\mathbb{Z}}_n(X;\underline{\mathbb{Z}})$ for $n \in \mathbb{Z}$. The maps in the final result are the induced maps on homology arising from the restrictions and transfers of the chain complex. Multiplication-by-zero maps have been suppressed.

$$\begin{array}{ccc}
H\underline{\mathbb{Z}}_2(X) & H\underline{\mathbb{Z}}_1(X) & H\underline{\mathbb{Z}}_0(X) \\
\mathbb{Z} & 0 & \mathbb{Z}/3 \\
1 \downarrow \uparrow 3 & \downarrow \uparrow & \downarrow \uparrow \\
\mathbb{Z} & 0 & 0
\end{array}$$

Finally, we record a fact about Bredon homology that will be useful in Chapter 6. This lemma will allow us to evaluate $H\underline{\mathbb{Z}}$ on proper subgroups more easily.

Lemma 3.4.3. *Let X be a G-space. Then*

$$H\underline{\mathbb{Z}}_n(X)(G/H) \cong H\underline{\mathbb{Z}}_n(i_H^* X)(H/H)$$

where $H\underline{\mathbb{Z}}$ on the left is a G-spectrum while $H\underline{\mathbb{Z}}$ on the right is the H-spectrum $i_H^* H\underline{\mathbb{Z}}$.

Proof. $H\underline{\mathbb{Z}}_n^H(X) \cong [S^n, H\underline{\mathbb{Z}} \wedge X \wedge G/H_+]^G \cong [G/H_+ \wedge S^n, H\underline{\mathbb{Z}} \wedge X]^G \cong [S^n, i_H^*(H\underline{\mathbb{Z}} \wedge X)]^H \cong H\underline{\mathbb{Z}}_n^H(i_H^* X)$ □

In Chapter 1 we mentioned that the RO(G)-graded homology of a point for a generic group was unknown and often difficult to compute. Yet we were able to compute homology groups in Example 3.4.2 fairly easily. This is because the example happens to be highly tractable - the group in question has an uncomplicated orbit category, the representation sphere that we looked at was low-dimensional, and the G-CW structure was small. For cases where our representation spheres have higher dimensions or our groups have other complications, the chain complex given in Definition 3.4.1 will be more difficult to obtain. We will need to leverage other methods when our spheres and groups become large and burdensome. This will be addressed in Chapter 7 when the issue arises.

Chapter 4

About the group $C_7 \rtimes C_3$

In this chapter we will introduce the family of nonabelian groups of order pq for primes p and q. Then we will restrict our attention to the group of order 21 and record several useful facts about this particular group, including its ring of real representations and its group homology.

4.1 Nonabelian groups of order pq

Let p, q both be primes, with $p > q$ and $p \equiv 1 \pmod{q}$. Let C_n be the cyclic group of order n and let the binary operation of C_n be written as multiplication. Then there is one nonabelian group of order pq up to isomorphism, and it is $C_p \rtimes C_q$.

More explicitly, let τ and σ be generators of C_q and C_p, respectively. Then the elements of $C_p \rtimes C_q$ are of the form $\sigma^s \tau^t$ and multiplication is subject to the relation $\tau^{-1}\sigma\tau = \sigma^k$ where σ^k is a fixed nonidentity element of C_p satisfying $k^q \equiv 1 \pmod{q}$. If there is more than one value of k satisfying $k^q \equiv 1 \pmod{q}$, we may choose any of them and the resulting group $= C_p \rtimes C_q$ will be unique, up to isomorphism.

The nonabelian group of order pq has $p+3$ subgroups. There is one subgroup isomorphic to C_p, and this subgroup is normal. There are p nontrivial, non-normal subgroups isomorphic to C_q, and they are conjugate to each other. Below is a subgroup diagram for $C_p \rtimes C_q$ and the corresponding orbit category. Since there are p subgroups isomorphic to C_q, all but one of them is suppressed for simplicity.

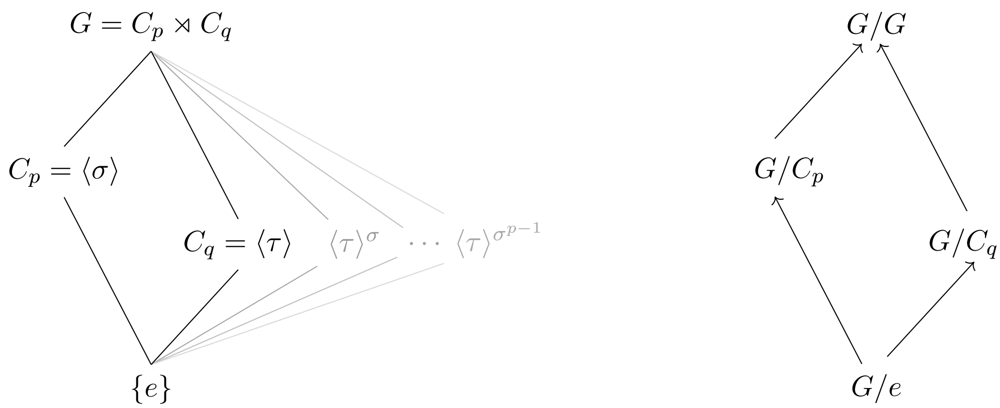


Figure 4.1: On the left, the subgroup lattice of $C_p \rtimes C_q$. On the right, the orbit category of $C_p \rtimes C_q$ with conjugate G/C_q suppressed

4.2 Irreducible representations of the nonabelian group of order 21

We now restrict our attention to the case where $p = 7$ and $q = 3$, and let G_{21} denote this nonabelian group of order 21. We choose the following presentation: let σ and τ be the generators of C_7 and C_3, respectively, and let $\tau\sigma = \sigma^2\tau$.

Since the subgroups of G_{21} are finite cyclic groups, we may use the method of "little groups" as described in [Ser77] Section 8.2. Applying this construction to G_{21} produces five irreducible complex representations. There are two degree-1 representations given by composing the projection $G_{21} \to C_3$ with the two nontrivial complex representations of C_3 (see Example 2.1.6). There are also two degree-3 representations given by inducing (see Definition 2.1.4) two degree-1 representations of C_7, namely $\sigma \mapsto \exp(i2\pi/7)$ and $\sigma \mapsto \exp(i2\pi 3/7)$. And finally there is the trivial representation.

The characters of these five complex representations are given below

	χ_0	χ_1	χ_2	χ_3	χ_4
1	1	1	1	3	3
σ	1	$\exp(i2\pi/3)$	$\exp(i4\pi/3)$	0	0
σ^3	1	$\exp(i4\pi/3)$	$\exp(i2\pi/3)$	0	0
τ	1	1	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$
τ^2	1	1	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$

Ultimately we will be interested in the ring $RO(G_{21})$, so if we observe that the $\Gamma_{\mathbb{R}}$-class of G_{21} partition the group into the sets $\{e\}$, $\{\sigma^i\}_{i\in\mathbb{Z}/7}$, and $\{\tau\sigma^i, \tau^2\sigma^i\}_{i\in\mathbb{Z}/7}$, then by Lemma (2.2.5) we must have three irreducible real representations of G.

Here is the character table for the three real irreducible representations of G_{21}. The elements on the left side of the table are representatives of their respective conjugacy classes, the three columns are characters for the trivial representation, the degree-2 representation λ, and the degree-6 representation Λ, respectively.

	χ_0	χ_λ	χ_Λ
e	1	2	6
σ	1	2	-1
$\tau\sigma$	1	-1	0

Figure 4.2: Real character table for $C_7 \rtimes C_3$. The three columns are characters for the trivial representation, the representation created via restriction, and the representation created via induction, respectively.

Using these three irreducible representations as the basis for $RO(G_{21})$, we can compute ring relations for $RO(G_{21})$ from the character table for G_{21}.

$$\lambda \cdot \lambda = \lambda + 2$$

$$\Lambda \cdot \Lambda = 2 + 2\,\lambda + 5\,\Lambda$$

$$\lambda \cdot \Lambda = 2\,\Lambda$$

Lemma 4.2.1. *The ring $RO(G_{21})$ is a quotient of the free abelian group on λ and Λ by three polynomial relations. Explicitly,*

$$RO(G_{21}) \cong \mathbb{Z}[\lambda, \Lambda]/\{\lambda^2 - \lambda - 2, \Lambda^2 - 5\Lambda - 2\lambda - 2, \Lambda\lambda - 2\Lambda\}$$

Proof. By Lemma (2.3.4), the relations between representations in $RO(G_{21})$ will be the same as the relations between irreducible characters of G_{21}, so the computations performed with characters above produce a full description of $RO(G_{21})$. □

We finish this subsection by defining an orientable representation. This will be used in the computations of Chapter 7

Definition 4.2.2. Let $\varphi : G \to \mathrm{O}(V)$ be a real orthogonal representation of G. If the determinant of $\varphi(g) = 1$ for all g in G, then we say that the representation φ is **orientable**.

Example 4.2.3. Let G be a group of odd order. Then every representation of G is orientable. To see this, note that for a representation φ, the composition $G \to \mathrm{O}(n) \to \{\pm 1\}$ that maps $g \in G$ to the determinant of $\varphi(g)$ is a group homomorphism. If this composition were surjective, then the kernel must have order $|G|/2$. Since G is odd, we know the composition maps every element of G to 1.

4.3 Representation spheres of G_{21}

In section 3.3, we mentioned that suspension isomorphism for $RO(G)$-graded homology uses representation spheres. Let us now consider the representation spheres for the group G_{21}.

Let λ and Λ be the representations of G_{21} whose characters are given in figure 4.2. First, the representation λ has a representation sphere S^{λ} with real dimension 2. The representation λ gives a G_{21}-action on S^{λ} where the generator τ of C_3 acts by rotating S^2 by an angle of $2\pi/3$ while the generator σ of C_7 acts trivially. The representation sphere S^{λ} and its G-CW structure can be seen in Example 3.1.3.

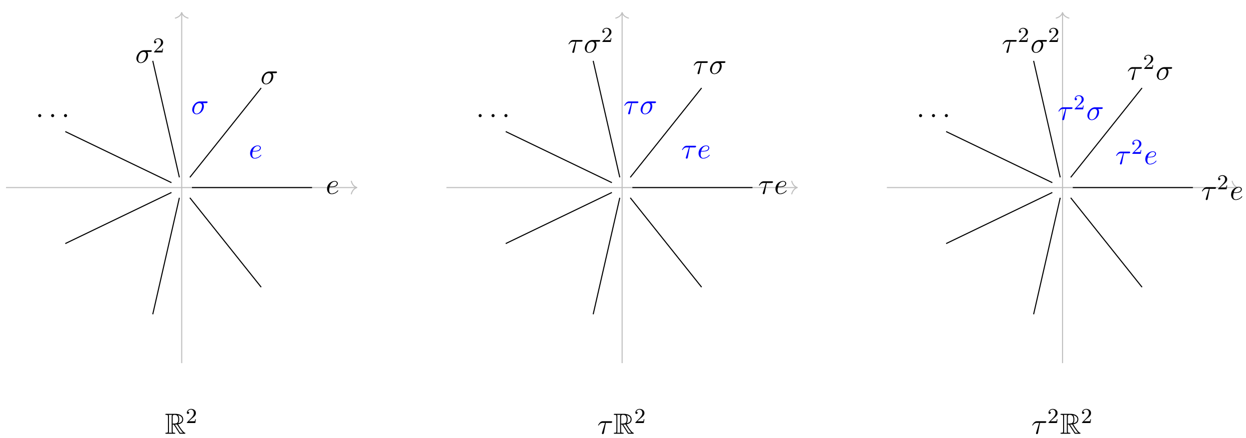
Second, the representation Λ has a representation sphere S^{Λ} with real dimension 6. Recall that this representation was formed via the method of "little groups" described in [Ser77] Section 8.2. In this method, we used a degree-2 representation of the subgroup C_7 to induce a representation of G_{21}. The underlying vector space of the induced representation is the direct sum of $|G_{21} : C_7| = 3$ copies of $\mathbb{R}^2$, and G_{21} acts on the resulting vector space $\mathbb{R}^2 \oplus \tau\mathbb{R}^2 \oplus \tau^2\mathbb{R}^2$ using the action given in the definition of an induced representation (See definition 2.1.4). Then the one-point compactification of the induced representation produces S^{Λ}. The sphere is the smash product of three copies S^2, and the action of G_{21} is given by 2.1.

It is possible to find an explicit cell structure for S^{Λ} as well. Since Λ was formed via induction from a degree-2 representation of C_7 (call this λ_7), we may be tempted to believe that we can take a G-CW cell structure on λ_7 and making an "induced cell structure" on Λ. In Example 3.1.4, we saw that it is sometimes possible to obtain a cell structure for a smash product by taking the product of cells. But the cell structure produced by smashing together copies of λ_7 suffers some flaws. The discussion of these flaws and methods for correcting them is the content of Chapter 5

Even though we will not give an explicit G-CW complex structure for S^{Λ} in this section, the sphere S^{Λ} contains an important subcomplex, which we will describe now and use later in Chapter 7. We will construct a subcomplex $Y \subseteq S^{\Lambda}$ that differs from S^{Λ} by cells of the form $G_+ \wedge D^k_+$ for some non-negative integer k. In other words, we will have a subcomplex Y such that $S^{\Lambda} = Y \cup (\bigcup G_+ \wedge D^{d_i}_+)$ for disks of various dimensions d_i.

Let a generic point in $\Lambda = \mathbb{R}^2 \oplus \tau\mathbb{R}^2 \oplus \tau^2\mathbb{R}^2$ be denoted (u, v, w) where u, v, w are vectors in $\mathbb{R}^2$. We let the origin $(0, 0, 0)$ be denoted by 0, and within each copy of $\mathbb{R}^2$, identify the seventh roots of unity with seven evenly-spaced vectors in $\mathbb{R}^2$. These vectors are denoted with a power of τ to indicate which copy of $\mathbb{R}^2$ they hail from, and then are labelled $e, \sigma, \sigma^2, \cdots \sigma^6$ (see drawing below). Using the notational convention described in 3.1.2, extend each vector to the point at infinity to form a 1-cell the begins at 0 and ends at ∞, and let this 1-cell share the name of the vector that it contains.

Thus each copy of $\mathbb{R}^2$ is divided into seven equal "panels," shown below. Let each panel be named by the 1-cell found on its counterclockwise boundary.



For any subgroup $H \subseteq G$, we can find points with orbit type G/H by looking for points fixed by H. The only point fixed by G_{21} is 0, and the same is true for the subgroup C_7. On the other hand, the subgroup $C_3 = \langle \tau \rangle$ fixes all points of the form (v, v, v), and these points form a subspace of dimension 2 in Λ. Similarly, each conjugate subgroup $\langle \sigma^{-i} \tau \sigma^i \rangle$ fixes a 2-dimensional subspace given by $(\sigma^{-i} v, \sigma^{-4i} v, \sigma^{-2i} v)$ for all $v \in \mathbb{R}^2$. So the set of points of Λ that are fixed by some subgroup of G_{21} (excluding the trivial subgroup that fixes every point) comprises seven 2-planes that intersect only at 0. Taking the one-point compatification of Λ to produce the representation sphere S^Λ turns these seven planes into seven 2-spheres, all joined together at 0 and also at ∞. Let this subspace be denoted Y. Here is one possible G-CW structure on Y that elevates it from just a subspace to a subcomplex of S^Λ.

- one fixed 0-cells and a basepoint at ∞: $G/G_+ \wedge (0, 0, 0)$

- a single 1-cell: $G/C_{3+} \wedge (e, e, e)$, where e is the 1-cell containing $(1, 0)$
- a single 2-cell: $G/C_{3+} \wedge (v, v, v)$

Note that since the subcomplex Y was built from points of Λ whose stabilizer subgroup in G_{21} is nontrivial, the remaining points of S^Λ must be fixed by only the identity of G_{21}, and so we know that this subcomplex Y satisfies $S^\Lambda = Y \cup (\bigcup G_+ \wedge e_+)$.

4.4 Group homology of G_{21}

We will now record the group homology of G_{21} here. This computation will be used later in Lemma 7.2.1 and elsewhere.

Recall that for a Serre fibration $f : X \to B$ and its fiber F, the Serre spectral sequence gives the following relationship among the homologies of the three spaces:

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(X).$$

Analogously, for a group quotient $\pi : G \to G/H$ and its normal subgroup H, the Lyndon-Hochschild-Serre spectral sequence gives a relationship between the homologies of the three groups. The statement of the spectral sequence below is quoted from [Wei94].

Lemma 4.4.1 (Weibel 6.8.2)**.** *For every normal subgroup H of a group G, there are two convergent first quadrant spectral sequences:*

$$E^2_{pq} = H_p(G/H; H_q(H; A)) \Rightarrow H_{p+q}(G; A);$$
$$E_2^{pq} = H^p(G/H; H^q(H; A)) \Rightarrow H^{p+q}(G; A).$$

Remark: There is an action of G/H on the (co)homology of H that must be accounted for. We detail the origin of this action for the homology case here. Let $F_\bullet \to \mathbb{Z} \to 0$ be a right, free, $\mathbb{Z}G$ resolution of $\mathbb{Z}$. Observe that $F_\bullet$ and A are right and left $\mathbb{Z}H$-modules via restriction, respectively. Let each $g \in G$ define a map $g : F_\bullet \otimes_{\mathbb{Z}H} A \to F_\bullet \otimes_{\mathbb{Z}H} A$ given by $x \otimes a \mapsto xg^{-1} \otimes ga$. Notice that the map g is the identity map for all $g \in H$. Also, the map g commutes with the differentials of $F_\bullet \otimes_{\mathbb{Z}H} A$. Combining these last two observations, g induces an action on the homology of H and the action is trivial when $g \in H$, so we have an induced action of G/H on $H_\bullet(H, A)$.

Let us use the homological spectral sequence to compute the group homology $H_*(G_{21}, \mathbb{Z})$. For our group of interest, we have

$$E^2_{pq} = H_p(G_{21}/C_7; H_q(C_7; \mathbb{Z})) \Rightarrow H_{p+q}(G_{21}; \mathbb{Z}).$$

Notice that the homology groups of C_7 serve as the coefficients in $H_p(G_{21}/C_7; H_q(C_7; \mathbb{Z}))$, so we begin by stating $H_q(C_7; \mathbb{Z})$, which is a well-known result (one place it can be found is as Theorem 9.48 in [Rot09]).

$$H_q(C_7, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/7\mathbb{Z} & q > 0,\ q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Combining the information about the homology of C_7 and the action of G_{21}/C_7, we can fill in the E^2 page of the spectral sequence as follows.

q						
$\vdots$						
$\mathbb{Z}/7\mathbb{Z}$	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	
$\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\cdots$
						p

Since the nonzero entries are very sparse, the spectral sequence collapses right away and we arrive at the group homology of G.

$$H_n(G_{21}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/3\mathbb{Z} & n > 0,\ n \equiv 1, 3 \pmod 6 \\ \mathbb{Z}/21\mathbb{Z} & n > 0,\ n \equiv 5 \pmod 6 \\ 0 & \text{otherwise} \end{cases}$$

Chapter 5

Cell structure on S^Λ

In this chapter, we present an explicit cell structure for the representation sphere S^Λ, where Λ is the 6-dimensional real representation of G_{21} whose character is given in table 4.2. The method used for generating this explicit G-CW cell structure is also suitable for dealing with cell structures for other representation sphere as well, so we will present more general examples at the end.

In section 4.2, we saw that the representation Λ is induced from a degree-2 representation λ_7 of its subgroup C_7. We might hope that a cell structure on λ would induce a corresponding cell structure on Λ. In the non-equivariant setting, this turns out to be true. Let X be a space (not a G-space) with a CW-cell structure composed of cells d_X and attaching maps φ_X, and let it also be so for space Y, d_Y, and φ_Y. Then the cells of $X \times Y$ are indeed all the pairwise Cartesian products $d_X \times d_Y$ with attaching maps $\varphi_X \times \varphi_Y$.

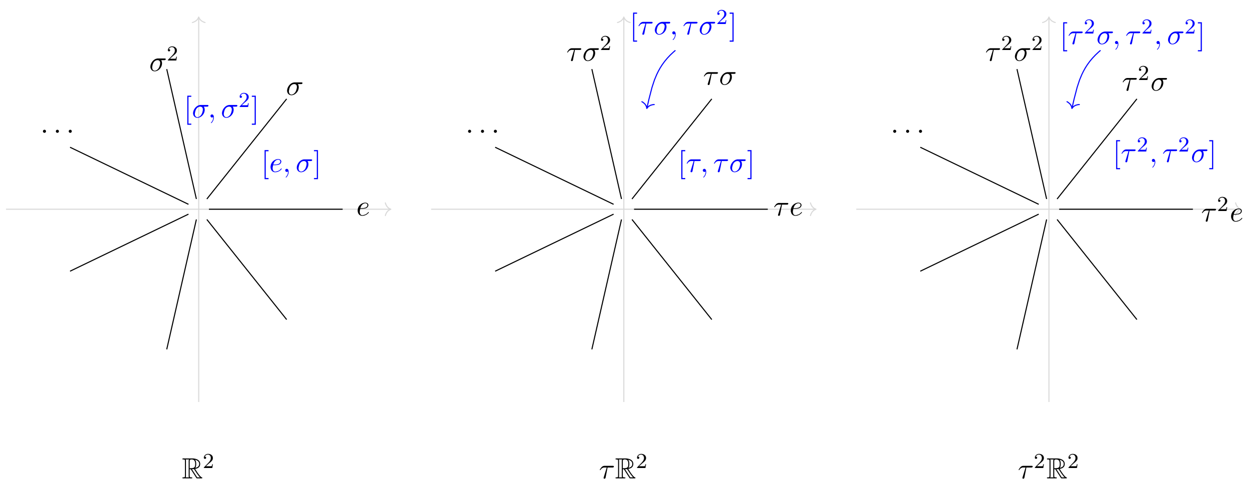
This process is less simple when X and Y are G-spaces, so let us examine a complication that arises when trying to replicate the non-equivariant case. Let λ_7 denote the degree-2 representation of $C_7 = \langle \sigma \rangle$ where σ rotates $\mathbb{R}^2$ by $2\pi/7$. Then using example 3.1.3 (which gives a G-CW structure for the C_3 analogue) as a model, one possible cell structure for λ_7 consists of these cells.

- a fixed 0-cell plus basepoint at ∞: $G/G_+ \wedge 0$
- a single 1-cell: $G/e_+ \wedge [e]$
- a single 2-cell: $G/e_+ \wedge [e, \sigma]$, where $[e, \sigma]$ is the 2-dimensional panel between the 1-cells $[e]$ and $[\sigma]$

Remark: Remember, the symbol σ is somewhat abusive as it is used to denote both the unit vector $\langle \cos 2\pi i/7, \sin 2\pi i/7\rangle$ in $\mathbb{R}^2$ as well as the 1-cell of λ_7 (or S^{λ}_7) with endpoints 0 and ∞ that contains said unit vector. We accept this conflation because we are never interested in the unit vector itself except to identify the ray that contains the unit vector.

Next we take inspiration from the non-equivariant case and form n-cells for Λ by taking products of cells (one from each copy of λ) with dimension summing to n. This naïve attempt will ultimately require corrections, and we will address those issues in Section 5.1. For now we detail the formation of these product cells.

Let τ and σ be fixed generators of subgroups isomorphic to C_3 and C_7, respectively. Let the three copies of λ that form Λ be indexed by the set $\{e, \tau, \tau^2\}$. These copies are drawn below along with labels that show the "seven-paneled" cell structure on λ_7.



Even though this drawing is the same as the one in section 4.3, the naming convention for cells is slightly different. Here we are referring to cells of dimension 2 or greater by listing the 1-cells that form its edges, e.g. $[\sigma, \sigma^2]$ is the 2-cell bordered by σ and σ^2. We will need this expanded notation to describe the product cells.

The cells in the product cell structure will range in dimension from 0 to 6. We will denote an n-cell in the product cell structure by listing the cells in each of the three copies of λ_7 whose product forms that n-cell. Since λ_7 only contains cells up to dimension 2 and its 2-cells are described by listing consecutive 1-cells, these restrictions must be respected when forming product cells. We give examples of cells in the product cell structure to illustrate their notation.

(a) The product of the 2-cell $[\sigma, \sigma^2]$ in λ, the 1-cell $[\tau\sigma^3]$ in $\tau\lambda$, and the 1-cell $[\tau^2]$ in $\tau^2\lambda$ is the 4-cell $[\sigma, \sigma^2, \tau\sigma^3, \tau^2]$ in Λ. This 4-cell is a 4-dimensional region of $\mathbb{R}^6$ that consists of all $\mathbb{R}_{\geq 0}$-linear combinations of the four listed vectors. This region in $\mathbb{R}^6$ becomes a 4-cell of S^Λ through one-point compactification of $\mathbb{R}^6$.

(b) $[\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}]$ for $i, j, k \in \mathbb{Z}/7\mathbb{Z}$ denotes a six-cell in the product cell structure of S^Λ. It was formed as the product of three pairs of 2-cells, one from each copy of S^λ. As in the previous example, this cell is a region of $\mathbb{R}^6$ consisting of $\mathbb{R}_{\geq 0}$-linear combinations of the listed vectors, and it becomes a cell of S^Λ through the one-point compactification of $\mathbb{R}^6$. It was formed as the product of two 2-cells from λ and $\tau\lambda$ and one 1-cell from $\tau^2\lambda$.

(c) $[\sigma, \tau\sigma^3, \tau^2]$ denotes a three-cell in the product cell structure of S^Λ. It was formed as the product of 3 one-cells, one from each copy of S^λ.

(d) The following is **not** a cell in the product cell structure: $[\sigma, \sigma^2, \sigma^3, \tau\sigma^5]$. There is no cell of the form $[\sigma, \sigma^2, \sigma^3]$ in λ because λ does not contain any three-cells.

The product cell structure on S^Λ contains $7^3 = 343$ 6-cells because there are seven 2-cells in each of the three copies of λ. The number of 5-cells in the product cell structure is $(3)7^3 = 1029$ because a 5-cell can only be formed from the product of a two 2-cells and one 1-cell, one from each of the three copies of λ and each with seven choices. Here is a list of how many cells of each dimension exist in the product cell structure.

Dimension	Cells	
6	$7^3 =$	343
5	$(3)7^3 =$	1029
4	$(3)7^2 + (3)7^3 =$	1176
3	$(6)7^2 + 7^3 =$	637
2	$(3)7 + (3)7^2 =$	168
1		21
0		2

Under the action of G, these cells are partitioned into orbits. The following list is a transversal, i.e. a complete list of orbit representatives, one for each of the 21 orbits formed by the action of G on 6-cells:

$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$$

$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$$

$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i+1}, \tau^2\sigma^{i+2}]$$

where $i \in \mathbb{Z}/7$. The following lemma shows that the list is indeed a transversal.

Lemma 5.0.1. *Let C be a generic 6-cell of the product cell structure. It is of the form*

$$C = [\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}]$$

Let I_C be the following set of elements of $\mathbb{Z}/7$:

$$I_C = \{k - 2j, j - 2i, i - 2k\}.$$

Then 6-cells C and D have the same orbit if and only if $I_C = I_D$.

Proof. First suppose that cells C, D are in the same orbit. Then to show that $I_C = I_D$, it suffices to show that the generators τ and σ of G preserve I_C when acting on C. The generator τ of C_3

acts on C in the following way.

$$\begin{aligned}\tau C &= \tau[\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}] \\ &= [\tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^j, \tau^2\sigma^{j+1}, \sigma^k, \sigma^{k+1}] \\ &= [\sigma^k, \sigma^{k+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^j, \tau^2\sigma^{j+1}].\end{aligned}$$

Since the elements of I_C are symmetric with respect to cyclic permutations, we know $I_C = I_{\tau C}$. The generator σ of C_7 acts on C in the following way.

$$\begin{aligned}\sigma C &= \sigma[\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}] \\ &= [\sigma^{i+1}, \sigma^{i+2}, \tau\sigma^{j+2}, \tau\sigma^{j+3}, \tau^2\sigma^{k+4}, \tau^2\sigma^{k+5}]\end{aligned}$$

That action of σ does not alter I_C, as seen below.

$$\begin{aligned}I_{\sigma C} &= \{(k+4) - 2(j+2), (j+2) - 2(i+1), (i+1) - 2(k+4)\} \\ &= \{k - 2j, j - 2i, i - 2k - 7\} \\ &= \{k - 2j, j - 2i, i - 2k\} \\ &= I_C\end{aligned}$$

For the other direction, suppose C and D are 6-cells such that $I_C = I_D$ and denote the cells by

$$\begin{aligned}C &= [\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}] \\ D &= [\sigma^r, \sigma^{r+1}, \tau\sigma^s, \tau\sigma^{s+1}, \tau^2\sigma^t, \tau^2\sigma^{t+1}].\end{aligned}$$

Since r, s, t exhibit symmetry under the action of τ, we may assume without loss of generality that

$$k - 2j = t - 2s \qquad j - 2i = s - 2r \qquad i - 2k = r - 2t.$$

These equations imply

$$r = i \qquad s = j - 2i + 2r \qquad t = k + 4r - 4i$$

and so

$$\begin{aligned}\sigma^{r-i}C &= \sigma^{r-i}[\sigma^i, \sigma^{i+1}, \tau\sigma^j, \tau\sigma^{j+1}, \tau^2\sigma^k, \tau^2\sigma^{k+1}] \\ &= [\sigma^r, \sigma^{r+1}, \tau\sigma^{j+2r-2i}, \tau\sigma^{j+2r-2i+1}, \tau^2\sigma^{k+4r-4i}, \tau^2\sigma^{k+4r-4i+1}] \\ &= [\sigma^r, \sigma^{r+1}, \tau\sigma^s, \tau\sigma^{s+1}, \tau^2\sigma^t, \tau^2\sigma^{t+1}] \\ &= D\end{aligned}$$

Thus we see C and D have the same orbit. □

The above lemma allows us to determine that the $7^3 = 343$ cells of dimension 6 are partitioned into twenty-one orbits, with seven orbits of size 7 and fourteen orbits of size 21.

5.1 Method of subdivision

We will now discuss why some cells in the product cell structure fail to be G-cells. Since a typical G-cell is of the form $G/H \times D^n$ for some subgroup $H \leq G$, every point in the interior of D^n should have an orbit of size $|G/H|$. The following example demonstrates that some cells in the product cell structure fail to have this property.

Example 5.1.1. Consider the 3-cell from the product cell structure $[\sigma, \tau\sigma, \tau^2\sigma]$. A point in this cell is a positive linear combination of the vectors associated to each 1-cell, so a point is given by

$$\alpha(\sigma) + \beta(\tau\sigma) + \gamma(\tau^2\sigma) \text{ for } \alpha, \beta, \gamma \in \mathbb{R}_{\geq 0}$$

A point in the interior of the cell is a point with $\alpha, \beta, \gamma \neq 0$. First consider a point with α, β, and γ not all equal. Then under the action of G, this point generates an orbit of size 21. Now let $\alpha = \beta = \gamma$, and notice that this point is fixed under the action of C_3 and only generates an orbit of size 7. So different points in the interior have different isotropy, and this 3-cell in the product cell structure fails to be a G-cell.

We introduce vocabulary for specifying how product cells can fail to be G-cells

Definition 5.1.2. A **k-face** of an n-cell is a k-dimensional cell given by positive linear combinations of k of the n 1-cells that generate the n-cell.

Definition 5.1.3. A cell with **non-uniform isotropy** is a non-equivariant cell in the product cell structure whose interior points do not all have the same isotropy group.

In the naïve product cell structure for S^{Λ}, cells with non-uniform isotropy are those formed by 1-cells that comprise a complete orbit. The nontrivial orbits of G are G/e, G/C_3, and G/C_7. But since the dimension of our cells is at most six, we have no need to consider orbit types G/e and G/C_3 because those orbits are larger than six. In a sense, all of the failure of the product cell structure lies with 1-cells of orbit type G/C_7. So now we will identify precisely where in each cell we can find these problematic 1-cells.

Since we are concerned with 1-cells with orbit size three, i.e those of orbit type G/C_7, no cells formed by fewer than three 1-cells can have non-uniform isotropy. The 3-cells with non-uniform isotropy are those of the form

$$[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$$

and the 6-cells with non-uniform isotrophy are those of the form

$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}].$$

Cells of dimension between 3 and 6 can have k-faces with non-uniform isotropy. For example, the 5-cell

$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i]$$

contains a 3-face with non-uniform isotropy.

All the cells discussed above with some form of non-uniform isotropy, either in its interior or on a k-face, fail to be G-cells and will need to be subdivided to form a G-cell structure on S^{Λ}. We will now give a method of subdivision for doing so. The method of subdivision is most clear for 3-cells, so we will begin by discussing 3-cells of the form $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$. Let σ^i, $\tau\sigma^i$, and $\tau^2\sigma^i$ represent vectors in $\mathbb{R}^2$, $\tau\mathbb{R}^2$, and $\tau^2\mathbb{R}^2$, respectively. Then the ray of positive multiples of the vector $\sigma^i + \tau\sigma^i + \tau^2\sigma^i$ is fixed under the action of τ. Let i be an element of $\mathbb{Z}/7$ and let $[a_i]$ denote the following 1-cell:

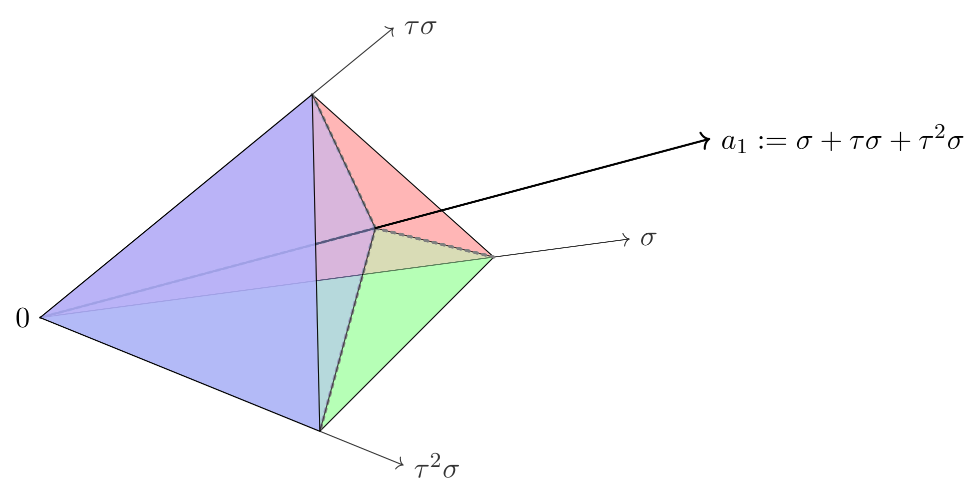
$$[a_i] = \{\alpha(\sigma^i + \tau\sigma^i + \tau^2\sigma^i) \in \mathbb{R}^2 \oplus \tau\mathbb{R}^2 \oplus \tau^2\mathbb{R}^2 \mid \alpha \in \mathbb{R}_{\geq 0}\}.$$

The seven new 1-cells $[a_i]$ can be used to subdivide cells with non-uniform isotropy. We will start with an example of dimension 3 to demonstrate the subdivision.

Example 5.1.4. Consider the 3-cell $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$ discussed in example 5.1.1. The newly-created 1-cell $[a_i]$ contains all the interior points that are fixed under the action of C_3. Thus we can use it to create the following three 3-cells:

$$[a_i, \tau\sigma^i, \tau^2\sigma^i]$$
$$[\sigma^i, a_i, \tau^2\sigma^i]$$
$$[\sigma^i, \tau\sigma^i, a_i]$$

These new cells were formed by replacing each 1-cell in the original cell by a_i in turn. The union of these three 3-cells is the original cell and the new 3-cells intersect only on their boundaries. Here is a drawing of the 3-cell $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$ subdivided into three different colors of new 3-cells



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Recall that cells of dimension 2 or lower do not require subdivision. For cells of dimension higher than 3, we can extend the method of subdivision. Again we illustrate with examples.

Example 5.1.5. The 5-cell $[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i]$ contains $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$ as a 3-face. We can divide it into three 5-cells by replacing each of σ^i, $\tau\sigma^i$, and $\tau^2\sigma^i$ by a_i in turn. The three new cells are

$$[a_i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i]$$
$$[\sigma^i, \sigma^{i+1}, a_i, \tau\sigma^{i+1}, \tau^2\sigma^i]$$
$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, a_i]$$

This subdivision makes three 5-cells whose union is the original 5-cell and the three new 5-cells intersect only on their boundaries. Notice that these three new 5-cells each generate three disjoint orbits of size 21. ★

Example 5.1.6. The 6-cell $[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$ contains the 3-cell $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$. We can divide it into the following three 6-cells

$$[a_i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$$
$$[\sigma^i, \sigma^{i+1}, a_i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$$
$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, a_i].$$

As in the example above, these new cells were formed by replacing three of the 1-cells with a_i in turn, their union forms the original cell and the new cells only intersect on their boundaries. Notice that the three new 6-cells each generate three disjoint orbits of size 21. ★

Example 5.1.7. The 6-cell $[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$ contains two non-uniformly isotropic 3-faces, namely $[\sigma^i, \tau\sigma^i, \tau^2\sigma^i]$ and $[\sigma^{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^{i+1}]$. Since there are three choices of 1-cells to replace with a_i and three choices of 1-cells to replace with a_{i+1}, subdividing this 6-cell will result in the following nine 6-cells:

$$[a_i, a_{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$$
$$[\sigma^i, \sigma^{i+1}, a_i, a_{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$$
$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, a_i, a_{i+1}]$$

$$[\sigma^i, a_{i+1}, a_i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$$
$$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, a_{i+1}, a_i, \tau^2\sigma^{i+1}]$$
$$[a_i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, a_{i+1}]$$

$$[a_i, \sigma^{i+1}, \tau\sigma^i, a_{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$$
$$[\sigma^i, a_{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, a_i, \tau^2\sigma^{i+1}]$$
$$[\sigma^i, \sigma^{i+1}, a_i, \tau\sigma^{i+1}, \tau^2\sigma^i, a_{i+1}]$$

The union of these nine cells is the original 6-cell, and the new cells intersect only on their boundaries. Unlike the previous examples though, these nine cells do not generate nine distinct orbits. Each new cell has the same orbit as two other cells in this list. In particular, the first three new cells form a C_3-orbit and are elements in an orbit of size 21. The same goes for the next three new cells, and the last three new cells. ★

Let us summarize the method of subdivision. We create a cell structure on a vector space whose 1-cells are rays and whose top dimension cells are positive linear combinations of those rays. For each subgroup, we partition the set of 1-cells into subsets and consider whether that subset is part of a higher-dimensional cell. If so, we create new cells out of the fixed points and subdivide. If not, nothing more needs to be done.

5.2 Cell structure for S^Λ

We can apply the method of subdivision to the naïve product cell structure of S^Λ. This gives a proper G-CW structure on S^Λ, which we list as a large table on the following pages. Since every G-cell is of the form $G/H \times D^n$, we will list all the cells by giving an orbit representative for each G-cell. For brevity, let i range over the elements of $\mathbb{Z}/7$.

Dimension	Orbit representative	Orbit type	No. of G-cells
6	$[a_i, \sigma^{i-1}, \tau\sigma^{i-1}, \tau\sigma^i, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^{i-1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^{i-1}, \tau\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$	G/e	7
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}, \tau^2\sigma^i]$	G/e	7
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \sigma^{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^i]$	G/e	7
5	$[a_i, a_{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \tau\sigma^i, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \tau\sigma^i, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^i, \tau\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^i, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau\sigma^{i-1}, \tau\sigma^i, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^{i-1}, \sigma^i, \tau\sigma^{i-1}, \tau^2\sigma^{i+1}]$	G/e	7
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^{i-1}, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \sigma^{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^{i-1}]$	G/e	7
	$[\sigma^{i+j}, \tau\sigma^i, \tau\sigma^{i+1}, \tau^2\sigma^i, \tau^2\sigma^{i+1}]$ for $j \in \{2, 4, 6\}$	G/e	21

Cell structure for S^{Λ}, continued. Let i ranges over the elements of $\mathbb{Z}/7$.

Dimension	Orbit representative	Orbit type	No. of G-cells
4	$[a_i, a_{i+1}, \sigma^i, \sigma^{i+1}]$	G/e	7
	$[a_i, a_{i+1}, \sigma^i, \tau\sigma^{i+j}]$ for $j \in \{0, 1\}$	G/e	14
	$[a_i, a_{i+1}, \sigma^{i+1}, \tau\sigma^{i+j}]$ for $j \in \{0, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau\sigma^i, \tau^j\sigma^{i+1}]$ for $j \in \{0, 1, 2\}$	G/e	21
	$[a_i, \sigma^i, \tau\sigma^i, \tau^j\sigma^{i-1}]$ for $j \in \{0, 1, 2\}$	G/e	21
	$[a_i, \sigma^i, \sigma^{i+j}, \tau\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \sigma^{i+j}, \tau\sigma^{i-j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \sigma^{i+j}, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \sigma^{i+j}, \tau^2\sigma^{i-j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau^2\sigma^{i+j}, \tau\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau^2\sigma^{i+j}, \tau\sigma^{i-j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[a_i, \sigma^{i+1}, \tau\sigma^{i+j}, \tau^2\sigma^{i-j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau\sigma^{i+1}]$	G/e	7
	$[\sigma^i, \sigma^{i+1}, \tau\sigma^i, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[\sigma^i, \sigma^{i+1}, \tau\sigma^{i+1}, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 0\}$	G/e	14
	$[\sigma^i, \sigma^{i+1}, \tau\sigma^{i-1}, \tau^2\sigma^{i-1}]$	G/e	7
3	$[a_i, a_{i+1}, \sigma^{i+j}]$ for $j \in \{0, 1\}$	G/e	14
	$[a_i, \sigma^i, \tau\sigma^{i+j}]$ for $j \in \{-1, 0, 1\}$	G/e	21
	$[a_i, \sigma^{i-1}, \tau\sigma^{i+j}]$ for $j \in \{-1, 0, 1\}$	G/e	21
	$[a_i, \sigma^{i+1}, \tau\sigma^{i+j}]$ for $j \in \{-1, 0, 1\}$	G/e	21
	$[a_i, \sigma^i, \sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
	$[\sigma^i, \sigma^{i+1}, \tau\sigma^i]$	G/e	7
	$[\sigma^i, \sigma^{i+1}, \tau^2\sigma^i]$	G/e	7
	$[\sigma^i, \tau\sigma^i, \tau^2\sigma^{i+j}]$ for $j \in \{-1, 1\}$	G/e	14
2	$[a_i, a_{i+1}]$	G/C_3	7
	$[a_i, \sigma^{i+j}]$ for $j \in \{-1, 0, 1\}$	G/e	21
	$[1, \tau\sigma^i]$	G/e	7
	$[1, \sigma]$	G/e	1
1	$[a_i]$	G/C_3	7
	$[1]$	G/e	1
0	[O]	G/G	1
	$[\infty]$	G/G	1

It is tedious, but straightforward to verify that this gives a CW-structure. It is also an impractical cell structure for hand computations.

Notice that in this large list of cells, most of cells are free. The only non-free cells are in dimensions 2 and 1, with orbit type G/C_3, and fixed points in dimension 0. This is consistent with the analysis of S^Λ toward the end of Section 4.3, where we saw that the only non-free cells of S^Λ are in dimension 2 or lower and can be assembled into the subcomplex called Y.

5.3 Other groups

The method of subdivision can be applied to other groups and G-CW structure. We give two examples of such use cases. This first example is a straightforward application of the method of subdivision to produce a cell structure on a representation γ of the symmetric group S_3. To see a related subdivision scheme that applies to $S^{m\gamma}$, see [KL20].

Example 5.3.1. Let S_3 be the symmetric group on n elements and C_3 be the cyclic subgroup of order 3. Let λ_3 be the degree-2 irreducible representation of C_3 and endow it with a cell structure given by 3 "beach ball panels" (see Example 3.1.3). The order-2 subgroups of S_3 will generate non-uniform isotropy in the 2-cells since a given order-2 subgroup will swap two elements of C_3 while leaving the third fixed. We can add three new rays between the three existing rays to produce a cell structure with six panels. This new structure will be S_3 equivariant. ★

This second example shows that looking for non-uniform isotropy one subgroup at a time can help determine that a given cell structure is a proper G-CW structure.

Example 5.3.2. Recall that the quaternions $\mathbb{H}$ are a real algebra of dimension 4 generated as a vector space by the elements 1, i, j, and k, and also that the quaternion group Q_8 is a subgroup of the units of this algebra. Then $\mathbb{H}$ is a 4-dimensional irreducible real representation of Q_8. Let the eight rays generated by the positive multiples of $\pm 1, \pm i, \pm j$ and $\pm k$ be 1-cells. The eight rays divide $\mathbb{H}$ into sixteen sectors of dimension 4, and the lower dimension cells are the k-faces of these sectors. To see if this cell structure requires subdivision, consider the action of each subgroup of Q_8. The action of the subgroup generated by -1 takes a ray to its opposite ray, but $[j, -j]$ is not a 2-cell in the proposed cell structure, and the same can be said of the other 1-cells. So the subgroup generated by -1 does not create any non-uniform isotropy. Next consider the action of the subgroup generated by i. The eight 1-cells are partitioned into orbits in the following way:

$$\{\pm 1, \pm i\}, \{\pm j, \pm k\}$$

None of these sets are 4-cells in the proposed cell structure. The same argument applies to the subgroups generated by j and k, so we conclude that the proposed cell structure is a proper G-CW cell decomposition. ★

Chapter 6

$H\underline{\mathbb{Z}}_n(S^\Lambda)$ for proper subgroups of G_{21}

In this chapter, we begin the computation of $H\underline{\mathbb{Z}}_n(S^\Lambda)$. In particular, we will focus on $H\underline{\mathbb{Z}}_n(S^\Lambda)(G_{21}/H)$ for proper subgroups H of G_{21} and leave the computation of $H\underline{\mathbb{Z}}_n(S^\Lambda)(G_{21}/G_{21})$ for the next chapter. The reason for this separation is that, for proper subgroups of G_{21}, we can reduce the computation to previously-known facts about C_3 and C_7. We will use Lemma 3.4.3 to do this, and we restate that lemma here.

Lemma 3.4.3 Let X be a G-space. Then

$$H\underline{\mathbb{Z}}_n(X)(G/H) \cong H\underline{\mathbb{Z}}_n(i_H^* X)(H/H)$$

where $H\underline{\mathbb{Z}}$ on the left is a G-spectrum while $H\underline{\mathbb{Z}}$ on the right is the H-spectrum $i_H^* H\underline{\mathbb{Z}}$ given by forgetting the action of elements of G not in H.

The upshot of lemma 3.4.3 is that when evaluating $H\underline{\mathbb{Z}}_n(S^\Lambda)$ at G/H for any proper subgroup H, we may restrict our attention to only the action of H on X. Since the nonabelian group of order 21 that we are interested in has three proper subgroups (discounting conjugates of C_3), we will examine the three cases in turn, beginning with the simplest case of the trivial subgroup.

For the trivial subgroup $\{e\}$, Lemma 3.4.3 becomes

$$H\underline{\mathbb{Z}}_n(S^\Lambda)(G/e) \cong H\underline{\mathbb{Z}}_n(i_e^* S^\Lambda)(e/e).$$

Forgetting all group actions on S^Λ except for that of the trivial subgroup means we have a sphere S^6 with trivial action. Thus we recover the ordinary homology of S^6 when $H\underline{\mathbb{Z}}_n(S^\Lambda)$ is evaluated at G/e.

The next two sections tackle the computation of $H\underline{\mathbb{Z}}_n(S^\Lambda)$ evaluated at C_7 and C_3.

6.1 Homology of S^Λ evaluated at C_7

By Lemma 3.4.3, we may compute $H\underline{\mathbb{Z}}_n(S^\Lambda)(G/C_7)$ by instead considering

$$H\underline{\mathbb{Z}}_n(i^*_{C_7}S^\Lambda)(C_7/C_7).$$

First we examine the G-space $i^*_{C_7}S^\Lambda$. Let λ_7 denote the degree-2 representation of $C_7 = \langle\sigma\rangle$ given by letting the generator σ act on $\mathbb{R}^2$ via rotation by $2\pi/7$. Recall from Section 4.3 that one possible G-CW cell structure for S^{λ_7} involves cutting $\mathbb{R}^2$ into seven "panels." Furthermore, three copies of S^{λ_7} together with an action from C_3 form a cell structure for S^Λ. From this construction, we can see that

$$i^*_{C_7}S^\Lambda \cong S^{3\lambda_7}.$$

Next we can find a cell structure for $S^{3\lambda_7}$. By using the "cell trick" described in the introduction of [HHR17], we arrive at this cell structure:

- two fixed 0-cells: $C_7/C_{7+} \wedge 0$ and $C_7/C_{7+} \wedge \infty$
- a single n-cell: $C_7/e_+ \wedge e^n$ for $n = 1, 2, \ldots, 6$

Lemma 3.4.1 instructs us as to how to form a cellular chain complex using this data, and the complex can be seen below. For simplicity, the index $i = 0, \ldots, 6$ has been omitted in some places, but we should interpret each summation as a sum over seven terms and $\mathbb{Z}[\sigma^i]$ as the free abelian group generated by the seven σ^i. The omitted differentials alternate following the pattern of the first two differentials shown. Notice that the "lower level" is the chain complex corresponding to the non-equivariant homology of the sphere S^6, and the "upper level" is formed by taking C_7-fixed points of the lower level. The diagonal map Δ is given by $e+\sigma+\cdots+\sigma^6 \to e+\sigma+\cdots+\sigma^6$, while the fold map ∇ takes generators $e, \sigma, \ldots, \sigma^6$ to $e+\sigma+\cdots+\sigma^6$.

$$\begin{array}{ccccccccc}
n=6 & & n=5 & & n=4 & \cdots & n=1 & & n=0 \\
\mathbb{Z}\left[\sum\sigma^i\right] & \xrightarrow{\;0\;} & \mathbb{Z}\left[\sum\sigma^i\right] & \xrightarrow{\;7\;} & \mathbb{Z}\left[\sum\sigma^i\right] & \cdots & \mathbb{Z}\left[\sum\sigma^i\right] & \xrightarrow{7(\infty-0)} & \mathbb{Z}[\infty,0] \xrightarrow{\varepsilon} \mathbb{Z} \\
\nabla \uparrow\downarrow \Delta & & \nabla \uparrow\downarrow \Delta & & \nabla \uparrow\downarrow \Delta & & \nabla \uparrow\downarrow \Delta & & \mathrm{id} \uparrow\downarrow \mathrm{id} \\
\mathbb{Z}[\sigma^i] & \xrightarrow{1-\sigma} & \mathbb{Z}[\sigma^i] & \xrightarrow{\sum\sigma^i} & \mathbb{Z}[\sigma^i] & \cdots & \mathbb{Z}[\sigma^i] & \xrightarrow{\infty-0} & \mathbb{Z}[\infty,0] \xrightarrow{\varepsilon} \mathbb{Z}
\end{array}$$

We seek the C_7/C_7 part, i.e. the top level, so passing to homology gives the following result.

$$\begin{aligned}
H\underline{\mathbb{Z}}_n(S^\Lambda)(G/C_7) &\cong H\underline{\mathbb{Z}}_n(S^{3\lambda_7})(C_7/C_7) \\
&\cong \begin{cases} \mathbb{Z} & n=6 \\ \mathbb{Z}/7 & n=0,2,4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

6.2 Homology of S^Λ evaluated at C_3

Just as in the case for C_7, we use Lemma 3.4.3 to deduce that

$$H\underline{\mathbb{Z}}_n(S^\Lambda)(G/C_3) \cong H\underline{\mathbb{Z}}_n(i^*_{C_3}S^\Lambda)(C_3/C_3).$$

We again examine the G-CW cell structure of S^Λ given in Section 4.3 to gain insight into $i^*_{C_3}S^\Lambda$. Noticed that the actio of $\tau \in C_3$ permutes the three copies of $\mathbb{R}^2$ cyclically. This demonstrates that

$$i^*_{C_3}S^\Lambda \cong S^{2\rho_3}$$

i.e. that $i^*_{C_3}S^\Lambda$ consists of two copies of the regular representation ρ_3 of C_3. Observe that ρ_3 can be decomposed as $1 \oplus \lambda_3$ where 1 denotes the degree-1 trivial representation and λ_3 is given via rotation by $2\pi/3$. Thus our desired homology group can be simplified in the following way.

$$\begin{aligned}
H\underline{\mathbb{Z}}_n(S^\Lambda)(G/C_3) &\cong H\underline{\mathbb{Z}}_n(S^{2\rho_3})(C_3/C_3) \\
&\cong H\underline{\mathbb{Z}}_n(S^2 \wedge S^{2\lambda_3})(C_3/C_3) \\
&\cong H\underline{\mathbb{Z}}_{n-2}(S^{2\lambda_3})(C_3/C_3)
\end{aligned}$$

Next we use the "cell trick" as before to arrive at the following cellular chain complex. The index $i = 0, 1, 2$ has been omitted for simplicity in some places, but we should interpret each summation as a sum over three terms and $\mathbb{Z}[\tau^i]$ the free abelian group generated by the τ^i.

$$\begin{array}{ccccccccccc}
n=4 & & n=3 & & n=2 & & n=1 & & n=0 & & \\
\mathbb{Z}\left[\sum \tau^i\right] & \xrightarrow{0} & \mathbb{Z}\left[\sum \tau^i\right] & \xrightarrow{3} & \mathbb{Z}\left[\sum \tau^i\right] & \xrightarrow{0} & \mathbb{Z}\left[\sum \tau^i\right] & \xrightarrow{3(\infty - 0)} & \mathbb{Z}[\infty, 0] & \xrightarrow{\varepsilon} & \mathbb{Z} \\
\nabla \downarrow \uparrow \Delta & & \nabla \downarrow \uparrow \Delta & & \nabla \downarrow \uparrow \Delta & & \nabla \downarrow \uparrow \Delta & & \mathrm{id} \downarrow \uparrow \mathrm{id} & & \\
\mathbb{Z}[\tau^i] & \xrightarrow{1-\tau} & \mathbb{Z}[\tau^i] & \xrightarrow{\sum \tau^i} & \mathbb{Z}[\tau^i] & \xrightarrow{1-\tau} & \mathbb{Z}[\tau^i] & \xrightarrow{\infty - 0} & \mathbb{Z}[\infty, 0] & \xrightarrow{\varepsilon} & \mathbb{Z}
\end{array}$$

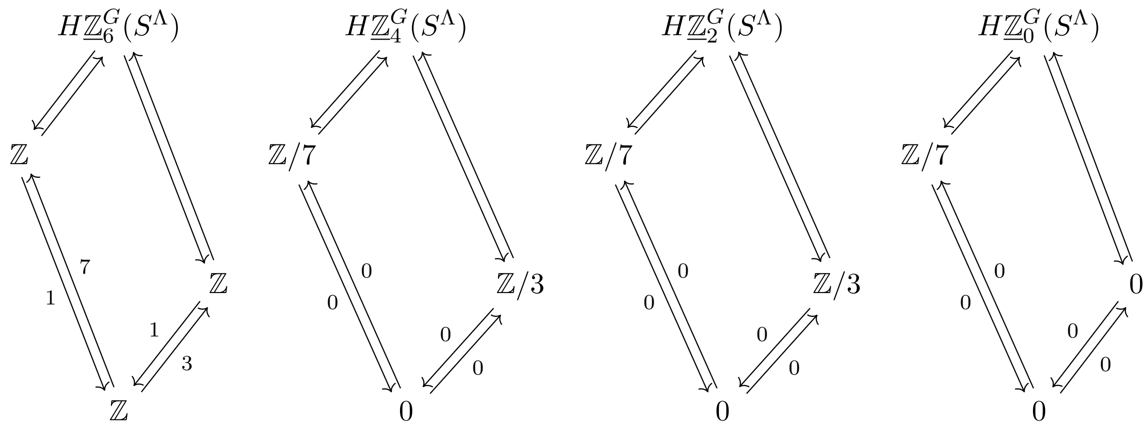
Again we are interested in the top level of this stacked chain complex, so passing to homology, we arrive at the final result.

$$\begin{aligned}
H\underline{\mathbb{Z}}_n(S^{\Lambda})(G/C_3) &\cong H\underline{\mathbb{Z}}_{n-2}(S^{2\lambda_3})(C_3/C_3) \\
&\cong \begin{cases} \mathbb{Z} & n = 6 \\ \mathbb{Z}/3 & n = 4, 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

To summarize the results of this chapter, we can arrange the homology groups that have been computed up to this point into the shape of the orbit category of G_{21}, leaving blank some of the computations that will be completed in a later chapter.

Summary of progress toward computing $H\underline{\mathbb{Z}}_n(S^\Lambda)$

$H\underline{\mathbb{Z}}_6(S^\Lambda)$ $H\underline{\mathbb{Z}}_4(S^\Lambda)$ $H\underline{\mathbb{Z}}_2(S^\Lambda)$ $H\underline{\mathbb{Z}}_0(S^\Lambda)$



Chapter 7

Computing $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$ via Isotropy Separation

In Chapter 6, we computed $H\underline{\mathbb{Z}}_n^H(S^\Lambda)$ for proper subgroups of G, and in this chapter, we complete the work by computing $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$. The strategy is to generate a long exact sequence whose terms include $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$, use various methods to compute all other terms, and then leverage the exactness of the sequence to find $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$. This method is based on the concept of isotropy separation, which can be found in [Hil20].

To begin, we can take the smash product of S^Λ with the cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$ to produce the new cofiber sequence below.

$$EG_+ \wedge S^\Lambda \to S^\Lambda \to \widetilde{EG} \wedge S^\Lambda \tag{7.1}$$

This cofiber sequences gives the following long exact sequence on homology:

$$\begin{aligned} \cdots \to H\underline{\mathbb{Z}}_{n+1}(EG_+ \wedge S^\Lambda) \to H\underline{\mathbb{Z}}_{n+1}(S^\Lambda) \to H\underline{\mathbb{Z}}_{n+1}(\widetilde{EG} \wedge S^\Lambda) \\ \to H\underline{\mathbb{Z}}_n(EG_+ \wedge S^\Lambda) \to H\underline{\mathbb{Z}}_n(S^\Lambda) \to H\underline{\mathbb{Z}}_n(\widetilde{EG} \wedge S^\Lambda) \to \cdots \end{aligned} \tag{7.2}$$

Since the above sequence is exact and repeats every three terms with a drop in degree, we can compute $H\underline{\mathbb{Z}}_*(EG_+ \wedge S^\Lambda)$ and $H\underline{\mathbb{Z}}_*(\widetilde{EG} \wedge S^\Lambda)$, and then use those results to determine $H\underline{\mathbb{Z}}_*(S^\Lambda)$. This is the content of Sections 7.2 and 7.3. First we devote one section to presenting the Atiyah-Hirzebruch spectral sequence, because it is used extensively in the computations that follow.

7.1 Atiyah-Hirzebrush Spectral Sequence

The spectral sequences used in the proof of Lemma 7.2.1 and the final computation toward the end of Section 7.3 are both examples of the Atiyah-Hirzebruch spectral sequence for homology. We will describe the technique for generating these spectral sequences in this section. The idea behind the technique is that cofibrations generate long exact sequences, and the long exact sequences link together to form exact couples. That exact couple will contain the data for a spectral sequence that computes the homology groups we are ultimately interested in.

To begin, let X and Y be spaces and let $X \to Y$ be a cofibration with cofiber F. Then applying a homology functor $H_*(-)$ generates the following long exact sequence:

$$\cdots \to H_m(X) \to H_m(Y) \to H_m(F) \to H_{m-1}(X) \to H_{m-1}(Y) \to H_{m-1}(F) \to \cdots$$

We will be interested in the specific homology functor whose output is Mackey functor-valued, $RO(G)$-graded homology evaluated at G/G, so henceforth we will use the specific functor $H\underline{\mathbb{Z}}_*^G(-)$ instead of the general $H_*(-)$.

Now suppose X is a space with a filtration $X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq X$. Then each inclusion $X^{(n)} \hookrightarrow X^{(n+1)}$ is a cofibration with cofiber $X^{(n+1)}/X^{(n)}$, and together they form a tower of cofibrations. The tower of cofibrations is shown below with one specific cofibration and its cofiber highlighted in blue.

$$\begin{array}{ccccccccccc} X^{(0)} & \longrightarrow & X^{(1)} & \longrightarrow & \cdots & \longrightarrow & X^{(n)} & \longrightarrow & X^{(n+1)} & \longrightarrow & \cdots \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ & & X^{(1)}/X^{(0)} & & & & X^{(n)}/X^{(n-1)} & & X^{(n+1)}/X^{(n)} & & \end{array}$$

Applying the homology functor $H\underline{\mathbb{Z}}_*^G(-)$ generates long exact sequences that link up to form the following unrolled exact couple,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H\underline{\mathbb{Z}}_m^G(X^{(n-1)}) & \xrightarrow{i} & H\underline{\mathbb{Z}}_m^G(X^{(n)}) & \xrightarrow{i} & H\underline{\mathbb{Z}}_m^G(X^{(n+1)}) & \longrightarrow & \cdots \\ & & & \nwarrow{\scriptstyle k} & \downarrow{\scriptstyle j} & \nwarrow{\scriptstyle k} & \downarrow{\scriptstyle j} & & \\ & & & & H\underline{\mathbb{Z}}_m^G(X^{(n)}/X^{(n-1)}) & & H\underline{\mathbb{Z}}_m^G(X^{(n+1)}/X^{(n)}) & & \end{array}$$

where the maps i and j are the induced maps stemming from the fibration and its fiber, highlighted in blue above, and and k is the connecting homomorphism that changes the degree m by -1. In other words, k is the composition of the induced map on $\left(X^{(n+1)} \big/ X^{(n)} \to \Sigma X^{(n)} \right)$ followed by the suspension isomorphism $H\underline{\mathbb{Z}}^G_m(\Sigma X^{(n)}) \cong H\underline{\mathbb{Z}}^G_{m-1}(X^{(n)})$

It will be convenient to use Adams grading in our desired spectral sequence, so let us re-index our terms by defining $p := n$ and $q := m - p$. Then let the $E^1_{p,q}$-page be given by

$$E^1_{p,q} := H\underline{\mathbb{Z}}^G_{p+q} \left(X^{(p)} \big/ X^{(p-1)} \right)$$

and let the $E^1_{p,q}$ differential be given by the composite map

$$d^1 : H\underline{\mathbb{Z}}^G_{p+q} \left(X^{(p)} \big/ X^{(p-1)} \right) \xrightarrow{k} H\underline{\mathbb{Z}}^G_{(p-1)+q} \left(X^{(p-1)} \right) \xrightarrow{j} H\underline{\mathbb{Z}}^G_{(p-1)+q} \left(X^{(p-1)} \big/ X^{(p-2)} \right)$$

We will use this spectral sequence for several computations that follow.

7.2 Computing $H\underline{\mathbb{Z}}_n(EG_+ \wedge S^\Lambda)$

In the beginning of this chapter, we mentioned that the task of computing $H\underline{\mathbb{Z}}^G_n(S^\Lambda)$ can be reduced to the tasks of computing $H\underline{\mathbb{Z}}^G_n(EG_+ \wedge S^\Lambda)$ and $H\underline{\mathbb{Z}}^H_n(\widetilde{EG} \wedge S^\Lambda)$ instead. We address the former now. The cellular filtration of EG_+ generates a spectral sequence whose E_1 page has terms that coincide with the terms of the bar resolution for computing group homology $H_*(G_{21}, \mathbb{Z})$. This is the content of the following lemma.

Lemma 7.2.1. *Suppose that V is orientable and* $\dim(V) = d$. *There is an isomorphism*

$$H\underline{\mathbb{Z}}^G_n(EG_+ \wedge S^V) \cong H_{n-d}(G, \mathbb{Z})$$

where the right-hand side is group homology.

Proof. Let the p-cells of EG_+ be ordered tuples of G^{p+1} and let the p-cells be denoted by symbols $[g_0 \otimes g_1 \otimes \cdots \otimes g_p]$, with each $g_i \in G$. Let G act on a p-cell via left multiplication in the first component, that is,

$$g \cdot [g_0 \otimes g_1 \otimes \cdots \otimes g_p] = [(gg_0) \otimes g_1 \otimes \cdots \otimes g_p]$$

The inclusion of the p-skeleton $EG_+^{(p)}$ into the $(p+1)$-skeleton is a cofibration with cofiber $\mathrm{Cof}_p := EG_+^{(p)}/EG_+^{(p-1)}$. The inclusion maps and their respective cofibers join to form a tower of cofibrations. Taking the smash product of each space in the tower with S^V produces a new tower of cofibrations. Then we can apply the method described in Appendix 7.1 to this tower to arrive at the spectral sequence

$$E^1_{p,q} = H\underline{\mathbb{Z}}^G_{p+q}(C_p \wedge S^V)$$

with the differential on the $E^1_{p,q}$ induced by

$$C_p \wedge S^V \to \Sigma EG_+^{(p-1)} \wedge S^V \to \Sigma C_{p-1} \wedge S^V$$

The cofiber Cof_p is a wedge of spheres S^p with a G-action inherited from the G-action on cells of EG. In particular, Cof_p is a wedge of $|G|^p$ copies of $G_+ \wedge S^p$ where each copy is indexed by a symbol of the form $[e \otimes g_1 \otimes \cdots \otimes g_p]$. Thus it will be useful to view Cof_p in the following way.

$$\mathrm{Cof}_p := EG_+^{(p)}/EG_+^{(p-1)} \quad \cong \bigvee_{[g_0 \otimes \cdots \otimes g_p]} S^p \quad \cong \bigvee_{[e \otimes g_1 \otimes \cdots \otimes g_p]} S^p \wedge G_+ \quad \cong \bigvee_{[g_1 \otimes \cdots \otimes g_p]} S^p \wedge G_+$$

for $p > 0$ and $\mathrm{Cof}_0 \cong G_+$. Applying this view of Cof_p to the terms of the E^1 page gives

$$\begin{aligned} E^1_{p,q} &= H\underline{\mathbb{Z}}^G_{p+q}\left(\bigvee_{[g_1 \otimes \cdots \otimes g_p]} S^p \wedge G_+ \wedge S^V \right) \\ &\cong \bigoplus_{[g_1 \otimes \cdots \otimes g_p]} H\underline{\mathbb{Z}}^e_q(S^V) \\ &\cong \begin{cases} \mathbb{Z}G^p & \text{if } q = \dim V \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This interpretation of Cof_p as a wedge of spheres produces the differential $d^1 : E^1_{p,q} \cong \mathbb{Z}G^p \to E^1_{p-1,q} \cong \mathbb{Z}G^{p-1}$. Explicitly, the map is

$$[g_1 \otimes \cdots \otimes g_p] \mapsto [g_2 \otimes \cdots g_p] + \sum_{i=1}^{p-1} (-1)^i [g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_p] + (-1)^{p-1}[g_1 \otimes \cdots \otimes g_{p-1}]$$

We observe that the differential $d^1 : \mathbb{Z}G^p \to \mathbb{Z}G^{p-1}$ is precisely the differential in the bar resolution for computing group homology $H_*(G; M)$ for $M = H\underline{\mathbb{Z}}^u_q(S^V)$ when $q = \dim V$. In other words, we have the following commutative diagram between the row of $E^1_{p,q}$ with $q = \dim V$ comprising the row above and the bar resolution below:

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & E^1_{0,q} & \longleftarrow & \cdots & \longleftarrow & E^1_{p-1,q} & \xleftarrow[d^1_{p,q}]{} & E^1_{p,q} & \longleftarrow & \cdots \\
\downarrow & & \cong\downarrow & & & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longleftarrow & B_0 \otimes M & \longleftarrow & \cdots & \longleftarrow & B_{p-1} \otimes M & \xleftarrow[d]{} & B_p \otimes M & \longleftarrow & \cdots
\end{array}$$

Then we can conclude that the only nonzero part of the E^2 page is the group homology of G on the row $q = \dim(V)$.

$$E^2_{p,q} \cong \begin{cases} H_{p+q}(G; \mathbb{Z}) & \text{if } q = \dim V \\ 0 & \text{otherwise} \end{cases}$$

Since all nonzero terms are concentrated in the row $q = \dim V$, the spectral sequence collapses to

$$H\underline{\mathbb{Z}}^G_{p+q}(EG_+ \wedge S^V) \cong H_p(G; \mathbb{Z})$$

$\square$

Recall that in Example 4.2.3, we showed that the degree-6 real representation Λ was orientable. Then by applying Lemma 7.2.1 to the representation Λ, we have an isomorphism

$$H\underline{\mathbb{Z}}^G_n(EG_+ \wedge S^\Lambda) \cong H_{n-6}(G_{21}; \mathbb{Z}). \tag{7.3}$$

Since group homology is always zero in negative degrees, we use this data in the long exact sequence (7.2) to produce the following isomorphisms.

$$H\underline{\mathbb{Z}}^G_n(S^\Lambda) \cong H\underline{\mathbb{Z}}^G_n(\widetilde{EG} \wedge S^\Lambda),\ n \leq 5 \tag{7.4}$$

So to determine $H\underline{\mathbb{Z}}^{G_{21}}_n(S^\Lambda)$ for $n \leq 5$, we now turn our attention to $H\underline{\mathbb{Z}}^{G_{21}}_n(\widetilde{EG} \wedge S^\Lambda)$.

7.3 Computing $H\underline{\mathbb{Z}}_n(\widetilde{EG} \wedge S^\Lambda)$

In the previous section, we used Lemma 7.2.1 to produce the isomorphism in equation (7.4). Thus to find $H\underline{\mathbb{Z}}_n^{G_{21}}(S^\Lambda)$ for $0 \leq n \leq 5$, we may instead compute $H\underline{\mathbb{Z}}_n^{G_{21}}(\widetilde{EG} \wedge S^\Lambda)$. This can be accomplished first by using an important property of $\widetilde{EG}$ to restrict our attention to a smaller subcomplex within S^Λ, creating a filtration of this subcomplex, and then using a spectral sequence built from this filtration to finish the computation.

Lemma 7.3.1. *Suppose that X is a G-CW complex and Y is a subcomplex such that X differs from Y by attaching finitely many free cells, i.e.*

$$X = Y \cup \left(\bigcup_{i \in I} G/e_+ \wedge D_+^{d_i} \right)$$

where D^{d_i} is a disk of dimension d_i. Then the inclusion $Y \to X$ induces an isomorphism

$$H\underline{\mathbb{Z}}_\bullet^G(\widetilde{EG} \wedge Y) \xrightarrow{\cong} H\underline{\mathbb{Z}}_\bullet^G(\widetilde{EG} \wedge X).$$

Proof. Because X can be built from Y by attaching free cells one at a time, it suffices to prove the desired isomorphism in the case that X differs from Y by attaching a single free cell of dimension d. In this case, the inclusion $Y \to X$ is a cofibration with cofiber $X/Y \simeq G/e_+ \wedge S^d$. This cofibration induces a long exact sequence on homology. In particular, the homology functor $H\underline{\mathbb{Z}}_*(\widetilde{EG} \wedge -)(G/G)$ produces the following long exact sequence:

$$\begin{aligned} \cdots \to & H\underline{\mathbb{Z}}_{n+1}^G(\widetilde{EG} \wedge Y) \to H\underline{\mathbb{Z}}_{n+1}^G(\widetilde{EG} \wedge X) \to H\underline{\mathbb{Z}}_{n+1}^G(\widetilde{EG} \wedge G/e_+ \wedge S^d) \\ & \to H\underline{\mathbb{Z}}_n^G(\widetilde{EG} \wedge Y) \to H\underline{\mathbb{Z}}_n^G(\widetilde{EG} \wedge X) \to H\underline{\mathbb{Z}}_n^G(\widetilde{EG} \wedge G/e_+ \wedge S^d) \to \cdots \end{aligned}$$

We can compute the homology terms associated with the cofiber. By Lemma 3.4.3, we know

$$H\underline{\mathbb{Z}}_\bullet^G(\widetilde{EG} \wedge G/e_+ \wedge S^d) = H\underline{\mathbb{Z}}_\bullet^e(i_e^*\widetilde{EG} \wedge S^d).$$

Since $i_e^*\widetilde{EG}$ is contractible, $i_e^*\widetilde{EG} \wedge S^d \simeq *$, and we are now looking to compute the underlying reduced homology of a point. Thus

$$H\underline{\mathbb{Z}}_\bullet^e(i_e^*\widetilde{EG} \wedge S^d) = 0.$$

Substituting these zeroes into the abovementioned long exact sequence, we see that the inclusion $Y \to X$ produces the desired isomorphisms

$$H\underline{\mathbb{Z}}^G_\bullet(\widetilde{EG} \wedge Y) \xrightarrow{\cong} H\underline{\mathbb{Z}}^G_\bullet(\widetilde{EG} \wedge X).$$

□

In order to use Lemma 7.3.1 to compute $H\underline{\mathbb{Z}}^G_n(\widetilde{EG} \wedge S^\Lambda)$, we take a brief detour to establish the following fact.

Lemma 7.3.2. *Let H be a finite group, EH be the universal cover of the classifying space of H, and $\widetilde{EH}$ be the homotopy cofiber of the map $EH_+ \to S^0$ that sends EH to 0 and sends + to the basepoint ∞ of S^0. Then*

(a) the map

$$H\underline{\mathbb{Z}}_n(\widetilde{EH}) \to H\underline{\mathbb{Z}}_{n-1}(EH_+)$$

induced by $\widetilde{EH} \to \Sigma EH_+$ is an isomorphism for $n \neq 0, 1$ and

(b) for any subgroup $K \subseteq H$, there is an exact sequence

$$0 \longrightarrow H\underline{\mathbb{Z}}^K_1(\widetilde{EH}) \longrightarrow \mathbb{Z} \xrightarrow{|K|} \mathbb{Z} \longrightarrow H\underline{\mathbb{Z}}^K_0(\widetilde{EH}) \longrightarrow 0$$

so that $H\underline{\mathbb{Z}}^K_1(\widetilde{EH}) = 0$ and $H\underline{\mathbb{Z}}^K_0(\widetilde{EH}) = \mathbb{Z}/|K|$.

Proof. The cofiber sequence $EH_+ \to S^0 \to \widetilde{EH}$ produces the following long exact sequence:

$$\cdots \to H\underline{\mathbb{Z}}_1(EH_+) \to H\underline{\mathbb{Z}}_1(S^0) \to H\underline{\mathbb{Z}}_1(\widetilde{EH})$$
$$\to H\underline{\mathbb{Z}}_0(EH_+) \to H\underline{\mathbb{Z}}_0(S^0) \to H\underline{\mathbb{Z}}_0(\widetilde{EH})$$
$$\to H\underline{\mathbb{Z}}_{-1}(EH_+) \to H\underline{\mathbb{Z}}_{-1}(S^0) \to H\underline{\mathbb{Z}}_{-1}(\widetilde{EH}) \to \cdots$$

Then statement (a) holds because $H\underline{\mathbb{Z}}_n(S^0) = 0$ for all $n \neq 0$. To see statement (b), notice that the map $H\underline{\mathbb{Z}}^K_0(EH_+) \to H\underline{\mathbb{Z}}^H_0(S^0)$ is induced by the map of spaces where the 0-cells of EH_+ (there are $|H|$ of them) are mapped to a point. In other words, this is equivalent to the map of G-sets

$H/K \to H/H$. The induced map on homology must be the transfer map in the constant Mackey functor, i.e. $\underline{\mathbb{Z}}(H/H) \xrightarrow{|K|} \underline{\mathbb{Z}}(H/K)$. Finally EH_+ is a CW-complex with cells only in non-negative dimensions, we must have $H\underline{\mathbb{Z}}^K_{-1}(EH_+) = 0$. The relevant portion of our long exact sequence becomes

$$0 \longrightarrow H\underline{\mathbb{Z}}^K_1(\widetilde{EH}) \longrightarrow \mathbb{Z} \xrightarrow{|K|} \mathbb{Z} \longrightarrow H\underline{\mathbb{Z}}^K_0(\widetilde{EK}) \longrightarrow 0$$

and by exactness, we can see that $H\underline{\mathbb{Z}}^K_1(\widetilde{EH}) = 0$ and $H\underline{\mathbb{Z}}^K_0(\widetilde{EH}) = \mathbb{Z}/|K|$. □

We now resume our work of computing $H\underline{\mathbb{Z}}^G_n(\widetilde{EG} \wedge S^\Lambda)$ by using Lemma 7.3.1 to restrict our attention to a suitable subcomplex of S^Λ instead. At the end of Section 4.3, we established that the representation sphere S^Λ contains the subcomplex Y and that S^Λ can be formed from Y by attaching free cells. Recall from Section 4.3 that one possible cell structure of Y consists of:

- two 0-cells: $G/G_+ \wedge 0$ and $G/G_+ \wedge \infty$
- a single 1-cell: $G/C_{3+} \wedge a_0$
- a single 2-cell: $G/C_{3+} \wedge \hat{a}_0$

Let $Y^{(p)}$ be the p-skeleton of Y under this cell structure. Take $H\underline{\mathbb{Z}}^G_n(\widetilde{EG} \wedge -)$ to be our homology functor of interest, and apply the spectral sequence method of Subsection 7.1 to produce a spectral sequence with E^1 page given by

$$E^1_{p,q} = H\underline{\mathbb{Z}}^G_{p+q}\left(\widetilde{EG} \wedge {}^{Y^{(p)}}\!/\!_{Y^{(p-1)}}\right) \Rightarrow H\underline{\mathbb{Z}}^G_{p+q}(\widetilde{EG} \wedge Y).$$

The quotient $Y^{(p)}/Y^{(p-1)}$ is a wedge of p-spheres, so we have the following equivalences.

$$\begin{aligned}
E^1_{p,q} &= H\underline{\mathbb{Z}}^G_{p+q}\left(\widetilde{EG}\wedge {Y^{(p)}}\big/{Y^{(p-1)}}\right)\\
&\cong H\underline{\mathbb{Z}}^G_{p+q}\left(\widetilde{EG}\wedge\left(\bigvee_{i\in I} G/H_{i_+}\wedge S^p\right)\right)\\
&\cong \bigoplus_{i\in I} H\underline{\mathbb{Z}}^G_{p+q}\left(\widetilde{EG}\wedge G/H_{i_+}\wedge S^p\right)\\
&\cong \bigoplus_{i\in I} H\underline{\mathbb{Z}}^G_{q}\left(\widetilde{EG}\wedge G/H_{i_+}\right)\\
&\cong \bigoplus_{i\in I} H\underline{\mathbb{Z}}^{H_i}_{q}\left(\widetilde{EH_i}\right) && (7.5)\\
&\cong \bigoplus_{i\in I} H\underline{\mathbb{Z}}^{H_i}_{q-1}\left(EH_{i+}\right) \quad \text{for } q\neq 0,1 && (7.6)\\
&\cong \bigoplus_{i\in I} H_{q-1}(H_i,\mathbb{Z}) \quad \text{for } q\neq 0,1 && (7.7)
\end{aligned}$$

The isomorphisms in the last two lines are consequences of earlier statements. The isomorphism that produces line 7.6 is due to Lemma 7.3.2(a). The isomorphism that produces line (7.7) is discussed in lemma 7.2.1. So with the exception of the bottom two rows where $q = 0, 1$, the first column is populated with the group homology of G_{21}, the second and third columns with the group homology of C_3, and all other columns are zero.

It remains to determine the appropriate groups for $q = 0, 1$ corresponding to the two bottom rows of the E^1 page. As described in line (7.5), these two rows are given by

$$E^1_{p,q} \cong \bigoplus_{i\in I} H\underline{\mathbb{Z}}^{H_i}_{1}\left(\widetilde{EH_i}\right), \quad q = 0,1$$

so using the cell structure for our specific complex Y, we conclude that for $q = 0, 1$, the E^1 page of our desired spectral sequence is given by

$$E^1_{p,q} \cong \begin{cases} H\underline{\mathbb{Z}}^G_q(\widetilde{EG}) & p = 0\\ H\underline{\mathbb{Z}}^{C_3}_q(\widetilde{EC_3}) & p = 1,2\\ 0 & \text{otherwise}\end{cases}$$

We can refer to Lemma 7.3.2(b) to see what these groups will be. The E^1 page is now complete, and the populated page is shown below.

$$
\begin{array}{l|ccccc}
q & & & & & \\
& \mathbb{Z}/21 & \xleftarrow{\times 7} & \mathbb{Z}/3 & & \mathbb{Z}/3 \\
& 0 & & 0 & & 0 \\
& \mathbb{Z}/3 & \xleftarrow{\times 1} & \mathbb{Z}/3 & & \mathbb{Z}/3 \\
& 0 & & 0 & & 0 \\
& \mathbb{Z}/3 & \xleftarrow{\times 1} & \mathbb{Z}/3 & & \mathbb{Z}/3 \\
& 0 & & 0 & & 0 \\
& \mathbb{Z}/21 & \xleftarrow{\times 7} & \mathbb{Z}/3 & & \mathbb{Z}/3 \\
\hline
& & & & & \rightarrow p
\end{array}
$$

The differential on the E^1 page is the composition

$$H\underline{\mathbb{Z}}^G_{p+q}\left(\widetilde{EG} \wedge {}^{Y^{(p)}}\!\big/_{Y^{(p-1)}}\right) \xrightarrow{k} H\underline{\mathbb{Z}}^G_{p-1+q}\left(Y^{(p-1)}\right) \xrightarrow{j} H\underline{\mathbb{Z}}^G_{p-1+s}\left(\widetilde{EG} \wedge {}^{Y^{(p-1)}}\!\big/_{Y^{(p-2)}}\right)$$

where the map k is the connecting homomorphism in the long exact sequence on homology for the inclusion $Y^{(p-1)} \to Y^{(p)}$ and the map j is the map on homology induced by the quotient $Y^{(p-1)} \to Y^{(p-1)}/Y^{(p-2)}$. The nonzero differentials have been labelled on the E^1 page above. Then passing to homology, we arrive at the E^2 page, which has only zero differentials. The result of the computation is

$$H\underline{\mathbb{Z}}^G_n(\widetilde{EG} \wedge S^\Lambda) \cong \begin{cases} \mathbb{Z}/7 & n = 0 \\ \mathbb{Z}/3 & n = 2, 4 \pmod 6,\ n > 0 \\ \mathbb{Z}/21 & n = 0 \pmod 6,\ n > 0 \\ 0 & \text{otherwise} \end{cases}$$

The long exact sequence (line 7.2) containing $H\underline{\mathbb{Z}}^G_n(S^\Lambda)$ can now be filled in with the completed computations, and the updated sequence is shown below. The content of the orange box below is the result of Lemma 7.2.1. The blue box contains the computations of Section 7.3.

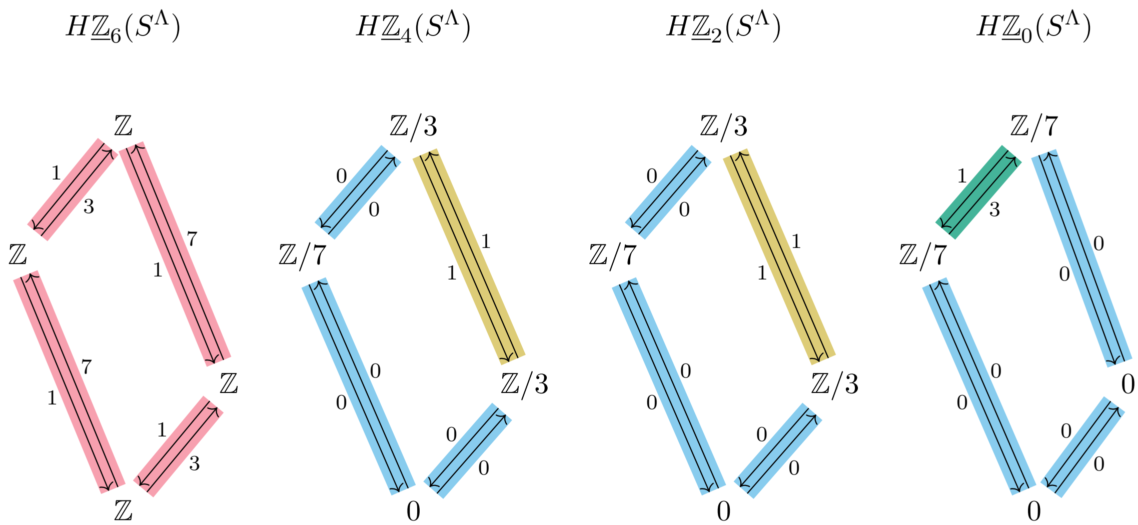
$$
\begin{array}{cccccc}
 & & \cdots & \longrightarrow & H\underline{\mathbb{Z}}_7(\widetilde{EG}\wedge S^\Lambda)=0 & \\
\longrightarrow & H\underline{\mathbb{Z}}_6(S^\Lambda\wedge EG_+)=\mathbb{Z} \longrightarrow & H\underline{\mathbb{Z}}_6(S^\Lambda) & \longrightarrow & H\underline{\mathbb{Z}}_6(\widetilde{EG}\wedge S^\Lambda)=\mathbb{Z}/21 & \\
\longrightarrow & H\underline{\mathbb{Z}}_5(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_5(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_5(\widetilde{EG}\wedge S^\Lambda)=0 & \\
\longrightarrow & H\underline{\mathbb{Z}}_4(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_4(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_4(\widetilde{EG}\wedge S^\Lambda)=\mathbb{Z}/3 & \\
\longrightarrow & H\underline{\mathbb{Z}}_3(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_3(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_3(\widetilde{EG}\wedge S^\Lambda)=0 & \\
\longrightarrow & H\underline{\mathbb{Z}}_2(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_2(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_2(\widetilde{EG}\wedge S^\Lambda)=\mathbb{Z}/3 & \\
\longrightarrow & H\underline{\mathbb{Z}}_1(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_1(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_1(\widetilde{EG}\wedge S^\Lambda)=0 & \\
\longrightarrow & H\underline{\mathbb{Z}}_0(S^\Lambda\wedge EG_+)=0 \longrightarrow & H\underline{\mathbb{Z}}_0(S^\Lambda) & \xrightarrow{\cong} & H\underline{\mathbb{Z}}_0(\widetilde{EG}\wedge S^\Lambda)=\mathbb{Z}/7 & \longrightarrow \cdots
\end{array}
$$

From this we can draw conclusions about $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$.

$$
H\underline{\mathbb{Z}}_n^G(S^\Lambda) \cong \begin{cases} \mathbb{Z}/7 & n=0 \\ \mathbb{Z}/3 & n=2,4 \\ \mathbb{Z} & n=6 \\ 0 & \text{otherwise} \end{cases}
$$

Lastly we wish to assemble the results and present the data in the form of Mackey functors. We will need to determine the maps that connect the various groups. We present the completed Mackey functors below first, and give explanations for the maps immediately after.

Theorem. *Let G_{21} be the non-abelian group of order 21 and Λ the irreducible degree-6 representation of G_{21}. Then the Mackey functor-valued* [1] *integer-graded Bredon homology of S^Λ is*



Remark: The restrictions and transfers given above may vary by an automorphism depending on one's choice of generator. Details for the choices made here are given below.

We used four approaches to determine the maps in the above Mackey functors, and the explanations below are grouped into four parts accordingly.

- Twelve pairs of restriction and transfer maps, highlighted in blue, are forced to be zero by the domain and codomain

- Example 3.10 in [HHR16] demonstrates that all restrictions in $H\underline{\mathbb{Z}}_6(S^\Lambda)$ are the identity map. Using the fact that $H\underline{\mathbb{Z}}_n(-)$ is a cohomological Mackey functor (see second remark below Lemma 3.4.1), we deduce that the transfers must be multiplication by the index. These maps are the maps highlighted in pink.

- To determine the maps between $H\underline{\mathbb{Z}}_0^G(S^\Lambda)$ and $H\underline{\mathbb{Z}}_0^{C_7}(S^\Lambda)$ (highlighted in green), we examine the behavior of the poles of S^Λ. Let the map $a_V : S^0 \to S^V$ (known as the Euler class) be given by inclusion at the poles. Notice that if S^V has no G-fixed points except for the poles, then a_V is not homotopic to the constant map. On the other hand if S^V has

[1] Parts of the orbit category corresponding to the subgroups conjugate to C_3 have been suppressed for succinctness

any G-fixed points, there must be a path of G-fixed points connecting the poles and a_V is null-homotopic.

The sphere S^Λ has G-fixed points only at the poles, so let the nonzero class of a_V be the generator of $H\underline{\mathbb{Z}}_0^G(S^\Lambda) \cong \mathbb{Z}/7$. Then the restriction to $H\underline{\mathbb{Z}}_0^{C_7}(S^\Lambda) \cong \mathbb{Z}/7$ takes a_V to

$$i_{C_7}^* a_V : S^0 \to S^{3\lambda_7}$$

since forgetting the action of G for elements not in C_7 produces $S^{3\lambda_7}$ (see Section 6.1 for details). The sphere $S^{3\lambda_7}$ has G-fixed points only at the poles, so $i_{C_7}^* a_V$ also produces a nonzero class, which we designate as the generator. Thus the restriction $H\underline{\mathbb{Z}}_0^G(S^\Lambda) \to H\underline{\mathbb{Z}}_0^{C_7}(S^\Lambda)$ is the identity. The composition of restriction and transfer must be multiplication by the index, so the transfer is multiplication by 3.

- Lastly, the maps between $H\underline{\mathbb{Z}}_n^G(S^\Lambda)$ and $H\underline{\mathbb{Z}}_n^{C_3}(S^\Lambda)$ for $n = 2, 4$ (highlighted in yellow) are isomorphisms. Since the restriction and transfer must compose to multiplication by 7, which is the identity map when interpreted modulo 3, the two maps must be inverses of each other. Depending on one's choice of generator, we can specify the maps as either the identity or multiplication by -1, and for convenience we choose the identity.

Remark: The "bottom half" maps to and from $H\underline{\mathbb{Z}}_n(S^\Lambda)(G/e)$ can also be determined by examining the chain complexes in Sections 6.1 and 6.2. Those chain complexes contain maps Δ and ∇, and these maps become the restrictions and transfers in our desired Mackey functor after passing to homology.

This computation of $H\underline{\mathbb{Z}}_n(S^\Lambda)$ is consistent with the result presented in Theorem 6.5 of [Ang22]. In said theorem, our computation is the case $t, r = 0$, $s = 1$, $p = 3$ and $q = 7$.

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