Leading Coefficients of Kazhdan–Lusztig Polynomials in Type D

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Kazhdan–Lusztig polynomials arise in the context of Hecke algebras associated to Coxeter groups. The computation of these polynomials is very difficult for examples of even moderate rank. In type A it is known that the leading coefficient, $\mu(x, w)$ of a Kazhdan–Lusztig polynomial $P_{x,w}$ is either 0 or 1 when x is fully commutative and w is arbitrary. In type D Coxeter groups there are certain "bad" elements that make μ-value computation difficult.

The Robinson–Schensted correspondence between the symmetric group and pairs of standard Young tableaux gives rise to a way to compute cells of Coxeter groups of type A. A lesser known correspondence exists for signed permutations and pairs of so-called domino tableaux, which allows us to compute cells in Coxeter groups of types B and D. I will use this correspondence in type D to compute μ-values involving bad elements. I will conclude by showing that $\mu(x, w)$ is 0 or 1 when x is fully commutative in type D.

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Contents

Chapter 1

Coxeter groups

In their seminal paper [13], Kazhdan and Lusztig defined remarkable polynomials, $P_{x,w}$ indexed by elements x and w of an arbitrary Coxeter group W. These polynomials are called Kazhdan–Lusztig polynomials, and are important in algebra and geometry. For example, they give rise to representations of both the Coxeter group and its corresponding Hecke algebra. Unfortunately, these polynomials are particularly difficult to compute, even for relatively small Coxeter groups. A bound on the degree of $P_{x,w}$ is known, but it unknown when this bound is achieved, in general. Of particular importance are the coefficients $\mu(x, w)$ of the highest possible degree term. The polynomials $P_{x,w}$ and the μ-values are defined by recurrence relations, but there is no known algorithm that allows for their efficient computation, even in groups of relatively small rank.

For many years computational evidence suggested that the values $\mu(x, w)$ were always 0 or 1 in Coxeter groups of type A. This conjecture, known as the *0-1 Conjecture*, was shown to be false by McLarnan and Warrington in [19]. However, empirical evidence suggests that $\mu(x, w) \in \{0, 1\}$ in many cases. For example, in type A_n it is known that $\mu(x, w) \in \{0, 1\}$ if one of the following holds:

(1) $n \leq 8$ [19];

(2) $a(x) < a(w)$ [24], where a is Lusztig's a-function, to be discussed in Section 1.3;

(3) x is fully commutative [11].

In this thesis we work in type D. Coxeter groups of type D have so-called *bad* elements whose descent sets have undesirable properties. These properties make computing $\mu(x, w)$ difficult when w is bad and x is fully commutative in the sense of Stembridge [23]. We compute μ-values involving these bad elements and use these calculations to prove our main result in Theorem 4.5.11:

Theorem. *Let $x, w \in W(D_n)$ be such that x is fully commutative. Then $\mu(x, w) \in \{0, 1\}$.*

We will only rely on computer calculations for two computations of μ-values in Coxeter groups of small rank.

1.1 Basic properties

We begin with a short overview of the basic properties of Coxeter groups. The following definitions are from [2] and [12].

Definition 1.1.1. A **Coxeter system** is an ordered pair (W, S) consisting of a **Coxeter group** W generated by a set S with presentation

$$\langle S \mid (st)^{m(s,t)} = 1, s, t \in S, m(s,t) \in \mathbb{N} \cup \{\infty\}\rangle,$$

where $m(s,t) = 1$ if $s = t$ and $m(s,t) = m(t,s) \geq 2$ if $s \neq t$. If there is no relation between a pair $s, t \in S$ we say $m(s,t) = \infty$. If $m(s,t) \leq 3$ for all $s, t \in S$ we say that (W, S) is **simply laced**.

If S is finite then we say that (W, S) is a Coxeter system of **rank** $|S|$.

Example 1.1.2. The dihedral group of order 8 is a Coxeter group with presentation

$$\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1\rangle.$$

Definition 1.1.3. Let $s, t \in S$ be such that $s \neq t$. We call each relation $(st)^{m(s,t)} = 1$ a **braid relation**. Note that each braid relation may be rewritten as

$$\underbrace{sts\cdots}_{m(s,t)\text{ factors}} = \underbrace{tst\cdots}_{m(s,t)\text{ factors}}.$$

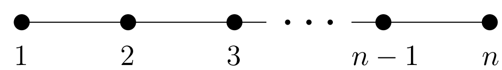
In particular, if $m(s,t) = 2$ then $st = ts$, so s and t commute. If $m(s,t) = 2$ we call the relation a **short braid relation**. If $m(s,t) \geq 3$ we call the relation a **long braid relation**.

We encode the information contained in the presentation of a Coxeter system into a picture called a Coxeter graph. We will use $\mathbf{n}$ to denote the set

$$\{1, 2, 3, 4, \dots, n\}.$$

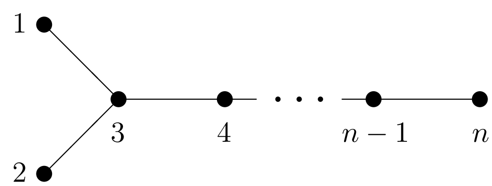
Definition 1.1.4. Let (W, S) be a Coxeter system. A **Coxeter graph** is a graph Γ with vertex set S. We join $s, t \in S$ by an edge labeled $m(s,t)$ whenever $m(s,t) \geq 3$. As a convention we omit the label when $m(s,t) = 3$.

Example 1.1.5. Let $n \in \mathbb{N}$ and let (W, S) be a Coxeter system with $S = \{s_1, s_2, \dots, s_n\}$, and Coxeter diagram shown below.



Such a Coxeter group is said to be of type A_n, and we write $W = W(A_n)$. The symmetric group on $n+1$ elements, S_{n+1} is a Coxeter group of type A_n since if we let $W = S_{n+1}$, $s_i = (i, i+1)$ and $S = \{s_i\}_{i=1}^n$, then (W, S) is a Coxeter system of type A_n [2, Proposition 1.5.4].

Example 1.1.6. Let (W, S) be a Coxeter system with $S = \{s_1, s_2, \dots, s_n\}$, and with Coxeter diagram given below.



Such a Coxeter group is said to be of type D_n, and we write $W = W(D_n)$. The wreath product $\mathbb{Z}_2 \wr S_n$ consists of all bijections σ of the set $\{i \mid \pm i \in \mathbf{n}\}$ such that $\sigma(-a) = -\sigma(a)$. This is called the signed permutation group, and is isomorphic to the Coxeter group of type B_n [2, Proposition 8.1.3]. The group $W(D_n)$ is an index 2 subgroup of the signed permutation group, consisting of all elements with an even number of sign changes, under the embedding

$$s_i \mapsto \begin{cases} (1,\ -2)(-1,\ 2) & \text{if } i = 1; \\ (i-1,\ i)(-(i-1),\ -i) & \text{if } i \geq 2, \end{cases}$$

[2, Proposition 8.2.3].

Remark 1.1.7. From Example 1.1.5 we see that $W(A_{n-1})$ consists of all permutations of $\mathbf{n}$. Then we have a canonical inclusion

$$\iota : W(A_{n-1}) \to W(D_n)$$

that sends a permutation in $W(A_{n-1})$ to the same permutation in $W(D_n)$. From Examples 1.1.5 and 1.1.6 we see that $\iota(s_i) = s_{i+1}$.

Definition 1.1.8. Let (W, S) be a Coxeter system. Any element $w \in W$ can be written as a product of generators $w = s_1 s_2 \cdots s_r$, $s_i \in S$.

(1) If r is minimal for all expressions of w, we call r the **length** of w, denoted $\ell(w)$.

(2) Any expression of w as a product of $\ell(w)$ generators is called a **reduced expression** for w.

(3) The set of all $s \in S$ that appear in a reduced expression of w is called the **support** of w, denoted $\mathrm{supp}(w)$. Note that each element in a Coxeter group can have many different reduced expressions. However, if $s \in S$ appears in a particular reduced expression for w, it must appear in each reduced expression for w as a consequence of [2, Theorem 3.3.1]. Thus, to determine $\mathrm{supp}(w)$ we only need to consider a particular reduced expression for w, so $\mathrm{supp}(w)$ is well-defined.

(4) Let $v_i \in W$ for $1 \leq i \leq k$. We say that the product $v = v_1 v_2 \cdots v_k$ is **reduced** if $\ell(v) = \sum_{i=1}^{k} \ell(v_i)$.

Example 1.1.9. Let $W = W(A_4)$, let $w = s_1 s_2 s_3$, and let $x = s_1 s_3 s_1$. Then the expression given for w is reduced, and $\ell(w) = 3$. However, we see that $x = s_1 s_3 s_1 = s_1 s_1 s_3 = s_3$, so the above expression for x is not reduced.

Proposition 1.1.10. *Let W be a finite Coxeter group. Then there is a unique element, w_0, of maximal length in W.*

Proof. This is [2, Proposition 2.3.1]. □

Example 1.1.11. Let $W = W(A_2)$. Then $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$.

We now distinguish between elements of a Coxeter groups to which long braid relations may be applied. The following definition is due to Stembridge [23].

Definition 1.1.12. Let (W, S) be a simply laced Coxeter system. We call an element $w \in W$ **complex** if there exist $w_1, w_2 \in W$ and $s, t \in S$ with $m(s,t) = 3$ such that $w = w_1 \cdot sts \cdot w_2 = w_1 \cdot tst \cdot w_2$ reduced. An element $w \in W$ that is not complex is called **fully commutative**. We denote the set of fully commutative elements by W_c.

Example 1.1.13. Let $W = W(D_6)$, let $x = s_1 s_2 s_4 s_3 s_4$ and let $y = s_1 s_2 s_6 s_3 s_5 s_4$. Then x is not fully commutative since we can apply a long braid relation to the product $s_4 s_3 s_4$. However, y is fully commutative: we cannot apply any long braid relations to y because there are no repeated generators.

Definition 1.1.14. Let (W, S) be a Coxeter system and let $w \in W$. We define the **left descent set**, $\mathcal{L}(w)$, and the **right descent set**, $\mathcal{R}(w)$, as follows:

$$\mathcal{L}(w) = \{s \in S \mid \ell(sw) < \ell(w)\};$$
$$\mathcal{R}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}.$$

A left or right descent set is **commutative** if it consists of mutually commuting generators.

Example 1.1.15. Let $W = W(D_4)$ and let $w = s_1 s_4 s_3 s_2 s_3$. Then $\mathcal{L}(w) = \{s_1, s_2, s_4\}$ is commutative and $\mathcal{R}(w) = \{s_2, s_3\}$ is not commutative.

It is known that $s \in \mathcal{L}(w)$ if and only if w has a reduced expression beginning in s [2, Corollary 1.4.6]. Similarly, $s \in \mathcal{R}(w)$ if and only if w has a reduced expression ending in s.

Proposition 1.1.16. *Let (W, S) be a finite Coxeter system with $w \in W$. Then $\mathcal{L}(w) = S$ if and only if $w = w_0$. Similarly, $\mathcal{R}(w) = S$ if and only if $w = w_0$.*

Proof. This is [2, Proposition 2.3.1 (ii)]. □

Definition 1.1.17. Let (W, S) be a Coxeter system and let $I \subset S$. Define W_I to be the subgroup of W generated by I and define the set

$$W^I = \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}.$$

Proposition 1.1.18. *Let (W, S) be a Coxeter system and let $I \subset S$. Then (W_I, I) is a Coxeter system.*

Proof. This is [12, Theorem 5.12(a)]. □

The Coxeter group W_I is called a **parabolic subgroup** of W, with presentation

$$\langle I \mid (st)^{m(s,t)} = 1, s, t \in I\rangle.$$

The set W^I is called the set of **distinguished coset representatives**, and consists of minimal representatives of left cosets wW_I.

Proposition 1.1.19. *Let (W, S) be a Coxeter system and let $I \subset S$. For each $w \in W$ there exist unique elements $w^I \in W^I$ and $w_I \in W_I$ such that $w = w^I w_I$ reduced.*

Proof. This is [2, Proposition 2.4.4]. □

Definition 1.1.20. Let (W, S) be a Coxeter system and let $I \subset S$. Write $w = w^I w_I$ as in Proposition 1.1.19. We call this the **reduced decomposition** of w with respect to I.

Lemma 1.1.21. *Let (W, S) be a simply laced Coxeter system. Let $w \in W$ suppose that $s, t \in S$ are such that s and t do not commute.*

(1) If $s, t \in \mathcal{R}(w)$ then w can be written $w = w' \cdot sts$ reduced for some $w' \in W$.

(2) If $s, t \in \mathcal{L}(w)$ then w can be written $w = sts \cdot w'$ reduced for some $w' \in W$.

Proof. Let $I = \{s, t\}$ and suppose that $s, t \in \mathcal{R}(w)$. Then by Proposition 1.1.19 we can find a reduced decomposition $ws = x^I x_I$ reduced such that $x^I \in W^I$ and $x_I \in W_I$. Then $w = x^I x_I s$ reduced, and since $x_I s \in W_I$ this is the unique reduced decomposition of w according to Proposition 1.1.19.

Write a reduced decomposition $w = w^I w_I$. Since $s, t \in \mathcal{R}(w)$ we can use the above argument to show $s, t \in R(w_I)$. Then by Proposition 1.1.16 we have $w_I = sts$, so $w = w^I \cdot sts$ reduced.

Now suppose $s, t \in \mathcal{L}(w)$. Then we can repeat the above argument with w^{-1} and the result follows. $\square$

Corollary 1.1.22. *Let (W, S) be a simply laced Coxeter system. If $w \in W_c$ then $\mathcal{R}(w)$ and $\mathcal{L}(w)$ are both commutative.*

Proof. Let $w \in W$ be such that $\mathcal{R}(w)$ is not commutative. Then by Lemma 1.1.21 we can write $w = w' \cdot sts$ reduced for some $w' \in W$ and noncommuting generators $s, t \in S$, so $w \notin W_c$. $\square$

We now examine a partial ordering on a Coxeter group called the **Bruhat order**.

Definition 1.1.23. Let (W, S) be a Coxeter system and let $w \in W$. Pick a reduced expression $w = s_1 s_2 \cdots s_r$. Then $x \leq w$ if and only if x is a subword of w; that is, $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq r$.

Since the definition allows for the choice of any reduced expression for w, it is not immediately clear that the Bruhat order is well defined. However, the Bruhat order is indeed well defined; see [2, Theorem 2.2.2] for a proof.

Example 1.1.24. Let $W = W(D_4)$, let $w = s_1 s_2 s_3 s_2$, let $x = s_1 s_2$ and let $y = s_3 s_4$. It is easy to see that the given expressions for w, x, and y are reduced. Then we have $x \leq w$ since x is a subword of w, but $y \not\leq w$ since $s_4 \in \mathrm{supp}(y) \setminus \mathrm{supp}(w)$, so no reduced expression for y is a subword of w.

We now introduce star operations, which were developed by Kazhdan and Lusztig in [13].

Definition 1.1.25. Let (W, S) be a simply laced Coxeter system. Let $s, t \in S$ such that $m(s,t) = 3$. Define

$$D_{\mathcal{L}}(s,t) = \left\{ w \in W \;\middle|\; |\mathcal{L}(w) \cap \{s,t\}| = 1 \right\};$$

$$D_{\mathcal{R}}(s,t) = \left\{ w \in W \;\middle|\; |\mathcal{R}(w) \cap \{s,t\}| = 1 \right\}.$$

If $w \in D_{\mathcal{L}}(s,t)$, then exactly one of sw, tw, is in $D_{\mathcal{L}}(s,t)$. We call the resulting element ${}^*w$, and define the **left star operation** with respect to $\{s,t\}$ to be the map ${}^* : W \to W : w \mapsto {}^*w$. We can define w^* in a similar way using $D_{\mathcal{R}}(s,t)$, resulting in a **right star operation** with respect to $\{s,t\}$. Note that each of these maps is an involution and partially defined.

Example 1.1.26. Let $x = s_3s_4s_5s_6s_5$ and let $w = s_4$. Note that the given expression for x is reduced. Then $x, w \in D_{\mathcal{L}}(s_3, s_4)$ so we can apply the operations ${}^*x = s_4s_5s_6s_5$ and ${}^*w = s_3s_4$. However, $x, w \notin D_{\mathcal{R}}(s_5, s_6)$, so the right star operation with respect to $\{s_5, s_6\}$ is not defined on either x or w.

Definition 1.1.27. If $w, y \in D_{\mathcal{L}}(s,t)$ are such that ${}^*w = y$ and $\ell(w) - 1 = \ell(y)$ then we say that w is **left star reducible** to y. If $w, y \in D_{\mathcal{R}}(s,t)$ are such that $w^* = y$ and $\ell(w) - 1 = \ell(y)$ then we say that w is **right star reducible** to y. If there is a sequence

$$w = w_{(0)}, w_{(1)}, \ldots, w_{(k-1)}, w_{(k)} = y$$

such that $w_{(i)}$ is left or right star reducible to $w_{(i+1)}$ then we say that w is **star reducible** to y.

Proposition 1.1.28. *Let W be a Coxeter group of type A or D and let $w \in W_c$. Then w is star reducible to a product of commuting generators.*

Proof. This is [10, Theorem 6.3]. □

Proposition 1.1.29. *Let (W, S) be a simply laced Coxeter system. If $x \in W_c$ and $s, t \in S$ are such that $x \in D_{\mathcal{L}}(s,t)$, then ${}^*w \in W_c$.*

Proof. This is [22, Proposition 2.10]. □

1.2 Kazhdan–Lusztig theory

We will now use a Coxeter system (W, S) to construct an algebra $\mathcal{H} = \mathcal{H}(W, S)$ called the **Hecke algebra**. The following definitions can be found in [13].

Definition 1.2.1. Let $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the ring of Laurent polynomials over $\mathbb{Z}$. Let $\mathcal{A}^+ = \mathbb{Z}[q^{\frac{1}{2}}]$. Then $\mathcal{H}(W, S)$ is the algebra over $\mathcal{A}$ with linear basis $\{T_w \mid w \in W\}$ with multiplication determined by the following relations:

(1) $T_s T_w = T_{sw}$ if $\ell(sw) > \ell(w)$;

(2) $T_s^2 = (q-1)T_s + qT_1$.

Using the above relations, we can compute that

$$T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_1.$$

Let $w \in W$ have reduced expression $w = s_1 s_2 \cdots s_r$. Then using the first multiplication rule above we see that $T_w = T_{s_1} T_{s_2} \cdots T_{s_r}$, so $T_w^{-1} = T_{s_r}^{-1} \cdots T_{s_1}^{-1}$, thus each T_w is invertible in $\mathcal{H}$.

We now define a ring homomorphism $\iota : \mathcal{H} \to \mathcal{H}$ by $\iota\left(q^{\frac{1}{2}}\right) = q^{-\frac{1}{2}}$ and $\iota(T_w) = T_{w^{-1}}^{-1}$. Note that ι is an involution. This involution gives rise to an interesting basis for $\mathcal{H}$.

Proposition 1.2.2. *For each $w \in W$ we have a unique element $C_w \in \mathcal{H}$ with the following properties:*

(1) $\iota(C_w) = C_w$*,*

(2) $C_w = \sum_{x \leq w} (-1)^{\ell(w)+\ell(x)} q^{\frac{1}{2}(\ell(w)-\ell(x))} \iota\left(P_{x,w}\right) T_x$*, where* $P_{w,w} = 1$ *and* $P_{x,w}(q) \in \mathbb{Z}[q]$ *has degree* $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ *if* $x < w$*.*

Proof. This is [13, Theorem 1.1]. $\square$

These C_w form a basis for $\mathcal{H}$ [2, 6.1] called the **Kazhdan–Lusztig basis**.

The polynomials $P_{x,w}$ in Proposition 1.2.2 are called **Kazhdan–Lusztig polynomials**, and are particularly difficult to calculate. For example, the degree of a particular polynomial $P_{x,w}$ is not even known in general. We do, however, have an upper bound for the degree of Kazhdan–Lusztig polynomials when $x < w$:

$$\deg(P_{x,w}) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$$

[13, (1.1.c)].

Definition 1.2.3. We denote by $\mu(x, w)$ the coefficient of the term of degree $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ in $P_{x,w}$. Note that if $\ell(w) \equiv \ell(x) \bmod 2$ then $\mu(x, w) = 0$ since $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ is not an integer; in particular, $\mu(w, w) = 0$ since $\ell(w) - \ell(w) = 0$ is even. If $\mu(x, w) \neq 0$ we write $x \prec w$.

As we will soon see, calculating μ-values for Kazhdan–Lusztig polynomials is very difficult, even for Coxeter groups of small rank. In [13] Kazhdan and Lusztig proved the following helpful elementary properties of μ-values.

Proposition 1.2.4.

(1) If $x, w \in D_{\mathcal{L}}(s, t)$ then $\mu(x, w) = \mu({}^*x, {}^*w)$;

(2) If $x, w \in D_{\mathcal{R}}(s, t)$ then $\mu(x, w) = \mu(x^*, w^*)$;

(3) If there exists $s \in \mathcal{L}(w) \setminus \mathcal{L}(x)$ then either

(a) $\mu(x, w) = 0$, or

(b) $x = sw$ and $\mu(x, w) = 1$;

(4) If there exists $s \in \mathcal{R}(w) \setminus \mathcal{R}(x)$ then either

(a) $\mu(x, w) = 0$, or

(b) $x = ws$ and $\mu(x, w) = 1$.

Proof. Parts (1) and (2) are [13, Theorem 4.2]. Parts (3) and (4) are [13, (2.3.e)] and [13, (2.3.f)], respectively. □

Corollary 1.2.5. *Let $x \in W_c$ and let w be a product of mutually commuting generators. Then $\mu(x, w) \in \{0, 1\}$.*

Proof. If $x \not< w$ then $\mu(x,w) = 0$ so we are done. If $x < w$ then $\mathcal{L}(x) \subsetneq \mathcal{L}(w)$, so there exists some $s \in \mathcal{L}(w) \setminus \mathcal{L}(x)$ and we are done by Proposition 1.2.4 part (3). □

Understanding μ-values helps us to calculate Kazhdan–Lusztig polynomials in general due to the following recurrence relation.

Proposition 1.2.6. *Let (W,S) be a Coxeter system. Let $x, w \in W$ and let $s \in S$ be such that $s \in \mathcal{L}(w)$. Then*

$$P_{x,w} = q^{1-c} P_{sx,sw} + q^c P_{x,sw} - \sum_{\substack{sz<z \\ z \prec sw}} \mu(z,sw) q^{\frac{1}{2}(\ell(w)-\ell(z))} P_{x,z},$$

where

$$c = \begin{cases} 1, & \text{if } s \in \mathcal{L}(x); \\ 0, & \text{else.} \end{cases}$$

Proof. This is [13, (2.2.c)]. □

Unfortunately, this is the only obvious way to compute Kazhdan–Lusztig polynomials. If $W = W(D_n)$ we have $|W| = 2^{n-1}n!$, which is very large even in groups of moderate rank. In order to compute $P_{x,w}$ using the above recurrence relation, we must compute intervals of the Bruhat order in W. In large groups such a computation is very expensive in terms of either processor time or memory, depending on the algorithm used. As a result, using the above recurrence relation to compute Kazhdan–Lusztig polynomials is computationally infeasible in all but groups of small rank.

We can use Proposition 1.2.6 to deduce several simple facts about Kazhdan–Lusztig polynomials.

Proposition 1.2.7. *Let $x, w \in W$ be such that $x < w$ and let $s \in S$.*

(1) If $sw < w$ and $sx > x$ then $P_{x,w} = P_{sx,w}$.

(2) If $w < ws$ and $xs \not\leq w$ (thus $xs > x$) then $P_{x,w} = P_{xs,ws}$.

Proof. Part (1) is [13, (2.3.g)]. Part (2) is [20, Lemma 1.4.5(v)]. □

Lemma 1.2.8. *Let $x, w \in W$ and let $u = s_1 \cdots s_r$ be a product of mutually commuting generators such that $s_i \notin \mathrm{supp}(x) \cup \mathrm{supp}(w)$ for each $1 \leq i \leq r$. Then $P_{x,w} = P_{xu,wu}$, and $\mu(x, w) = \mu(xu, wu)$.*

Proof. We will show $P_{x,w} = P_{xu,wu}$ by induction on $r = \ell(u)$. If $r = 1$ then we are done by Proposition 1.2.7 (2). Suppose that the statement holds for all values less than r. Then by induction, we have $P_{x,w} = P_{xus_r,wus_r}$. Since u is a product of mutually commuting generators, $s_r \notin \mathrm{supp}(xus_r) \cup \mathrm{supp}(wus_r)$, so by Proposition 1.2.7 (1) we have $P_{x,w} = P_{xus_r,wus_r} = P_{xu,wu}$.

Then since

$$\frac{1}{2}\left(\ell(wu) - \ell(xu) - 1\right) = \frac{1}{2}\left(\ell(w) + r - \ell(x) - r - 1\right) = \frac{1}{2}\left(\ell(w) - \ell(x) - 1\right)$$

we have $\mu(x, w) = \mu(xu, wu)$. □

Surprisingly little is known about μ-values, even for finite Coxeter groups. Previously, computer computations suggested that $\mu(x, w) \in \{0, 1\}$ in Coxeter systems of type A. This was shown to be egregiously false by McLarnan and Warrington in [19] using computer calculations. Billey and Warrington have developed a more efficient recursive way to compute Kazhdan–Lusztig polynomials in certain cases in type A [1, Lemma 39].

The group $W(\widetilde{A}_n)$ is an infinite Coxeter group which, like $W(D_n)$, contains the symmetric group as a parabolic subgroup. Since the 0-1 conjecture fails in type A, it must therefore also fail in types $\widetilde{A}$ and D. However, in [11] Green showed that $\mu(x, w) \in \{0, 1\}$ for $x, w \in W(\widetilde{A}_n)$ as long as x is fully commutative. This proof relies on the fact that Coxeter groups of type $\widetilde{A}$ do not contain certain elements called "bad elements," which will be discussed in Chapter 2. Green remarks that there may be many other types of Coxeter groups for which $\mu(x, w) \in \{0, 1\}$ if x is fully commutative [11, Introduction]. We will prove this result in Theorem 4.5.11 for Coxeter groups of type D.

We can partition a Coxeter group into sets called Kazhdan–Lusztig cells, first defined in [13], which behave nicely with regard to calculations involving μ-values.

Definition 1.2.9. Recall from Definition 1.2.3 that we write $x \prec w$ if $\mu(x, w) \neq 0$. Define $x \leq_L w$

if there is a (possibly trivial) chain

$$x = x_0, x_1, \ldots, x_r = w$$

such that either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$ and $\mathcal{L}(x_i) \not\subset \mathcal{L}(x_{i+1})$. Define $x \sim_L w$ if and only if $x \leq_L w$ and $w \leq_L x$. Then $\sim_L$ is an equivalence relation that partitions W into **left Kazhdan–Lusztig cells**, or left cells. There is an analogous definition for **right Kazhdan–Lusztig cells**.

Definition 1.2.10. We define $x \leq_{LR} w$ if there is a (possibly trivial) chain

$$x = x_0, x_1, \ldots, x_r = w$$

such that either $x_i \leq_L x_{i+1}$ or $x_i \leq_R x_{i+1}$ for each $i < r$. Define $x \sim_{LR} w$ if and only if $x \leq_{LR} w$ and $w \leq_{LR} x$. As above, $\sim_{LR}$ is an equivalence relation that partitions W into **two-sided Kazhdan–Lusztig cells**, or two-sided cells.

1.3 Lusztig's a-function

In [14], Lusztig defined a function that behaves nicely with respect to Kazhdan–Lusztig cells. As we will see, this function will help us to bound the degree of certain Kazhdan–Lusztig polynomials. We begin with a series of definitions and lemmas from [14] leading to the definition of Lusztig's a-function. Although the a-function may be defined for affine Coxeter groups, we will simplify our calculations by assuming that W is a finite Coxeter group.

Lemma 1.3.1. *We may define polynomials $Q_{x,w}$ for each $x \leq w$ using the following identity:*

$$\sum_{x \leq z \leq w} (-1)^{\ell(z)-\ell(x)} Q_{x,z}(q) P_{z,w}(q) = \begin{cases} 1 & \text{if } x = w; \\ 0 & \text{if } x < w. \end{cases}$$

Then $Q_{x,w}$ is a polynomial of degree $\leq \frac{1}{2}(\ell(x) - \ell(x) - 1)$ if $y < w$ and $Q_{w,w} = 1$.

Proof. This is [14, (1.3.1)]. □

The polynomials $Q_{x,w}$ are sometimes called **inverse Kazhdan–Lusztig polynomials**. Like Kazhdan–Lusztig polynomials, they are difficult to compute.

Definition 1.3.2. For $w \in W$ set $\widetilde{T}_w = q^{-\ell(w)/2} T_w$. We define

$$D_x = \sum_{x \leq w} Q_{x,w}\left(q^{-1}\right) q^{\frac{1}{2}(\ell(w)-\ell(x))} \widetilde{T}_w.$$

Definition 1.3.3. Recall that $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ is the ring of Laurent polynomials over $\mathbb{Z}$. Let $\tau : \mathcal{H} \to \mathcal{A}$ be the $\mathcal{A}$-linear map defined by $\tau\left(\sum_w \alpha_w \widetilde{T}_w\right) = \alpha_e$. (The map τ turns out to be a trace map.)

Definition 1.3.4. Let $w \in W$ and define the set

$$\mathscr{S}_w = \left\{ i \in \mathbb{N} \;\middle|\; q^{\frac{i}{2}} \tau\left(\widetilde{T}_x \widetilde{T}_y D_w\right) \in \mathcal{A}^+ \text{ for all } x, y \in W \right\}.$$

If $\mathscr{S}_w$ is nonempty we denote $a(w) = \min(\mathscr{S}_w)$, otherwise set $a(w) = \infty$. Then we have a function

$$a : W \to \mathbb{N} \cup \{\infty\}.$$

We now observe some properties of a.

Proposition 1.3.5. *We have*

$$\deg(P_{e,w}) \leq \frac{1}{2}(\ell(w) - a(w)).$$

Proof. This is [15, 1.3(a)]. □

It will later be very useful to compute $a(w)$ to find a bound for $\deg(P_{e,w})$. However, from the definition, we can see that calculating $a(w)$ can be very difficult. To make calculations easier, we can use the fact that the a-function is known to be constant on Kazhdan–Lusztig cells when W is a Weyl group. Note that Coxeter groups of types A and D are Weyl groups.

Lemma 1.3.6. *Let W be a finite Weyl group and let $x, w \in W$ be such that $x \sim_{LR} w$. Then $a(x) = a(w)$.*

Proof. This is [14, Theorem 5.4]. □

In [16] Lusztig introduces the a-function for Hecke algebras with unequal parameters. He develops a series of conjectures about how the a-function relates to the Coxeter group W, the structure of the Hecke algebra, and Kazhdan–Lusztig cells. These conjectures are not known to hold for general Coxeter groups in the unequal parameter case. However, these conjectures are known to hold for finite Coxeter groups in the equal parameter case.

Lemma 1.3.7. *Let W be a finite Coxeter group with longest element w_0. Then $a(w_0) = \ell(w_0)$.*

Proof. This is [16, Proposition 13.8]. □

Lemma 1.3.8. *Let (W, S) be a Coxeter system of type D_n and let $I \subset S$. If $w \in W_I$ then $a(w)$ calculated in terms of W_I is equal to $a(w)$ calculated in terms of W.*

Proof. This is [16, Conjecture 14.2 P12]. In [16, 15.1] Lusztig proves that Conjecture 14.2 P12 holds in our case. □

Lemma 1.3.9. *Let (W, S) be a Coxeter system with Coxeter graph Γ that can be decomposed into a disjoint union of connected components $\Gamma = \Gamma_1 \cup \Gamma_2$. Then every $w \in W$ has a unique expression $w = w_1 w_2$ reduced where $w_1 \in W(\Gamma_1)$ and $w_2 \in W(\Gamma_2)$. Furthermore $a(w) = a(w_1) + a(w_2)$.*

Proof. This is [21, Lemma 1.8 (1)]. □

Chapter 2

Bad elements

There are elements called bad elements whose reduced expressions have certain unfavorable properties which complicate computing $\mu(x, w)$ where x is fully commutative and w is bad. As we will see in Section 2.1, there are no bad elements in Coxeter groups of type A. This fact was used in [11] by Green to show that $\mu(x, w) \in \{0, 1\}$ when x is fully commutative. However in Section 2.2 we will see that Coxeter groups of type D do contain bad elements. We conclude by finding a general form for reduced expressions of bad elements in Section 2.3.

2.1 Type A

Definition 2.1.1. Let W be a simply laced Coxeter group and let $w \in W$. We say that w is **bad** if w is not a product of commuting generators and if w has no reduced expressions beginning or ending in two noncommuting generators. We say w is **weakly bad** if w has no reduced expressions beginning or ending in two noncommuting generators.

Example 2.1.2. Let $W = W(D_4)$ and consider the elements $x = s_1s_2s_3$, $y = s_1s_2s_4$, and $w = s_1s_2s_4s_3s_1s_2s_4$. We see that x is not bad since it has a reduced expression ending in s_2s_3. Since y is a product of mutually commuting generators we see that y is weakly bad, but not bad. However, if we compute all reduced expressions for w we see that none of them begin or end in two noncommuting generators. We can easily see that w is not a product of commuting generators, so w is bad.

Lemma 2.1.3. *If w is bad then so is w^{-1}.*

Proof. This is an immediate consequence of the symmetry of the definition. □

Recall from Example 1.1.5 that $W(A_n) \cong S_{n+1}$. For each element $w \in W(A_n)$ we may use one-line notation to represent w:

$$w = (w(1), w(2), w(3), \ldots, w(n), w(n+1)).$$

We can use this one-line notation to help find left and right descent sets.

Proposition 2.1.4. *Let $w \in W(A_n)$. Then*

$$\mathcal{R}(w) = \{s_i \in S \mid w(i) > w(i+1)\}$$

is the right descent set of w and

$$\mathcal{L}(w) = \left\{s_i \in S \mid w^{-1}(i) > w^{-1}(i+1)\right\}$$

is the left descent set of w.

Proof. This is proven in [2, Proposition 1.5.3]. □

We will now use this correspondence between descent sets and one-line notation to find and classify bad elements in terms of pattern avoidance.

Definition 2.1.5. Let $w \in W(A_n)$ and let a, b, and c be positive integers. We say that w **has the consecutive pattern** abc if there is some $i \in \mathbf{n-1}$ such that $(w(i), w(i+1), w(i+2))$ is in the same relative order as (a, b, c). If w does not have the consecutive pattern abc then we say that w **avoids the consecutive pattern** abc.

Example 2.1.6. Let $w \in W(A_5)$ have the one-line notation

$$w = (5, 3, 2, 1, 6, 4).$$

Then w has the consecutive pattern 321 since $(w(1), w(2), w(3)) = (5, 3, 2)$ are in the same relative order as $(3, 2, 1)$. However, w avoids the consecutive pattern 123 since there is no i such that $(w(i), w(i+1), w(i+2))$ is in the same relative order as $(1, 2, 3)$.

Lemma 2.1.7. *Let (W, S) be a Coxeter system of type A_n and let $w \in W$. Then*

(1) w has a reduced expression ending in two noncommuting generators if and only if w has at least one of the consecutive patterns 321*,* 231*, or* 312*, and*

(2) $w \in W$ has a reduced expression beginning in two noncommuting generators if and only if w^{-1} has at least one of the consecutive patterns 321*,* 231*, or* 312*.*

Proof. Let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in Proposition 1.1.19. We first observe that if w has a reduced expression ending in two noncommuting generators, s_i, s_{i+1}, in some order, then we have $w_I \in \{s_i s_{i+1} s_i, s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that w has the consecutive pattern 321. Then there is some i such that $w(i) > w(i+1) > w(i+2)$, so by Proposition 2.1.4 we have $s_i, s_{i+1} \in \mathcal{R}(w)$, thus w has a reduced expression ending in $s_i s_{i+1} s_i$ by Lemma 1.1.21. Conversely, suppose that $w_I = s_i s_{i+1} s_i$. Then $s_i, s_{i+1} \in \mathcal{R}(w)$, so $w(i) > w(i+1) > w(i+2)$ by Proposition 2.1.4, thus $(w(i), w(i+1), w(i+2))$ has the consecutive pattern 321.

Next suppose that w has the consecutive pattern 231. Then there is some i such that $w(i+1) > w(i) > w(i+2)$, so $s_{i+1} \in \mathcal{R}(w)$ by Proposition 2.1.4. If we multiply on the right by s_{i+1} then we get $ws_{i+1}(i+1) = w(i+2) < w(i) = ws_{i+1}(i)$, so $s_i \in \mathcal{R}(ws_{i+1})$. Then w has a reduced expression ending in $s_i s_{i+1}$. Conversely, if $w_I = s_i s_{i+1}$ then $w(i+2) < w(i+1)$ and $w(i) < w(i+1)$. Furthermore, since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+2) = ws_{i+1}(i+1) < ws_{i+1}(i) = w(i)$, so $(w(i), w(i+1), w(i+2))$ has the consecutive pattern 231.

Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i) > w(i+2) > w(i+1)$. Then $s_i \in \mathcal{R}(w)$. If we multiply on the right by s_i then we get $ws_i(i+1) = w(i) > w(i+2) = ws_i(i+2)$, so $s_{i+1} \in \mathcal{R}(ws_i)$. Then w has a reduced expression ending in $s_{i+1} s_i$. Conversely, if $w_I = s_{i+1} s_i$ then $w(i) > w(i+1)$ and $w(i+2) > w(i+1)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+2) = ws_i(i+2) < ws_i(i+1) = w(i)$, so $(w(i), w(i+1), w(i+2))$ has the consecutive pattern 312.

Finally, we know that w has no reduced expressions beginning in two noncommuting generators if and only if w^{-1} has no reduced expressions ending in two noncommuting generators, which by the above discussion occurs if and only if w^{-1} avoids the consecutive patterns 321, 231, and 312. $\square$

The following theorem was originally proven by Green in the more general case of a Coxeter group of type $\tilde{A}$. We will state the result in type A.

Proposition 2.1.8. *Let $W = W(A_n)$. Then there are no bad elements in W.*

Proof. This is a consequence of [11, Propositions 2.3, 2.4]

$\square$

Corollary 2.1.9. *If $w \in W(D_n)$ is such that all entries in w are positive then w is not bad.*

Proof. Let $w \in W(D_n)$ be such that all entries in w are positive. Recall the embedding ι from Remark 1.1.7. Since all entries in w are positive, we can find an element $w' \in W(A_{n-1})$ such that $\iota(w') = w$. If w' has a reduced expression

$$w' = s_{i_1} s_{i_2} \cdots s_{i_r}$$

in $W(A_{n-1})$ then w has the reduced expression

$$w = s_{i_1+1} s_{i_2+1} \cdots s_{i_r+1}$$

in $W(D_n)$. By Proposition 2.1.8 we see that w' is not bad, thus w is not bad. $\square$

Lemma 2.1.10. *Let $w \in W(A_n)$ be such that both w and w^{-1} avoid the consecutive patterns 321, 231, and 312. Then w is a product of commuting generators.*

Proof. By Lemma 2.1.7 we know that w has no reduced expressions beginning or ending in two noncommuting generators. By Proposition 2.1.8 we know that w is not bad, so w must be a product of commuting generators. $\square$

2.2 Type D

Recall from Example 1.1.6 that we can represent each element $w \in W(D_n)$ as a member of the signed permutation group. We write $w \in W$ using one-line notation

$$w = (w(1), w(2), w(3), \ldots, w(n)),$$

where we write a bar underneath a number in place of a negative sign in order to simplify notation.

Example 2.2.1. Let $w = s_2 s_3 s_1 s_2 s_4 \in W(D_4)$. Then we write

$$w = (\underline{2}, \underline{3}, 4, 1).$$

As in type A, we can use the one-line notation of an element to find its length.

Proposition 2.2.2. *Let $w \in W(D_n)$. Then*

$$\ell(w) = |\{(i,j) \in \mathbf{n} \times \mathbf{n} \mid i < j, w(i) > w(j)\}| + |\{(i,j) \in \mathbf{n} \times \mathbf{n} \mid i < j, w(-i) > w(j)\}|.$$

Proof. This is [2, Proposition 8.2.1]. □

Proposition 2.2.3. *Let (W, S) be a Coxeter system of type D_n, and let $w \in W$ have signed permutation*

$$w = (w(1), \ldots, w(n)).$$

Suppose $s_i \in S$. If $i \geq 2$ then multiplying w by s_i on the right has the effect of interchanging $w(i)$ and $w(i+1)$. Multiplying w by s_i on the left has the effect of interchanging the entries in w whose absolute values are i and $i+1$.

If $i = 1$ then multiplying w by s_1 on the right has the effect of interchanging $w(1)$ and $w(2)$ and switching their signs. Multiplying w by s_1 on the left has the effect of interchanging the entries in w whose absolute values are 1 and 2 and changing their signs.

Proof. This follows from the discussion in [2, Sections 8.1 and A3.1]. □

As in type A, in type D we can easily find the descent sets of an element written in one-line notation.

Proposition 2.2.4. *Let $w \in W(D_n)$. Then*

$$\mathcal{R}(w) = \{s_i \in S \mid w(i-1) > w(i)\}$$

and

$$\mathcal{L}(w) = \left\{s_i \in S \mid w^{-1}(i-1) > w^{-1}(i)\right\},$$

where $w(0) \stackrel{\text{def}}{=} -w(2)$.

Proof. This is [2, Propositions 8.2.1 and 8.2.2]. □

The following lemmas will help us to classify the signed permutations of bad elements in Theorem 2.2.18. We will first introduce the notion of *signed pattern avoidance*, which is not known to be found in other sources, to begin to describe how to find bad elements in type D.

Definition 2.2.5. Let $w \in W(D_n)$. As in type A, we say that w **avoids the consecutive pattern** abc if there is no $i \in \mathbf{n-2}$ such that $(w(i), w(i+1), w(i+2))$ is in the same relative order as (a, b, c). We say that w **avoids the signed consecutive pattern** abc if there is no $i \in \mathbf{n-2}$ such that $(|w(i)|, |w(i+1)|, |w(i+2)|)$ is in the same relative order as $(|a|, |b|, |c|)$ and such that $\text{sign}(a) = \text{sign}(w(i))$, $\text{sign}(b) = \text{sign}(w(i+1))$, and $\text{sign}(c) = \text{sign}(w(i+2))$.

Example 2.2.6. Let $w \in W(D_6)$ have the following one line notation

$$w = (4, \underline{3}, 1, 5, 6, \underline{2}).$$

Then w has the signed consecutive pattern $3\underline{2}1$ since $(|w(1)|, |w(2)|, |w(3)|)$ are in the same relative order as $(|3|, |-2|, |1|)$ and $\text{sign}(3) = \text{sign}(w(1))$, $\text{sign}(-2) = \text{sign}(w(2))$, and $\text{sign}(1) = \text{sign}(w(3))$. However, w avoids the signed consecutive pattern $1\underline{2}3$.

Lemma 2.2.7. *Let $s, t \in S$ such that $m(s,t) = 3$ and $s_1 \notin \{s, t\}$. Then*

(1) w has a reduced expression ending in st or ts if and only if w has at least one of the consecutive patterns 321, 231, or 312, and

(2) w has a reduced expression beginning in st or ts if and only if w^{-1} has at least one of the consecutive patterns 321, 231, or 312.

Proof. Let $i \geq 2$, let $I = \{s_i, s_{i+1}\}$ and write $w = w^I w_I$ as in Proposition 1.1.19. We first observe that if w has a reduced expression ending in two noncommuting generators, s_i, s_{i+1}, in some order, then we have $w_I \in \{s_i s_{i+1} s_i, s_i s_{i+1}, s_{i+1} s_i\}$.

Suppose that w has the consecutive pattern 321. Then there is some i such that $w(i-1) > w(i) > w(i+1)$, so by Proposition 2.2.4 we have $s_i, s_{i+1} \in \mathcal{R}(w)$, thus w has a reduced expression ending in $s_i s_{i+1} s_i$ by Lemma 1.1.21. Conversely, suppose that $w_I = s_i s_{i+1} s_i$. Then $s_i, s_{i+1} \in \mathcal{R}(w)$, so $w(i-1) > w(i) > w(i+1)$ by Proposition 2.2.4, thus $(w(i-1), w(i), w(i+1))$ has the consecutive pattern 321.

Next suppose that w has the consecutive pattern 231. Then there is some i such that $w(i) > w(i-1) > w(i+1)$, so $s_{i+1} \in \mathcal{R}(w)$ by Proposition 2.2.4. If we multiply on the right by s_{i+1} then we get $ws_{i+1}(i) = w(i+1) < w(i-1) = ws_{i+1}(i-1)$, so $s_i \in \mathcal{R}(ws_{i+1})$. Then w has a reduced expression ending in $s_i s_{i+1}$. Conversely, if $w_I = s_i s_{i+1}$ then $w(i+1) < w(i)$ and $w(i-1) < w(i)$. Furthermore, since $s_i \in \mathcal{R}(ws_{i+1})$ we have $w(i+1) = ws_{i+1}(i) < ws_{i+1}(i-1) = w(i-1)$, so $(w(i-1), w(i), w(i+1))$ has the consecutive pattern 231.

Suppose that w has the consecutive pattern 312. Then there is some i such that $w(i-1) > w(i+1) > w(i)$, so $s_i \in \mathcal{R}(w)$. If we multiply on the right by s_i then we get $ws_i(i) = w(i-1) > w(i+1) = ws_i(i+1)$, so $s_{i+1} \in \mathcal{R}(ws_i)$. Then w has a reduced expression ending in $s_{i+1}s_i$. Conversely, if $w_I = s_{i+1}s_i$ then $w(i-1) > w(i)$ and $w(i+1) > w(i)$. Since $s_{i+1} \in \mathcal{R}(ws_i)$, we have $w(i+1) = ws_i(i+1) < ws_i(i) = w(i-1)$, so $(w(i-1), w(i), w(i+1))$ has the consecutive pattern 312.

Finally, we know that w has no reduced expressions beginning in two noncommuting generators s, t with $s_1 \notin \{s, t\}$ if and only if w^{-1} has no reduced expressions ending in st or ts. By the above discussion, this occurs if and only if w^{-1} avoids the consecutive patterns 321, 231, and 312. $\square$

Lemma 2.2.8. *Let $w \in W(D_n)$. Then*

(1) w has a reduced expression ending in s_1s_3 or s_3s_1 if and only if $-w(1) > w(3)$, and

(2) w has a reduced expression beginning in s_1s_3 or s_3s_1 if and only if $-w^{-1}(1) > w^{-1}(3)$.

Proof. Let $w \in W$ be such that $-w(1) > w(3)$. Then we either have $-w(2) > w(1)$ or $-w(2) \leq w(1)$.

If $-w(2) > w(1)$ then $s_1 \in \mathcal{R}(w)$. Multiplying on the right by s_1, we see that $ws_1(2) = -w(1) > w(3) = ws_1(3)$, so $s_3 \in \mathcal{R}(ws_1)$. Then w has a reduced expression ending in s_3s_1.

On the other hand, if $-w(2) \leq w(1)$ we must have $w(2) \geq -w(1) > w(3)$, so $s_3 \in \mathcal{R}(w)$. Multiplying on the right by s_3, we see that $-ws_3(2) = -w(3) > w(1) = ws_3(1)$, so $s_1 \in \mathcal{R}(ws_3)$. Then w has a reduced expression ending in s_1s_3.

Conversely, let $w \in W$ be such that w has a reduced expression ending in s_3s_1 or s_1s_3. If an expression for w ends in s_3s_1 then we have $s_1 \in \mathcal{R}(w)$ and $s_3 \in \mathcal{R}(ws_1)$, so $-w(1) = ws_1(2) > ws_1(3) = w(3)$. If an expression for w ends in s_1s_3 then we have $s_3 \in \mathcal{R}(w)$ and $s_1 \in \mathcal{R}(ws_3)$, so $-w(1) = -ws_3(1) > ws_3(2) = w(3)$.

Finally, we know that w has no reduced expressions beginning in s_1s_3 or s_3s_1 if and only if w^{-1} has no reduced expressions ending in s_1s_3 or s_3s_1. By the above discussion, this occurs if and only if $-w^{-1}(1) > w^{-1}(3)$. $\square$

Lemma 2.2.9. *Let $w \in W(D_n)$ be such that each entry in the one-line notation for w is positive and both w and w^{-1} avoid the consecutive patterns* 321*,* 231*, and* 312*. Then w is a product of commuting generators.*

Proof. Recall the embedding ι from Remark 1.1.7. Since all entries in w are positive, we can find an element $w' \in W(A_{n-1})$ such that $\iota(w') = w$. If w' has a reduced expression

$$w' = s_{i_1}s_{i_2}\cdots s_{i_r}$$

in $W(A_{n-1})$ then w has the reduced expression

$$w = s_{i_1+1}s_{i_2+1}\cdots s_{i_r+1}$$

in $W(D_n)$. By Lemma 2.1.10 the s_{i_j} are mutually commuting generators, so the s_{i_j+1} are also mutually commuting, thus w is a product of commuting generators in $W(D_n)$. $\square$

Lemma 2.2.10. *Let $w \in W(D_n)$ be weakly bad and let $i \in \mathbf{n}$. Then w satisfies the following conditions:*

(1) $w(j) > \min(\{w(i-1), w(i)\})$ for all $j > i$;

(2) $w(k) < \max(\{w(i-1), w(i)\})$ for all $k < i-1$;

(3) if $w(i), w(i+1) > 0$ then $w(j) > 0$ for all $j \geq i$;

(4) if $w(i), w(i+1) < 0$ then $w(j) < 0$ for all $j \leq i+1$.

Proof. Suppose that there is some least $j > i$ such that $w(j) \leq \min(\{w(i-1), w(i)\})$. Note that since $j > i$ we cannot have $w(j) = w(i)$ or $w(j) = w(i-1)$, so $w(j) < \min(\{w(i-1), w(i)\})$. Then $w(j-2) \geq \min(\{w(i-1), w(i)\}) > w(j)$ and $w(j-1) \geq \min(\{w(i-1), w(i)\}) > w(j)$, so we see that $(w(j-2), w(j-1), w(j))$ must have the consecutive pattern 321 or 231, which is impossible by Lemma 2.2.7, proving (1).

Suppose that there is some greatest $k < i-1$ such that $w(k) \geq \max(\{w(i-1), w(i)\})$. Note that since $k < i-1$ we cannot have $w(k) = w(i)$ or $w(k) = w(i-1)$, so $w(k) > \max(\{w(i-1), w(i)\})$. Then $w(k+1) \leq \max(\{w(i-1), w(i)\}) < w(k)$ and $w(k+2) \leq \max(\{w(i-1), w(i)\}) < w(k)$, so we see that $(w(k), w(k+1), w(k+2))$ must have the consecutive pattern 321 or 312, which is impossible by Lemma 2.2.7, proving (2).

It is easy to see that assertion (1) implies (3) and (2) implies (4). $\square$

Example 2.2.11. Let $w \in W(D_7)$ have the one-line notation given below

$$w = (2, \underline{3}, \underline{6}, 1, 4, \underline{5}, 7).$$

Then $(w(1), w(2), w(3)) = (2, -3, -6)$ has the consecutive pattern 321, and w is not bad by Lemma 2.2.7. Similarly, $(w(4), w(5), w(6)) = (1, 4, -5)$ has the consecutive pattern 231, and w is not bad by Lemma 2.2.7.

Lemma 2.2.12. *Let $w \in W(D_n)$ be a bad element. Then $(w(1), w(2), w(3))$ has one of the following consecutive signed patterns:*

$$1\underline{2}3,\ \underline{12}3,\ 1\underline{3}2,\ \underline{13}2,\ 2\underline{1}3,\ \underline{21}3.$$

In particular, we have $|w(1)| < w(3)$.

Proof. There are $2^3 \cdot 3! = 48$ possible choices of signed consecutive patterns for $(w(1), w(2), w(3))$.

123	$\underline{1}23$	$\underline{12}3$	$\boxed{\underline{1}2\underline{3}}$	$\boxed{\boxed{\underline{123}}}$	$1\underline{2}3$	$\boxed{12\underline{3}}$	$\boxed{\boxed{1\underline{23}}}$
132	$\underline{1}32$	$\underline{13}2$	$\boxed{\underline{1}3\underline{2}}$	$\underline{132}$	$1\underline{3}2$	$\boxed{13\underline{2}}$	$1\underline{32}$
213	$\underline{2}13$	$\underline{21}3$	$\boxed{\underline{2}1\underline{3}}$	$\boxed{\underline{213}}$	$2\underline{1}3$	$\boxed{\boxed{21\underline{3}}}$	$\boxed{\boxed{2\underline{13}}}$
$\boxed{231}$	$\colorbox{lightgray}{$\underline{2}31$}$	$\colorbox{lightgray}{$\underline{23}1$}$	$\colorbox{lightgray}{$\underline{2}3\underline{1}$}$	$\colorbox{lightgray}{$\underline{231}$}$	$2\underline{3}1$	$\boxed{23\underline{1}}$	$2\underline{31}$
312	$\colorbox{lightgray}{$\underline{3}12$}$	$\colorbox{lightgray}{$\underline{31}2$}$	$\colorbox{lightgray}{$\underline{3}1\underline{2}$}$	$\colorbox{lightgray}{$\underline{312}$}$	$3\underline{1}2$	$\boxed{\boxed{31\underline{2}}}$	$\boxed{\boxed{3\underline{12}}}$
$\boxed{\boxed{321}}$	$\colorbox{lightgray}{$\underline{3}21$}$	$\colorbox{lightgray}{$\underline{32}1$}$	$\colorbox{lightgray}{$\underline{3}2\underline{1}$}$	$\colorbox{lightgray}{$\underline{321}$}$	$3\underline{2}1$	$\boxed{\boxed{32\underline{1}}}$	$3\underline{21}$

We can use Lemma 2.2.7 to eliminate the possibilities that have the consecutive patterns $\boxed{\boxed{321}}$, $\boxed{231}$, or 312, and Lemma 2.2.8 to eliminate the possibilities in which $\colorbox{lightgray}{$-w(1) > w(3)$}$. This leaves us with 12 possible choices.

123	$\underline{1}23$	$\underline{12}3$	$1\underline{2}3$
132	$\underline{1}32$	$\underline{13}2$	$1\underline{3}2$
213	$\underline{2}13$	$\underline{21}3$	$2\underline{1}3$

Then we can use Lemma 2.2.10 to see that if $w(1) < 0$ and $w(2), w(3) > 0$, then $w(i) > 0$ for all $i \geq 2$, so $w \notin W(D_n)$ since w has an odd number of negative signs. Furthermore, if $w(1), w(2), w(3) > 0$ then $w(i) > 0$ for each i by Lemma 2.2.10, so w is not bad by Corollary 2.1.9. This eliminates the first and second column of possibilities, leaving us with the desired choices. $\square$

Lemma 2.2.13. *Let $w \in W(D_n)$ be bad and define l to be the maximum integer i such that $w(i)$ is negative, or 0 if no such integer exists. Then l is even and we can write*

$$w = \begin{cases} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{n/2}, b_{n/2}) & \textit{if } n \textit{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}, a_{(n+1)/2}) & \textit{if } n \textit{ is odd;} \end{cases}$$

where $k = l/2$, (a_i) and (b_i) are increasing sequences, $a_i > 0$ for $i \geq 2$, $b_i < 0$ if $i \leq k$, and $b_i > 0$ if $i > k$.

Proof. Let $w \in W(D_n)$ be bad. Then by Lemma 2.2.12 we see that $w(3)$ must be positive, so by Lemma 2.2.10 we can write

$$w = \begin{cases} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{n/2}, b_{n/2}) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}, a_{(n+1)/2}) & \text{if } n \text{ is odd;} \end{cases}$$

where $a_i > 0$ for $i \geq 2$, $b_i < 0$ if $i \leq k$, and $b_i > 0$ if $i > k$. Now we see that (a_i) must be increasing, since if $a_{i+1} < a_i$, then (a_i, b_i, a_{i+1}) would have one of the consecutive patterns 321, 231, or 312, contradicting Lemma 2.2.7. Similarly, (b_i) must be increasing, since if $b_{i+1} < b_i$, then (b_i, a_{i+1}, b_{i+1}) would have one of the consecutive patterns 321, 231, or 312, contradicting Lemma 2.2.7. $\square$

Lemma 2.2.14. *Let $w \in W_n$ be bad, let n be even, and write w as in Lemma 2.2.13. If $n = 2k$ we have*

$$w = \left((-1)^{n/2}, \underline{n}, 3, \underline{n-2}, 5, \ldots, \underline{4}, n-1, \underline{2}\right),$$

and thus $w = w^{-1}$.

Proof. First suppose that $n \equiv 0 \bmod 4$. Then k is even, so $a_1 > 0$, else w would have an odd number of negative entries. By Lemma 2.1.3 we know that w^{-1} is also bad, so we can write

$$w^{-1} = (a'_1, b'_1, a'_2, b'_2, \ldots, a'_{k'}, b'_{k'}, a'_{k'+1}, b'_{k'+1}, \ldots, a'_{n/2}, b'_{n/2})$$

as in Lemma 2.2.13. Note that w and w^{-1} each must have k negative entries, so either $k = k'$ or $k' = k - 1$ and $a'_1 < 0$.

Suppose, towards a contradiction, that $k' = k - 1$. Then

$$\{b_i\} = \{-1, -2, -4, -6, \ldots, -(n-2)\}.$$

These are the only choices for b_i since these are the places of the negative entries in w^{-1}. In other words, if $i \in \{-1, -2, -4, -6, \ldots, -(n-2)\}$ we know that $w^{-1}(i) < 0$, so the entry with absolute value i in w must be negative. Since (b_i) is increasing, we have $b_k = w(n) = -1$. But then $w^{-1}(1) = -n$, so $n = |w^{-1}(1)| > w^{-1}(3)$, which is impossible by Lemma 2.2.12. Thus, $k = k'$ and $a'_1 > 0$.

This means that $(b_i) = (b'_i) = (-n, -(n-2), \ldots, -4, -2)$. Then since (a_i) is increasing, we have $(a_i) = (a'_i) = (1, 3, 5, \ldots, n-1)$, so w has the desired form.

Next suppose that $n \equiv 2 \bmod 4$. Then k is odd, so $a_1 < 0$ else w would have an odd number of negative entries. By Lemma 2.1.3 we know that w^{-1} is also bad, so we can write

$$w^{-1} = (a'_1, b'_1, a'_2, b'_2, \ldots, a'_{k'}, b'_{k'}, a'_{k'+1}, b'_{k'+1}, \ldots, a'_{n/2}, b'_{n/2})$$

as in Lemma 2.2.13. Note that w and w^{-1} must have $k+1$ negative entries. Now we must have $k = k'$, since there is no other way for w^{-1} to have $k+1$ negative entries.

This means that $\{b_i\} \cup \{a_1\} = \{b'_i\} \cup \{a'_1\} = \{-1, -2, -4, -6, \ldots, -n\}$. Then since (a_i) is increasing we have $(a_i)_{i=2}^k = (a'_i)_{i=2}^k = (3, 5, \ldots, n-1)$.

By Lemma 2.2.12 we must have $|a_1| = |w(1)| < w(3) = 3$, so either $a_1 = -1$ or $a_1 = -2$. If $a_1 = -2$ then we must have $w(n) = b_k = -1$ since (b_i) is increasing. But this means that $w^{-1}(1) = -n$, so $n = |w^{-1}(1)| > w^{-1}(3)$, contradicting Lemma 2.2.12. Then we must have $a_1 = -1$, so $a'_1 = -1$. Then $(b_i) = (b'_i) = (-n, -(n-2), \ldots, -4, -2)$, so w has the desired form.

Once w is in the desired form we can easily see that $w = w^{-1}$. □

Lemma 2.2.15. *Let $w \in W_n$ be bad, let n be odd, and write w as in Lemma 2.2.13. If $n - 2k = 1$ we have*

$$w = \left((-1)^{(n-1)/2}, \underline{n-1}, 3, \underline{n-3}, 5, \ldots, \underline{4}, n-2, \underline{2}, n\right),$$

and thus $w = w^{-1}$.

Proof. By Lemma 2.1.3 we know that w^{-1} is also bad, so we can write

$$w^{-1} = (a'_1, b'_1, a'_2, b'_2, \ldots, a'_{k'}, b'_{k'}, a'_{k'+1}, b'_{k'+1}, \ldots, a'_{(n-1)/2}, b'_{(n-1)/2}, a'_{(n+1)/2})$$

as in Lemma 2.2.13. We see that $a'_{(n+1)/2} = w^{-1}(n) > 0$, so since all $b_i < 0$ there must be some j

such that $a_j = n$. Then since (a_i) is increasing we must have $a_{(n+1)/2} = n$. Then we have

$$w = (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}, n),$$

and the result follows by applying Lemma 2.2.14 to

$$(a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}).$$

□

Lemma 2.2.16. *Let $w \in W(D_n)$ be bad and write w as in Lemma 2.2.13. If $n - 2k > 1$ then $w^{-1}(n-1)$ and $w^{-1}(n)$ are both positive.*

Proof. We see that $w(n-1)$ and $w(n)$ are clearly positive since $n - 2k > 1$. Write w^{-1} as in Lemma 2.2.13, and suppose that $b'_i < 0$ for $i \leq k'$. If either $w^{-1}(n-1)$ or $w^{-1}(n)$ were negative then we would have $n - 2k' \leq 1$. Then by Lemma 2.2.14 or 2.2.15 we would have $w = w^{-1}$, which cannot be true since $k \neq k'$, thus $w^{-1}(n-1)$ and $w^{-1}(n)$ are positive. □

Lemma 2.2.17. *Let $w \in W(D_n)$ be bad and write w as in Lemma 2.2.13. Then $a_1 = \pm 1$, $(a_i)_{i=2}^k = (3, 5, 7, \ldots, 2k-1)$, and $(b_i)_{i=1}^k = (-2k, -(2k-2), \ldots, -6, -4, -2)$.*

Proof. We will induct on $n - 2k$. If $n - 2k = 0$ then we are done by Lemma 2.2.14 and if $n - 2k = 1$ then we are done by Lemma 2.2.15.

If $w(n) = n$ then we have

$$w = \begin{cases} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{n/2}, n) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}, n) & \text{if } n \text{ is odd,} \end{cases}$$

so we can apply the inductive hypothesis to

$$\begin{array}{ll} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{n/2}) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}) & \text{if } n \text{ is odd,} \end{array}$$

and obtain the desired result.

Suppose $w(n) \neq n$. Recall that $n-2k > 1$, so by Lemma 2.2.16, we have $w^{-1}(n) > 0$, and since (a_i) and (b_i) are increasing we must have $w(n-1) = n$.

If $w(n) \neq n-1$ and $w(n) \neq n$ then by Lemma 2.2.16 we have $w^{-1}(n-1) > 0$, so we can use the fact that (a_i) and (b_i) are increasing to show that $w(n-3) = n-1$. Then $w^{-1}(n) > 0$ and $w^{-1}(n) = n-1$, so w^{-1} must have consecutive pattern 312 or 321 where the 3 is at position $w(n)$, contradicting Lemma 2.2.7. Thus we must have have $w(n) = n-1$, so

$$w = \begin{cases} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, b_{n/2-1}, n, n-1) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, n, n-1) & \text{if } n \text{ is odd,} \end{cases}$$

and we can apply the inductive hypothesis to

$$\begin{array}{ll} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, b_{n/2-1}) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}) & \text{if } n \text{ is odd,} \end{array}$$

and obtain the desired result. □

Theorem 2.2.18. *Let $w \in W(D_n)$ be bad and define*

$$w_m = \begin{cases} \left((-1)^{m/2}, \underline{m}, 3, \underline{m-2}, 5, \ldots, \underline{4}, m-1, \underline{2}, m+1, m+2, \ldots, n\right) & \textit{if } m \textit{ is even;} \\ \left((-1)^{(m-1)/2}, \underline{m-1}, 3, \underline{m-3}, 5, \ldots, \underline{4}, m-2, \underline{2}, m, m+1, \ldots, n\right) & \textit{if } m \textit{ is odd.} \end{cases}$$

Then we must have $w = w_m u$ reduced for some $m \leq n$, where u is a product of mutually commuting generators such that $\operatorname{supp}(u) \subset \{s_{m+2}, s_{m+3}, s_{m+4}, \ldots, s_n\}$.

Proof. Let $w \in W(D_n)$ be bad. Then by Lemma 2.2.17 we can write

$$w = \begin{cases} (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{n/2}, b_{n/2}) & \text{if } n \text{ is even;} \\ (a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots, a_{(n-1)/2}, b_{(n-1)/2}, a_{(n+1)/2}) & \text{if } n \text{ is odd;} \end{cases}$$

where $a_1 = \pm 1$, $(a_i)_{i=2}^k = (1, 3, 5, 7, \ldots, 2k-1)$, and $(b_i)_{i=1}^k = (-2k, -(2k-2), \ldots, -6, -4, -2)$. Then for $j > k$ we have $a_j > 2k$ and $b_j > 2k$, so we can write

$$w = w_{2k} \cdot u$$

where

$$u = \begin{cases} (1,2,3,4,\ldots,2k-1,2k,a_{k+1},b_{k+1},\ldots,a_{n/2},b_{n/2}) & \text{if } n \text{ is even;} \\ (1,2,3,4,\ldots,2k-1,2k,a_{k+1},b_{k+1},\ldots,a_{(n-1)/2},b_{(n-1)/2},a_{(n+1)/2}) & \text{if } n \text{ is odd.} \end{cases}$$

It follows that u is a product of commuting generators by Lemma 2.2.9 because the original (a_i) and (b_i) sequences were increasing. We see that $s_1, s_2, \ldots, s_{2k+1} \notin \text{supp}(u)$, so $\text{supp}(u) \subset \{s_{2k+2}, s_{2k+3}, \ldots, s_n\}$.

□

Corollary 2.2.19. *If n is even we have*

$$w_n(i) = \begin{cases} (-1)^{n/2} & \text{if } i = 1; \\ i & \text{if } i > 1 \text{ and } i \text{ is odd;} \\ -(n+2-i) & \text{if } i \text{ is even.} \end{cases}$$

If n is odd we have

$$w_n(i) = \begin{cases} (-1)^{(n-1)/2} & \text{if } i = 1; \\ i & \text{if } i > 1 \text{ and } i \text{ is odd;} \\ -(n+1-i) & \text{if } i \text{ is even.} \end{cases}$$

Proof. This is immediate from Theorem 2.2.18 □

Eventually, we will find a reduced expression for w_n. In order to ensure that the expression that we find is reduced, we must first find the length of w_n.

Lemma 2.2.20. *We have*

$$\ell(w_n) = \begin{cases} \dfrac{3n^2}{8} + \dfrac{n}{4} & \text{if } n \text{ is even;} \\ \dfrac{3(n-1)^2}{8} + \dfrac{n-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $W = W(D_n)$ and consider $w_n \in W$. Suppose until further notice that n is even. By Theorem 2.2.18 we can write

$$w_n = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n),$$

where

$$a_1 = \pm 1, \quad a_i = 2i - 1\,(i \geq 2), \quad \text{and} \quad b_i = -(n + 2 - 2i).$$

Using Proposition 2.2.2 we see that

$$\begin{aligned}\ell(w_n) = &\sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, a_i > a_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, -a_i > a_j\}| \\ &+ \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, b_i > b_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, b_i > a_j\}| \\ &+ \sum_{i=1}^{\frac{n}{2}} |\{j \mid i \leq j, a_i > b_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i \leq j, -a_i > b_j\}| \\ &+ \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, -b_i > b_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, -b_i > a_j\}| .\end{aligned}$$

Now since $(a_i)_{i=1}^{\frac{n}{2}}$ and $(b_i)_{i=1}^{\frac{n}{2}}$ are both increasing sequences with $a_i \geq -1$ and $b_i \leq -2$ for all i, we see that the first four terms above are all equal to zero, so

$$\begin{aligned}\ell(w_n) = &\sum_{i=1}^{\frac{n}{2}} |\{j \mid i \leq j, a_i > b_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i \leq j, -a_i > b_j\}| \\ &+ \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, -b_i > b_j\}| + \sum_{i=1}^{\frac{n}{2}} |\{j \mid i < j, -b_i > a_j\}| .\end{aligned}$$

Now since $a_i > b_j$ for all $i, j \in \mathbb{N}$, we see that $a_i > b_j$ holds for all $j \geq i$, so

$$|\{j \mid i \leq j, a_i > b_j\}| = \frac{n}{2} + 1 - i.$$

Similarly, since all $b_i < 0$ and since $(b_i)_{i=1}^{\frac{n}{2}}$ is increasing we see that $-b_i > b_j$ holds for all $j > i$, so

$$|\{j \mid i \leq j, -b_i > b_j\}| = \frac{n}{2} - i.$$

Then using the above expressions for a_i and b_i we see that $i \leq j$ and $-a_i > b_j$ if and only if $i \leq j \leq \frac{n}{2} + 1 - i$, so

$$|\{j \mid i \leq j, -a_i > b_j\}| = \begin{cases} \frac{n}{2} + 2 - 2i & \text{if } i \leq \frac{n+2}{4}; \\ 0 & \text{else.} \end{cases}$$

Finally we can again use the expressions for a_i and b_i to show $i < j$ and $-b_i > a_j$ if and only if $i < j \leq \frac{n}{2} + 1 - i$, so

$$|\{j \mid i \leq j, -b_i > a_j\}| = \begin{cases} \frac{n}{2} + 1 - 2i & \text{if } i \leq \frac{n+2}{4}; \\ 0 & \text{else.} \end{cases}$$

Then we have

$$\ell(w_n) = \begin{cases} \sum_{i=1}^{\frac{n}{2}} \left(\left(\frac{n}{2} + 1 - i\right) + \left(\frac{n}{2} - i\right)\right) + \sum_{i=1}^{\frac{n}{4}} \left(\left(\frac{n}{2} + 2 - 2i\right) + \left(\frac{n}{2} + 1 - 2i\right)\right) & \text{if } n \equiv 0 \bmod 4; \\ \sum_{i=1}^{\frac{n}{2}} \left(\left(\frac{n}{2} + 1 - i\right) + \left(\frac{n}{2} - i\right)\right) + \sum_{i=1}^{\frac{n+2}{4}} \left(\left(\frac{n}{2} + 2 - 2i\right) + \left(\frac{n}{2} + 1 - 2i\right)\right) & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

If $n \equiv 0 \bmod 4$ then we have

$$\begin{aligned} \ell(w_n) &= \sum_{i=1}^{\frac{n}{2}} \left(\left(\frac{n}{2} + 1 - i\right) + \left(\frac{n}{2} - i\right)\right) + \sum_{i=1}^{\frac{n}{4}} \left(\left(\frac{n}{2} + 2 - 2i\right) + \left(\frac{n}{2} + 1 - 2i\right)\right) \\ &= \sum_{i=1}^{\frac{n}{2}} (n + 1 - 2i) + \sum_{i=1}^{\frac{n}{4}} (n + 3 - 4i) \\ &= \frac{n^2}{2} + \frac{n}{2} - 2\sum_{i=1}^{\frac{n}{2}} i + \frac{n^2}{4} + \frac{3n}{4} - 4\sum_{i=1}^{\frac{n}{4}} i \\ &= \frac{3n^2}{4} + \frac{5n}{4} - 2\left(\frac{\frac{n}{2}\left(\frac{n}{2} + 1\right)}{2}\right) - 4\left(\frac{\frac{n}{4}\left(\frac{n}{4} + 1\right)}{2}\right) \\ &= \frac{3n^2}{8} + \frac{n}{4}. \end{aligned}$$

If $n \equiv 2 \bmod 4$ then we can use a nearly identical argument to show that

$$\ell(w_n) = \frac{3n^2}{8} + \frac{n}{4}.$$

Now suppose that n is odd. Then

$$w_n = (a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, n),$$

where the a_i and b_i are as above. We see that $-a_i, a_i, -b_i, b_i < n$ for each i, so

$$\ell(w_n) = \ell(w_{n-1}) = \frac{3(n-1)^2}{8} + \frac{n-1}{4}.$$

□

2.3 Properties of bad elements

We will now find a reduced expression for the bad elements w_n. To do this it will be convenient to use interval notation.

Definition 2.3.1. For $2 \leq i \leq j$, denote the element $s_i s_{i+1} \cdots s_{j-1} s_j$ by $[i,j]$. For $i \geq 3$, denote $s_1 s_3 s_4 \cdots s_i$ by $[1,i]$ and for $j \geq 2$ denote $s_1 s_2 s_3 \cdots s_j$ by $[0,j]$. If $0 \leq j < i$ and $i \geq 2$ define $[j,i] = [i,j]^{-1}$. Finally for $i \leq -3$ and $j \geq 3$, denote $s_i s_{i-1} s_{i-2} \cdots s_4 s_3 s_1 s_2 s_3 s_4 \cdots s_j$ by $[-i,j]$.

The following two lemmas help describe how these intervals act as signed permutations.

Lemma 2.3.2. *Let $i,j,k \in \mathbb{N}$ be such that $j \geq i \geq 2$. Then as a signed permutation, we have*

$$[j,i] = (1,2,\ldots,i-2,j,i-1,i,\ldots,j-2,j-1,j+1,j+2,\ldots,n).$$

Proof. We will prove the lemma using induction on $j-i$. If $j=i$ the lemma is true by Proposition 2.2.3. Now assume that, as a signed permutation, we have

$$[j-1,i] = (1,2,\ldots,i-2,j-1,i-1,i,\ldots,j-3,j-2,j,j+1,\ldots,n).$$

Then by Proposition 2.2.3 multiplying on the left by s_j has the effect of interchanging the entries with values j and $j-1$, so we have

$$[j,i] = s_j[j-1,i] = (1,2,\ldots,i-2,j,i-1,i,\ldots,j-2,j-1,j+1,j+2,\ldots,n).$$

$\square$

Lemma 2.3.3. *Let $j,k \in \mathbb{N}$ be such that $j \geq 2$. Then as a signed permutation, we have*

$$[j,0] = \begin{cases} (\underline{1},\underline{2},3,4,\ldots,n) & \textit{if } j=2; \\ (\underline{1},\underline{j},2,3,\ldots,j-2,j-1,j+1,j+2,\ldots,n) & \textit{else.} \end{cases}$$

Proof. We will prove the lemma using induction on j. If $j=2$ the lemma is true by Proposition 2.2.3. Now assume that, as a signed permutation, we have

$$[j-1,0] = (\underline{1},\underline{j-1},2,3,\ldots,j-3,j-2,j,j+1,\ldots,n).$$

Then by Proposition 2.2.3 multiplying on the left by s_j has the effect of interchanging the entries with values j and $j-1$, so we have

$$[j,0] = s_j[j-1,0] = (\underline{1}, \underline{j}, 2, 3, \ldots, j-2, j-1, j+1, j+2, \ldots, n).$$

$\square$

Now we are ready to write down a reduced expression for w_n. We begin by finding a (not necessarily reduced) expression for w_n in terms of generators of the Coxeter group.

Lemma 2.3.4. *Using the above notation, we have*

$$w_n = \begin{cases} [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n] & \text{if } n \text{ is even;} \\ [2,0][4,0]\cdots[m-2,0][m,0][m-k,m-2k]\cdots[m-1,m-2][m,m] & \text{if } n \text{ is odd;} \end{cases}$$

where $m = n-1$ and

$$k = \begin{cases} \frac{n}{2} - 2 & \text{if } n \text{ is even;} \\ \frac{n-1}{2} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose n is even. Define

$$w_n' = [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n].$$

There are $\frac{n}{2}$ intervals in w_n' that end in 0, and all other intervals fix 1, so $w(1) = (-1)^{n/2}$.

Now suppose that $i \in \mathbf{n} \setminus \{1\}$ is odd. Then using Lemma 2.3.2 we have

$$\begin{aligned} w_n'(i) &= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n](i) \\ &= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots\left[\frac{n+i-1}{2}, i-1\right]\left[\frac{n+i+1}{2}, i+1\right](i) \\ &= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots\left[\frac{n+i-1}{2}, i-1\right]\left(\frac{n+i+1}{2}\right) \\ &= [2,0][4,0]\cdots[n-4,0][n-2,0][n,0]\left(\frac{n+i+1}{2}\right) \\ &= [2,0][4,0]\cdots[n-4,0][n-2,0]\left(\frac{n+i+1}{2} - 1\right) \\ &= [2,0][4,0]\cdots[n-4,0]\left(\frac{n+i+1}{2} - 2\right) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= [2,0][4,0]\cdots[i-1,0][i+1,0]\,(i+1) \\
&= [2,0][4,0]\cdots[i-1,0]\,(i) \\
&= i.
\end{aligned}
$$

If $i = 2$ then

$$
\begin{aligned}
w_n'(2) &= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n](2) \\
&= [2,0][4,0]\cdots[n-2,0][n,0]\,(2) \\
&= [2,0][4,0]\cdots[n-2,0]\,(-n) \\
&\vdots \\
&= -n.
\end{aligned}
$$

Next suppose that $i > 2$ is even. Then

$$
\begin{aligned}
w_n'(i) &= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots[n-1,n-2][n,n](i) \\
&= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots\left[\frac{n+i-2}{2}, i-2\right]\left[\frac{n+i}{2}, i\right](i) \\
&= [2,0][4,0]\cdots[n-2,0][n,0][n-k,n-2k]\cdots\left[\frac{n+i-2}{2}, i-2\right](i-1) \\
&\vdots \\
&= [2,0][4,0]\cdots[n-2,0][n,0]\left(\frac{i+2}{2}\right) \\
&\vdots \\
&= [2,0][4,0]\cdots[n-i,0][n+2-i,0](2) \\
&= [2,0][4,0]\cdots[n-i,0](-(n+2-i)) \\
&= -(n+2-i).
\end{aligned}
$$

Then $w_n'(i) = w_n(i)$ for all $i \in \mathbf{n}$ by Corollary 2.2.19, so $w_n' = w_n$.

If n is odd we see that $w_n = w_{n-1}$ in $W(D_n)$, so they must have the same reduced expression.

$\square$

Lemma 2.3.5. *The expression for w_n in Lemma 2.3.4 is reduced.*

Proof. Suppose that n is even. By Lemma 2.2.20 we have $\ell(w_n) \leq \frac{3}{8}n^2 + \frac{1}{4}n$. Let r be the number of generators in the given expression for w_n. Then we have

$$\begin{aligned} r &= \sum_{i=1}^{\frac{n}{2}} 2i + \sum_{i=1}^{\frac{n}{2}-1} i \\ &= 2\sum_{i=1}^{\frac{n}{2}} i + \sum_{i=1}^{\frac{n}{2}} i - \frac{n}{2} \\ &= \frac{3}{2} \cdot \frac{n}{2}\left(\frac{n}{2}+1\right) - \frac{n}{2} \\ &= \frac{3}{8}n^2 + \frac{1}{4}n. \end{aligned}$$

Then $\ell(w_n) = \frac{3}{8}n^2 + \frac{1}{4}n$ and the above expression for w_n is reduced. We can use an identical argument to show that the expression for w_n is reduced if n is odd. □

Recall that a bad element is an element that is not a product of commuting generators and that has no reduced expressions beginning or ending in two noncommuting generators.

Theorem 2.3.6. *Let $W = W(D_n)$. Then there is a unique longest bad element, $w_n \in W$. Every other bad element in W is of the form $w_k \cdot u$ where $k < n$ and where u is a product of mutually commuting generators not in* $\operatorname{supp}(w_k)$. *Furthermore, if n is odd then $w_n = w_{n-1}$.*

Proof. Let $n \in \mathbb{N}$ and let $w \in W$ be bad. Then by Theorem 2.2.18 we must have $w = w_k \cdot u$ reduced where $k \leq n$ and u is a product of mutually commuting generators that are not in $\operatorname{supp}(w_k)$. Write $i = n - k$. Since $\operatorname{supp}(u) \subset \{s_{k+2}, s_{k+3}, \ldots, s_n\}$ we see $\ell(u) < (i/2)$. Then $\ell(w) = \ell(w_k) + \ell(u)$ so by Lemma 2.2.20

$$\begin{aligned} \ell(w) &\leq \frac{3}{8}(n-i)^2 + \frac{1}{4}(n-i) + \frac{i}{2} \\ &\leq \frac{3}{8}n^2 + \frac{1}{4}n + \frac{1}{8}i \cdot (-6n + 3i + 2). \end{aligned}$$

Since $n \geq 4$ and $n \geq i$ we see that $(-6n + 3i + 2) < 0$, so $l(w) \leq (3/8)n^2 + (1/4)n = l(w_n)$. □

Corollary 2.3.7. *If $w \in W(D_n)$ is bad then $w \notin W_c$.*

Proof. By Theorem 2.3.6 we can write $w = w_m \cdot u$ reduced where $4 \leq m \leq n$ and u is a product of mutually commuting generators that are not in $\mathrm{supp}(w_m)$. Since $w_m = w_{m-1}$ when m is odd we may assume that m is even. By Lemma 2.3.4 we have

$$w_m = [2,0][4,0]\cdots[m-2,0][m,0][m-k,m-2k]\cdots[m-1,m-2][m,m]$$

where $k = \frac{m}{2} - 2$. We can write $[m-k,m-2k] = s_{m-k}[m-k-1,m-2k]$ and commute s_{m-k} to the left to obtain

$$w_m = [2,0]\cdots[m,m-k+1]\cdot s_{m-k}s_{m-k-1}s_{m-k}\cdot[m-k-2,0][m-k-1,m-2k]\cdots[m,m],$$

so $w = w' \cdot s_{m-k}s_{m-k-1}s_{m-k} \cdot w''u$ reduced for some $w', w'' \in W$, and thus $w \notin W_c$. $\square$

In [11], Green uses the fact that Coxeter groups of type $\widetilde{A}$, and therefore of type A, have no bad elements to show that $\mu(x,w) \in \{0,1\}$ if x is fully commutative. In type D we can use the proof of [11, Theorem 3.1] to show that $\mu(x,w) \in \{0,1\}$ if x is fully commutative and w is not bad. However, the case where w is bad is much harder. We will show that we can reduce the case where w is bad to calculating $\mu(x_n, w_n)$, where w_n is the longest bad element in $W(D_n)$ and where $x_n = \prod_{s\in\mathcal{L}(w)} s$. We see that since $w_n = w_{n+1}$ for even n we only need to calculate $\mu(x_n, w_n)$ for even n. Furthermore, if $n \equiv 2$ or $4 \bmod 8$, then $\ell(w_n) - \ell(x_n)$ is even, so $\mu(x_n, w_n) = 0$. However, if $n \equiv 0$ or $6 \bmod 8$ the calculation is much harder.

As we will see, Lusztig's a-function gives us a way to bound the degree of $P_{e,w}$ for an element $w \in W$. With this in mind, we first look at P_{e,w_n}.

Definition 2.3.8. Let $n \geq 4$. Then define

$$x_n = \begin{cases} s_1s_2s_4s_6\cdots s_{n-2}s_n & \text{if } n \text{ is even;} \\ s_1s_2s_4s_6\cdots s_{n-3}s_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Note that x_n is a product of mutually commuting generators.

Lemma 2.3.9. *We have $P_{x_n,w_n} = P_{e,w_n}$.*

Proof. We repeatedly apply Proposition 1.2.7 (1), starting with $x = e$, $w = w_n$, and taking s from the set $\mathcal{L}(x_n) = \{s_1, s_2, s_4, s_6, \ldots, s_{n-2}, s_n\}$. □

This tells us that $\mu(x_n, w_n)$ is equal to the coefficient of $q^{\frac{1}{2}(\ell(w_n)-\ell(x_n)-1)}$ in P_{e,w_n}. Using Proposition 1.3.5, if we calculate $a(w_n)$ we can find a bound for the degree of P_{e,w_n}. This will allow us to calculate $\mu(x_n, w_n)$ in certain cases.

In order to simplify the calculation of $a(w_n)$, we will next find an element u_n in the same two-sided cell as w_n. The calculation of $a(u_n)$ will be made easy by lemmas 1.3.7 and 1.3.8. To find such an element we will use domino tableaux in order to calculate the Kazhdan–Lusztig cells of $W(D_n)$.

Lemma 2.3.10. *Recall the definition of star reducible from Definition 1.1.27. Let $w \in W$ be such that $w \notin W_c$. Then w is star reducible to either*

(1) a bad element; or

(2) an element x such that either $\mathcal{L}(x)$ or $\mathcal{R}(x)$ is not commutative.

Proof. We will proceed by induction on $\ell(w)$. We see that since $w \notin W_c$ we must have $\ell(w) \geq 3$ and we know that w is not a product of commuting generators. If $\ell(w) = 3$ then $w = sts$ where s and t are noncommuting generators, so we are done. Suppose that $\ell(w) = r$. If w is not bad and both $\mathcal{L}(w)$ and $\mathcal{R}(w)$ are commutative then we have either $w \in st \cdot w'$ reduced or $w = w' \cdot ts$ reduced for some pair of noncommuting generators s and t and some $w' \in W$. Without loss of generality suppose $w = st \cdot w'$ reduced. Then ${}^*w = t \cdot w'$, so $\ell({}^*w) = r - 1$, and by induction ${}^*w$ is star reducible to either a bad element or an element x such that either $\mathcal{L}(x)$ or $\mathcal{R}(x)$ is not commutative, thus w is star reducible to either a bad element or an element x such that either $\mathcal{L}(x)$ or $\mathcal{R}(x)$ is not commutative. □

Corollary 2.3.11. *Let $w \in W$. Then w is star reducible to either*

(1) a product of mutually commuting generators,

(2) a bad element, or

(3) an element x such that either $\mathcal{L}(x)$ or $\mathcal{R}(x)$ is not commutative.

Proof. This is immediate from Proposition 1.1.28 and Lemma 2.3.10. $\square$

Chapter 3

Kazhdan–Lusztig cells and domino tableaux

If $W = W(A_n)$, we can use the Robinson–Schensted algorithm to describe the left and right Kazhdan–Lusztig cells. Each element of of w may be viewed as a permutation of $\{1, 2, \ldots, n+1\}$. We can use the Robinson–Schensted algorithm to construct an $(n+1)$-tableau, $T(w)$, corresponding to w as in [4, 4.1]. Then $x, w \in W$ are in the same left (respectively, right) cell if and only if $T(x) = T(w)$ (respectively, $T(x^{-1}) = T(w^{-1})$) [20, Theorem 1.7.2].

If $W = W(D_n)$, we can use an algorithm similar to the Robinson–Schensted algorithm to construct tableaux with dominoes. This algorithm was developed by D. Garfinkle in [5]. We assign each element $w \in W$ a so-called domino tableau, $T_L(w)$. Unlike in type A, two elements of a single cell W may have two different domino tableaux. However, we define the notion of cycles of domino tableaux, and use the notion of moving a tableau through a cycle to create a new domino tableau. Then we introduce an equivalence class, $\approx$, on tableaux such that $T(w) \approx T(y)$ if and only if we can obtain $T_L(w)$ from $T_L(y)$ by moving through cycles. Once we do this, we will see that the partition of W induced by $\approx$ is exactly the left cells of W. We will then use these domino tableaux to find an element, u_n, in the same two-sided cell as w_n with a relatively easy to calculate a-value.

In Section 3.1 we begin with an account of Garfinkle's definitions corresponding to domino tableaux as seen in [5]. In Section 3.2 we define the algorithm developed by Garfinkle in [5] used to assign a pair of tableaux to an element of W. In Section 3.3 we follow Garfinkle's methods in [5] to define the notion of cycles of a domino tableau and use these cycles to define the equivalence relation $\approx$.

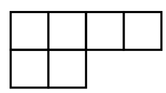
In Chapter 4 we will apply Garfinkle's notion of cycles to tableaux of our bad elements. This allows us to use $\approx$ to find an new element u_n whose a-value is relatively easy to calculate. We conclude by calculating $a(w_n)$.

3.1 Definitions

Let $\mathbb{N}$ be the set of natural numbers starting with 1 and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ define $\mathbf{n} = \{i \in \mathbb{N} \mid i \leq n\}$. Consider the set $\{S_{i,j} \mid i, j \in \mathbb{N}_0\}$ of positions in the quadrant, where i denotes the row, increasing left to right, and j denotes the column, increasing top to bottom.

Definition 3.1.1. Let $\mathcal{F} = \{S_{i,j}\}_{i,j \in \mathbb{N}}$, and let $\mathcal{F}^0 = \{S_{i,j}\}_{i,j \in \mathbb{N}_0}$. We call the elements of $\mathcal{F}$ and $\mathcal{F}^0$ **squares**.

Example 3.1.2. Let $J = \{S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}, S_{2,1}, S_{2,2}\}$. We can visualize J as a box diagram:

$J =$  .

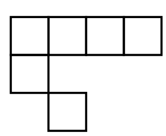
Definition 3.1.3. A subset $J \subseteq \mathcal{F}$ is a **Young diagram** if it satisfies all of the following conditions:

(1) J is finite;

(2) for each i there exists i_j such that $S_{i,k} \in J$ if and only if $0 \leq k \leq i_j$;

(3) for each j there exists j_i such that $S_{j,k} \in J$ if and only if $0 \leq k \leq j_i$.

Example 3.1.4. In Example 3.1.2, above, J is a Young diagram. However, if we let

$J' =$ 

then we see that J' is not a Young diagram.

Definition 3.1.5. Let $J \subset \mathcal{F}$ be a Young diagram. Then define

(1) $\rho_i(J) = \max\{0, \max\{j \mid S_{i,j} \in J\}\}$, and

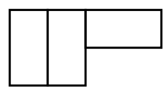
(2) $\kappa_j(J) = \max\{0, \max\{i \mid S_{i,j} \in J\}\}$.

Remark 3.1.6. We see that $\rho_i(J)$ is the number of boxes in the ith row of J, and $\kappa_j(J)$ is the number of boxes in the jth column of J.

Example 3.1.7. If J is as in Example 3.1.2, then $\rho_1(J) = 4$ and $\kappa_1(J) = 2$.

Definition 3.1.8. A subset $D \subset \mathcal{F}$ is a called a **domino** if $D = \{S_{i,j}, S_{i,j+1}\}$ or $D = \{S_{i,j}, S_{i+1,j}\}$ for some $i, j \in \mathbb{N}$.

Example 3.1.9. Let J be as in Example 3.1.2. The pairs $\{S_{1,1}, S_{2,1}\}$, $\{S_{1,2}, S_{2,2}\}$, and $\{S_{1,3}, S_{1,4}\}$ are all dominoes. With these pairings in mind we can write J as a disjoint union of dominoes:

$J =$  .

Definition 3.1.10. Let $M \subseteq \mathbb{N}$ be finite. Define projections

$$\pi_1 : \mathcal{F} \times M \to \mathcal{F} : (f, m) \mapsto f$$

and

$$\pi_2 : \mathcal{F} \times M \to M : (f, m) \mapsto m.$$

Let $T \subseteq \mathcal{F} \times M$ satisfy the following conditions:

(1) $\pi_1|_T$ is injective;

(2) $\pi_1(T)$ is a Young diagram;

(3) $(\pi_2)^{-1}(k)$ is a domino for all $k \in M$.

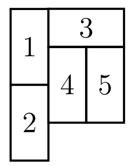
(4) Suppose $(S_{i,j}, k) \in T$. Then if $(S_{i,j+1}, k_1) \in T$ or $(S_{i+1,j}, k_2) \in T$ we have $k \leq k_1, k_2$.

Then we call T a **domino tableau**. Let $\mathcal{T}(M)$ be the set of all domino tableaux.

Example 3.1.11. Using the above definition we see that each element $T \in \mathcal{T}(M)$ is a disjoint union of labeled dominoes in the shape of a Young diagram, such that the labels increase along rows and columns. For example, let $M = \mathbf{5}$, and let

$$T = \{(S_{1,1}, 1), (S_{1,2}, 3), (S_{1,3}, 3), (S_{2,1}, 1), (S_{2,2}, 4), (S_{2,3}, 5), (S_{3,1}, 2), (S_{3,2}, 4), (S_{3,3}, 5), (S_{4,1}, 2)\}.$$

Then $T \in \mathcal{T}(M)$ and we write

$T =$  .

Definition 3.1.12. Let $M \subset \mathbb{N}$, let $T \in \mathcal{T}(M)$ and let $k \in M$.

(1) The **domino with label** k **in** T is given by $D(T, k) = (\pi_2)^{-1}(k)$.

(2) The **position of domino with label** k **in** T is given by $P(T, k) = \pi_1(D(T, k))$.

(3) The **shape** of T is $\mathrm{Shape}(T) = \pi_1(T)$.

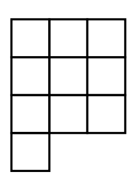
(4) Let $i, j \in \mathbb{N}$. Then define a map

$$N : \mathcal{T}(M) \times \mathcal{F}^0 \to M \cup \{0, \infty\}$$

by

$$N(T, S_{i,j}) = \begin{cases} k & \text{if } (S_{i,j}, k) \in T; \\ 0 & \text{if } i = 0 \text{ or } j = 0; \\ \infty & \text{else.} \end{cases}$$

Example 3.1.13. Let T be as in Example 3.1.11. We see that $D(T, 3) = \{(S_{1,2}, 3), (S_{1,3}, 3)\}$ and $P(T, 3) = \{S_{1,2}, S_{1,3}\}$. Also, $\mathrm{Shape}(T)$ is the unlabeled tableau with the same outline as T. Thus

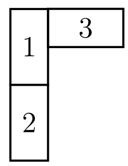
$\mathrm{Shape}(T) =$  .

Finally, $N(T, S_{i,j})$ is the label of block $S_{i,j}$ in T, if it exists. Otherwise, we say that $N(T, S_{i,j}) = 0$ if either $i = 0$ or $j = 0$, and $N(T, S_{i,j}) = \infty$ otherwise. Then, $N(T, S_{2,3}) = 5$, $N(T, S_{0,2}) = 0$, and $N(T, S_{4,2}) = \infty$.

Definition 3.1.14. Let $M = \{e_1, e_2, \ldots, e_k\}$ and let $T \in \mathcal{T}(M)$. Then we define

$$T(j) = T \setminus \left(\bigcup_{e_i > j} D(T, e_i) \right).$$

Example 3.1.15. We note that $T(j)$ is obtained from T by removing all dominoes with label strictly greater than j. Then using Example 3.1.11 we see that

$T(3) =$  .

We are eventually working towards a way to add a domino to a tableau to create a new tableau. To do this, we will next define a way to shuffle a domino that overlaps with a particular tableau into a position that allows it to fit into the tableau without overlapping.

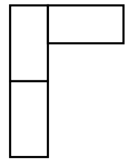
Definition 3.1.16. Let $J \subset \mathcal{F}$ be a Young diagram and let $P = \{S_{i,j}, S_{i,j+1}\}$ (respectively, $P = \{S_{i,j}, S_{i,j+1}\}$) be a domino. We define $A(J, P)$ in the following cases:

(1) If $j = \rho_i(J) + 1$ (respectively, $i = \kappa_j(J) + 1$) then $A(J, P) = P$.

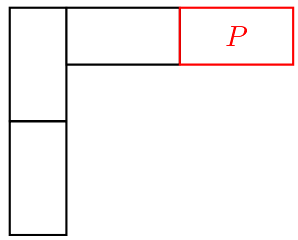
(2) If $j = \rho_i(J) - 1$ (respectively, $i = \kappa_j(J) - 1$) and $\rho_{i+1}(J) < j$ (respectively, $\kappa_{j+1}(J) < i$) then $A(J, P) = \{S_{i+1,r} S_{i+1,r+1}\}$ (respectively, $A(J, P) = \{S_{r,j+1} S_{r+1,j+1}\}$) where $r = \rho_{i+1}(J) + 1$ (respectively, $r = \kappa_{j+1}(J) + 1$).

(3) If $j = \rho_i(J)$ (respectively, $i = \kappa_j(J)$) and $\rho_{i+1}(J) = j$ (respectively, $\kappa_{j+1}(J) = i$) then $A(J, P) = \{S_{i,j+1}, S_{i+1,j+1}\}$ (respectively, $A(J, P) = \{S_{i+1,j}, S_{i+1,j+1}\}$).

Example 3.1.17. Let J be given as below, and let $P = \{S_{1,4}, S_{1,5}\}$, $Q = \{S_{1,2}, S_{1,3}\}$, and $R = \{S_{2,1}, S_{2,2}\}$:

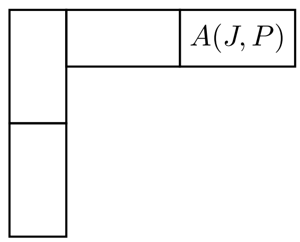
$J =$  .

Then

$J \cup P =$ 

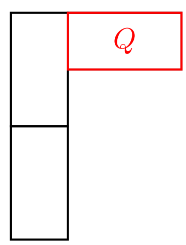
,

so we are in case (1) of Definition 3.1.16. Thus

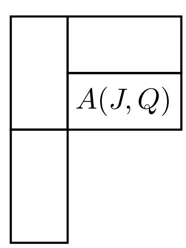
$J \cup A(J, P) =$ 

.

We also see that

$J \cup Q =$ 

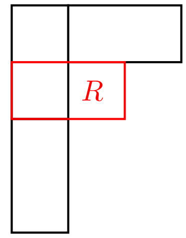
so we are in case (2) of Definition 3.1.16. Thus

$J \cup A(J, Q) =$ 

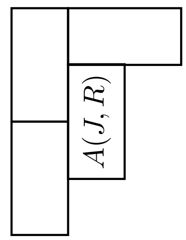
.

Note that the tableau $J \cup A(J, Q) =$ would not be well-defined if we had $S_{2,2} \in J$. However this would violate the condition that $\rho_{i+1}(J) < j$ for $i = 1$ and $j = 2$, so the resulting tableau is guaranteed to be well-defined.

Similarly,

$J \cup R =$ 

so we are in case (3) of Definition 3.1.16. Thus

$J \cup A(J, R) =$ 

.

As before, the tableau $J \cup A(J, Q) =$ would not be well-defined if we had $S_{3,2} \in J$. However this would violate the condition that $\rho_i(J) = j$ for $i = 2$ and $j = 1$, so the resulting tableau is well-defined.

3.2 Constructing tableaux

We will now define an algorithm that assigns a domino tableau to each element in $W(D_n)$. Let $W = W(D_n)$ and let $w \in W$. We will construct a domino tableau from w using the signed permutation representation of w. Using a method similar to row insertion in standard tableaux, we add labeled dominoes to a tableau. If $w^{-1}(k)$ is positive we initially add a horizontal domino with label k. If $w^{-1}(k)$ is negative then we initially add a vertical domino with label k. When dominoes overlap, we employ a shuffling technique that allows us to create a valid domino tableau. We begin by defining a way to write each w as a set of ordered triples,

$$\{(k, |w(k)|, \operatorname{sign}(w(k))) \mid k \in \mathbf{n}\}.$$

Let $M_1, M_2 \subset \mathbb{N}$ be finite with $|M_1| = |M_2|$ and let p_1, p_2, and p_3 be the projections of $M_1 \times M_2 \times \{\pm 1\}$ onto the first, second, and third coordinate, respectively. Let

$$\mathcal{W}(M_1, M_2) = \left\{ w \subset M_1 \times M_2 \times \{\pm 1\} \;\middle|\; \begin{array}{c} p_1|_w \text{ and } p_2|_w \text{ are bijections and} \\ \left|\left(p_3^{-1}(-1) \cap w\right)\right| \equiv 0 \bmod 2 \end{array} \right\}.$$

Let $W = W(D_n)$. Then we can realize each element of W as a signed permutation. Define the map

$$\delta : W \to \mathcal{W}(\mathbf{n}, \mathbf{n})$$

by $\delta(w) = \{(i, |w(i)|, \operatorname{sign}(w(i))) \mid i \in \mathbf{n}\}$.

Example 3.2.1. Let $w = w_4 = (1, \underline{4}, 3, \underline{2}) \in W(D_4)$. Then

$$\delta(w) = \{(1,1,1), (2,4,-1), (3,3,1), (4,2,-1)\}.$$

Definition 3.2.2. Let $M \subset \mathbb{N}$ be finite. Define

$$\mathcal{A}(M) = \{(T, v, \epsilon) \mid v \in M, T \in \mathcal{T}(M \setminus \{v\}), \epsilon \in \{\pm 1\}\}.$$

We can think of an element of $\mathcal{A}(M)$ as a domino tableau on the set $M \setminus \{v\}$ paired with a horizontally aligned domino with label v if $\epsilon = 1$, or a vertically aligned domino with label v if $\epsilon = -1$. We proceed by defining a map from $\mathcal{A}(M)$ to $\mathcal{T}(M)$. This will allow us to add the domino with label v to the tableau to form a new tableau using all elements of M. We will use the idea of shuffling introduced in Definition 3.1.16.

Definition 3.2.3. We define a map $\alpha : \mathcal{A}(M) \to \mathcal{T}(M)$ inductively. Let $M = \{e_1, \cdots, e_n\} \subset \mathbb{N}$ be such that $e_1 < e_2 < \cdots < e_n$ and let $v = e_j$. Suppose that α is defined for all M' with $|M'| < n$. Let $(T, v, \epsilon) \in \mathcal{A}(M)$. We have two cases.

Case 1. Suppose $v = e_n$. Then

$$\alpha(T, v, \epsilon) = \begin{cases} T \cup \{(S_{1,\rho_1(T)+1}, e_n), (S_{1,\rho_1(T)+2}, e_n)\}, & \text{if } \epsilon = 1, \\ T \cup \{(S_{\kappa_1(T)+1,1}, e_n), (S_{\kappa_1(T)+2,1}, e_n)\}, & \text{if } \epsilon = -1. \end{cases}$$

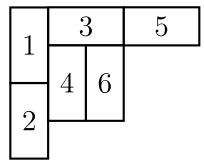
Case 2. Suppose $v < e_n$. Then let $T' = \alpha(T(e_{n-1}), v, \epsilon)$, which is defined by the inductive hypothesis. Then define

$$\alpha(T, v, \epsilon) = T' \cup \{(S, e_n) \mid S \in A(T', P(T, e_n))\}.$$

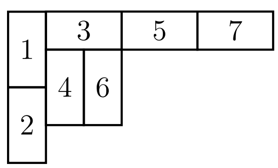
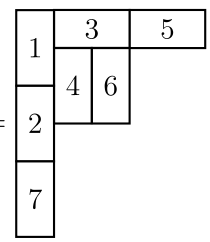
Lemma 3.2.4. *The map α is well defined on all of $\mathcal{A}(M)$, and maps into $\mathcal{T}(M)$.*

Proof. This is a consequence of [5, Proposition 1.3.4, Definition 1.2.5]. $\square$

Example 3.2.5. Let $M = \mathbf{7}$ and consider $T \in \mathcal{T}(M \setminus \{7\})$ as defined below

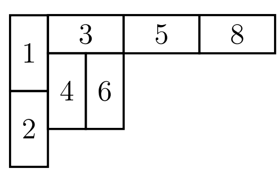
$T =$  .

Then $(T, 7, 1), (T, 7, -1) \in \mathcal{A}(M)$, and since $7 = \max(M)$ we can calculate $\alpha(T, 7, 1)$ and $\alpha(T, 7, -1)$ using case (1) of Definition 3.2.3.

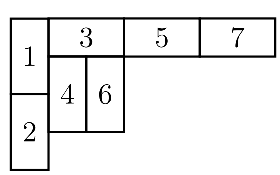
$\alpha(T, 7, 1) =$  , $\alpha(T, 7, -1) =$  .

Note that we obtained the tableaux above by simply adding a domino with label 7 to the first row or column of T.

Example 3.2.6. Let $M = \mathbf{8}$ and consider $T \in \mathcal{T}(M \setminus \{7\})$, below

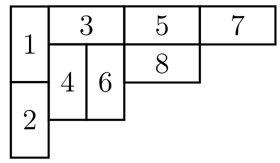
$T =$  .

As above, we see that $(T, 7, 1), (T, 7, -1) \in \mathcal{A}(M)$. Now since $7 < \max(M)$, we will use case (2) of Definition 3.2.3 to find $\alpha(T, 7, 1)$. First we will remove all dominoes with labels strictly greater than 7 and add a horizontal domino with label 7 at the end of the first row. Next, we shuffle in the removed dominoes. We first use case (1) of Definition 3.2.3 to see that

$T' = \alpha(T(6), 7, 1) =$  .

Then $P(T,8) = \{S_{1,6}, S_{1,7}\}$, and $A(T', P(T,8)) = \{S_{2,4}, S_{2,5}\}$, so

$\alpha(T', 7, 1) =$



.

Definition 3.2.7. Let $M_1, M_2 \subset \mathbb{N}$ with $|M_1| = |M_2| = m$. Let $u = \max(M_2)$, and let $w \in \mathcal{W}(M_1, M_2)$. Then there exists $v \in M_1$ and $\epsilon \in \{\pm 1\}$ such that $(v, u, \epsilon) \in w$, and we define

$$w_{(m)} = w \setminus \{(v, u, \epsilon)\}.$$

Example 3.2.8. Let $w = \{(1,1,1), (2,4,-1), (3,3,1), (4,2,-1)\}$. Then

$$w_{(4)} = \{(1,1,1), (3,3,1), (4,2,-1)\}.$$

Now we are ready to assign a tableau, $T(w)$ to each element, $w \in W = W(D_n)$. We will start by assuming that $T\left(w_{(n)}\right)$ is defined by induction, and then use α to add a domino with label n to obtain $T(w)$.

Definition 3.2.9. Let $M_1, M_2 \subset \mathbb{N}$ be such that $|M_1| = |M_2| = m$. We will define a map,

$$\widehat{T}(M_1, M_2) : \mathcal{W}(M_1, M_2) \to \mathcal{T}(M_1),$$

by induction. Suppose that $\widehat{T}(M_1', M_2')$ is defined when $|M_1'| = |M_2'| < m$. Let $w \in \mathcal{W}(M_1, M_2)$ and let $(v, u, \epsilon) = w \setminus w_{(m)}$. Then

$$\widehat{T}(w) = \alpha\left(\widehat{T}\left(M_1 \setminus \{v\}, M_2 \setminus \{u\}\right)\left(w_{(m)}\right), v, \epsilon\right).$$

Let $w \in W(D_n)$, then define $T_L(w) = \widehat{T}(\mathbf{n}, \mathbf{n})(\delta(w))$.

We further define $T_R(w) = T_L(w^{-1}) \in \mathcal{T}(M_2)$.

Remark 3.2.10. Given $w \in W(D_n)$ we can calculate $T_L(w)$ in the following way. Write

$$\delta(w) = \{(w^{-1}(i), i, \epsilon_i) \mid i \in \mathbf{n}\}.$$

Suppose that we have constructed a tableau, $T_L^{j-1}(w)$, with dominoes $w^{-1}(1), \ldots, w^{-1}(j-1)$. We can obtain a new tableau that includes domino $w^{-1}(j)$ by setting $T_L^j(w) = \alpha\left(T_L^{j-1}(w), w^{-1}(j), \epsilon_j\right)$.

To construct $T_L^j(w)$, first write down $T_L^{j-1}(w)(w^{-1}(i))$. Now add $w^{-1}(i)$ as a vertical domino at the end of the first column if $\epsilon_i = -1$ and a horizontal domino at the end of the first row if $\epsilon_i = 1$. Finally, shuffle in the remaining $w^{-1}(k)$-labeled dominoes in increasing order in the $A\left(T_L^{j-1}(w)(k-1), P\left(T^{j-1}, w^{-1}(k)\right)\right)$ position.

Example 3.2.11. Let $w = w_6$. Then as a signed permutation we have

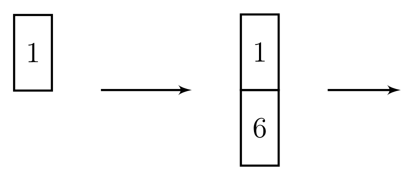
$$w = (\underline{1}, \underline{6}, 3, \underline{4}, 5, \underline{2}),$$

so

$$\delta(w) = \{(1,1,-1), (6,2,-1), (3,3,1), (4,4,-1), (5,5,1), (2,6,-1)\}.$$

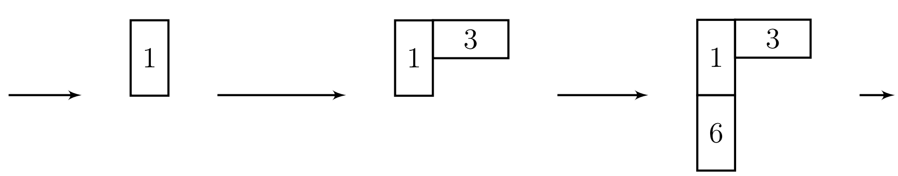
We construct $T_L(w)$ using the process outlined in Remark 3.2.10.

Since $w^{-1}(1) = -1$, we begin by adding a vertical domino with label 1 to obtain $T_L^1(w)$. Then since $w^{-1}(2) = -6$ and since 6 is larger than all labels in $T_L^1(w)$ we add a vertical domino with label 6 at the end of the first column to obtain $T_L^2(w)$.



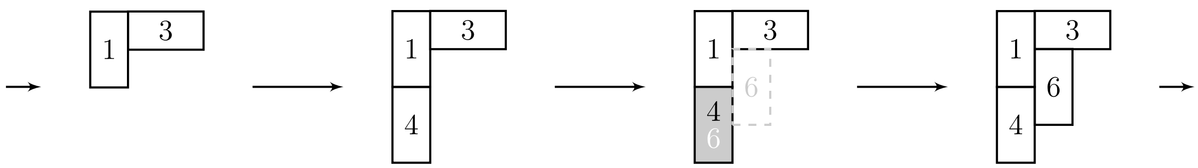
$T_L^1(w)$ $T_L^2(w)$

Now $w^{-1}(3) = 3$ so we must next add a horizontal domino with label 3. However, we must first remove all dominoes with labels larger than 3. Once we have done this we are able to add a horizontal domino with label 3 to the end of the first row. We place the domino with label 6 back in its original position because there is no overlap.



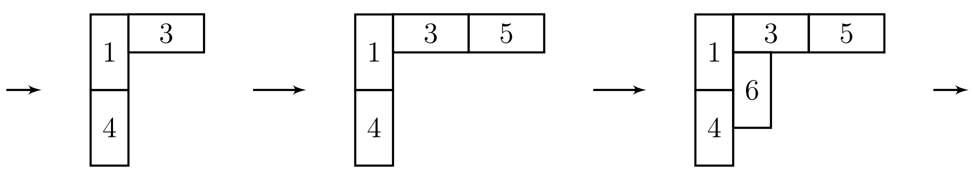
$T_L^2(w)(3)$ $T_L^3(w)$

Next, we see that $w^{-1}(4) = -4$, so we will add a vertical domino with label 4. As before, we must first remove all dominoes with label greater than 4, and then we are free to add a vertical domino with label 4 to the end of the first column. However, when we try to replace the domino with label 6 it now overlaps with the domino with label 4. As a result, we must use the A map from Definition 3.1.16 to add a domino with label 6. We see that since the domino with label 6 overlaps completely with the tableau, we are in case (2) of the definition, so the A map has the effect of bumping the domino with label 6 to the right.



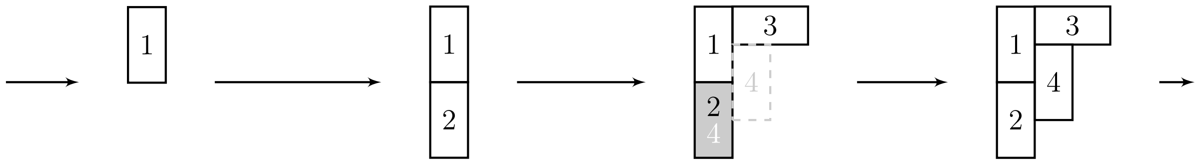
$T_L^3(w)(4)$ $\qquad$ $T_L^4(w)$

Now $w^{-1}(5) = 5$, so we will next add a horizontal domino with label 5. We remove all dominoes with label greater than 5, then add a domino with label 5 at the end of the first row. We are then able to replace the remaining dominoes without overlap.



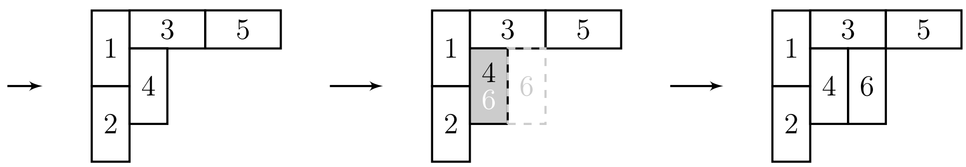
$T_L^4(w)(5)$ $\qquad$ $T_L^5(w)$

Finally, $w^{-1}(6) = -2$, so the last domino that we will add is a vertical domino with label 2. We remove all dominoes with label greater than 2 and add a domino with label 2 at the end of the first column. Now we must add in the dominoes that we removed. The domino with label 3 does not overlap with any other dominoes, so we may add it in its former position. The former position of the domino with label 4 is now fully occupied by another domino, so we must use the A map from Definition 3.1.16. As above, we are in case (2) of the definition, so the domino with label 4 gets bumped to the right.



$T_L^5(w)(2)$

When we try to replace the domino with label 6 we see that its former position is fully occupied by the domino with label 4. We again use the A map from Definition 3.1.16, which has the effect of bumping the domino with label 6 to the right.



$T_L^6(w) = T_L(w)$

At this point we have added all dominoes to the tableau, so $T_L^6(w) = T_L(w)$.

Proposition 3.2.12. *Let $W = W(D_n)$. Then the map*

$$W \to \mathcal{T}(\mathbf{n}) \times \mathcal{T}(\mathbf{n}) : w \mapsto (T_L(w), T_R(w))$$

is an injection.

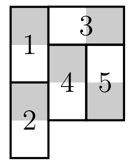
Proof. This is proved in [5, Theorem 1.2.13]. $\square$

3.3 Cycles of tableaux

The partition of $W = W(D_n)$ into sets with the same left tableau is finer than the partition of W into left cells [17]. However, following the work of Garfinkle in [5], we can use the notion of cycles to define an equivalence relation on tableaux that corresponds to the partition of W into left cells. We begin with some preliminary definitions.

Definition 3.3.1. Let $S_{i,j} \in \mathcal{F}$. If $i+j$ is even then we say that the square $S_{i,j}$ is **fixed**. If $S_{i,j} \in \mathcal{F}$ is not fixed then we say that $S_{i,j}$ is **variable**.

Example 3.3.2. Let T be as in Example 3.1.11. Then the fixed squares of T are those that are shaded below

$T =$  .

It is easy to see that if $T \in \mathcal{T}(M)$ and $k \in M$ then $P(T,k)$ contains exactly one fixed square. We will now introduce a way to move dominoes within a tableau in such a way that the fixed squares are not affected.

Definition 3.3.3. Let $M \subset \mathbb{N}$ be finite and let $T \in \mathcal{T}(M)$. Pick $k \in M$. Let $S_{i,j}$ be the fixed square in $P(T,k)$, and find $l, m \in \mathbb{N}$ such that $P(T,k) = \{S_{i,j}, S_{l,m}\}$. Let

$$r = \begin{cases} N(T, S_{i-1,j+1}) & \text{if } l > i \text{ or } m < j; \\ N(T, S_{i+1,j-1}) & \text{if } l < i \text{ or } m > j. \end{cases}$$

Then we define a new domino, $P'(T,k)$, as follows. If $l > i$ or $m < j$ then

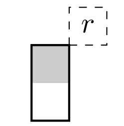
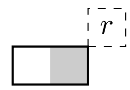
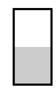
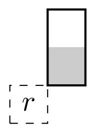
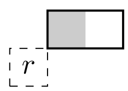
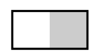
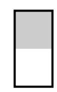
$$P'(T,k) = \begin{cases} \{S_{i,j}, S_{i-1,j}\} & \text{if } r > k; \\ \{S_{i,j}, S_{i,j+1}\} & \text{if } r < k. \end{cases}$$

If $l < i$ or $m > j$ then

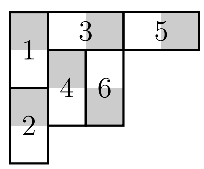
$$P'(T,k) = \begin{cases} \{S_{i,j}, S_{i+1,j}\} & \text{if } r < k; \\ \{S_{i,j}, S_{i,j-1}\} & \text{if } r > k. \end{cases}$$

Note that $P(T,k) \cap P'(T,k) = S_{i,j}$ is a fixed square. Define $D'(T,k) = \{(S,k) \mid S \in P'(T,k)\}$.

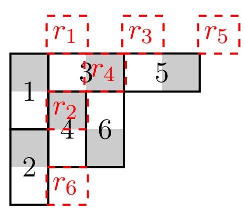
Remark 3.3.4. Let $M \subset \mathbb{N}$ be finite, let $k \in M$, let $T \in \mathcal{T}(M)$, and let r be defined as above. Then the following table summarizes the relationship between $P(T,k)$ and $P'(T,k)$. The shaded squares in the table below correspond to fixed squares.

Position	$P(T,k)$	$P'(T,k)$	
		$r > k$	$r < k$
$l > i$ or $m < j$	 		
$l < i$ or $m > j$	 		

Example 3.3.5. Let T be defined as in Example 3.2.11. The shaded squares in the diagram below correspond to fixed squares.

$T =$  .

The squares containing the r-values associated to each $k \in \mathbf{6}$ are outlined below,

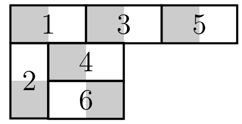


.

We list the r-values in the following table:

k	1	2	3	4	5	6
r_k	0	4	0	3	0	∞

Now we can use these values to help calculate $P'(T,k)$ for each $k \in \mathbf{6}$. For example, if we consider $P(T,1)$ we see that $r = 0$, so $r < k$ and thus $P'(T,1) = \{S_{1,1}, S_{1,2}\}$. Similarly, considering $P(T,6)$ we see that $r = \infty$, so $r > k$, and thus $P'(T,6) = \{S_{3,3}, S_{3,2}\}$. We can repeat this process to create a new tableau

$$T' = \{D'(T,k) \mid k \in \mathbf{6}\} =$$  .

Remarkably, by [5, Proposition 1.5.27] if $T \in \mathcal{T}(M)$ then we are guaranteed $T' \in \mathcal{T}(M)$. Observe that each fixed square has the same label in both T and T'.

We can now these new P' dominoes to define an equivalence relation, $\sim$, on M.

Definition 3.3.6. Let $M \subset \mathbb{N}$ be finite, let $T \in \mathcal{T}(M)$, and let $a, b \in M$. Then $\sim$ is the equivalence relation generated by $a \sim b$ if $P(T,a) \cap P'(T,b)$ is nonempty.

The equivalence relation $\sim$ partitions M into sets called **cycles**. We call a cycle C **closed** if $N(T, P(T,k) \setminus S_{i,j}) \in C$ for all $k \in C$, where $S_{i,j}$ is the fixed square of $P(T,k)$. If a cycle is not closed we call the cycle **open**.

Example 3.3.7. Let T be as in Example 3.3.5. Then since each of $P'(T,2) \cap P(T,1)$, $P'(T,1) \cap P(T,3)$, and $P'(T,3) \cap P(T,5)$ is nonempty, we know that $C_1 = \{1,2,3,5\}$ is a cycle. However since the variable square in $P'(T,5)$ is not in T, we have $N(T, P'(T,5) \setminus P_{1,6}) = \infty \notin C_1$, so C_1 is an open cycle.

Similarly, $P(T,4) \cap P'(T,6)$ is nonempty, so $C_2 = \{4,6\}$ is a cycle. In this case, the variable squares in $P'(T,4)$ and $P'(T,6)$ both overlap with T. We have $N(T, P'(T,4) \setminus P_{2,3}) = 6 \in C_2$ and $N(T, P'(T,6) \setminus P_{3,2}) = 4 \in C_2$, so C_2 is a closed cycle.

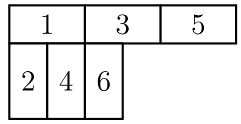
Definition 3.3.8. Let $T \in \mathcal{T}(M)$ be a domino tableau, and let C be a cycle. Define

$$E(T,C) = (T \setminus \{D(T,k) \mid k \in C\}) \cup \{D'(T,k) \mid k \in C\}.$$

Moreover, if $C_1, C_2, \ldots, C_n \subset M$ are cycles, then we define

$$E(T, C_1, C_2, \ldots, C_n) = E(\cdots E(E(T,C_1), C_2), \cdots, C_n).$$

Example 3.3.9. Again, let T be as in Example 3.3.5. Let $C = \{1,2,3,5\}$. To construct $E(T,C)$ we replace $D(T,k)$ with $D'(T,k)$ for each $k \in C$. Then

$E(T,C) =$  .

Proposition 3.3.10. *If $T \in \mathcal{T}(M)$ and $C \subset M$ is a cycle then $E(T,C) \in \mathcal{T}(M)$. In addition, $E(T,C_1,C_2) = E(T,C_2,C_1)$.*

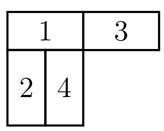
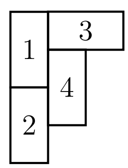
Proof. This follows from [5, Proposition 1.5.27, Proposition 1.5.31]. □

Definition 3.3.11. Let $T, T' \in \mathcal{T}(M)$. If $T' = E(T, C_1, C_2, \ldots, C_n)$ for cycles C_i of T then we say that we can move from T to T' through the sequence of cycles $C_1, C_2, \ldots, C_n$.

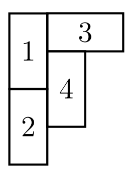
We define $T \approx T'$ if and only if we can move from T to T' through a (possibly empty) sequence of open cycles.

Remark 3.3.12. The relation $\approx$ is an equivalence relation on elements of $\mathcal{T}(M)$. It is immediately apparent that $\approx$ is reflexive and transitive, and symmetry follows from [5, Proposition 1.5.28].

Example 3.3.13. Let T and T' be given below

$T =$  , $T' =$  .

Then it can be shown that $C = \{1,2,3\}$ is an open cycle in T, and if we move T through C we obtain

$E(T,C) =$  $= T'$,

so $T \approx T'$.

Remarkably, these domino tableau can help us calculate the cells of a Coxeter group. As we will see in the following theorem, the partition of a Coxeter group W into cells corresponds to the partition of $\mathcal{T}(M)$ generated by $\sim$. Two elements $x, w \in W$ are in the same cell if and only if we can move $T_L(x)$ through open cycles to obtain $T_L(w)$.

Theorem 3.3.14. *Let $x, w \in W$. Then $x \sim_L w$ if and only if $T_L(x) \approx T_L(w)$.*

Proof. See the discussion in [17, Section 3]. □

In [5], [6] and [7] Garfinkle proves a version of Theorem 3.3.14 for Coxeter groups of type B using the following method. Let $W = W(B_n)$ be a Coxeter group with simple root system Π. For each adjacent $\alpha, \beta \in \Pi$, Garfinkle defines operators $T_{\alpha\beta}$. These operators are defined both on certain subsets of W and on the corresponding type B domino tableaux. Applying a sequence of $T_{\alpha\beta}$ operators to an element of W is equivalent to moving the corresponding domino tableau through a sequence of open cycles [7, Theorem 3.2.2 and Proposition 3.2.3].

As defined in [7, Definition 3.4.1], let $\mathbf{T} = \{T_{\alpha\beta} | \alpha, \beta \in \Pi \text{ are adjacent}\}$ and let $\{T_i\}_{i=0}^{k} \subset \mathbf{T}$. In [7, Theorem 3.5.11] Garfinkle proved that two elements $x, w \in W$ lie in the same left cell if and only if the following two conditions hold:

(1) $T_k(T_{k-1}(\cdots T_0(x)\cdots))$ is defined if and only if $T_k(T_{k-1}(\cdots T_0(w)\cdots))$ is defined;

(2) the resulting elements must have the same generalized τ-invariant.

Most of the proof in type D follows as in type B. However, in type D we do not have to worry about defining the $T_{\alpha\beta}$ operator when α and β have different lengths, but the branch node introduces complications. We have to define a new operator, T_D, that corresponds to the four simple roots in the Dynkin diagram that form a system of type D_4 [18, Discussion preceding Lemma 3.1]. The definition of T_D is given in [9, Theorem 2.15] and [18, Discussion preceding Lemma 3.1].

The set of operators $\{T_{\alpha\beta} | \alpha, \beta \in \Pi \text{ are adjacent}\} \cup \{T_D\}$ then preserve left cells. As in type B, applying a sequence of operators from $\mathbf{T}$ to an element of W is equivalent to moving the corresponding domino tableau through a sequence of open cycles [8], [9, 4.1].

Corollary 3.3.15. *Let $x, w \in W$. Then $x \sim_R w$ if and only if $T_R(x) \approx T_R(w)$.*

Proof. By definition $x \sim_R w$ if and only if $x^{-1} \sim_L w^{-1}$, which by Theorem 3.3.14 happens if and only if $T_L(x^{-1}) \approx T_L(w^{-1})$, so $T_R(x) \approx T_R(w)$ by Definition 3.2.9. □

Chapter 4

Calculating a-values of bad elements

4.1 Constructing $T_L(w_n)$

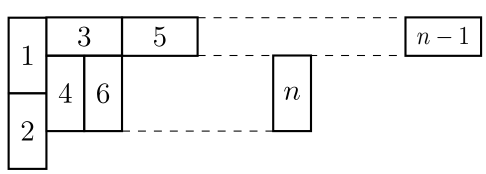
We can use Theorem 3.3.14 to better understand the two sided cells of bad elements in type D by computing their domino tableaux. Since the unique longest bad element, w_n, in $W(D_n)$ is an involution, we have $T_L(w_n) = T_R(w_n)$, so it suffices to calculate $T_L(w_n)$. Furthermore, by Lemma 2.3.4, w_n and w_{n+1} have the same reduced expressions for even n, so we will only consider the case where n is even.

Recall the signed permutation representation of w_n from Theorem 2.2.18:

$$w_n = \left((-1)^{n/2}, \underline{n}, 3, \underline{n-2}, 5, \ldots, \underline{4}, n-1, \underline{2}\right).$$

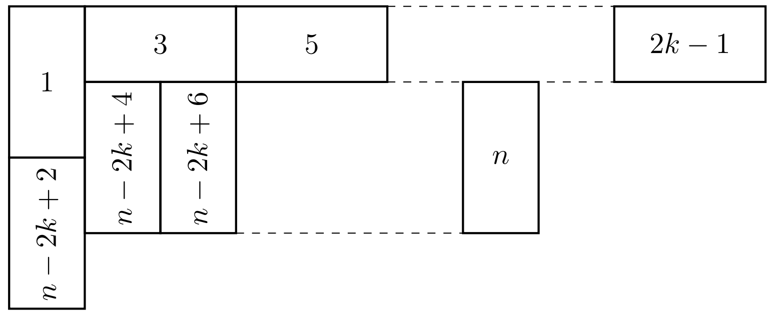
Lemma 4.1.1. *Let* $n \equiv 2 \bmod 4$ *and let* $w_n \in W(D_n)$ *be the unique longest bad element. Then we have*

$T_L(w_n) =$



.

Proof. We will build up $T_L(w_n)$ following the method used in Remark 3.2.10. It will be enough to show that after $2k$ steps we obtain

$T_L^{2k}(w_n) =$ 

.

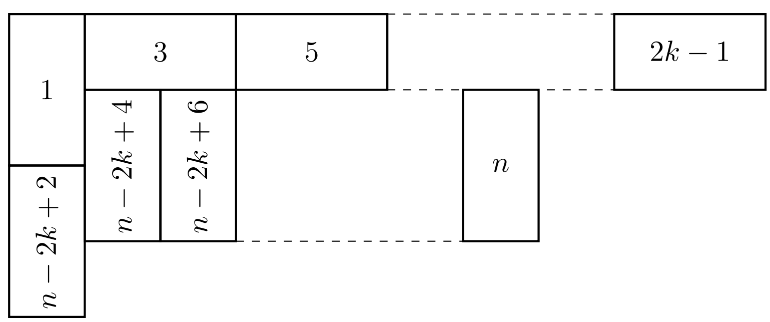
Note that $k \leq \frac{n}{2}$, so $T_L^{2k}(w_n)$ is always a valid domino tableau. Then $T_L(w_n)$ will take n steps to build, so if we set $k = \frac{n}{2}$ we obtain the desired result.

We will proceed by induction on k. For the base case, let $k = 1$. Then since $w_n^{-1}(1) = -1$ and $w_n^{-1}(2) = -n$ we have

$$T_L^2(w_n) = \begin{array}{|c|}\hline 1 \\ \hline n \\ \hline \end{array},$$

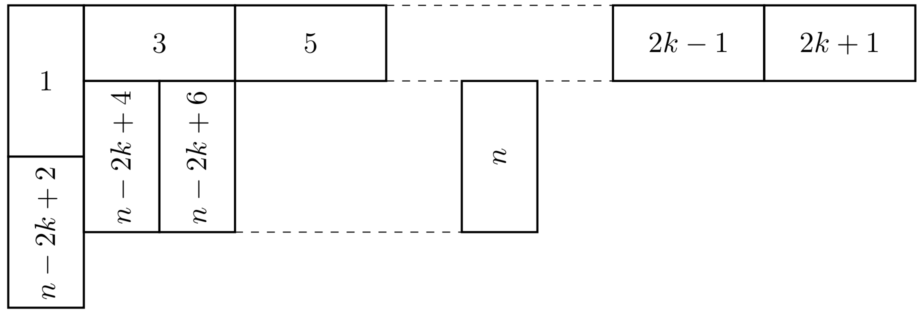
and the hypothesis holds when $k = 1$.

Suppose that $k > 1$ and we have created the first $2k$ steps of $T_L(w_n)$, $T_L^{2k}(w_n)$, using Remark 3.2.10, resulting in

$T_L^{2k}(w_n) =$ 

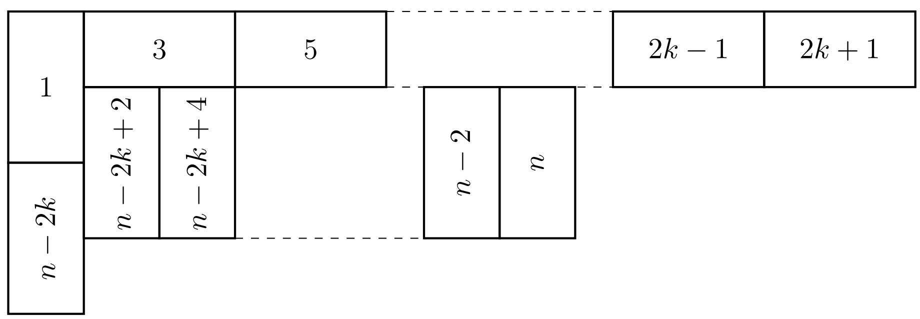
.

Now since $w_n^{-1}(2k+1) = 2k+1$ when $k > 1$, we must next add a domino with label $2k+1$. We first add the $2k+1$ domino, which simply gets added on to the right end of the top row:

$T_L^{2k+1}(w_n) =$ 

.

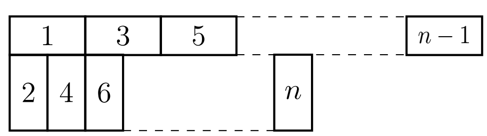
Then $w_n^{-1}(2k+2) = -(n-2k)$, so we next add a vertical domino with label $n-2k$. We first remove all dominoes with labels greater than $n-2k$. Then the domino with label $n-2k$ is inserted in the place of the $n-2k+2$ domino, which has the effect of bumping the even dominoes to the right:

$T_L^{2k+2}(w_n) =$ 

.

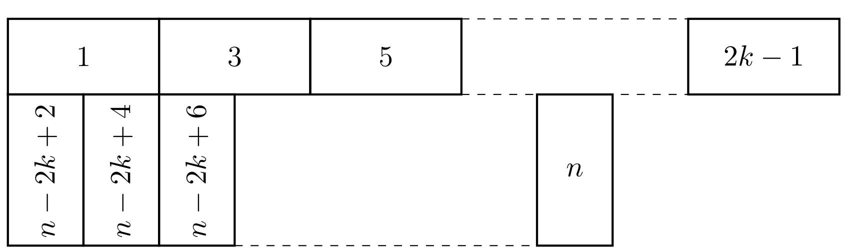
□

Lemma 4.1.2. *Let $n \equiv 0 \bmod 4$ and let $w_n \in W(D_n)$ be the unique longest bad element. Then we have*

$T_L(w_n) =$ 

.

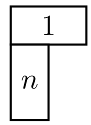
Proof. Now suppose that $n \equiv 0 \bmod 4$. As in Lemma 4.1.1, we will build up $T_L(w_n)$ following the method used in Remark 3.2.10. It will be enough to show that after $2k$ steps we obtain

$T_L^{2k}(w_n) =$ 

.

Note that $k \leq \frac{n}{2}$, so $T_L^{2k}(w_n)$ is always a valid domino tableau. Then $T_L(w_n)$ will take n steps to build, so if we set $k = \frac{n}{2}$ we obtain the desired result.

We will proceed by induction on k. For the base case, let $k = 1$. Then since $w_n^{-1}(1) = 1$ and $w_n^{-1}(2) = -n$ we have

$T_L^2(w_n) =$ 

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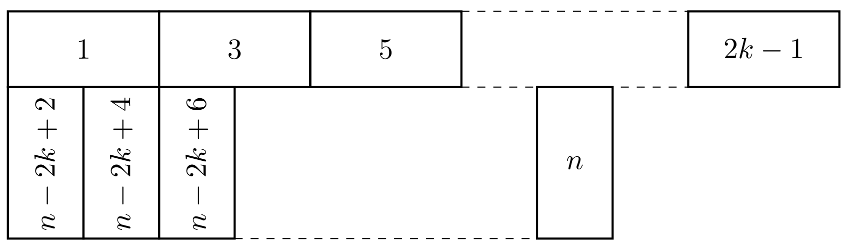
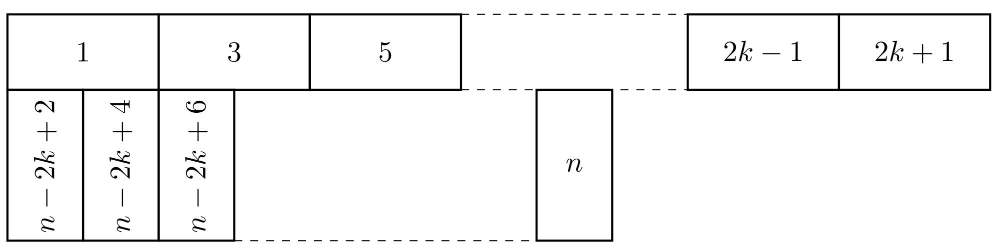
so the hypothesis holds when $k = 1$.

Suppose that $k > 1$ and we have created the first $2k$ steps of $T_L(w_n)$, $T_L^{2k}(w_n)$, using Remark 3.2.10, resulting in

$T_L^{2k}(w_n) =$ 

.

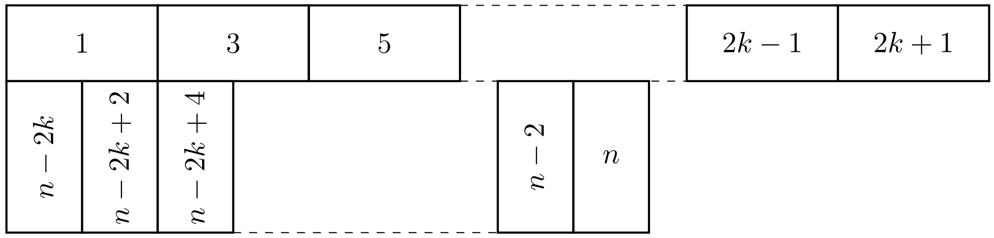
Now since $w_n^{-1}(2k+1) = 2k+1$ when $k > 1$ we must next add a domino with label $2k+1$. We first add the $2k+1$ domino, which simply gets added on to the right end of the top row:

$T_L^{2k+1}(w_n) =$ 

.

Then $w_n^{-1}(2k+2) = -(n-2k)$, so we next add a vertical domino with label $n-2k$. We first remove all dominoes with labels greater than $n-2k$. Then the domino with label $n-2k$ is inserted in the place of the $n-2k+2$ domino, which has the effect of bumping the even dominoes to the right:

$T_L^{2k+2}(w_n) =$



.

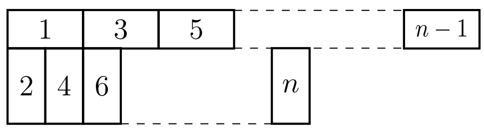
$\square$

4.2 An element in the left cell of w_n

Now that we have computed $T_L(w_n)$ we can use Theorem 3.3.14 to find an element, v_n, in the same left cell as w_n. We will move $T_L(w_n)$ through cycles to help us find v_n.

Lemma 4.2.1. *Let $n \equiv 2 \bmod 4$. Each $T_L(w_n)$ contains an open cycle $C_1 = \{1, 2, 3, 5, 7, 9 \ldots, n-1\}$. If we move $T_L(w_n)$ through C_1 then we obtain*

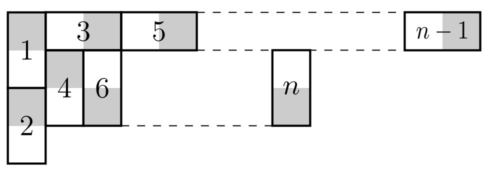
$E(T_L(w_n), C_1) =$



.

Proof. By Lemma 4.1.1

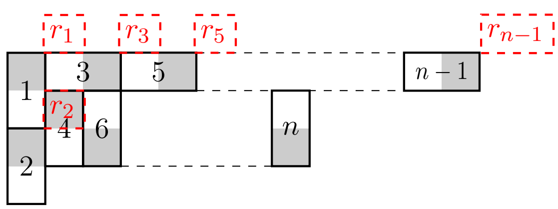
$T_L(w_n) =$



,

where the shaded squares above correspond to fixed squares. We now label r_k for $k \in C_1$ as in Definition 3.3.3.

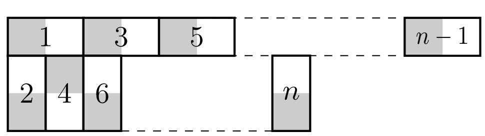
$T_L(w_n) =$



.

Now for $k \in \{1, 3, 5, \ldots, n-1\}$ we have $0 = r_k < k$, and $4 = r_2 > 2$. Then if we replace $D(T_L(w_n), k)$ with $D'(T_L(w_n), k)$ for $k \in C_1$ in $T_L(w_n)$, we obtain

$(T_L(w_n))' =$



.

Observe that each intersection

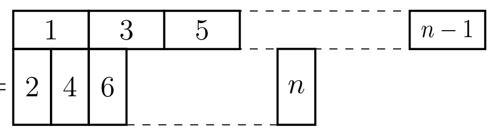
$$P'(T_L(w_n), 2) \cap P(T_L(w_n), 1)$$

$$P'(T_L(w_n), 1) \cap P(T_L(w_n), 3)$$

$$\vdots$$

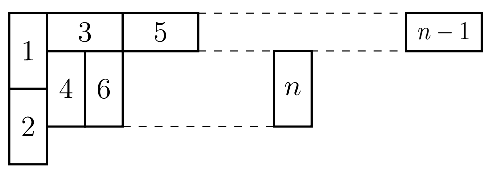
$$P'(T_L(w_n), n-3) \cap P(T_L(w_n), n-1)$$

is nonempty. Furthermore, the variable square in $P(T_L(w_n), 2)$ does not lie in $(T_L(w_n))'$ and the variable square of $P'(T_L(w_n), n-1)$ does not lie in $T_L(w_n)$, so C_1 is an open cycle, and

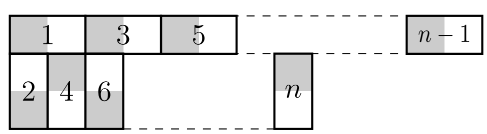
$E(T_L(w_n), C_1) =$  .

∎

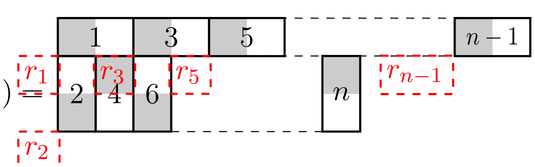
Lemma 4.2.2. *Let $n \equiv 0 \bmod 4$. Each $T_L(w_n)$ contains an open cycle $C_1 = \{1, 2, 3, 5, 7, 9 \ldots, n-1\}$. If we move $T_L(w_n)$ through C_1 then we obtain*

$E(T_L(w_n), C_1) =$  .

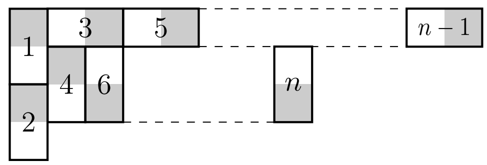
Proof. By Lemma 4.1.2

$T_L(w_n) =$  ,

where the shaded squares above correspond to fixed squares. We now label r_k for $k \in C_1$ as in Definition 3.3.3.

$T_L(w_n) =$  .

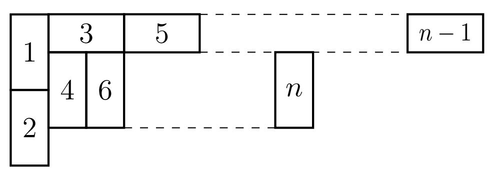
Now for $k \in \{1,2\}$ we have $0 = r_k < k$, and otherwise we have $r_k > k$. Then if we replace $D(T_L(w_n),k)$ with $D'(T_L(w_n),k)$ for $k \in C_1$ in $T_L(w_n)$, we obtain

$(T_L(w_n))' =$  .

Observe that each intersection

$$P(T_L(w_n),2) \cap P'(T_L(w_n),1)$$
$$P(T_L(w_n),1) \cap P'(T_L(w_n),3)$$
$$\vdots$$
$$P(T_L(w_n),n-3) \cap P'(T_L(w_n),n-1)$$

is nonempty. Furthermore, the variable square in $P'(T_L(w_n),2)$ does not lie in $T_L(w_n)$ and the variable square of $P(T_L(w_n),n-1)$ does not lie in $(T_L(w_n))'$, so C_1 is an open cycle, and

$E(T_L(w_n),C_1) =$  .

□

Now we can move $T_L(w_n)$ through C_1 and use Theorem 3.3.14 to find another element in the same left cell as w_n.

Lemma 4.2.3. *Suppose that $n \geq 6$ is even and let $v_n = s_n s_{n-1} s_n w_{n-2}$. Then we have the following:*

(1) $v_n = \left((-1)^{(n-2)/2}, \underline{n}, 3, \underline{n-4}, 5, \underline{n-6}, 7, \ldots, n-3, \underline{2}, n-1, n-2\right)$*;*

(2) $T_L^{n-3}(v_n) = T_L^{n-3}(w_{n-2})$.

Proof. We will prove the lemma using domino tableaux. We first find the signed permutation representation of v_n. Consider $w_{n-2} \in W(D_n)$. By Theorem 2.2.18 we have

$$w_{n-2} = \left((-1)^{(n-2)/2}, \underline{n-2}, 3, \underline{n-4}, 5, \ldots, \underline{4}, n-3, \underline{2}, n-1, n\right).$$

Now we can use Proposition 2.2.3 to find v_n. We have

$$\begin{aligned} s_n w_{n-2} &= \left((-1)^{(n-2)/2}, \underline{n-2}, 3, \underline{n-4}, 5, \ldots, \underline{4}, n-3, \underline{2}, n, n-1\right), \\ s_{n-1}s_n w_{n-2} &= \left((-1)^{(n-2)/2}, \underline{n-1}, 3, \underline{n-4}, 5, \ldots, \underline{4}, n-3, \underline{2}, n, n-2\right), \text{ and} \\ s_n s_{n-1}s_n w_{n-2} &= \left((-1)^{(n-2)/2}, \underline{n}, 3, \underline{n-4}, 5, \ldots, \underline{4}, n-3, \underline{2}, n-1, n-2\right), \end{aligned}$$

so

$$v_n = \left((-1)^{(n-2)/2}, \underline{n}, 3, \underline{n-4}, 5, \underline{n-6}, 7, \ldots, n-3, \underline{2}, n-1, n-2\right).$$

Now since w_{n-2} is an involution, we have $v_n^{-1} = w_{n-2}s_n s_{n-1} s_n$ so by Proposition 2.2.3 we have

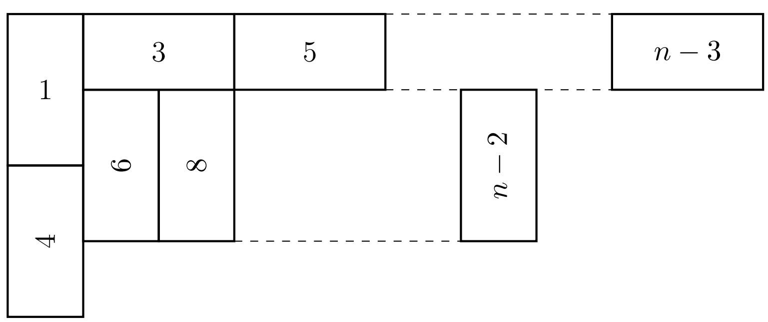
$$\left(v_n^{-1}(1), v_n^{-1}(2), \ldots, v_n^{-1}(n-3)\right) = \left(w_{n-2}^{-1}(1), w_{n-2}^{-1}(2), \ldots, w_{n-2}^{-1}(n-3)\right),$$

and $T_L^{n-3}(v_n) = T_L^{n-3}(w_{n-2})$. □

Lemma 4.2.4. *If $n \geq 6$ and $n \equiv 0 \bmod 4$, then $w_n \sim_L v_n$.*

Proof. We see that $n-2 \equiv 2 \bmod 4$, so by Lemmas 4.1.1 and 4.2.3 we have

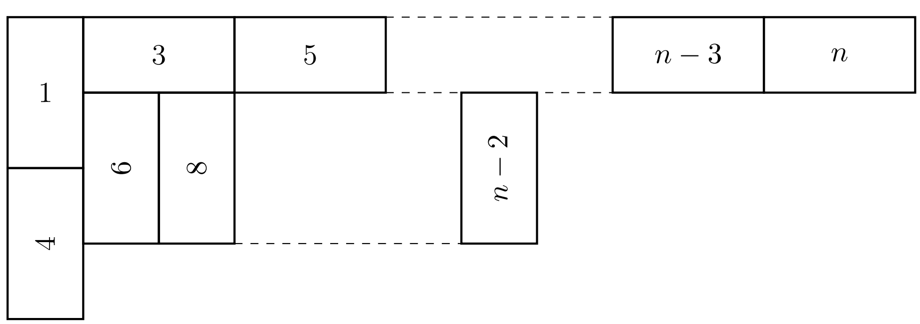
$T_L^{n-3}(v_n) =$



.

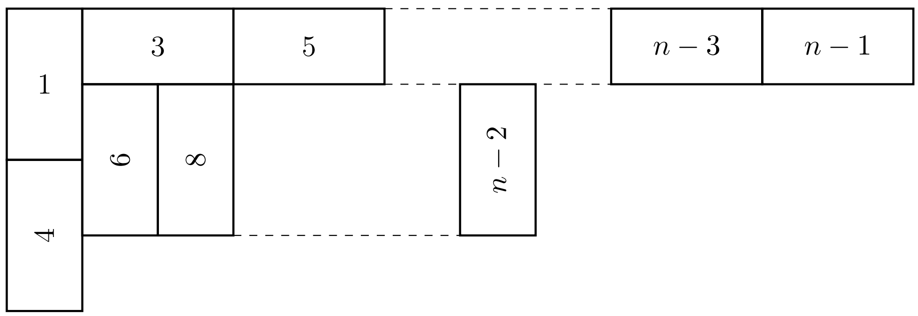
We have $v_n^{-1}(n-2) = n$, so to obtain $T_L^{n-2}(v_n)$ we see that we add a horizontal domino with label n on the end of the first row to get

$T_L^{n-2}(v_n) =$



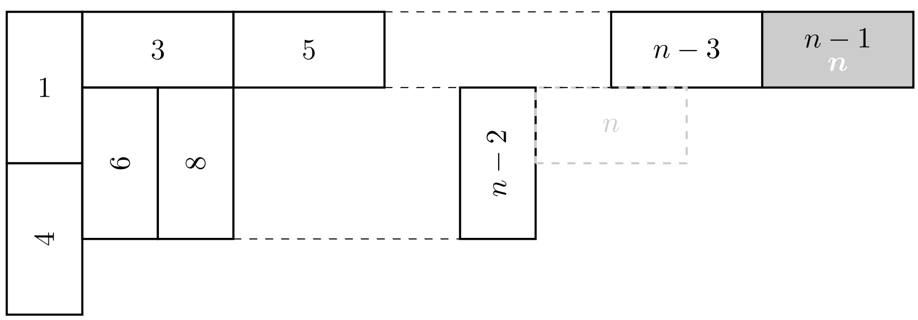
.

Next, $v_n^{-1}(n-1) = n-1$, so we must next add a horizontal domino with label $n-1$. To do this we must first remove all dominoes with labels greater than $n-1$, then add a horizontal domino with label $n-1$ to the end of the first row.



.

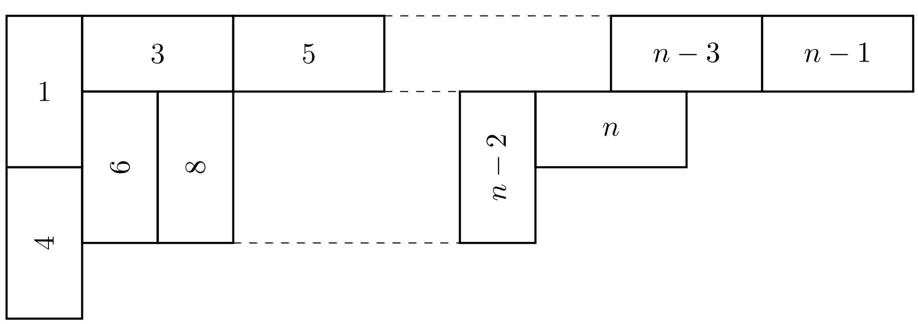
When we try to replace the domino with label n, we see that it fully overlaps with the domino with label $n-1$,



,

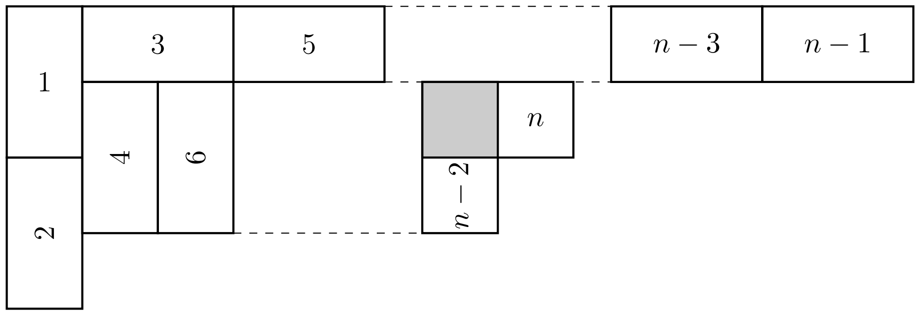
so we use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the second row

$T_L^{n-1}(v_n) =$



.

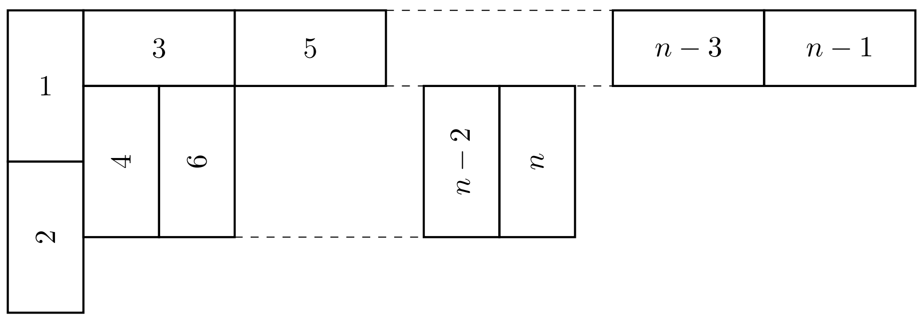
Finally, $v_n^{-1}(n) = -2$, so to complete the calculation of $T_L(v_n)$ we must add a vertical domino with label 2. As in Example 3.2.11, this bumps all dominoes with even labels less than n to the right. However, when we try to replace the domino with label n it partially overlaps with the domino with label $n-2$.



.

Now we use case (3) of Definition 3.1.16 to shuffle the domino with label n into position that allows it to fit into the tableau

$T_L(v_n) =$



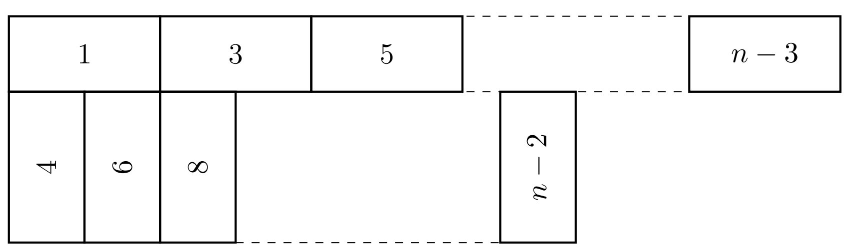
.

Then by Lemma 4.2.1 we have $T_L(v_n) = E(T_L(w_n), C_1)$, so $T_L(v_n) \approx T_L(w_n)$, thus by Theorem 3.3.14, we have $w_n \sim_L v_n$. ☐

Lemma 4.2.5. *If $n \geq 6$ and $n \equiv 2 \bmod 4$, then $w_n \sim_L v_n$.*

Proof. We see that $n - 2 \equiv 0 \bmod 4$, so by Lemmas 4.1.2 and 4.2.3 we have

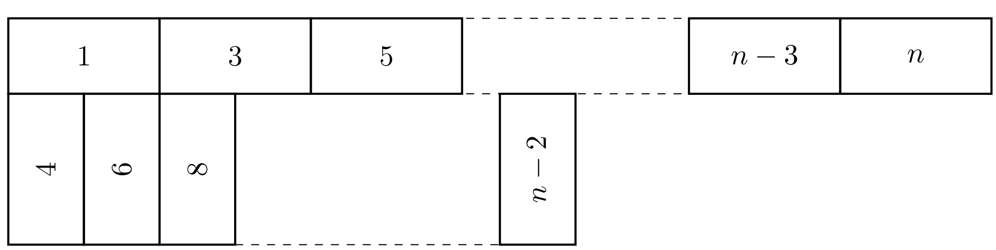
$T_L^{n-3}(v_n) =$



.

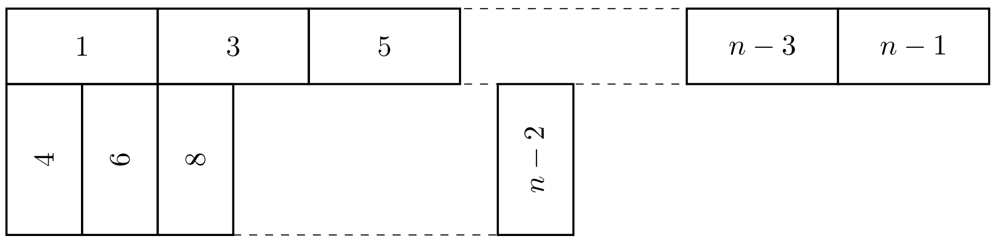
We have $v_n^{-1}(n-2) = n$, so to obtain $T_L^{n-2}(v_n)$ we see that we add a horizontal domino with label n on the end of the first row to get

$T_L^{n-2}(v_n) =$



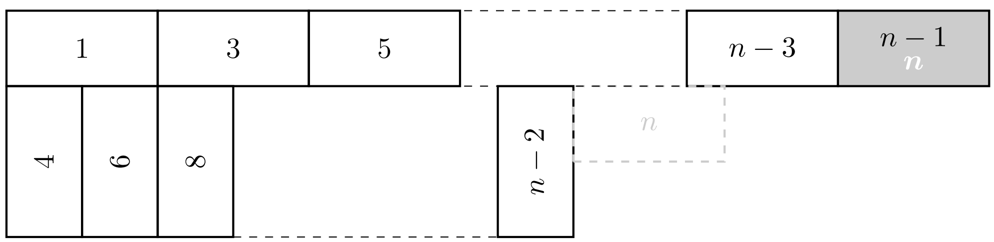
.

Next, $v_n^{-1}(n-1) = n-1$, so we must next add a horizontal domino with label $n-1$. To do this we must first remove all dominoes with labels greater than $n-1$, then add a horizontal domino with label $n-1$ to the end of the first row.



.

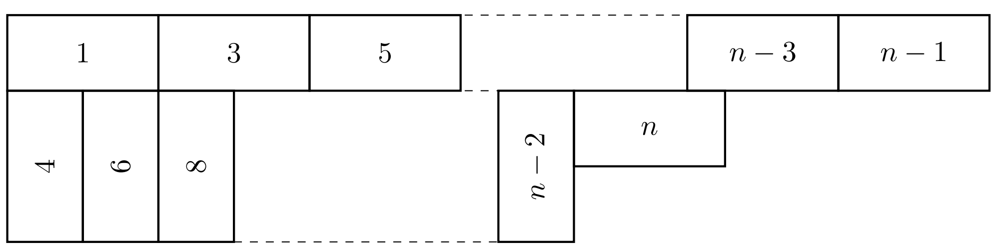
When we try to replace the domino with label n, we see that it fully overlaps with the domino with label $n-1$,



,

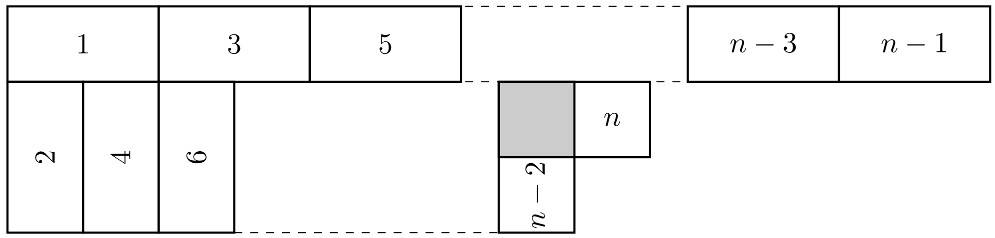
so we use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the second row

$T_L^{n-1}(v_n) =$



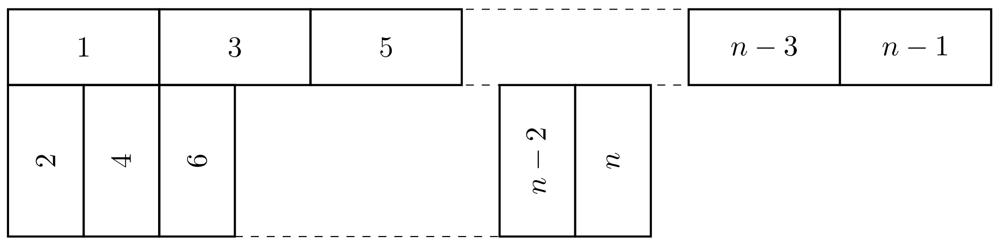
.

Finally, $v_n^{-1}(n) = -2$, so to complete the calculation of $T_L(v_n)$ we must add a vertical domino with label 2. As in Lemma 4.2.4, this bumps all dominoes with even labels less than n to the right. However, when we try to replace the domino with label n it partially overlaps with the domino with label $n-2$.



Now we use case (3) of Definition 3.1.16 to shuffle the domino with label n into position that allows it to fit into the tableau

$T_L(v_n) =$



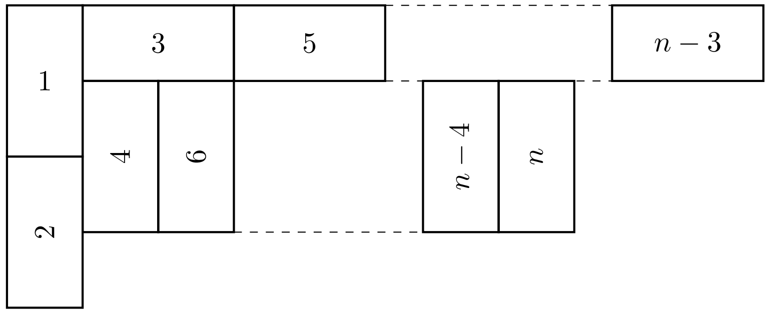
Then by Lemma 4.2.2 we have $T_L(v_n) = E(T_L(w_n), C_1)$, so $T_L(v_n) \approx T_L(w_n)$, thus by Theorem 3.3.14, we have $w_n \sim_L v_n$. $\square$

Now we have an element, v_n, in the same left cell as w_n. Unfortunately, $a(v_n)$ is still difficult to calculate. However, we can find an element in the same right cell as v_n that will allow us to calculate $a(w_n)$.

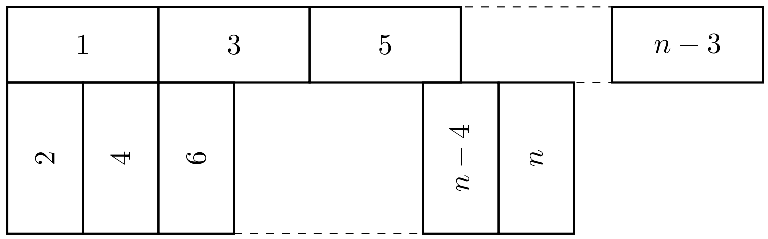
4.3 An element in the two-sided cell of w_n

We will now find an element, u_n in the same right cell as v_n, and therefore in the same two-sided cell as w_n. To do this we will use the function T_R. Recall that $T_R(w) = T_L(w^{-1})$, so to construct $T_R(w)$ we follow the method outlined in Remark 3.2.10, but we switch the roles of w and w^{-1}. We begin by constructing $T_R(v_n)$ and moving $T_R(v_n)$ through a cycle.

Lemma 4.3.1. *Recall that $v_n = s_n s_{n-1} s_n w_{n-2}$. The domino tableau $T_R^{n-2}(v_n)$ is constructed in the same way as $T_R^{n-2}(w_{n-2})$. That is, the dominoes in $T_R^{n-2}(v_n)$ lie in the same positions and relative order as those in $T_R^{n-2}(w_{n-2})$, but the labels correspond to the first $n-2$ entries in the signed permutation of v_n. In particular we have*

$T_R^{n-2}(v_n) =$ 

when $n \equiv 0 \bmod 4$ *and*

$T_R^{n-2}(v_n) =$ 

when $n \equiv 2 \bmod 4$.

Proof. In Theorem 2.2.18 and Lemma 4.2.3 we found that

$$w_{n-2} = \left((-1)^{(n-2)/2}, \underline{n-2}, 3, \underline{n-4}, 5, \ldots, \underline{4}, n-3, \underline{2}, n-1, n\right), \text{ and}$$
$$v_n = \left((-1)^{(n-2)/2}, \underline{n}, 3, \underline{n-4}, 5, \underline{n-6}, 7, \ldots, n-3, \underline{2}, n-1, n-2\right),$$

so

$$\left(|v_n(1)|, |v_n(2)|, \ldots, |v_n(n-2)|\right)$$

are in the same relative order as

$$\left(|w_{n-2}(1)|, |w_{n-2}(2)|, \ldots, |w_{n-2}(n-2)|\right),$$

and

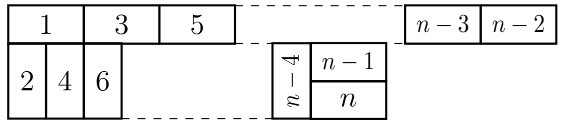
$$\left(\operatorname{sign}(v_n(1)), \ldots, \operatorname{sign}(v_n(n-2))\right) = \left(\operatorname{sign}(w_{n-2}(1)), \ldots, \operatorname{sign}(w_{n-2}(n-2))\right).$$

Thus, $T_R^{n-2}(v_n)$ is constructed in the same way as $T_R^{n-2}(w_{n-2})$. $\square$

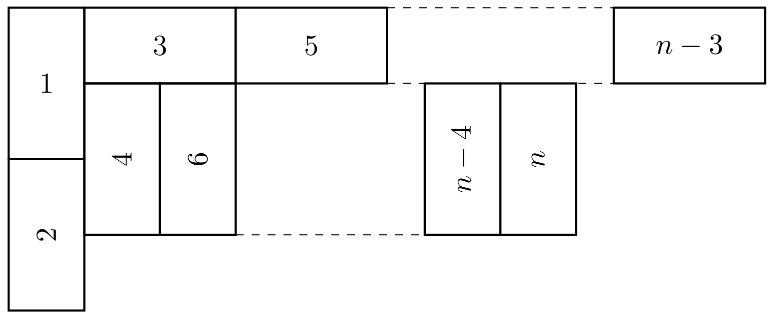
Lemma 4.3.2. *Let* $n \equiv 0 \bmod 4$. *Then* $T_R(v_n)$ *contains an open cycle*

$$C_2 = \{1, 2, 3, 5, 7, 9 \ldots, n-5, n-3, n-2\}.$$

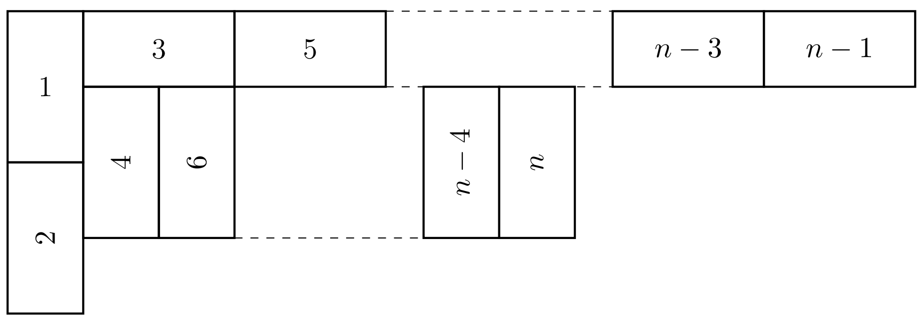
If we move $T_R(v_n)$ *through* C_2 *we obtain*

$E\left(T_R(v_n), C_2\right) =$  .

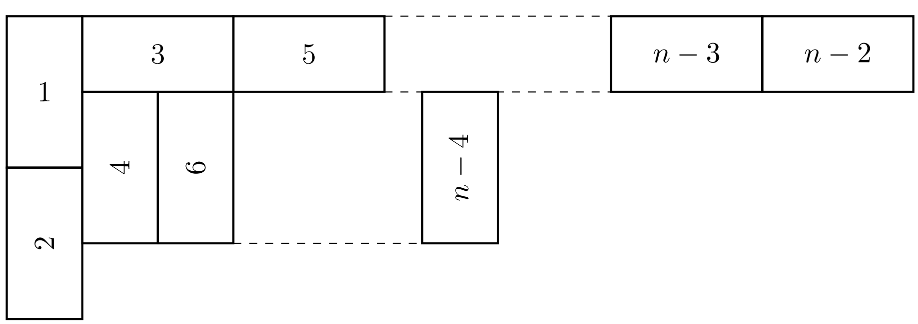
Proof. By Lemma 4.3.1 we have

$T_R^{n-2}(v_n) =$  .

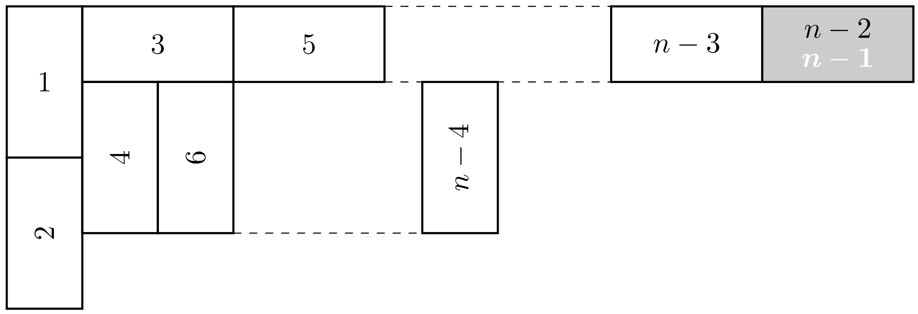
By Lemma 4.2.3 we have $v_n(n-1) = n-1$, so to obtain $T_R^{n-1}(v_n)$ we see that we add $n-1$ as a horizontal domino on the end of the first row:

$T_R^{n-1}(v_n) =$  .

Finally, $v_n(n) = n-2$, so we finish by adding a horizontal domino with label $n-2$. We first remove all dominoes with labels greater than $n-2$, then add a domino with label $n-2$ to the end of the first row.

 .

Now when we try to replace the domino with label $n-1$, we see that it fully overlaps with the domino with label $n-2$:



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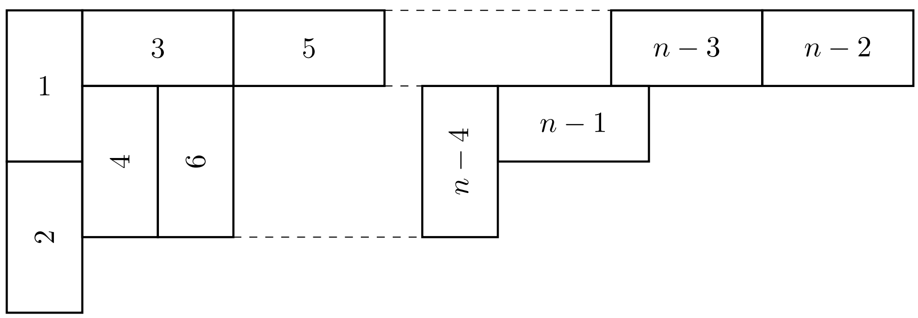
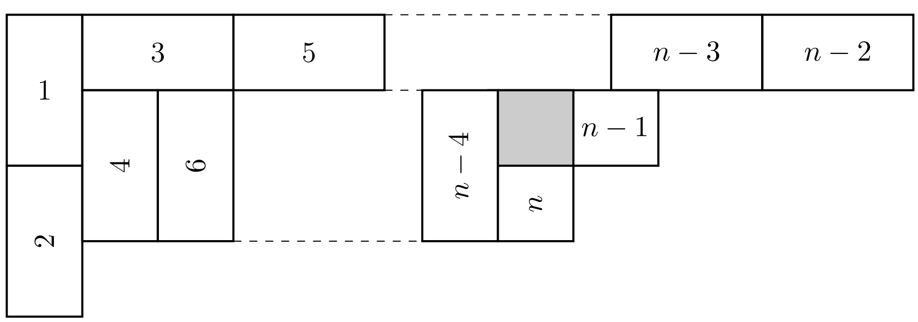
so we use case (2) of Definition 3.1.16 to bump the domino with label $n-1$ to the end of the second row:



.

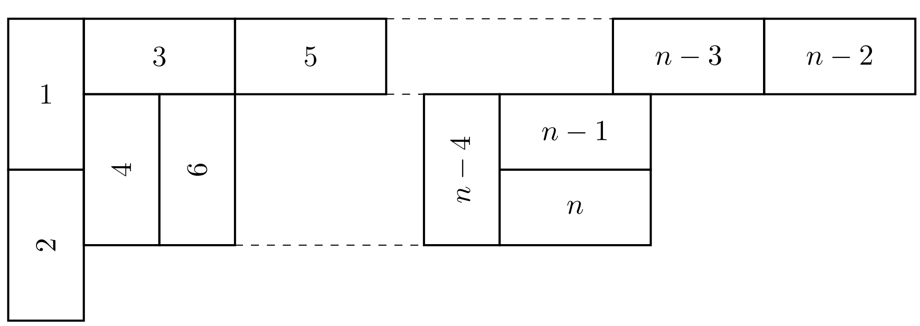
We finally must replace the domino with label n, which we see partially overlaps with the domino with label $n-1$:



,

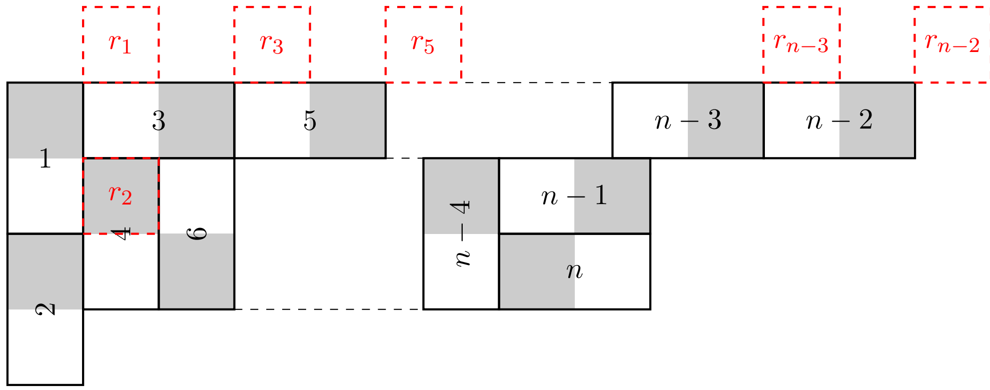
so we use case (3) of Definition 3.1.16 to shuffle the domino with label n into position that allows it to fit into the tableau

$T_R(v_n) =$



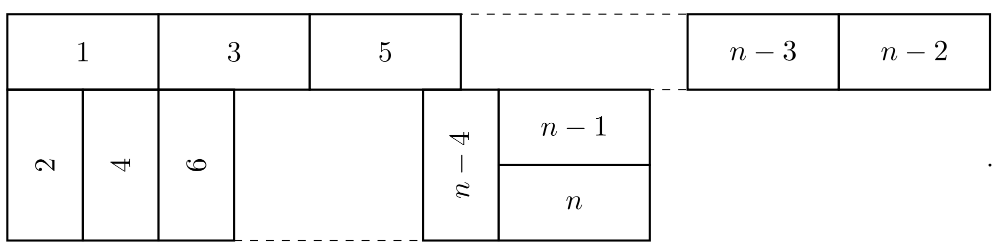
.

As in Lemma 4.2.2, we can shade the fixed squares and compute the r_k for $k \in \{1, 2, 3, 5, 7, 9, \ldots, n-5, n-3, n-2\}$ in order to help us find cycles.



.

We see that $r_2 = 4 > 2$ and $r_k = 0 < k$ for $k = 1, 3, 5, 7, \ldots, n-3, n-2$, so we can use Remark 3.3.4 to shuffle each of the corresponding dominoes. Now, as in Lemma 4.2.2, $T_R(v_n)$ contains an open cycle $C_2 = \{1, 2, 3, 5, 7, 9, \ldots, n-5, n-3, n-2\}$. If we move $T_R(v_n)$ through this open cycle we obtain

$E(T_R(v_n), C_2) =$ 

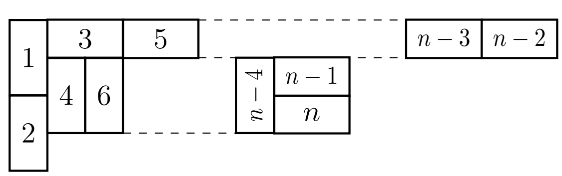
.

$\square$

Lemma 4.3.3. *Let $n \equiv 2 \bmod 4$. Then $T_R(v_n)$ contains an open cycle*

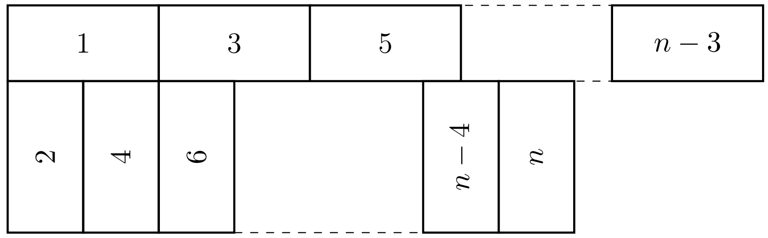
$$C_2 = \{1, 2, 3, 5, 7, 9 \ldots, n-5, n-3, n-2\}.$$

If we move $T_R(v_n)$ through C_2 we obtain

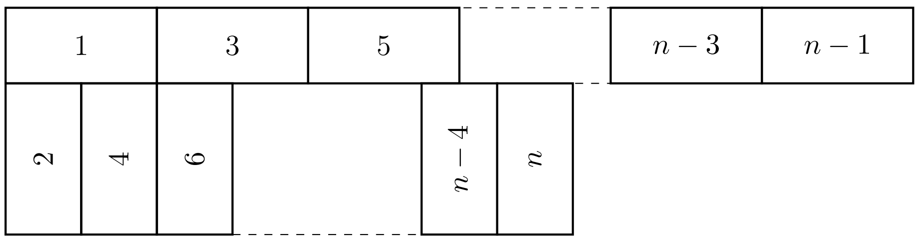
$E(T_R(v_n), C_2) =$ 

.

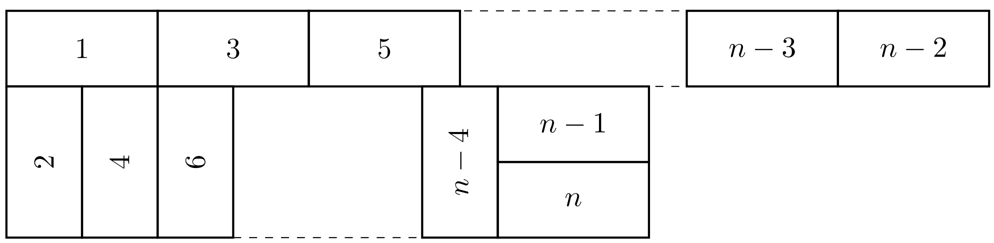
Proof. By Lemma 4.3.1 we have

$T_R^{n-2}(v_n) =$  .

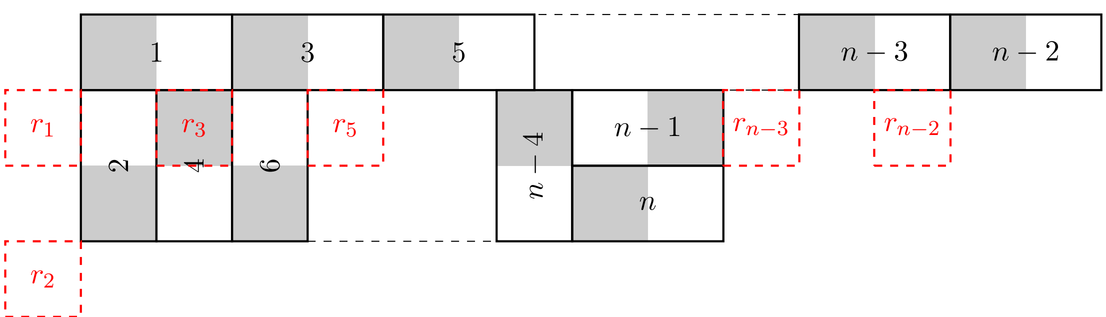
By Lemma 4.2.3 we have $v_n(n-1) = n-1$, so to obtain $T_R^{n-1}(v_n)$ we see that we add $n-1$ as a horizontal domino on the end of the first row:

$T_R^{n-1}(v_n) =$  .

Finally, $v_n(n) = n-2$, so we finish by adding a horizontal domino with label $n-2$. The shuffling that occurs is the same as in the 0 mod 4 case, so we obtain

$T_R(v_n) =$  .

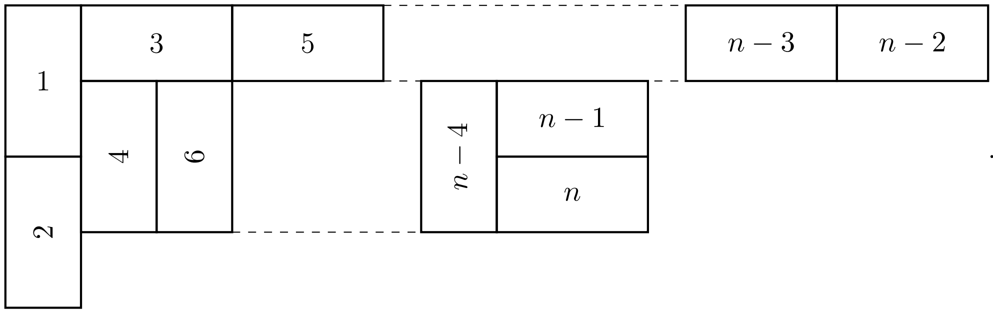
As in Lemma 4.3.2, we can shade the fixed squares and compute the r_k for $k \in \{1, 2, 3, 5, 7, 9, \ldots, n-5, n-3, n-2\}$ in order to help us find cycles.

 .

We see that $r_1 = 0 < 1$, $r_2 = 0 < 2$ and $r_k > k$ for $k = 3, 5, 7, \ldots, n-3, n-2$, so we can use Remark 3.3.4 to shuffle each of the corresponding dominoes. Now, as in Lemma 4.3.2, $T_R(v_n)$

contains an open cycle $C_2 = \{1, 2, 3, 5, 7, 9, \ldots, n-5, n-3, n-2\}$. If we move $T_R(v_n)$ through this open cycle we obtain

$E(T_R(v_n), C_2) =$

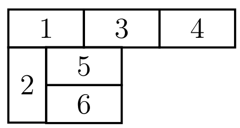


.

$\square$

Remark 4.3.4. If $n = 6$ then $T_R(v_6)$ is given by

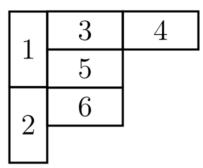
$T_R(v_6) =$



,

and contains the open cycle $C_2 = \{1, 2, 3, 4\}$. If we move $T_R(v_6)$ through C_2 we obtain

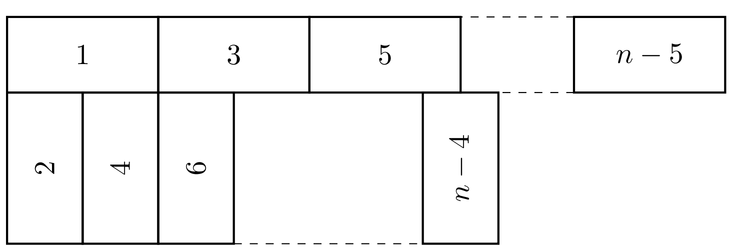
$E\left(T_R(v_n), C_2\right) =$



.

Lemma 4.3.5. *Let $n \geq 8$ be even and let $u_n = w_{n-4} s_n s_{n-1} s_n$. The domino tableau $T_R^{n-4}(u_n)$ is constructed in the same way as $T_R(v_{n-4})$. That is, the dominoes in $T_R^{n-4}(u_n)$ lie in the same positions and relative order as those in $T_R(v_{n-4})$, but the labels correspond to the first $n-4$ entries in the signed permutation of u_n. Then we obtain*

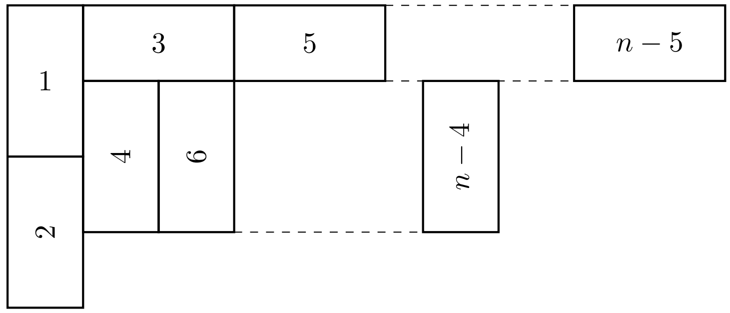
$T_R^{n-4}(u_n) =$



.

if $n \equiv 0 \bmod 4$ *and*

$T_R^{n-4}(u_n) =$



if $n \equiv 2 \bmod 4$.

Proof. We will first find the signed permutation representation of u_n. By Theorem 2.2.18 we have

$$w_{n-4} = \left((-1)^{(n-4)/2}, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n-2, n-1, n\right).$$

Now we can use Proposition 2.2.3 to find u_n. We have

$$\begin{aligned} w_{n-4}s_n &= \left((-1)^{(n-4)/2}, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n-2, n, n-1\right), \\ w_{n-4}s_ns_{n-1} &= \left((-1)^{(n-4)/2}, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n, n-2, n-1\right), \text{ and} \\ w_{n-4}s_ns_{n-1}s_n &= \left((-1)^{(n-4)/2}, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n, n-1, n-2\right), \end{aligned}$$

so

$$u_n = \begin{cases} (1, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n, n-1, n-2) & \text{if } n \equiv 0 \bmod 4; \\ (\underline{1}, \underline{n-4}, 3, \underline{n-6}, 5, \ldots, \underline{4}, n-5, \underline{2}, n-3, n, n-1, n-2) & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

We next compute $T_R(u_n)$. We first notice that

$$(|u_n(1)|, |u_n(2)|, \ldots, |u_n(n-4)|)$$

are in the same relative order as

$$(|w_{n-4}(1)|, |w_{n-4}(2)|, \ldots, |w_{n-4}(n-4)|),$$

and that

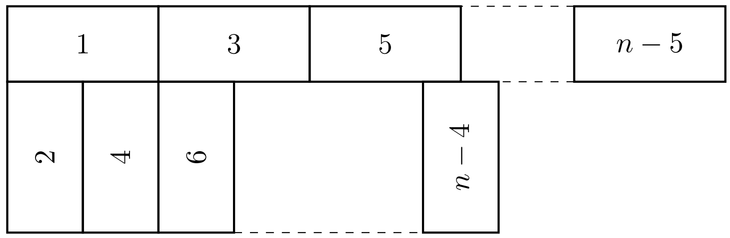
$$(\operatorname{sign}(u_n(1)), \ldots, \operatorname{sign}(u_n(n-4))) = (\operatorname{sign}(w_{n-4}(1)), \ldots, \operatorname{sign}(w_{n-4}(n-4))),$$

so $T_R^{n-4}(u_n)$ is constructed in the same way as $T_R(w_{n-4})$. $\square$

Lemma 4.3.6. *If $n \geq 8$ is such that $n \equiv 0 \bmod 4$, then $v_n \sim_R u_n$.*

Proof. By Lemma 4.3.5 we have

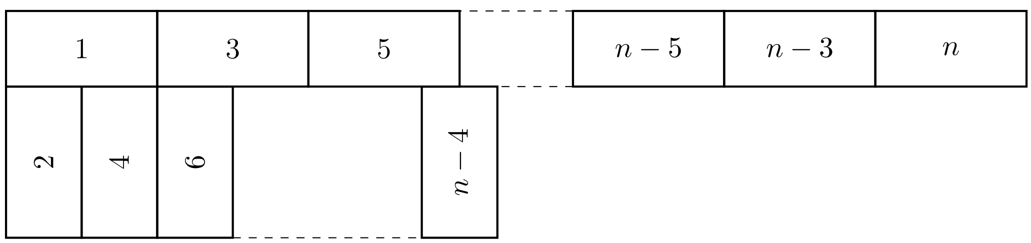
$T_R^{n-4}(u_n) =$



.

Now $u_n(n-3) = n-3$, and since $n-3$ is larger than any of the labels of dominoes in $T_R^{n-4}(u_n)$ we simply add a horizontal domino with label $n-3$ to the end of the first row. Similarly, $u_n(n-2) = n$, so we add a horizontal domino with label n to the end of the first row, thus obtaining

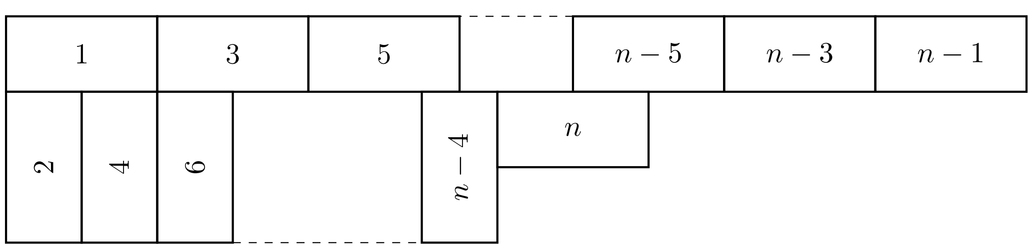
$T_R^{n-2}(u_n) =$



.

Next, $u_n(n-1) = n-1$, so we must add a horizontal domino with label $n-1$. We first remove then domino with label n and place a horizontal domino with label $n-1$ at the end of the first row. When we replace the domino with label n we see that it overlaps fully with the domino with label $n-1$, so we use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the second row:

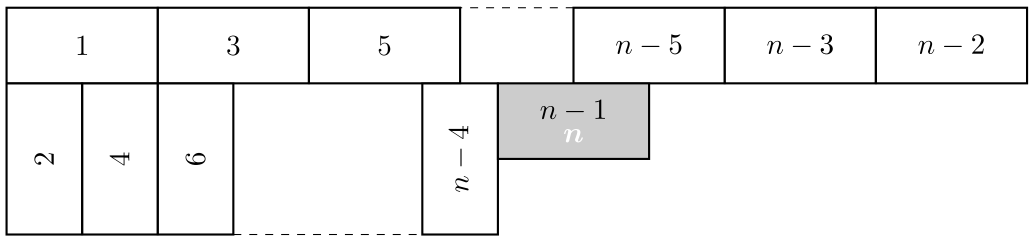
$T_R^{n-1}(u_n) =$



.

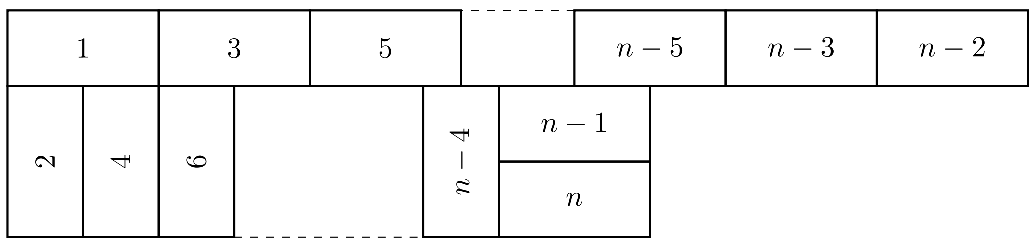
Finally, $u_n(n) = n-2$, so we conclude by adding a domino with label $n-2$. We remove the dominoes with labels $n-1$ and n and add a horizontal domino with label $n-2$ to the end of the first row. When we replace the domino with label $n-1$ it fully overlaps with the domino with label

$n-2$, so we use case (2) of Definition 3.1.16 to bump the domino with label $n-1$ to the end of the second row:



.

Then when the domino with label n is replaces it fully overlaps with the domino with label $n-1$, so we again use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the third row. Then we obtain

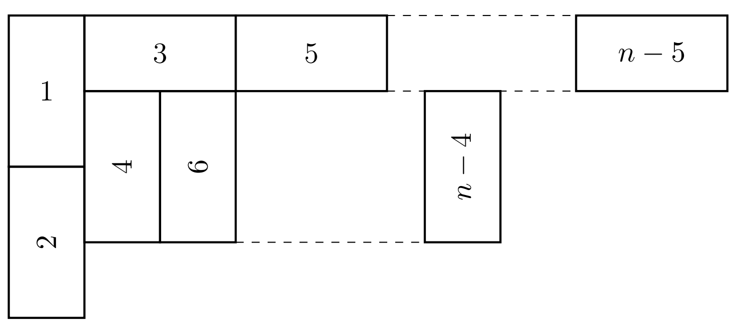
$T_R(u_n) =$ 

.

Now by Lemma 4.3.2 we see that $T_R(u_n) = E(T_R(v_n), C_2)$, so $T_R(u_n) \approx T_R(v_n)$, thus by Corollary 3.3.15 we see that $u_n \sim_R v_n$. $\square$

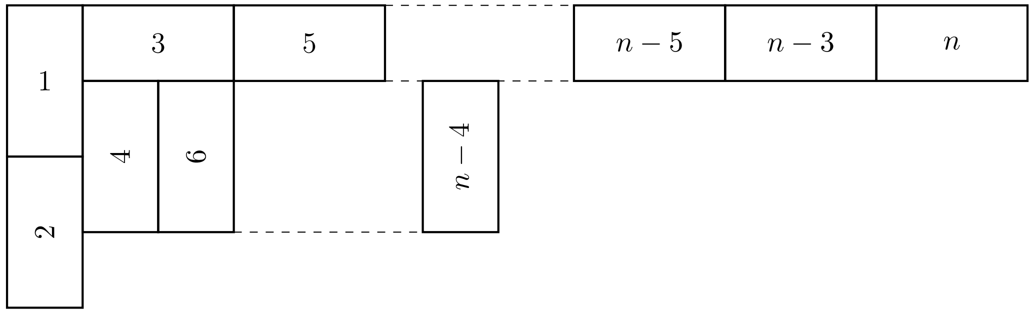
Lemma 4.3.7. *If $n \geq 8$ is such that $n \equiv 2 \bmod 4$, then $v_n \sim_R u_n$.*

Proof. By Lemma 4.3.5 we have

$T_R^{n-4}(u_n) =$ 

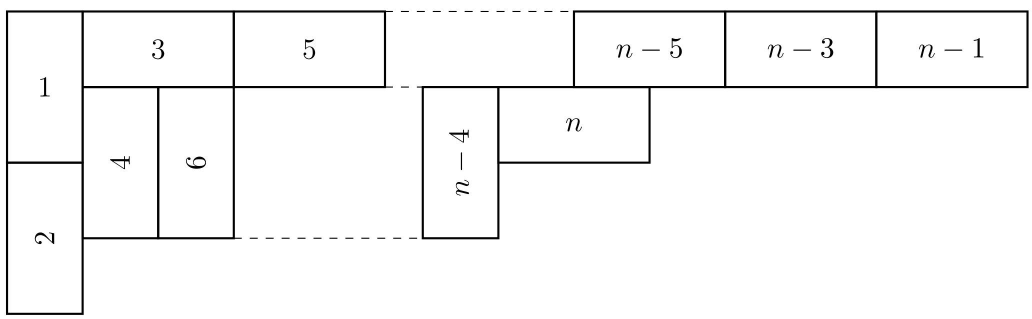
.

Now $u_n(n-3) = n-3$, and since $n-3$ is larger than any of the labels of dominoes in $T_R^{n-4}(u_n)$ we simply add a horizontal domino with label $n-3$ to the end of the first row. Similarly, $u_n(n-2) = n$, so we add a horizontal domino with label n to the end of the first row, thus obtaining

$T_R^{n-2}(u_n) =$ 

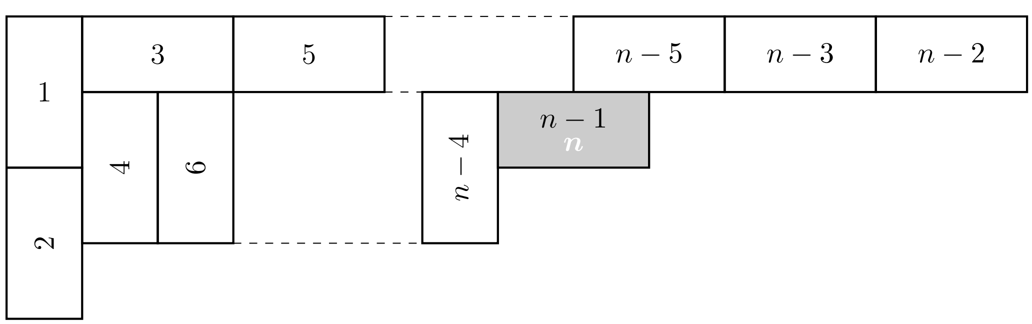
.

Next, $u_n(n-1) = n-1$, so we must add a horizontal domino with label $n-1$. We first remove then domino with label n and place a horizontal domino with label $n-1$ at the end of the first row. When we replace the domino with label n we see that it overlaps fully with the domino with label $n-1$, so we use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the second row, obtaining

$T_R^{n-1}(u_n) =$ 

.

Finally, $u_n(n) = n-2$, so we conclude by adding a domino with label $n-2$. We remove the dominoes with labels $n-1$ and n and add a horizontal domino with label $n-2$ to the end of the first row. When we replace the domino with label $n-1$ it fully overlaps with the domino with label $n-2$, so we use case (2) of Definition 3.1.16 to bump the domino with label $n-1$ to the end of the second row

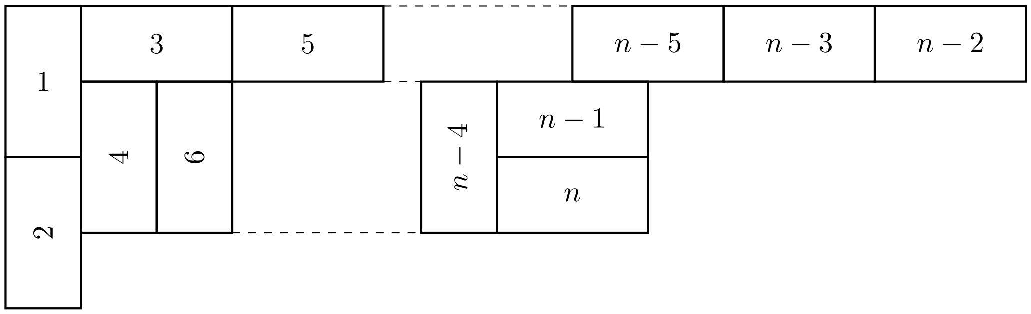


.

Then when the domino with label n is replaces it fully overlaps with the domino with label

$n-1$, so we again use case (2) of Definition 3.1.16 to bump the domino with label n to the end of the third row. Then we obtain

$T_R(u_n) =$



.

Now by Lemma 4.3.3 we see that $T_R(u_n) = E(T_R(v_n), C_2)$, so $T_R(u_n) \approx T_R(v_n)$, thus by Corollary 3.3.15 we see that $u_n \sim_R v_n$. $\square$

Proposition 4.3.8. *If $n \geq 8$ is even, then $w_n \sim_{LR} u_n$.*

Proof. It follows from Lemmas 4.2.4 and 4.2.5 that $w_n \sim_L v_n$, and it follows from Lemmas 4.3.6 and 4.3.7 that $v_n \sim_R u_n$, thus we have $w_n \sim_{LR} u_n$. $\square$

4.4 Two-sided cells of w_n for small values of n

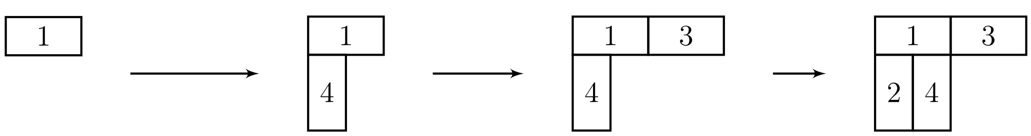
We have now found elements, u_n, in the same Kazhdan–Lusztig cell as w_n for $n \geq 8$. As we will see in Section 4.5 these will allow us to calculate $a(w_n)$ for $n \geq 8$. We are left to calculate simpler elements in the same two-sided cell as w_4 and w_6.

Proposition 4.4.1. *We have $w_4 \sim_{LR} s_1s_2s_4$.*

Proof. We know that

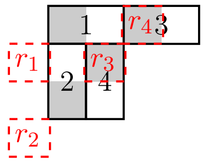
$$w_4 = (1, \underline{4}, 3, \underline{2}),$$

so using Remark 3.3.4 we can calculate $T_L(w_4)$.



$T_L^1(w_4)$ $T_L^2(w_4)$ $T_L^3(w_4)$ $T_L(w_4)$

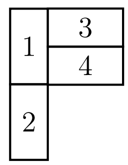
Now we shade the fixed squares and label the r-values



.

so using Remark 3.3.4 we can calculate

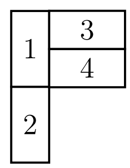
$$(T_L(w_4))' =$$



.

Then $C_1 = \{1, 2, 3\}$ and $C_2 = \{4\}$ are open cycles, and

$$E(T_L(w_4), C_1, C_2) =$$

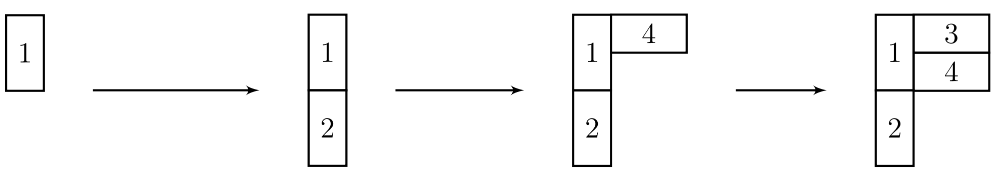


.

Now as a signed permutation we have

$$s_1 s_2 s_4 = (\underline{1}, \underline{2}, 4, 3),$$

so we can calculate $T_L(s_1 s_2 s_4)$.

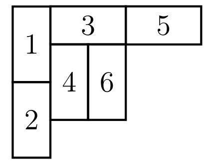


$T_L^1(s_1 s_2 s_4)$ $\quad$ $T_L^2(s_1 s_2 s_4)$ $\quad$ $T_L^3(s_1 s_2 s_4)$ $\quad$ $T_L(s_1 s_2 s_4)$

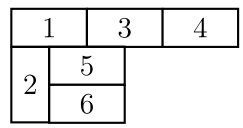
Then $T_L(s_1 s_2 s_3) = E(T_L(w_4), C_1, C_2)$, thus by Theorem 3.3.14 $w_4 \sim_L s_1 s_2 s_4$, so $w_4 \sim_{LR}$ $s_1 s_2 s_4$. $\square$

Lemma 4.4.2. *We have* $w_6 \sim_{LR} s_1 s_2 s_6 s_5 s_6$.

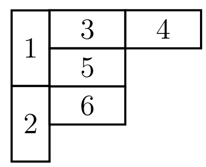
Proof. We found that

$T_L(w_6) =$ 

in Example 3.2.11. Now $v_6 = s_6s_5s_6w_4$ and by Lemma 4.2.5 we have $w_6 \sim_L v_6$. By Lemma 4.3.3 and Remark 4.3.4 we have

$T_R(v_6) =$  ,

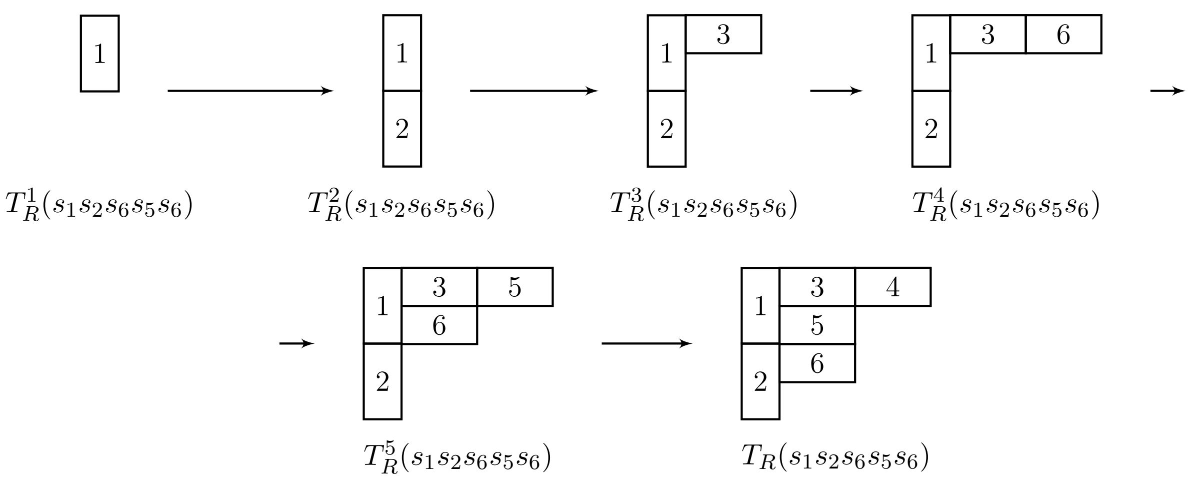
and $C_2 = \{1, 2, 3, 4\}$ is an open cycle in $T_R(v_6)$. If we move $T_R(v_6)$ through C_2 we get

$E(T_R(v_6), C_2) =$  .

Now we will calculate $T_R(s_1s_2s_6s_5s_6)$. First we can calculate that as a signed permutation we have

$$s_1s_2s_6s_5s_6 = (s_1s_2s_6s_5s_6)^{-1} = (\underline{1}, \underline{2}, 3, 6, 5, 4).$$

Then we have



so $E(T_R(v_6), C_2) = T_R(s_1s_2s_6s_5s_6)$. Then $v_6 \sim_R s_1s_2s_6s_5s_6$ by Corollary 3.3.15, thus $w_6 \sim_L v_6 \sim_R s_1s_2s_6s_5s_6$, so $w_6 \sim_{LR} s_1s_2s_6s_5s_6$. □

4.5 Proof of main result

Recall Lusztig's a-function, from Definition 1.3. We now have sufficient information to calculate Lusztig's a-function on bad elements. This will allow us to bound the degree of key Kazhdan–Lusztig polynomials and to eventually prove our main result in Theorem 4.5.11:

Theorem. *Let $x, w \in W(D_n)$ be such that x is fully commutative. Then $\mu(x, w) \in \{0, 1\}$.*

Proposition 4.5.1. *If $n \in \mathbb{N}$ is even, then*

$$a(w_n) = \begin{cases} \dfrac{3n}{4} & \text{if } n \equiv 0 \bmod 4; \\ \dfrac{3n+2}{4} & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Proof. By Proposition 4.4.1, $a(w_4) = a(s_1 s_2 s_4) = 3$ and by Lemmas 1.3.8 and 4.4.2 $a(w_6) = a(s_1 s_2 s_6 s_5 s_6) = 5$.

By Proposition 4.3.8 and Lemma 1.3.6 we know that $a(w_n) = a(u_n)$ when $n \geq 8$. By Lemma 4.3.5, $u_n = w_{n-4} s_n s_{n-1} s_n$, we can use Lemmas 1.3.7, 1.3.8, and 1.3.9 to see that $a(w_n) = a(w_{n-4} s_n s_{n-1} s_n) = a(w_{n-4}) + 3$. Then we have

$$a(w_n) = \begin{cases} 3 + \dfrac{3}{4}(n-4) = \dfrac{3n}{4} & \text{if } n \equiv 0 \bmod 4; \\ 5 + \dfrac{3}{4}(n-4) = \dfrac{3n+2}{4} & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

□

Lemma 4.5.2. *If $n > 8$ is such that $n \equiv 0 \bmod 4$, then $\mu(x_n, w_n) = 0$.*

Proof. By Proposition 4.5.1 and Lemma 2.2.20 we have

$$a(w_n) = \frac{3n}{4}$$
$$\ell(w_n) = \frac{3n^2}{8} + \frac{n}{4}.$$

Then by Proposition 1.3.5

$$\deg(P_{e,w_n}) \leq \frac{1}{2}\left(\frac{3n^2}{8} - \frac{n}{2}\right) = \frac{3n^2}{16} - \frac{n}{4}.$$

We see that $\ell(x_n) = \frac{n}{2} + 1$, so

$$\frac{1}{2}(\ell(w_n) - \ell(x_n) - 1) = \frac{3n^2}{16} - \frac{n}{8} - 1.$$

Then as long as $n > 8$ we have $\deg(P_{e,w_n}) < \frac{1}{2}(\ell(w_n) - \ell(x_n) - 1)$, so by Lemma 2.3.9 we have $\mu(x_n, w_n) = 0$. ☐

Lemma 4.5.3. *If $n > 8$ is such that $n \equiv 2 \bmod 4$, then $\mu(x_n, w_n) = 0$.*

Proof. By Proposition 4.5.1 and Lemma 2.2.20 we have

$$a(w_n) = \frac{3n+2}{4}$$
$$\ell(w_n) = \frac{3n^2}{8} + \frac{n}{4}.$$

Then by Proposition 1.3.5

$$\deg(P_{e,w_n}) \leq \frac{1}{2}\left(\frac{3n^2}{8} - \frac{n}{2} - \frac{1}{2}\right) = \frac{3n^2}{16} - \frac{n}{4} - \frac{1}{4}.$$

As in Lemma 4.5.2, $\ell(x_n) = \frac{n}{2} + 1$, so

$$\frac{1}{2}(\ell(w_n) - \ell(x_n) - 1) = \frac{3n^2}{16} - \frac{n}{8} - 1.$$

Then as long as $n > 6$ we have $\deg(P_{e,w_n}) < \frac{1}{2}(\ell(w_n) - \ell(x_n) - 1)$, so by Lemma 2.3.9 we have $\mu(x_n, w_n) = 0$. ☐

Lemma 4.5.4. *We have $\mu(x_4, w_4) = 0$.*

Proof. We see that $\ell(w_4) = 7$ and $\ell(x_4) = 3$, so $\ell(w_4) \equiv \ell(x_4) \bmod 2$. Proposition 1.2.4 shows that $\mu(x_4, w_4) = 0$. ☐

Lemma 4.5.5. *We have $\mu(x_6, w_6) = 1$ and $\mu(x_8, w_8) = 0$.*

Proof. These values were calculated using a program called `coxeter` developed by du Cloux [3]. ☐

Lemma 4.5.6. *Let $n \geq 5$ be odd. Then $\mu(x_n, w_n) = \mu(x_{n-1}, w_{n-1})$.*

Proof. We know that x_n and x_{n-1} have the same reduced expressions, and by Lemma 2.3.4, w_n and w_{n-1} have the same reduced expressions, so $\mu(x_n, w_n) = \mu(x_{n-1}, w_{n-1})$. $\square$

Lemma 4.5.7. *Let $n \geq 4$. Then $\mu(x_n, w_n) \in \{0, 1\}$.*

Proof. We have shown this for all possibilities of n in Lemmas 4.5.2, 4.5.3, 4.5.4, 4.5.5, and 4.5.6. $\square$

Lemma 4.5.8. *Let n be even and let $x \in W_c$ be such that*

$$\mathcal{L}(x) = \mathcal{R}(x) = \mathcal{L}(x_n) = \mathcal{R}(x_n) = \{s_1, s_2, s_4, s_6, s_8, \ldots, s_n\}.$$

Then $x = x_n$.

Proof. Let $x \in W_c$ be such that $\mathcal{L}(x) = \mathcal{R}(x) = \{s_1, s_2, s_4, s_6, s_8, \ldots, s_n\}$. We see that each generator in $S \setminus \mathcal{L}(w) = S \setminus \mathcal{R}(w) = \{s_3, s_5, s_7, \ldots, s_{n-1}\}$ fails to commute with at least two of the generators in $\mathcal{L}(w) = \mathcal{R}(w)$. Since $x \in W_c$ it follows that x has no reduced expressions beginning or ending in two noncommuting generators, thus x is either a product of commuting generators or bad. By Corollary 2.3.7, bad elements cannot be fully commutative, so x must be a product of commuting generators. Then we have $x = x_n$.

$\square$

Lemma 4.5.9. *Let $W = W(D_n)$ and let $x, w \in W$ be such that $x \in W_c$ and w is bad. Then $\mu(x, w) \in \{0, 1\}$.*

Proof. Let $x, w \in W$ be such that x is fully commutative and w is bad. If there exists $s \in \mathcal{L}(w) \setminus \mathcal{L}(x)$, or $s \in \mathcal{R}(w) \setminus \mathcal{R}(x)$ then we are done by Proposition 1.2.4.

Now suppose $\mathcal{L}(w) \subseteq \mathcal{L}(x)$ and $\mathcal{R}(w) \subseteq \mathcal{R}(x)$. By Theorem 2.3.6 we can write $w = w_k \cdot u$, where $k \leq n$ and u is a product of commuting generators not in $\mathrm{supp}(w_k)$. Suppose $s_i \in \mathcal{L}(x) \setminus \mathcal{L}(w)$. Since $s_k \in \mathcal{L}(w)$ we see that either $1 < i < k$ and i is odd, or $i > k$. In the first case we must have $s_{i+1} \in \mathcal{L}(x)$ since $s_{i+1} \in \mathcal{L}(w)$, so x is not fully commutative by Corollary 1.1.22, contradicting our assumption. If $i > k$ then $x \not\leq w$ and thus $\mu(x, w) = 0$. Then $s_i \in \mathcal{L}(x) \setminus \mathcal{L}(w)$ implies that

$\mu(x, w) \in \{0, 1\}$, so we may assume that $\mathcal{L}(x) = \mathcal{L}(w)$. We can use a similar argument to show we may assume that $\mathcal{R}(x) = \mathcal{R}(w)$.

Now since x is fully commutative with $\mathcal{L}(x) = \mathcal{L}(w)$ and $\mathcal{R}(x) = \mathcal{R}(w)$ we must have $x = x_k \cdot u$ by Lemma 4.5.8. By Lemma 1.2.8 we have $\mu(x_k \cdot u, w_k \cdot u) = \mu(x_k, w_k)$, and by Lemma 4.5.7 we have $\mu(x, w) = \mu(x_k, w_k) \in \{0, 1\}$. □

Corollary 4.5.10. *Let $x, w \in W$ be such that $x \in W_c$ and such that w has one of the following properties:*

(1) w is a product of commmuting generators;

(2) w is bad; or

(3) either $\mathcal{L}(w)$ or $\mathcal{R}(w)$ is not commutative.

Then $\mu(x, w) \in \{0, 1\}$.

Proof. If either w is a product of commmuting generators or w is bad then we are done by Corollary 1.2.5 or Lemma 4.5.9, respectively. If either $\mathcal{L}(w)$ or $\mathcal{R}(w)$ is not commutative, then there is some generator s in $\mathcal{L}(w) \setminus \mathcal{L}(x)$ or $\mathcal{R}(w) \setminus \mathcal{R}(x)$, so we are done by Proposition 1.2.4. □

We are now ready to prove our main result.

Theorem 4.5.11. *Let $x, w \in W(D_n)$ be such that x is fully commutative. Then $\mu(x, w) \in \{0, 1\}$.*

Proof. Let $w \in W$. If either $\mathcal{L}(w)$ or $\mathcal{R}(w)$ is not commutative, then the result follows from Proposition 1.2.4.

By Corollary 2.3.11 we see that w is star reducible to an element $y \in W$ such that y has one of the following properties:

(1) y is a product of commmuting generators;

(2) y is bad; or

(3) either $\mathcal{L}(y)$ or $\mathcal{R}(y)$ is not commutative.

Then we can write a sequence

$$w = w_{(0)}, w_{(1)}, \ldots, w_{(k-1)}, w_{(k)} = y$$

such that $w_{(i)}$ is left or right star reducible to $w_{(i+1)}$. We will complete the proof by induction on k. If $k = 0$ then we are done by Corollary 4.5.10.

Suppose that $w_{(1)} = {}^*w$ and let s, t be the pair of generators such that $w_{(1)} = {}^*w$, $s \in \mathcal{L}(w)$, and $t \notin \mathcal{L}(w)$. If $s \notin \mathcal{L}(x)$ then we are done by Proposition 1.2.4, so suppose that $s \in \mathcal{L}(x)$. Then $t \notin \mathcal{L}(x)$ by Corollary 1.1.22 since x is fully commutative, so $x \in D_{\mathcal{L}}(s,t)$, and thus ${}^*x$ is defined. By Proposition 1.2.4, $\mu(x, w) = \mu({}^*x, {}^*w)$, and by Proposition 1.1.29 we have ${}^*x \in W_c$. Then ${}^*w$ is star reducible to ${}^*x$ using a sequence of length less than k, so $\mu(x, w) = \mu({}^*x, {}^*w) \in \{0, 1\}$ by induction.

If $w_{(1)} = w^*$ we can use a symmetric argument to show $\mu(x, w) \in \{0, 1\}$.

$\square$

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