Verification of interconnected systems via parametric assume-guarantee contracts

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Abstract— **The paper verifies interconnected systems via parametric assume-guarantee contracts (AGC), which encode behaviors of a system in a parameter domain. In our approach to solve this verification problem, and by assuming that each component of the system satisfies its own parametric AGC separately, we define a mapping that generates the sequence of parameters for which the corresponding contracts are satisfied after interconnecting the components. Then if a small gain condition on the sequence of parameters holds, a new parametric AGC is declared for the interconnected system. A small gain theorem on bounded input bounded output (BIBO) stability is recovered by the obtained results showing the relation between the assume-guarantee reasoning and the small gain approach. We also provide an example of a large-scale transportation system to illustrate the significance of our results.**

I. Introduction

The computational complexity of verifying monolithically interconnected systems may be exponential in the number of interacting components. One technique to address this state-explosion problem is by using assume-guarantee contracts. AGCs enable a "divide-and-conquer" approach for verifying complex properties of an interconnected system compositionally, by verifying the system's components separately while making assumptions on each component's environment [1], [2]. In this context we consider the notion of parametric AGCs [3], encoding behaviors of a system in a parameter domain. Indeed, parametrization of contracts allows for tighter guarantees on the system's behavior, while a contract with a different form has a coarse guarantee since it is designed based on the worst case assumption on the system's environment.

Then we study the parametric AGC verification problem: given an interconnected system composed of $N \geq 2$ components, and by assuming that each component's behavior satisfies separately a complex property encoded by a parametric AGC, provide conditions on the contracts' elements so that the interconnected system's behavior satisfies a global property.

In our approach to solve the parametric AGC verification problem, we impose conditions on the contracts' guarantees and assumptions. Moreover, we define a mapping that generates the sequence of parameters for which the parametric AGCs are satisfied after interconnecting the system's components together. If this sequence of parameters converges to a limit point, a new parametric AGC is declared for the interconnected system. The result is shown to recover a small gain theorem for guaranteeing the bounded input bounded output stability of an interconnection of dynamical systems using classical trajectory-based small gain theorems [4], [5]. In addition to BIBO stability, more complex specifications for verifying an interconnected system, such as a fragment of linear temporal logic (LTL) specifications, can be embedded in the framework of parametric AGCs as shown in [3]. Then, using a temporal logic specification, we demonstrate our results by verifying the behavior of a large-scale transportation system.

Various compositional approaches are developed in literature to verify properties of an interconnected system. In the traditional control theory literature which is concerned mainly with stability properties, compositional methods in the form of small gain theorems are established in [6], [5] for continuous systems and in [7] for discrete systems. In the formal verification and symbolic controller synthesis literature, where the desired properties on the system's behavior become much more complicated, such as temporal logic properties [8], several compositional results are presented. See, e.g., [9], [10], [11], [12], [13], [14], [15], [16] for abstraction-based approaches and [17], [18], [19], [20], [21], [22] for methods relying on assume-guarantee reasoning. Our approach is mostly related to the work in [3], where the authors consider verification of interconnected systems via parametric AGC, instead of fixed AGCs as in [19], [20], but just for the case of two components.

This manuscript is organized as the following: after the introduction, the parametric AGC verification problem is formulated in Section II for an interconnection of systems. The problem's solution is presented in Section III where the main result is established in Theorem 1. Section IV re-establishes a classical small gain theorem using the main result. An example of a transportation network illustrates the significance of our approach in Section V before concluding our work.

Notations

Let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{N}^+$ denote the sets of reals, non-negative reals, positive reals, non-negative integers and positive integers, respectively. For $I \subseteq \mathbb{R}_0^+$, let $\mathbb{N}_I = \mathbb{N} \cap I$. For $N \in \mathbb{N}$ we denote by $\overline{N}(i)$ the set $\mathbb{N}_{[1,N]} \setminus \{i\}$. For a set $\mathcal{S}$, we denote the set of all subsets of $\mathcal{S}$ by $2^{\mathcal{S}}$. We denote by $\mathbf{cl}(\mathcal{S})$ the closure of the set $\mathcal{S}$. We denote by $M =$

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$(M_{ij}) = \mathbf{diag}(a_1, \ldots, a_N)$ a matrix with diagonal elements $M_{ii} = a_i, i \in \mathbb{N}_{[1,N]}$, and zero off-diagonal elements.

II. Problem Formulation

In this paper, a dynamical system is defined as a relation between internal input signals, external input signals, and output signals.

Definition 1: A dynamical system $\Sigma(\mathcal{U}_e[\cdot], \mathcal{U}_f[\cdot], \mathcal{Y}[\cdot])$ is a relation

$$\Sigma(\mathcal{U}_e[\cdot], \mathcal{U}_f[\cdot], \mathcal{Y}[\cdot]) \subseteq \mathcal{U}_e[\cdot] \times \mathcal{U}_f[\cdot] \times \mathcal{Y}[\cdot], \tag{1}$$

where $\mathcal{U}_e[\cdot]$, $\mathcal{U}_f[\cdot]$, and $\mathcal{Y}[\cdot]$ are the external input, internal input, and output set of signals respectively.

We assume that any input $(u_e[\cdot], u_f[\cdot]) \in \mathcal{U}_e[\cdot] \times \mathcal{U}_f[\cdot]$ is paired with at least one $y[\cdot] \in \mathcal{Y}[\cdot]$ using the relation $\Sigma(\mathcal{U}_e[\cdot], \mathcal{U}_f[\cdot], \mathcal{Y}[\cdot])$. Such $y[\cdot]$ is unique if (1) is deterministic, and we write $y[\cdot] = \Sigma(u_e[\cdot], u_f[\cdot])$, otherwise $y[\cdot]$ is not unique where in this case we say that $y[\cdot] \in \Sigma(u_e[\cdot], u_f[\cdot])$. If a dynamical system does not have internal inputs then Definition 1 reduces to $\Sigma(\mathcal{U}_e[\cdot], \mathcal{Y}[\cdot]) \subseteq \mathcal{U}_e[\cdot] \times \mathcal{Y}[\cdot]$. We use the latter definition when we define the interconnected system in Section II-A. Note that in some cases and for the sake of brevity we just use the notion Σ for a dynamical system.

A. Interconnected system

The notion of interconnection is given here similar to [23, Definition 3.1]:

Definition 2: Given an output set $\mathcal{Y}[\cdot]$ and an internal input set $\mathcal{U}_f[\cdot]$, an interconnection $\mathcal{I}$ is a tuple $\mathcal{I} = (\mathcal{Y}[\cdot], \mathcal{U}_f[\cdot], \mathcal{G})$, where $\mathcal{G} : \mathcal{Y}[\cdot] \mapsto \mathcal{U}_f[\cdot]$ maps output signals to internal input signals.

Now we formally define an interconnected system Σ as:

Definition 3: Consider $N \in \mathbb{N}^+$ subsystems $\Sigma^i(\mathcal{U}_e^i[\cdot], \mathcal{U}_f^i[\cdot], \mathcal{Y}^i[\cdot])$, $i \in \mathbb{N}_{[1,N]}$, and an interconnection

$$\mathcal{I} = (\prod_{i=1}^{N} \mathcal{Y}^i[\cdot], \prod_{i=1}^{N} \mathcal{U}_f^i[\cdot], \mathcal{G}), \tag{2}$$

defining the coupling between these subsystems. We define an interconnected system $\mathcal{I}(\Sigma^1, \ldots, \Sigma^N)$ as a relation $\Sigma(\mathcal{U}_e[\cdot], \mathcal{Y}[\cdot])$, with $\mathcal{U}_e[\cdot] = \prod_{i=1}^{N} \mathcal{U}_e^i[\cdot]$, $\mathcal{Y}[\cdot] = \prod_{i=1}^{N} \mathcal{Y}^i[\cdot]$, and internal inputs of $\Sigma^1, \ldots, \Sigma^N$ constrained by

$$[u_f^1[\cdot]; \ldots; u_f^N[\cdot]] \in \mathcal{G}([y^1[\cdot]; \ldots; y^N[\cdot]]). \tag{3}$$

.

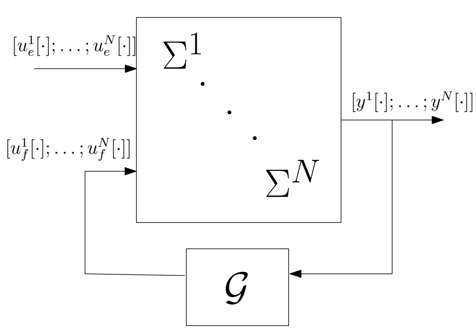

Fig. 1: The interconnected system $\mathcal{I}(\Sigma^1, \ldots, \Sigma^N)$.

B. Assume-guarantee contracts

In the sequel, a specification Φ over the set $\mathcal{Z}[\cdot]$ describes a set of desirable input-output behaviors, and in the set point of view it satisfies the set inclusion $\Phi \subseteq \mathcal{Z}[\cdot]$. For example an input (or output) specification Φ, for system $\Sigma(\mathcal{U}_e[\cdot], \mathcal{Y}[\cdot])$, satisfies $\Phi \subseteq \mathcal{U}_e[\cdot]$ (or $\Phi \subseteq \mathcal{Y}[\cdot]$). In the Boolean point of view, $z[\cdot] \models \Phi$ means that a signal $z[\cdot] \in \mathcal{Z}[\cdot]$ satisfies the specification Φ. Consequently, projection from a Boolean point of view of a specification to a set point of view is possible, where $\Phi = \{z[\cdot] \in \mathcal{Z}[\cdot] : z[\cdot] \models \Phi\}$. It is obvious then that $z[\cdot] \in \Phi$ if and only if $z[\cdot] \models \Phi$. We note that it will be clear from the context whether a specification is interpreted from the set or Boolean point of view.

Next, we define an assume-guarantee contract as given in [3]:

Definition 4: (Assume-Guarantee Contract) An assume-guarantee contract $\mathcal{C}$ is a pair (Φ_a, Φ_g) consisting of an assumption specification Φ_a and a guarantee specification Φ_g such that $\Phi_a \Rightarrow \Phi_g$ holds true.

We say that a system Σ satisfies $\mathcal{C} = (\Phi_a, \Phi_g)$, if $\Sigma \cap \Phi_a \subseteq \Phi_g$. It can also be easily shown that $(\Phi_a, \Phi_g) = (\Phi_a, \Phi_a \Rightarrow \Phi_g)$. This transformation of the contract is useful when dealing with parametric assume-guarantee contracts for a system $\Sigma(\mathcal{U}_e[\cdot], \mathcal{U}_f[\cdot], \mathcal{Y}[\cdot])$, which are given as:

Definition 5: (Parametric Assume-Guarantee Contract) An assume-guarantee contract $\mathcal{C} = (\Phi_a, \Phi_g)$ is in parametric form if there exists an external parametric assumption specification $\Psi_{ae} : \mathcal{P}_{ae} \mapsto 2^{\mathcal{U}_e[\cdot]}$, an internal parametric assumption specification $\Psi_{af} : \mathcal{P}_{af} \mapsto 2^{\mathcal{U}_f[\cdot]}$, a parametric guarantee specification $\Psi_g : \mathcal{P}_g \mapsto 2^{\mathcal{Y}[\cdot]}$, and parameter map $\lambda : \mathcal{P}_{ae} \times \mathcal{P}_{af} \mapsto \mathcal{P}_g$ such that:

$$\Phi_a = \bigvee_{(p_e, p_f) \in \mathcal{P}_{ae} \times \mathcal{P}_{af}} \Big(\Psi_{ae}(p_e) \wedge \Psi_{af}(p_f) \Big) \tag{4a}$$

$$\Phi_g = \bigwedge_{(p_e, p_f) \in \mathcal{P}_{ae} \times \mathcal{P}_{af}} \Bigg(\Big(\Psi_{ae}(p_e) \quad \wedge \quad \Psi_{af}(p_f) \Big) \Rightarrow \quad \Psi_g(\lambda(p_e, p_f)) \Bigg), \tag{4b}$$

where $\mathcal{P}_{ae}$, $\mathcal{P}_{af}$, and $\mathcal{P}_g$ are the external input parametric set, internal input parametric set, and output parametric set.

Remark 1: In case a system does not have an internal input then Φ_a and Φ_g in (4) reduces to:

$$\Phi_a = \bigvee_{p_e \in \mathcal{P}_{ae}} \Psi_{ae}(p_e) \tag{5a}$$

$$\Phi_g = \bigwedge_{p_e \in \mathcal{P}_{ae}} \Big(\Psi_{ae}(p_e) \Rightarrow \Psi_g(\lambda(p_e)) \Big). \tag{5b}$$

Note that we can write a parametric assume-guarantee contract as a conjunction of smaller contracts: $\mathcal{C}$ =

$\bigwedge_{(p_e,p_f)\in\mathcal{P}_{ae}\times\mathcal{P}_{af}} \mathcal{C}(p_e,p_f)$ where $\mathcal{C}(p_e,p_f) = \Big(\Psi_{ae}(p_e) \wedge \Psi_{af}(p_f), \big(\Psi_{ae}(p_e) \wedge \Psi_{af}(p_f)\big) \Rightarrow \Psi_g(\lambda(p_e,p_f)\Big)$ and it is clear that system Σ satisfies $\mathcal{C}$ if for all $(p_e,p_f) \in \mathcal{P}_{ae}\times\mathcal{P}_{af}, \Sigma \cap \big(\Psi_{ae}(p_e) \cap \Psi_{af}(p_f)\big) \subseteq \Psi_g(\lambda(p_e,p_f))$. Definition 5 says that a system under a given parametric AGC must satisfy only the guarantee specifications whose corresponding assumption specifications are triggered. In this paper, we are interested in using parametric assume-guarantee contracts in order to verify an interconnected system, or equivalently we provide a solution to the following problem:

Problem 1: Consider an interconnection $\mathcal{I} = (\Sigma^1,\dots,\Sigma^N)$ and a set of parametric assume-guarantee contracts $\{\mathcal{C}^1,\dots,\mathcal{C}^N\}$, such that each system Σ^i satisfies the contract $\mathcal{C}^i$, $i \in \mathbb{N}_{[1,N]}$. Derive small gain conditions guaranteeing that $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$ satisfies a parametric AGC $\mathcal{C}$ to be determined.

We derive in Section III a generalized small gain theorem for parametric contracts and recover later, in Section IV, results on a small gain theorem that ensures BIBO stability of an interconnected system.

III. A General SGT for Parametric Assume-Guarantee Contracts

This section presents the main results in this manuscript by solving Problem 1. The parametric AGCs $\mathcal{C}^i = (\Phi^i_a,\Phi^i_g)$, $i \in \mathbb{N}_{[1,N]}$, are given where Φ^i_a and Φ^i_g are defined by (4) with specifications Ψ^i_{af}, Ψ^i_{ae}, and Ψ^i_g and parameter sets $\mathcal{P}^i_{af}$, $\mathcal{P}^i_{ae}$, and $\mathcal{P}^i_g$.

Also each subsystem $\Sigma^i(\mathcal{U}^i_e[\cdot],\mathcal{U}^i_f[\cdot],\mathcal{Y}^i[\cdot])$, $i \in \mathbb{N}_{[1,N]}$, has an internal input, an external input, and an output satisfy

Assumption 1: The input sets $\mathcal{U}^i_f[\cdot]$, $i \in \mathbb{N}_{[1,N]}$, are given by:

$$\mathcal{U}^i_f[\cdot] = \mathcal{Y}^{i_1}[\cdot]\times\dots\times\mathcal{Y}^{i_{N-1}}[\cdot]. \tag{6}$$

where $i_1,\dots,i_{N-1} \in \overline{N}(i)$ and $i_j \neq i_k$ for $j \neq k$.

The internal assumption parameter sets are further partitioned:

Assumption 2: The internal assumption parameter sets $\mathcal{P}^i_{af}$, $i \in \mathbb{N}_{[1,N]}$, satisfy

$$\mathcal{P}^i_{af} = \mathcal{P}^{ii_1}_{af}\times\dots\times\mathcal{P}^{ii_N}_{af}, \tag{7}$$

where $i_1,\dots,i_{N-1} \in \overline{N}(i)$ and $i_j \neq i_k$ for $j \neq k$.

Following Assumptions 1 and 2, we assume that Ψ^i_{af} is a conjunction of specifications:

Assumption 3: The specifications $\Psi^i_{af} : \mathcal{P}^{ii_1}_{af}\times\dots\times\mathcal{P}^{ii_N}_{af} \mapsto 2^{\mathcal{Y}^{i_1}[\cdot]\times\dots\times\mathcal{Y}^{i_{N-1}}[\cdot]}$, $i \in \mathbb{N}_{[1,N]}$, $i_1,\dots,i_{N-1} \in \overline{N}(i)$ and $i_j \neq i_k$ for $j \neq k$, satisfy:

$$\Psi^i_{af}(p^i_f) = \bigwedge_{j\in\overline{N}(i)} \Psi^{ij}_{af}(p^{ij}_f), \tag{8}$$

where $\Psi^{ij}_{af} : \mathcal{P}^{ij}_{af} \mapsto 2^{\mathcal{Y}^j[\cdot]}$ and $p^i_f = (p^{ii_1}_f,\dots,p^{ii_{N-1}}_f)$.

The parametric guarantee specification for Σ^i, $i \in \mathbb{N}_{[1,N]}$, is also given by $\Psi^i_g : \mathcal{P}^i_g \mapsto 2^{\mathcal{Y}^i[\cdot]}$ and the parameter map in Definition 5, associated to system Σ^i, is given by

$$\lambda^i : \mathcal{P}^i_{ae}\times\mathcal{P}^i_{af} \mapsto \mathcal{P}^i_g. \tag{9}$$

One last assumption is made on the assumption and guarantee specifications.

Assumption 4: Consider Assumptions 1, 2, 3, $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$, and a set of parametric assume-guarantee contracts $\{\mathcal{C}^1,\dots,\mathcal{C}^N\}$. Then:

1) Every system Σ^i satisfies its parametric assume-guarantee contract $\mathcal{C}^i = (\Phi^i_a,\Phi^i_g)$ with Φ^i_a and Φ^i_g defined as

$$\Phi^i_a = \bigwedge_{(p^i_e,p^i_f)\in\mathcal{P}^i_{ae}\times\mathcal{P}^i_{af}} \Big(\Psi^i_{ae}(p^i_e)\wedge\Psi^i_{af}(p^i_f)\Big), \tag{10a}$$

$$\Phi^i_g = \bigvee_{(p^i_e,p^i_f)\in\mathcal{P}^i_{ae}\times\mathcal{P}^i_{af}} \Big(\Psi^i_{ae}(p^i_e)\wedge\Psi^i_{af}(p^i_f)\Big) \Rightarrow \Psi^i_g(\lambda^i(p^i_e,p^i_f)). \tag{10b}$$

2) The parameter sets satisfy $\mathcal{P}^{i_j}_g \subseteq \mathcal{P}^{ii_j}_{af}$, $j \in \mathbb{N}_{[1,N-1]}$, $i_j \in \overline{N}(i)$, and $i \in \mathbb{N}_{[1,N]}$. Also, the guarantee specifications must imply the internal assumption specifications:

$$\Psi^{i_j}_g(p) \Rightarrow \Psi^{ii_j}_{af}(p), \tag{11}$$

for $p \in \mathcal{P}^{i_j}_g$, $i_j \in \overline{N}(i)$, $j \in \mathbb{N}_{[1,N-1]}$, and $i \in \mathbb{N}_{[1,N]}$.
3) There exists an external parameter $p_e = (p^1_e,\dots,p^N_e) \in \mathcal{P}^1_{ae}\times\dots\times\mathcal{P}^N_{ae}$ such that for all $i \in \mathbb{N}_{[1,N]}$, $\Psi^i_{ae}(p^i_e)$ is satisfied.
4) There exists an internal parameter $p_f[0] = (p^1_f[0],\dots,p^N_f[0]) \in \mathcal{P}^1_{af}\times\dots\times\mathcal{P}^N_{af}$ such that $\Psi^i_{af}(p^i_f)$ is satisfied for all $i \in \mathbb{N}_{[1,N]}$.

Remark 2: The first item of Assumption 4 or equivalently verifying whether or not subsystem Σ^i satisfies the parametric AGC $\mathcal{C}^i$ is checked by following the falsification procedure as explained in [3, Section 6].

Before presenting the main result, we state a lemma which will be used in the former's proof.

Lemma 1: Consider Assumptions 1, 2, 3, $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$, and a set of parametric assume-guarantee contracts $\{\mathcal{C}^1,\dots,\mathcal{C}^N\}$. Suppose that Assumption 4 holds with internal and external parameters $p_f[0]$ and p_e respectively. In addition, define for every contract $\mathcal{C}^i$, $i \in \mathbb{N}_{[1,N]}$, and as a function of λ^i in (9), a new internal parameter map $\hat\lambda^i(\cdot) = \lambda^i(p^i_e,\cdot)$ and define guarantee parameter iterations

$$p_g[k+1] = \Gamma(p_g[k]), \quad k \in \mathbb{N}, \tag{12}$$

with $p_g[k] = [p^1_g[k];\dots;p^N_g[k]]$, $\Gamma = [\hat\lambda^1;\dots;\hat\lambda^N]$, and $p^i_g[0] = \hat\lambda^i(p^1_f[0],\dots,p^N_f[0])$, $i \in \mathbb{N}_{[1,N]}$. Then the guarantee simplifies to

$$\bigwedge_{i\in\mathbb{N}_{[1,N]}} \bigwedge_{k\in\mathbb{N}} \Psi^i_g(p^i_g[k]). \tag{13}$$

Proof: External parameter p_e and $p_f[0]$ in addition to (10b) and (11) implies that the following conjunction is true

$$\bigwedge_{i\in\mathbb{N}_{[1,N]}} \Big(\Psi^i_{ae}(p^i_e) \wedge \Psi^i_{af}(p^i_f[0])\Big)$$
$$\wedge \bigwedge_{i\in\mathbb{N}_{[1,N]}} \Big(\Big(\Psi^i_{ae}(p^i_e) \wedge \Psi^i_{af}(p^i_f[0])\Big) \Rightarrow \Psi^i_g(p^i_g[0])\Big)\Big)$$
$$\wedge\Big(\bigwedge_{i\in\mathbb{N}_{[1,N]}} \big(\Psi^i_g(p^i_g[0])\big) \Rightarrow \bigwedge_{i\in\mathbb{N}_{[1,N]}} \big(\Psi^i_{af}(p^i_g[0])\big)\Big).$$

Now using the sequence of parameters generated by (12) as well, further guarantees hold true

$$\bigwedge_{\mathbf{i}\in\mathbb{N}_{[\mathbf{1},\mathbf{N}]}} \Big(\mathbf{\Psi^i_g}(\mathbf{p^i_g}[\mathbf{0}])\Big) \wedge \bigwedge_{i\in\mathbb{N}_{[1,N]}} \Big(\Psi^i_{ae}(p^i_e) \wedge \Psi^i_{af}(p^i_g[0])\Big)$$
$$\wedge \bigwedge_{i\in\mathbb{N}_{[1,N]}} \Big(\Big(\Psi^i_{ae}(p^i_e) \wedge \Psi^i_{af}(p^i_g[0])\Big) \Rightarrow \Psi^i_g(p^i_g[1])\Big)\Big)$$
$$\wedge\Big(\bigwedge_{\mathbf{i}\in\mathbb{N}_{[\mathbf{1},\mathbf{N}]}} \big(\mathbf{\Psi^i_g}(\mathbf{p^i_g}[\mathbf{1}])\big) \Rightarrow \bigwedge_{i\in\mathbb{N}_{[1,N]}} \big(\Psi^i_{af}(p^i_g[1])\big)\Big) \wedge \dots$$

It is obvious then that the guarantee (13) is a subsequence (components are in bold) within the latter infinite sequence of conjunctions. ∎

By exploiting some additional assumptions on the map Γ in (12) the guarantee in (13) could be further simplified.

Theorem 1: (SGT for parametric AGCs) Consider $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$ and a set of parametric assume-guarantee contracts $\{\mathcal{C}^1,\dots,\mathcal{C}^N\}$. Suppose that the assumptions of Lemma 1, with an external parameter $p_e = (p^1_e,\dots,p^N_e)$, hold. Assume also that

1) For every $i \in \mathbb{N}_{[1,N]}$ there exists a metric $\mathbf{d^i} : \mathcal{P}^i_g \times \mathcal{P}^i_g \mapsto \mathbb{R}^+_0$ on $\mathcal{P}^i_g$. The Hausdorff distance $\mathbf{d_H}$ is also a metric on $\Psi^i_g(\cdot)$, $i \in \mathbb{N}_{[1,N]}$.
2) The specification Ψ^i_g varies continuously with parameters in $\mathcal{P}^i_g$, $i \in \mathbb{N}_{[1,N]}$. In other words, for every $\varepsilon^i > 0$ and $p_1 \in \mathcal{P}^i_g$ there exists a $\delta^i > 0$ such that
$$\mathbf{d^i}(p_1,p) < \delta^i \Rightarrow d_H(\Psi^i_g(p_1),\Psi^i_g(p)) < \varepsilon^i,$$
for $i \in \mathbb{N}_{[1,N]}$.
3) The sequence $(p_g[k])_{k\in\mathbb{N}}$ satisfying (12) converges for any $p_g[0] \in \mathcal{P}^1_g \times \dots \times \mathcal{P}^N_g$ to a parameter $\hat{p}_g = [\hat{p}^1_g;\dots;\hat{p}^N_g]$.

Then $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$ satisfies the parametric AGC $\mathcal{C} = (\Phi_a,\Phi_g)$ given by (5) with $\mathcal{P}_{ae} = \prod_{i=1}^N \mathcal{P}^i_{ae}$, $\mathcal{P}_g = \prod_{i=1}^N \mathcal{P}^i_g$, $\Psi_{ae}(p_e) = \bigwedge_{i\in\mathbb{N}_{[1,N]}} \Psi^i_{ae}(p^i_e)$, $\lambda(p_e) = \hat{p}_g$, and

$$\Psi_g(\hat{p}_g) = \bigwedge_{i\in\mathbb{N}_{[1,N]}} \Big(\mathbf{cl}\big(\Psi^i_g(\hat{p}^i_g)\big)\Big). \quad (14)$$

Proof: The proof follows from [3, Theorem 2] using the fact that the sequence $(p_g[k])_{k\in\mathbb{N}}$ converges to a limit point $\hat{p}_g = [\hat{p}^1_g;\dots;\hat{p}^N_g]$ where we can conclude that the guarantee in (13) simplifies to (14). ∎

Remark 3: In [3], it is shown that a fragment of linear temporal logic (LTL) specifications satisfies Assumption 2) in Theorem 1 and thus allows for considering contracts defined by LTL specifications.

IV. SGT on Bounded Input Bounded Output Stability

Using the proposed results in the previous sections and under an additional assumption on the map Γ in (12), we recover here a small gain theorem on bounded input bounded output stability of an interconnected system which is analogous to the asymptotic gain property (AG) in [5]. Given a norm $|\cdot|$, we denote by $\mathcal{L}$ the set of norm bounded signals. A function $\gamma : \mathbb{R}^+_0 \mapsto \mathbb{R}^+_0$ is said to be of class $\mathcal{K}$, or $\gamma \in \mathcal{K}$, if it is continuous, increasing, and $\gamma(0) = 0$. We say $\gamma \in \mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. For later derivations, we make the following assumption on a map $\Gamma_s : (\mathbb{R}^+_0)^N \mapsto (\mathbb{R}^+_0)^N$, which is used in Corollary 1 to define the map Γ.

Assumption 5: Consider a map $\Gamma_s : (\mathbb{R}^+_0)^N \mapsto (\mathbb{R}^+_0)^N$. Γ_s is irreducible and there exist $\alpha^i \in \mathcal{K}_\infty$, $i \in \mathbb{N}_{[1,N]}$, such that

$$(\Gamma_s \circ D_s)(s) \ngeq s, \quad \forall s \in (\mathbb{R}^+)^N, \quad (15)$$

where $D : (\mathbb{R}^+_0)^N \mapsto (\mathbb{R}^+_0)^N$ is defined by $D_s(s^1,\dots,s^N) = [(Id+\alpha^1)(s^1);\dots;(Id+\alpha^N)(s^N)]$. The next result follows from Theorem 1.

Corollary 1: (SGT on BIBO stability) Consider an interconnected system $\mathcal{I}(\Sigma^1,\dots,\Sigma^N)$, with $\Sigma^i(\mathcal{L},\mathcal{L}^{N-1},\mathcal{L})$, $i \in \mathbb{N}_{[1,N]}$. Assume that for every $i \in \mathbb{N}_{[1,N]}$, there exist $\gamma^{i1},\dots,\gamma^{iN},\gamma^{u_i} \in \mathcal{K}\cup\{0\}$, $\gamma^{ii} = 0$, such that

$$|y^i[\cdot]| \le \sum_{j\in\mathbb{N}} \gamma^{ij}(|y^j[\cdot]|) + \gamma^{u_i}(|u^i[\cdot]|). \quad (16)$$

If $\Gamma_s : (\mathbb{R}^+_0)^N \mapsto (\mathbb{R}^+_0)^N$, defined by

$$\Gamma_s(s_1,\dots,s_N) = \Big[\sum_{j=1}^N \gamma^{1j}(s_j);\dots;\sum_{j=1}^N \gamma^{Nj}(s_j)\Big], \quad (17)$$

satisfies Assumption 5, then there exists a $\beta \in \mathcal{K}_\infty$ such that:

$$||y[\cdot]|_v| \le \beta(|D(|u[\cdot]|_v)|), \quad (18)$$

where $|y[\cdot]|_v = [|y^1[\cdot]|;\dots;|y^N[\cdot]|]$, $D = \mathbf{diag}(\gamma^{u_1},\dots,\gamma^{u_N})$, and $|u[\cdot]|_v = [|u^1[\cdot]|;\dots;|u^N[\cdot]|]$.

Proof: We define parameter sets by

$$\mathcal{P}^i_{ae},\mathcal{P}^i_g \in \mathbb{R}^+_0\cup\{\infty\}, \text{ and } \mathcal{P}^i_{af} \in (\mathbb{R}^+_0\cup\{\infty\})^{N-1}, \quad (19)$$

$i \in \mathbb{N}_{[1,N]}$. Also, for all $i \in \mathbb{N}_{[1,N]}$ parametric assumption and parametric guarantee specifications are defined as:

$$\Psi^i_{ae}(p) = |u^i[\cdot]| \le p, \quad (20a)$$
$$\Psi^{ij}_{af}(r) = |y^j[\cdot]| \le r, \ \ j \in \overline{N}(i), \quad (20b)$$
$$\Psi^i_g(r) = |y^i[\cdot]| \le r, \quad (20c)$$

and parameter maps as

$$\lambda^i(p,r^i) = \gamma^{u_i}(p) + \sum_{j\in\overline{N}(i)} \gamma^{ij}(r^{ij}) \quad (21)$$

with $r^i = [r^{ii_1}; \ldots ; r^{ii_{N-1}}]$, $i_1, \ldots, i_{N-1} \in \overline{N}(i)$ and $i_j \neq i_k$ for $j \neq k$. Consequently, we can reformulate bounds (16) using the parametric assume-guarantee contracts $\mathcal{C}^i = (\Phi_a^i, \Phi_g^i)$, with Φ_a^i, Φ_g^i defined as in (10) with Ψ_{ae}^i, Ψ_{af}^{ij}, Ψ_g^i given by (20) and with parameter map given by (21). Therefore, Assumptions 1, 2, and 3 are satisfied as well as the first item of Assumption 4. The second item of Assumption 4 is also satisfied because the guarantees and internal assumptions are of the same form. The third and fourth items of Assumption 4 follow from the boundedness of the internal and external input signals which guarantees the existence of internal and external parameters such that the assumption specifications are satisfied. The first and second conditions of Theorem 1 are also satisfied with $\mathcal{P}_g^i$, $i \in \mathbb{N}_{[1,N]}$, and Ψ_g^i, $i \in \mathbb{N}_{[1,N]}$, as in (19) and (20c) respectively.

Now for fixed $|u^i[\cdot]|$, $i \in \mathbb{N}_{[1,N]}$, the internal iteration maps are given by

$$\hat{\lambda}^i(r^i) = \gamma^{u_i}(|u^i[\cdot]|) + \sum_{j \in \overline{N}(i)} \gamma^{ij}(r^{ij}), \tag{22}$$

with $r^i = [r^{ii_1}; \ldots ; r^{ii_{N-1}}]$, $i_1, \ldots, i_{N-1} \in \overline{N}(i)$ and $i_j \neq i_k$ for $j \neq k$. It follows from (22) that the map Γ in (12) is given by $\Gamma(s) = \Gamma_s(s) + D(|u[\cdot]|_v)$ for any $s \in (\mathbb{R}^+)^N$. Since Γ_s satisfies Assumption 5, then, using [5, Theorem 23], Γ_s is a decreasing operator with $\lim_{k \to \infty} \Gamma_s^k(s) = 0$ for any $s \in (\mathbb{R}^+)^N$. This implies that Γ is indeed decreasing and converges to a limit point $\hat{p}_g$ satisfying:

$$(Id - \Gamma_s)\hat{p}_g \leq D(|u[\cdot]|_v). \tag{23}$$

Therefore, using [5, Lemma 13], there exists $\beta \in \mathcal{K}_\infty$ such that:

$$|\hat{p}_g| \leq \beta(|D(|u[\cdot]|_v)|). \tag{24}$$

Using Theorem 1, inequality (18) is satisfied which completes the proof. ■

In the next section, we present a large-scale transportation system to demonstrate the results obtained by Theorem 1.

V. Illustrative Example

We consider a large-scale transportation system $\mathcal{I}(\Sigma^1, \ldots, \Sigma^N)$ consisting of $N \in \mathbb{N}_{[3,+\infty)}$ interconnected segments Σ^i, depicted each by Figure 2. Subsystem Σ^i consists of 10 links $l_1^i, \ldots, l_{10}^i$ and is given by the discrete dynamics, as in [24]:

$$\begin{aligned}
x_1^i[k+1] &= x_1^i[k] - f_1^{out_i}[k] + f_1^{in_i}[k] \\
x_2^i[k+1] &= x_2^i[k] - f_2^{out_i}[k] + f_2^{in_i}[k] \\
x_3^i[k+1] &= x_3^i[k] - f_3^{out_i}[k] + f_1^{out_i}[k] + f_2^{out_i}[k] \\
x_4^i[k+1] &= x_4^i[k] - f_4^{out_i}[k] + f_4^{in_i}[k] \\
x_5^i[k+1] &= x_5^i[k] - f_5^{out_i}[k] + f_3^{out_i}[k] + f_4^{out_i}[k] \\
x_j^i[k+1] &= x_j^i[k] - f_j^{out_i}[k] + f_{j-1}^{out_i}[k], \quad j \in \mathbb{N}_{[6,10]}
\end{aligned}$$

where $x_j^i \in \mathbb{R}_0^+$ represents the average number of vehicles in link l_j^i, $j \in \mathbb{N}_{[1,10]}$ and the output of Σ^i is $[x_1^i[\cdot]; x_{10}^i[\cdot]]$.

The interconnection $\mathcal{I}$, as in Figure 3, is given by (2) with $\mathcal{G} \in \mathbb{R}^{2N(N-1) \times 2N}$:

$$\mathcal{G} = \begin{bmatrix}
0 & G_1^1 & 0 & \ldots & \ldots & 0 & G_2^1 \\
G_2^2 & 0 & G_1^2 & 0 & \ldots & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & \ddots & \ddots & G_1^{N-1} \\
G_1^N & 0 & \ldots & \ldots & 0 & G_2^N & 0
\end{bmatrix},$$

where $G_1^1 = \begin{bmatrix} \overline{G}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $G_1^2 = \begin{bmatrix} 0 \\ \overline{G}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $G_1^N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{G}_1 \end{bmatrix}$ with $\overline{G}_1 = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2^1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{G}_2 \end{bmatrix}$, $G_2^2 = \begin{bmatrix} \overline{G}_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $G_2^N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{G}_2 \end{bmatrix}$ with $\overline{G}_2 = \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}$.

The flows out of links $l_1^i, \ldots, l_9^i$ are given, for $i \in \mathbb{N}_{[1,N]}$, by:

$$\begin{aligned}
f_1^{out_i}[k] &= \min\left(0.8(40 - x_3^i[k]), 10, x_1^i[k]\right), \\
f_2^{out_i}[k] &= \min\left(0.2(40 - x_3^i[k]), 5, x_2^i[k]\right), \\
f_3^{out_i}[k] &= \min\left(0.8(40 - x_5^i[k]), 10, x_3^i[k]\right), \\
f_4^{out_i}[k] &= \min\left(0.2(40 - x_5^i[k]), 5, x_4^i[k]\right), \\
f_j^{out_i}[k] &= \min\left(0.8(40 - x_{j+1}^i[k]), 10, x_j^i[k]\right), j \in \mathbb{N}_{[5,9]},
\end{aligned}$$

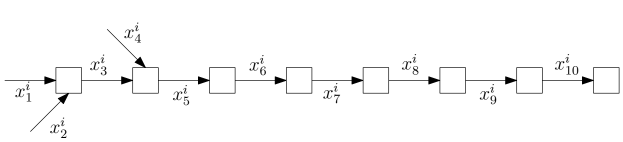

Fig. 2: Model of subsystem Σ^i.

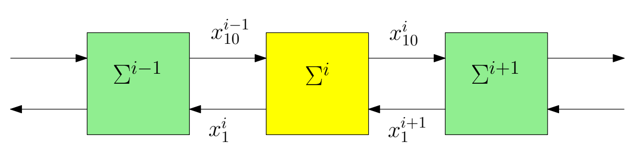

Fig. 3: Interconnections of subsystem Σ^i $i \notin \{1, N\}$.

As for links l_{10}^i, $i \in \mathbb{N}_{[1,N]}$, the output flows are given by:

$$f_{10}^{out_i}[k] = \min\left(0.2(40 - x_1^{i+1}[k]), 10, x_{10}^i[k]\right), i \in \mathbb{N}_{[1,N-1]},$$
$$f_{10}^{out_N}[k] = \min\left(0.2(40 - x_1^1[k]), 10, x_{10}^N[k]\right).$$

Furthermore, the flows into the links are given by:

$$\begin{aligned}
&f_1^{in_1}[k] = f_{10}^{out_N}[k],\\
&f_1^{in_i}[k] = \min\left(0.8(40 - x_1^i[k]), 10, x_{10}^{i-1}[k]\right), i \in \mathbb{N}_{[2,N]},\\
&f_2^{in_i}[k] = \min\left((20 - x_2^i[k]), d_2^i[k]\right), i \in \mathbb{N}_{[1,N]},\\
&f_4^{in_i}[k] = \min\left((20 - x_4^i[k]), d_4^i[k]\right), i \in \mathbb{N}_{[1,N]}.
\end{aligned}$$

A. Certifying assume-guarantee contracts

For all subsystems, the onramp demands are limited to be always less than 3:

$$\Box(d_j^i \le 3), j \in \{2, 4\}, i \in \mathbb{N}_{[1,N]}. \tag{25}$$

All links are assumed to have an initial number of vehicles less than 4. Using signal temporal logic formulas [25], we consider the parametric assume-guarantee contracts $\mathcal{C}^i = (\Phi_a^i, \Phi_g^i)$, $i \in \mathbb{N}_{[1,N]}$ with:

$$\begin{aligned}
\Phi_a^i = \bigvee_{s^i \ge 0, d^i \ge 0} \Big(&\Box_{[0,3]}\Diamond_{[0,2]}(s(x_1^{i+1}) \ge 10 - s^i)\\
&\wedge \Box_{[0,3]}\Diamond_{[0,2]}(\min(10, x_{10}^{i-1}) \le d^i)\Big)
\end{aligned} \tag{26a}$$

$$\begin{aligned}
\Phi_g^i = \bigwedge_{s^i \ge 0, d^i \ge 0} \bigg(\Big(&\Box_{[0,3]}\Diamond_{[0,2]}(s(x_1^{i+1}) \ge 10 - s^i)\\
&\wedge \Box_{[0,3]}\Diamond_{[0,2]}(\min(10, x_{10}^{i-1}) \le d^i)\Big)\\
&\Rightarrow \Big(\Box_{[0,3]}\Diamond_{[0,2]}(s(x_1^i) \ge 10 - \lambda_2(d^i))\\
&\wedge \Box_{[0,3]}\Diamond_{[0,2]}(\min(10, x_{10}^i) \le \lambda_1(s^i))\Big)\bigg),
\end{aligned} \tag{26b}$$

for $i \in \mathbb{N}_{[2,N-1]}$, where $s(x) = 0.8(40 - x)$, $\lambda_1(s) = 0.9s + 6.5$, and $\lambda_2(d) = 0.2d$. We note that the assumption and guarantee specifications for $\mathcal{C}^1$ and $\mathcal{C}^N$ are similar to (26) but for $\mathcal{C}^1$ the assumptions are made on the states x_1^2 and x_{10}^N whereas for $\mathcal{C}^N$ the assumptions are made on x_1^1 and x_{10}^{N-1}. Following Remark 2, we reformulate the falsification problem having signal temporal logic formulas into mixed integer linear programs [26]. Using the Gurobi optimization tool [27] the latter problem was not feasible and thus failed to violate (ϕ_a^i, ϕ_g^i), for any $s^i, d^i > 0$, $i \in \mathbb{N}_{[1,N]}$. Therefore Σ^i satisfies the parametric assume-guarantee contract (ϕ_a^i, ϕ_g^i), $i \in \mathbb{N}_{[1,N]}$. The conditions for the small gain theorem (i.e. Theorem 1) hold as the following:

- The parametric contracts are satisfied for each network.
- The internal assumptions of any network are implied by guarantees from neighboring networks because they are of the same form.
- The external assumptions are satisfied via (25).
- For any $i \in \mathbb{N}_{[1,N]}$ for a large enough $d^i \ge 0$ and $s^i \ge 0$, the internal assumption $(\Box_{[0,3]}\Diamond_{[0,2]}(s(x_1^i) \ge 10 - s^i) \wedge \Box_{[0,3]}\Diamond_{[0,2]}(\min(10, x_{10}^{i-1}) \le d^i))$ is satisfied because $\min(10, x_{10}^{i-1})$ has a maximum value of 10 and $s(x)$ has a minimum value of 0.

In addition, it can be shown that the sequence of parameters in (12) converges to a limit point having $[d^i; s^i] = [7.92, 1.585]$, $i \in \mathbb{N}_{[1,N]}$. Thus we conclude that the interconnected system $\mathcal{I}(\Sigma^1, \ldots, \Sigma^N)$ is guaranteed to satisfy the following specification:

$$\Box_{[0,3]}\Diamond_{[0,2]}(s(x_1^i) \ge 8.41) \wedge \Box_{[0,3]}\Diamond_{[0,2]}(\min(10, x_{10}^i) \le 7.92),$$

for $i \in \mathbb{N}_{[1,N]}$.

VI. Conclusion

In this work, small gain conditions were derived for parametric assume-guarantee contracts allowing for the compositional analysis of a large-scale system based on a fragment of LTL specifications. Using these conditions we recovered a classical small gain theorem guaranteeing BIBO stability for an interconnected system. Also, the validity of our approach was illustrated by a large-scale transportation system. Further investigations are carried to solve the controller synthesis problem in order to enforce a set of parametrized LTL specifications on an interconnected system.

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References

[1] C. S. Păsăreanu, D. Giannakopoulou, M. G. Bobaru, J. M. Cobleigh, and H. Barringer, "Learning to divide and conquer: applying the l* algorithm to automate assume-guarantee reasoning," *Formal Methods in System Design*, vol. 32, no. 3, pp. 175–205, 2008.

[2] A. Benveniste, B. Caillaud, A. Ferrari, L. Mangeruca, R. Passerone, and C. Sofronis, "Multiple viewpoint contract-based specification and design," in *International Symposium on Formal Methods for Components and Objects*. Springer, 2007, pp. 200–225.

[3] E. Kim, M. Arcak, and S. Seshia, "A small gain theorem for parametric assume-guarantee contracts," in *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*. ACM, 2017, pp. 207–216.

[4] S. Dashkovskiy and M. Kosmykov, "Input-to-state stability of interconnected hybrid systems," *Automatica*, vol. 49, no. 4, pp. 1068–1074, 2013.

[5] S. Dashkovskiy, B. Rüffer, and F. Wirth, "An iss small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.

[6] A. R. Teel, "A nonlinear small gain theorem for the analysis of control systems with saturation," *IEEE transactions on Automatic Control*, vol. 41, no. 9, pp. 1256–1270, 1996.

[7] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

[8] A. Pnueli, "The temporal logic of programs," in *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*. IEEE, 1977, pp. 46–57.

[9] M. Rungger and M. Zamani, "Compositional construction of approximate abstractions of interconnected control systems," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 1, pp. 116–127, 2018.

[10] F. Kerber and A. van der Schaft, "Compositional analysis for linear control systems," in *Proceedings of the 13th ACM international conference on Hybrid systems: computation and control*. ACM, 2010, pp. 21–30.

[11] P. Tabuada, G. Pappas, and P. Lima, "Compositional abstractions of hybrid control systems," *Discrete event dynamic systems*, vol. 14, no. 2, pp. 203–238, 2004.

[12] G. Pola, P. Pepe, and M. Di Benedetto, "Symbolic models for networks of control systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3663–3668, 2016.
[13] Y. Tazaki and J. Imura, "Bisimilar finite abstractions of interconnected systems," in *International Workshop on Hybrid Systems: Computation and Control*. Springer, 2008, pp. 514–527.
[14] D. Boskos and D. V. Dimarogonas, "Decentralized abstractions for feedback interconnected multi-agent systems," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 282–287.
[15] O. Hussien, A. Ames, and P. Tabuada, "Abstracting partially feedback linearizable systems compositionally," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 227–232, 2017.
[16] P.-J. Meyer, A. Girard, and E. Witrant, "Compositional abstraction and safety synthesis using overlapping symbolic models," *IEEE Transactions on Automatic Control*, vol. 63, no. 6, pp. 1835–1841, 2018.
[17] E. Dallal and P. Tabuada, "On compositional symbolic controller synthesis inspired by small-gain theorems," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 6133–6138.
[18] A. Le Coënt, L. Fribourg, N. Markey, F. De Vuyst, and L. Chamoin, "Distributed synthesis of state-dependent switching control," in *International Workshop on Reachability Problems*. Springer, 2016, pp. 119–133.
[19] A. Saoud, A. Girard, and L. Fribourg, "Contract based design of symbolic controllers for interconnected multiperiodic sampled-data systems," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 773–779.
[20] E. Kim, M. Arcak, and S. Seshia, "Directed specifications and assumption mining for monotone dynamical systems," in *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*. ACM, 2016, pp. 21–30.
[21] M. Al Khatib, A. Girard, and T. Dang, "Stability verification and timing contract synthesis for linear impulsive systems using reachability analysis," *Nonlinear Analysis: Hybrid Systems*, vol. 25, pp. 211–226, 2017.
[22] S. Sadraddini, J. Rudan, and C. Belta, "Formal synthesis of distributed optimal traffic control policies," in *2017 ACM/IEEE 8th International Conference on Cyber-Physical Systems (ICCPS)*. IEEE, 2017, pp. 15–24.
[23] E. Kim, M. Arcak, and M. Zamani, "Constructing control system abstractions from modular components," in *Proceedings of the 21st International Conference on Hybrid Systems: Computation and Control (part of CPS Week)*. ACM, 2018, pp. 137–146.
[24] C. Daganzo, "The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory," *Transportation Research Part B: Methodological*, vol. 28, no. 4, pp. 269–287, 1994.
[25] V. Raman, A. Donzé, D. Sadigh, R. Murray, and S. Seshia, "Reactive synthesis from signal temporal logic specifications," in *Proceedings of the 18th international conference on hybrid systems: Computation and control*. ACM, 2015, pp. 239–248.
[26] W. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, no. 7, pp. 1085–1091, 2001.
[27] G. OPTIMIZATION, "Inc. gurobi optimizer reference manual, 2015," *URL: http://www. gurobi. com*, 2014.