Stochastic Analysis for Problems in Mathematical Finance and Economics

by

Saeed Khalili
B.S., Shahid Beheshti University, 2009
M.S., Sharif University of Technology, 2012
M.A., University of Colorado Boulder, 2016

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics

2019

This thesis entitled:
Stochastic Analysis for Problems in Mathematical Finance and Economics
written by Saeed Khalili
has been approved for the Department of Mathematics

Associate Professor Sergei Kuznetsov

Assistant Professor Yu-Jui Huang

Date:______

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Khalili, Saeed (Ph.D., Mathematics Department)
Stochastic Analysis for Problems in Mathematical Finance and Economics
Thesis directed by Prof. Yu-Jui Huang

Abstract

This thesis is the study of two different problems in mathematical finance. In the first chapter, we investigate optimal consumption in the stochastic Ramsey problem with the Cobb-Douglas production function. Contrary to prior studies, we allow for general consumption processes, without any a priori boundedness constraint. A non-standard stochastic differential equation, with neither Lipschitz continuity nor linear growth, specifies the dynamics of the controlled state process. A mixture of probabilistic arguments are used to construct the state process, and establish its non-explosiveness and strict positivity. This leads to the optimality of a feedback consumption process, defined in terms of the value function and the state process. Based on additional viscosity solutions techniques, we characterize the value function as the unique classical solution to a nonlinear elliptic equation, among an appropriate class of functions. This characterization involves a condition on the limiting behavior of the value function at the origin, which is the key to dealing with unbounded consumptions. Finally, relaxing the boundedness constraint is shown to increase, strictly, the expected utility at all wealth levels.

In the second chapter, in a discrete-time financial market, a generalized duality is established for model-free superhedging, given marginal distributions of the underlying asset. Contrary to prior studies, we do not require contingent claims to be upper semicontinuous, allowing for upper semi-analytic ones. The generalized duality stipulates an extended version of risk-neutral pricing. To compute the model-free superhedging price, one needs to find the supremum of expected values of a contingent claim, evaluated *not* directly under martingale (risk-neutral) measures, but along sequences of measures that converge, in an appropriate sense, to martingale ones. To derive the main result, we first establish a portfolio-constrained duality for upper semi-analytic contingent claims, relying on Choquet's capacitability theorem. As we gradually fade out the portfolio constraint, the generalized duality emerges through delicate probabilistic estimations.

Dedicated to my parents.

Acknowledgements

First, I want to express my deep gratitude towards my advisor Prof. Yu-Jui Huang for his support during my Ph.D research and composition. He has put in his valuable time to guide me during the process. Without our weekly meetings of teaching me innovative methods in finance and economics, I would not have been able to navigate the process of research. His dedication in my research allowed me to learn valuable skills and knowledge to use in my work. I owe him many thanks.

I would also like to thank the rest of my thesis committee: Prof. Sergei Kuznetsov, Prof. Judith Packer, and Prof. Sean O'Rourke, Prof. Manuel Lladser for serving as my committee members during their busy semester. I would like to specifically thank Prof. Sergei Kuznetsov for meeting and educating me about the graduate process, and supporting me through my research.

Finally, I am indebted to my family for supporting me spiritually throughout writing this thesis and my life in general. I would also like to thank all of my friends and colleagues who supported me in writing, and motivated me to reach my goals.

Contents

1 Introduction **1**

2 Optimal Consumption in the Stochastic Ramsey Problem without Boundedness Constraints **3**

2.1 Introduction . . . 3
2.2 The Model . . . 7
2.3 The Capital per Capita Process . . . 9
2.4 Properties of the Value Function . . . 15
2.5 Optimal Consumption . . . 21
2.5.1 Comparison with [26, 27] . . . 30
2.6 Comparison with Bounded Consumption in [28] . . . 31

3 Generalized Duality for Model-Free Superhedging given Marginals **41**

3.1 Introduction . . . 41
3.1.1 Notation . . . 46
3.2 The Main Result . . . 47
3.2.1 Preliminaries . . . 47
3.2.2 The Generalized Duality . . . 49
3.3 Complete Duality under Portfolio Constraints . . . 52
3.3.1 Continuity of P^N . . . 53
3.3.2 Continuity of D^N . . . 55
3.3.3 Complete Duality . . . 60
3.4 Derivation of Theorem 3.2.1 . . . 62

4 Conclusion **73**

Bibliography **77**

Appendices **78**

A **79**

A.1 Derivation of Proposition 2.4.1 . . . 79

Figures

3.1 Support of $\mathbb{Q}_M$. 51

Chapter 1

Introduction

This thesis is devoted to studying two different problems in Financial Mathematics and Economics. The first problem, which we discuss in Chapter 2, is about finding an optimal control for the stochastic Ramsey problem. Later, we consider Generalized Duality for Model-Free Superhedging given Marginals in Chapter 3.

Our ultimate goal in Chapter 2 is to find an optimal control in a growth model for the Cobb-Douglas production function. In contrast to prior studies, we have no bound on the set of consumption processes. The unboundedness assumption makes our problem very challenging to solve. Especially, the dynamic programming equation (HJB equation) related to this becomes a nonlinear elliptic equation. Hence, to find the optimal control, one needs to find a unique smooth solution to the HJB equation related to this problem. The optimal control can be found by having the value function. The main difficulty here is that the dynamic programming equation is a nonlinear elliptic equation, and the existence and uniqueness of a solution to this PDE is highly nontrivial. Based on our knowledge, there is no classical results which can be applied to this PDE. Therefore, we tackle this problem from a different angle. We define a related problem, and then study this problem. This related problem helps us to understand the main problem, and by the help of the related problem we can show the HJB equation has a unique smooth solution.

In chapter 3, we let $S_1, S_2, \ldots, S_T$ represents the price of a fixed asset S, and fix an exotic option with payoff $\Phi(S_1, S_2, \ldots, S_T)$. Indeed, the payoff of the option depends on the asset price at times $t = 1, 2, \ldots, T$. Assuming one sells this option at time zero, she then needs to know how much she expects to pay at the maturity of the option. Therefore, she constructs a semi-static superhedge portfolio consisting of the sum of a static vanilla portfolio and a delta strategy. The payoff of this portfolio is $\Psi_{u,\Delta}(x) := \sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T$ at time $t = T$. The corresponding model-free superhedging price of Φ is defined by

$$D(\Phi) := \inf \left\{ \mu(u) : u \in L^1(\mu) \text{ satisfies } \exists \Delta \in \mathcal{H} \text{ s.t. } \Psi_{u,\Delta}(x) \geq \Phi(x)\ \forall x \in \Omega \right\}.$$

Indeed, $D(\Phi)$ is equivalent to the smallest amount necessary for her to have at time zero to make sure she does not default at time T, the maturity time of the option. This amount $D(\Phi)$ lets her construct a portfolio $\Psi_{u,\Delta}(x)$ at the current time such the value of this portfolio is at least as great as the value of the option at time T. Recently, Beiglböck, Henry-Labordère, and Penkner [3] showed that if the cost function is an upper semi-continuous function, and is bounded by a linear function from above, then there is no duality gap, but without upper semi-continuous assumption, there is a duality gap. To restore the duality, they introduce a new concept called quasi-surely, and under this modification they prove $D_{qs}(\Phi) = P(\Phi)$. We approach the failure of $D(\Phi) = P(\Phi)$ from an opposite angle. We keep the definition of $D(\Phi)$ as it is, and modify $P(\Phi)$ in order to get a general duality for Borel measurable Φ.

Chapter 2

Optimal Consumption in the Stochastic Ramsey Problem without Boundedness Constraints[1]

2.1 Introduction

In the economic growth theory, capital stock of a society amounts to the total value of assets that can be used to produce goods and services, such as factories, equipment, and monetary resources. Whereas capital can be consumed to give individuals immediate welfare, it can also be used to generate more capital and thus sustain economic growth, which enhances future welfare. As Ramsey [32] pointed out in a deterministic model, sensible financial planning, regarding consumption and saving of capital, is imperative to strike a balance between current and future welfare. In a continuous-time setting, Merton [25] enriched the problem by considering stochastic evolution of the population in a society.

The stochastic Ramsey problem, coined by Merton [25], has been investigated in the stochastic control literature through viscosity solution techniques, Banach's fixed-point argument, and the combination of both; see e.g. Morimoto and Zhou [28], Morimoto [26, 27], and Liu [24], among others. Surprisingly, many of these works require an *a priori* uniform upper bound, usually the constant 1, for consumption processes $\{c_t\}_{t\geq 0}$. This is implicitly suggested in the problem formulation of [25], and explicitly stated as $0 \leq c_t \leq 1$ in [28]

[1] [17]

and [24]. While this uniform upper bound provides technical conveniences, it can *not* be fully justified economically in continuous time. After all, for each $t \geq 0$, c_t represents the consumption ratio *per unit of time* instantly at time t, which does not admit any natural upper bound. This is in contrast to the discrete-time setting where the upper bound 1 can be easily justified. Morimoto [26, 27] consider general, unbounded consumption processes, but not without a cost. There, the production function in the Ramsey model is required to have finite first derivatives, along a boundary of its domain. This particularly rules out the standard Cobb-Douglas production function, commonly used in economic modeling.

In other words, a tradeoff exists between the viscosity solutions approach in [28, 24] and Banach's fixed-point argument in [26, 27]. The former accommodates the classical Cobb-Douglas production function, but is limited to uniformly bounded consumption processes; the latter allows for general consumptions, but fails to cover the Cobb-Douglas production function. We aim to resolve this tradeoff: this paper considers *both* unbounded consumption processes and the Cobb-Douglas production function, in the stochastic Ramsey problem. The goal is to characterize the associated value function V, as well as a (possibly unbounded) optimal consumption process $\hat{c}$.

The upfront challenge of our studies is the non-standard stochastic differential equation (SDE) of the state process X, which represents *capital per capita*; see (2.2.8) below. On the one hand, the Cobb-Douglas production function renders the drift coefficient of X non-Lipschitz (see Section 2.5.1 for a comparison with the Lipschitz case [26, 27]). On the other hand, the unboundedness of consumptions may induce superlinear growth in the same drift coefficient, in contrast to [28, 24] where linear growth is guaranteed (see Remark 2.3.1). With *neither* Lipschitz *nor* linear growth condition, standard techniques for SDEs cannot be applied. Instead, we investigate the existence and uniqueness of X, by constructing solutions directly. In Proposition 2.3.1 and Corollary 2.3.1, we establish the existence of X, yet observe that the uniqueness fails in general. Based on the construction of X, we also derive moment estimates in Proposition 2.3.2, without resorting to linear growth condition.

With the state process X constructed, we proceed to relate our value function V to a differential equation. Our strategy is to approximate V by V_L, the value function when one is restricted to consumption processes uniformly bounded by $L > 0$. By generalizing arguments in [28] to infinite horizon, V_L is shown to be a classical solution to a nonlinear elliptic equation (Proposition 2.4.1). As $L \to \infty$, we prove that V_L converges to V desirably, such that V is a classical solution to the limiting nonlinear elliptic equation (Proposition 2.4.2 and Theorem 2.4.1).

There are two remaining tasks: (i) to find an optimal consumption process $\hat{c}$, and (ii) to characterize V further as the *unique* classical solution among a certain class of functions.

While $\hat{c}$ can be heuristically derived in feedback form (i.e. $\hat{c}_t = \hat{c}(X_t)$), it is highly nontrivial whether the controlled state process $X^{\hat{c}}$ is well-defined. First, whether $X^{\hat{c}}$ exists is unclear: The aforementioned existence result of X does not apply here, as the current control process $\hat{c}$ is not *a priori* given, but depends on the unknown X. Second, even if $X^{\hat{c}}$ exists, it is in question whether the dire situation "$X^{\hat{c}}_t = 0$ for some $t > 0$" (i.e. the society using up all its capital at time t) can be avoided. A careful construction of $X^{\hat{c}}$, along with a detailed analysis on its explosion and pathwise uniqueness, is carried out in Proposition 2.5.1. It shows that $X^{\hat{c}}$ is indeed a well-defined strictly positive process, on the strength of Feller's test for explosion and a mixture of probabilistic arguments in Nakao [29] and Yamada [36]. Now, with $X^{\hat{c}}$ well-defined and V solving a nonlinear elliptic equation, a standard verification argument establishes the optimality of $\hat{c}$.

Note that the construction of $X^{\hat{c}}$ was done with much more ease in [28], through a change of measure. This works, however, only with bounded consumptions and finite time horizon. That is, Proposition 2.5.1 complements [28], by providing a new, different construction that is general enough to accommodate both unbounded consumptions and infinite horizon; see Remark 2.5.1 for details.

In fact, the construction in Proposition 2.5.1 can be made much more general. For any $u \in C^1((0,\infty))$ that is strictly increasing, concave, and whose behavior at $0+$ satisfies

(2.5.12) below, we can construct from u a candidate optimal consumption $\hat{c}^u$, and show that the state process $X^{\hat{c}^u}$ is well-defined and strictly positive; see Corollary 2.5.1 and (2.5.15). With the aid of a verification argument, this leads to the *full* characterization: V is the unique classical solution to a nonlinear elliptic equation among the class of functions $u \in C^2((0,\infty)) \cap C([0,\infty))$ that are strictly increasing, concave, satisfying (2.5.12) and the linear growth condition; see Theorem 3.2.1.

In [28], where consumptions are uniformly bounded, the value function is only shown to be a classical solution, with no further characterization. Theorem 3.2.1 fills this void, in a more general setting with unbounded consumptions; see Remark 2.5.3. Specifically, the identification of (2.5.12) in Theorem 3.2.1 is the key to dealing with unbounded consumptions. If one is restricted to bounded consumptions (as in [28]), there is no need to impose (2.5.12); see Remark 2.5.2.

Finally, we compare our no-constraint optimal consumption $\hat{c}$ with the optimal $\hat{c}^L$ in [28], bounded by $L > 0$. Two questions are particularly of interest. First, by switching from the bounded strategy $\hat{c}^L$ to the possibly unbounded $\hat{c}$, can we truly increase our expected utility? An affirmative answer is provided in Proposition 2.6.1: expected utility rises at *all* levels of wealth (capital per capita), whenever $\hat{c}$ is truly unbounded. This justifies economically the use of unbounded strategies. Second, for each $L > 0$, do agents following $\hat{c}^L$ simply chop the no-constraint optimal strategy $\hat{c}$ at the bound $L > 0$? Corollary 2.6.1 shows that the relation "$\hat{c}^L = \hat{c} \wedge L$" fails in general, suggesting a more structural change from $\hat{c}^L$ to $\hat{c}$. For the isoelastic utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $0 < \gamma < 1$, we demonstrate the above two results fairly explicitly.

The paper is organized as follows. Section 2.2 introduces the stochastic Ramsey problem with general unbounded consumptions. Section 2.3 investigates the existence and uniqueness of the state process X, and derives moment estimates of it. Section 2.4 shows that the value function V is a classical solution to a nonlinear elliptic equation. Section 2.5 finds an optimal consumption $\hat{c}$, and establishes a full characterization of V. Section 2.6 compares our results with previous literature with bounded consumptions. Appendix A.1 generalizes arguments in [28] to infinite horizon.

2.2 The Model

Consider the canonical space $\Omega := \{\omega \in C([0,\infty);\mathbb{R}) \mid \omega_0 = 0\}$ of continuous paths starting with value 0. Let W be the canonical process on Ω, $\mathbb{P}$ be the Wiener measure, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by W. Given $t > 0$ and $\omega \in \Omega$, for any $\bar{\omega} \in \Omega$, we define the concatenation of ω and $\bar{\omega}$ at time t as

$$(\omega \otimes_t \bar{\omega})_r := \omega_r 1_{[0,t]}(r) + (\bar{\omega}_{r-t} + \omega_t)1_{(t,\infty)}(r), \quad r \geq 0. \tag{2.2.1}$$

Note that $\omega \otimes_t \bar{\omega}$ again belongs to Ω.

Consider a society in which the labor supply is equal to total population. The capital stock K of the society accumulates from economic output, generated by the capital itself and the labor force. At the same time, K may decrease due to capital depreciation and consumption from the population. Specifically, we assume that K follows the dynamics

$$dK_t = [F(K_t, Y_t) - \lambda K_t - c_t K_t]dt \quad \text{for } t > 0, \qquad K_0 = k > 0.$$

Here, $F : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a production function, Y is the labor supply process, $\lambda \geq 0$ is the constant rate of depreciation, and c is the consumption rate process chosen by the population. Throughout this paper, we take F to be the Cobb-Douglas form, i.e.

$$F(k, y) := k^\alpha y^{1-\alpha}, \quad \text{for some } \alpha \in (0,1). \tag{2.2.2}$$

Also, we assume that the labor supply process Y is stochastic, modeled as a geometric

Brownian motion:

$$dY_t = nY_t dt + \sigma Y_t dW_t \quad \text{for } t > 0, \qquad Y_0 = y > 0.$$

where $n \in \mathbb{R}$ and $\sigma > 0$ are two given constants. In addition, we consider general consumption processes c without any a priori boundedness condition, as opposed to most previous studies in the literature. Specifically, the set $\mathcal{C}$ of admissible consumption processes is taken as

$$\mathcal{C} := \Big\{ c : \Omega \times [0,\infty) \to \mathbb{R}_+ : c \text{ is progressively measurable,} \\ \text{with } \int_0^t c_s ds < \infty \ \ \forall t > 0 \ \text{ a.s.} \Big\}. \tag{2.2.3}$$

At each time $t \geq 0$, every individual is allotted the capital K_t/Y_t, which can be consumed immediately or saved for future production. An individual is then faced with an optimal consumption problem: he/she intends to choose an appropriate consumption process $\hat{c} \in \mathcal{C}$, so that the expected discounted utility from consumption can be maximized. Specifically, the corresponding value function is given by

$$v(k,y) := \sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U\left(c_t \frac{K_t^k}{Y_t^y} \right) dt \right], \tag{2.2.4}$$

where $\beta \geq 0$ is the discount rate and $U : [0,\infty) \to \mathbb{R}$ is a utility function. We will assume that

$$U \text{ is strictly increasing and strictly concave,} \tag{2.2.5}$$

$$U'(0+) = U(\infty) = \infty \quad \text{and} \quad U'(\infty) = U(0) = 0. \tag{2.2.6}$$

The dimension of the problem can be reduced, by introducing the variable $x := k/y$ and the process $X_t := K_t/Y_t$, i.e. the *capital per capita* process. Specifically, the value function in (2.2.4) can be re-written as

$$V(x) := \sup_{c\in\mathcal{C}} \mathbb{E}\bigg[\int_0^\infty e^{-\beta t} U\left(c_t X_t^x\right) dt\bigg], \tag{2.2.7}$$

where the process X satisfies, thanks to Itô's formula,

$$dX_t = (X_t^\alpha - \mu X_t - c_t X_t)dt - \sigma X_t dW_t \quad t > 0, \qquad X_0 = x \geq 0, \tag{2.2.8}$$

with $\mu := \lambda + n - \sigma^2$. As in [28], we will assume throughout the paper that

$$\mu > 0. \tag{2.2.9}$$

The goal of this paper is to provide characterizations for the value function V in (2.2.7), as well as the associated optimal consumption process $\hat{c}$.

2.3 The Capital per Capita Process

In this section, we analyze the capital per capita process X, formulated as the stochastic differential equation (SDE) (2.2.8). We will investigate the existence and uniqueness of solutions to (2.2.8), and derive several moment estimates for X, useful in Sections 2.4 and 2.5 for characterizing V in (2.2.7).

The SDE (2.2.8) is non-standard: the drift coefficient is *neither* Lipschitz *nor* of linear growth. Indeed, Lipschitz continuity fails due to the term X_t^α, and the unboundedness of c may lead to superlinear growth. Consequently, standard techniques to establish existence and uniqueness of solutions (requiring both "Lipschitz" and "linear growth") and to derive moment estimates (requiring "linear growth") cannot be applied here.

Remark 2.3.1. *In [28],* (2.2.8) *is studied in a simpler setting, where c is assumed to be uniformly bounded (in fact, $c_t \leq 1$ for all $t \geq 0$). This ensures linear growth of the drift* *coefficient of* (2.2.8)*, such that some standard techniques and estimates can still be used.*

Without the aid of standard results, we investigate existence and uniqueness of solutions to (2.2.8), by constructing solutions directly. As shown in Proposition 2.3.1 and Corollary 2.3.1 below, existence can be established in general, yet uniqueness need not always hold.

Proposition 2.3.1. *For any $c \in \mathcal{C}$ and $x > 0$, there exists a unique strong solution to* (2.2.8)*, which is strictly positive a.s.*

Proof. Fix $c \in \mathcal{C}$ and $x > 0$. Consider $Z_t := X_t^{1-\alpha}$, with $Z_0 = z := x^{1-\alpha} > 0$. Since the function $f(y) := y^{1-\alpha}$ is well-defined on $[0, \infty)$ and differentiable on $(0, \infty)$, we can apply Itô's formula to Z only up to the stopping time

$$\tau := \inf\{t \geq 0 : X_t^x = 0\} = \inf\{t \geq 0 : Z_t^z = 0\}.$$

This gives the dynamics of Z up to time τ:

$$dZ_t = (1-\alpha)\left(1 - (\mu + c_t + \frac{1}{2}\sigma^2\alpha)Z_t\right)dt - \sigma(1-\alpha)Z_t dW_t, \quad \text{for } 0 < t < \tau. \tag{2.3.1}$$

We claim that this SDE admits a unique strong solution. For simplicity, let $a := 1 - \alpha$ and $b_t := -(1-\alpha)(\mu + c_t + \frac{1}{2}\sigma^2\alpha)$, and define

$$G_t := \exp\left(\int_0^t (-b_s + \frac{\sigma^2 a^2}{2})ds + \sigma a W_t\right) > 0, \qquad \text{for } t \geq 0. \tag{2.3.2}$$

Note that G is well-defined a.s. thanks to $c \in \mathcal{C}$; recall (2.2.3). By definition, G satisfies the dynamics $dG_t = (-b_t + \sigma^2 a^2)G_t dt + \sigma a G_t dW_t$, for all $t > 0$. By applying Itô's formula to the product process GZ up to time τ, we get

$$\begin{aligned} d(G_t Z_t) &= G_t(a + b_t Z_t)dt - \sigma a G_t Z_t dW_t \\ &\qquad + G_t Z_t(-b_t + \sigma^2 a^2)dt + \sigma a G_t Z_t dW_t - \sigma^2 a^2 G_t Z_t dt \\ &= a G_t dt, \quad \text{for } 0 < t < \tau. \end{aligned}$$

This implies that

$$Z_t = \frac{1}{G_t}\left(z + (1-\alpha)\int_0^t G_s ds\right) \tag{2.3.3}$$

is the unique strong solution to (2.3.1), given that $Z_0 = z$. Now, in view of (2.3.3) and $G_t > 0$ for all $t \geq 0$, we conclude that $Z_t > 0$ for all $t \geq 0$ a.s., and thus $\tau = \infty$ a.s.

With $\tau = \infty$ a.s., the construction in the proof above implies that the process $X_t = Z_t^{1/(1-\alpha)}$, $t \geq 0$, with Z given by (2.3.3), is the unique strong solution to (2.2.8), and it is strictly positive a.s. □

For the case $x = 0$ in (2.2.8), uniqueness of solutions fails.

Corollary 2.3.1. *For any $c \in \mathcal{C}$, if $x = 0$ in* (2.2.8)*, then $X \equiv 0$ and*

$$\widetilde{X}_t := \begin{cases} 0 & \text{if } t = 0, \\ \left(\frac{1-\alpha}{G_t}\int_0^t G_s ds\right)^{\frac{1}{1-\alpha}} > 0 & \text{if } t > 0, \end{cases}$$

are two dinstinct strong solutions to (2.2.8)*. Here, G is defined as in* (2.3.2)*.*

Proof. Since $X \equiv 0$ trivially solves (2.2.8), we focus on showing that $\widetilde{X}$ is a strong solution to (2.2.8). First, since $G_0 = 1 \neq 0$, $\widetilde{X}$ is continuous at $t = 0$, i.e. $\lim_{t \downarrow 0} \widetilde{X}_t = 0 = \widetilde{X}_0$. Now, consider the SDE (2.3.1), with $Z_0 = 0$. Due to the term $(1-\alpha)dt$, Z will immediately go up from 0, such that $\tau' := \inf\{t > 0 : Z_t^0 = 0\} > 0$. We can then apply Itô's formula to the process GZ over the interval $(0, \tau')$. Similarly to the proof of Proposition 2.3.1, we find that $Z_t = \frac{1-\alpha}{G_t}\int_0^t G_s ds$ is the unique strong solution to (2.3.1) up to time τ', given that $Z_0 = 0$.

But the formula of Z entails $Z_t > 0$ for all $t > 0$ a.s., and thus $\tau' = \infty$ a.s. Observe that $\widetilde{X}_t = (Z_t)^{1/(1-\alpha)}$ for all $t \geq 0$. With $\tau' = \infty$ a.s., we can apply Itô's formula to $\widetilde{X}$ over $(0, \infty)$, which shows that it is a strong solution to (2.2.8). $\square$

Remark 2.3.2. *Recall V in* (2.2.7)*. By Corollary 2.3.1, $V(0)$ is not well-defined. Indeed, one has $V(0) = 0$ with $X \equiv 0$ in* (2.2.7)*, but $V(0) > 0$ with $X = \widetilde{X}$ in* (2.2.7)*.*

Remark 2.3.3. *According to the boundary classification in Karlin and Taylor [19, Chapter 15], $x = 0$ is an "entrance boundary" of the state space $[0, \infty)$ of $\widetilde{X}$ in Corolloary 2.3.1: beginning at the boundary $x = 0$, $\widetilde{X}$ quickly moves to the interior and never returns to the boundary.*

Classical moment estimates of SDEs rely on linear growth of coefficients, along with an application of Gronwall's lemma; see e.g. Krylov [22, Chapter 2], especially Corollary 2.5.12. As mentioned before, the drift coefficient of (2.2.8) does not necessarily have linear growth, unless c is known *a priori* a bounded process (as in [28]). The explicit formula of X via (2.3.3) turns out to be handy here. Detailed analysis on such a formula yields desirable moment estimates, without requiring any linear growth condition.

Proposition 2.3.2. *Let $\eta := \frac{1}{1-\alpha}$. Given $c \in \mathcal{C}$, the unique strong solution X of* (2.2.8) *satisfies*

$$\mathbb{E}[X_t^x] \leq 2^{\eta-1}(x + t^\eta), \quad \mathbb{E}[(X_t^x)^2] \leq 2^{2\eta-1} e^{\sigma^2 t} \left(x^2 + \frac{t^{2\eta-1}}{\sigma^2} \right), \quad \forall x > 0, t \geq 0. \qquad (2.3.4)$$

Moreover, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\mathbb{E}[|X_t^x - X_t^y|] \leq C_\varepsilon |x - y| + \varepsilon(x + y + t^\eta) \qquad \forall x, y > 0. \qquad (2.3.5)$$

Proof. Fix $c \in \mathcal{C}$ and $x > 0$. Consider $Z_t := (X_t^x)^{1-\alpha}$. Then, as shown in the proof of Proposition 2.3.1, Z satisfies (2.3.1), which can be solved to get the formula (2.3.3). It follows that

$$X_t = Z_t^\eta = G_t^{-\eta}\bigg(x^{1-\alpha} + (1-\alpha)\int_0^t G_s ds\bigg)^\eta \le 2^{\eta-1} G_t^{-\eta}\bigg[x + (1-\alpha)^\eta\bigg(\int_0^t G_s ds\bigg)^\eta\bigg], \quad (2.3.6)$$

where the inequality follows from $(u+v)^k \le 2^{k-1}(u^k + v^k)$ for $u, v \ge 0$ and $k > 1$. Observe from (2.3.2) that

$$G_t = \exp\bigg(\int_0^t (1-\alpha)(\mu + c_t + \frac{\sigma^2}{2})ds + (1-\alpha)\sigma W_t\bigg) > 0, \qquad t \ge 0. \quad (2.3.7)$$

This, together with $c_t \ge 0$, implies that

$$\mathbb{E}[G_t^{-\eta}] \le \mathbb{E}\bigg[\exp\bigg((-\mu - \frac{\sigma^2}{2})t - \sigma W_t\bigg)\bigg] = e^{-\mu t} < 1. \quad (2.3.8)$$

Now, for any $0 \le s \le t$, we introduce

$$G_{s,t} := \exp\bigg(\int_s^t (1-\alpha)(\mu + c_r + \frac{\sigma^2}{2})dr + (1-\alpha)\sigma(W_t - W_s)\bigg) > 0. \quad (2.3.9)$$

Then, observe that

$$\mathbb{E}\bigg[G_t^{-\eta}\bigg(\int_0^t G_s ds\bigg)^\eta\bigg] = \mathbb{E}\bigg[\bigg(\int_0^t G_{s,t}^{-1} ds\bigg)^\eta\bigg].$$

By applying Jensen's inequality to $\big(\int_0^t G_{s,t}^{-1} ds\big)^\eta$, we deduce from the above equality that

$$\mathbb{E}\bigg[G_t^{-\eta}\bigg(\int_0^t G_s ds\bigg)^\eta\bigg] \le \mathbb{E}\bigg[t^{\eta-1}\int_0^t G_{s,t}^{-\eta} ds\bigg] = t^{\eta-1}\int_0^t \mathbb{E}[G_{s,t}^{-\eta}]ds \le t^\eta, \quad (2.3.10)$$

where the last inequality follows from $\mathbb{E}[G_{s,t}^{-\eta}] \le 1$, which can be proved as in (2.3.8). Now, by (2.3.8) and (2.3.10), we conclude from (2.3.6) that $\mathbb{E}[X_t] \le 2^{\eta-1}(x + t^\eta)$, as desired. To prove the second part of (2.3.4), we replace η by 2η in the above arguments. First, (2.3.8) becomes

$$\mathbb{E}[G_t^{-2\eta}] \le \mathbb{E}\Big[\exp\big((-2\mu-\sigma^2)t - 2\sigma W_t\big)\Big] = e^{-(2\mu-\sigma^2)t} \le e^{\sigma^2 t}. \tag{2.3.11}$$

Then, (2.3.10) becomes

$$\begin{aligned}\mathbb{E}\bigg[G_t^{-2\eta}\Big(\int_0^t G_s ds\Big)^{2\eta}\bigg] &= \mathbb{E}\bigg[\Big(\int_0^t G_{s,t}^{-1} ds\Big)^{2\eta}\bigg]\\ &\le \mathbb{E}\bigg[t^{2\eta-1}\int_0^t G_{s,t}^{-2\eta} ds\bigg] = t^{2\eta-1}\int_0^t \mathbb{E}[G_{s,t}^{-2\eta}]ds \le \frac{t^{2\eta-1}}{\sigma^2}(e^{\sigma^2 t}-1),\end{aligned} \tag{2.3.12}$$

where the first inequality follows from applying Jensen's inequality to $\big(\int_0^t G_{s,t}^{-1} ds\big)^{2\eta}$ and the second inequality is due to $\mathbb{E}[G_{s,t}^{-2\eta}] \le e^{\sigma^2(t-s)}$, which can be proved as in (2.3.11). Finally, using the same calculation in (2.3.6) with η replaced by 2η, along with (2.3.11) and (2.3.12), we conclude that $\mathbb{E}[(X_t^x)^2] \le 2^{2\eta-1}e^{\sigma^2 t}(x^2 + t^{2\eta-1}/\sigma^2)$, as desired.

To prove (2.3.5), consider the process Z defined above, as well as $\bar{Z}_t := (X_t^y)^{1-\alpha}$. As above, Z and $\bar{Z}$ take the form (2.3.3), with initial values $z = x^{1-\alpha}$ and $\bar{z} = y^{1-\alpha}$, respectively. Thus, by (2.3.8),

$$\mathbb{E}[|Z_t - \bar{Z}_t|^\eta] \le |z-\bar{z}|^\eta\ \mathbb{E}[G_t^{-\eta}] \le |z-\bar{z}|^\eta = |x^{1-\alpha} - y^{1-\alpha}|^{\frac{1}{1-\alpha}} \le |x-y|, \tag{2.3.13}$$

where the last inequality follows from the observation $|u^r - v^r| \le |u-v|^r$ for any $u, v \ge 0$ and $0 < r < 1$. Indeed, we may assume without loss of generality that $u \ge v$ and define $\lambda := u/v \ge 1$. Thus, the observation is equivalent to $\lambda^r - 1 \le (\lambda-1)^r$ for any $\lambda \ge 1$ and $0 < r < 1$. The latter is true because $f(\lambda) := (\lambda-1)^r - \lambda^r + 1$ satisfies $f(1) = 0$ and $f'(\lambda) = r\big((\frac{1}{\lambda-1})^{1-r} - (\frac{1}{\lambda})^{1-r}\big) > 0$ for all $\lambda > 1$.

Next, for any $a, b \geq 0$ and $\varepsilon > 0$, observe that

$$\begin{aligned} |a^\eta - b^\eta| &= \left| \int_a^b \eta r^{\eta-1} dr \right| \leq \eta |a-b| (a^{\eta-1} + b^{\eta-1}) \\ &\leq \frac{1}{\varepsilon^\eta} \ |a-b|^\eta + (\eta - 1) \ \varepsilon^{\frac{\eta}{\eta-1}} (a^{\eta-1} + b^{\eta-1})^{\frac{\eta}{\eta-1}} \\ &\leq \frac{1}{\varepsilon^\eta} \ |a-b|^\eta + (\eta - 1) \ (2\varepsilon)^{\frac{\eta}{\eta-1}} (a^\eta + b^\eta), \end{aligned} \tag{2.3.14}$$

where the second line follows from Young's inequality with $p = \eta$ and $q = \frac{\eta}{\eta-1}$, and the third line is due to $(u+v)^k \leq 2^{k-1}(u^k + v^k)$ for $u, v \geq 0$ and $k > 1$. Now, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{E}[|X_t^x - X_t^y|] = \mathbb{E}[|Z_t^\eta - \bar{Z}_t^\eta|] &\leq \frac{1}{\varepsilon^\eta} |x-y| + (\eta-1)(2\varepsilon)^{\frac{\eta}{\eta-1}} (\mathbb{E}[Z_t^\eta] + \mathbb{E}[\bar{Z}_t^\eta]) \\ &\leq \frac{1}{\varepsilon^\eta} |x-y| + 2^\eta (\eta-1)(2\varepsilon)^{\frac{\eta}{\eta-1}} (x + y + t^\eta), \end{aligned}$$

where the first inequality follows from (2.3.14) and (2.3.13), and the second inequality is due to the first part of (2.3.4). Now, in the last line of the previous inequality, by taking $\varepsilon' := 2^\eta(\eta-1)(2\varepsilon)^{\frac{\eta}{\eta-1}}$ and $C_{\varepsilon'} := \frac{1}{\varepsilon^\eta} = 2^{\eta^2} \left(\frac{\eta-1}{\varepsilon'}\right)^{\eta-1}$, we see that (2.3.5) holds. □

2.4 Properties of the Value Function

In this section, we introduce, for each $L > 0$, the auxiliary value function

$$V_L(x) := \sup_{c \in \mathcal{C}_L} \mathbb{E} \left[\int_0^\infty e^{-\beta t} U(c_t X_t^x) dt \right] \quad x \geq 0, \tag{2.4.1}$$

where

$$\mathcal{C}_L := \{c \in \mathcal{C} : c_t \leq L \text{ for all } t \geq 0\}. \tag{2.4.2}$$

We will first derive useful properties of V_L. As $L \to \infty$, we will see that V_L converges desirably to V in (2.2.7), so that V inherits many properties of V_L.

Morimoto and Zhou [28] studied a similar problem to V_L: they took $L = 1$ and the time horizon to be finite in (2.4.1). Extending their arguments to infinite horizon gives properties of V_L as below.

Proposition 2.4.1. *(i) There exists $\varphi_0 > 0$ such that $V_L(x) \leq x + \varphi_0$ for all $x > 0$ and $L > 0$.*

(ii) For any $L > 0$, $V_L \in C^2((0,\infty))$ is a concave classical solution to

$$\beta v(x) = \frac{1}{2}\sigma^2 x^2 v''(x) + (x^\alpha - \mu x)v'(x) + \tilde{U}_L(x, v'(x)) \quad \text{for } x \in (0,\infty), \tag{2.4.3}$$

where $\tilde{U}_L : (0,\infty)^2 \to (0,\infty)$ is defined by

$$\tilde{U}_L(x,p) := \sup_{0 \leq c \leq L} \{U(cx) - cxp\}.$$

The proof of Proposition 2.4.1 is relegated to Appendix A.1, where arguments in [28] are extended to infinite horizon. While this extension can mostly be done in a straightforward way, there are technicalities that require detailed, nontrivial analysis. This includes, particularly, the derivation of the dynamic programming principle for V_L; see Lemma A.1.2 for details.

Given that $\{V_L\}_{L>0}$ is by definition a nondecreasing sequence of functions, we define

$$V_\infty(x) := \lim_{L\to\infty} V_L(x) \quad \text{for } x > 0. \tag{2.4.4}$$

Remark 2.4.1. *V_∞ immediately inherits many properties from V_L's.*

(i) Thanks to Proposition 2.4.1, V_∞ is concave, nondecreasing, and satisfies

$$0 \leq V_\infty(x) \leq x + \varphi_0 \quad \forall x > 0. \tag{2.4.5}$$

(ii) The concavity of V_∞ implies that it is continuous on $(0,\infty)$. Hence, by Dini's theorem, V_L converges uniformly to V_∞ on any compact subset of $(0,\infty)$.

Lemma 2.4.1. *V_∞ is a continuous viscosity solution to*

$$\beta v(x) = \frac{1}{2}\sigma^2 x^2 v''(x) + (x^\alpha - \mu x)v'(x) + \tilde{U}(v'(x)) \quad \text{for } x \in (0,\infty), \tag{2.4.6}$$

where $\tilde{U} : (0,\infty) \to (0,\infty)$ is defined by

$$\tilde{U}(p) := \sup_{y\geq 0}\{U(y) - yp\}.$$

Proof. By (2.2.5) and (2.2.6), for any $p > 0$, there exists a unique maximizer $y^*(p) > 0$ such that $\tilde{U}(p) = U(y^*(p)) - y^*(p)p$, and the map $p \mapsto y^*(p)$ is continuous. It follows that $\tilde{U}_L(x,p) = U(c^*(x,p)x) - c^*(x,p)xp$, where $c^*(x,p) := \min\{y^*(p)/x, L\}$. From these forms of $\tilde{U}$ and $\tilde{U}_L$, we see that $\tilde{U}_L$ converges uniformly to $\tilde{U}$ on any compact subset of $(0,\infty)^2$. This, together with Remark 2.4.1 (ii), implies that we can invoke the stability result of viscosity solutions (see e.g. [27, Theorem 4.5.1]). We then conclude from the stability and Proposition 2.4.1 (ii) that V_∞ is a viscosity solution to (2.4.6). □

In fact, the convergence of V_L to V_∞ is highly desirable. As the next result demonstrates, not only V_L but also V_L' and V_L'' converge uniformly. This readily implies smoothness of the limiting function V_∞.

Proposition 2.4.2. *V_L' and V_L'' converge uniformly, up to a subsequence, on any compact subset of $(0,\infty)$. Hence, V_∞ is $C^2((0,\infty))$ with $V_\infty'(x) = \lim_{L\to\infty} V_L'(x)$ and $V_\infty''(x) = \lim_{L\to\infty} V_L''(x)$, up to a subsequence, for each $x > 0$. Furthermore, V_∞ is a classical solution to (2.4.6).*

Proof. Fix a compact subset E of $(0,\infty)$. Let $a := \inf E > 0$ and $b := \sup E$. For any $L > 0$, since V_L is nonnegative, nondecreasing, concave, and bounded above by $x + \varphi_0$ (Proposition 2.4.1),

$$0 \leq V_L'(x) \leq \frac{V_L(x) - V_L(0^+)}{x} \leq \frac{x+\varphi_0}{x} = 1 + \frac{\varphi_0}{x} \leq 1 + \frac{\varphi_0}{a}, \quad \forall x \in E.$$

Thus, $\{V_L'(x)\}_{L>0}$ is uniformly bounded on E.

Next, we claim that $\left\{\tilde{U}_L(x, V_L'(x))\right\}_{L>0}$ is also uniformly bounded on E. To this end, we will show that there exists $C_E > 0$ such that $V_L'(b) \geq C_E$ for all $L > 0$. Assume to the contrary that there exits a subsequence $\{L_n\}_{n\in\mathbb{N}}$ such that $V_{L_n}'(b) \downarrow 0$. For any $x > b$, by the concavity of V_{L_n}, we have $V_{L_n}'(u) \leq V_{L_n}'(b)$ for $u \in [b, x]$, for all $n \in \mathbb{N}$. Taking integrals on both sides from b to x yields

$$V_{L_n}(x) - V_{L_n}(b) \leq V_{L_n}'(b)(x - b) \quad \forall n \in \mathbb{N}.$$

As $n \to \infty$, we obtain $V_\infty(x) \leq V_\infty(b)$. Since V_∞ is nondecreasing (Remark 2.4.1 (i)), we conclude that $V_\infty(x) = V_\infty(b)$ for all $x > b$, which in particular implies $V_\infty'(x) = V_\infty''(x) = 0$ for all $x > b$. By the viscosity solution property of V_∞ (Lemma 2.4.1), for any $x > b$ we have $\beta V_\infty(x) = \tilde{U}(0) = \infty$, a contradiction. Now, with $V_L'(b) \geq C_E$ for all $L > 0$, we have

$$0 \leq \tilde{U}_L(x, V_L'(x)) \leq \tilde{U}_L(x, V_L'(b)) \leq \tilde{U}_L(x, C_E) \leq \tilde{U}(C_E) < \infty, \quad \forall x \in E \text{ and } L > 0,$$

where the second and the third inequalities follow from $V_L'(x) \geq V_L'(b) \geq C_E$ and $p \mapsto \tilde{U}_L(x, p)$ is by definition nonincreasing. This shows that $\left\{\tilde{U}_L(x, V_L'(x))\right\}_{L>0}$ is uniformly bounded on E.

Recall from Proposition 2.4.1 that each V_L satisfies

$$\beta V_L(x) = \frac{1}{2}\sigma^2 x^2 V_L''(x) + (x^\alpha - \mu x)V_L'(x) + \tilde{U}_L(x, V_L'(x)), \quad \forall x > 0. \tag{2.4.7}$$

By the uniform boundedness on E of $\{(x^\alpha - \mu x)V_L'(x)\}_{L>0}$, $\{\tilde{U}_L(x, V_L'(x))\}_{L>0}$, and $\{V_L(x)\}_{L>0}$ (thanks to Proposition 2.4.1), (2.4.7) entails the uniform boundedness of $\{V_L''(x)\}_{L>0}$ on E. By the Arzela Ascoli Theorem, this implies V_L' converges uniformly, up to some subsequence, on E. With V_L, V_L', and $\tilde{U}_L$ all converging uniformly on E (recall from the proof of Lemma 2.4.1 that $\tilde{U}_L$ converges uniformly to $\tilde{U}$), (2.4.7) implies that V_L'' also converges uniformly on E.

Now, with V_L converging to V_∞ and V_L' converging uniformly on E, V_∞ must be continuously differentiable with $V_\infty' = \lim_{L\to\infty} V_L'$ (up to some subsequence) in the interior of E. This, together with V_L'' converging uniformly on E, shows that V_∞' is continuously differentiable with $V_\infty'' = \lim_{L\to\infty} V_L''$ (up to some subsequence) in the interior of E. Since E is arbitrarily chosen, we conclude that $V_\infty \in C^2((0,\infty))$. In view of Lemma 2.4.1, V_∞ is a classical solution to (2.4.6). $\square$

Remark 2.4.2. *In deriving the uniform boundedness of* $\{\tilde{U}^L(x, V_L'(x))\}_{L>0}$ *in the proof above, we particularly show that* V_∞ *is strictly increasing on* $(0,\infty)$*, otherwise the viscosity solution property of* V_∞ *(Lemma 2.4.1) would be violated.*

Now, a verification argument connects V_∞ to our value function V.

Theorem 2.4.1. *The value function* V *in* (2.2.7) *coincides with* V_∞ *on* $(0,\infty)$*. Thus,* V *is concave, strictly increasing, satisfies* (2.4.5)*, and solves* (2.4.6) *in the classical sense.*

Proof. Since V_∞ is nonnegative, concave, and nondecreasing (Remark 2.4.1 (i)), $0 \le V_\infty'(x) \le V_\infty(x)/x$ for all $x > 0$. Fix $x > 0$. Then, for any $T > 0$ and $c \in \mathcal{C}$,

$$\begin{aligned}\mathbb{E}\left[\int_0^T (e^{-\beta s} V_\infty'(X_s) X_s)^2 ds\right] &\le \mathbb{E}\left[\int_0^T (e^{-\beta s} V_\infty(X_s))^2 ds\right] \\ &\le \mathbb{E}\left[\int_0^T (e^{-\beta s}(X_s + \varphi_0))^2 ds\right] < \infty,\end{aligned}$$

where the second line follows from Remark 2.4.1 (i) and the finiteness is due to (2.3.4). It follows that $\int_0^t e^{-\beta s} V_x(X_s) X_s dW_s$ is a martingale on $[0,T]$, for any $T > 0$ and $c \in \mathcal{C}$. Now, fix $c \in \mathcal{C}$. By using Ito's formula, for any $T > 0$,

$$\begin{aligned}
\mathbb{E}[e^{-\beta T} V_\infty(X_T)] &= V_\infty(x) + \mathbb{E}\bigg[\int_0^T e^{-\beta t}\Big(- \beta V_\infty(X_t) \\
&\qquad + V_\infty'(X_t)(X_t^\alpha - \mu X_t - c_t X_t) + \frac{\sigma^2}{2} X_t^2 V_\infty''(X_t)\Big) dt\bigg] \\
&\le V_\infty(x) - \mathbb{E}\bigg[\int_0^T e^{-\beta t} U(c_t X_t) dt\bigg],
\end{aligned} \tag{2.4.8}$$

where the inequality follows from V_∞ satisfying (2.4.6) (Proposition 2.4.2). As $T \to \infty$, we deduce from Remark 2.4.1 (i) and (2.3.4) that

$$\mathbb{E}[e^{-\beta T} V_\infty(X_T)] \le \mathbb{E}\left[e^{-\beta T}(X_T + \varphi_0)\right] \le e^{-\beta T}(2^{\eta-1}(x + T^\eta) + \varphi_0) \to 0, \quad \text{as } T \to \infty.$$

Thus, we conclude from (2.4.8) that $V_\infty(x) \ge \mathbb{E}\big[\int_0^\infty e^{-\beta t} U(c_t X_t) dt\big]$ for all $c \in \mathcal{C}$, and thus $V_\infty(x) \ge V(x)$. On the other hand, by definition $V(x) \ge V_L(x)$ for all $L > 0$, and thus $V(x) \ge V_\infty(x)$. We therefore conclude that $V(x) = V_\infty(x)$. The remaining assertions follow from Remark 2.4.1 (i), Remark 2.4.2, and Proposition 2.4.2. □

While Theorem 2.4.1 associates V with the nonlinear elliptic equation (2.4.6), this is *not* a full characterization of V, as there may be multiple solutions to (2.4.6). To further characterize V as the *unique* classical solution to (2.4.6) among a certain class of functions, the standard approach is to stipulate an optimal control of feedback form, by which one can complete the verification argument; note that the proof of Theorem 2.4.1 amounts to the first half of the verification argument.

As detailed in Section 2.5 below, although the form of a candidate optimal consumption process $\hat{c}$ can be readily read out from the equation (2.4.6), it is highly nontrivial whether $\hat{c}$ is a well-defined stochastic process, due to the unboundedness of $\hat{c}$. This entails additional analysis of the value function V and the capital per capita process X, as we will now introduce.

2.5 Optimal Consumption

In view of (2.4.6), one can heuristically stipulate the form of an optimal consumption process as

$$\hat{c}_t := \hat{c}(X_t) \quad \text{for } t \geq 0, \qquad \text{with} \quad \hat{c}(x) := \frac{(U')^{-1}(V'(x))}{x} \quad \text{for } x > 0, \tag{2.5.1}$$

where X is the solution to the SDE (2.2.8) with c_t replaced by $\hat{c}_t$, i.e. the solution to

$$dX_t = \left(X_t^\alpha - \mu X_t - (U')^{-1}\left(V'(X_t)\right)\right) dt - \sigma X_t dW_t, \quad X_0 = x > 0. \tag{2.5.2}$$

For $\hat{c}$ in (2.5.1) to be well-defined, two questions naturally arise. First, it is unclear whether (2.5.2) admits a solution: Proposition 2.3.1 is an existence result for (2.2.8), specifically when c is an *a priori* given process, without X_t involved. Second, even if a solution X to (2.5.2) exists, it is in question whether X is strictly positive, so that one does not need to worry about the problematic case "$X_t = 0$" in (2.5.1).

For (2.5.2) to admit a solution, we first observe that it is necessary to have $V'(0+) = \infty$. Indeed, if $c := V'(0+) < \infty$, when X is close enough to zero, the drift coefficient of (2.5.2) will approach the constant $-(U')^{-1}(c) < 0$, while the diffusion coefficient will tend to zero. This will eventually bring X down to zero. When this happens, the drift and the diffusion coefficients will be precisely $-(U')^{-1}(c) < 0$ and 0 respectively, which will move X further to take negative values. The drift coefficient of (2.5.2), however, is not well-defined for negative values of X_t. A solution to (2.5.2), as a result, cannot exist if $V'(0+) < \infty$.

The next result analyzes the behavior of V as $x \downarrow 0$, and particularly establishes $V'(0+) = \infty$.

Lemma 2.5.1. *The function V defined in* (2.2.7) *satisfies the following:*

(i) $V(0+) > 0$.

(ii) Assume $U \in C^2((0,\infty))$. As $x \downarrow 0$, V' explodes and is of the order of $x^{-\alpha}$. Specifically,

$$V'(0+) = \infty \quad \textit{and} \quad \lim_{x\to 0+} x^\alpha V'(x) = \beta V(0+) > 0.$$

Furthermore,

$$\lim_{x\downarrow 0} \frac{(U')^{-1}(V'(x))}{x^\alpha} = 0. \tag{2.5.3}$$

Proof. (i) Consider $\bar{c} \in \mathcal{C}$ with $\bar{c} \equiv 1$. For any $x > 0$, in view of (2.3.3), the corresponding capital per capita process X_t^x is given by

$$X_t^x = G_t^{-\frac{1}{1-\alpha}} \left(x^{1-\alpha} + (1-\alpha) \int_0^t G_s ds \right)^{\frac{1}{1-\alpha}},$$

where G_t is given as in (2.3.7) with c_t replaced by the constant 1. Then, by the definition of V,

$$\begin{aligned} V(x) &\geq \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(X_t^x) dt \right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\beta t} U \left(G_t^{-\frac{1}{1-\alpha}} \left(x^{1-\alpha} + (1-\alpha) \int_0^t G_s ds \right)^{\frac{1}{1-\alpha}} \right) dt \right]. \end{aligned}$$

As $x \downarrow 0$, Fatou's lemma gives $V(0+) \geq \mathbb{E}\big[\int_0^\infty e^{-\beta t} U(((1-\alpha) \int_0^t G_{s,t}^{-1} ds)^{\frac{1}{1-\alpha}} dt \big] > 0$, where $G_{s,t}$ is given as in (2.3.9) with c_t replaced by the constant 1.

(ii) By contradiction, assume that $c := V'(0+) < \infty$. Note that $c > 0$ must hold, as V is concave and strictly increasing (Theorem 2.4.1). Consider $I(y) := (U')^{-1}(y)$ for $y \in (0,\infty)$. With $U \in C^2((0,\infty))$, the inverse function theorem implies that $I \in C^1((0,\infty))$ with $I'(y) = 1/U''(y)$. Thanks again to Theorem 2.4.1, we have

$$\beta V(x) = \frac{1}{2}\sigma^2 x^2 V''(x) + (x^\alpha - \mu x) V'(x) + U(I(V'(x))) - I(V'(x)) V'(x), \quad \forall x > 0. \tag{2.5.4}$$

We can then express $V''(x)$ in terms of the functions x, $V(x)$, $V'(x)$, $I(V'(x))$, and $U(I(V'(x)))$.

Since each of these functions is continuously differentiable, we have $V \in C^3((0,\infty))$. By using L'Hospital's rule,

$$c = \lim_{x \downarrow 0} V'(x) = \lim_{x \downarrow 0} \frac{xV'(x)}{x} = \lim_{x \downarrow 0} \left(V'(x) + xV''(x)\right), \tag{2.5.5}$$

which implies $\lim_{x \downarrow 0} xV''(x) = 0$. The same argument in turn gives

$$0 = \lim_{x \downarrow 0} xV''(x) = \lim_{x \downarrow 0} \frac{x^2 V''(x)}{x} = \lim_{x \downarrow 0} \left(2xV''(x) + x^2 V'''(x)\right),$$

leading to $\lim_{x \downarrow 0} x^2 V'''(x) = 0$. Now, by differentiating both sides of (2.5.4) and multiplying them by $x^{1-\alpha}$, we get

$$\begin{aligned}\beta x^{1-\alpha} V'(x) &= \sigma^2 x^{2-\alpha} V''(x) + \frac{1}{2}\sigma^2 x^{3-\alpha} V'''(x) + xV''(x) + \alpha V'(x) \\ &\quad - \mu x^{1-\alpha} V'(x) - \mu x^{2-\alpha} V''(x) - x^{1-\alpha} I(V'(x)) V''(x),\end{aligned} \tag{2.5.6}$$

where the last term is obtained by noting that $U' \circ I$ is the identity map. As $x \downarrow 0$ in (2.5.6), we get

$$0 = \alpha c + \lim_{x \downarrow 0} x^{1-\alpha} I(V'(x))(-V''(x)).$$

This is a contradiction by noting that $\alpha c > 0$ and the limit above is nonnegative (as I is a positive function and V is concave). We therefore conclude that $V'(0+) = \infty$.

Now, since V satisfies (2.4.5) (Theorem 2.4.1), we have $\limsup_{x \downarrow 0} xV'(x) < \infty$. Take an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \downarrow 0$ and $x_n V'(x_n)$ converges as $n \to \infty$. Let $\ell := \lim_{n \to \infty} x_n V'(x_n) < \infty$. Similarly to (2.5.5),

$$\begin{aligned}\ell = \lim_{n \to \infty} x_n V'(x_n) = \lim_{n \to \infty} \frac{x_n^2 V'(x_n)}{x_n} &= \lim_{n \to \infty} \left(2x_n V'(x_n) + x_n^2 V''(x_n)\right) \\ &= 2\ell + \lim_{n \to \infty} x_n^2 V''(x_n),\end{aligned}$$

which yields $\lim_{n\to\infty} x_n^2 V''(x_n) = -\ell$. Recalling that V is a classical solution to (2.4.6), we have

$$\beta V(x_n) = \frac{1}{2}\sigma^2 x_n^2 V''(x_n) + (x_n^\alpha - \mu x_n)V'(x_n) + \tilde{U}(V'(x_n)) \quad \text{for all } n \in \mathbb{N}.$$

As $n \to \infty$, since $V'(0+) = \infty$ implies $\tilde{U}(V'(x_n)) \to 0$, we obtain

$$\beta V(0+) = -\left(\frac{1}{2}\sigma^2 + \mu\right)\ell + \lim_{n\to\infty} x_n^\alpha V'(x_n).$$

If $\ell > 0$, then $\lim_{n\to\infty} x_n^\alpha V'(x_n) = \ell \lim_{n\to\infty} x_n^{\alpha-1} = \infty$, which would violate the above equality. Thus, $\ell = 0$ must hold. Since $\{x_n\}_{n\in\mathbb{N}}$ above is arbitrarily chosen, we conclude that $\lim_{x\downarrow 0} x^\alpha V'(x) = \beta V(0+) > 0$, where the inequality follows from (i).

Finally, observe that $0 \le \tilde{U}(V'(x)) = U((U')^{-1}(V'(x))) - V'(x)(U')^{-1}(V'(x))$ for all $x > 0$, leading to

$$0 \le V'(x)(U')^{-1}(V'(x)) \le U((U')^{-1}(V'(x))) \quad \forall x > 0.$$

As $x \downarrow 0$, since $V'(0+) = \infty$ and $U(0) = 0$, the right hand side above approaches zero, which implies

$$\lim_{x\downarrow 0} V'(x)(U')^{-1}(V'(x)) = 0.$$

This, together with $\lim_{x\downarrow 0} x^\alpha V'(x) = \beta V(0+) > 0$, gives (2.5.3). □

On the strength of Lemma 2.5.1, we are ready to present the existence result for (2.5.2).

Proposition 2.5.1. *Suppose $U \in C^2((0,\infty))$. For any $x > 0$, there exists a unique strong solution to* (2.5.2)*, which is strictly positive a.s.*

Proof. We will first establish the existence of a weak solution to (2.5.2), which is strictly positive a.s. Then, we will prove that pathwise uniqueness holds for (2.5.2). By [18, Section 5.3.D], this gives the desired result that a unique strong solution exists and it is strictly positive a.s.

Step 1: Construct a weak solution to (2.5.2) **that is strictly positive a.s.** Thanks to the argument in [18, Theorem 5.5.15], with $\mathbb{R}$ replaced by $(0,\infty)$, there exists a weak solution X to (2.5.2) up to the explosion time

$$S := \lim_{n\to\infty} S_n, \quad \text{where} \quad S_n := \inf\{t \geq 0 : X_t \notin (1/n, n)\}.$$

We will show that $\mathbb{P}(S=\infty)=1$. In view of Feller's test for explosion (see e.g. [18, Theorem 5.5.29]), as well as [18, Theorem 5.5.27], it suffices to prove that for any $\ell \in (0,\infty)$,

$$A_1 := \int_\ell^\infty \exp\left(-2\int_r^\ell \frac{y^\alpha - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right) dr = \infty, \tag{2.5.7}$$

and

$$A_2 := \int_{0+}^\ell \exp\left(2\int_r^\ell \frac{y^\alpha - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right) dr = \infty. \tag{2.5.8}$$

Let $C_1 := \exp\left(-\frac{2}{\sigma^2}\left(\frac{\ell^{\alpha-1}}{1-\alpha} + \mu \log(\ell)\right)\right) > 0$. Observe that

$$\begin{aligned} A_1 &\geq \int_\ell^\infty \exp\left(-2\int_r^\ell \frac{y^\alpha - \mu y}{\sigma^2 y^2} dy\right) dr = C_1 \int_\ell^\infty \exp\left(\frac{2}{\sigma^2(1-\alpha)}\left(\frac{1}{r}\right)^{1-\alpha}\right) r^{\frac{2\mu}{\sigma^2}} dr \\ &\geq C_1 \int_\ell^\infty r^{\frac{2\mu}{\sigma^2}} dr = \infty, \end{aligned}$$

which gives (2.5.7). On the other hand, by (2.5.3), there exists $0 < \delta < \ell$ such that

$(U')^{-1}(V'(y)) < \frac{1}{2}y^{\alpha}$ for $0 < y < \delta$. It follows that

$$
\begin{aligned}
A_2 &\geq \int_{0+}^{\delta} \exp\left(2\int_r^{\ell} \frac{y^{\alpha} - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right) dr \\
&= \int_{0+}^{\delta} \exp\left(2\int_r^{\delta} \frac{y^{\alpha} - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy + 2\int_{\delta}^{\ell} \frac{y^{\alpha} - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right) dr \\
&= C_2 \int_{0+}^{\delta} \exp\left(2\int_r^{\delta} \frac{y^{\alpha} - \mu y - (U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right) dr \\
&\geq C_2 \int_{0+}^{\delta} \exp\left(\frac{2}{\sigma^2}\int_r^{\delta} \frac{1}{2}y^{\alpha-2} - \mu y^{-1} dy\right) dr \\
&\geq C_2 C_3 \int_{0+}^{\delta} \exp\left(\frac{1}{\sigma^2(1-\alpha)}\left(\frac{1}{r}\right)^{1-\alpha}\right) r^{\frac{2\mu}{\sigma^2}} dr = \infty,
\end{aligned}
$$

where $C_2 := \exp\left(2\int_{\delta}^{\ell} \frac{y^{\alpha}-\mu y-(U')^{-1}(V'(y))}{\sigma^2 y^2} dy\right)$, $C_3 := \exp\left(\frac{-\delta^{\alpha-1}}{\sigma^2(1-\alpha)}\right)\delta^{-\frac{2\mu}{\sigma^2}}$, and the fourth line above follows from $(U')^{-1}(V'(y)) < \frac{1}{2}y^{\alpha}$ for $0 < y < \delta$. This readily shows (2.5.8). We therefore conclude that the weak solution X takes values in $(0,\infty)$ a.s.

Step 2: Show that pathwise uniqueness holds for (2.5.2)**.** Let $x^* > 0$ be the unique maximizer of $\sup_{x\geq 0}\{x^{\alpha} - \mu x\}$. Observe that $x \mapsto x^{\alpha} - \mu x$ is strictly increasing on $(0, x^*)$ and strictly decreasing on (x^*, ∞). Also, the concavity of V (Theorem 2.4.1) implies that V' is nonincreasing. Since U is strictly concave, U' is strictly decreasing, and so is $(U')^{-1}$. It follows that $x \mapsto (U')^{-1}(V'(x))$ is nondecreasing. We then conclude that the drift coefficient $b(x) := x^{\alpha} - \mu x - (U')^{-1}(V'(x))$ of (2.5.2) is strictly decreasing on (x^*, ∞).

Besides the weak solution X in Step 1, let $\overline{X}$ be another weak solution to (2.5.2), with $(\Omega, \mathcal{F}, \mathbb{P})$, W, and the initial value $x > 0$ all the same as those of X. By the same argument in Step 1, $\overline{X}$ takes values in $(0,\infty)$ a.s. For each $N \in \mathbb{N}$, consider

$$\tau_N := \inf\{t \geq 0 : X_t \leq 1/N\}.$$

We claim that for any $x > 0$,

$$\mathbb{P}\left(X^x_{t\wedge\tau_N} = \overline{X}^x_{t\wedge\tau_N},\ \forall t \geq 0\right) = 1, \quad \forall N \in \mathbb{N}. \tag{2.5.9}$$

Pick an arbitrary $\varepsilon > 0$, and let $x_0 := x^* + \varepsilon$. Fix $N \in \mathbb{N}$. If the initial value $x < x_0$, since the diffusion coefficient $a(u) := \sigma u$ of (2.5.2) is bounded away from zero on $[1/N, x_0]$, the argument in [29, Theorem] (with c and M therein replaced by σ/N and σx_0 in our case) implies

$$\mathbb{P}\left(X^x_{t\wedge\tau_N\wedge\tau_{x_0}} = \overline{X}^x_{t\wedge\tau_N\wedge\tau_{x_0}},\ \forall t \geq 0\right) = 1, \tag{2.5.10}$$

where $\tau_0 := \inf\{t \geq 0 : X^x_t \geq x_0\}$. On the other hand, if the initial value $x \geq x_0$, since the drift coefficient $b(u)$ of (2.5.2) is strictly decreasing on (x^*, ∞), [36, Example 1.1] asserts that

$$\mathbb{P}\left(X^x_{t\wedge\tau_{x^*}} = \overline{X}^x_{t\wedge\tau_{x^*}},\ \forall t \geq 0\right) = 1, \tag{2.5.11}$$

where $\tau_{x^*} := \inf\{t \geq 0 : X^x_t \leq x^*\}$. Note that (2.5.10) and (2.5.11) already imply the desired result (2.5.9). Indeed, if the initial value $x < x_0$, we can define a sequence of stopping times recursively as follows: $\tau_0 := 0$,

$$\tau_{2n-1} := \inf\{t \geq \tau_{2n-2} : X^x_t \geq x_0\}, \quad \tau_{2n} := \inf\{t \geq \tau_{2n-1} : X^x_t \leq x^*\}, \quad \forall n \in \mathbb{N}.$$

Then, by using (2.5.10) and (2.5.11) alternately on the time intervals $[\tau_{n-1}, \tau_n]$, $n = 1, 2, ...$, we obtain (2.5.9). If the initial value $x \geq x_0$, we can similarly define a sequence of stopping times recursively as follows: $\tau_0 := 0$,

$$\tau_{2n-1} := \inf\{t \geq \tau_{2n-2} : X^x_t \leq x^*\}, \quad \tau_{2n} := \inf\{t \geq \tau_{2n-1} : X^x_t \geq x_0\}, \quad \forall n \in \mathbb{N}.$$

By applying (2.5.11) and (2.5.10) alternately on the time intervals $[\tau_{n-1}, \tau_n]$, $n = 1, 2, ...$, we again obtain (2.5.9).

Finally, since X is strictly positive a.s., $\tau_N \to \infty$ a.s. as $N \to \infty$. We then conclude from (2.5.9) that $\mathbb{P}\left(X^x_t = \overline{X}^x_t,\ \forall t \geq 0\right) = 1$, for all $x > 0$. That is, pathwise uniqueness holds for (2.5.2), as desired. $\square$

Remark 2.5.1. *With bounded consumptions and finite horizon $T > 0$, [28, Lemma 6.1]* *constructs a strictly positive solution to* (2.5.2) *easily, through a change of measure and using Girsanov's theorem. This does not work in our case. With unbounded consumptions, the same change of measure is not well-defined. Also, applying Girsanov's theorem requires some finite horizon. In view of this, Proposition 2.5.1 complements [28, Lemma 6.1], by providing a new, different construction that accommodates both unbounded consumptions and infinite horizon.*

Proposition 2.5.1 deals with the SDE (2.5.2), induced by the value function V. In fact, the same arguments can be applied to SDEs induced by a much larger class of functions.

Corollary 2.5.1. *Suppose* $U \in C^2((0,\infty))$. *Let* $u \in C^1((0,\infty))$ *be strictly increasing, concave, and satisfy*

$$\lim_{x\downarrow 0} \frac{(U')^{-1}(u'(x))}{x^\alpha} = 0. \tag{2.5.12}$$

Then, for any $x > 0$*, the SDE*

$$dX_t = \left(X_t^\alpha - \mu X_t - (U')^{-1}\left(u'(X_t)\right)\right) dt - \sigma X_t dW_t, \quad X_0 = x \tag{2.5.13}$$

admits a unique strong solution, which is strictly positive a.s.

Proof. The result can be established by following the proof of Proposition 2.5.2, with V replaced by u. Specifically, Step 1 in the proof can be carried out thanks to $u'(x) > 0$ and (2.5.12), while Step 2 relies on the concavity of u. $\square$

Let $\mathcal{U}$ denote the class of functions $u \in C^2((0,\infty)) \cap C([0,\infty))$ that are nonnegative, strictly increasing, concave, satisfying (2.5.12) and the following linear growth condition: there exists $C > 0$ such that

$$u(x) \leq C(1+x) \quad \text{for all } x \geq 0. \tag{2.5.14}$$

Now, we are ready to present the main result of this paper.

Theorem 2.5.1. *Suppose $U \in C^2((0,\infty))$. The function V defined in* (2.2.7) *is the unique classical solution to* (2.4.6) *among functions in $\mathcal{U}$. Moreover, $\hat{c} \in \mathcal{C}$ defined by* (2.5.1)*, with X being the unique strong solution to* (2.5.2)*, is an optimal consumption process for* (2.2.7)*.*

Proof. We know from Theorem 2.4.1 and Lemma 2.5.1 that $V \in \mathcal{U}$ and it solves (2.4.6) in the classical sense. By following the arguments in Theorem 2.4.1, with V_∞ and c therein replaced by V and $\hat{c}$, we note that the inequality in (2.4.8) now becomes equality, leading to $V(x) = \mathbb{E}\big[\int_0^\infty e^{-\beta t} U(\hat{c}_t X_t^x) dt\big]$ for all $x > 0$. This readily shows that $\hat{c} \in \mathcal{C}$ is an optimal consumption process for (2.2.7).

For any $u \in \mathcal{U}$ that solves (2.4.6) in the classical sense, we can again follow the arguments in Theorem 2.4.1 to show that $u \geq V$. On the other hand, consider the consumption process

$$\hat{c}_t^u := \hat{c}^u(X_t) \quad \text{for } t \geq 0, \qquad \text{with} \quad \hat{c}^u(x) := \frac{(U')^{-1}(u'(x))}{x} \quad \text{for } x > 0, \tag{2.5.15}$$

where X is the unique strong solution to (2.5.13), whose existence is guaranteed by Corollary 2.5.1. Now, in (2.4.8), if we replace V_∞ and c therein by u and $\hat{c}^u$, the inequality becomes equality, leading to $u(x) = \mathbb{E}\big[\int_0^\infty e^{-\beta t} U(\hat{c}_t^u X_t^x) dt\big] \leq V(x)$ for all $x > 0$. Thus, we conclude that $u = V$. □

Remark 2.5.2. *In the characterization of V in Theorem 3.2.1, condition* (2.5.12) *is the key to dealing with unbounded consumptions (recall that* (2.5.12) *is part of the definition of $\mathcal{U}$). If we restrict ourselves to $\mathcal{C}_L$ in* (2.4.2) *for some $L > 0$ (as in [28]), there is no need to impose* (2.5.12)*.*

To see this, note that (2.5.12) *can be re-written as*

$$\lim_{x \downarrow 0} \hat{c}^u(x) x^{1-\alpha} = 0, \quad \textit{with } \hat{c}^u \textit{ as in } (2.5.15).$$

That is, we require the optimal consumption to be dominated by $x^{1-\alpha}$ as $x \downarrow 0$. When we are restricted to $\mathcal{C}_L$, this requirement holds trivially, thanks to the bound $L > 0$ for each $c \in \mathcal{C}_L$.

Thus, for V_L defined in (2.4.1)*, the same arguments in Proposition 2.5.1, Corollary 2.5.1, and Theorem 3.2.1 can be carried out, without the need to impose* (2.5.12)*. This leads to the characterization: V_L is the unique classical solution to* (2.4.3) *among the class of functions $u \in C^2((0,\infty)) \cap C([0,\infty))$ that are nonnegative, strictly increasing, concave, and satisfying* (2.5.14)*.*

Remark 2.5.3. *In [28], one is restricted to $\mathcal{C}_L$ in* (2.4.2)*. The main results, [28, Theorems 4.2 and 6.2], only show that the value function V_L is a classical solution and that a feedback optimal consumption exists; there is no further characterization of V_L. At the end of [28], the authors very briefly mention, without a proof, that V_L is the unique solution. However, the class of functions among which V_L is unique, the key ingredient of any PDE characterization, is missing. Theorem 3.2.1, along with the resulting characterization of V_L in Remark 2.5.2, fills this void.*

We will demonstrate the use of Theorem 3.2.1 explicitly in Proposition 2.6.3 below.

2.5.1 Comparison with [26, 27]

To the best of our knowledge, Morimoto [26, 27] are the only prior works that consider unbounded consumptions in the stochastic Ramsey problem. Our studies complement [26, 27] in two ways.

First, [26, 27] require the production function $F(k, y)$ to satisfy $F_k(0+, y) < \infty$ for all $y > 0$. This provides technical conveniences: (i) The drift coefficient of the capital per capita process is Lipschitz (see e.g. (11) and (12) in [26]), such that the SDE has uniqueness of solutions even when the initial condition is 0. The value function V is thus well-defined at $x = 0$, with $V(0) = 0$. (ii) The continuity of V at $x = 0$ is ensured, with $V(0+) = V(0) = 0$, which leads to a short simple proof for $V'(0+) = \infty$ (see the last two lines in the proof of [26, Theorem 4.1]).

Our contribution here is taking into account the classical, widely-used Cobb-Douglas production function (2.2.2), which violates $F_k(0+, y) < \infty$. In contrast to [26, 27], the drift coefficient of (2.2.8) is non-Lipschitz, such that (2.2.8) admits multiple solutions when the initial condition is 0 (see Corollary 2.3.1), leaving the value function V undefined at $x = 0$ (see Remark 2.3.2). Moreover, proving $V'(0+) = \infty$ now requires much more involved analysis, as shown in Lemma 2.5.1.

Second, with unbounded consumptions considered, the framework in [26, 27], like ours, suffers the potential issue that the solution X to (2.5.2) may reach 0 in finite time. The author of [26, 27] does not analyze whether or not, or how likely, X will reach 0 in finite time, but simply restricts the Ramsey problem to the random horizon $[0, \tau_X]$, where τ_X is the first time X reaches 0. However, it is hard to imagine that in practice individuals would allow X, the capital per capita, to reach 0, and enjoy no consumption at all afterwards (This is, nonetheless, what [26, (36)] prescribes).

In a reasonable economic model, an optimal consumption process should by itself prevents X from reaching 0, so that there is no need to artificially introduce τ_X. In this aspect, our paper complements [26, 27], by providing a framework in which $\tau_X = \infty$ is ensured under optimal consumption behavior.

2.6 Comparison with Bounded Consumption in [28]

For each $L > 0$, one can solve the problem (2.4.1) by modifying the arguments in [28], with an optimal consumption process given by

$$\hat{c}_t^L := \hat{c}^L(X_t) \quad \text{for } t \geq 0, \qquad \text{with} \quad \hat{c}^L(x) := \min\left\{\frac{(U')^{-1}(V_L'(x))}{x}, L\right\} \quad \text{for } x > 0, \tag{2.6.1}$$

where X is the unique strong solution to (2.2.8) with c_t replaced by $\hat{c}_t^L$.

Two questions are particularly of interest here. First, by switching from the bounded strategy $\hat{c}^L$, however large $L > 0$ may be, to the possibly unbounded $\hat{c}$ in (2.5.1), can we truly raise our expected utility? An affirmative answer will be provided below, which justifies economically the use of unbounded strategies. Second, for each $L > 0$, do agents following

$\hat{c}^L$ simply chop the no-constraint optimal strategy $\hat{c}$ at the bound $L > 0$? In other words, does "$\hat{c}^L = \hat{c} \wedge L$" hold? As we will see, this fails in general, suggesting a more structural change from $\hat{c}^L$ to $\hat{c}$.

Our first result shows that switching from $\hat{c}^L$ to $\hat{c}$ strictly increases expected utility at *all* levels of wealth (capital per capita) $x > 0$, whenever $\hat{c}$ is truly unbounded.

Proposition 2.6.1. *Suppose $U \in C^2((0,\infty))$. Let $M := \sup_{x>0} \hat{c}(x)$.*

(i) If $M < \infty$, then for any $L \geq M$, $V_L(x) = V(x)$ for all $x > 0$.

(ii) If $M = \infty$, then for any $L > 0$, $V_L(x) < V(x)$ for all $x > 0$.

Proof. (i) Since $\hat{c}$ in (2.5.1) is optimal for V (Theorem 3.2.1) and bounded by $M < \infty$, the definitions of V and V_L in (2.2.7) and (2.4.1) directly imply $V_L = V$ for $L \geq M$.

(ii) Fix $L > 0$. First, we claim that there exists $x^* \in (0,\infty)$ with $V(x^*) > V_L(x^*)$. Suppoe $V = V_L$ on $(0,\infty)$. With $M = \infty$, we can take $x > 0$ with $\hat{c}(x) > L$. This implies $\tilde{U}(V'(x)) = U(\hat{c}(x)x) - \hat{c}(x)xV'(x) > U(Lx) - LxV'(x) = \tilde{U}_L(x, V'(x))$. By this and Theorem 2.4.1,

$$\begin{aligned}
0 &= -\beta V(x) + \frac{1}{2}\sigma^2 x^2 V''(x) + (x^\alpha - \mu x)V'(x) + \tilde{U}(V'(x)) \\
&> -\beta V(x) + \frac{1}{2}\sigma^2 x^2 V''(x) + (x^\alpha - \mu x)V'(x) + \tilde{U}_L(x, V'(x)) \\
&= -\beta V_L(x) + \frac{1}{2}\sigma^2 x^2 V_L''(x) + (x^\alpha - \mu x)V_L'(x) + \tilde{U}_L(x, V_L'(x)),
\end{aligned}$$

where the last line follows from $V = V_L$ on $(0,\infty)$. This, however, contradicts Proposition 2.4.1 (ii).

With $V(x^*) > V_L(x^*)$ for some $x^* > 0$, we will show that $V(x) > V_L(x)$ for *all* $x > 0$. Recall the dynamic programming principle of V_L in (A.1.4). By using the same arguments in Lemma A.1.2, one can derive the corresponding principle for V, i.e. for any $x > 0$,

$$V(x) \geq \sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau} V(X_\tau^x)\right], \quad \forall \tau \in \mathcal{T}. \tag{2.6.2}$$

Now, for any $x > 0$ with $x \neq x^*$, let X denote the unique strong solution to (2.2.8), with c_t replaced by $\hat{c}^L_t$. Consider $\tau^* := \inf\{t \geq 0 : X^x_t = x^*\} \in \mathcal{T}$. Thanks to (2.6.2),

$$\begin{aligned}
V(x) &\geq \mathbb{E}\left[\int_0^{\tau^*} e^{-\beta t} U(\hat{c}^L_t X^x_t) dt + e^{-\beta\tau^*} V(X^x_{\tau^*})\right] \\
&> \mathbb{E}\left[\int_0^{\tau^*} e^{-\beta t} U(\hat{c}^L_t X^x_t) dt + e^{-\beta\tau^*} V_L(X^x_{\tau^*})\right] \\
&\geq \mathbb{E}\left[\int_0^{\tau^*} e^{-\beta t} U(\hat{c}^L_t X^x_t) dt + e^{-\beta\tau^*} \mathbb{E}\left[\int_{\tau^*}^{\infty} e^{-\beta(t-\tau^*)} U(\hat{c}^L_t X^x_t) dt \,\middle|\, \mathcal{F}_{\tau^*}\right]\right] \\
&= \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} U(\hat{c}^L_t X^x_t) dt\right] = V_L(x),
\end{aligned}$$

where the second inequality is due to $V(X^x_{\tau^*}) = V(x^*) > V_L(x^*) = V_L(X^x_{\tau^*})$, the third inequality follows from the same calculation as in (A.1.3), and the last equality holds as $\hat{c}^L$ is optimal for V_L. Hence, we conclude that $V(x) > V_L(x)$ for all $x > 0$. □

Proposition 2.6.1 provides an answer to whether "$\hat{c}^L = \hat{c} \wedge L$" holds.

Corollary 2.6.1. *Suppose $\sup_{x>0} \hat{c}(x) = \infty$. Given $L > 0$, for any $x > 0$ with $\hat{c}(x) < L$, and any $\delta > 0$, there exists $x^* > 0$ such that $|x^* - x| < \delta$ and $\hat{c}^L(x^*) \neq \hat{c}(x^*) \wedge L$. Hence, for any $L > \inf_{x>0} \hat{c}(x)$, there exists $x^* > 0$ such that $\hat{c}^L(x^*) \neq \hat{c}(x^*) \wedge L$.*

Proof. Take $L > 0$ such that there exists $x > 0$ with $\hat{c}(x) < L$. For any $\delta > 0$, by the continuity of $\hat{c}$, there exists $0 < \delta' \leq \delta$ such that $\hat{c}(y) < L$ for all $y \in (x - \delta', x + \delta')$. We claim that there exists $y^* \in (x - \delta', x + \delta')$ such that $\hat{c}^L(y^*) \neq \hat{c}(y^*) \wedge L$. By contradiction, suppose $\hat{c}^L = \hat{c} \wedge L$ on $(x - \delta', x + \delta')$. It follows that $\hat{c}^L = \hat{c}$ on $(x - \delta', x + \delta')$. By (2.5.1) and (2.6.1), this implies $V'_L = V'$ on $(x - \delta', x + \delta')$, which in turn entails $V''_L = V''$ on $(x - \delta', x + \delta')$. Hence, for any $y \in (x - \delta', x + \delta')$,

$$\begin{aligned}
\beta V(y) &= \frac{1}{2}\sigma^2 y^2 V''(y) + (y^\alpha - \mu y)V'(y) + U(\hat{c}(y)y) - \hat{c}(y)yV'(y) \\
&= \frac{1}{2}\sigma^2 y^2 V''_L(y) + (y^\alpha - \mu y)V'_L(y) + U(\hat{c}^L(y)y) - \hat{c}^L(y)yV'_L(y) = \beta V_L(y),
\end{aligned}$$

where the first and the last equalities follows from Theorem 2.4.1 and Proposition 2.4.1. This implies $V = V_L$ on $(x-\delta', x+\delta')$, a contradiction to Proposition 2.6.1 (ii). $\square$

To concretely illustrate the above results, in the following we focus on the utility function

$$U(x) := \frac{x^{1-\gamma}}{1-\gamma} \quad \text{for } x > 0, \qquad \text{with } 0 < \gamma < 1. \tag{2.6.3}$$

Lemma 2.6.1. *Assume* (2.6.3)*. Then, there exist $C_1, C_2 > 0$ such that*

$$C_1 x^{1-\gamma} \le V(x) \quad \forall x > 0 \qquad \textit{and} \qquad V(x) \le C_2(1 + x^{1-\gamma}) \quad \textit{as } x \to \infty. \tag{2.6.4}$$

In particular, we have

$$\lim_{x\to\infty} x^\gamma V'(x) = \left(\frac{\gamma}{\beta + \mu(1-\gamma) + \frac{1}{2}\sigma^2\gamma(1-\gamma)}\right)^\gamma > 0. \tag{2.6.5}$$

Proof. Consider the constant consumption process $\bar{c}_t \equiv 1$. For any $x > 0$, let X denote the unique strong solution to (2.2.8) with $c = \bar{c}$. By the definition of V and (2.6.3),

$$V(x) \ge \frac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta t}(X_t^x)^{1-\gamma} dt\right].$$

Recall from Section 2.3 that $X_t = (Z_t)^{1/(1-\alpha)}$, with Z explicitly given in (2.3.3). It follows that

$$\begin{aligned} V(x) &\ge \frac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(G_t^{-1}\left(x^{1-\alpha} + (1-\alpha)\int_0^t G_s ds\right)\right)^{\frac{1-\gamma}{1-\alpha}} dt\right] \\ &\ge \frac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(G_t^{-1}x^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}} dt\right] = \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta t} G_t^{\frac{\gamma-1}{1-\alpha}} dt\right], \end{aligned}$$

where G is defined as in (2.3.2), with $c_t = \bar{c}_t \equiv 1$, and the second inequality follows from $G_t > 0$ for all $t \ge 0$, $1-\alpha > 0$, and $\frac{1-\gamma}{1-\alpha} > 0$. Noting that the process G is independent of x, we conclude from the above inequality that the first part of (2.6.4) holds.

By Theorem 2.4.1 and (2.6.3), V satisfies

$$\beta V(x) = \frac{1}{2}\sigma^2 x^2 V''(x) + (x^\alpha - \mu x)V'(x) + \frac{\gamma}{1-\gamma}\left(V'(x)\right)^{\frac{\gamma-1}{\gamma}}, \quad \forall x > 0. \tag{2.6.6}$$

Recall from Theorem 2.4.1 that $V'(x) > 0$ and $V''(x) \leq 0$ for all $x > 0$. Also, by the standing assumption $\mu > 0$ in (2.2.9), $x^\alpha - \mu x < 0$ for $x > 0$ large enough. Hence, (2.6.6) implies the existence of $x_0 > 0$ such that

$$\beta V(x) \leq \frac{\gamma}{1-\gamma}\left(V'(x)\right)^{\frac{\gamma-1}{\gamma}}, \quad \text{for } x \geq x_0.$$

Note that V being nonnegative, concave, and nondecreasing entails $V'(x) \leq \frac{V(x)}{x}$ for all $x > 0$. The above inequality then yields $\beta x V'(x) \leq \frac{\gamma}{1-\gamma}\left(V'(x)\right)^{\frac{\gamma-1}{\gamma}}$ for $x \geq x_0$, which is equivalent to

$$V'(x) \leq \left(\frac{\gamma}{\beta(1-\gamma)}\right)^\gamma x^{-\gamma}, \quad \text{for } x \geq x_0.$$

Integrating both sides from x_0 to $x \geq x_0$ gives

$$V(x) \leq V(x_0) + \left(\frac{\gamma}{\beta}\right)^\gamma \left(\frac{1}{1-\gamma}\right)^{\gamma+1} (x^{1-\gamma} - x_0^{1-\gamma}), \quad \text{for } x \geq x_0.$$

This shows that the second part of (2.6.4) is true.

By (2.6.4), $0 < \liminf_{x\to\infty} \frac{V(x)}{x^{1-\gamma}} \leq \limsup_{x\to\infty} \frac{V(x)}{x^{1-\gamma}} < \infty$. Hence, for any $\{x_n\}_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $x_n \to \infty$ and $\frac{V(x_n)}{x_n^{1-\gamma}}$ converges, we must have $\lim_{n\to\infty} \frac{V(x_n)}{x_n^{1-\gamma}} = c$ for some $0 < c < \infty$. Taking $x = x_n$ in (2.6.6) and dividing the equation by $x_n^{1-\gamma}$, we get

$$\beta \frac{V(x_n)}{x_n^{1-\gamma}} = \frac{1}{2}\sigma^2 x_n^{1+\gamma} V''(x_n) + (x_n^{\alpha-1} - \mu) x_n^\gamma V'(x_n) + \frac{\gamma}{1-\gamma}\left(x_n^\gamma V'(x_n)\right)^{\frac{\gamma-1}{\gamma}}, \quad \forall n \in \mathbb{N}.$$

With $c = \lim_{n\to\infty} \frac{V(x_n)}{x_n^{1-\gamma}}$, L'Hospital's rule implies $c(1-\gamma) = \lim_{n\to\infty} x_n^\gamma V'(x_n)$. Using L'Hospital's rule again yields $-c\gamma(1-\gamma) = \lim_{n\to\infty} x_n^{\gamma+1} V''(x_n)$. Thus, as $n \to \infty$, the above equation gives

$$\beta c = -\frac{1}{2}\sigma^2 c\gamma(1-\gamma) - \mu c(1-\gamma) + \frac{\gamma}{1-\gamma}\left(c(1-\gamma)\right)^{\frac{\gamma-1}{\gamma}},$$

which has a unique solution $c = \frac{1}{1-\gamma}\left(\frac{\gamma}{\beta+\mu(1-\gamma)+\frac{1}{2}\sigma^2\gamma(1-\gamma)}\right)^{\gamma} > 0$. With $\{x_n\}_{n\in\mathbb{N}}$ arbitrarily chosen, $\lim_{x\to\infty}\frac{V(x)}{x^{1-\gamma}}$ exists and must equal c as above. L'Hospital's rule then gives the result (2.6.5). □

Proposition 2.6.2. *Assume* (2.6.3). *Then,*

$$\lim_{x\to\infty}\hat{c}(x) = \frac{\beta}{\gamma} + (1-\gamma)\left(\frac{\mu}{\gamma}+\frac{\sigma^2}{2}\right).$$

Moreover,

$$\lim_{x\downarrow 0}\hat{c}(x) = \begin{cases} 0, & \text{if } \gamma < \alpha; \\ (\beta V(0+))^{-1/\gamma} > 0, & \text{if } \gamma = \alpha; \\ \infty, & \text{if } \gamma > \alpha. \end{cases} \tag{2.6.7}$$

Proof. Under (2.6.3), $\hat{c}(x) = \frac{(V'(x))^{-1/\gamma}}{x}$. It follows that

$$\lim_{x\to\infty}\hat{c}(x) = \lim_{x\to\infty}\left(x^{\gamma}V'(x)\right)^{-\frac{1}{\gamma}} = \frac{\beta}{\gamma} + (1-\gamma)\left(\frac{\mu}{\gamma}+\frac{\sigma^2}{2}\right),$$

where the second equality follows from (2.6.5). On the other hand, by Lemma 2.5.1 (ii),

$$\lim_{x\downarrow 0}\hat{c}(x) = \lim_{x\downarrow 0}\frac{(V'(x))^{-1/\gamma}}{x} = \lim_{x\downarrow 0}\frac{(\beta V(0+)x^{-\alpha})^{-1/\gamma}}{x} = (\beta V(0+))^{-1/\gamma}\lim_{x\downarrow 0}x^{\frac{\alpha}{\gamma}-1},$$

which directly implies (2.6.7). □

Proposition 2.6.2 admits interesting economic interpretation. An agent's consumption behavior is determined by two competing effects, captured by the parameters γ and α respectively. First, as in the literature of mathematical finance, γ in (2.6.3) measures the agent's relative risk aversion: the larger γ, the stronger the agent's intention to consume capital right away (to get immediate, riskless utility), as opposed to saving capital in the form of X, subject to risky, stochastic evolution. On the other hand, α in (2.2.8) measures how efficient capital is used in an economy to produce new capital: the larger α, the stronger the upward potential of X, and thus the more willing the agent to save capital (i.e. consume less). Now, as in (2.6.7), when capital per capita X dwindles near 0, (i) if risk aversion of the agent is not so strong relative to the efficiency of capital production (i.e. $\gamma < \alpha$), the effect of α prevails, so that the agent (in the limit) saves all capital to fully exploit the upward potential of X; (ii) if risk aversion of the agent is very strong relative to the efficiency of capital production (i.e. $\gamma > \alpha$), the effect of γ prevails, so that the agent consumes capital as fast as possible, to reduce risky position in X; (iii) if risk aversion of the agent is comparable to the efficiency of capital production (i.e. $\gamma = \alpha$), the effects of α and γ are balanced, leading to bounded, positive consumption of the agent.

Note that while γ measures relative risk aversion, $1/\gamma$ characterizes the elasticity of intertemporal substitution (EIS) of the agent. Using the concept of the EIS, one can possibly derive a related economic interpretation for Proposition 2.6.2.

Corollary 2.6.2. *Assume* (2.6.3)*. If $\gamma \leq \alpha$, as long as $L > 0$ is large enough, $V_L(x) = V(x)$ for all $x > 0$. If $\gamma > \alpha$, then for any $L > 0$, $V_L(x) < V(x)$ for all $x > 0$; moreover, for $L > \frac{\beta}{\gamma} + (1-\gamma)\left(\frac{\mu}{\gamma} + \frac{\sigma^2}{2}\right)$, there exists $x^* > 0$ such that $\hat{c}^L(x^*) \neq \hat{c}(x^*) \wedge L$.*

Proof. Since $\hat{c}(x)$ is by definition continuous on $(0,\infty)$, whether it is bounded on $(0,\infty)$ depends on its limiting behavior as $x \downarrow 0$ and $x \to \infty$. Thus, Proposition 2.6.2 implies (i) $\hat{c}(x)$ is bounded on $(0,\infty)$ if and only if $\gamma \leq \alpha$, and (ii) $\inf_{x>0} \hat{c}(x) \leq \frac{\beta}{\gamma} + (1-\gamma)\left(\frac{\mu}{\gamma} + \frac{\sigma^2}{2}\right)$. The result then follows from Proposition 2.6.1 and Corollary 2.6.1. $\square$

The next two results focus on the specific case $\gamma = \alpha$. The purpose is twofold. First, we demonstrate that the value function V and optimal consumption $\hat{c}$ can be solved explicitly. Second, as we will see, $\hat{c}$ is constant (and thus bounded), so that Corollary 2.6.1 is inconclusive on the failure of "$\hat{c}^L = \hat{c} \wedge L$". Explicit calculation shows that "$\hat{c}^L = \hat{c} \wedge L$" holds for some, but not all, $L > 0$.

Proposition 2.6.3. *Assume* (2.6.3) *with* $\gamma = \alpha$. *Then,* $V(x) = \zeta \cdot \left(\frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{\beta}\right)$, *with*

$$\zeta := \left(\frac{\alpha}{\beta + \mu(1-\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha)}\right)^{\alpha} > 0. \tag{2.6.8}$$

Moreover, the optimal consumption (2.5.1) *is a constant process given by*

$$\hat{c}_t \equiv \frac{\beta}{\alpha} + (1-\alpha)\left(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\right) > 0. \tag{2.6.9}$$

Proof. By Theorem 2.4.1, (2.6.3), and $\gamma = \alpha$, V is a classical solution to

$$\beta v(x) = \frac{1}{2}\sigma^2 x^2 v''(x) + (x^\alpha - \mu x)v'(x) + \frac{\alpha}{1-\alpha}\left(v'(x)\right)^{\frac{\alpha-1}{\alpha}}, \quad \forall x > 0.$$

We plug the ansatz $v(x) = ax^{1-\alpha} + b$, for some $a, b \in \mathbb{R}$, in the above equation. Equating the $x^{1-\alpha}$ terms on both sides leads to

$$\beta a = \frac{1}{2}\sigma^2 a(1-\alpha)(-\alpha) - \mu a(1-\alpha) + \frac{\alpha}{(1-\alpha)^{\frac{1}{\alpha}}} a^{\frac{\alpha-1}{\alpha}},$$

which implies $a = \frac{\zeta}{1-\alpha}$, with ζ as in (2.6.8). Similarly, equating the constant terms on both sides yields $\beta b = a(1-\alpha)$, which implies $b = \frac{\zeta}{\beta}$, with ζ as in (2.6.8). By construction, $v(x) = \zeta \cdot \left(\frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{\beta}\right)$ is nonnegative, concave, strictly increasing, and satisfies the linear growth condition (2.5.14). Moreover,

$$\lim_{x \downarrow 0} \frac{(U')^{-1}(v'(x))}{x^\alpha} = \lim_{x \downarrow 0} \frac{(\zeta x^{-\alpha})^{-1/\alpha}}{x^\alpha} = \lim_{x \downarrow 0} \zeta^{-1/\alpha} x^{1-\alpha} = 0,$$

i.e. (2.5.12) is satisfied. Hence, we conclude from Theorem 3.2.1 that $V(x) = v(x)$ for all

$x > 0$, and the optimal consumption process $\hat{c}$ is given by

$$\hat{c}_t = \frac{(U')^{-1}(v'(X_t))}{X_t} = \frac{(\zeta X_t^{-\alpha})^{-1/\alpha}}{X_t} = \zeta^{-1/\alpha} = \frac{\beta}{\alpha} + (1-\alpha)\left(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\right), \quad \forall t \geq 0.$$

$\square$

The constant consumption (2.6.9) turns out to be the threshold, uniform in $x > 0$, for "$\hat{c}^L = \hat{c} \wedge L$" to hold.

Proposition 2.6.4. *Assume* (2.6.3) *with* $\gamma = \alpha$. *Then,* $\hat{c}^L(x) = \hat{c}(x) \wedge L$ *for all* $x > 0$ *if and only if* $L \geq \frac{\beta}{\alpha} + (1-\alpha)\big(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\big)$.

Proof. By Proposition 2.6.3, $\hat{c}(x) \equiv \frac{\beta}{\alpha} + (1-\alpha)\big(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\big)$. If $L \geq \frac{\beta}{\alpha} + (1-\alpha)\big(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\big)$, by Proposition 2.6.1 we have $V_L = V$ on $(0,\infty)$, which in turn implies $\hat{c}^L = \hat{c} = \hat{c} \wedge L$ on $(0,\infty)$. On the other hand, if $\hat{c}^L(x) = \hat{c}(x) \wedge L$ for all $x > 0$, assume to the contrary that $L < \frac{\beta}{\alpha} + (1-\alpha)\big(\frac{\mu}{\alpha} + \frac{\sigma^2}{2}\big)$. Then, $\hat{c}^L(x) = L$ for all $x > 0$. By Proposition 2.4.1, V_L is a classical solution to

$$\beta v(x) = \frac{1}{2}\sigma^2 x^2 v''(x) + (x^\alpha - \mu x)v'(x) + \frac{(Lx)^{1-\alpha}}{1-\alpha} - Lxv'(x) \quad \forall x > 0.$$

Take the ansatz $v(x) = ax^{1-\alpha} + b$ for some $a, b \in \mathbb{R}$. By the same argument in Proposition 2.6.3, we obtain $v(x) = \zeta_L \cdot \left(\frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{\beta}\right)$, where

$$\zeta_L := \frac{L^{1-\alpha}}{\beta + (1-\alpha)\big(\mu + L + \frac{\alpha\sigma^2}{2}\big)}$$

By construction, $v(x) = \zeta_L \cdot \big(\frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{\beta}\big)$ is nonnegative, concave, strictly increasing, and satisfies the linear growth condition (2.5.14). In view of the characterization of V_L in Remark 2.5.2, we have $V_L(x) = v(x)$ for all $x > 0$. Now, for any $x > 0$,

$$\frac{(U')^{-1}(V_L'(x))}{x} = \frac{(\zeta_L x^{-\alpha})^{\frac{-1}{\alpha}}}{x} = \zeta_L^{\frac{-1}{\alpha}} \leq \left(\frac{L^{1-\alpha}}{L(1-\alpha)}\right)^{\frac{-1}{\alpha}} = L(1-\alpha)^{\frac{1}{\alpha}} < L,$$

which implies $\hat{c}^L(x) = \min\left\{\frac{(U')^{-1}(V_L'(x))}{x}, L\right\} = \frac{(U')^{-1}(V_L'(x))}{x} < L$, a contradiction to $\hat{c}^L(x) = L$. $\square$

Chapter 3

Generalized Duality for Model-Free Superhedging given Marginals[1]

3.1 Introduction

Given a finite time horizon $T \in \mathbb{N}$ with $T \geq 2$, let $\Omega := \mathbb{R}_+^T = [0,\infty)^T$ be the path space and S be the canonical process, i.e. $S_t(x_1, x_2, ..., x_T) = x_t$ for all $(x_1, x_2, ..., x_T) \in \Omega$. We denote by $\mathfrak{P}(\Omega)$ the set of all probability measures on Ω. For all $t = 1, ..., T$, let μ_t be a probability measure on $\mathbb{R}_+$ that has finite first moment; namely,

$$m(\mu_t) := \int_{\mathbb{R}_+} y d\mu_t(y) < \infty. \tag{3.1.1}$$

The set of admissible probability measures on Ω is given by

$$\Pi := \left\{ \mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{Q} \circ (S_t)^{-1} = \mu_t,\ \forall t = 1, ..., T \right\}, \tag{3.1.2}$$

which is known to be nonempty, convex, and compact under the topology of weak convergence, thanks to [20, Proposition 1.2]. We further consider

$$\mathcal{M} := \{\mathbb{Q} \in \Pi : S \text{ is a } \mathbb{Q}\text{-martingale}\}. \tag{3.1.3}$$

[1] Available @ arXiv:1909.06036.

Note that $\mathcal{M} \neq \emptyset$ if and only if $\mu_1, ..., \mu_T$ possess the same finite first moment and increase in the convex order (i.e. $\int_{\mathbb{R}_+} f d\mu_1 \leq \int_{\mathbb{R}_+} f d\mu_2 \leq ... \leq \int_{\mathbb{R}_+} f d\mu_T$, for convex $f : \mathbb{R}_+ \to \mathbb{R}$); see [34]. We will assume $\mathcal{M} \neq \emptyset$ throughout this paper.

The current setup is motivated by a financial market that involves a risky asset, represented by S, and abundant tradable options written on it. For instance, if the tradable options at time 0 include vanilla call options, with payoff $(S_t - K)^+$, for all $t = 1, \cdots, T$ and $K \geq 0$, then the current market prices $C(t, K)$ of these call options already prescribe the distribution of S_t, for each $t = 1, ..., T$, under any pricing (martingale) measure.[2]

A path-dependent contingent claim $\Phi : \Omega \to \mathbb{R}$ can be superhedged by trading the underlying S and holding options available at time 0. Specifically, let $\mathcal{H}$ be the set of $\Delta = \{\Delta_t\}_{t=1}^{T-1}$ with $\Delta_t : \mathbb{R}_+^t \to \mathbb{R}$ Borel measurable for all $t = 1, ..., T-1$. Each $\Delta \in \mathcal{H}$ represents a self-financing (dynamic) trading strategy. The resulting change of wealth over time along a path $x = (x_1, ..., x_T) \in \Omega$ is given by

$$(\Delta \cdot x)_t := \sum_{i=1}^{t-1} \Delta_i(x_1, ..., x_i) \cdot (x_{i+1} - x_i), \qquad \text{for } t = 2, ..., T.$$

In addition, by writing $\mu = (\mu_1, ..., \mu_T)$, we denote by $L^1(\mu)$ the set of $u = (u_1, ..., u_T)$ where $u_t : \mathbb{R}_+ \to \mathbb{R}$ is μ_t-integrable for all $t = 1, ..., T$. Each $u \in L^1(\mu)$ represents a collection of options with different maturities. A semi-static superhedge of Φ consists of some $\Delta \in \mathcal{H}$ and $u \in L^1(\mu)$ such that

$$\Psi_{u,\Delta}(x) := \sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T \geq \Phi(x), \quad \text{for all } x = (x_1, ..., x_T) \in \Omega. \tag{3.1.4}$$

Such superhedging is *model-free*: the terminal wealth $\Psi_{u,\Delta}$ is required to dominate Φ on *every* path $x \in \Omega$, instead of $\mathbb{P}$-a.e. $x \in \Omega$ for some probability $\mathbb{P}$. This is distinct from the

[2]By [14, Proposition 2.1], for each fixed t, as long as $K \mapsto C(t, K)$ is convex and nonnegative, $\lim_{K \downarrow 0+} \partial_K C(t, K) \geq -1$, and $\lim_{K \to \infty} C(t, K) = 0$, the relation "$\mathbb{E}_{\mathbb{Q}}[(S_t - K)^+] = C(t, K)$ for all $K \geq 0$" determines the distribution of S_t. That is, Π in (3.1.2) can be expressed as $\{\mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{E}_{\mathbb{Q}}[(S_t - K)^+] = C(t, K),\ \forall t = 1, \cdots, T \text{ and } K \geq 0\}$.

standard model-based approach: classically, one first specifies a model, or physical measure, $\mathbb{P}$ for the financial market, and then superhedges a contingent claim $\mathbb{P}$-a.s. With the *pointwise* relation (3.1.4), no matter which $\mathbb{P}$ materializes, $\Psi_{u,\Delta} \geq \Phi$ must hold $\mathbb{P}$-a.s. There is then no need to specify a physical measure $\mathbb{P}$ a priori, which prevents any model misspecification.

The corresponding model-free superhedging price of Φ is defined by

$$D(\Phi) := \inf \left\{ \mu(u) : u \in L^1(\mu) \text{ satisfies } \exists \Delta \in \mathcal{H} \text{ s.t. } \Psi_{u,\Delta}(x) \geq \Phi(x) \ \forall x \in \Omega \right\}, \qquad (3.1.5)$$

where $\mu(u) := \sum_{t=1}^{T} \int_{\mathbb{R}_+} u_t d\mu_t$. To characterize $D(\Phi)$, the minimal cost to achieve (3.1.4), Beiglböck, Henry-Labordére, and Penkner [3] introduce the martingale optimal transport problem

$$P(\Phi) := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\Phi]. \qquad (3.1.6)$$

When Φ is upper semicontinuous, denoted by $\Phi \in \text{USC}(\Omega)$, and grows linearly, $D(\Phi)$ coincides with $P(\Phi)$.

Proposition 3.1.1 (Corollary 1.1, [3])**.** *Given $\Phi \in \text{USC}(\Omega)$ for which there exists $K > 0$ such that*

$$\Phi(x) \leq K(1 + x_1 + \cdots + x_T), \quad \forall x = (x_1, \cdots, x_T) \in \Omega, \qquad (3.1.7)$$

we have $D(\Phi) = P(\Phi)$.

Model-free superhedging given marginals, pioneered by Hobson [15], has traditionally focused on several specific forms of contingent claims; see e.g. [6], [16], [23], [7], and [10]. The main contribution of [3] is to allow for general, albeit upper semicoutinuous, contingent claims, via the superhedging duality stated in Proposition 3.1.1. In deriving this duality, [3] uses upper semicontinuity only once for a minimax argument. It is tempting to believe that upper semicontinuity is only a technical condition that can eventually be relaxed.

This is, however, *not* the case. While the model-free duality given marginals in [3] has been widely studied and enriched by now (see [11], [1], [12], and [9], among others), the requirement of upper semicontinuity stands still. Recently, Beiglböck, Nutz, and Touzi [4] has shown that, in fact, upper semicontinuity *cannot* be relaxed. They provide a counterexample where Φ is lower, but not upper, semicontinuous and the duality $D(\Phi) = P(\Phi)$ fails. To restore the duality, [4] modifies the definition of $D(\Phi)$ in (3.1.5) in a quasi-sure way: the inequality $\Psi_{u,\Delta} \geq \Phi$ is required to hold *not* pointwise, but $\mathcal{M}$-quasi surely; that is, $\Psi_{u,\Delta} \geq \Phi$ holds outside of a set that is $\mathbb{P}$-null for all $\mathbb{P} \in \mathcal{M}$. This quasi-sure modification successfully yields the duality $D_{\text{qs}}(\Phi) = P(\Phi)$ for Borel measurable Φ, where $D_{\text{qs}}(\Phi)$ denotes the modified $D(\Phi)$ as described above. This is done in [4] for the two-period model (i.e. $T = 2$), and in Nutz, Stebegg, and Tan [30] for the multi-period case (i.e. $T \in \mathbb{N}$).

In this paper, we approach the failure of $D(\Phi) = P(\Phi)$ from an opposite angle. We keep the definition of $D(\Phi)$ as in (3.1.5), and investigate how $P(\Phi)$ should be modified to get a general duality for Borel measurable Φ. This has two motivations in terms of both theory and applications.

From the theoretical point of view, the pointwise relation (3.1.4) is inherited from the optimal transport theory: the dual problem in the Monge-Kantorovich duality is almost identical to $D(\Phi)$, except that it involves the simpler pointwise relation $\sum_{t=1}^{T} u_t(x_t) \geq \Phi(x)$ (i.e. without the term $(\Delta \cdot x)_T$ in (3.1.4)); see [20]. That is, $D(\Phi)$ naturally extends the classical dual problem from optimal transport to the more general setting we focus on. Finding the primal problem corresponding to this extended dual is of great theoretical interest in itself.

More crucially, as $D(\Phi)$ represents precisely the minimal cost for model-free superhedging, if we modify its definition, although a duality can be obtained (as in [4] and [30]), it will no longer adhere to the model-free superhedging context, thereby losing its financial relevance. In fact, there are two different applications here. In the context of optimal transport, Φ is a payoff function that assigns a reward to each transportation path $x = (x_1, ..., x_T) \in \Omega$, and every $\mathbb{Q} \in \mathcal{M}$ is an admissible transportation plan. The goal is to maximize reward from transportation, i.e. to attain $P(\Phi)$ in (3.1.6)—the perspective taken by [4] and [30].

Our goal, by contrast, is to minimize the cost of model-free superhedging; all developments should then be centered around $D(\Phi)$ in (3.1.5).

Instead of dealing with $D(\Phi)$ directly, we impose, somewhat artificially, portfolio constraints. For any $N \in \mathbb{N}$, we consider

$$\mathcal{H}^N := \{\Delta \in \mathcal{H} : \ |\Delta_t| \leq N, \ \forall\, t = 1, \cdots, T-1\}, \tag{3.1.8}$$

and define $D^N(\Phi)$ as in (3.1.5), with $\Delta \in \mathcal{H}$ replaced by $\Delta \in \mathcal{H}^N$. That is, $D^N(\Phi)$ is a portfolio-constrained model-free superhedging price. Thanks to the general duality in Fahim and Huang [12], the corresponding primal problem $P^N(\Phi)$ can be identified, and there is no duality gap (i.e. $D^N(\Phi) = P^N(\Phi)$) when Φ is upper semicontinuous. The first major contribution of this paper, Theorem 3.3.1, shows that this portfolio-constrained duality actually holds generally for upper semi-analytic Φ. Specifically, by treating D^N and P^N as functionals, we derive appropriate upward and downarrow continuity (Sections 3.3.1 and 3.3.2). Choquet's capacitability theorem can then be invoked to extend $D^N(\Phi) = P^N(\Phi)$ from upper semicontinuous Φ to upper semi-analytic ones.

Note that the portfolio bound $N \in \mathbb{N}$ is indispensable here. In the technical result Lemma 3.3.2, the compactness of the space of semi-static strategies $(u, \Delta) \in L^1(\mu) \times \mathcal{H}^N$ is extracted from the bound $N \in \mathbb{N}$, under an appropriate weak topology. Such compactness then gives rise to the upward continuity of D^N; see Proposition 3.3.4. As opposed to this, D in (3.1.5), when viewed as a functional, does not possess the desired upward continuity. This prevents a direct application of Choquet's capacitability theorem to the unconstrained duality $D(\Phi) = P(\Phi)$ in Proposition 3.1.1; see Remark 3.3.3 for details.

By taking $N \to \infty$ in the constrained duality $D^N(\Phi) = P^N(\Phi)$, we obtain a new characterization of $D(\Phi)$, for upper semi-analytic Φ; see Theorem 3.2.1, the main result of this paper. This new characterization asserts a generalized version of risk-neutral pricing. To find the model-free superhedging price $D(\Phi)$, we need to compute expected values of Φ, but *not* merely under risk-neutral (martingale) measures $\mathbb{Q} \in \mathcal{M}$. As prescribed by Theorem 3.2.1, we should consider sequences of measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ that converge to $\mathcal{M}$ appropriately, and compute the limiting expected values of Φ, i.e. $\limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\Phi]$. The supremum of these limiting expected values then characterizes $D(\Phi)$. For the special case where Φ is upper semicontinuous, these limiting expected values can be attained by measures $\mathbb{Q} \in \mathcal{M}$, as shown in Proposition 3.2.1. The generalized duality in Theorem 3.2.1 thus reduces to one that involves solely measures in $\mathcal{M}$, recovering the classical duality in Proposition 3.1.1.

In deriving the generalized duality in Theorem 3.2.1 from the constrained one $D^N(\Phi) = P^N(\Phi)$, one needs the relation $\lim_{N\to\infty} D^N(\Phi) = D(\Phi)$. This turns out to be highly non-trivial, and is established through delicate probabilistic estimations; see Proposition 3.4.2 for details. Such a relation is economically intriguing in itself, as it states that restricting to bounded trading strategies does not increase the cost of model-free superhedging.

The rest of the paper is organized as follows. Section 3.2 introduces the main result of this paper, a generalized duality that characterizes $D(\Phi)$, for upper semi-analytic Φ. Section 3.3 establishes a portfolio-constrained duality for upper semi-analytic contingent claims, by using Choquet's capacity theory. Section 3.4 derives an unconstrained duality for upper semi-analytic contingent claims, as the limiting case of the constrained one in Section 3.3; this completes the proof of the main result.

3.1.1 Notation

Let $Y = \mathbb{R}_+^t$ for some $t = 1, 2, ..., T$. We denote by $\mathcal{G}(Y)$ the set of all functions from Ω to $\mathbb{R}$. Moreover, let USA(Y), $\mathcal{B}(Y)$, and USC(Y) be the sets of functions in $\mathcal{G}(Y)$ that are upper semi-analytic, Borel measurable, and upper semicontinuous, respectively. Throughout this paper, for any $\Phi \in \mathcal{G}(\Omega)$ and $\mathbb{Q} \in \Pi$, we will interpret $\mathbb{E}_{\mathbb{Q}}[\Phi]$ as the *outer expectation* of Φ. When Φ is actually Borel measurable, it reduces to the standard expectation of Φ.

For any $u \in L^1(\mu)$, we will write $\oplus u(x) := \sum_{t=1}^T u_t(x_t)$ for $x = (x_1, ..., x_T) \in \Omega$ and $\mu(u) := \sum_{t=1}^T \int_{\mathbb{R}_+} u_t d\mu_t$, as specified below (3.1.5).

3.2 The Main Result

3.2.1 Preliminaries

Given $N \in \mathbb{N}$, recall $\mathcal{H}^N$ defined in (3.1.8). For each $\mathbb{Q} \in \Pi$, we introduce

$$\mathbf{A}_T^N(\mathbb{Q}) := \sup_{\Delta \in \mathcal{H}^N} \mathbb{E}_{\mathbb{Q}}[(\Delta \cdot S)_T] = \sup_{\Delta \in \mathcal{H}_c^N} \mathbb{E}_{\mathbb{Q}}[(\Delta \cdot S)_T], \tag{3.2.1}$$

where

$$\mathcal{H}_c^N := \{\Delta \in \mathcal{H}^N : \ \Delta_t \text{ is continuous}, \ \forall t = 1, \cdots, T-1\}.$$

Note that the reduction to continuous trading strategies in (3.2.1) is justified by Fahim and Huang [12, Lemma 3.3]. The set $\mathcal{M}$ in (3.1.3) can be fully characterized by $\mathbf{A}_T^N(\mathbb{Q})$ as follows.

Lemma 3.2.1. $\mathbb{Q} \in \mathcal{M} \iff \mathbf{A}_T^1(\mathbb{Q}) = 0 \iff \mathbf{A}_T^N(\mathbb{Q}) = 0$ *for all* $N \in \mathbb{N}$.

Proof. By definition, $\mathbf{A}_T^N(\mathbb{Q}) = N\mathbf{A}_T^1(\mathbb{Q})$. Thus, $\mathbf{A}_T^1(\mathbb{Q}) = 0$ if and only if $\mathbf{A}_T^N(\mathbb{Q}) = 0$ for all $N \in \mathbb{N}$. Now, by (3.2.1), "$\mathbf{A}_T^N(\mathbb{Q}) = 0$ for all $N \in \mathbb{N}$" is equivalent to "$\mathbb{E}_{\mathbb{Q}}[(\Delta \cdot S)_T] = 0$ for all $\Delta \in \mathcal{H}_c^N$, for any $N \in \mathbb{N}$". The latter condition holds if and only if $\mathbb{Q} \in \mathcal{M}$, by [3, Lemma 2.3]. $\square$

Lemma 3.2.1 indicates that a pseudometric on Π can be defined by

$$d(\mathbb{Q}_1, \mathbb{Q}_2) := \left|\mathbf{A}_T^1(\mathbb{Q}_1) - \mathbf{A}_T^1(\mathbb{Q}_2)\right|, \quad \forall \mathbb{Q}_1, \mathbb{Q}_2 \in \Pi. \tag{3.2.2}$$

It is only a pseudometric, but not a metric, because $d(\mathbb{Q}_1, \mathbb{Q}_2) = 0$ does not necessarily imply $\mathbb{Q}_1 = \mathbb{Q}_2$. We can turn it into a metric by considering equivalent classes induced by d. Specifically, we say $\mathbb{Q}_1, \mathbb{Q}_2 \in \Pi$ are equivalent (denoted by $\mathbb{Q}_1 \sim \mathbb{Q}_2$) if $d(\mathbb{Q}_1, \mathbb{Q}_2) = 0$, or $\mathbf{A}_T^1(\mathbb{Q}_1) = \mathbf{A}_T^1(\mathbb{Q}_2)$. Equivalent classes are then defined by $[\mathbb{Q}] := \{\mathbb{Q}' \in \Pi : d(\mathbb{Q}', \mathbb{Q}) = 0\}$ for all $\mathbb{Q} \in \Pi$. On the quotient space $\Pi^* := \Pi/\sim\, = \{[\mathbb{Q}] : \mathbb{Q} \in \Pi\}$,

$$\rho([\mathbb{Q}_1], [\mathbb{Q}_2]) := d(\mathbb{Q}_1, \mathbb{Q}_2) \tag{3.2.3}$$

defines a metric.

Remark 3.2.1. *In view of Lemma 3.2.1,* $\mathcal{M} = [\mathbb{Q}]$ *for any* $\mathbb{Q} \in \mathcal{M}$*.*

Remark 3.2.2. *Instead of the pseudometric on* Π *in* (3.2.2)*, one can consider the semi-norm*

$$\|Q\| := \sup_{\Delta \in \mathcal{H}^1} \int_\Omega (\Delta \cdot S)_T \; dQ$$

defined on the vector space $\mathcal{K} := \{Q : Q \text{ is a signed measure on } \Omega\}$*. When we restrict the semi-norm to* $\Pi \subset \mathcal{K}$*, we have* $\|\mathbb{Q}\| = 0$ *if and only* $\mathbb{Q} \in \mathcal{M}$ *(thanks to Lemma 3.2.1). This can be used to define a metric equivalent to* (3.2.3)*.*

To state the main result of this paper, Theorem 3.2.1 below, we need to consider a sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ in Π that converge to $\mathcal{M}$ under the metric ρ; that is, by Remark 3.2.1,

$$\rho([\mathbb{Q}_N], \mathcal{M}) = \rho([\mathbb{Q}_N], [\mathbb{Q}]) \to 0 \quad \text{as } N \to \infty, \quad \forall \mathbb{Q} \in \mathcal{M}.$$

For simplicity, this will be denoted by $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$. As $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$ is equivalent to $\mathbf{A}^1_T(\mathbb{Q}_N) \to 0$, by (3.2.3) and (3.2.2), they will be used interchangeably throughout the paper.

Crucially, $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$ entails weak convergence to $\mathcal{M}$ (up to a subsequence).

Lemma 3.2.2. *Consider* $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ *in* Π *such that* $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$*. For any subsequence* $\{\mathbb{Q}_{N_k}\}_{k\in\mathbb{N}}$ *that converges weakly, it must converge weakly to some* $\mathbb{Q}_* \in \mathcal{M}$*.*

Proof. Let $\mathbb{Q}_* \in \Pi$ denote the probability measure to which $\mathbb{Q}_{N_k}$ converges weakly. First, recall that $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$ is equivalent to $\mathbf{A}^1_T(\mathbb{Q}_N) \to 0$, which in turn implies $\mathbf{A}^1_T(\mathbb{Q}_{N_k}) \to 0$. Next, for any $\Delta \in \mathcal{H}^1_c$, since $|(\Delta \cdot x)_T| \le h(x) := x_1 + 2(x_2 + ... + x_{T-1}) + x_T$, we deduce from [35, Lemma 4.3] that $\mathbb{Q} \mapsto \mathbb{E}_\mathbb{Q}[(\Delta \cdot S)_T]$ is continuous under the topology of weak convergence. It follows that

$$\mathbb{Q} \mapsto \mathbf{A}^1_T(\mathbb{Q}) = \sup_{\Delta \in \mathcal{H}^1_c} \mathbb{E}_\mathbb{Q}[(\Delta \cdot S)_T]$$

is lower semicontinuous under the topology of weak convergence. Hence,

$$\mathbf{A}_T^1(\mathbb{Q}_*) \leq \liminf_{k\to\infty} \mathbf{A}_T^1(\mathbb{Q}_{N_k}) = 0.$$

We then conclude $\mathbf{A}_T^1(\mathbb{Q}_*) = 0$, which implies $\mathbb{Q}_* \in \mathcal{M}$ thanks to Lemma 3.2.1. $\square$

3.2.2 The Generalized Duality

Now, we are ready to present the main result of this paper.

Theorem 3.2.1. *For any $\Phi \in \mathrm{USA}(\Omega)$ for which there exists $K > 0$ such that*

$$|\Phi(x)| \leq K(1 + x_1 + \cdots + x_T) \quad \forall x = (x_1, \cdots, x_T) \in \Omega, \tag{3.2.4}$$

we have

$$D(\Phi) = \widetilde{P}(\Phi) := \sup\left\{\limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi] : \mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}\right\}. \tag{3.2.5}$$

When Φ is additionally upper semicontinuous, Theorem 3.2.1 recovers the classical duality in Proposition 3.1.1, as the next result demonstrates.

Proposition 3.2.1. *For any $\Phi \in \mathrm{USC}(\Omega)$ that satisfies* (3.2.4)*, $\widetilde{P}(\Phi)$ reduces to $P(\Phi)$ in* (3.1.6)*.*

Proof. For any $\mathbb{Q} \in \mathcal{M}$, by taking $\mathbb{Q}_N := \mathbb{Q}$ for all $N \in \mathbb{N}$, the definition of $\widetilde{P}(\Phi)$ in (3.2.5) directly implies $\widetilde{P}(\Phi) \geq \mathbb{E}_{\mathbb{Q}}[\Phi]$. Taking supremum over $\mathbb{Q} \in \mathcal{M}$ yields $\widetilde{P}(\Phi) \geq P(\Phi)$.

On the other hand, take an arbitrary $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ in Π such that $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$. For any $\varepsilon > 0$, there exists a subsequence $\{\mathbb{Q}_{N_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathbb{E}_{\mathbb{Q}_{N_k}}[\Phi] \geq \limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi] - \varepsilon. \tag{3.2.6}$$

As Π is compact (recall the explanation below (3.1.2)), there is a further subsequence, which will still be denoted by $\{\mathbb{Q}_{N_k}\}_{k\in\mathbb{N}}$, that converges weakly to some $\mathbb{Q}_* \in \Pi$. By Lemma 3.2.2,

$\mathbb{Q}_*$ must belong to $\mathcal{M}$. Now, as Φ is upper semicontinuous and satisfies (3.2.4), we deduce from [35, Lemma 4.3] and $\{\mathbb{Q}_{N_k}\}$ converging weakly to $\mathbb{Q}_* \in \mathcal{M}$ that

$$\lim_{k\to\infty} \mathbb{E}_{\mathbb{Q}_{N_k}}[\Phi] \leq \mathbb{E}_{\mathbb{Q}_*}[\Phi] \leq P(\Phi).$$

This, together with (3.4.10) and the arbitrariness of $\varepsilon > 0$, shows that $\limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi] \leq P(\Phi)$. As $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ such that $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$ is arbitrarily chosen, we conclude that $\widetilde{P}(\Phi) \leq P(\Phi)$. $\square$

Theorem 3.2.1 extends the standard wisdom for risk-neutral pricing. To find the model-free superhedging price $D(\Phi)$, one needs to compute expected values of Φ, but not merely under risk-neutral (martingale) measures $\mathbb{Q} \in \mathcal{M}$. Instead, one should consider, more generally, sequences of measures $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ in Π that converge appropriately to $\mathcal{M}$, and compute the limiting expected values of Φ. Only when Φ is continuous enough (i.e. upper semicontinuous) can we restrict our attention to solely martingale measures in $\mathcal{M}$, as Proposition 3.2.1 indicates.

The next example demonstrates explicitly that despite $D(\Phi) > P(\Phi)$, the generalized duality $D(\Phi) = \widetilde{P}(\Phi)$ holds.

Example 3.2.1. *Let $T = 2$ and $\mu_1 = \mu_2$ be the Lebesgue measure on $[0,1]$. Then $\mathcal{M}$ contains one single measure $\mathbb{P}_0$, under which (S_1, S_2) is uniformly distributed on $\{(x,y) \in [0,1]^2 : x = y\}$. For the lower semicontinuous $\Phi(x_1,x_2) := 1_{\{x_1\neq x_2\}}$, it is shown in [4, Example 8.1] that $0 = P(\Phi) < D(\Phi) = 1$; in addition, $(u_1^*, u_2^*, \Delta_1^*) \equiv (1,0,0)$ is an optimizer of $D(\Phi)$.*

We will show that $\widetilde{P}(\Phi) = 1$. Consider a collection of probability measures $\{\mathbb{Q}_M\}_{M\in\mathbb{N}}$ on $[0,1]^2$, with the density function of each $\mathbb{Q}_M$ given by

$$g(x_1,x_2) = M \sum_{i=0}^{M-1} 1_{[\frac{i}{M},\frac{i+1}{M})^2}(x_1,x_2); \tag{3.2.7}$$

see Figure 3.1. It can be checked by definition that $\mathbb{Q}_M \in \Pi$. Observe that

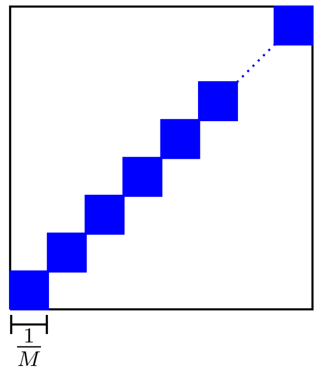

Figure 3.1: Support of $\mathbb{Q}_M$.

$$\begin{aligned}\mathbf{A}_2^1(\mathbb{Q}_M) = \sup_{\Delta_1 \in \mathcal{H}^1} \mathbb{E}_{\mathbb{Q}_M}[\Delta_1 \cdot (S_2 - S_1)] &= \sup_{\Delta_1 \in \mathcal{H}^1} \mathbb{E}_{\mathbb{Q}_M}\left[\Delta_1 \cdot (\mathbb{E}_{\mathbb{Q}_M}[S_2 \mid \mathcal{F}_1] - S_1)\right] \\ &= \mathbb{E}_{\mathbb{Q}_M}[|\mathbb{E}_{\mathbb{Q}_M}[S_2 \mid \mathcal{F}_1] - S_1|],\end{aligned}$$

where $\mathcal{F}_1$ *denotes the* σ*-algebra generated by* S_1*, and the second line holds as the supremum is attained by taking* $\Delta_1 = sgn(\mathbb{E}_{\mathbb{Q}_M}[S_2 \mid \mathcal{F}_1] - S_1)$*. As*

$$\mathbb{E}_{\mathbb{Q}_M}[S_2 \mid S_1 = x] = \sum_{i=0}^{M-1} \frac{2i+1}{2M} 1_{\{[\frac{i}{M}, \frac{i+1}{M})\}}(x).$$

we obtain

$$\mathbf{A}_2^1(\mathbb{Q}_M) = \mathbb{E}_{\mathbb{Q}_M}[|\mathbb{E}_{\mathbb{Q}_M}[S_2 \mid S_1] - S_1|] = \sum_{i=0}^{M-1} \frac{1}{4M^2} = \frac{1}{4M} \to 0, \quad \text{as } M \to \infty. \tag{3.2.8}$$

That is, $\mathbb{Q}_M \xrightarrow{\rho} \mathcal{M}$*. It follows that* $\widetilde{P}(\Phi) \geq \limsup_{M\to\infty} \mathbb{E}_{\mathbb{Q}_M}[\Phi] = 1$*, where the equality stems from* $\mathbb{E}_{\mathbb{Q}_M}[\Phi] = 1$ *for all* $M \in \mathbb{N}$*, by the definition of* Φ *and* (3.2.7)*. As* $\Phi \leq 1$ *readily implies* $\widetilde{P}(\Phi) \leq 1$*, we conclude* $\widetilde{P}(\Phi) = 1 = D(\Phi)$*.*

This paper is devoted to the derivation of Theorem 3.2.1. It will be done through a delicate two-step plan, to be carried out in detail in Sections 3.3 and 3.4. We give a brief outline as follows.

For any $N \in \mathbb{N}$, recall $D^N(\Phi)$, the portfolio-constrained model-free superhedging price defined below (3.1.8). Also, consider

$$P^N(\Phi) := \sup_{\mathbb{Q}\in\Pi}\{\mathbb{E}_{\mathbb{Q}}[\Phi] - \mathbf{A}_T^N(\mathbb{Q})\}. \tag{3.2.9}$$

As a direct consequence of Fahim and Huang [12, Theorem 3.14], $D^N(\Phi)$ can be characterized, in the same spirit of Proposition 3.1.1, as follows.

Proposition 3.2.2. *Given* $\Phi \in \mathrm{USC}(\Omega)$ *that satisfies* (3.1.7), $D^N(\Phi) = P^N(\Phi)$ *for all* $N \in \mathbb{N}$.

Section 3.3 focuses on extending this portfolio-constrained duality to one that allows for *upper semi-analytic* Φ. Intriguingly, by using Choquet's capacity theory, we will show that the same duality $D^N(\Phi) = P^N(\Phi)$ simply holds for upper semi-analytic Φ; there is no need to adjust $P^N(\Phi)$. By taking $N \to \infty$, Section 3.4 elaborates how $D^N(\Phi) = P^N(\Phi)$ turns into the desired duality (3.2.5).

3.3 Complete Duality under Portfolio Constraints

Given $N \in \mathbb{N}$, the goal of this section is to establish the complete duality $D^N(\Phi) = P^N(\Phi)$ for upper semi-analytic Φ. As such a duality is known to hold for upper semicontinuous Φ (Proposition 3.2.2), our strategy is to treat P^N and D^N as functionals, and exploit their continuity properties.

Let us first recall the notion of a Choquet capacity. Recall also the notation in Section 3.1.1.

Definition 3.3.1. *A functional* $C : \mathcal{G}(\Omega) \to \mathbb{R}^*$ *is called a Choquet capacity associated with* $\mathrm{USC}(\Omega)$ *(or simply capacity) if it satisfies*

(i) $C(\phi) \le C(\psi)$ *if* $\phi \le \psi$*;*

(ii) if $\phi_i \uparrow \phi$*, then* $\sup_{i\in\mathbb{N}} C(\Phi_i) = C(\Phi)$*;*

(iii) for any sequence $\{\phi_i\}$ *in* $\mathrm{USC}(\Omega)$ *such that* $\phi_i \downarrow \phi$, $\inf_{i\in\mathbb{N}} C(\Phi_i) = C(\Phi)$.

Choquet's capacitability theorem (see [20, Proposition 2.11] or [8, Section 3]) asserts a desirable continuity property of a capacity.

Lemma 3.3.1. *Let* $C : \mathcal{G}(\Omega) \to \mathbb{R}$ *be a Choquet capacity associated with* $\mathrm{USC}(\Omega)$*. Then, for any* $\Phi \in \mathrm{USA}(\Omega)$*,*

$$C(\Phi) = \sup\{C(\phi) \ : \ \phi \leq \Phi \quad \textit{with } \phi \in \mathrm{USC}(\Omega)\}.$$

Hence, if two capacities C_1 *and* C_2 *coincide on* $\mathrm{USC}(\Omega)$*, they coincide on* $\mathrm{USA}(\Omega)$*.*

Remark 3.3.1. *The original Choquet's capacitability theorem gives a more general result: if* C_1 *and* C_2 *are two Choquet capacities associated with a set of functions* $\mathcal{A}$ *and they coincide on functions in* $\mathcal{A}$*, then they coincide on* $\mathcal{A}$*-Suslin functions. Here, we take* $\mathcal{A} = \mathrm{USC}(\Omega)$ *in Definition 3.3.1 and Lemma 3.3.1, and note that "*$\mathrm{USC}(\Omega)$*-Suslin functions" are simply "upper semi-analytic functions (*$\mathrm{USA}(\Omega)$*)"; see [20, Proposition 2.13] and [5, Definition 7.21].*

3.3.1 Continuity of P^N

Proposition 3.3.1. *Consider* $\{\Phi_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{G}(\Omega)$ *for which there exists* $K > 0$ *such that for each* $i \in \mathbb{N}$*,*

$$\Phi_i(x) \geq -K(1 + x_1 + \cdots + x_T), \quad \forall x = (x_1, \cdots, x_T) \in \Omega. \tag{3.3.1}$$

If $\Phi_i \uparrow \Phi$*, then*

$$\sup_{i\in\mathbb{N}} P^N(\Phi_i) = P^N(\Phi), \quad \forall N \in \mathbb{N}.$$

Proof. Since Φ_1 satisfies (3.3.1), the monotone convergence theorem for outer expectation gives $\mathbb{E}_{\mathbb{Q}}[\Phi_i] \uparrow \mathbb{E}_{\mathbb{Q}}[\Phi]$, for all $\mathbb{Q} \in \Pi$. By changing the order of two supremums, we get

$$\sup_{i\in\mathbb{N}} P^N(\Phi_i) = \sup_{\mathbb{Q}\in\Pi} \sup_{i\in\mathbb{N}} \left(\mathbb{E}_{\mathbb{Q}}[\Phi_i] - \mathbf{A}_T^N(\mathbb{Q})\right) = \sup_{\mathbb{Q}\in\Pi} \left(\mathbb{E}_{\mathbb{Q}}[\Phi] - \mathbf{A}_T^N(\mathbb{Q})\right) = P^N(\Phi),$$

for each $N \in \mathbb{N}$. $\square$

Proposition 3.3.2. *Consider* $\{\Phi_i\}_{i\in\mathbb{N}}$ *in* $\mathrm{USC}(\Omega)$ *for which there exists* $K > 0$ *such that* (3.1.7) *is satisfied for each* Φ_i. *If* $\Phi_i \downarrow \Phi$, *then*

$$\inf_{i\in\mathbb{N}} P^N(\Phi_i) = P^N(\Phi), \quad \forall N \in \mathbb{N}.$$

Proof. Fix $N \in \mathbb{N}$. As $\Phi_i \downarrow \Phi$ clearly implies $\inf_{i\in\mathbb{N}} P^N(\Phi_i) \geq P^N(\Phi)$, we focus on proving the "$\leq$" relation. Assume $\inf_{i\in\mathbb{N}} P^N(\Phi_i) > -\infty$, otherwise the proof would be trivial. For any $\delta < \inf_{i\in\mathbb{N}} P^N(\Phi_i)$, define

$$\mathcal{M}^N(\Phi_i, \delta) := \{\mathbb{Q} \in \Pi : \mathbb{E}_{\mathbb{Q}}[\Phi_i] - \mathbf{A}_T^N(\mathbb{Q}) \geq \delta\} \quad \text{for all } N \in \mathbb{N}.$$

We intend to show that $\mathcal{M}^N(\Phi_i, \delta)$ is compact under the topology of weak convergence. As Π is compact (recall the explanation below (3.1.2)), it suffices to prove that $\mathcal{M}^N(\Phi_i, \delta)$ is closed. Since Φ_i is upper semicontinuous and satisfies (3.1.7), we deduce from [35, Lemma 4.3] that $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[\Phi_i]$ is upper semicontinuous under the topology of weak convergence. On the other hand, by the same argument in the proof of Lemma 3.2.2, $\mathbb{Q} \mapsto \mathbf{A}_T^N(\mathbb{Q})$ is lower semicontinuous under the topology of weak convergence. As a result, $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[\Phi_i] - \mathbf{A}_T^N(\mathbb{Q})$ is upper semicontinuous, which gives the desired closedness of $\mathcal{M}^N(\Phi_i, \delta)$.

Now, since $\{\mathcal{M}^N(\Phi_i, \delta)\}_{i\in\mathbb{N}}$ is a nonincreasing sequence of compact sets, $\bigcap_{i=1}^\infty \mathcal{M}^N(\Phi_i, \delta) \neq \emptyset$. Take $\tilde{\mathbb{Q}} \in \bigcap_{i=1}^\infty \mathcal{M}^N(\Phi_i, \delta)$, and observe that

$$P^N(\Phi) \geq \mathbb{E}^{\tilde{\mathbb{Q}}}[\Phi] - \mathbf{A}_T^N(\tilde{\mathbb{Q}}) = \lim_{i\to\infty} \mathbb{E}^{\tilde{\mathbb{Q}}}[\Phi_i] - \mathbf{A}_T^N(\tilde{\mathbb{Q}}) \geq \delta,$$

where the equality follows from the reverse monotone convergence theorem, applicable here as (3.1.7) is satisfied for each Φ_i, and the last inequality results from the definition of $\mathcal{M}^N(\Phi_i, \delta)$. With $\delta < \inf_{i\in\mathbb{N}} P^N(\Phi_i)$ arbitrarily chosen, we conclude $\inf_{i\in\mathbb{N}} P^N(\Phi_i) \leq P^N(\Phi)$. $\square$

3.3.2 Continuity of D^N

The downward continuity of D^N is a consequence of Propositions 3.2.2 and 3.3.2.

Proposition 3.3.3. *Consider* $\{\Phi_i\}_{i\in\mathbb{N}}$ *in* $\mathrm{USC}(\Omega)$ *for which there exists* $K > 0$ *such that* (3.1.7) *is satisfied for each* Φ_i. *If* $\Phi_i \downarrow \Phi$, *then*

$$\inf_{i\in\mathbb{N}} D^N(\Phi_i) = D^N(\Phi), \quad \forall N \in \mathbb{N}.$$

Proof. As the infimum of a sequence of upper semicontinuous functions satisfying (3.1.7), Φ is again upper semicontinuous and satisfies (3.1.7). It then follows from Proposition 3.2.2 that

$$\inf_{i\in\mathbb{N}} D^N(\Phi_i) = \inf_{i\in\mathbb{N}} P^N(\Phi_i) = P^N(\Phi) = D^N(\Phi),$$

where the second equality is due to Proposition 3.3.2. □

The upward continuity of D^N, by contrast, is much more obscure. We need the following technical result, Lemma 3.3.2, to construct certain compactness for the space of semi-static strategies (u, Δ), which will facilitate the derivation of the upward continuity of D^N in Proposition 3.3.4 below. This lemma can be viewed as a generalization of [20, Lemma 1.27] to the case of martingale optimal transport. The main idea involved is to extract additional compactness from the portfolio bound $N > 0$ through Tychonoff's theorem.

In Lemma 3.3.2 below, let $\mathcal{B}(\mathbb{R}^t_+)$ be equipped with the topology of pointwise convergence. In addition, consider the product measure $\nu := \mu_1 \otimes \cdots \otimes \mu_T$ on Ω, and denote by $L^1(\mu_t)$ (resp. $L^1(\nu)$) the set of μ_t-integrable (resp. ν-integrable) functions. Also recall $m(\mu_t)$, $t = 1, ..., T$, from (3.1.1).

Lemma 3.3.2. *Fix* $N \in \mathbb{N}$ *and* $\Phi \in \mathcal{G}(\Omega)$ *that satisfies* (3.3.1) *and* $D^N(\Phi) < \infty$. *For any* $\delta > D^N(\Phi)$, *define* $\mathcal{L}(\Phi, \delta, N)$ *as the collection of all pairs* (Θ, Δ), *with*

$$\Theta := \left\{(u^k_1, ..., u^k_T, W^k)\right\}_{k\in\mathbb{N}} \in \left(\Pi^T_{t=1} L^1(\mu_t) \times L^1(\nu)\right)^{\mathbb{N}} \quad \text{and} \quad \Delta \in \mathcal{H}^N, \tag{3.3.2}$$

satisfying

(i) *For each* $k \in \mathbb{N}$, $0 \le u_t^k \le 2k$, $\forall t = 1, \cdots, T$*;*

(ii) $u_t^1 \le u_t^2 \le \cdots$, $\forall t = 1, \cdots, T$*;*

(iii) *For each* $k \in \mathbb{N}$, $\mu(u^k) \le \delta + (K+2N)(1 + m(\mu_1) + \cdots + m(\mu_T))$*;*

(iv) *For each* $k \in \mathbb{N}$, $W^k \in L^1(\nu)$ *with* $0 \le W^k \le \Lambda$, *where* $\Lambda \in L^1(\nu)$ *is defined by*

$$\Lambda(x) := 2N(x_1 + \cdots + x_T);$$

moreover, $W^k = 0$ *on the set* $\{x : \Lambda(x) < k\}$*;*

(v) *For each* $k \in \mathbb{N}$, $\oplus u^k \ge (\Phi + \Gamma) \wedge k + (\Delta \cdot x)_T - W^k$, *where* $\Gamma \in L^1(\nu)$ *is defined by*

$$\Gamma(x) := (K+2N)(1 + x_1 + \cdots + x_T).$$

Here, the constant $K > 0$ *in (iii) and (v) comes from* (3.3.1).

The set $\mathcal{L}(\Phi, \delta, N)$ *is a nonempty compact subset of* $\left(\Pi_{t=1}^T L^1(\mu_t) \times L^1(\nu)\right)^{\mathbb{N}} \times \Pi_{t=1}^{T-1} \mathcal{B}(\mathbb{R}_+^t)$, *under the product of the weak topologies of the spaces* $L^1(\mu_t)$, $L^1(\nu)$, *and* $\mathcal{B}(\mathbb{R}_+^t)$.

Proof. **Step 1:** We show that $\mathcal{L}(\Phi, \delta, N)$ is *nonempty.* As $\delta > D^N(\Phi)$, there exist $u = (u_1, ..., u_T) \in L^1(\mu)$ and $\Delta \in \mathcal{H}^N$ such that

$$\mu(u) \le \delta \quad \text{and} \quad \oplus u + (\Delta \cdot x)_T \ge \Phi.$$

As Φ satisfies (3.3.1) and $|(\Delta \cdot x)_T| \le \Lambda(x)$, we have $\oplus u(x) \ge \Phi(x) - (\Delta \cdot x)_T \ge -\Gamma(x)$. This implies that we can find constants $a_1, a_2, ..., a_T$ such that $\sum_{t=1}^T a_t = 0$ and $a_t + u_t \ge -(K+2N)(1/T + x_t)$ for all $t = 1, ..., T$. Now, define $\bar{u}_t := a_t + u_t + (K+2N)(1/T + x_t) \ge 0$

for all $t = 1, ..., T$. Then, one can write

$$\oplus \bar{u} \geq \Phi + \Gamma + (\bar{\Delta} \cdot x)_T, \quad \text{with } \bar{\Delta} := -\Delta \in \mathcal{H}^N.$$

On the other hand, by the concavity of $x \mapsto x \wedge (2k)$,

$$(\oplus \bar{u}) \wedge (2k) \geq \big((\Phi + \Gamma) + (\bar{\Delta} \cdot x)_T\big) \wedge (2k) \geq (\Phi + \Gamma) \wedge k + (\bar{\Delta} \cdot x)_T \wedge k.$$

Since $\bar{u}_t \geq 0$ for all $t = 1, ..., T$, it can be checked that $\oplus(\bar{u} \wedge (2k)) \geq (\oplus \bar{u}) \wedge (2k)$. This, together with the previous inequality, gives

$$\oplus \left(\bar{u} \wedge (2k)\right) \geq (\Phi + \Gamma) \wedge k + (\bar{\Delta} \cdot x)_T \wedge k. \tag{3.3.3}$$

We claim that $u_t^k := \bar{u}_t \wedge (2k)$, $W^k := (\bar{\Delta} \cdot x)_T - (\bar{\Delta} \cdot x)_T \wedge k$, and $\bar{\Delta}$ form an element of $\mathcal{L}(\Phi, \delta, N)$. By construction, it is straightforward to verify conditions (i), (ii), and (v). Since $\oplus \bar{u}^k \leq \oplus \bar{u} = \oplus u + \Gamma$, we have $\mu(\bar{u}^k) \leq \delta + (K + 2N)\,(1 + m(\mu_1) + \cdots + m(\mu_T))$, i.e. condition (iii) is satisfied. For each $k \in \mathbb{N}$, observe that $0 \leq W^k \leq |(\bar{\Delta} \cdot x)_T| \leq \Lambda(x)$. In particular, if $\Lambda(x) \leq k$, then $|(\bar{\Delta} \cdot x)_T| \leq k$ and thus $W^k = 0$ by definition. This shows that condition (iv) is satisfied.

Step 2: We prove that $\mathcal{L}(\Phi, \delta, N)$ is contained in a *weakly compact* space of functions. Observe that the following collections of functions

$$U(t, k) := \{u \in L^1(\mu_t) : 0 \leq u \leq 2k\} \quad t = 1, ..., T \text{ and } k \in \mathbb{N},$$
$$V := \{W \in L^1(\nu) : 0 \leq W \leq \Lambda\}$$

are all uniformly integrable, and thus relatively weakly compact thanks to the Dunford-Pettis theorem. It follows that the countable product $(\Pi_{t,k} U(t, k)) \times V^{\mathbb{N}}$ is also relatively weakly compact. On the other hand, for each $t = 1, \cdots, T-1$,

$$F_t := \{f : \mathbb{R}^t_+ \to \mathbb{R} : |f| \leq N\} = \Pi_{x \in \mathbb{R}^t_+}[-N, N]^x$$

is compact under the topology of pointwise convergence, as a consequence of Tychonoff's theorem. The space F_t is therefore weakly compact, and this carries over to the product space $\mathcal{H}^N = \Pi_t F_t$. We then conclude that $\Pi_{t,k} U(t,k) \times V^{\mathbb{N}} \times \mathcal{H}^N$ is a weakly compact set containing $\mathcal{L}(\Phi, \delta, N)$.

Step 3: We prove that $\mathcal{L}(\Phi, \delta, N)$ is *strongly closed*. Take a sequence

$$\left\{\{(u_1^{k,m}, \cdots, u_T^{k,m}, W^{k,m})\}_{k \in \mathbb{N}}, \Delta^m\right\}_{m \in \mathbb{N}}$$

in $\mathcal{L}(\Phi, \delta, N)$ such that it converges to $(\{(u_1^k, \cdots, u_T^k, W^k)\}_{k \in \mathbb{N}}, \Delta)$ in the strong sense. That is, $u_t^{k,m} \to u_t^k$ in $L^1(\mu_t)$, $W^{k,m} \to W^k$ in $L^1(\nu)$, and $\Delta^m \to \Delta$ pointwise in $\mathcal{H}^N$. We intend to show that $(\{(u_1^k, \cdots, u_T^k, W^k)\}_{k \in \mathbb{N}}, \Delta)$ also lies in $\mathcal{L}(\Phi, \delta, N)$.

The convergence in $L^1(\mu_t)$ (resp. $L^1(\nu)$) implies the existence of a subsequence that converges μ_t-a.e (resp. ν-a.e.). Then, as $m \to \infty$, we conclude from $\oplus u^{k,m} \geq (\Phi + \Gamma) \wedge k + (\Delta^m \cdot x)_T - W^{k,m}$ that

$$\oplus u^k \geq (\Phi + \Gamma) \wedge k + (\Delta \cdot x)_T - W^k \tag{3.3.4}$$

holds outside a ν-null set $\mathcal{N}$. We can then modify $(u_t^k)_{t=1}^T$ and W^k on $\mathcal{N}$ such that (3.3.4) holds everywhere, i.e. condition (vi) is satisfied. Also, we see from the convergence $u_t^{k,m} \to u_t^k$ and $\Delta^m \to \Delta$ that conditions (i), (ii), and (v) are satisfied, and Fatou's lemma implies the validity of (iii). From the convergence $W^{k,m} \to W^k$, we have $0 \leq W^k \leq \Lambda$. Moreover, $W^k = 0$ on $\{x : \Lambda(x) < k\}$ because $W^{k,m} = 0$ on $\{x : \Lambda(x) < k\}$ for all $m \in \mathbb{N}$. This shows that condition (iv) is satisfied. We therefore conclude that $(\{(u_1^k, \cdots, u_T^k, W^k)\}_{k \in \mathbb{N}}, \Delta) \in \mathcal{L}(\Phi, \delta, N)$, and thus $\mathcal{L}(\Phi, \delta, N)$ is closed under the strong topology.

Step 4: We prove the desired compactness of $\mathcal{L}(\Phi, \delta, N)$. Observe that $\mathcal{L}(\Phi, \delta, N)$ is convex. Since a strongly closed convex set is also weakly closed, and the weak topology of a product space coincides with the product of the weak topologies, we conclude that $\mathcal{L}(\Phi, \delta, N)$ is closed under the product of the weak topologies in the spaces $L^1(\mu_t)$, $L^1(\nu)$, and $\mathcal{B}(\mathbb{R}^t_+)$. It is therefore weakly compact in view of Step 2. $\square$

Remark 3.3.2. *While the motivation of Lemma 3.3.2 is to construct some compactness for the space of semi-static strategies (u, Δ), we have to introduce the auxiliary random variable W^k in (3.3.2) to ensure the convexity of $\mathcal{L}(\Phi, \delta, N)$, needed in the last step of the proof.*

Proposition 3.3.4. *Consider $\{\Phi_i\}_{i\in\mathbb{N}}$ in $\mathcal{G}(\Omega)$ for which there exists $K > 0$ such that (3.3.1) is satisfied for all $i \in \mathbb{N}$. If $\Phi_i \uparrow \Phi$, then*

$$\sup_{i\in\mathbb{N}} D^N(\Phi_i) = D^N(\Phi), \quad \forall N \in \mathbb{N}.$$

Proof. Fix $N \in \mathbb{N}$. As $\Phi_i \uparrow \Phi$ clearly implies $\sup_{i\in\mathbb{N}} D^N(\Phi_i) \leq D^N(\Phi)$, we focus on proving the "$\geq$" relation. Assume $\sup_{i\in\mathbb{N}} D^N(\Phi) < \infty$, otherwise the proof would be trivial. Pick an arbitrary $\delta > \sup_{i\in\mathbb{N}} D^N(\Phi_i)$. By Lemma 3.3.2, $\{\mathcal{L}(\Phi_i, \delta, N)\}_{i\in\mathbb{N}}$ is a nonincreasing sequence of nonempty compact sets. We can therefore choose some $\big(\{(u_1^k, ..., u_T^k, W^k)\}_{k\in\mathbb{N}}, \Delta\big) \in \bigcap_{i\in\mathbb{N}} \mathcal{L}(\Phi_i, \delta, N_i)$. In view of conditions (i), (ii), and (iii) in Lemma 3.3.2, $u_t := \lim_{k\to\infty} \uparrow u_t^k \in L^1(\mu_t)$ is well-defined, and $u = (u_1, ..., u_T)$ satisfies

$$\mu(u) \leq \delta + (K + 2N)(1 + m(\mu_1) + \cdots + m(\mu_T)). \tag{3.3.5}$$

Moreover, condition (v) in Lemma 3.3.2 implies that for each k and i,

$$\oplus u^k \geq (\Phi_i + \Gamma) \wedge k + (\Delta \cdot x)_T - W^k.$$

Recall from condition (iv) in Lemma 3.3.2 that $W^k = 0$ on $\{x : \Lambda(x) < k\}$. This in particular implies $W^k(x) \to 0$ for all $x \in \Omega$ as $k \to \infty$. Therefore, by taking $k \to \infty$ in the previous inequality, we get $\oplus u \geq \Phi_i + \Gamma + (\Delta \cdot x)_T$. As $i \to \infty$, this yields

$$\oplus u \geq \Phi + \Gamma + (\Delta \cdot x)_T. \tag{3.3.6}$$

Now, define $\bar{u}_t := u_t - (K + 2N)(1/T + x_t)$ for all $t = 1, \cdots, T$. By (3.3.5) and (3.3.6),

$$\begin{aligned}\mu(\bar{u}) &= \mu(u) - (K + 2N)(1 + m(\mu_1) + \cdots + m(\mu_T)) \leq \delta, \\ \oplus\bar{u} &= \oplus u - \Gamma \geq \Phi + (\Delta \cdot x)_T.\end{aligned}$$

This readily implies $D^N(\Phi) \leq \delta$. With $\delta > \sup_{i\in\mathbb{N}} D^N(\Phi_i)$ arbitrarily chosen, we conclude $\sup_{i\in\mathbb{N}} D^N(\Phi_i) \geq D^N(\Phi)$. $\square$

3.3.3 Complete Duality

Theorem 3.3.1. *For any* $\Phi \in \mathrm{USA}(\Omega)$ *that satisfies* (3.2.4),

$$D^N(\Phi) = P^N(\Phi), \quad \forall N \in \mathbb{N}. \tag{3.3.7}$$

Moreover, there exists an optimizer $(u, \Delta) \in L^1(\mu) \times \mathcal{H}^N$ *for* $D^N(\Phi)$ *whenever* $D^N(\Phi) < \infty$.

Proof. Fix $N \in \mathbb{N}$. Define $\zeta^K(x) := K(1 + x_1 + ... + x_T)$, with $K > 0$ specified in (3.2.4). Consider the functionals $\bar{P}^N$ and $\bar{D}^N$ defined by

$$\bar{P}^N(\varphi) := P^N(-\zeta^K \vee (\varphi \wedge \zeta^K)) \quad \text{and} \quad \bar{D}^N(\varphi) := D^N(-\zeta^K \vee (\varphi \wedge \zeta^K)), \quad \text{for } \varphi \in \mathcal{G}(\Omega).$$

In view of Propositions 3.3.1 and 3.3.2 (resp. Propositions 3.3.3 and 3.3.4), $\bar{P}^N$ (resp. $\bar{D}^N$) is a Choquet capacity associated with $\mathrm{USC}(\Omega)$; recall Definition 3.3.1. Moreover, thanks to Proposition 3.2.2, $\bar{D}^N(\varphi) = \bar{P}^N(\varphi)$ for all $\varphi \in \mathrm{USC}(\Omega)$. We then conclude from Lemma 3.3.1 that $\bar{D}^N(\varphi) = \bar{P}^N(\varphi)$ for all $\varphi \in \mathrm{USA}(\Omega)$. That is to say, $D^N(\varphi) = P^N(\varphi)$ for all $\varphi \in \mathrm{USA}(\Omega)$ satisfying $|\varphi| \leq \zeta^K$, or (3.2.4).

It remains to prove the existence of an optimizer for $D^N(\Phi)$. If $D^N(\Phi) < \infty$, take a real sequence $\{\delta_i\}$ such that $\delta_i \downarrow D^N(\Phi)$. By Lemma 3.3.2, $\{\mathcal{L}(\Phi, \delta_i, N)\}_{i\in\mathbb{N}}$ is a nonincreasing sequence of nonempty compact sets. We can therefore choose some $\big(\{(u_1^k, ..., u_T^k, W^k)\}_{k\in\mathbb{N}}, \Delta\big) \in \bigcap_{i\in\mathbb{N}} \mathcal{L}(\Phi, \delta_i, N)$. In view of conditions (i), (ii), and (iii) in Lemma 3.3.2, $u_t := \lim_{k\to\infty} \uparrow u_t^k \in L^1(\mu_t)$ is well-defined, and $u = (u_1, ..., u_T)$ satisfies

$$\mu(u) \leq D^N(\Phi) + (K + 2N)(1 + m(\mu_1) + \cdots + m(\mu_T)). \tag{3.3.8}$$

Moreover, condition (v) in Lemma 3.3.2 implies that for each k and i,

$$\oplus u^k \geq (\Phi + \Gamma) \wedge k + (\Delta \cdot x)_T - W^k.$$

As shown in the proof of Proposition 3.3.4, $W^k(x) \to 0$ for all $x \in \Omega$ as $k \to \infty$. Thus, by taking $k \to \infty$ in the previous inequality, we get $\oplus u \geq \Phi + \Gamma + (\Delta \cdot x)_T$. Now, define $\bar{u}_t := u_t - (K + 2N)(1/T + x_t)$ for all $t = 1, \cdots, T$. Then, $\oplus\bar{u} = \oplus u - \Gamma \geq \Phi + (\Delta \cdot x)_T$. Moreover, by (3.3.8),

$$\mu(\bar{u}) = \mu(u) - (K + 2N)(1 + m(\mu_1) + \cdots + m(\mu_T)) \leq D^N(\Phi).$$

This implies that, $(\bar{u}, -\Delta) \in L^1(\mu) \times \mathcal{H}^N$ is an optimizer of $D^N(\Phi)$. $\square$

Remark 3.3.3. *When we view D and P, defined in* (3.1.5) *and* (3.1.6)*, as functionals, arguments similar to (and simpler than) those in Sections 3.3.1 and 3.3.2 yield the upward and downward continuity of P, as well as the downward continuity of D. However, the upward continuity of D is obscure. Without the portfolio bound $N > 0$, it is unclear how the space of semi-static strategies $(u, \Delta) \in L^1(\mu) \times \mathcal{H}$ can be made compact under any topology, so that the upward continuity does not follow from the arguments in Proposition 3.3.4.*

In fact, since $D(\Phi) \neq P(\Phi)$ for some Borel measurable Φ (as shown in [4, Example 3.1]), the upward continuity of D must not hold. Otherwise, we could apply Choquet's capacitability *theorem directly to the classical duality $D(\Phi) = P(\Phi)$ in Proposition 3.1.1, extending it from upper semicontinuous Φ to upper semi-analytic ones (which include Borel measurable ones).*

Remark 3.3.4. *Recall Example 3.2.1, where $0 = P(\Phi) < D(\Phi) = 1$. We will show that $P^N(\Phi) = D^N(\Phi)$ for all $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. Recall that $(u_1^*, u_2^*, \Delta_1^*) \equiv (1, 0, 0)$ is an optimizer of $D(\Phi)$. As $\Delta_1^* \in \mathcal{H}^N$, $(u_1^*, u_2^*, \Delta_1^*)$ is also an optimizer of $D^N(\Phi)$, and thus $D^N(\Phi) = D(\Phi) = 1$. On the other hand, consider $\{\mathbb{Q}_M\}_{M\in\mathbb{N}}$ in Π constructed in (3.2.7). By (3.2.8), $\mathbf{A}_2^N(\mathbb{Q}_M) = N\mathbf{A}_2^1(\mathbb{Q}_M) = \frac{N}{4M}$. It follows that*

$$P^N(\Phi) = \sup_{\mathbb{Q}\in\Pi} \left\{\mathbb{E}_{\mathbb{Q}}[\Phi] - \mathbf{A}_2^N(\mathbb{Q})\right\} \geq \lim_{M\to\infty} \left\{\mathbb{E}_{\mathbb{Q}_M}[\Phi] - \mathbf{A}_2^N(\mathbb{Q}_M)\right\} = 1.$$

As $\Phi \leq 1$ already implies $P^N(\Phi) \leq 1$, we conclude $P^N(\Phi) = 1 = D^N(\Phi)$.

Remark 3.3.5. *$P^N(\Phi)$ in general does not admit an optimizer, unless Φ is upper semi-continuous. To illustrate, in Example 3.2.1, suppose that there exists $\mathbb{Q}_* \in \Pi$ such that $\mathbb{E}_{\mathbb{Q}_*}[\Phi] - \mathbf{A}_2^N(\mathbb{Q}_*) = P^N(\Phi) = 1$ for some $N \in \mathbb{N}$. Then, $0 \leq \mathbf{A}_2^N(\mathbb{Q}_*) = \mathbb{E}_{\mathbb{Q}_*}[\Phi] - 1 \leq 0$, which yields $\mathbf{A}_2^N(\mathbb{Q}_*) = 0$. By Proposition 3.2.1, $\mathbb{Q}_*$ must belong to $\mathcal{M}$ and thus coincide with $\mathbb{P}_0$. This, however, entails $\mathbb{E}_{\mathbb{Q}_*}[\Phi] - \mathbf{A}_2^N(\mathbb{Q}_*) = 0$, a contradiction.*

3.4 Derivation of Theorem 3.2.1

This section is devoted to proving Theorem 3.2.1. First, we define $D^\infty(\Phi)$ as in (3.1.5), with $\Delta \in \mathcal{H}$ replaced by $\Delta \in \mathcal{H}^\infty$, where

$$\mathcal{H}^\infty := \{\Delta \in \mathcal{H} : \ \Delta_t \text{ is bound, } \forall\, t = 1, \cdots, T-1\}.$$

To connect the portfolio-constrained duality (3.3.7) to the desired (unconstrained) duality (3.2.5), it is natural to relax the constraint $N > 0$ by taking $N \to \infty$, leading to the following result. Recall $\widetilde{P}(\Phi)$ defined in (3.2.5).

Proposition 3.4.1. *For any* $\Phi \in \mathrm{USA}(\Omega)$ *that satisfies* (3.2.4), $D^\infty(\Phi) = \widetilde{P}(\Phi)$.

Proof. First, we show that $D^\infty(\Phi) \geq \widetilde{P}(\Phi)$. Fix $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ in Π such that $\mathbb{Q}_N \xrightarrow{\rho} \mathcal{M}$ (or equivalently, $\mathbf{A}_T^1(\mathbb{Q}_N) \to 0$). We can choose some nonnegative function h such that $h(N) \to \infty$ and $h(N)\mathbf{A}_T^1(\mathbb{Q}_N) \to 0$ (for instance, take $h(N) := 1/\sqrt{\mathbf{A}_T^1(\mathbb{Q}_N)}$). For each $N \in \mathbb{N}$, there exist $(u, \Delta) \in L^1(\mu) \times \mathcal{H}^{h(N)}$ with $\mu(u) < D^{h(N)}(\Phi) + 1/N$ such that $\oplus u + (\Delta \cdot S)_T \geq \Phi$. If follows that

$$D^{h(N)}(\Phi) + 1/N + \mathbf{A}_T^{h(N)}(\mathbb{Q}_N) \geq \mu(u) + \mathbb{E}_{\mathbb{Q}_N}[(\Delta \cdot S)_T] \geq \mathbb{E}_{\mathbb{Q}_N}[\Phi],$$

where the first inequality follows from the definition of $\mathbf{A}_T^{h(N)}(\mathbb{Q}_N)$ in (3.2.1). As $N \to \infty$ in the above inequality, since $\mathbf{A}_T^{h(N)}(\mathbb{Q}_N) = h(N)\mathbf{A}_T^1(\mathbb{Q}_N) \to 0$ and $D^{h(N)}(\Phi) \to D^\infty(\Phi)$ by definition, we get $D^\infty(\Phi) \geq \limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi]$. With $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ arbitrarily chosen, we obtain $D^\infty(\Phi) \geq \widetilde{P}(\Phi)$.

On the other hand, for any $N \in \mathbb{N}$, by the definition of $P^N(\Phi)$, we can take $\mathbb{Q}_N \in \Pi$ such that

$$P^N(\Phi) \geq \mathbb{E}_{\mathbb{Q}_N}[\Phi] - \mathbf{A}_T^N(\mathbb{Q}_N) > P^N(\Phi) - 1/N. \tag{3.4.1}$$

This, together with $\mathbf{A}_T^N(\mathbb{Q}_N) = N\mathbf{A}_T^1(\mathbb{Q}_N)$, shows that

$$\mathbf{A}_T^1(\mathbb{Q}_N) < \frac{\mathbb{E}_{\mathbb{Q}_N}[\Phi] - P^N(\Phi) + 1/N}{N} \leq \frac{C}{N}, \quad \forall N \in \mathbb{N}. \tag{3.4.2}$$

Here, the constant $C > 0$ can be chosen to be independent of N, thanks to (3.2.4) and (3.1.1). This in particular implies $\mathbf{A}_T^1(\mathbb{Q}_N) \to 0$. In view of (3.4.1), this yields

$$\lim_{N\to\infty} P^N(\Phi) = \lim_{N\to\infty} \{\mathbb{E}_{\mathbb{Q}_N}[\Phi] - \mathbf{A}_T^N(\mathbb{Q}_N)\} \leq \limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi] \leq \widetilde{P}(\Phi). \tag{3.4.3}$$

Finally, by taking $N \to \infty$ in the constrained duality (3.3.7) and using the above inequality, we obtain $D^\infty(\Phi) = \lim_{N\to\infty} P^N(\Phi) \leq \widetilde{P}(\Phi)$. $\square$

In view of Proposition 3.4.1, to obtain the desired duality (3.2.5), it remains to show $D^\infty(\Phi) = D(\Phi)$ for all $\Phi \in \mathcal{B}(\Omega)$ satisfying (3.2.4). That is, restricting to bounded trading strategies does not increase the cost of model-free superhedging. To this end, we need the following technical result.

Lemma 3.4.1. *Given $\Phi \in \mathcal{G}(\Omega)$ that satisfies (3.2.4), we define $\Phi_n \in \mathcal{G}(\Omega)$, for each $n \in \mathbb{N}$, by*

$$\Phi_n(x_1, ..., x_T) := \Phi(x_1, ..., x_T) 1_{\{x_1 \le n, ..., x_T \le n\}}(x_1, ..., x_T), \quad \forall x = (x_1, ..., x_T) \in \Omega. \tag{3.4.4}$$

For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ large enough such that

$$|\mathbb{E}_{\mathbb{Q}}[\Phi] - \mathbb{E}_{\mathbb{Q}}[\Phi_n]| < \varepsilon, \quad \forall \mathbb{Q} \in \Pi. \tag{3.4.5}$$

Proof. Fix $\varepsilon > 0$. Let $\delta := \frac{\varepsilon}{K(T+T^2)}$. Thanks to (3.1.1), we can take $n \in \mathbb{N}$ large enough such that

$$\mu_t((n, \infty)) < \delta \quad \text{and} \quad \int_{\{y > n\}} y d\mu_t(y) < \delta, \quad \forall t = 1, ..., T. \tag{3.4.6}$$

For simplicity, we will write $\mathcal{A} = \{x \in \Omega : x_1 \le n, ..., x_T \le n\}$. Observe that

$$\mathcal{A}^c \subseteq \bigcup_{t \in \{1,...,T\}} \{x \in \Omega : x_t > n\}. \tag{3.4.7}$$

Moreover, for each fixed $t = 1, ..., T$,

$$\mathcal{A}^c = \{x \in \Omega : x_t > n\} \cup \bigcup_{i \in \{1,...,T\} \setminus \{t\}} \{x \in \Omega : x_t \le n \text{ and } x_i > n\}. \tag{3.4.8}$$

Now, for any $\mathbb{Q} \in \Pi$, by (3.2.4),

$$|\mathbb{E}_{\mathbb{Q}}[\Phi] - \mathbb{E}_{\mathbb{Q}}[\Phi_n]| \le \mathbb{E}_{\mathbb{Q}}[|\Phi| 1_{\mathcal{A}^c}] \le K\bigg(\mathbb{E}_{\mathbb{Q}}[1_{\mathcal{A}^c}] + \sum_{t=1,...,T} \mathbb{E}_{\mathbb{Q}}[x_t 1_{\mathcal{A}^c}(x)]\bigg). \tag{3.4.9}$$

The first inequality above requires the linearity of outer expectations; recall from Section 3.1.1 that $\mathbb{E}_{\mathbb{Q}}[\cdot]$ denotes an outer expectation if the integrad need not be Borel measurable. While the linearity of outer expectations does not hold in general, it holds specifically here thanks to the definition of Φ_n. Indeed, by [21, Lemma 6.3], there exists $\Phi^* \in \mathcal{B}(\Omega)$, a minimal Borel measurable majorant of Φ, such that $\mathbb{E}_{\mathbb{Q}}[\Phi^*] = \mathbb{E}[\Phi]$ and $\mathbb{E}_{\mathbb{Q}}[\Phi^* 1_B] = \mathbb{E}_{\mathbb{Q}}[\Phi 1_B]$ for any Borel subset B of Ω. It follows that

$$\mathbb{E}_{\mathbb{Q}}\left[\Phi\right] - \mathbb{E}_{\mathbb{Q}}\left[\Phi_n\right] = \mathbb{E}_{\mathbb{Q}}\left[\Phi^*\right] - \mathbb{E}_{\mathbb{Q}}\left[\Phi^* 1_{\mathcal{A}}\right] = \mathbb{E}_{\mathbb{Q}}\left[\Phi^* 1_{\mathcal{A}^c}\right] = \mathbb{E}_{\mathbb{Q}}\left[\Phi 1_{\mathcal{A}^c}\right],$$

where the second equality follows from the linearity of standard expectations, as Φ^* and $\Phi^* 1_{\mathcal{A}}$ are both Borel measurable.

Thanks to (3.4.7),

$$\mathbb{E}_{\mathbb{Q}}[1_{\mathcal{A}^c}(x)] \leq \sum_{t=1,\ldots,T} \mathbb{E}_{\mathbb{Q}}[1_{\{x_t > n\}}(x)] = \sum_{t=1,\ldots,T} \mu_t((n,\infty)) < T\delta,$$

where the last inequality follows from (3.4.6). On the other hand, for any $t = 1, ..., T$, (3.4.8) implies

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[x_t 1_{\mathcal{A}^c}(x)] &= \mathbb{E}_{\mathbb{Q}}[x_t 1_{\{x_t > n\}}(x)] + \sum_{i \in \{1,\ldots,T\} \setminus \{t\}} \mathbb{E}_{\mathbb{Q}}[x_t 1_{\{x_t \leq n,\ x_i > n\}}(x)] \\
&\leq \mathbb{E}_{\mathbb{Q}}[x_t 1_{\{x_t > n\}}(x)] + \sum_{i \in \{1,\ldots,T\} \setminus \{t\}} \mathbb{E}_{\mathbb{Q}}[x_i 1_{\{x_i > n\}}(x)] \\
&= \sum_{i=1,\ldots,T} \mathbb{E}_{\mathbb{Q}}[x_i 1_{\{x_i > n\}}(x)] = \sum_{i=1,\ldots,T} \int_{\{y > n\}} y \, d\mu_i(y) < T\delta,
\end{aligned}$$

where the last inequality follows from (3.4.6). Hence, we conclude from (3.4.9) that $|\mathbb{E}_{\mathbb{Q}}\left[\Phi(x)\right] - \mathbb{E}_{\mathbb{Q}}\left[\Phi_n(x)\right]| \leq K(T + T^2)\delta = \varepsilon$, as desired. □

Corollary 3.4.1. *If $D^\infty(\Phi) = D(\Phi)$ for all bounded $\Phi \in \mathrm{USA}(\Omega)$, then the same equality holds for all $\Phi \in \mathrm{USA}(\Omega)$ satisfying* (3.2.4).

Proof. First, we show that $D^\infty(\Phi) = D(\Phi)$ for all nonnegative $\Phi \in \mathrm{USA}(\Omega)$ satisfying (3.2.4). Given $\Phi \in \mathrm{USA}(\Omega)$ that is nonnegative and satisfies (3.2.4), consider Φ_n, $n \in \mathbb{N}$, defined in (3.4.4). As a product of $\Phi \in \mathrm{USA}(\Omega)$ and a nonnegative Borel measurable function, Φ_n also belongs to $\mathrm{USA}(\Omega)$, thanks to [5, Lemma 7.30]. In view of the estimate (3.4.5) and the definition of $\widetilde{P}$ in (3.2.5), we deduce from Proposition 3.4.1 that $D^\infty(\Phi_n) = \widetilde{P}(\Phi_n) \to \widetilde{P}(\Phi) = D^\infty(\Phi)$. Now, note that every Φ_n is bounded, thanks to the fact that Φ satisfies (3.2.4). As the boundness of $\Phi_n \in \mathrm{USA}(\Omega)$ implies $D^\infty(\Phi_n) = D(\Phi_n)$ for all $n \in \mathbb{N}$, we have

$$D^\infty(\Phi) = \lim_{n\to\infty} D^\infty(\Phi_n) = \lim_{n\to\infty} D(\Phi_n) \le D(\Phi),$$

where the inequality stems from $\Phi_n \uparrow \Phi$, thanks to the fact that Φ is nonnegative. Since $D^\infty(\Phi) \ge D(\Phi)$ by definition, we conclude that $D^\infty(\Phi) = D(\Phi)$.

Now, take an arbitrary $\Phi \in \mathrm{USA}(\Omega)$ that satisfies (3.2.4) (which need not be nonnegative). Consider $v = (v_1, ..., v_T) \in L^1(\mu)$ defined by $v_t(y) := K(\frac{1}{T} + y)$ for $t = 1, ..., T$, where $K > 0$ is taken from (3.2.4). As Φ satisfies (3.2.4), $\Phi + \oplus v$ is nonnegative. Indeed, $(\Phi + \oplus v)(x) \ge -K(1 + x_1 + ... + x_T) + \sum_{t=1}^T K(\frac{1}{T} + x_t) = 0$, for all $x \in \Omega$. Moreover, $\Phi + \oplus v$ again satisfies (3.2.4), with a possibly larger $K > 0$. Hence, we have

$$D^\infty(\Phi + \oplus v) = D(\Phi + \oplus v). \tag{3.4.10}$$

Note that $\mathbb{E}_{\mathbb{Q}}[\Phi + \oplus v] = \mathbb{E}_{\mathbb{Q}}[\Phi] + \mu(v)$ for all $\mathbb{Q} \in \Pi$. This, together with Proposition 3.4.1, implies

$$D^\infty(\Phi + \oplus v) = \widetilde{P}(\Phi + \oplus v) = \widetilde{P}(\Phi) + \mu(v) = D^\infty(\Phi) + \mu(v). \tag{3.4.11}$$

On the other hand, by definition

$$
\begin{aligned}
D(\Phi + \oplus v) &= \inf\{\mu(u) : u \in L^1(\mu) \text{ satisfying } \exists \Delta \in \mathcal{H} \text{ s.t. } \oplus u + (\Delta \cdot S)_T \geq \Phi + \oplus v \text{ on } \Omega\} \\
&= \inf\{\mu(u) : u \in L^1(\mu) \text{ satisfying } \exists \Delta \in \mathcal{H} \text{ s.t. } \oplus (u - v) + (\Delta \cdot S)_T \geq \Phi \text{ on } \Omega\} \\
&= \inf\{\mu(\tilde{u}) + \mu(v) : \tilde{u} \in L^1(\mu) \text{ satisfying } \exists \Delta \in \mathcal{H} \text{ s.t. } \oplus \tilde{u} + (\Delta \cdot S)_T \geq \Phi \text{ on } \Omega\} \\
&= D(\Phi) + \mu(v). \qquad (3.4.12)
\end{aligned}
$$

On the strength of (3.4.11) and (3.4.12), (3.4.10) yields $D^\infty(\Phi) = D(\Phi)$. $\square$

Now, we are ready to establish $D^\infty(\Phi) = D(\Phi)$ for all upper semi-analytic Φ satisfying (3.2.4).

Proposition 3.4.2. *For any* $\Phi \in \mathrm{USA}(\Omega)$ *that satisfies* (3.2.4)*,* $D^\infty(\Phi) = D(\Phi)$.

Proof. First, by Corollary 3.4.1, we can assume without loss of generality that $\Phi \in \mathrm{USA}(\Omega)$ is bounded. We take $C > 0$ such that $|\Phi| \leq C$ on Ω.

As $D^\infty(\Phi) \geq D(\Phi)$ by definition, we focus on proving the opposite inequality. Fix $\delta > 0$. There exist $u = (u_1, ..., u_T) \in L^1(\mu)$ and $\Delta \in \mathcal{H}$ such that

$$\mu(u) < D(\Phi) + \delta/2 \quad \text{and} \quad \oplus u(x) + (\Delta \cdot x)_T \geq \Phi(x) \quad \forall x \in \Omega. \qquad (3.4.13)$$

Step 1: We replace $u \in L^1(\mu)$ by nonnegative functions. By the Vitali-Carathéodory theorem, there exists $v = (v_1, ..., v_T) \in L^1(\mu)$, with $u_t \leq v_t$ and v_t bounded from below for all $t = 1, ..., T$, such that $\mu(u) \leq \mu(v) \leq \mu(u) + \delta/2$. Take $\ell > 0$ large enough such that $v_t \geq -\ell$ for all $t = 1, ..., T$. By setting $\bar{v}_t := v_t + \ell \geq 0$, we deduce from (3.4.13) that

$$\mu(v) < D(\Phi) + \delta \quad \text{and} \quad \oplus \bar{v}(x) + (\Delta \cdot x)_T \geq \Phi(x) + T\ell \quad \forall x \in \Omega. \qquad (3.4.14)$$

Step 2: We construct a bounded trading strategy $\bar{\Delta} \in \mathcal{H}^\infty$ and replace (3.4.14) by a superhedging relation involving $\bar{\Delta}$. Fix arbitrary $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{T-1} > 0$ that are sufficiently small. As $\bar{v}_1$ is μ_1-integrable, by [33, Problem 14, p.63], there exists $M_1 \in \mathcal{B}(\mathbb{R}_+)$ such that $\mu_1(\mathbb{R}_+ \setminus M_1) < \varepsilon_1$ and $\bar{v}_1$ is bounded on M_1. We can assume without loss of generality that M_1 contains $\{0\}$. Indeed, if $\mu_1(\{0\}) = 0$, adding $\{0\}$ to M_1 does not change the above statement; if $\mu_1(\{0\}) > 0$, then M_1 has to contain $\{0\}$ as long as $\varepsilon_1 < \mu_1(\{0\})$. For any $m_1 > 1$, define

$$\widetilde{M}_1 := M_1 \cap (\{0\} \cup (1/m_1, m_1)).$$

Note that $\mu_1(\widetilde{M}_1) \uparrow \mu_1(M_1)$ as $m_1 \to \infty$. Now, we claim that

$$\Delta_1 \text{ is bounded on } \widetilde{M}_1, \quad \forall m_1 > 1.$$

By contradiction, suppose that there exist $\{x_1^n\}_{n\in\mathbb{N}}$ in $\widetilde{M}_1$ such that $\Delta_1(x_1^n) \to \infty$ or $-\infty$. By taking $x_1 = x_1^n$ and $x_2 = x_3 = ... = x_T \in \mathbb{R}_+$ in the second part of (3.4.15) and using the fact $|\Phi| \le C$, we get

$$\bar{v}_1(x_1^n) + \bar{v}_2(x_2) + \ldots + \bar{v}_T(x_2) + \Delta_1(x_1^n)(x_2 - x_1^n) \ge -C + T\ell. \tag{3.4.15}$$

For the case $\Delta_1(x_1^n) \to \infty$ (resp. $\Delta_1(x_1^n) \to -\infty$), we take $x_2 = \frac{1}{2m_1}$ (resp. $x_2 = m_1 + 1$) in (3.4.15). As $n \to \infty$, by the boundedness of $\bar{v}_1$ on $\widetilde{M}_1$, the left hand side of (3.4.15) tends to $-\infty$, a contradiction. Similarly to the above, by [33, Problem 14, p.63], there exists $M_2 \in \mathcal{B}(\mathbb{R}_+)$, containing $\{0\}$, such that $\mu_2(\mathbb{R}_+ \setminus M_2) < \varepsilon_2$ and $\bar{v}_2$ is bounded on M_2. For any $m_2 > 1$, define

$$\widetilde{M}_2 := M_2 \cap (\{0\} \cup (1/m_2, m_2)),$$

and note that $\mu_2(\widetilde{M}_2) \uparrow \mu_2(M_2)$ as $m_2 \to \infty$. We claim that

$$\Delta_2 \text{ is bounded on } \widetilde{M}_1 \times \widetilde{M}_2, \quad \forall m_1, m_2 > 1.$$

By contradiction, suppose that there exist $\{(x_1^n, x_2^n)\}_{n\in\mathbb{N}}$ in $\widetilde{M}_1 \times \widetilde{M}_2$ such that $\Delta_2(x_1^n, x_2^n) \to$

∞ or $-\infty$. By taking $(x_1, x_2) = (x_1^n, x_2^n)$ and $x_3 = x_4 = ... = x_T \in \mathbb{R}_+$ in the second part of (3.4.14) and using the fact $|\Phi| \leq C$, we get

$$\bar{v}_1(x_1^n) + \bar{v}_2(x_2^n) + \bar{v}_3(x_3) + ... + \bar{v}_T(x_3) + \Delta_1(x_1^n)(x_2^n - x_1^n) + \Delta_2(x_1^n, x_2^n)(x_3 - x_2^n) \geq -C + T\ell. \tag{3.4.16}$$

For the case $\Delta_2(x_1^n, x_2^n) \to \infty$ (resp. $\Delta_2(x_1^n, x_2^n) \to -\infty$), we take $x_3 = \frac{1}{2m_2}$ (resp. $x_3 = m_2 + 1$) in (3.4.16). As $n \to \infty$, by the boundedness of $\bar{v}_1$ (on $\widetilde{M}_1$), $\bar{v}_2$ (on $\widetilde{M}_2$), and Δ_1 (on $\widetilde{M}_1$), the left hand side of (3.4.16) tends to $-\infty$, a contradiction. By repeating the same argument for all $t = 3, 4, ..., T-1$, we obtain $\{M_t\}_{t=1}^{T-1}$ in $\mathcal{B}(\mathbb{R}_+)$ such that for each $t = 1, ..., T-1$,

(i) $\mu_t(M_t^c) = \mu_t(\mathbb{R}_+ \setminus M_t) < \varepsilon_t$;

(ii) $\mu_t(\widetilde{M}_t) \uparrow \mu_t(M_t)$ as $m_t \to \infty$, where $\widetilde{M}_t := M_t \cap (\{0\} \cup (1/m_t, m_t))$ for $m_t > 1$;

(iii) $\Delta_t(x_1, x_2, \ldots, x_t)$ is bounded on $\widetilde{M}_1 \times \widetilde{M}_2 \times ... \times \widetilde{M}_t$.

We also consider

$$a_t := \sup_{\widetilde{M}_1 \times \widetilde{M}_2 \times ... \times \widetilde{M}_t} |\Delta_t(x_1, x_2, \ldots, x_t)| < \infty, \tag{3.4.17}$$

for all $t = 1, ..., T-1$, which will be used in Step 3 of the proof.

Now, let us define the bounded strategy $\bar{\Delta} = \{\bar{\Delta}_t\}_{t=1}^{T-1} \in \mathcal{H}^\infty$ by

$$\bar{\Delta}_t(x_1, x_2, \ldots, x_t) := \Delta_t(x_1, x_2, \ldots, x_t) 1_{\widetilde{M}_1 \times ... \times \widetilde{M}_t}(x_1, x_2, ..., x_t), \quad \forall t = 1, ..., T-1.$$

Also, for any $x = (x_1, ..., x_T) \in \Omega$, we introduce

$$\bar{\Phi}(x) := (\Phi(x) + T\ell) 1_{\widetilde{M}_1 \times ... \times \widetilde{M}_{T-1}}(x) + \sum_{t=2}^{T-1} (\Delta \cdot S)_t 1_{\widetilde{M}_1 \times ... \times \widetilde{M}_{t-1} \times \widetilde{M}_t^c}(x_1, x_2, ..., x_t). \tag{3.4.18}$$

We claim that

$$\oplus \bar{v}(x) + (\bar{\Delta} \cdot x)_T \geq \bar{\Phi}(x), \quad \forall x \in \Omega. \tag{3.4.19}$$

Indeed, for any $x \in \Omega$ such that $x_t \in \widetilde{M}_t$ for all $t = 1, ..., T-1$, the above inequality simply reduces to the second part of (3.4.14). For any $x \in \Omega$ such that $x_t \notin \widetilde{M}_t$ for some $t = 1, ..., T-1$, consider $t^* := \inf\{t \in \{1, 2, ..., T-1\} : x_t \notin \widetilde{M}_t\}$. Observe that

$$\begin{aligned}
\oplus \bar{v}(x) + (\bar{\Delta} \cdot x)_T &= \oplus \bar{v}(x) + \Delta_1(x_1)(x_2 - x_1) + \cdots + \Delta_{t^*-1}(x_1, x_2, \ldots, x_{t^*-1})(x_{t^*} - x_{t^*-1}) \\
&= \oplus \bar{v}(x) + (\Delta \cdot x)_{t^*} \geq (\Delta \cdot x)_{t^*} = \bar{\Phi}(x),
\end{aligned}$$

where the inequality follows from $\bar{v}_t \geq 0$, and the last equality is deduced from the definitions of $\bar{\Phi}$ and t^*. We therefore conclude that (3.4.19) holds.

Step 3: We show that for any $\varepsilon > 0$, $\{\widetilde{M}_t\}_{t=1}^{T-1}$ can be constructed appropriately so that $\mathbb{E}_{\mathbb{Q}}[\bar{\Phi}] \geq \mathbb{E}_{\mathbb{Q}}[\Phi + T\ell] - \varepsilon$ for all $\mathbb{Q} \in \Pi$. For any $\mathbb{Q} \in \Pi$, by (3.4.18),

$$\mathbb{E}_{\mathbb{Q}}[\bar{\Phi}] = \mathbb{E}_{\mathbb{Q}}\left[(\Phi + T\ell)1_{\widetilde{M}_1 \times \ldots \times \widetilde{M}_{T-1}}\right] + \sum_{t=2}^{T-1} \mathbb{E}_{\mathbb{Q}}\left[(\Delta \cdot S)_t 1_{\widetilde{M}_1 \times \ldots \times \widetilde{M}_{t-1} \times \widetilde{M}_t^c}\right]. \tag{3.4.20}$$

Note that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\Phi + T\ell)\left(1 - 1_{\widetilde{M}_1 \times \ldots \times \widetilde{M}_{T-1}}\right)\right] \\
&\quad \leq (C + T\ell)\mathbb{E}_{\mathbb{Q}}\left[1_{\widetilde{M}_1^c}(x_1) + 1_{\widetilde{M}_2^c}(x_2) + \ldots + 1_{\widetilde{M}_{T-1}^c}(x_{T-1})\right] \\
&\quad = (C + T\ell)\left(\mu_1(\widetilde{M}_1^c) + \mu_2(\widetilde{M}_2^c) + \cdots + \mu_{T-1}(\widetilde{M}_{T-1}^c)\right).
\end{aligned} \tag{3.4.21}$$

On the other hand, for any $t = 2, ..., T-1$,

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[(\Delta \cdot S)_t 1_{\widetilde{M}_1 \times ... \times \widetilde{M}_{t-1} \times \widetilde{M}_t^c}(x_1, x_2, ..., x_t)] \\
&\quad = \mathbb{E}_{\mathbb{Q}}\left[\{\Delta_1(x_2 - x_1) + ... + \Delta_{t-1}(x_t - x_{t-1})\} 1_{\widetilde{M}_1}(x_1) ... 1_{\widetilde{M}_{t-1}}(x_{t-1}) 1_{\widetilde{M}_i^c}(x_t)\right] \\
&\quad \geq -\mathbb{E}_{\mathbb{Q}}\left[\{|\Delta_1|(x_2 + x_1) + \cdots + |\Delta_{t-1}|(x_t + x_{t-1})\} 1_{\widetilde{M}_1}(x_1) ... 1_{\widetilde{M}_{t-1}}(x_{t-1}) 1_{\widetilde{M}_i^c}(x_t)\right] \\
&\quad \geq -\mathbb{E}_{\mathbb{Q}}\left[\{a_1(m_2 + m_1) + ... + a_{t-1}(x_t + m_{t-1})\} 1_{\widetilde{M}_1}(x_1) ... 1_{\widetilde{M}_{t-1}}(x_{t-1}) 1_{\widetilde{M}_t^c}(x_t)\right] \\
&\quad \geq -\mathbb{E}_{\mathbb{Q}}\left[\{a_1(m_2 + m_1) + ... + a_{t-1}(x_t + m_{t-1})\} 1_{\widetilde{M}_t^c}(x_t)\right] \\
&\quad = -\left[a_1(m_2 + m_1) + ... + a_{t-2}(m_{t-1} + m_{t-2}) + a_{t-1} m_{t-1}\right] \mu_i(\widetilde{M}_t^c) - a_{t-1} \int_{\widetilde{M}_t^c} y d\mu_t(y),
\end{aligned}
$$

where the first inequality follows from $x_i \geq 0$ and the second inequality is due to $y < m_i$ for all $y \in \widetilde{M}_i$ and $|\Delta_i| \leq a_i$ on $\widetilde{M}_i$, for all $i = 1, ..., T-1$. We then deduce from (3.4.20), (3.4.21), and the previous inequality that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\bar{\Phi}] \geq \mathbb{E}_{\mathbb{Q}}[\Phi + T\ell] &- (C + T\ell)\left(\mu_1(\widetilde{M}_1^c) + \mu_2(\widetilde{M}_2^c) + \mu_{T-1}(\widetilde{M}_{T-1}^c)\right) \\
&- \sum_{t=2}^{T-1} \Bigg(\left[a_1(m_2 + m_1) + ... + a_{t-2}(m_{t-1} + m_{t-2}) + a_{t-1} m_{t-1}\right] \mu_t(\widetilde{M}_t^c) \\
&\qquad + a_{t-1} \int_{\widetilde{M}_t^c} y d\mu_t(y) \Bigg). \qquad (3.4.22)
\end{aligned}
$$

The above inequality particularly requires the linearity of outer expectations, which holds here for $\Phi + T\ell$ and $(\Phi + T\ell) 1_{\widetilde{M}_1 \times ... \times \widetilde{M}_{T-1}}$. This can be proved as in the discussion below (3.4.9). We will show that every term on the right hand side of (3.4.22), except $\mathbb{E}_{\mathbb{Q}}[\Phi + T\ell]$, can be made arbitrarily small, by choosing m_t and a_t appropriately for all $t = 1, ..., T-1$.

Fix $\varepsilon > 0$, and define $\eta := \frac{\varepsilon}{(C+T\ell)(T-1)+(T-2)} > 0$. Taking $\varepsilon_1 = \eta$ in Step 2 gives $\mu_1(M_1^c) < \eta$. Since $\mu_1(\widetilde{M}_1) \uparrow \mu_1(M_1)$ as $m_1 \to \infty$, we can pick $m_1 > 1$ large enough such that $\mu_1(\widetilde{M}_1^c) < \eta$. With m_1 chosen, $a_1 \geq 0$ in (3.4.17) is then determined. Given the fixed m_1 and a_1, we can take $\varepsilon_2 \in (0, \varepsilon_1)$ small enough such that $a_1 m_1 \varepsilon_2 + a_1 \int_A y d\mu_2(y) < \eta$ for all $A \in \mathcal{B}(\mathbb{R}_+)$ with $\mu_2(A) < \varepsilon_2$. Using this $\varepsilon_2 > 0$ in Step 2 gives $\mu_2(M_2^c) < \varepsilon_2$. Since

$\mu_2(\widetilde{M}_2) \uparrow \mu_2(M_2)$ as $m_2 \to \infty$, we can pick $m_2 > 1$ large enough such that the first term in the summation of (3.4.22) is less than η, i.e.

$$a_1 m_1 \mu_2(\widetilde{M}_2^c) + a_1 \int_{\widetilde{M}_2^c} y d\mu_2(y) < \eta.$$

With m_1, m_2 chosen, $a_2 \geq 0$ in (3.4.17) is then determined. Given the fixed m_t and a_t for $t = 1, 2$, we can take $\varepsilon_3 \in (0, \varepsilon_2)$ small enough such that $[a_1(m_2 + m_1) + a_2 m_2]\,\varepsilon_3 + a_2 \int_A y d\mu_3(y) < \eta$ for all $A \in \mathcal{B}(\mathbb{R}_+)$ with $\mu_3(A) < \varepsilon_3$. Using this $\varepsilon_3 > 0$ in Step 2 gives $\mu_3(M_3^c) < \varepsilon_3$. Since $\mu_3(\widetilde{M}_3) \uparrow \mu_3(M_3)$ as $m_3 \to \infty$, we can pick $m_3 > 1$ large enough such that the second term in the summation of (3.4.22) is less than η, i.e.

$$[a_1(m_2 + m_1) + a_2 m_2]\,\mu_3(\widetilde{M}_3^c) + a_2 \int_{\widetilde{M}_3^c} y d\mu_3(y) < \eta.$$

By repeating the same argument for all $t = 4, ..., T-1$, we have $\mu_t(\widetilde{M}_t^c)$, $t = 1, ..., T-1$, and every term in summation of (3.4.22) less than η. We then conclude from (3.4.22) that

$$\mathbb{E}_{\mathbb{Q}}[\bar{\Phi}] \geq \mathbb{E}_{\mathbb{Q}}[\Phi + T\ell] - ((C + T\ell)(T-1) + (T-2))\eta = \mathbb{E}_{\mathbb{Q}}[\Phi + T\ell] - \varepsilon. \tag{3.4.23}$$

Step 4: We establish $D(\Phi) \geq D^\infty(\Phi)$. For any $\varepsilon > 0$, as (3.4.23) holds for all $\mathbb{Q} \in \Pi$, the definition of $\widetilde{P}(\Phi)$ in (3.2.5) implies $\widetilde{P}(\bar{\Phi}) \geq \widetilde{P}(\Phi + T\ell) - \varepsilon$. By Proposition 3.4.1, this in turn implies $D^\infty(\bar{\Phi}) \geq D^\infty(\Phi + T\ell) - \varepsilon$. Now, observe that

$$D(\Phi) + \delta + T\ell \geq \mu(v) + T\ell = \mu(\bar{v}) \geq D^\infty(\bar{\Phi}) \geq D^\infty(\Phi + T\ell) - \varepsilon = D^\infty(\Phi) + T\ell - \varepsilon,$$

where the first inequality follows from the first part of (3.4.14), the second inequality is due to (3.4.19), and the last equality is a direct consequence of Proposition 3.4.1. As $\delta, \varepsilon > 0$ are arbitrarily chosen, we conclude $D(\Phi) \geq D^\infty(\Phi)$. $\square$

Thanks to Propositions 3.4.1 and 3.4.2, the proof of Theorem 3.2.1 is complete.

Chapter 4

Conclusion

In this thesis, we considered two problems in Financial Mathematics and Economics. In our first problem, we studied the stochastic Ramsey problem with the Cobb-Douglass production function. We first reduce the dimension of the problem, then our problem turns into the stochastic Ramsey problem. In contrast to prior studies, we have no boundedness assumptions on consumption processes. This assumption make our problem very challenging. In particular, no classical result could apply to the HJB equation. To resolve this issue, we define a related problem and then by the help of the related problem, we were able to show there exists a unique smooth solution to the HJB equation. Then, we could find a candidate for an optimal consumption $\hat{c}$. For this optimal candidate, we showed there exists a unique positive strong solution to the nonstandard stochastic differential equation, which no classical techniques could apply to. Hence, by using the optimal stochastic control method, we proved the candidate is indeed an optimal control.

In the second problem, we studied Generalized Duality for Model-Free Superhedging given Marginals. It is known for some upper semi-analytic cost functions Φ, the duality $D(\Phi) = P(\Phi)$ fails. We note $D(\Phi)$ is the minimum cost for model-free superhedging. Therefore, in contrast to prior studies that they modify the definition of $D(\Phi)$ to restore the duality which will no longer comply with the superhedging context, we revise the definition of $P(\Phi)$, and keep the definition of $D(\Phi)$ as it is to get the duality. Indeed, we consider sequences of measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ that approaches $\mathcal{M}$, and take the limiting expected values of Φ under these measures i.e. $\limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\Phi]$ to characterize $D(\Phi)$ i.e.

$$D(\Phi) = \widetilde{P}(\Phi) := \sup\left\{\limsup_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}[\Phi] : \mathbb{Q}_N \overset{\rho}{\to} \mathcal{M}\right\}.$$

Bibliography

[1] B. Acciaio, M. Beiglböck, F. Penkner, and W. Schachermayer, *A model-free version of the fundamental theorem of asset pricing and the super-replication theorem*, Math. Finance, 26 (2016), pp. 233–251.

[2] E. Bayraktar and Y.-J. Huang, *On the multidimensional controller-and-stopper games*, SIAM Journal on Control and Optimization, 51 (2013), pp. 1263?–1297.

[3] M. Beiglböck, P. Henry-Labordère, and F. Penkner, *Model-independent bounds for option prices—a mass transport approach*, Finance Stoch., 17 (2013), pp. 477–501.

[4] M. Beiglböck, M. Nutz, and N. Touzi, *Complete duality for martingale optimal transport on the line*, Ann. Probab., 45 (2017), pp. 3038–3074.

[5] D. P. Bertsekas and S. E. Shreve, *Stochastic optimal control*, vol. 139 of Mathematics in Science and Engineering, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. The discrete time case.

[6] H. Brown, D. Hobson, and L. C. G. Rogers, *Robust hedging of barrier options*, Math. Finance, 11 (2001), pp. 285–314.

[7] X. Chen, G. Deelstra, J. Dhaene, and M. Vanmaele, *Static super-replicating strategies for a class of exotic options*, Insurance Math. Econom., 42 (2008), pp. 1067–1085.

[8] G. Choquet, *Forme abstraite du téorème de capacitabilité*, Ann. Inst. Fourier. Grenoble, 9 (1959), pp. 83–89.

[9] A. M. G. Cox, Z. Hou, and J. Obł ój, *Robust pricing and hedging under trading restrictions and the emergence of local martingale models*, Finance Stoch., 20 (2016), pp. 669–704.

[10] A. M. G. Cox and J. Obłój, *Robust pricing and hedging of double no-touch options*, Finance Stoch., 15 (2011), pp. 573–605.

[11] Y. Dolinsky and H. M. Soner, *Robust hedging with proportional transaction costs*, Finance Stoch., 18 (2014), pp. 327–347.

[12] A. Fahim and Y.-J. Huang, *Model-independent superhedging under portfolio constraints*, Finance Stoch., 20 (2016), pp. 51–81.

[13] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, vol. 25 of Stochastic Modelling and Applied Probability, Springer, New York, second ed., 2006.

[14] F. Hirsch and B. Roynette, *A new proof of Kellerer's theorem*, ESAIM: Probability and Statistics, to appear, (2012).

[15] D. Hobson, *Robust hedging of the lookback option*, Finance & Stochastics, 2 (1998), pp. 329šC–347.

[16] D. Hobson, P. Laurence, and T.-H. Wang, *Static-arbitrage upper bounds for the prices of basket options*, Quant. Finance, 5 (2005), pp. 329–342.

[17] Y.-J. Huang and S. Khalili, *Optimal consumption in the stochastic Ramsey problem without boundedness constraints*, SIAM J. Control Optim., 57 (2019), pp. 783–809.

[18] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, vol. 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1991.

[19] S. Karlin and H. M. Taylor, *A second course in stochastic processes*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[20] H. G. Kellerer, *Duality theorems for marginal problems*, Z. Wahrsch. Verw. Gebiete, 67 (1984), pp. 399–432.

[21] M. R. Kosorok, *Introduction to empirical processes and semiparametric inference*, Springer Series in Statistics, Springer, New York, 2008.

[22] N. V. Krylov, *Controlled diffusion processes*, vol. 14 of Applications of Mathematics, Springer-Verlag, New York, 1980. Translated from the Russian by A. B. Aries.

[23] P. Laurence and T.-H. Wang, *Sharp upper and lower bounds for basket options*, Applied Mathematical Finance, 12 (2005), pp. 253–282.

[24] C. Liu, *Optimal consumption of the stochastic Ramsey problem for non-Lipschitz diffusion*, J. Inequal. Appl., (2014), pp. 2014:391, 15.

[25] R. Merton, *An asymptotic theory of growth under uncertainty*, Review of Economic Studies, 42 (1975), pp. 375–393.

[26] H. Morimoto, *Optimal consumption models in economic growth*, J. Math. Anal. Appl., 337 (2008), pp. 480–492.

[27] ——, *Stochastic control and mathematical modeling*, vol. 131 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2010. Applications in economics.

[28] H. Morimoto and X. Y. Zhou, *Optimal consumption in a growth model with the Cobb-Douglas production function*, SIAM J. Control Optim., 47 (2008), pp. 2991–3006.

[29] S. Nakao, *On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations*, Osaka J. Math., 9 (1972), pp. 513–518.

[30] M. Nutz, F. Stebegg, and X. Tan, *Multiperiod martingale transport*, Stochastic Processes and their Applications, (2019). Forthcoming, available at https://arxiv.org/abs/1703.10588.

[31] H. Pham, *Continuous-time stochastic control and optimization with financial applications*, vol. 61 of Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2009.

[32] F. P. Ramsey, *A mathematical theory of saving*, The Economic Journal, 38 (1928), pp. 543–559.

[33] H. L. Royden, *Real analysis*, Macmillan Publishing Company, New York, third ed., 1988.

[34] V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist., 36 (1965), pp. 423–439.

[35] C. Villani, *Optimal transport, old and new*, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009.

[36] T. Yamada, *On a comparison theorem for solutions of stochastic differential equations and its applications*, J. Math. Kyoto Univ., 13 (1973), pp. 497–512.

Appendices

Appendix A

A.1 Derivation of Proposition 2.4.1

In this appendix, we will establish Proposition 2.4.1 by generalizing arguments in [28] to infinite horizon. As mentioned in Section 2.4, [28] studies a similar problem to V_L in (2.4.1), yet under finite horizon and with the specific bound $L = 1$. As we will see, many arguments in [28] can be modified without much difficulty to infinite horizon. A distinctive exception is the derivation of the dynamic programming principle for V_L; see Lemma A.1.2 below for details.

Lemma A.1.1. *(i) For any $L > 0$, V_L is concave on $(0, \infty)$.*

(ii) There exists $\varphi_0 > 0$ such that $V_L(x) \leq x + \varphi_0$ for all $x > 0$ and $L > 0$.

Proof. (i) This follows from the same argument in [28, Theorem 5.1].

(ii) We will prove this result by modifying the argument in the first part of [28, Lemma 3.2]. Define $\varphi(x) := x + \varphi_0$ with $\varphi_0 > 0$ to be determined later. Fix $L > 0$. For any $c \in \mathcal{C}_L$, $x > 0$, and $T > 0$, Itô's formula implies

$$0 \leq \mathbb{E}[e^{-\beta T}\varphi(X_T^x)] = \varphi(x) + \mathbb{E}\left[\int_0^T e^{-\beta s}(-\beta\varphi(X_s^x) + (X_s^x)^\alpha - \mu X_s^x - c_s X_s^x)ds\right]. \quad \text{(A.1.1)}$$

The term $-\mathbb{E}[\int_0^T e^{-\beta s}\sigma X_s \; dW_s]$ vanishes in the above inequality as $\int_0^{\cdot} e^{-\beta s}\sigma X_s dW_s$ is a martingale, thanks to the second part of (2.3.4). By (2.2.6) and $\mu > 0$, we have $\sup_{y\geq 0}\{U(y)-$

$y\} < \infty$ and $A := \sup_{x\geq 0}\{x^\alpha - \mu x\} < \infty$. We can therefore take $\varphi_0 > 0$ large enough such that

$$-\beta\varphi(x) + (x^\alpha - \mu x) + \sup_{y\geq 0}\{U(y) - y\} \leq -\beta\varphi_0 + A \ + \sup_{y\geq 0}\{U(y) - y\} < 0, \quad x \geq 0. \ \text{(A.1.2)}$$

This, together with (A.1.1), yields

$$0 \leq \mathbb{E}[e^{-\beta T}\varphi(X_T^x)] \leq \varphi(x) - \mathbb{E}\bigg[\int_0^T e^{-\beta s}U(c_s X_s^x)ds\bigg],$$

Hence, by using Fatou's lemma as $T \to \infty$ and then taking supremum over $c \in \mathcal{C}_L$, we get the desired result $V_L(x) \leq \varphi(x)$. Finally, note that our choice of $\varphi_0 > 0$ can be made independent of both $L > 0$ and $x > 0$. Indeed, the right hand side of (A.1.2), which involves φ_0, does not depend on either L or x. □

Next, we derive the dynamic programming principle for V_L, to show that it is a viscosity solution. As explained in detail under (A.1.5), arguments in [28] only lead us to a *weak* dynamic programming principle. Additional probabilistic arguments are invoked to upgrade this weak principle.

Lemma A.1.2. *For any $L > 0$, V_L is a continuous viscosity solution to* (2.4.3).

Proof. Fix $L > 0$. The continuity of V_L on $(0, \infty)$ is a direct consequence of Lemma A.1.1 (i). In view of [13, Chapter V] and [31, Chapter 4], to prove the viscosity solution property, it suffices to show the following dynamic programming principle: for any $x > 0$,

$$V_L(x) = \sup_{c\in\mathcal{C}_L} \mathbb{E}\left[\int_0^\tau e^{-\beta t}U(c_t X_t^x)dt + e^{-\beta\tau}V_L(X_\tau^x)\right], \quad \forall \tau \in \mathcal{T},$$

where $\mathcal{T}$ denotes the set of all stopping times. The "$\leq$" relation is straightforward to derive.

Indeed, given $c \in \mathcal{C}_L$, we have, for any $\tau \in \mathcal{T}$, that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t} U(c_t X_t^x) dt\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau}\mathbb{E}\left[\int_\tau^\infty e^{-\beta(t-\tau)} U(c_t X_t^x) dt \;\middle|\; \mathcal{F}_\tau\right]\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau}\mathbb{E}\left[\int_{\tau(\omega)}^\infty e^{-\beta(t-\tau(\omega))} U(c^{\tau,\omega}_{t-\tau(\omega)} X^{X^x_\tau(\omega)}_{t-\tau(\omega)}) dt\right]\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau}\mathbb{E}\left[\int_0^\infty e^{-\beta t} U(c^{\tau,\omega}_t X^{X^x_\tau(\omega)}_t) dt\right]\right] \\
&\le \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau} V_L(X_\tau^x)\right].
\end{aligned}
\tag{A.1.3}
$$

Here, the third line follows from [2, Proposition A.1], with $c^{\tau,\omega} \in \mathcal{C}_L$ defined by $c^{\tau,\omega}_s(\bar{\omega}) := c_{\tau(\omega)+s}(\omega \otimes_{\tau(\omega)} \bar{\omega})$, $s \ge 0$, for each fixed $\omega \in \Omega$; recall (2.2.1). The last line, on the other hand, follows from the definition of V_L. Now, taking supremum over $c \in \mathcal{C}_L$ gives the desired "$\le$" relation.

The rest of the proof focuses on deriving the converse inequality

$$
V_L(x) \ge \sup_{c\in\mathcal{C}_L} \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta\tau} V_L(X_\tau^x)\right], \quad \forall \tau \in \mathcal{T}. \tag{A.1.4}
$$

Following the arguments in [28, Theorem 3.3] and using the estimates in (2.3.4) and (2.3.5), we can derive a weaker version of (A.1.4):

$$
V_L(x) \ge \sup_{c\in\mathcal{C}_L} \mathbb{E}\left[\int_0^r e^{-\beta t} U(c_t X_t^x) dt + e^{-\beta r} V_L(X_r^x)\right], \quad \forall r \ge 0. \tag{A.1.5}
$$

Note that the arguments in [28, Theorem 3.3] directly give the stronger statement (A.1.4) under finite horizon $T > 0$, with $\mathcal{T}$ replaced by $\mathcal{T}_T$, the set of stopping times taking values in $[0, T]$ a.s. The same arguments, however, only render the weaker statement (A.1.5) under infinite horizon. This is because with finite horizon $T > 0$, one can derive an estimate for $\mathbb{E}[\sup_{0\le t\le T} X_t^2]$, i.e. (2.7) in [28], which ensures that (3.14) in [28] holds *simultaneously* for all $\tau \in \mathcal{T}_T$. When the time horizon is infinite, one would need a corresponding estimate for

$\mathbb{E}[\sup_{0\le t<\infty} X_t^2]$, which is often unavailable. In our case, we only have the estimates (2.3.4) and (2.3.5), which ensure that (3.14) in [28] holds only for *each* deterministic time $r \ge 0$.

In the following, we will show that the weaker statement (A.1.5) in fact implies (A.1.4). First, we claim that for any $c \in \mathcal{C}_L$ and $x > 0$, the process $\int_0^t e^{-\beta s} U(c_s X_s^x) ds + e^{-\beta t} V_L(X_t^x)$, $t \ge 0$, is a supermartingale. Given $0 \le r \le t$, it holds for a.e. $\omega \in \Omega$ that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-\beta s} U(c_s X_s^x) ds + e^{-\beta t} V_L(X_t^x) \,\middle|\, \mathcal{F}_r\right](\omega) \\
&= \int_0^r e^{-\beta s} U(c_s X_s^x) ds(\omega) \\
&\qquad + e^{-\beta r} \mathbb{E}\left[\int_r^t e^{-\beta(s-r)} U(c_s X_s^x) ds + e^{-\beta(t-r)} V_L(X_t^x) \,\middle|\, \mathcal{F}_r\right](\omega) \\
&= \int_0^r e^{-\beta s} U(c_s X_s^x) ds(\omega) \\
&\qquad + e^{-\beta r} \mathbb{E}\left[\int_r^t e^{-\beta(s-r)} U(c_{s-r}^{r,\omega} X_{s-r}^{X_r^x(\omega)}) ds + e^{-\beta(t-r)} V_L\left(X_{t-r}^{X_r^x(\omega)}\right)\right] \\
&= \int_0^r e^{-\beta s} U(c_s X_s^x) ds(\omega) \\
&\qquad + e^{-\beta r} \mathbb{E}\left[\int_0^{t-r} e^{-\beta s} U\left(c_s^{r,\omega} X_s^{X_r^x(\omega)}\right) ds + e^{-\beta(t-r)} V_L\left(X_{t-r}^{X_r^x(\omega)}\right)\right],
\end{aligned}
$$

where the third line follows from [2, Proposition A.1], with $c^{r,\omega} \in \mathcal{C}_L$ defined by $c_s^{r,\omega}(\bar\omega) := c_{r+s}(\omega \otimes_r \bar\omega)$, $s \ge 0$, for each fixed $\omega \in \Omega$; recall (2.2.1). This, together with (A.1.5), yields

$$
\mathbb{E}\left[\int_0^t e^{-\beta s} U(c_s X_s^x) ds + e^{-\beta t} V_L(X_t^x) \,\middle|\, \mathcal{F}_r\right] \le \int_0^r e^{-\beta s} U(c_s X_s^x) ds + e^{-\beta r} V_L(X_r^x) \text{ a.s.}
$$

This shows the desired supermartingale property. Now, for any $x > 0$ and $\tau \in \mathcal{T}$, by the optional sampling theorem,

$$
V_L(x) \ge \mathbb{E}\left[\int_0^{\tau\wedge T} e^{-\beta s} U(c_s X_s^x) ds + e^{-\beta(\tau\wedge T)} V_L(X_{\tau\wedge T}^x)\right], \quad \forall\, T > 0.
$$

As $T \to \infty$, thanks to Fatou's lemma and the continuity of V_L, we obtain (A.1.4). $\square$

The next comparison result follows directly from the argument in [28, Theorem 4.1].

The argument is in fact slightly simpler here, as the time variable is not involved in our infinite-horizon setup; see also a very similar proof in [24, Proposition 4.1] for a related infinite-horizon problem.

Lemma A.1.3. *Fix $L > 0$. For any $0 < a < b$, if $u_1, u_2 \in C([a,b])$ are two viscosity solutions to*

$$\beta v(x) = \frac{1}{2}\sigma^2 x^2 v''(x) + (x^\alpha - \mu x)v'(x) + \tilde{U}_L(x, v'(x)) \quad \text{for } x \in (a,b), \tag{A.1.6}$$

with $u_1(a) = u_2(a)$ and $u_1(b) = u_2(b)$, then $u_1 \equiv u_2$.

Now, we are ready to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. In view of Lemma A.1.1, it remains to show that V_L belongs to $C^2((0,\infty))$ and solves (2.4.3). For any $0 < a < b$, consider the boundary value problem (A.1.6) with $v(a) = V_L(a)$ and $v(b) = V_L(b)$. Thanks to the boundedness of $c \in \mathcal{C}_L$, the same estimate for $|\tilde{U}_L(x_1, p_1) - \tilde{U}_L(x_2, p_2)|$ in [28, Theorem 4.2] still holds, which means that the condition (5.18) in [27] is true under current setting. We then conclude from [27, Theorem 5.3.7] that there exists a classical solution $v \in C^2((a,b)) \cap C([a,b])$ to (A.1.6). Since v is also a viscosity solution, Lemmas A.1.2 and A.1.3 imply that $V_L = v$ on $[a,b]$, and thus $V_L \in C^2((a,b))$. With $0 < a < b$ arbitrarily chosen, we have $V_L \in C^2((0,\infty))$ and solves (2.4.3) in the classical sense. $\square$