Generalizing the Kantorovich Metric to Projection-Valued Measures: With an Application to Iterated Function Systems

by

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Generalizing the Kantorovich Metric to Projection-Valued Measures: With an Application to Iterated Function Systems

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Given a compact metric space X, the collection of Borel probability measures on X can be made into a compact metric space via the Kantorovich metric [14]. We partially generalize this well known result to projection-valued measures. In particular, given a Hilbert space $\mathcal{H}$, we consider the collection of projection-valued measures from X into the projections on $\mathcal{H}$. We show that this collection can be made into a complete and bounded metric space via a generalized Kantorovich metric. However, we add that this metric space is not compact, thereby identifying an important distinction from the classical setting. We have seen recently that this generalized metric has been previously defined by F. Werner in the setting of mathematical physics in 2004 [26]. We develop new properties and applications of this metric. Indeed, we use the Contraction Mapping Theorem on this complete metric space of projection-valued measures to provide an alternative method for proving a fixed point result due to P. Jorgensen (see [18] and [17]). This fixed point, which is a projection-valued measure, arises from an iterated function system on X, and is related to Cuntz algebras. We conclude this document with a discussion of unitary representations of the Baumslag-Solitar group which arise from the Cantor set. We identify a family of partial isometries which can be used to construct the unitary operators which realize the representation of the Baumslag-Solitar group. These partial isometries satisfy relations similar to the Cuntz algebra relations.

Dedication

To my parents, who have instilled in me a love of learning.

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Contents

Chapter 1

Overview

In this opening chapter, we will provide an overview of what is to come. To begin, let (Y, d) be a complete and separable metric space.

Definition 1.0.1. *A Lipschitz contraction, L, on Y is a map $L : Y \to Y$ such that*

$$d(L(x), L(y)) \leq rd(x, y)$$

for all $x, y, \in Y$, where $0 < r < 1$.

Let $L : Y \to Y$ be a Lipschitz contraction on Y. It is well known that L has a unique fixed point $y \in Y$, meaning that $L(y) = y$. This result is known as the Contraction Mapping Principle, or the Banach Fixed Point Theorem. In 1981, J. Hutchinson published a seminal paper (see [14]), where he generalized the Contraction Mapping Theorem to a finite family, $\mathcal{S} = \{\sigma_0, ..., \sigma_{N-1}\}$, of Lipschitz contractions on Y, where $N \in \mathbb{N}$ is such that $N \geq 2$. Unless otherwise specified, this N will be the same throughout the document. Indeed, one can associate to $\mathcal{S}$ a unique compact subset $X \subseteq Y$ which is invariant under the $\mathcal{S}$, meaning that

$$X = \bigcup_{i=0}^{N-1} \sigma_i(X).$$

A finite family of Lipschitz contractions on Y is called an iterated function system (IFS) with respect to Y, and the compact invariant subset X described above is called the fractal set associated to the IFS. We will briefly describe the different methods that have been discovered for realizing the fractal set. The reader will note that the Contraction Mapping Theorem is used at some point in every method. Therefore, it is appropriate to view the fractal set as a generalization of the Contraction Mapping Theorem.

(1) Closure of the Fixed Points:

We first note that since each $\sigma_i : Y \to Y$ is a Lipschitz contraction, $\sigma_{i_1} \circ ... \circ \sigma_{i_k} : Y \to Y$ is a Lipschitz contraction as well, where $i_j \in \{0, ..., N-1\}$ for $1 \leq j \leq k$, and where $k \in \mathbb{N}$. We note that there can be repetitions of the indices i_j; that is, it is possible to have $i_j = i_k$ when $j \neq k$. By the Contraction Mapping Theorem, $\sigma_{i_1} \circ ... \circ \sigma_{i_k}$ has a unique fixed point in Y, which we call $s_{i_1 ... i_k}$. Hutchinson showed in [14] that the fractal set X is the closure of the set of fixed points $s_{i_1 ... i_k}$, for any finite length tuple $(i_1, ..., i_k)$. This gives a first characterization of the fractal set.

(2) Hutchinson-Barnsley Operator:

If $A, B \subseteq Y$, the Hausdorff metric, δ, is defined by

$$\delta(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\}.$$

Denote by $\mathcal{K}$ the collection of compact subsets of Y. It is known that the metric space $(\mathcal{K}, \delta)$ is complete. The following result is due to J. Hutchinson and M. Barnsley.

Theorem 1.0.2. *[14][3][Hutchinson, Barnsley] The Hutchinson-Barnsley operator $F : \mathcal{K} \to \mathcal{K}$ given by*

$$K \mapsto \bigcup_{i=0}^{N-1} \sigma_i(K)$$

is a Lipschitz contraction in the δ metric. By the Contraction Mapping Theorem, there exists a unique compact $X \subseteq Y$ such that $F(X) = X$. That is,

$$X = \bigcup_{i=0}^{N-1} \sigma_i(X).$$

(3) The Markov Operator:

The fractal set can be realized as the support of a Borel probability measure, μ, on Y, called the Hutchinson measure. In particular, this measure will be the unique fixed point of a Lipschitz contraction, T, on a complete metric space of Borel probability measures on Y. The metric H, which we will refer to as the classical Kantorovich metric, is given by

$$H(\mu,\nu) = \sup_{f\in \mathrm{Lip}_1(Y)} \left\{ \left| \int_Y f d\mu - \int_Y f d\nu \right| \right\}, \tag{1.1}$$

where $\mathrm{Lip}_1(Y) = \{f : Y \to \mathbb{R} : |f(x) - f(y)| \leq d(x, y) \text{ for all } x, y \in Y\}$, and where μ and ν are Borel probability measures on Y. The map T is given by

$$T(\nu) = \sum_{i=0}^{N-1} \frac{1}{N} \nu(\sigma_i^{-1}(\cdot)). \tag{1.2}$$

The goal of Chapter 1 is to discuss the classical Kantorovich metric. In the case that (Y, d) is an arbitrary complete and separable metric space, this metric is not necessarily well defined (finite) on all Borel probability measures on Y. Hence, there has been research to determine which sub-collection of measures will make the Kantorovich metric well defined and complete. The new results that we present in Chapter 1 are metric space completion results (see Theorem 2.2.1 and Theorem 2.4.3).

Let us restrict the Hutchinson measure, μ, to its support, which is the fractal set X. Consider the Hilbert space $L^2(X, \mu)$. Further, assume that there exists a measurable function $\sigma : X \to X$ such that $\sigma \circ \sigma_i = \mathrm{id}_X$ for all $0 \leq i \leq N - 1$. On this Hilbert space, define the following operator:

$$S_i : L^2(X,\mu) \to L^2(X,\mu) \text{ by } \phi \mapsto (\phi\circ\sigma)\sqrt{N}\mathbf{1}_{\sigma_i(X)}$$

for all $0 \leq i \leq N - 1$, and the adjoint

$$S_i^* : L^2(X,\mu) \to L^2(X,\mu) \text{ by } \phi \mapsto \frac{1}{\sqrt{N}}(\phi\circ\sigma_i)$$

for all $0 \leq i \leq N - 1$.

P. Jorgensen showed (see [17], [18]) that there exists a unique projection-valued measure, $E(\cdot)$, defined on the Borel subsets of X taking values in the projections on $L^2(X, \mu)$ such that

(I) $E(\cdot) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*$, and

(II) $E(\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X)) = S_{i_1} \circ ... \circ S_{i_k} S_{i_k}^* \circ ... \circ S_{i_1}^*$ for all $k \in \mathbb{Z}_+$ and $(i_1, ..., i_k) \in \Gamma_N^k$, where $\Gamma_N = \{0, ..., N-1\}$.

We make several observations about this result:

- The projection-valued measure E is the canonical projection-valued measure given by multiplication by the indicator function. That is, if Δ is a Borel subset of X, $E(\Delta) = M_{\mathbf{1}_{\boldsymbol{\Delta}}}$.

- E can be thought of as a functional analytic generalization of the Hutchinson measure, because it satisfies a fixed point relation (item (I) above) which is similar to equation (1.2), and because one can recover the Hutchinson measure from E. Indeed, if we consider the the element $1 \in L^2(X, \mu)$, we can define the positive measure $E_{1,1}(\cdot) := \langle E(\cdot)1, 1\rangle_{L^2(X,\mu)}$. For all $(i_1, ..., i_k) \in \Gamma_N^k$,

$$E_{1,1}(\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X)) = \int_X \mathbf{1}_{\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X)} d\mu = \mu(\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X)).$$

  This equality can be extended to all Borel subsets of X, and therefore, $E_{1,1} = \mu$.

Jorgensen proved this result by showing the existence and uniqueness of a projection-valued measure E which satisfies item (II) above. In particular, he used a standard technique of extending the map $\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X) \mapsto S_{i_1} \circ ... \circ S_{i_k} S_{i_k}^* \circ ... \circ S_{i_1}^*$ to all Borel subsets of X, and therefore by construction, obtaining the desired projection-valued measure. As a consequence of item (II), he showed that E also satisfies item (I).

In the following chapters, we will develop an alternative approach to proving this result. In particular, we will realize the map

$$F \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot)) S_i^*$$

as a Lipschitz contraction on a complete metric space of projection-valued measures from the Borel subsets of X into the projections on $L^2(X, \mu)$. We will then be able to use the Contraction Mapping Theorem to guarantee the existence and uniqueness of a projection-valued measure satisfying item (I) above. By induction, we will obtain item (II).

The contents of Chapter 3 include defining a generalized Kantorovich metric on the space of projection-valued measures from the Borel subsets of a compact metric space to the projections on a fixed Hilbert space (see Theorem 3.1.9). Importantly, we would like to note that this metric was defined previously by F. Werner in the setting of mathematical physics (see [26]) in 2004. We will develop new properties and applications of this metric. In particular, our application for this metric is discussed in the above paragraph; we will put X to be the (compact) fractal set, and $\mathcal{H}$ to be the Hilbert space $L^2(X, \mu)$ (or more generally, a Hilbert space which admits a representation of the Cuntz algebra on N generators). The main result of Chapter 3 is that this metric space is complete (see Theorem 3.1.15). In addition, we will define a weak topology on the space of projection-valued measures, and show that this topology coincides with the topology induced by the generalized Kantorovich metric (see Theorem 3.2.2). We will conclude the chapter by showing that this metric space is not compact, thereby identifying an important distinction from the classical setting (see Proposition 2.3.3).

There is a well known generalization of a projection-valued measure to that of a positive operator-valued measure. Positive operator-valued measures share many of the properties of projection-valued measures, except that they take values in the positive operators on a Hilbert space, rather than the projections on a Hilbert space. In Chapter 4, we will extend the generalized Kantorovich metric to positive operator-valued measures. We will show that this metric space is complete (see Theorem 4.1.7). Since we know by the above discussion that this metric space is not compact, we will introduce a topology on this collection of positive operator-valued measures which is compact (see Corollary 4.2.5). We will call this topology the WOT-weak topology. Importantly, this topology was previously introduced by S. Ali (see [1]), and he proved the compactness result using more general theory. Our proof will generalize a diagonalization argument that is used for the proof of an analogous fact in the classical setting (see Proposition 2.3.3).

In Chapter 5, we will consider the situation that the underlying metric space, say (Y, d), is complete and separable, but not necessarily compact. We will extend the results discussed in Chapter 2 to the generalized Kantorovich metric. In particular, we will show that a certain sub-collection of projection-valued measures from the Borel subsets of Y to the projections on a fixed Hilbert space forms a complete metric space with respect to the generalized Kantorovich metric (see Theorem 5.2.2).

In Chapter 6, we will present the main application of our above developed theory. Specifically, we will consider the situation that the underlying metric space (X, d) is the fractal set corresponding to an iterated function system, and that $\mathcal{H} = L^2(X, \mu)$, or more generally that $\mathcal{H}$ is a Hilbert space which admits a representation of the Cuntz algebra on N generators. In this setting, we will show that the map

$$F \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot)) S_i^*,$$

is indeed a Lipschitz contraction on the space of projection-valued measures with respect to the generalized Kantorovich metric (see Theorem 6.2.2). Therefore, this map will have a unique fixed point E satisfying item (I) above. The fact that E satisfies item (II) will follow with an induction argument. We will conclude this chapter with a brief foray into the topic of multifunctions, which are set-valued functions.

In 1996, A. Edalat introduced the notion of a weak hyperbolic iterated function system (see [13]), which we will abbreviate as whIFS. In Chapter 7, we will discuss a specific example of a whIFS which is closely related to the classical Cantor set. The members of this whIFS are $\tau_0 : [0,1] \to [0,1]$ by $\tau_0(x) = \frac{1}{3}x^3$ and $\tau_1 : [0,1] \to [0,1]$ by $\tau_1(x) = \frac{1}{3}x^3 + \frac{2}{3}$. The reader will note that these functions are 'almost' Lipschitz contractions; their derivative is less than 1 except at $x = 1$, where their derivative is equal to 1. Indeed, we will show that these functions satisfy the conditions of a whIFS, and moreover, we extend the results of Chapter 6 to this example.

In Chapter 8, we will discuss unitary representations of the Baumslag-Solitar group associated to the Cantor set, which is the fractal set constructed from the IFS $\sigma_0(x) = \frac{1}{3}x$ and $\sigma_1(x) = \frac{1}{3}x + \frac{2}{3}$ on $[0,1]$. The Baumslag-Solitar group, denoted $BS(1, N)$, is the group on two generators, a and b, with the relation $aba^{-1} = b^N$ where $N \in \mathbb{N}$. For the Cantor set, we take $N = 3$. Specifically, we will discuss two Hilbert spaces which admit isomorphic unitary representations of $BS(1, 3)$, the first being an L^2-space on a so called inflated fractal space (see [12]), and the second being an L^2-space on a compact topological group called the 3-solenoid (see [11]). The main result of this chapter is to construct a family of partial isometries $\{T_i\}_{i=0}^{N-1}$ defined on the latter Hilbert space which satisfy the following Cuntz-like relations:

- $\displaystyle\sum_{i=0}^{N-1} T_iT_i^* = \sum_{i=0}^{N-1} T_i^*T_i = \mathbf{1}_{\mathcal{H}}$, and
- $T_i^*T_j = T_iT_j^* = 0$ if $i \neq j$ where $0 \leq i, j \leq N-1$.

It will turn out that the two unitary operators on the latter Hilbert space which satisfy the relation of $BS(1,3)$ can be derived from these partial isometries. We note that this construction can be generalized to any $N \in \mathbb{N}$ with $N \geq 2$.

In the final chapter, we will calculate Fourier transform formulas for several measures on the solenoid. These measures will be constructed from so called generating filters. One of these filters will be related to the Cantor set.

We note that some of the results below can be found in a recent paper by the author (see [10]). Accordingly, we will use the citation [10] next to some of the results.

Chapter 2

The Classical Kantorovich Metric

In this chapter, we will discuss the classical Kantorovich metric, which provides a way of measuring the distance between Borel probability measures on a metric space. In the case that the underlying metric space is compact, the Kantorovich metric is defined on the collection of all Borel probability measures on the metric space, and the resulting metric space of measures is compact (in particular, complete). However, if the metric space is not compact, one must restrict the Kantorovich metric to a sub-collection of Borel probability measures on the metric space, where it is defined. The choice for this sub-collection may or may not yield a complete metric space of measures. To our knowledge, there are two standard choices for a sub-collection which have been studied, and which we will discuss; one which is not complete in the Kantorovich metric and one which is complete. We will show that the completion of the former metric space is the latter.

2.1 Defining the Kantorovich Metric

Let (Y, d) be a complete and separable metric space, and let μ and ν be two Borel probability measures on Y. The Kantorovich metric, H, between the two measures, μ and ν, is given by

$$H(\mu, \nu) = \sup_{f \in \mathrm{Lip}_1(Y)} \left\{ \left| \int_Y f d\mu - \int_Y f d\nu \right| \right\}, \tag{2.1}$$

where $\mathrm{Lip}_1(Y) = \{f : Y \to \mathbb{R} : |f(x) - f(y)| \leq d(x,y) \text{ for all } x, y \in Y\}$. Consider the following two collections of Borel probability measures on Y:

- $M_{\mathrm{loc}}(Y)$ is the collection of Borel probability measures on Y with bounded support. The support of a measure μ, denoted $\mathrm{supp}(\mu)$, is $\mathrm{supp}(\mu) = Y \backslash \cup \{A \subseteq Y : A \text{ is open and } \mu(A) = 0\}$.

- $M(Y)$ is the collection of Borel probability measures μ on Y such that $\int_Y |f| d\mu < \infty$ for all $f \in \mathrm{Lip}(Y)$, where $\mathrm{Lip}(Y)$ is the collection of all real valued Lipschitz functions on Y.

It was first claimed in [14] that $(M_{\mathrm{loc}}(Y), H)$ is a complete metric space. However, we will briefly outline an example, presented in [21], which shows this not to be true.

Claim 2.1.1. *[21][Kravchenko] Let (Y, d) be an unbounded metric space. Then $(M_{loc}(Y), H)$ is not complete.*

Proof. Choose a sequence of points $x_k \in Y$ for $k = 0, 1, 2, ...$, such that $d(x_0, x_k) \leq k$ for all k, and $d(x_k, x_0) \to \infty$. For a point $x \in Y$, define the delta measure at x by

$$\delta_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

For $n = 1, 2, 3, ...$, define the sequence of measures $\nu_n = 2^{-n}\delta_{x_0} + \Sigma_{k=1}^{n} 2^{-k}\delta_{x_k} \in M_{\mathrm{loc}}(Y)$. This sequence is Cauchy in $(M_{\mathrm{loc}}(Y), H)$. However, it can be shown that it does not converge to a measure in $(M_{\mathrm{loc}}(Y), H)$.

□

Since $(M_{\mathrm{loc}}(Y), H)$ is not a complete metric space (when Y is unbounded), we consider the larger sub-collection of measures, $M(Y)$, equipped with the H metric. Indeed, we will review that $M_{\mathrm{loc}}(Y) \subseteq M(Y)$.

Definition 2.1.2. *A measure μ on the metric space Y is said to be regular if for every Borel subset $A \subseteq Y$, and every $\epsilon > 0$, there exists a closed set F and an open set G such that $F \subseteq A \subseteq G$ and $\mu(G \setminus F) < \epsilon$.*

Definition 2.1.3. *A measure* μ *on the metric space* Y *is said to be tight if for every* $\epsilon > 0$*, there exists a compact set* K *such that* $\mu(Y \setminus K) < \epsilon$*.*

Remark 2.1.4. *Since* Y *is a complete and separable metric space, every Borel probability measure on* Y *is regular and tight (see Ch. 1, Section 1 in [5]). In particular, the measures in* $M(Y)$ *and* $M_{loc}(Y)$ *are all regular and tight.*

Lemma 2.1.5. *[5][Ch. 1, Section 1 in Billingsley] A Borel probability measure* μ *is tight on the metric space* Y *if and only if for each Borel subset* $A \subseteq Y$*,* $\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \text{ compact }\}$*.*

Corollary 2.1.6. *If* μ *is a Borel probability measure which is tight on the metric space* Y*, then* $\mu(Y \setminus supp(\mu)) = 0$*.*

Proof. Note that $Y \setminus \text{supp}(\mu) = \cup\{A \subseteq Y : A \text{ is open and } \mu(A) = 0\}$ which is a Borel set in Y. Therefore by Lemma 2.1.5, $\mu(Y \setminus \text{supp}(\mu)) = \sup\{\mu(K) : K \subseteq Y \setminus \text{supp}(\mu) \text{ and } K \text{ compact }\}$. Now if $K \subseteq Y \setminus \text{supp}(\mu)$, then since K is compact, it has a finite subcovering by μ-measure zero open sets. Hence, $\mu(K) = 0$, and therefore $\mu(Y \setminus \text{supp}(\mu)) = 0$. □

Proposition 2.1.7. $M_{loc}(Y) \subseteq M(Y)$*.*

Proof. Let $\mu \in M_{\text{loc}}(Y)$. To show that $\mu \in M(Y)$, we need to show that $\int_Y |f| d\mu < \infty$ for all $f \in \text{Lip}(Y)$. Choose $f \in \text{Lip}(Y)$ with Lipschitz constant γ, and choose a point $x_0 \in Y$. Since μ has bounded support, we can assume that there exists a $K \geq 0$ such that $\text{supp}(\mu) \subseteq B_K(x_0)$, where $B_K(x_0) = \{x \in Y : d(x, x_0) \leq K\}$. Moreover, $\mu(Y \setminus B_K(x_0)) = 0$ by Corollary 2.1.6. This implies that

$$\begin{aligned} \int_Y |f| d\mu &= \int_{B_K(x_0)} |f| d\mu + \int_{Y \setminus B_K(x_0)} |f| d\mu \\ &= \int_{B_K(x_0)} |f| d\mu. \end{aligned}$$

Continuing, observe that

$$\begin{aligned}
\int_{B_K(x_0)} |f(x)|d\mu(x) &\leq \int_{B_K(x_0)} |f(x) - f(x_0)|d\mu(x) + \int_{B_K(x_0)} |f(x_0)|d\mu(x) \\
&\leq \int_{B_K(x_0)} \gamma d(x, x_0)d\mu(x) + \int_{B_K(x_0)} |f(x_0)|d\mu(x) \\
&\leq \gamma K \mu(B_K(x_0)) + |f(x_0)|\mu(B_K(x_0)) \\
&< \infty.
\end{aligned}$$

This shows that $M_{\text{loc}}(Y) \subseteq M(Y)$.

□

Up to now we have been tacitly assuming that H is a metric on $M(Y)$. This was shown by C. Akerlund-Bistrom in [6]. Akerlund-Bistrom also proved that if $Y = \mathbb{R}^n$, then $(M(Y), H)$ is a complete metric space. The author A. S. Kravchenko recently proved in [21] the general case (Y an arbitrary complete and separable metric space).

Theorem 2.1.8. *[21][Kravchenko] The metric space* $(M(Y), H)$ *is complete.*

2.2 The Completion of $M_{\text{loc}}(Y)$

In this section, we will show that the metric space completion of $(M_{\text{loc}}(Y), H)$ is $(M(Y), H)$. This question was posed by A. Gorokhovsky during a seminar talk that the author presented at the University of Colorado in November 2012.

Theorem 2.2.1. *[Davison]* $(M(Y), H)$ *is the completion of the metric space* $(M_{loc}(Y), H)$.

Proof. Suppose that μ is a Borel probability measure in $M(Y)$. We need to find a sequence of measures $\{\mu_n\}_{n=1}^{\infty} \subseteq M_{\text{loc}}(Y)$ such that $\mu_n \to \mu$ in the H metric. We know from earlier, namely Lemma 2.1.5, that there exists a sequence of compact subsets $\{K_n\}_{n=1}^{\infty}$ of Y such that $\lim_{n\to\infty} \mu(K_n) = 1$. We can choose this sequence of compact sets such that $K_1 \subseteq K_2 \subseteq K_3 \subseteq ...$, because the union of finitely many compact sets is compact, and because measures are monotone. Next, choose some $x_0 \in K_1$. Since each K_n is compact, it is bounded so there exists a positive integer

k_n such that $K_n \subseteq B_{k_n}(x_0)$, where $B_{k_n}(x_0) = \{x \in Y : d(x, x_0) \leq k_n\}$. For each $n = 1, ..., \infty$, define a Borel measure μ_n on Y by $\mu_n(\Delta) = \dfrac{\mu(\Delta \cap K_n)}{\mu(K_n)}$ for all Borel subsets $\Delta \subseteq Y$. Furthermore for $f \in C(Y)$, note that

$$\int_Y f d\mu_n = \frac{1}{\mu(K_n)} \int_Y f \mathbf{1}_{K_n} d\mu.$$

We claim that each μ_n has bounded support. Consider the open set $Y \setminus K_n$.

$$\mu_n(Y \setminus K_n) = \frac{\mu((Y \setminus K_n) \cap K_n)}{\mu(K_n)} = 0,$$

and hence the support of μ_n is contained within the bounded set K_n. Also, observe that

$$\mu_n(Y) = \frac{\mu(Y \cap K_n)}{\mu(K_n)} = \frac{\mu(K_n)}{\mu(K_n)} = 1,$$

so that μ_n is a Borel probability measure on Y. We have shown that for all $n = 1, 2, ...$, $\mu_n \in M_{\text{loc}}(Y)$. It remains to show that $\mu_n \to \mu$ in the H metric. For this we use the alternate formulation for the H metric which is shown in [6]; namely

$$H(\mu_n, \mu) = \sup_{f \in \text{Lip}_1(x_0)} \left\{ \left| \int_Y f d\mu_n - \int_Y f d\mu \right| \right\},$$

where $\text{Lip}_1(x_0)$ are the $\text{Lip}_1(Y)$ functions which vanish at x_0. Let $\epsilon > 0$. Choose some $f \in \text{Lip}_1(x_0)$. Then

$$\left| \int_Y f d\mu_n - \int_Y f d\mu \right| = \left| \frac{1}{\mu(K_n)} \int_Y f \mathbf{1}_{K_n} d\mu - \int_Y f d\mu \right|$$

$$= \frac{1}{\mu(K_n)} \left| \int_Y f \mathbf{1}_{K_n} - \mu(K_n) f d\mu \right|$$

$$\leq \frac{1}{\mu(K_n)} \left| \int_{K_n} (f \mathbf{1}_{K_n} - \mu(K_n) f) d\mu \right| + \frac{1}{\mu(K_n)} \left| \int_{Y \setminus K_n} (f \mathbf{1}_{K_n} - \mu(K_n) f) d\mu \right|$$

$$\leq \left(\frac{1 - \mu(K_n)}{\mu(K_n)} \int_{K_n} |f| d\mu \right) + \int_{Y \setminus K_n} |f| d\mu$$

$$\leq \left(\frac{1 - \mu(K_n)}{\mu(K_n)} \int_{K_n} d(x, x_0) d\mu \right) + \int_{Y \setminus K_n} d(x, x_0) d\mu := I(n),$$

where the last inequality is because $|f(x)| = |f(x) - f(x_0)| \leq d(x, x_0)$.

Since $\mu \in M(Y)$ and $d(x, x_0) \in \text{Lip}_1(Y) \subseteq \text{Lip}(Y)$

$$0 \leq \int_Y d(x, x_0) d\mu := L < \infty.$$

Because $d(x, x_0)$ is a non-negative function, we note that for all n, $0 \leq \int_{K_n} d(x, x_0) d\mu \leq L < \infty$ and $0 \leq \int_{Y \setminus K_n} d(x, x_0) d\mu \leq L < \infty$.

Since $\lim_{n\to\infty} \mu(K_n) = \mu(Y) = 1$, and $K_1 \subseteq K_2 \subseteq ...$, observe that $\mathbf{1}_{Y\setminus K_n} d(x, x_0)$ decreases pointwise to 0 μ-almost everywhere. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{Y\setminus K_n} d(x, x_0) d\mu = \lim_{n\to\infty} \int_Y \mathbf{1}_{Y\setminus K_n} d(x, x_0) d\mu = \int_Y \lim_{n\to\infty} \mathbf{1}_{Y\setminus K_n} d(x, x_0) d\mu = 0.$$

Also, $\lim_{n\to\infty} \left(\frac{1 - \mu(K_n)}{\mu(K_n)} \right) = 0$. Choose an N such that for $n \geq N$,

$$\left(\frac{1 - \mu(K_n)}{\mu(K_n)} \right) \leq \frac{\epsilon}{2L},$$

and

$$\int_Y \mathbf{1}_{Y\setminus K_n} d(x, x_0) d\mu \leq \frac{\epsilon}{2}.$$

For $n \geq N$, $I(n) \leq \frac{\epsilon}{2L}(L) + \frac{\epsilon}{2} = \epsilon$. Since the choice of N is independent of the choice of $f \in \text{Lip}_1(x_0)$, we can conclude that $H(\mu_n, \mu) \leq I(n) \leq \epsilon$. Therefore, we have shown that $M(Y)$ is the completion of the metric space $M_{\text{loc}}(Y)$ in the H metric.

□

2.3 The Compact Case

In this section, we consider the case that the underlying metric space is compact, and record the known facts. Accordingly for this section, assume that (X, d) is a compact (and therefore separable) metric space.

Remark 2.3.1. *In this case, $M_{loc}(X) = M(X)$. Indeed, since compact metric spaces are bounded $M_{loc}(X)$ is the collection of all Borel probability measures on X. Since Lipschitz functions on a* *compact metric space are bounded, $M(X)$ also is the collection of all Borel probability measures on X.*

Definition 2.3.2. *A sequence of measures $\{\mu_n\}_{n=1}^{\infty} \subseteq M(X)$ converges weakly to a measure $\mu \in M(X)$, written $\mu_n \Rightarrow \mu$, if for all $f \in C_{\mathbb{R}}(X)$, $\int_X f d\mu_n \to \int_X f d\mu$, where $C_{\mathbb{R}}(X)$ is the set of all continuous real valued functions on X.*

The motivation behind the above definition is the following. For each $f \in C_{\mathbb{R}}(X)$, define a mapping $\hat{f} : M(X) \to \mathbb{R}$ by $\mu \mapsto \int_X f d\mu$. For each $\nu \in M(X)$, for each $\epsilon > 0$, and for any finite collection of functions $\{f_1, ..., f_k\} \subseteq C_{\mathbb{R}}(X)$, consider the subset $\{\mu \in M(X) : |\hat{f}_j(\mu) - \hat{f}_j(\nu)| < \epsilon \text{ for all } 1 \leq j \leq k\}$ of $M(X)$. If we consider the collection of all finite intersections of such sets, we obtain a basis for a topology on $M(X)$ which is called the weak topology on $M(X)$. Note that since $C_{\mathbb{R}}(X)$ is a separable metric space, the weak topology is first countable, and hence can be characterized by sequences.

This leads us to the following facts, presented by F. Latremoliere in [20].

Proposition 2.3.3. *[20][Latremoliere]*

(1) $(M(X), H)$ is a compact metric space.

(2) The H metric on $M(X)$ induces a topology which coincides with the weak topology on $M(X)$.

(3) The map $\iota : X \to M(X)$ given by $x \mapsto \delta_x$ is an injective metric space isometry (where δ_x is the delta measure at x).

In the following chapter, we will generalize the classical Kantorovich metric to projection-valued measures. We will refer back to Proposition 2.3.3 in order to observe what elements of this proposition are retained or lost in the generalized setting.

2.4 A Modified Kantorovich Metric

We return to the situation that (Y, d) is an arbitrary complete and separable metric space. Let $Q(Y)$ denote the collection of all Borel probability measures on Y. Hence, we have the containment

$$M_{\text{loc}}(Y) \subseteq M(Y) \subseteq Q(Y).$$

Define a modified Kantorovich metric, MH, on $Q(Y)$ as follows. For $\mu, \nu \in Q(Y)$,

$$MH(\mu, \nu) = \sup\left\{\left|\int_Y f d\mu - \int_Y f d\nu\right| : f \in \text{Lip}_1(Y) \text{ and } ||f||_\infty \leq 1\right\}. \tag{2.2}$$

The condition $||f||_\infty \leq 1$ guarantees that MH will be well defined (finite) on $Q(Y)$. We note that the MH metric is equivalent to the metric, D, given by

$$D(\mu, \nu) = \sup\left\{\left|\int_Y f d\mu - \int_Y f d\nu\right| : ||f||_{\text{Lip}_b(Y)} \leq 1\right\}, \tag{2.3}$$

where $|| \cdot ||_{\text{Lip}_b(Y)}$ is a norm defined on $\text{Lip}_b(Y)$, the collection of real valued bounded Lipschitz functions on Y. The norm is given by

$$||f||_{\text{Lip}_b(Y)} = ||f||_\infty + \text{Lip}(f),$$

where $\text{Lip}(f)$ denotes the Lipschitz constant of f (see Section 8.3 of [7]).

We can equip $Q(Y)$ with the weak topology. Indeed, a net of measures $\{\mu_\lambda\}_{\lambda \in \Lambda} \subseteq Q(Y)$ converges weakly to a measure $\mu \in Q(Y)$, if for all $f \in C_b(Y)$, $\int_Y f d\mu_\lambda \to \int_Y f d\mu$, where $C_b(Y)$ is the set of all bounded continuous real valued functions on Y. The following result can be found in Section 8.3 of [7].

Theorem 2.4.1. *[7][Section 8.3 in Bogachev] The weak topology on $Q(Y)$ coincides with the topology induced by the MH metric on Y.*

We now state a result recently proved in [21]. We will use Proposition 2.4.2 in later chapters.

Proposition 2.4.2. *[21] [Kravchenko] Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of Borel measures on the complete and separable metric space Y such that $\mu_n(Y) = K < \infty$ for all $n = 1, 2, ...,$ and such that for all*

$f \in Lip_b(Y)$, the sequence $\{\int_Y f d\mu_n\}_{n=1}^{\infty}$ of real numbers is Cauchy. Then there exists a Borel measure μ on Y such that $\mu(Y) = K$, and such that the sequence $\{\mu_n\}_{n=1}^{\infty}$ converges in the weak topology to μ.

We note that this proposition will imply that $(Q(Y), MH)$ is a complete metric space. Indeed, if $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence of measures in $Q(Y)$, one can show that for all $f \in \mathrm{Lip}_b(Y)$, the sequence $\{\int_Y f d\mu_n\}_{n=1}^{\infty}$ of real numbers is Cauchy. Therefore, by the above proposition there will exist a Borel probability measure μ such that μ_n converges to μ in the weak topology, or equivalently, in the MH metric. We now adapt Theorem 2.2.1 to this setting.

Theorem 2.4.3. *[Davison] The completion of the metric space $(M_{loc}(Y), MH)$ is $(Q(Y), MH)$.*

Proof. The proof of this theorem is similar to the earlier proof of Theorem 2.2.1. Suppose that $\mu \in Q(Y)$. We need to find a sequence of measures $\{\mu_n\}_{n=1}^{\infty} \subseteq M_{\mathrm{loc}}(Y)$ such that $\mu_n \to \mu$ in the MH metric. Define, exactly as before, a sequence of measures $\{\mu_n\}_{n=1}^{\infty} \subseteq M_{\mathrm{loc}}(Y)$. In particular, μ_n satisfies $\int f d\mu_n = \frac{1}{\mu(K_n)} \int_Y f \mathbf{1}_{K_n} d\mu$ for all $f \in C(Y)$. Choose $f \in \mathrm{Lip}_1(X)$ such that $||f||_\infty \leq 1$. Then

$$\begin{aligned}
\left| \int_Y f d\mu_n - \int_Y f d\mu \right| &= \left| \int_Y f d\mu_n - \int_Y f d\mu \right| \\
&= \left| \frac{1}{\mu(K_n)} \int_Y f \mathbf{1}_{K_n} d\mu - \int_Y f d\mu \right| \\
&= \frac{1}{\mu(K_n)} \left| \int_Y f \mathbf{1}_{K_n} - \mu(K_n) f d\mu \right| \\
&\leq \frac{1}{\mu(K_n)} \left| \int_{K_n} (f \mathbf{1}_{K_n} - \mu(K_n) f) d\mu \right| \\
&+ \frac{1}{\mu(K_n)} \left| \int_{Y \setminus K_n} (f \mathbf{1}_{K_n} - \mu(K_n) f) d\mu \right| \\
&\leq \left(\frac{1 - \mu(K_n)}{\mu(K_n)} \int_{K_n} |f| d\mu \right) + \int_{Y \setminus K_n} |f| d\mu \\
&\leq \left(\frac{1 - \mu(K_n)}{\mu(K_n)} \int_{K_n} 1 d\mu \right) + \int_{Y \setminus K_n} 1 d\mu \\
&\leq (1 - \mu(K_n)) + \mu(Y \setminus K_n) \\
&= 2\mu(Y \setminus K_n).
\end{aligned}$$

The last line of the above expression is independent of the choice of f and goes to zero as n goes to infinity. Hence, $\mu_n \to \mu$ in the MH metric.

$\square$

Chapter 3

Generalizing the Kantorovich Metric

In this chapter, we will generalize the classical Kantorovich metric to projection-valued measures from a compact metric space into the projections on a fixed Hilbert space. In particular, we will discuss what properties of the metric space are retained or lost in the generalized setting. Recall that Proposition 2.3.3 reviews the main properties of the classical Kantorovich metric space, in the case that the underlying metric space is compact.

3.1 A Metric Space of Projection-Valued Measures on X:

Let (X, d) be a compact (and therefore separable) metric space, and let $\mathcal{H}$ be an arbitrary Hilbert space. We take the convention that the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is linear in the first coordinate and conjugate linear in the second coordinate. Let $\mathscr{B}(X)$ denote the σ-algebra of Borel subsets of X, and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded operators on $\mathcal{H}$. We first will recall several basic facts about projection-valued measures.

Definition 3.1.1. *A projection $P \in \mathcal{B}(\mathcal{H})$ satisfies $P^* = P$ and $P^2 = P$.*

For the next definition, we denote the empty set by $\emptyset$.

Definition 3.1.2. *A projection-valued measure with respect to the pair $(X, \mathcal{H})$ is a map $F : \mathscr{B}(X) \to \mathcal{B}(\mathcal{H})$ such that:*

- *$F(\Delta)$ is a projection in $\mathcal{B}(\mathcal{H})$ for all $\Delta \in \mathscr{B}(X)$;*

- $F(\emptyset) = 0$ *and* $F(X) = id_{\mathcal{H}}$ *(the identity operator on* $\mathcal{H}$*);*

- $F(\Delta_1 \cap \Delta_2) = F(\Delta_1)F(\Delta_2)$ *for all* $\Delta_1, \Delta_2 \in \mathscr{B}(X)$ *(where the product* $F(\Delta_1)F(\Delta_2)$ *is in* $\mathcal{B}(\mathcal{H})$*);*

- *If* $\{\Delta_n\}_{n=1}^{\infty}$ *is a sequence of pairwise disjoint sets in* $\mathscr{B}(X)$*, and if* $g, h \in \mathcal{H}$*, then*

$$\left\langle F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) g, h \right\rangle = \sum_{n=1}^{\infty} \langle F(\Delta_n)g, h\rangle.$$

Lemma 3.1.3. *[8][Lemma IX.1.9 in Conway] Let* F *be a projection-valued measure with respect to the pair* $(X, \mathcal{H})$*, and let* $g, h \in \mathcal{H}$*. For all* $\Delta \in \mathscr{B}(X)$ *define*

$$F_{g,h}(\Delta) = \langle F(\Delta)g, h\rangle.$$

Then $F_{g,h}(\cdot)$ *defines a countably additive (complex-valued) measure on* $\mathscr{B}(X)$ *with total variation less than or equal to* $||g||\ ||h||$*. Moreover,* $F_{g,h}(\cdot) = \overline{F_{h,g}(\cdot)}$*.*

Remark 3.1.4. *If* $h \in \mathcal{H}$*, then* $F_{h,h}(\cdot)$ *is a positive measure with* $F_{h,h}(X) = ||h||^2$*. This follows from the fact that projections are positive operators, and that* $F(X) = id_{\mathcal{H}}$*. Also,* $F_{h,h}(\cdot)$ *is a regular measure on* $\mathscr{B}(X)$*. This follows from Remark 2.1.4.*

Lemma 3.1.5. *[8][Proposition IX.1.10 in Conway] Let* F *be a projection-valued measure with respect to the pair* $(X, \mathcal{H})$*. Let* $\psi : X \to \mathbb{C}$ *be a bounded Borel measurable function. Then there exists a unique bounded operator,* $\int \psi dF$*, that satisfies*

$$\left\langle \left(\int \psi dF\right) g, h \right\rangle = \int_X \psi dF_{g,h},$$

for all $g, h \in \mathcal{H}$*. Moreover,* $||\int \psi dF|| \leq ||\psi||_\infty$*, where* $||\cdot||$ *denotes the operator norm, and* $||\cdot||_\infty$ *denotes the supremum norm.*

Lemma 3.1.6. *Let* F *be a projection-valued measure with respect to the pair* $(X, \mathcal{H})$*, and let* $\psi : X \to \mathbb{R}$ *be continuous. Then* $\int \psi dF$ *is a self-adjoint operator on* $\mathcal{B}(\mathcal{H})$*.*

Proof. Since ψ is continuous on the compact space X, it is a bounded Borel measurable function. Therefore, one can define $\int \psi dF$ according to Lemma 3.1.5. If $g, h \in \mathcal{H}$, then

$$\begin{aligned}
\left\langle \left(\int \psi dF \right) g, h \right\rangle &= \int_X \psi dF_{g,h} \\
&= \int_X \overline{\psi} d\overline{F_{h,g}} \\
&= \overline{\int_X \psi dF_{h,g}} \\
&= \overline{\left\langle \left(\int \psi dF \right) h, g \right\rangle} \\
&= \left\langle g, \left(\int \psi dF \right) h \right\rangle,
\end{aligned}$$

where the second equality depends on the fact that ψ is real-valued.

□

Lemma 3.1.7. *If $\lambda \in \mathbb{C}$ is a constant, then*

$$\int \lambda dF = \lambda id_{\mathcal{H}}.$$

Proof. Let $g, h \in \mathcal{H}$. Then

$$\begin{aligned}
\left\langle \left(\int \lambda dF \right) g, h \right\rangle &= \int_X \lambda dF_{g,h}(x) \\
&= \lambda \int_X dF_{g,h}(x) \\
&= \lambda \langle F(X) g, h \rangle \\
&= \lambda \langle \mathrm{id}_{\mathcal{H}} g, h \rangle \\
&= \langle \lambda \mathrm{id}_{\mathcal{H}} g, h \rangle.
\end{aligned}$$

□

Let $P(X)$ be the collection of all projection-valued measures with respect to the pair $(X, \mathcal{H})$. As we mentioned in Chapter 1, the author recently discovered that the below metric was also defined by F. Werner in [26] in 2004. We will develop new properties and applications of the metric.

Definition 3.1.8. *[10][26][Davison, Werner] Define a metric ρ on $P(X)$ by*

$$\rho(E,F) = \sup_{f \in Lip_1(X)} \left\{ \left|\left| \int f dE - \int f dF \right|\right| \right\}, \tag{3.1}$$

where $||\cdot||$ denotes the operator norm in $\mathcal{B}(\mathcal{H})$, and E and F are arbitrary members of $P(X)$.

This metric directly generalizes the Kantorovich metric discussed in Chapter 2.

Theorem 3.1.9. *[10][Davison] ρ is a metric on $P(X)$.*

Proof.

(1) Let $E, F \in P(X)$. We will show that ρ is well defined (i.e. $\rho(E,F) < \infty$). Let $f \in \mathrm{Lip}_1(X)$ and $x_0 \in X$. By Lemma 3.1.7, with $\lambda = f(x_0)$,

$$\begin{aligned} \left|\left| \int f dE - \int f dF \right|\right| &= \left|\left| \int f dE - f(x_0)\mathrm{id}_{\mathcal{H}} + f(x_0)\mathrm{id}_{\mathcal{H}} - \int f dF \right|\right| \\ &= \left|\left| \int f dE - \int f(x_0) dE - \left(\int f dF - \int f(x_0) dF \right) \right|\right| \\ &\leq \left|\left| \int (f - f(x_0)) dE \right|\right| + \left|\left| \int (f - f(x_0))\, dF \right|\right| \end{aligned} \tag{3.2}$$

By Lemma 3.1.6, since $f - f(x_0)$ is a real-valued continuous function on X, $\int (f - f(x_0)) dE$ is a self-adjoint operator, and therefore

$$\left|\left| \int (f - f(x_0)) dE \right|\right| = \sup_{h \in \mathcal{H}, ||h||=1} \left\{ \left| \left\langle \left(\int (f(x) - f(x_0)) dE \right) h, h \right\rangle \right| \right\}.$$

Let $h \in \mathcal{H}$ with $||h|| = 1$. Then

$$\left| \left\langle \left(\int (f(x) - f(x_0)) dE \right) h, h \right\rangle \right| = \left| \int_X (f(x) - f(x_0)) dE_{h,h}(x) \right| \leq$$

$$\int_X |f(x) - f(x_0)| dE_{h,h}(x) \leq \int_X d(x, x_0) dE_{h,h}(x) \leq$$

$$\mathrm{diam}(X) \int_X dE_{h,h}(x) = \mathrm{diam}(X) \langle E(X)h, h \rangle = \mathrm{diam}(X) ||h||^2 = \mathrm{diam}(X) < \infty,$$

where $\mathrm{diam}(X)$ denotes the diameter of the metric space X. This quantity is finite because X is compact. Hence

$$\left|\left| \int (f - f(x_0)) dE \right|\right| \leq \mathrm{diam}(X) < \infty,$$

and by the same argument,

$$\left|\left|\int (f - f(x_0))dF\right|\right| \leq \text{diam}(X) < \infty,$$

which implies that the last line of (3.2) is less than or equal to 2 diam$(X) < \infty$. Since diam(X) is independent of the choice of $f \in \text{Lip}_1(X)$, $\rho(E, F) \leq 2\ \text{diam}(X) < \infty$.

(2) Let $E, F \in P(X)$. It is clear from the definition of ρ that $\rho(E, F) = \rho(F, E)$.

(3) Let $E, F \in P(X)$. We will show that $\rho(E, F) = 0$ if and only if $E = F$. The backwards direction is clear from the definition of ρ. For the forwards direction, suppose that $\rho(E, F) = 0$. We need to show that $E = F$. That is, for all $\Delta \in \mathscr{B}(X)$, we need to show that $E(\Delta) = F(\Delta)$. Choose a closed subset $C \subseteq X$. Define $f_n : X \to \mathbb{R}$ for $n = 1, ...\infty$ by $f_n(x) = \max\{1 - nd(x, C), 0\}$. Note that $f_n \in \text{Lip}_n(X) = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq nd(x, y)$ for all $x, y \in X\}$. Therefore, $\frac{1}{n} f_n \in \text{Lip}_1(X)$. Since $\rho(E, F) = 0$

$$\int \frac{1}{n} f_n dE = \int \frac{1}{n} f_n dF$$

for all n, which implies

$$\int f_n dE = \int f_n dF \tag{3.3}$$

for all n. Note that $f_n \downarrow \mathbf{1}_C$ pointwise. Also, f_1 is bounded and therefore integrable with respect to Borel probability measures on X. Choose $h \in \mathcal{H}$ with $||h|| = 1$. By the dominated convergence theorem,

$$E_{h,h}(C) = \int_X \mathbf{1}_C dE_{h,h} = \lim_{n\to\infty} \int_X f_n dE_{h,h}$$

and,

$$F_{h,h}(C) = \int_X \mathbf{1}_C dF_{h,h} = \lim_{n\to\infty} \int_X f_n dF_{h,h}.$$

By (3.3),

$$\int_X f_n dE_{h,h} = \int_X f_n dF_{h,h},$$

for all n, and hence, $E_{h,h}(C) = F_{h,h}(C)$ for all closed sets $C \subseteq X$. Since $E_{h,h}(\cdot)$ and $F_{h,h}(\cdot)$ are regular measures (see Remark 3.1.4), $E_{h,h}(\Delta) = F_{h,h}(\Delta)$ for all $\Delta \in \mathscr{B}(X)$. Equivalently, $\langle (E(\Delta) - F(\Delta))h, h\rangle = 0$ for all $\Delta \in \mathscr{B}(X)$. Since $E(\Delta) - F(\Delta)$ is a self-adjoint operator (being the difference of two projections),

$$||E(\Delta) - F(\Delta)|| = \sup_{h \in \mathcal{H}, ||h||=1} |\langle (E(\Delta) - F(\Delta))h, h\rangle| = 0.$$

Therefore, $E(\Delta) = F(\Delta)$ for all $\Delta \in \mathscr{B}(X)$.

(4) Let $E, F, G \in P(X)$. We will show that ρ satisfies:

$$\rho(E, G) \leq \rho(E, F) + \rho(F, G). \tag{3.4}$$

Choose $f \in \mathrm{Lip}_1(X)$. Then

$$\left|\left|\int f dE - \int f dG\right|\right| \leq \left|\left|\int f dE - \int f dF\right|\right| + \left|\left|\int f dF - \int f dG\right|\right|.$$

By taking the supremum of both sides over all $\mathrm{Lip}_1(X)$ functions, inequality (3.4) follows.

□

Corollary 3.1.10. *[10][Davison] The metric space $(P(X), \rho)$ is bounded.*

Proof. In (1) of the above proof, we showed that for any $E, F \in P(X)$, $\rho(E, F) \leq 2\mathrm{diam}(X) < \infty$. □

We will now show that the metric space $(P(X), \rho)$ is complete.

Definition 3.1.11. *Let $C(X)$ denote the C^*-alegbra of continuous functions from X to $\mathbb{C}$, with pointwise operations. A representation $\pi : C(X) \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism that preserves the identity.*

Theorem 3.1.12. *[8][Proposition IX.1.12 in Conway] Let E be a projection-valued measure with respect to the pair $(X, \mathcal{H})$. The map $\pi : C(X) \to \mathcal{B}(\mathcal{H})$ given by*

$$f \mapsto \int f dE$$

is a representation.

Theorem 3.1.13. *[8][Theorem IX.1.14 in Conway] Let $\pi : C(X) \to \mathcal{B}(\mathcal{H})$ be a representation. There exists a unique projection-valued measure $E : \mathscr{B}(X) \to \mathcal{B}(\mathcal{H})$ such that*

$$\pi(f) = \int f dE$$

for all $f \in C(X)$.

Lemma 3.1.14. *[20] Lip(X) is dense in $C_{\mathbb{R}}(X)$, where Lip(X) is the collection of real-valued Lipschitz functions on X.*

Theorem 3.1.15. *[10][Davison] The metric space $(P(X), \rho)$ is complete.*

Proof. Let $\{E_n\}_{n=1}^{\infty} \subseteq P(X)$ be a Cauchy sequence of projection-valued measures in the ρ metric. For each $n = 1, 2, ...$, use Theorem 3.1.12 to define a representation $\pi_n : C(X) \to \mathcal{B}(\mathcal{H})$ by

$$f \mapsto \int f dE_n.$$

Claim 3.1.16. *Let $f \in C(X)$. The sequence of operators $\{\pi_n(f)\}_{n=1}^{\infty}$ is Cauchy in the operator norm.*

Proof of claim: Let $\epsilon > 0$. Let $f = f_1 + if_2$, where $f_1, f_2 \in C_{\mathbb{R}}(X)$. By Lemma 3.1.14, choose $g_1, g_2 \in \text{Lip}(X)$ such that $||f_1 - g_1||_\infty \leq \frac{\epsilon}{6}$ and $||f_2 - g_2||_\infty \leq \frac{\epsilon}{6}$.

There is a $K > 0$ such that $\frac{1}{K}g_1 \in \text{Lip}_1(X)$ and $\frac{1}{K}g_2 \in \text{Lip}_1(X)$. Since $\{E_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the ρ metric, the sequence $\{\pi_n(\frac{1}{K}g_1)\}_{n=1}^{\infty}$ is Cauchy in the operator norm, and hence the sequence $\{\pi_n(g_1)\}_{n=1}^{\infty}$ is Cauchy in the operator norm. Similarly, $\{\pi_n(g_2)\}_{n=1}^{\infty}$ is Cauchy in the operator norm. Therefore, choose N such that for $n, m \geq N$,

$$||\pi_n(g_1) - \pi_m(g_1)|| \leq \frac{\epsilon}{6} \text{ and } ||\pi_n(g_2) - \pi_m(g_2)|| \leq \frac{\epsilon}{6}.$$

If $m, n \geq N$,

$$\begin{aligned} ||\pi_n(f_1) - \pi_m(f_1)|| &\leq ||\pi_n(f_1) - \pi_n(g_1)|| + ||\pi_n(g_1) - \pi_m(g_1)|| + ||\pi_m(g_1) - \pi_m(f_1)|| \\ &\leq ||\pi_n(f_1 - g_1)|| + \frac{\epsilon}{6} + ||\pi_m(f_1 - g_1)|| \\ &\leq \frac{\epsilon}{2}, \end{aligned}$$

where the third inequality is because $||\pi_n(f_1 - g_1)|| \leq ||f_1 - g_1||_\infty$ and $||\pi_m(f_1 - g_1)|| \leq ||f_1 - g_1||_\infty$. Similarly, $||\pi_n(f_2) - \pi_m(f_2)|| \leq \frac{\epsilon}{2}$. Then if $n, m \geq N$,

$$\begin{aligned} ||\pi_n(f) - \pi_m(f)|| &= ||\pi_n(f_1 + if_2) - \pi_m(f_1 + if_2)|| \\ &= ||(\pi_n(f_1) - \pi_m(f_1)) + i(\pi_n(f_2) + \pi_m(f_2))|| \\ &\leq ||\pi_n(f_1) - \pi_m(f_1)|| + ||\pi_n(f_2) - \pi_m(f_2)|| \\ &\leq \epsilon. \end{aligned}$$

This proves the claim.

Define $\pi : C(X) \to \mathcal{B}(\mathcal{H})$ by $f \mapsto \lim_{n\to\infty} \pi_n(f)$. This map is well defined by Claim 3.1.16, and the fact that $\mathcal{B}(\mathcal{H})$ is complete in the operator norm. We show that π is a representation.

(1) π is linear: Let $f, g \in C(X)$ and $\alpha \in \mathbb{C}$. Then $\pi(\alpha f + g) = \lim_{n\to\infty} \pi_n(\alpha f + g) = \lim_{n\to\infty}(\alpha\pi_n(f) + \pi_n(g)) = \alpha \lim_{n\to\infty} \pi_n(f) + \lim_{n\to\infty} \pi_n(g) = \alpha\pi(f) + \pi(g)$.

(2) π is an algebra homomorphism: Let $f, g \in C(X)$. Then $\pi(fg) = \lim_{n\to\infty} \pi_n(fg) = \lim_{n\to\infty} \pi_n(f) \lim_{n\to\infty} \pi_n(g) = \pi(f)\pi(g)$.

(3) π is a $*$-homomorphism: Let $f \in C(X)$. Then $\pi(\overline{f}) = \lim_{n\to\infty} \pi_n(\overline{f}) = \lim_{n\to\infty} \pi_n(f)^* = \pi(f)^*$, where the last equality is because $||\pi_n(f) - \pi(f)|| = ||(\pi_n(f) - \pi(f))^*|| = ||\pi_n(f)^* - \pi(f)^*||$.

(4) π preserves the identity: $\pi(1) = \lim_{n\to\infty} \pi_n(1) = \lim_{n\to\infty} \mathbf{1}_{\mathcal{H}} = \mathbf{1}_{\mathcal{H}}$.

By Theorem 3.1.13, there exists a unique projection-valued measure E with respect to the pair $(X, \mathcal{H})$ such that $\pi(f) = \int f dE$ for all $f \in C(X)$. We show that $E_n \to E$ in the ρ metric as $n \to \infty$. Let $\epsilon > 0$. Choose N such that for $n, m \geq N$, $\rho(E_n, E_m) \leq \epsilon$. Let $n \geq N$ and $f \in \mathrm{Lip}_1(X)$. Observe that

$$\begin{aligned} \left\| \int f dE_n - \int f dE \right\| &= \lim_{m \to \infty} \left\| \int f dE_n - \int f dE_m \right\| \\ &\leq \epsilon, \end{aligned}$$

where the equality is because $\lim_{m\to\infty} \int f dE_m = \lim_{m\to\infty} \pi_m(f) = \pi(f) = \int f dE$, and the inequality is because $\rho(E_n, E_m) \leq \epsilon$ for $m \geq N$. Since the choice of N is independent of the choice of f, we have for $n \geq N$,

$$\rho(E_n, E) = \sup_{f \in \mathrm{Lip}_1(X)} \left\{ \left\| \int f dE_n - \int f dE \right\| \right\} \leq \epsilon.$$

Hence, $E_n \to E$ in the ρ metric as $n \to \infty$ and the metric space $(P(X), \rho)$ is complete.

□

3.2 The Weak Topology on $P(X)$

In this section, we will define the weak topology on the space $P(X)$. As in the classical setting (see Proposition 2.3.3), it will turn out that the weak topology will coincide with the topology induced by the generalized Kantorovich metric, ρ, on $P(X)$ (see Definition 3.1.8).

Definition 3.2.1. *A sequence of projection-valued measures $\{F_n\}_{n=1}^{\infty} \subseteq P(X)$ converges weakly to a projection-valued measure $F \in P(X)$, written $F_n \Rightarrow F$, if for all $f \in C_{\mathbb{R}}(X)$, $\int_Y f dF_n \to \int_Y f dF$, where convergence is in the operator norm on $\mathcal{B}(\mathcal{H})$.*

Theorem 3.2.2. *[10][Davison] The weak topology on $P(X)$ coincides with the topology induced by the ρ metric on $P(X)$.*

Proof. Suppose that $\{E_n\}_{n=1}^{\infty} \subseteq P(X)$ converges to a projection-valued measure $E \in P(X)$ in the ρ metric. We will show that $E_n \Rightarrow E$. Toward this end, let $\epsilon > 0$ and choose $f \in C_{\mathbb{R}}(X)$. We use Lemma 3.1.14 to choose a function $g \in \mathrm{Lip}(X)$, with Lipschitz constant $K > 0$, such that

$||f-g||_\infty \leq \frac{\epsilon}{3}$. Since $E_n \to E$ in the ρ metric, we know there exists an N such that for $n \geq N$, $\rho(E_n, E) \leq \frac{\epsilon}{3K}$. In particular, for $n \geq N$

$$\left|\left|\int_X \frac{g}{K} dE_n - \int_X \frac{g}{K} dE\right|\right| \leq \rho(E_n, E) \leq \frac{\epsilon}{3K},$$

which implies that

$$\left|\left|\int_X g dE_n - \int_X g dE\right|\right| \leq \frac{\epsilon}{3}.$$

Combining this information, we get that for $n \geq N$

$$\begin{aligned}
\left|\left|\int_X f dE_n - \int_X f dE\right|\right| &\leq \left|\left|\int_X f dE_n - \int_X g dE_n\right|\right| + \left|\left|\int_X g dE_n - \int_X g dE\right|\right| \\
&+ \left|\left|\int_X g dE - \int_X f dE\right|\right| \\
&\leq ||f-g||_\infty + \frac{\epsilon}{3} + ||f-g||_\infty \\
&= \epsilon,
\end{aligned}$$

which implies that $E_n \Rightarrow E$.

Next, suppose that $E_n \Rightarrow E$. We show that E_n converges to E in the ρ metric. Choose $x_0 \in X$. Consider the set $B = \{f \in C_{\mathbb{R}}(X) : f \in \mathrm{Lip}_1(X) \text{ and } f(x_0) = 0\}$.

- B is closed in the supremum norm in $C_{\mathbb{R}}(X)$: Suppose that $\{f_n\}_{n=1}^\infty \subseteq B$ converges in the supremum norm to some $f \in C_{\mathbb{R}}(X)$. We need to show that $f \in B$. In particular, choose $x, y \in X$. Since $\{f_n\}_{n=1}^\infty \subseteq B$ converges in the supremum norm to f, $\{f_n\}_{n=1}^\infty$ converges pointwise to f. Therefore

  $$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq |x - y|,$$

  because each $f_n \in B$. Also

  $$f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0.$$

  Hence $f \in B$.

- B is pointwise bounded: If $x \in X$, and $f \in B$, then $|f(x)| = |f(x) - f(x_0)| \leq d(x, x_0) \leq \operatorname{diam}(X) < \infty$.

- B is equicontinuous: Let $x \in X$ and $\epsilon > 0$. If $y \in X$ such that $d(x,y) < \epsilon$, $|f(x) - f(y)| \leq d(x,y) < \epsilon$ for all $f \in B$.

The above facts show that by Ascoli's Thoerem, see [23], B is compact in the supremum norm. Accordingly, choose $\{f_1, ..., f_k\} \subseteq B$ such that $B \subseteq \cup_{j=1}^{k} \mathcal{O}_{\frac{\epsilon}{3}}(f_j)$, where $\mathcal{O}_{\frac{\epsilon}{3}}(f_j)$ represents the open ball of radius $\frac{\epsilon}{3}$ centered at f_j. Since $E_n \Rightarrow E$, and $f_j \in C_{\mathbb{R}}(X)$ for all $1 \leq j \leq k$, there exists an N such that for $n \geq N$

$$\left|\left|\int_X f_j dE_n - \int_X f_j dE\right|\right| \leq \frac{\epsilon}{3}$$

for all $1 \leq j \leq k$. Let $g \in \operatorname{Lip}_1(X)$. Define $f(x) = g(x) - g(x_0)$, and note that $f \in B$. There exists an f_j such that $||f - f_j||_\infty \leq \frac{\epsilon}{3}$. Observe that if $n \geq N$,

$$\begin{aligned}
\left|\left|\int_X g dE_n - \int_X g dE\right|\right| &= \left|\left|\int_X f dE_n - \int_X f dE\right|\right| \\
&\leq \left|\left|\int_X f dE_n - \int_X f_j dE_n\right|\right| + \left|\left|\int_X f_j dE_n - \int_X f_j dE\right|\right| \\
&+ \left|\left|\int_X f_j dE - \int_X f dE\right|\right| \\
&\leq ||f - f_j||_\infty + \frac{\epsilon}{3} + ||f - f_j||_\infty \\
&= \epsilon.
\end{aligned}$$

Since N does not depend on the choice of g, $\rho(E_n, E) \leq \epsilon$ if $n \geq N$.

□

3.3 Isometry of Metric Spaces

In this section, we will show that an isomorphism of Hilbert spaces induces a bijective isometry between the associated metric spaces of projection-valued measures.

Theorem 3.3.1. *[Davison] Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two isomorphic Hilbert spaces with isomorphism $S : \mathcal{H}_1 \to \mathcal{H}_2$. Consider the two associated complete metric spaces $(P_{\mathcal{H}_1}(X), \rho_1)$ and $(P_{\mathcal{H}_2}(X), \rho_2)$. Define $\Theta : (P_{\mathcal{H}_1}(X), \rho_1) \to (P_{\mathcal{H}_2}(X), \rho_2)$ by*

$$E(\cdot) \mapsto SE(\cdot)S^*.$$

Then Θ is a bijective isometry of metric spaces.

Remark 3.3.2. *We note that S satisfies the relation $S^*S = id_{\mathcal{H}_1}$ and $SS^* = id_{\mathcal{H}_2}$.*

Proof. We will first show that Θ is well defined. Choose some $E \in P_{\mathcal{H}_1}(X)$ and show that $\Theta(E) \in P_{\mathcal{H}_2}(X)$. By construction, $\Theta(E)(\Delta)$ is a bounded operator in $\mathcal{B}(\mathcal{H}_2)$ for all Borel subsets $\Delta \subseteq X$.

- $\Theta(E)(\Delta)^* = (SE(\Delta)S^*)^* = SE(\Delta)S^* = \Theta(E)(\Delta)$ for all $\Delta \in \mathscr{B}(X)$.
- $\Theta(E)(\Delta)\Theta(E)(\Delta) = SE(\Delta)S^*SE(\Delta)S^* = SE(\Delta)E(\Delta)S^* = SE(\Delta)S^* = \Theta(E)(\Delta)$ for all $\Delta \in \mathscr{B}(X)$.
- $\Theta(E)(\emptyset) = SE(\emptyset)S^* = 0$.
- $\Theta(E)(X) = SE(X)S^* = SS^* = \mathbf{1}_{\mathcal{H}_2}$.
- $\Theta(E)(\Delta_1 \cap \Delta_2) = SE(\Delta_1 \cap \Delta_2)S^* = SE(\Delta_1)E(\Delta_2)S^* =$ $SE(\Delta_1)S^*SE(\Delta_2)S^* = \Theta(E)(\Delta_1)\Theta(E)(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathscr{B}(X)$.
- Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint Borel subsets of X and let $h, k \in \mathcal{H}_2$. Then

$$\langle \Theta(E)(\cup_{n=1}^{\infty}\Delta_n)(h), k\rangle = \langle SE(\cup_{n=1}^{\infty}\Delta_n)S^*h, k\rangle = \langle E(\cup_{n=1}^{\infty}\Delta_n)S^*h, S^*k\rangle =$$

$$\sum_{n=1}^{\infty}\langle E(\Delta_n)S^*h, S^*k\rangle = \sum_{n=1}^{\infty}\langle SE(\Delta_n)S^*h, k\rangle = \sum_{n=1}^{\infty}\langle \Theta(E)(\Delta_n)h, k\rangle,$$

  where the third equality is because E is a projection-valued measure.

Hence, $\Theta(E)$ is a projection-valued measure. Now we will show that Θ preserves the metric. In particular, let $E, F \in (P_{\mathcal{H}_1}(X), \rho_1)$. We want to show that $\rho_2(\Theta(E), \Theta(F)) = \rho_1(E, F)$. To this end, choose $f \in \text{Lip}_1(X)$ and suppose $h \in \mathcal{H}_2$ with $||h|| = 1$. Observe

$$\left|\left\langle \left(\int f d\Theta(E) - \int f d\Theta(F)\right) h, h\right\rangle\right| =$$

$$\left|\left\langle \left(\int f dSE(\cdot)S^* - \int f dSF(\cdot)S^*\right) h, h\right\rangle\right| =$$

$$\left|\left\langle \left(\int f dE - \int f dF\right) S^*h, S^*h\right\rangle\right|.$$

Since S^* is a surjective isometry, $\{k \in \mathcal{H}_1 : ||k|| = 1\} = \{S^*h : h \in \mathcal{H}_2, ||h|| = 1\}$. Hence

$$\begin{aligned}
\left|\left|\int f d\Theta(E) - \int f d\Theta(F)\right|\right| &= \sup_{h\in\mathcal{H}_2, ||h||=1} \left|\left\langle \left(\int f d\Theta(E) - \int f d\Theta(F)\right) h, h\right\rangle\right| \\
&= \sup_{h\in\mathcal{H}_2, ||h||=1} \left|\left\langle \left(\int f dE - \int f dF\right) S^*h, S^*h\right\rangle\right| \\
&= \sup_{k\in\mathcal{H}_1, ||k||=1} \left|\left\langle \left(\int f dE - \int f dF\right) k, k\right\rangle\right| \\
&= \left|\left|\int f dE - \int f dF\right|\right|.
\end{aligned}$$

By taking the supremum over all $\text{Lip}_1(X)$ functions we get that, $\rho_2(\Theta(E), \Theta(F)) = \rho_1(E, F)$.

We will now show that Θ is surjective. Choose $E \in (P_{\mathcal{H}_2}(X), \rho_2)$. Consider $S^*E(\cdot)S \in (P_{\mathcal{H}_1}(X), \rho_1)$. Then, $\Theta(S^*E(\cdot)S) = SS^*E(\cdot)SS^* = E(\cdot)$, and Θ is surjective. To show Θ is injective, suppose $E, F \in P_{\mathcal{H}_1}(X)$ are such that $SE(\cdot)S^* = SF(\cdot)S^*$. By using the fact that $S^*S = \text{id}_{\mathcal{H}_1}$, we get that $E = F$.

□

3.4 An Injective Isometry

We will show in this section that item (3) of Proposition 2.3.3 generalizes to the metric space $(P(X), \rho)$.

Definition 3.4.1. *Let $x \in X$. Define the projection-valued measure $E_x \in P(X)$ by*

$$E_x(\Delta) = \begin{cases} id_{\mathcal{H}} \text{ if } x \in \Delta \\ 0 \text{ if } x \notin \Delta. \end{cases}$$

Proposition 3.4.2. *[Davison] The map $\iota : X \to P(X)$ by $x \mapsto E_x$ is an injective isometry.*

Proof. It is clear the ι is injective. Choose $x, y \in X$. We need to show that, $d(x,y) = \rho(E_x, E_y)$. Let $f \in \mathrm{Lip}_1(X)$. Then

$$\begin{aligned} \left|\left|\int_X f dE_x - \int_X f dE_y\right|\right| &= ||f(x)\mathrm{id}_{\mathcal{H}} - f(y)\mathrm{id}_{\mathcal{H}}|| \\ &\leq d(x,y). \end{aligned}$$

Hence, $\rho(E_x, E_y) \leq d(x,y)$.

Consider the $\mathrm{Lip}_1(X)$ function $d(s,y)$, a function of s. Then

$$\begin{aligned} \left|\left|\int_X d(s,y) dE_x - \int_X d(s,y) dE_y\right|\right| &= ||d(x,y)\mathrm{id}_{\mathcal{H}} - d(y,y)\mathrm{id}_{\mathcal{H}}|| \\ &= d(x,y). \end{aligned}$$

Therefore $d(x,y) \leq \sup_{g \in \mathrm{Lip}_1(X)} \left\{ \left|\left|\int_X g dE_x - \int_X g dE_y\right|\right| \right\} = \rho(E_x, E_y)$, which completes the proof.

□

3.5 Non-compactness of $(P(X), \rho)$

It turns out that item (1) of Proposition 2.3.3 does not generalize to $(P(X), \rho)$. That is, in this section we will describe a counterexample which shows that $(P(X), \rho)$ is not a compact metric space. We begin with several preliminary facts.

Definition 3.5.1. *Let $\mathcal{H}$ be a Hilbert space. A normal operator N on $\mathcal{H}$ satisfies $NN^* = N^*N$.*

Definition 3.5.2. *Let E be a projection-valued measure with respect to the pair $(X, \mathcal{H})$. The support of E, denoted supp(E), is the subset supp$(E) = X \setminus \bigcup\{U \subset X : U$ open and $E(U) = 0\}$.*

Proposition 3.5.3. *[8] [Proposition IX.1.12 in Conway] Let E be a projection-valued measure with respect to the pair $(X, \mathcal{H})$, and let $f, g \in C(X)$. Then*

$$\int fg dE = \left(\int f dE\right)\left(\int g dE\right),$$

where the right side of the equality is operator composition. We say that E is multiplicative to describe this property.

Theorem 3.5.4. *[8] [Theorem IX.2.2 in Conway] If N is a normal operator on $\mathcal{H}$, there exists a unique projection-valued measure E on the Borel subsets of $\mathbb{C}$ whose support is $\sigma(N)$ (the spectrum of N) such that*

$$N = \int_{\sigma(N)} z dE(z).$$

Let $\mathcal{H}$ be a Hilbert space and let $M > 0$ be some fixed constant. Let B_M be the collection of all normal operators on $\mathcal{H}$ such that $||N|| \leq M$, where $|| \cdot ||$ denotes the operator norm of N. We note that if $N \in B_M$, then $\sigma(N)$ is contained in $B_0(M)$ the closed ball of radius M in $\mathbb{C}$ centered at the origin. By Theorem 3.5.4, there exists a one to one correspondence between B_M and the collection of projection-valued measures with respect to the pair $(B_0(M), \mathcal{H})$. That is, the map $N \in B_M \mapsto E \in P(B_0(M))$, where E satisfies $N = \int_{\sigma(N)} z dE(z)$, is bijective.

We now note that $(P(B_0(M)), \rho)$ is a complete metric space since $B_0(M)$ is compact. Moreover, the ρ metric induces a metric, s, on B_M in the following way. If $N, A \in B_M$, define

$$s(N, A) = \rho(E, F),$$

where $N = \int_{\sigma(N)} z dE(z)$, and $A = \int_{\sigma(A)} z dF(z)$. The bijective correspondence $N \in B_M \mapsto E \in P(B_0(M))$ guarantees that s is a metric, and that the metric space (B_M, s) is complete. By definition, a sequence $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the s metric if and only if the corresponding sequence $\{E_k\}_{k=1}^{\infty} \subseteq B_0(M)$ converges to $E \in B_0(M)$ in the ρ metric.

Proposition 3.5.5. *[10][Davison] The topology induced by the s metric on B_M coincides with the topology induced by the operator norm on B_M.*

Proof. First, suppose that $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the s metric. We will show that $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the operator norm. That is, given $\epsilon > 0$, we will find a K such that $||N_k - N|| \leq \epsilon$ for $k \geq K$. Since $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the s metric, there exists a K such that $s(N_k, N) \leq \frac{\epsilon}{2}$ for $k \geq K$. That is, since $s(N_k, N) = \rho(E_k, E)$, we have that

$$\sup_{f \in \mathrm{Lip}_1(B_0(M))} \left\{ \left|\left| \int_{B_0(M)} f dE_k - \int_{B_0(M)} f dE \right|\right| \right\} \leq \frac{\epsilon}{2}$$

for $k \geq K$. Note that the maps $Re(f) : B_0(M) \to \mathbb{R}$ given by $x + iy \mapsto x$, and $Im(f) : B_0(M) \to \mathbb{R}$ given by $x + iy \mapsto y$ are both elements of $\text{Lip}_1(B_0(M))$. If $k \geq K$,

$$||N_k - N|| = \left|\left| \int_{B_0(M)} zdE_k(z) - \int_{B_0(M)} zdE(z) \right|\right| =$$

$$\left|\left| \left(\int_{B_0(M)} Re(f)dE_k(z) - \int_{B_0(M)} Re(f)dE(z) \right) + i \left(\int_{B_0(M)} Im(f)dE_k(z) - \int_{B_0(M)} Im(f)dE(z) \right) \right|\right| \leq$$

$$\left|\left| \int_{B_0(M)} Re(f)dE_k(z) - \int_{B_0(M)} Re(f)dE(z) \right|\right| +$$

$$\left|\left| \int_{B_0(M)} Im(f)dE_k(z) - \int_{B_0(M)} Im(f)dE(z) \right|\right| \leq 2\rho(E_k, E) \leq \epsilon.$$

Hence, $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the operator norm. Conversely, suppose that $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the operator norm. We need to show that$\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the s metric, which is equivalent to showing that $\{E_k\}_{k=1}^{\infty} \subseteq P(B_0(M))$ converges to $E \in P(B_0(M))$. Moreover, by Theorem 3.2.2, it is enough to show that $\{E_k\}_{k=1}^{\infty} \subseteq P(B_0(M))$ converges to $E \in P(B_0(M))$ in the weak topology.

Since $\{N_k\}_{k=1}^{\infty} \subseteq B_M$ converges to $N \in B_M$ in the operator norm,

$$\lim_{k\to\infty} \int_{B_0(M)} zdE_k(z) = \int_{B_0(M)} zdE(z),$$

where convergence is in the operator norm. Since projection-valued measures are multiplicative (see Proposition 3.5.3), and since integration with respect to projection-valued measures is linear, we can claim that

$$\lim_{k\to\infty} \int_{B_0(M)} p(z, \bar{z})dE_k(z) = \int_{B_0(M)} p(z, \bar{z})dE(z),$$

where $p(z, \bar{z})$ is a polynomial on $B_0(M)$ in the variables z and $\bar{z}$, and again, convergence is in the operator norm. Let $f \in C_{\mathbb{R}}(B_0(M))$. By the Stone-Weierstrass Theorem (see [8]), there exists a polynomial $p(z, \bar{z})$ on $B_0(M)$ such that $||f - p(z, \bar{z})||_\infty \leq \frac{\epsilon}{3}$. Choose K such that for $k \geq K$,

$$\left|\left| \int_{B_0(M)} p(z, \bar{z})dE_k(z) - \int_{B_0(M)} p(z, \bar{z})dE(z) \right|\right| \leq \frac{\epsilon}{3}.$$

If $k \geq K$,

$$\left|\left|\int_{B_0(M)} f dE_k(z) - \int_{B_0(M)} f dE(z)\right|\right| \leq$$

$$\left|\left|\int_{B_0(M)} f dE_k(z) - \int_{B_0(M)} p(z,\bar{z}) dE_k(z)\right|\right| + \left|\left|\int_{B_0(M)} p(z,\bar{z}) dE_k(z) - \int_{B_0(M)} p(z,\bar{z}) dE(z)\right|\right| +$$

$$\left|\left|\int_{B_0(M)} p(z,\bar{z}) dE(z) - \int_{B_0(M)} f dE(z)\right|\right| \leq 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

Hence, $\{E_k\}_{k=1}^{\infty} \subseteq P(B_0(M))$ converges to $E \in P(B_0(M))$ in the weak topology, and the proof of the proposition is done.

□

We now use Proposition 3.5.5 to show the non-compactness of the metric space $(P(X), \rho)$. Indeed, let $\mathcal{H} = L^2(\mathbb{R}, m)$ where m is Lebesgue measure, and let $M = 1$. Consider B_1 and the associated metric space (B_1, s). For $n = 1, 2, ...$, define $\phi_n = \mathbf{1}_{[n,n+1)} \in L^{\infty}(\mathbb{R}, m)$, and consider the self-adjoint (and therefore normal) operators M_n, which are given by multiplication by ϕ_n. Indeed, $M_n(h) = \phi_n h$ for all $h \in \mathcal{H}$. Note that for all n, $||M_n|| = 1$, and hence the sequence $\{M_n\}_{n=1}^{\infty} \subseteq B_1$.

Consider the function $\phi_{nm} = \frac{1}{\sqrt{2}}\mathbf{1}_{[n,n+1)} + \frac{1}{\sqrt{2}}\mathbf{1}_{[m,m+1)} \in L^2(\mathbb{R}, m)$ for $m \neq n$. Now,

$$||\phi_{nm}||_{L^2} = \left(\int_{\mathbb{R}} |\phi_{nm}|^2 dm\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} \frac{1}{2}\mathbf{1}_{[n,n+1)} + \frac{1}{2}\mathbf{1}_{[m,m+1)} dm\right)^{\frac{1}{2}} = 1.$$

Hence for $n \neq m$ $||M_n - M_m|| \geq ||(M_n - M_m)\phi_{nm}||_{L^2} = ||\phi_{nm}||_{L^2} = 1$. This shows that $\{M_n\}_{n=1}^{\infty} \subseteq B_1$ has no convergent subsequence in the operator norm. By Proposition 3.5.5, $\{M_n\}_{n=1}^{\infty}$ has no convergent subsequence in the s metric. Equivalently, the corresponding sequence of projection-valued measures $\{E_n\}_{n=1}^{\infty} \subseteq P(B_0(1), \rho)$ has no convergent subsequence in the ρ metric. This shows that $P(B_0(1), \rho)$ is not compact.

Chapter 4

Generalizing to Positive Operator-Valued Measures

In this section, we will generalize the Kantorovich metric to the space of positive operator-valued measures on a Hilbert space, which are operator-valued measures which take values in the positive operators. The positive operators on a Hilbert space contain the projections.

4.1 A Metric Space of Positive Operator-Valued Measures

Let (X, d) be a compact metric space, and let $\mathcal{H}$ be an arbitrary Hilbert space. We begin with some preliminary definitions and facts.

Definition 4.1.1. *A positive operator $L \in \mathcal{B}(\mathcal{H})$ satisfies $\langle Lh, h\rangle \geq 0$ for all $h \in \mathcal{H}$.*

Definition 4.1.2. *A positive operator-valued measure with respect to the pair $(X, \mathcal{H})$ is a map $A : \mathscr{B}(X) \to \mathcal{B}(\mathcal{H})$ such that:*

- *$A(\Delta)$ is a positive operator in $\mathcal{B}(\mathcal{H})$ for all $\Delta \in \mathscr{B}(X)$;*
- *$A(\emptyset) = 0$ and $A(X) = id_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$);*
- *If $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathscr{B}(X)$, and if $g, h \in \mathcal{H}$, then*

$$\left\langle A\left(\bigcup_{n=1}^{\infty} \Delta_n\right) g, h\right\rangle = \sum_{n=1}^{\infty} \langle A(\Delta_n) g, h\rangle.$$

Remark 4.1.3. *A projection-valued measure with respect to the pair $(X, \mathcal{H})$ is a positive operator-valued measure because projections are positive operators.*

Remark 4.1.4. *Let A be a positive operator-valued measure with respect to the pair $(X, \mathcal{H})$. The map $[g, h] \in \mathcal{H} \times \mathcal{H} \mapsto A_{g,h}(\cdot)$ is sesquilinear. This follows from the fact that the inner product on $\mathcal{H}$ is sesquilinear.*

Our below discussion will rely on the following two standard theorems of functional analysis, which are stated with the amount of generality we will need.

Theorem 4.1.5. *[8] [Theorem III.5.7 in Conway] Let X be a compact metric space, and $T : C(X) \to \mathbb{C}$ be a bounded linear functional. There exists a unique complex-valued regular Borel finite measure μ on X such that*

$$\int_X f d\mu = T(f),$$

for all $f \in C(X)$, and such that $||\mu|| = ||T||$ (where $||\mu||$ denotes the total variation norm of μ).

Theorem 4.1.6. *[8] [Theorem II.2.2 in Conway] Let $u : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bounded sesquilinear form with bound M. There exists a unique operator $A \in \mathcal{B}(\mathcal{H})$ such that $u(g, h) = \langle Ag, h\rangle$ for all $g, h \in \mathcal{H}$, and such that $||A|| \leq M$.*

Let $S(X)$ be the collection of all positive operator-valued measures with respect to the pair $(X, \mathcal{H})$. Note that the properties described in the beginning of Chapter 3 for projection-valued measures also hold true for positive operator-valued measures (i.e. integration is well defined). Accordingly, consider the metric ρ on $S(X)$. That is,

$$\rho(A, B) = \sup_{f \in \mathrm{Lip}_1(X)} \left\{ \left|\left| \int f dA - \int f dB \right|\right| \right\}, \tag{4.1}$$

where $|| \cdot ||$ denotes the operator norm in $\mathcal{B}(\mathcal{H})$, and A and B are arbitrary members of $S(X)$.

Theorem 4.1.7. *[Davison] The metric space $(S(X), \rho)$ is complete.*

Proof. Let $\{A_n\}_{n=1}^{\infty} \subseteq S(X)$ be a Cauchy sequence in the ρ metric. This assumption implies that for $f \in C(X)$, the sequence of operators $\{\int f dA_n\}_{n=1}^{\infty}$ is Cauchy in the operator norm. Indeed, the proof that $\{\int f dA_n\}_{n=1}^{\infty}$ is Cauchy in the operator norm follows the proof of the similar result for projection-valued measures (see Claim 3.1.16).

In particular, the fact that $\{\int f dA_n\}_{n=1}^{\infty}$ is Cauchy in the operator norm implies the following: if $g, h \in \mathcal{H}$ and $f \in C(X)$, the sequence of complex numbers of $\{\int_X f dA_{n_{g,h}}\}_{n=1}^{\infty}$ is Cauchy. This is because we have the bound

$$\left|\int_X f dA_{n_{g,h}} - \int_X f dA_{m_{g,h}}\right| = \left|\left\langle\left(\int f dA_n - \int f dA_m\right) g, h\right\rangle\right| \leq$$
$$\left|\left|\int f dA_n - \int f dA_m\right|\right| ||g||||h||,$$

and the last term goes to zero as m and n approach infinity.

For $g, h \in \mathcal{H}$, define $\mu_{g,h} : C(X) \to \mathbb{C}$ by $f \mapsto \lim_{n\to\infty} \int f dA_{n_{g,h}}$, which is well defined by the above discussion, and since $\mathbb{C}$ is complete. Observe that $\mu_{g,h}$ is a bounded linear functional. We will show that it is bounded, and leave the proof of linearity to the reader. Let $f \in C(X)$. Then

$$|\mu_{g,h}(f)| = \left|\lim_{n\to\infty} \int_X f dA_{n_{g,h}}\right| = \lim_{n\to\infty} \left|\int_X f dA_{n_{g,h}}\right|.$$

Now for all n

$$\left|\int_X f dA_{n_{g,h}}\right| \leq \int_X |f| d|A_{n_{g,h}}| \leq ||f||_\infty ||g||||h||,$$

and hence

$$\lim_{n\to\infty} \left|\int_X f dA_{n_{g,h}}\right| \leq ||f||_\infty ||g||||h||.$$

This shows that $\mu_{g,h}$ is bounded by $||g||||h||$. We can now invoke Theorem 4.1.5 to conclude that $\mu_{g,h}$ is a measure.

The map $[g, h] \in \mathcal{H} \times \mathcal{H} \mapsto \mu_{g,h}$ is sesquilinear. Indeed, we will show that $[g, h] \mapsto \mu_{g,h}$ is linear in the first coordinate. The remaining properties of sesquilinearity are proved with a similar approach, and are left to the reader.

Let $g, h, k \in \mathcal{H}$, and let $f \in C(X)$. Then

$$\begin{aligned}
\int_X f d\mu_{g+h,k} &= \lim_{n\to\infty} \int_X f dA_{n_{g+h,k}} \\
&= \lim_{n\to\infty} \left(\int_X f dA_{n_{g,k}} + \int_X f dA_{n_{h,k}}\right) \\
&= \int_X f d\mu_{g,k} + \int_X f d\mu_{h,k},
\end{aligned}$$

where the second equality is because of Remark 4.1.4.

Consider a closed subset $C \subseteq X$, and choose a sequence of functions $\{f_m\}_{m=1}^{\infty} \subseteq C(X)$ such that $f_m \downarrow \mathbf{1}_C$ pointwise. See Part (3) of Theorem 3.1.9 for a specific formula for these functions. By the dominated convergence theorem,

$$\begin{aligned}
\int_X \mathbf{1}_C d\mu_{g+h,k} &= \lim_{m\to\infty} \int_X f_m d\mu_{g+h,k} \\
&= \lim_{m\to\infty} \left(\int_X f_m d\mu_{g,k} + \int_X f_m d\mu_{h,k} \right) \\
&= \int_X \mathbf{1}_C d\mu_{g,k} + \int_X \mathbf{1}_C d\mu_{h,k}.
\end{aligned}$$

Hence, for any closed $C \subseteq X$

$$\mu_{g+h,k}(C) = \mu_{g,k}(C) + \mu_{h,k}(C). \tag{4.2}$$

By decomposing the measures $\mu_{g+h,k}, \mu_{g,k}, \mu_{h,k}$ into their real and imaginary parts, we can show that (4.2) is equivalent to the following:

$$\text{Re}\mu_{g+h,k}(C) = \text{Re}\mu_{g,k}(C) + \text{Re}\mu_{h,k}(C), \tag{4.3}$$

and

$$\text{Im}\mu_{g+h,k}(C) = \text{Im}\mu_{g,k}(C) + \text{Im}\mu_{h,k}(C). \tag{4.4}$$

By further decomposing $\text{Re}\mu_{g+h,k}, \text{Re}\mu_{g,k}, \text{Re}\mu_{h,k}$ into their positive and negative parts (denoted $\text{Re}\mu_{g+h,k}^{+}$ and $\text{Re}\mu_{g+h,k}^{-}$ respectively), we can show, by rearranging terms, that (4.3) is equivalent to

$$M_1(C) = M_2(C), \tag{4.5}$$

where $M_1 = \text{Re}\mu_{g+h,k}^{+} + \text{Re}\mu_{g,k}^{-} + \text{Re}\mu_{h,k}^{-}$, and $M_2 = \text{Re}\mu_{g+h,k}^{-} + \text{Re}\mu_{g,k}^{+} + \text{Re}\mu_{h,k}^{+}$.

Since M_1 and M_2 are positive Borel measures on a metric space, M_1 and M_2 are regular (see Remark 2.1.4). That is, we can conclude that $M_1(\Delta) = M_2(\Delta)$ for any Borel subset $\Delta \in \mathscr{B}(X)$. By invoking the equivalence of (4.3) and (4.5), we have that (4.3) is true for all $\Delta \in \mathscr{B}(X)$. A similar approach, will yield that (4.4) is true for all $\Delta \in \mathscr{B}(X)$. Hence, (4.2) is true for all $\Delta \in \mathscr{B}(X)$.

This shows linearity in the first coordinate. As mentioned above, the following additional properties listed below are proved similarly:

- Let $g, h, k \in \mathcal{H}$. Then $\mu_{g,h+k} = \mu_{g,h} + \mu_{g,k}$.
- Let $\alpha \in \mathbb{C}$ and $g, h \in \mathcal{H}$. Then $\mu_{\alpha g,h} = \alpha\mu_{g,h}$.
- Let $\beta \in \mathbb{C}$ and $g, h \in \mathcal{H}$. Then $\mu_{g,\beta h} = \overline{\beta}\mu_{g,h}$.

Hence, the map $[g, h] \mapsto \mu_{g,h}$ is sesquilinear. We also note that $\mu_{g,h}$ inherits the following three additional properties:

- For $h \in \mathcal{H}$, $\mu_{h,h}$ is a positive Borel measure on X.
- For $g, h \in \mathcal{H}$, $\mu_{g,h}$ has total variation less than or equal to $||g||||h||$.
- For $g, h \in \mathcal{H}$, $\overline{\mu_{g,h}} = \mu_{h,g}$.

We will spend a short time justifying the second item in the above list. Suppose that $\Delta_1, ..., \Delta_n$ is a collection of disjoint subsets of $\mathscr{B}(X)$. Then using a generalized Schwarz inequality for positive sesquilinear forms we calculate that

$$\sum_{k=1}^{n} |\mu_{g,h}(\Delta_k)| \leq \sum_{k=1}^{n} (\mu_{g,g}(\Delta_k)\mu_{h,h}(\Delta_k))^{\frac{1}{2}} \leq$$

$$\left(\sum_{k=1}^{n} \mu_{g,g}(\Delta_k) \sum_{k=1}^{n} \mu_{h,h}(\Delta_k)\right)^{\frac{1}{2}} = (\mu_{g,g}(X)\mu_{h,h}(X))^{\frac{1}{2}} = (||h||^2||g||^2)^{\frac{1}{2}} = ||g||||h||,$$

which shows that the total variation of $\mu_{g,h}$ is less than or equal to $||g||||h||$.

Let $\Delta \in \mathscr{B}(X)$. The map $[g, h] \mapsto \int_X \mathbf{1}_\Delta d\mu_{g,h}$ is a bounded sesquilinear form with bound 1. Indeed,

$$|[g, h]| \leq ||\mathbf{1}_\Delta||_\infty ||g||||h|| = ||g||||h||.$$

By Theorem 4.1.6, there exists a unique bounded operator, $A(\Delta) \in \mathcal{B}(\mathcal{H})$, such that for all $g, h \in \mathcal{H}$

$$\langle A(\Delta)g, h\rangle = \int_X \mathbf{1}_\Delta d\mu_{g,h},$$

with $||A(\Delta)|| \leq 1$. Accordingly, define $A : \mathscr{B}(X) \to \mathcal{B}(\mathcal{H})$ by $\Delta \mapsto A(\Delta)$, and note that for $g, h \in \mathcal{H}$, $A_{g,h} = \mu_{g,h}$.

Claim 4.1.8. *A is a positive operator-valued measure.*

Proof of claim:

(1) Let $\Delta \in \mathscr{B}(X)$, and $h \in \mathcal{H}$. Then

$$\langle A(\Delta)h, h\rangle = \int_X \mathbf{1}_\Delta d\mu_{h,h} \geq 0.$$

Hence, $A(\Delta)$ is a positive operator.

(2) Let $h \in \mathcal{H}$. Then

$$\langle A(X)h, h\rangle = \int_X d\mu_{h,h} = \mu_{h,h}(X) = \langle h, h\rangle,$$

and

$$\langle A(\emptyset)h, h\rangle = \int_X \mathbf{1}_\emptyset d\mu_{h,h} = \mu_{h,h}(\emptyset) = 0.$$

Hence, $A(X) = \mathrm{id}_\mathcal{H}$ and $A(\emptyset) = 0$.

(3) If $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint sets in $\mathscr{B}(X)$, then for all $g, h \in \mathcal{H}$,

$$\left\langle A\left(\bigcup_{n=1}^\infty \Delta_n\right) g, h\right\rangle = \int_X \mathbf{1}_{\bigcup_{n=1}^\infty \Delta_n} d\mu_{g,h} = \sum_{n=1}^\infty \mu_{g,h}(\Delta_n) =$$

$$\sum_{n=1}^\infty \int_X \mathbf{1}_{\Delta_n} d\mu_{g,h} = \sum_{n=1}^\infty \langle A(\Delta_n)g, h\rangle.$$

This completes the proof of the claim.

We will now show that $A_n \to A$ in the ρ metric. Let $\epsilon > 0$. Choose an N such that for $n, m \geq N$, $\rho(A_n, A_m) \leq \epsilon$. Let $f \in \mathrm{Lip}_1(X)$. If $n \geq N$, and $h \in \mathcal{H}$ with $||h|| = 1$,

$$\begin{aligned}
\left|\left\langle \left(\int f dA_n - \int f dA\right) h, h\right\rangle\right| &= \left|\int_X f dA_{n_{h,h}} - \int_X f dA_{h,h}\right| \\
&= \lim_{m\to\infty} \left|\int_X f dA_{n_{h,h}} - \int_X f dA_{m_{h,h}}\right| \\
&= \lim_{m\to\infty} \left|\left\langle \left(\int f dA_n - \int f dA_m\right) h, h\right\rangle\right|,
\end{aligned}$$

where the second equality is because $\int f dA_{h,h} = \mu_{h,h}(f) = \lim_{n\to\infty} \int f dA_{n_{h,h}}$. For $m \geq N$

$$\begin{aligned} \left|\left\langle\left(\int f dA_n - \int f dA_m\right) h, h\right\rangle\right| &\leq \left|\left|\int f dA_n - \int f dA_m\right|\right| ||h||^2 \\ &= \left|\left|\int f dA_n - \int f dA_m\right|\right| \\ &\leq \rho(A_n, A_m) \\ &\leq \epsilon. \end{aligned}$$

Hence

$$\lim_{m\to\infty} \left|\left\langle\left(\int f dA_n - \int f dA_m\right) h, h\right\rangle\right| \leq \epsilon,$$

and therefore

$$\left|\left|\int f dA_n - \int f dA\right|\right| \leq \epsilon.$$

Since the choice of N is independent of $f \in \text{Lip}_1(X)$, $\rho(A_n, A) \leq \epsilon$, which shows that the metric space $(S(X), \rho)$ is complete.

□

Since we have previously shown that $(P(X), \rho)$ is a complete metric space, and $P(X) \subseteq S(X)$, where $(S(X), \rho)$ is also complete, we can conclude that $P(X)$ is a closed subset of $S(X)$ in the ρ metric. We can also consider the weak topology on $S(X)$. Using the same argument as before, one can show that the the weak topology on $S(X)$ coincides with the topology induced by the ρ metric.

4.2 WOT-weak Topology

In Chapter 2, we showed that $(P(X), \rho)$ is not compact, and therefore, $(S(X), \rho)$ is also not compact. In this section, we will consider a topology on $(S(X), \rho)$ that is weaker than the topology induced by the ρ metric, which we call the WOT-weak topology. We will show that the WOT-weak topology on $(S(X), \rho)$ is compact, by directly generalizing the proof of compactness in the classical setting (Proposition 2.3.3 Part 1). Importantly, we note that this fact has been previously shown by Ali [1], using more general theory.

Definition 4.2.1. *Let* $\mathcal{H}$ *be a Hilbert space. The weak operator topology (WOT) on* $\mathcal{B}(\mathcal{H})$ *is the locally convex topology defined by the semi norms* $\{p_{h,k} : h,k \in \mathcal{H}\}$ *where* $p_{h,k} = |\langle Ah, k\rangle|$*. Accordingly, a net of operators* $\{L_i\}_{i\in I} \subseteq \mathcal{B}(\mathcal{H})$ *converges to an operator* $L \in \mathcal{B}(\mathcal{H})$ *in the weak operator topology if* $\langle L_i h, k\rangle \to \langle Lh, k\rangle$ *for all* $h, k \in \mathcal{H}$*.*

Theorem 4.2.2. *[8][Proposition IX.5.5 in Conway] If* $M > 0$*, the subset of* $\{L \in \mathcal{B}(\mathcal{H}) : ||L|| \leq M\} \subseteq \mathcal{B}(\mathcal{H})$ *is compact in the weak operator topology.*

Equip $\mathcal{B}(\mathcal{H})$ with the weak operator topology. For each $f \in C(X)$, define a mapping $\hat{f} : S(X) \to \mathcal{B}(\mathcal{H})$ by $A \mapsto \int_X f dA$. We note here that we will use the following equivalent notations:

$$\hat{f}(A) = \int_X f dA = A(f).$$

Let the WOT-weak topology be the weakest topology on $S(X)$ that makes the collection of maps $\{\hat{f} : f \in C_{\mathbb{R}}(X)\}$ continuous where we put the weak operator topology on $\mathcal{B}(\mathcal{H})$. In other words, a net of positive operator-valued measures $\{A_i\}_{i\in I} \subseteq S(X)$ converges to a positive operator-valued measure $A \in S(X)$, if for all $f \in C_{\mathbb{R}}(X)$, $\hat{f}(A_i)$ converges to $\hat{f}(A)$ in the weak operator topology. Since the weak operator topology is a weaker topology than the operator norm topology on $\mathcal{B}(\mathcal{H})$, the WOT-weak topology is a weaker topology than the weak topology (or equivalently the topology induced by the ρ metric) on the space of positive operator-valued measures $S(X)$.

Theorem 4.2.3. *[1][Ali, Davison] The WOT-weak topology is sequentially compact.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $S(X)$. Since X is compact, $C(X)$ is separable, and therefore choose a countable dense subset of functions $\{f_i\}_{i=1}^{\infty} \subseteq C(X)$. Consider the bounded operators $\{A_n(f_1)\}_{n=1}^{\infty}$. Note that for all $n = 1, ...,$ $||A_n(f_1)|| \leq ||f_1||_{\infty}$. Since the subset $\{L \in \mathcal{B}(\mathcal{H}) : ||L|| \leq ||f_1||_{\infty}\} \subseteq \mathcal{B}(\mathcal{H})$ is compact in the weak operator topology (see Theorem 4.2.2), the sequence $\{A_n(f_1)\}_{n=1}^{\infty}$ admits a convergent subsequence in the weak operator topology, which we call $\{A_n^1(f_1)\}_{n=1}^{\infty}$. Consider the sequence of bounded operators $\{A_n^1(f_2)\}_{n=1}^{\infty}$. Since for all $n = 1, ...\infty$, $||A_n^1(f_2)|| \leq ||f_2||_{\infty}$, the subsequence $\{A_n^1(f_2)\}_{n=1}^{\infty}$ admits a further subsequence $\{A_n^2(f_2)\}_{n=1}^{\infty}$ which is convergent in the weak operator topology. If we continue the process, we obtain for each

$i = 1, \ldots \infty$ a sequence $\{A_n^i(f_i)\}_{n=1}^{\infty}$ which is convergent in the weak operator topology, such that $\{A_n^{i+1}\}_{n=1}^{\infty}$ is a subsequence of $\{A_n^i\}_{n=1}^{\infty}$. Now choose some $f_i \in C(X)$ for $1 \leq i \leq \infty$, and consider the diagonal sequence, $\{A_n^n(f_i)\}_{n=1}^{\infty}$. For $n \geq i$, $\{A_n^n(f_i)\}$ is a subsequence of $\{A_n^i(f_i)\}$, and since $\{A_n^i(f_i)\}_{n=i}^{\infty}$ is convergent in the weak operator topology, so is $\{A_n^n(f_i)\}_{n=i}^{\infty}$, which implies that $\{A_n^n(f_i)\}_{n=1}^{\infty}$ is convergent in the weak operator topology.

Let $f \in C(X)$ and $g, h \in \mathcal{H}$. We will show that the sequence $\{\langle A_n^n(f)g, h\rangle\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$. If $g = 0$ or $h = 0$, then the result is clear because every term in the sequence is zero. Therefore, suppose that $g \neq 0$ and $h \neq 0$. Choose $f_i \in C(X)$ such that

$$||f - f_i||_\infty \leq \frac{\epsilon}{3||h||||g||}.$$

By above, we know that $\{A_n^n(f_i)\}_{n=1}^{\infty}$ is convergent in the weak operator topology. Therefore, there exists an N such that for $m, n \geq N$, $|\langle A_n^n(f_i)g, h\rangle - \langle A_m^m(f_i)g, h\rangle| \leq \frac{\epsilon}{3}$. Thus, if $m, n \geq N$

$$\begin{aligned}
|\langle A_n^n(f)g, h\rangle - \langle A_m^m(f)g, h\rangle| &\leq |\langle A_n^n(f)g, h\rangle - \langle A_n^n(f_i)g, h\rangle| \\
&+ |\langle A_n^n(f_i)g, h\rangle - \langle A_m^m(f_i)g, h\rangle| \\
&+ |\langle A_m^m(f_i)g, h\rangle - \langle A_m^m(f)g, h\rangle| \\
&\leq \int_X |f - f_i| dA^n_{n g,h} + \frac{\epsilon}{3} + \int_X |f - f_i| dA^m_{m g,h} \\
&\leq \epsilon.
\end{aligned}$$

Hence, for all $f \in C(X)$ and $g, h \in \mathcal{H}$, the sequence $\{\langle A_n^n(f)g, h\rangle\}_{n=1}^{\infty} = \left\{\int_X f dA^n_{n g,h}\right\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$. Define $\mu_{g,h} : C(X) \to \mathbb{C}$ by $f \mapsto \lim_{n\to\infty} \int_X f dA^n_{n g,h}$. Observe that $\mu_{g,h}$ is a bounded linear functional, and hence by Theorem 4.1.5, $\mu_{g,h}$ is a measure.

Using a similar approach as in the proof of Theorem 4.1.7, we note that the map $[g, h] \mapsto \mu_{g,h}$ is sesquilinear, and accordingly, there exists a positive operator-valued measure $A \in S(X)$ such that $\langle A(\Delta)g, h\rangle = \mu_{g,h}(\Delta)$ for all $\Delta \in \mathscr{B}(X)$.

It remains to show that $\{A_n^n\}_{n=1}^{\infty}$ converges to A in the weak operator topology. Choose $f \in C_{\mathbb{R}}(X)$, and $g, h \in \mathcal{H}$. By construction,

$$\langle A_n^n(f)g, h\rangle \to \langle A(f)g, h\rangle.$$

Hence, $\{A_n\}_{n=1}^{\infty}$ admits a convergent subsequence $\{A_n^n\}_{n=1}^{\infty}$ in the WOT-weak topology, which completes the proof.

□

Proposition 4.2.4. *[1][Ali, Davison] Let $\mathcal{H}$ be a separable Hilbert space. The WOT-weak topology on $S(X)$ is first countable.*

Proof. Since $\mathcal{H}$ is a separable Hilbert space, let $O = \{h_j : j = 1, ..., \infty\}$ be a countable orthonormal basis in $\mathcal{H}$. Since $C_{\mathbb{R}}(X)$ is separable, let P be a countable dense subset of $C_{\mathbb{R}}(X)$.

Let $A \in S(X)$, $f_1, ..., f_k \in P$, and $h_j, h_l \in O$. For $n \in \mathbb{N} = \{1, 2, ...\}$, consider the following subset of $S(X)$:

$$\{B \in S(X) : |\langle B(f_i)h_j, h_l\rangle - \langle A(f_i)h_j, h_l\rangle| < \frac{1}{n} \text{ for all } i = 1, ..., k\}.$$

Consider the collection of all finite intersections of subsets of $S(X)$ of the above form where $A \in S(X)$, $f_1, ..., f_k \in P$, $h_j, h_l \in O$, $n \in \mathbb{N}$ are all arbitrary. This forms a basis for a topology on $S(X)$ which is first countable, and let this topology be denoted ξ.

We claim that the the ξ topology and the WOT-weak topology coincide. To this end, put the weak operator topology on $\mathcal{B}(\mathcal{H})$, and let $f \in C_{\mathbb{R}}(X)$. We will show that the previously defined map $\hat{f} : S(X) \to \mathcal{B}(\mathcal{H})$ is continuous with respect to the ξ topology. Since the WOT-weak topology is the weakest topology making all of the maps of the form $\{\hat{f} : f \in C_{\mathbb{R}}(X)\}$ continuous, we will have shown that the WOT-weak topology is weaker than the ξ topology.

Since the ξ topology is first countable, it can be defined by sequences. Therefore, suppose $\{A_n\}_{n=1}^{\infty} \subseteq S(X)$ converges in the ξ topology to $A \in S(X)$. We need to show that $\hat{f}(A_n) \to \hat{f}(A)$ in the weak operator topology. Note that for all n

$$\left|\left|\int f dA_n\right|\right| \leq ||f||_{\infty}$$

and hence,

$$\sup_{n=1,...,\infty} \left|\left|\int f dA_n\right|\right| < \infty.$$

Therefore, by Proposition IX.1.3 in [8], it is enough to show that $\lim_{n\to\infty}\langle A_n(f)h_j, h_l\rangle = \langle A(f)h_j, h_l\rangle$ for all $h_j, h_l \in O$. Accordingly, let $h_j, h_l \in O$, and let $\epsilon > 0$. Choose $g \in P$ such that $||f-g||_\infty \le \frac{\epsilon}{3||h_j||||h_l||}$, and choose $s > 0$ such that $\frac{1}{s} \le \frac{\epsilon}{3}$. Consider

$$\mathcal{O} = \left\{ B \in S(X) : |\langle B(g)h_j, h_l\rangle - \langle A(g)h_j, h_l\rangle| < \frac{1}{s} \right\}.$$

Since $A_n \to A$ in the ξ topology, there exists an N such that for $n \ge N$, $A_n \in \mathcal{O}$. For $n \ge N$

$$\begin{aligned} |\langle A_n(f)h_j, h_l\rangle - \langle A(f)h_j, h_l\rangle| &\le |\langle A_n(f)h_j, h_l\rangle - \langle A(g)h_j, h_l\rangle| \\ &+ |\langle A_n(g)h_j, h_l\rangle - \langle A(g)h_j, h_l\rangle| \\ &+ |\langle A(g)h_j, h_l\rangle - \langle A(f)h_j, h_l\rangle| \\ &\le ||f-g||_\infty ||h_j||||h_l|| + \frac{1}{s} + ||f-g||_\infty ||h_j||||h_l|| \\ &\le \epsilon. \end{aligned}$$

Hence, $A_n(f) \to A(f)$ in the weak operator topology.

Let $A \in S(X)$ and let $\mathcal{W} = \{B \in S(X) : |\langle B(f_i)h_j, h_l\rangle - \langle A(f_i)h_j, h_l\rangle| < \frac{1}{n}$ for all $i = 1, ..., k\}$ be an arbitrary sub-basis element of the the ξ topology. We need to show that $\mathcal{W}$ is open in the WOT-weak topology. Define $\mathcal{O}_i = \hat{f_i}^{-1}(\{L \in \mathcal{B}(\mathcal{H}) : |\langle Lh_j, h_l\rangle - \langle A(f_i)h_j, h_l\rangle| < \frac{1}{n}\})$. Since the set $\{L \in \mathcal{B}(\mathcal{H}) : |\langle Lh_j, h_l\rangle - \langle A(f_i)h_j, h_l\rangle| < \frac{1}{n}\}$ is open in the weak operator topology, $\mathcal{O}_i$ is open in the WOT-weak topology. Notice that $\mathcal{O}_i = \{B \in S(X) : |\langle B(f_i)h_j, h_l\rangle - \langle A(f_i)h_j, h_l\rangle| < \frac{1}{n}\}$. Now observe that $\mathcal{W} = \bigcap_{i=1}^k \mathcal{O}_i$, which is an open element in the WOT-weak topology, because each $\mathcal{O}_i$ is open in the WOT-weak topology. Hence, the two topologies coincide. Since the ξ topology is first countable, the WOT-weak topology is first countable as well.

□

Corollary 4.2.5. *[1][Ali, Davison] Let $\mathcal{H}$ be a separable Hilbert space. Then the WOT-weak topology on $S(X)$ is compact.*

Proof. Since $\mathcal{H}$ is a separable Hilbert space, the above proposition shows that the WOT-weak topology on $S(X)$ is first countable. By Theorem 4.2.3, we know that $S(X)$ is sequentially compact.

In first countable topologies, sequential compactness and compactness are equivalent. Hence, $S(X)$ with the WOT-weak topology is compact. $\square$

Chapter 5

Generalizing to Non-Compact Metric Spaces

In this chapter, we will consider the generalized Kantorovich metric when the underlying metric space is complete and separable (not necessarily compact). In Chapter 1, we presented the known results for the classical Kantorovich metric over an underlying complete and separable metric space. Indeed, the reader may recall that for an appropriate collection of Borel probability measures, the resulting metric space of measures equipped with the Kantorovich metric is complete. It will turn out that the same result is true for an appropriate collection of projection-valued and positive operator-valued measures.

5.1 The Appropriate Collection of Projection Valued Measures

Let (Y,d) be a complete and separable metric space. Let $\mathcal{H}$ be a Hilbert space, and let $P_0(Y)$ be the collection of projection valued measures with respect to the pair $(Y,\mathcal{H})$ with the following additional property: If $E \in P_0(Y)$, then for all $f \in \mathrm{Lip}(Y)$, there exists an $0 \leq M_{f,E} < \infty$ such that

$$\left| \int_Y f dE_{g,h} \right| \leq M_{f,E} ||g|| ||h||,$$

for all $g,h \in \mathcal{H}$, and where $\mathrm{Lip}(Y)$ denotes the collection of all real-valued Lipschitz functions on Y. An example of an element in $P_0(Y)$ would be a projection-valued measure E such that $E(K) = \mathbf{1}_{\mathcal{H}}$, for K a compact subset of Y. In this case, note that $E(Y \setminus K) = 0$. If $f \in \mathrm{Lip}(Y)$, let $M_{f,E} = \max_{x\in K} |f(x)|$ and observe that

$$\begin{aligned}\left|\int_Y f dE_{g,h}\right| &\leq \left|\int_K f dE_{g,h}\right| + \left|\int_{Y\setminus K} f dE_{g,h}\right| \\ &\leq \int_K |f| d|E_{g,h}| \\ &\leq M_{f,E}||g||||h||,\end{aligned}$$

for all $g, h \in \mathcal{H}$. Hence, $E \in P_0(Y)$.

On this sub-collection of projection-valued measures we will consider the generalized Kantorovich metric. That is, for $E, F \in P_0(Y)$ define (exactly as before)

$$\rho(E,F) = \sup_{f\in \mathrm{Lip}_1(Y)} \left\{ \left|\left| \int f dE - \int f dF \right|\right| \right\}.$$

We will now show that this metric is well defined (finite) on $P_0(Y)$. To do this, we need to make a preliminary observation. In particular, if $E \in P_0(Y)$ and $f \in \mathrm{Lip}(Y)$, there exists by definition an $M_{f,E} \geq 0$ such that

$$\left|\int_Y f dE_{g,h}\right| \leq M_{f,E}||g||||h||,$$

for all $g, h \in \mathcal{H}$. This means that the map $[g, h] \mapsto \int_Y f dE_{g,h}$ is a bounded sesquilinear form. By Theorem 4.1.6 there exists a bounded operator $\int f dE \in \mathcal{B}(\mathcal{H})$ such that

$$\left\langle \left(\int f dE\right) g, h \right\rangle = \int_Y f dE_{g,h},$$

for all $g, h \in \mathcal{H}$, where $\left|\left|\int f dE\right|\right| \leq M_{f,E}$. With this observation, we will proceed to showing the finiteness of ρ. Let $E, F \in P_0(Y)$, $f \in \mathrm{Lip}_1(Y)$, and $x_0 \in Y$. Then

$$\begin{aligned}\left|\left|\int f dE - \int f dF\right|\right| &= \left|\left|\int f dE - f(x_0)\mathrm{id}_{\mathcal{H}} + f(x_0)\mathrm{id}_{\mathcal{H}} - \int f dF\right|\right| \\ &= \left|\left|\int f dE - \int f(x_0) dE - \left(\int f dF - \int f(x_0) dF\right)\right|\right| \\ &\leq \left|\left|\int (f - f(x_0)) dE\right|\right| + \left|\left|\int (f - f(x_0))\, dF\right|\right|\end{aligned}$$

Let $h \in \mathcal{H}$ with $||h|| = 1$. Then

$$\begin{aligned}\left|\left\langle\left(\int(f(x)-f(x_0))dE\right)h,h\right\rangle\right| &= \left|\int_Y(f(x)-f(x_0))dE_{h,h}(x)\right|\\ &\le \int_Y|f(x)-f(x_0)|dE_{h,h}(x)\\ &\le \int_Y d(x,x_0)dE_{h,h}(x)\\ &\le M_{d(x,x_0),E}||h||^2\\ &= M_{d(x,x_0),E},\end{aligned}$$

where $M_{d(x,x_0),E} \geq 0$ is the non-negative number associated to the Lip(Y) (in fact, Lip$_1(Y)$) function $d(x, x_0)$ and the projection valued measure E. Hence,

$$\left|\left|\int(f-f(x_0))dE\right|\right| \le M_{d(x,x_0),E}.$$

Similarly, there exists an $M_{d(x,x_0),F} \geq 0$ such that

$$\left|\left|\int(f-f(x_0))dF\right|\right| \le M_{d(x,x_0),F}.$$

Since $M_{d(x,x_0),E}$ and $M_{d(x,x_0),F}$ do not depend on the choice of $f \in \text{Lip}_1(Y)$, $\rho(E,F) \leq M_{d(x,x_0),E} + M_{d(x,x_0),F} < \infty$.

5.2 The Metric Space $(P_0(Y), \rho)$ is Complete

In this section, we will show that the metric space $(P_0(Y), \rho)$ is complete. We will rely on Proposition 2.4.2. We will also use the following lemma, which can be found in the proof of Proposition 1 in [4].

Lemma 5.2.1. *[4] [Proposition 1 in Berberian] Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of positive operators on the Hilbert space $\mathcal{H}$ such that $||B_n|| \leq M$ for all $n = 1, 2, ...,$ and such that for all $h \in \mathcal{H}$, $\lim_{n\to\infty}\langle B_n h, h\rangle = 0$. Then $\lim_{n\to\infty} ||B_n h|| = 0$.*

Proof. Let $h \in \mathcal{H}$. Note that $||B_n h||^2 = \langle B_n h, B_n h\rangle = |\langle B_n h, g\rangle|$ where $g = B_n h$. By a generalized Schwarz inequality,

$$0 \le |\langle B_n h, g\rangle| \le \langle B_n h, h\rangle^{\frac{1}{2}}\langle B_n g, g\rangle^{\frac{1}{2}} \le \langle B_n h, h\rangle^{\frac{1}{2}}(||B_n g||||g||)^{\frac{1}{2}} \le$$

$$\langle B_n h, h\rangle^{\frac{1}{2}}(||B_n||^3||h||^2)^{\frac{1}{2}} \leq \langle B_n h, h\rangle^{\frac{1}{2}} M^{\frac{3}{2}}||h||.$$

By assumtion, $\lim_{n\to\infty}\langle B_n h, h\rangle = 0$. Hence, $\lim_{n\to\infty}\langle B_n h, h\rangle^{\frac{1}{2}} M^{\frac{3}{2}}||h|| = 0$, which implies that $\lim_{n\to\infty} ||B_n h|| = 0$. □

Theorem 5.2.2. *[Davison] The metric space $(P_0(Y), \rho)$ is complete.*

Proof. We note that the proof of this result uses some of the same techniques as in the proof of the analogous result by Kravchenko [21] that $(M(Y), H)$ is complete (see Theorem 2.1.8). Suppose that $\{E_n\}_{n=1}^{\infty}$ is a Cauchy sequence of elements in $(P_0(Y), \rho)$. We want to find an $E \in (P_0(Y), \rho)$ such that $E_n \to E$ in the ρ metric.

Claim 5.2.3. *Let $h \in \mathcal{H}$ and $f \in Lip(Y)$. The sequence $\{\int_Y f dE_{n_{h,h}}\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers.*

Proof of Claim: This claim follows from the observation that there exists a $T > 0$ such that $\frac{f}{T} \in \text{Lip}_1(Y)$. Hence

$$0 \leq \left|\int_Y \frac{f}{T} dE_{n_{h,h}} - \int_Y \frac{f}{T} dE_{m_{h,h}}\right| = \left|\left\langle\left(\int \frac{f}{T} dE_n - \int \frac{f}{T} dE_m\right) h, h\right\rangle\right| \leq$$
$$\left|\left|\int \frac{f}{T} dE_n - \int \frac{f}{T} dE_m\right|\right| ||h||^2 \leq \rho(E_n, E_m)||h||^2 \to 0$$

as $m, n \to \infty$. Since $||h||$ and T are fixed,

$$\left|\int_Y f dE_{n_{h,h}} - \int_Y f dE_{m_{h,h}}\right| \to 0$$

as $m, n \to \infty$. This proves the claim.

Observe that $E_{n_{h,h}}(Y) = \langle E_n(Y)h, h\rangle = ||h||^2$ for all $n = 1, 2, ...$. Since $\text{Lip}_b(Y) \subseteq \text{Lip}(Y)$, we can use Proposition 2.4.2 to conclude that there exists a Borel measure $\mu_{h,h}$ on Y such that $\mu_{h,h}(Y) = ||h||^2$, and such that $E_{n_{h,h}}$ converges to $\mu_{h,h}$ in the weak topology. By convergence in the weak topology, we mean that for all $f \in C_b(Y)$, $\left\{\int_Y f dE_{n_{h,h}}\right\}_{n=1}^{\infty}$ converges to $\int_Y f d\mu_{h,h}$ (where $C_b(Y)$ are the real valued continuous functions on Y that are bounded). We note that Proposition 2.4.2 is the key result of Kravchenko that allowed him to prove the analogous result which is that $(M(Y), H)$ is a complete metric space.

We now want to define $\mu_{g,h}$ for $g, h \in \mathcal{H}$ such that $\left\{\int_Y f dE_{n_{g,h}}\right\}_{n=1}^{\infty}$ converges to $\int_Y f d\mu_{g,h}$ for all $f \in C_b(Y)$. Let $g, h \in \mathcal{H}$. If $E_{n_{g,h}} = \text{Re}E_{n_{g,h}} + i\text{Im}E_{n_{g,h}}$, we can calculate that $\text{Re}E_{n_{g,h}} = \frac{1}{2}(E_{n_{g+h,g+h}} - E_{n_{g,g}} - E_{n_{h,h}})$ and $\text{Im}E_{n_{g,h}} = -\frac{1}{2}(E_{n_{ig+h,ig+h}} - E_{n_{g,g}} - E_{n_{h,h}})$. Accordingly, define $\text{Re}\mu_{g,h} := \frac{1}{2}(\mu_{g+h,g+h} - \mu_{g,g} - \mu_{h,h})$ and $\text{Im}\mu_{g,h} := -\frac{1}{2}(\mu_{ig+h,ig+h} - \mu_{g,g} - \mu_{h,h})$. Hence, by the discussion in the above paragraph, we can conclude that $\left\{\int_Y f dE_{n_{g,h}}\right\}_{n=1}^{\infty}$ converges to $\int_Y f d\mu_{g,h}$ for all $f \in C_b(Y)$.

Using a similar method as in the proof of Theorem 4.1.7, we can conclude that the map $[g, h] \mapsto \mu_{g,h}$ is sesquilinear, and that $\mu_{g,h}$ inherits the following two additional properties:

- For $g, h \in \mathcal{H}$, $\mu_{g,h}$ has total variation less than or equal to $||g||||h||$.
- For $g, h \in \mathcal{H}$, $\overline{\mu_{g,h}} = \mu_{h,g}$.

Let $\Delta \in \mathscr{B}(Y)$. The map $[g, h] \mapsto \int_Y \mathbf{1}_\Delta d\mu_{g,h}$ is a bounded sesquilinear form with bound 1. By Theorem 4.1.6, there exists a unique bounded operator, $E(\Delta) \in \mathcal{B}(\mathcal{H})$, such that for all $g, h \in \mathcal{H}$

$$\langle E(\Delta)g, h\rangle = \int_Y \mathbf{1}_\Delta d\mu_{g,h},$$

with $||E(\Delta)|| \leq 1$. Define $E : \mathscr{B}(Y) \to \mathcal{B}(\mathcal{H})$ by $\Delta \mapsto E(\Delta)$, and note that for $g, h \in \mathcal{H}$, $E_{g,h} = \mu_{g,h}$. This map E is a positive operator valued measure (see the proof in Theorem 4.1.7). It remains to show that $E \in P_0(Y)$. That is, we need to show:

(1) For $f \in \text{Lip}(Y)$, there exists an $M_{f,E} < \infty$ such that $\left|\int_Y f dE_{g,h}\right| \leq M_{f,E}||g||||h||$ for all $g, h \in \mathcal{H}$.

(2) $\{E_n\}_{n=1}^{\infty}$ converges to E in the ρ metric.

(3) E is a projection-valued measure.

We will first show (1). Choose some $f \geq 0 \in \text{Lip}(Y)$. There exists a $T > 0$ such that $\frac{1}{T}f \in \text{Lip}_1(Y)$. Since we are assuming the sequence $\{E_n\}_{n=1}^{\infty}$ is Cauchy in the ρ metric, the sequence of self-adjoint operators $\left\{\int f dE_n\right\}_{n=1}^{\infty}$ is Cauchy in the operator norm topology on $\mathcal{B}(\mathcal{H})$. This implies that these operators are uniformly bounded. That is, there exists an $N > 0$ such that

$$\left|\left|\int f dE_n\right|\right| \leq N$$

for all $n = 1, ...\infty$.

For all $g, h \in \mathcal{H}$, we claim that$\left|\int_Y f dE_{g,h}\right| < \infty$. Indeed, choose $h \in \mathcal{H}$ and consider the sequence $\{E_{n_{h,h}}\}_{n=1}^{\infty}$ which converges to $E_{h,h}$ in the weak topology. Define $f_k(x) = \min_{x \in Y}\{k, f(x)\} \in C_b(Y) \cap \mathrm{Lip}(Y)$, and note that $f_k \uparrow f$ on Y. We note here that the idea of using the cutoff function, f_k, is also a central part of the proof of Theorem 2.1.8 by A. Kravchenko. By the monotone convergence theorem

$$\int_Y f_k dE_{h,h} \uparrow \int_Y f dE_{h,h}.$$

Suppose this is an unbounded increasing sequence. Then choose a k_l such that

$$\int_Y f_{k_l} dE_{h,h} > l,$$

where $l = 1, 2, ...$ For a fixed l,

$$\int_Y f_{k_l} dE_{n_{h,h}} \to \int_Y f_{k_l} dE_{h,h},$$

because $\{E_{n_{h,h}}\}_{n=1}^{\infty}$ converges to $E_{h,h}$ in the weak topology and $f_{k_l} \in C_b(Y)$. Hence choose an n_l such that

$$\int_Y f_{k_l} dE_{n_{l_{h,h}}} > l.$$

Again by the monotone convergence theorem

$$\int_Y f_{k_l} dE_{n_{l_{h,h}}} \uparrow \int_Y f dE_{n_{l_{h,h}}},$$

and hence,

$$\int_Y f dE_{n_{l_{h,h}}} > l.$$

This last line is a contradiction to the fact that the sequence $\left\{\int_Y f dE_{n_{h,h}}\right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers (because $f \in \mathrm{Lip}(Y)$ and Claim 5.2.3) . Hence $\int_Y f dA_{h,h} < \infty$ for all $h \in \mathcal{H}$. For

$g, h \in \mathcal{H}$, we can decompose $A_{g,h}$ into its positive measure parts, as we have done previously, to get that $\left|\int_Y f dA_{g,h}\right| < \infty$.

The next thing to note is that since $f_k(x) \leq f(x)$, $\int f_k dE_n \leq \int f dE_n$ for all n, as elements of $\mathcal{B}(\mathcal{H})$. Hence for any k and n,

$$\left|\left|\int f_k dE_n\right|\right| \leq \left|\left|\int f dE_n\right|\right| \leq N.$$

We are now prepared to show that there exists an $M_{f,E} < \infty$ such that

$$\left|\int_Y f dE_{g,h}\right| \leq M_{f,E}||g||||h||,$$

for all $g, h \in \mathcal{H}$. Let $\epsilon > 0$ and let $g, h \in \mathcal{H}$. Since $f_k \uparrow f$ and $\left|\int_Y f dE_{g,h}\right| < \infty$, there exists a k such that

$$\left|\int_Y (f - f_k) dE_{g,h}\right| < \epsilon.$$

Observe that

$$\begin{aligned} \left|\int_Y f dE_{g,h}\right| &\leq \left|\int_Y (f - f_k) dE_{g,h}\right| + \left|\int_Y f_k dE_{g,h}\right| \\ &\leq \epsilon + \lim_{n\to\infty} \left|\int_Y f_k dE_{n_{g,h}}\right|, \end{aligned}$$

where the second inequality is because $f_k \in C_b(Y)$. We know that for all n and k that

$$\left|\int_Y f_k dE_{n_{g,h}}\right| \leq \left|\left\langle\left(\int f_k dE_n\right) g, h\right\rangle\right| \leq N||g||||h||.$$

Therefore

$$\epsilon + \lim_{n\to\infty} \left|\int_Y f_k dE_{n_{g,h}}\right| \leq \epsilon + N||g||||h||.$$

Since ϵ is arbitrary,

$$\left|\int f dE_{g,h}\right| \leq N||g||||h||.$$

Note that N does not depend on the choice of $g, h \in \mathcal{H}$. It only depends on the choice of $f \geq 0 \in \text{Lip}(Y)$. Hence, we can let $M_{f,E} = N$.

For any arbitrary $f \in \mathrm{Lip}(Y)$, decompose f into its positive and negative parts; $f = f_+ - f_-$. Note that f_+ and f_- are both non-negative elements of $\mathrm{Lip}(Y)$. Let $M_{f,E} = M_{f_+,E} + M_{f_-,E}$. For $g, h \in \mathcal{H}$,

$$\begin{aligned} \left|\int f dE_{g,h}\right| &\leq \left|\int f_+ dE_{g,h}\right| + \left|\int f_- dE_{g,h}\right| \\ &\leq M_{f_+,E}||g||||h|| + M_{f_-,E}||g||||h|| \\ &= M_{f,E}|g||||h|| \end{aligned}$$

This complete the proof of (1). We will next show (2). We need to show that $E_n \to E$ in the ρ metric. Let $\epsilon > 0$. Since $\{E_n\}_{n=1}^{\infty}$ is Cauchy in the ρ metric, there exists an N such that for $n, m \geq N$, $\rho(E_n, E_m) \leq \frac{\epsilon}{6}$. Let $n \geq N$, let $f \in \mathrm{Lip}_1(Y)$ with $f \geq 0$, and define $f_k(x) = \min_{x \in Y}\{k, f(x)\} \in C_b(Y) \cap \mathrm{Lip}_1(Y)$. As before, observe that $f_k \uparrow f$ on Y. Let $h \in \mathcal{H}$ with $||h|| = 1$. By the monotone convergence theorem,

$$\int_Y f_k dE_{n_{h,h}} \uparrow \int_Y f dE_{n_{h,h}} < \infty$$

and

$$\int_Y f_k dE_{h,h} \uparrow \int_Y f dE_{h,h} < \infty,$$

where the finiteness of the limiting integrals is because $E_n \in P_0(Y)$, and because E satisfies part (1) above. Accordingly, choose k such that

$$\left|\int_Y f_k dE_{n_{h,h}} - \int_Y f dE_{n_{h,h}}\right| \leq \frac{\epsilon}{6}$$

and

$$\left|\int_Y f_k dE_{h,h} - \int_Y f dE_{h,h}\right| \leq \frac{\epsilon}{6}.$$

Since $\{E_{m_{h,h}}\}_{m=1}^{\infty}$ converges in the weak topology to $E_{h,h}$, and $f_k \in C_b(Y)$,

$$\lim_{m \to \infty} \int_Y f_k dE_{m_{h,h}} = \int_Y f_k dE_{h,h}.$$

Then

$$\left|\left\langle\left(\int f dE_n - \int f dE\right) h, h\right\rangle\right| = \left|\int_Y f dE_{n_{h,h}} - \int_Y f dE_{h,h}\right|$$

$$\le \left|\int_Y fdE_{n_{h,h}} - \int_Y f_k dE_{n_{h,h}}\right| + \left|\int_Y f_k dE_{n_{h,h}} - \int_Y f_k dE_{h,h}\right| + \left|\int_Y f_k dE_{h,h} - \int_Y fdE_{h,h}\right|$$

$$= \left|\int_Y fdE_{n_{h,h}} - \int_Y f_k dE_{n_{h,h}}\right| + \lim_{m\to\infty}\left|\int_Y f_k dE_{n_{h,h}} - \int_Y f_k dE_{m_{h,h}}\right| + \left|\int_Y f_k dE_{h,h} - \int_Y fdE_{h,h}\right|$$

$$\le \frac{\epsilon}{6} + \lim_{m\to\infty}\left|\int_Y f_k dE_{n_{h,h}} - \int_Y f_k dE_{m_{h,h}}\right| + \frac{\epsilon}{6}$$

$$= \frac{\epsilon}{3} + \lim_{m\to\infty}\left|\left\langle\left(\int f_k dE_n - \int f_k dE_m\right)h, h\right\rangle\right| \le \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{\epsilon}{2},$$

because of the inequality

$$\left|\left\langle\left(\int f_k dE_n - \int f_k dE_m\right)h, h\right\rangle\right| \le \left|\left|\int f_k dE_n - \int f_k dE_m\right|\right| ||h||^2 \le \rho(E_n, E_m)||h||^2 = \rho(E_n, E_m).$$

Hence for $n \ge N$ and $f \in \text{Lip}_1(Y)$ such that $f \ge 0$,

$$\left|\left|\int fdE_n - \int fdE\right|\right| \le \frac{\epsilon}{2}.$$

Now for arbitrary $f \in \text{Lip}_1(Y)$, decompose f into its positive and negative parts; $f = f_+ - f_-$. Note that f_+ and f_- are both non-negative elements of $\text{Lip}_1(Y)$. Then for $n \ge N$

$$\begin{aligned}\left|\left|\int fdE_n - \int fdE\right|\right| &\le \left|\left|\int f_+ dE_n - \int f_+ dE\right|\right| + \left|\left|\int f_- dE_n - \int f_- dE\right|\right| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

which shows that $\rho(E_n, E) \le \epsilon$. This is because the choice of N is independent of the choice of $f \in \text{Lip}_1(Y)$.

Lastly, we need to show (3). That is, we need to show that E is a projection-valued measure. Since we know that E is a positive operator-valued measure, $E(\Delta)$ is self adjoint for all $\Delta \in \mathscr{B}(Y)$. Hence, to show that E is a projection valued measure, it is enough to show that $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for $\Delta_1, \Delta_2 \in \mathscr{B}(Y)$. To this end, let C and D be closed subsets of Y. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\text{Lip}(Y)$ such that $f_n \downarrow \mathbf{1}_C$ and such that $||f_n||_\infty \le 1$ for all $n = 1, 2..$ (see the proof of Theorem 3.1.9 for a specific definition of this sequence of functions). Similarly, let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions in $\text{Lip}(Y)$ such that $g_n \downarrow \mathbf{1}_D$ and such that $||g_n||_\infty \le 1$ for all $n = 1, 2...$

For all $h \in \mathcal{H}$, by the dominated convergence theorem,

$$\left\langle \left(\int f_n dE \right) h, h \right\rangle \to \left\langle \left(\int \mathbf{1}_C dE \right) h, h \right\rangle$$

as $n \to \infty$. That is, for all $h \in \mathcal{H}$

$$\int_Y f_n - \mathbf{1}_C dE_{h,h} \downarrow 0$$

as $n \to \infty$. Also, note that for all $n = 1, 2, ...$,

$$\left|\left| \int f_n - \mathbf{1}_C dE \right|\right| \leq ||f_n - \mathbf{1}_C||_\infty \leq 1.$$

Moreover, since E is already known to be a positive operator valued measure, and since $f_n - \mathbf{1}_C \geq 0$ for all $n = 1, 2, ..$, the sequence of operators $\{\int f_n - \mathbf{1}_C dE\}_{n=1}^\infty$ are positive operators. By the above discussion, we see that the sequence of operators $\{\int f_n - \mathbf{1}_C dE\}_{n=1}^\infty$ satisfies Lemma 5.2.1. This means that for all $h \in \mathcal{H}$

$$\lim_{n \to \infty} \left|\left| \left(\int f_n - \mathbf{1}_C dE \right) h \right|\right| = 0,$$

which is equivalent to saying that

$$\lim_{n \to \infty} \left|\left| \left(\int f_n dE \right) h - E(C)h \right|\right| = 0. \tag{5.1}$$

Similarly,

$$\lim_{n \to \infty} \left|\left| \left(\int g_n dE \right) h - E(D)h \right|\right| = 0. \tag{5.2}$$

We now have the following claim.

Claim 5.2.4. *For all $n = 1, 2, ...$,*

$$\left(\int f_n dE \right) \left(\int g_n dE \right) = \int f_n g_n dE.$$

Proof of Claim: Choose some $n = 1, 2, ...$ Since $f_n \in \text{Lip}(Y)$ with $||f_n||_\infty \leq 1$, and since $g_n \in \text{Lip}(Y)$ with $||g_n||_\infty \leq 1$, $f_n g_n \in \text{Lip}(Y)$. Next, since $E_m \to E$ in the ρ metric, we have that $\int f_n dE_m \to \int f_n dE$, $\int g_n dE_m \to \int g_n dE$, and $\int f_n g_n dE_m \to \int f_n g_n dE$ as $m \to \infty$ where convergence is in the operator norm. Moreover, since E_m is a projection valued measure, and since

f_n and g_n are bounded, we have that $\left(\int f_n dE_m\right)\left(\int g_n dE_m\right) = \left(\int f_n g_n dE_m\right)$ for all $m = 1, 2, ...$ Combining all of this data, we get that

$$\int f_n g_n dE_m = \left(\int f_n dE_m\right)\left(\int g_n dE_m\right) \to \left(\int f_n dE\right)\left(\int g_n dE\right),$$

and

$$\int f_n g_n dE_m \to \int f_n g_n dE,$$

which shows that $\int f_n g_n dE = \left(\int f_n dE\right)\left(\int g_n dE\right)$. This completes the proof of the claim.

We will now show that $E(C)E(D) = E(C \cap D)$ as an operator on $\mathcal{H}$. Note that $f_n \downarrow \mathbf{1}_C$, $g_n \downarrow \mathbf{1}_D$, and moreover, $f_n g_n \downarrow \mathbf{1}_{C\cap D}$. Hence for $h \in \mathcal{H}$, we also have that

$$\left\langle \left(\int f_n g_n dE\right) h, h \right\rangle \to \left\langle \left(\int \mathbf{1}_{C\cap D} dE\right) h, h \right\rangle,$$

as $n \to \infty$. Since E is a positive operator-valued measure, we know that $E(C)$ is self adjoint. Therefore,

$$\langle E(C)E(D)h, h\rangle = \langle E(D)h, E(C)h\rangle = \lim_{n\to\infty} \left\langle \left(\int g_n dE\right) h, \left(\int f_n dE\right) h \right\rangle =$$

$$\lim_{n\to\infty} \left\langle \left(\int f_n g_n dE\right) h, h \right\rangle = \left\langle \left(\int \mathbf{1}_{C\cap D} dE\right) h, h \right\rangle = \langle E(C\cap D)h, h\rangle,$$

where the second equality is by equations (5.1) and (5.2), and the third equality is because of Claim 5.2.4.

Now let $\Delta_1, \Delta_2 \in \mathscr{B}(Y)$. If $h \in \mathcal{H}$, note that $E_{h,h}$ is a regular measure. Hence, there exists a sequence of compact subsets $\{C_k\}_{k=1}^{\infty} \subseteq \mathscr{B}(Y)$ such that $E_{h,h}(C_k) \uparrow E_{h,h}(\Delta_1)$, and a sequence of compact subsets $\{D_k\}_{k=1}^{\infty} \subseteq \mathscr{B}(Y)$ such that $E_{h,h}(D_k) \uparrow E_{h,h}(\Delta_2)$. Additionally, $E_{h,h}(C_k \cap D_k) \uparrow E_{h,h}(\Delta_1 \cap \Delta_2)$. Note that

$$\langle (E(\Delta_1) - E(C_k))h, h\rangle \to 0,$$

as $k \to \infty$. Next, note that since $C_k \subseteq \Delta_1$ for all $k = 1, 2, ...$, the operator $E(\Delta_1) - E(C_k)$ is a positive operator. Moreover, $||E(\Delta_1) - E(C_k)|| \leq 2$ for all $k = 1, 2, ...$ We can appeal to

Lemma 5.2.1 to conclude that $\lim_{k\to\infty} ||(E(\Delta_1)-E(C_k))h|| = 0$, or equivalently, $\lim_{k\to\infty} ||E(\Delta_1)h-E(C_k)h|| = 0$. Similarly, $\lim_{k\to\infty} ||E(\Delta_2)h - E(D_k)h|| = 0$. Then

$$\langle E(\Delta_1)E(\Delta_2)h, h\rangle = \langle E(\Delta_2)h, E(\Delta_1)h\rangle = \lim_{k\to\infty} \langle E(D_k)h, E(C_k)h\rangle =$$

$$\lim_{k\to\infty} \langle E(C_k)E(D_k)h, h\rangle = \lim_{k\to\infty} \langle E(C_k \cap D_k)h, h\rangle = \langle E(\Delta_1 \cap \Delta_2)h, h\rangle,$$

where the fourth equality is because C_k and D_k are compact (in particular, closed). Hence, E is a projection-valued measure, and this completes the proof of part (3), and the theorem.

□

5.3 A Modified Generalized Kantorovich Metric

As we did in Chapter 2, we will consider a modified Kantorovich metric. Indeed, let $P(Y)$ denote the collection of all projection-valued measures with respect to the pair $(Y, \mathcal{H})$.

Definition 5.3.1. *Define the modified generalized Kantorovich metric, $M\rho$, on $P(Y)$ by:*

$$M\rho(E,F) = \sup\left\{\left|\left|\int f dE - \int f dF\right|\right| : f \in Lip_1(Y) \text{ and } ||f||_\infty \le 1\right\}$$

for $E, F \in P(Y)$.

The condition $||f||_\infty \le 1$ guarantees that this metric will be finite on $P(Y)$.

Theorem 5.3.2. *[Davison] The metric space $(P(Y), M\rho)$ is complete.*

Proof. The proof of this theorem follows the proof of Theorem 5.2.2, with several differences that we will point out. Suppose that $\{E_n\}_{n=1}^\infty$ is a Cauchy sequence of elements in $(P(Y), M\rho)$. We want to find an $E \in (P(Y), M\rho)$ such that $E_n \to E$ in the $M\rho$ metric. Because $M\rho$ takes a supremum over $f \in \text{Lip}_1(Y)$ such that $||f||_\infty \le 1$, we obtain a version of Claim 5.2.3 only for $\text{Lip}_b(Y)$ functions. However, this is not an impediment, because Proposition 2.4.2 only considers $\text{Lip}_b(Y)$ functions. Hence, using the techniques of the proof of Theorem 5.2.2, we obtain a positive operator-valued measure E on Y. The proof that E is a projection-valued measure depends on the construction of a sequence $\{f_n\}_{n=1}^\infty \in \text{Lip}(Y)$, but one can see that actually this sequence of functions is contained in $\mathrm{Lip}_b(Y)$. Hence, the proof that E is a projection-valued measure carries over to the $M\rho$ metric.

Lastly, we need to show that $E_n \to E$ in the $M\rho$ metric. Let $\epsilon > 0$. Choose an N such that for $n, m \geq N$, $M\rho(E_n, E_m) \leq \epsilon$. Let $f \in \mathrm{Lip}_1(Y)$ with $||f||_\infty \leq 1$, and let $h \in \mathcal{H}$ with $||h|| = 1$. If $n \geq N$

$$\begin{aligned} \left|\left\langle\left(\int f dE_n - \int f dE\right) h, h\right\rangle\right| &= \left|\int_Y f dE_{n_{h,h}} - \int_X f dE_{h,h}\right| \\ &= \lim_{m\to\infty}\left|\int_X f dE_{n_{h,h}} - \int_X f dE_{m_{h,h}}\right|, \end{aligned}$$

where the last equality is because $f \in C_b(Y)$ and $E_{m_{h,h}}$ converges weakly to $E_{h,h}$. Observe that for all $m \geq N$,

$$\begin{aligned} \left|\int_X f dE_{n_{h,h}} - \int_X f dE_{m_{h,h}}\right| &= \left|\left\langle\left(\int f dE_n - \int f dE_m\right) h, h\right\rangle\right| \\ &\leq M\rho(E_n, E_m)||h||^2 \\ &\leq \epsilon. \end{aligned}$$

Therefore if $n \geq N$, $\left|\left|\int f dE_n - \int f dE\right|\right| \leq \epsilon$. Since N does not depend on the choice of f, $M\rho(E_n, E) \leq \epsilon$, and $(P(Y), M\rho)$ is complete. □

Chapter 6

A Fixed Projection-Valued Measure

In this chapter, we will introduce the well known notion of an iterated function system (IFS) on a compact metric space. In particular, we will use the Contraction Mapping Theorem on the complete metric space, $(P(X), \rho)$, of projection-valued measures to provide an alternative method for proving a fixed point result due to P. Jorgensen (see [17] and [18]). This fixed point, which is a projection-valued measure, is related to Cuntz algebras.

6.1 Preliminaries

Let (X, d) be a compact metric space and consider the compact metric space $(M(X), H)$, where we recall that $M(X)$ is the collection of Borel probability measures on X and H is the classical Kantorovich metric (see Proposition 2.3.3) . Let $\mathcal{S} = \{\sigma_0, ..., \sigma_{N-1}\}$ be an iterated function system (IFS) on (X, d). That is, for all $0 \leq i \leq N-1$, $\sigma_i : X \to X$ such that for all $x, y \in X$

$$d(\sigma_i(x), \sigma_i(y)) \leq r_i d(x, y),$$

where $0 < r_i < 1$. Indeed, each σ_i is a Lipschitz contraction on X, and r_i is the Lipschitz constant associated to σ_i.

Let $\sigma : X \to X$ be a Borel measurable function such that $\sigma \circ \sigma_i = \mathrm{id}_X$ for all $0 \leq i \leq N-1$. Assume further that

$$X = \bigcup_{i=0}^{N-1} \sigma_i(X), \tag{6.1}$$

where the above union is disjoint. We provide a standard example for the above scenario:

- Let $X =$ Cantor Set $\subseteq [0,1]$, with the standard metric on $\mathbb{R}$.
- Let $\sigma_0(x) = \frac{1}{3}x$ and $\sigma_1(x) = \frac{1}{3}x + \frac{2}{3}$.
- Let $\sigma(x) = 3x \mod 1$.

We now state the following important result due to Hutchinson.

Theorem 6.1.1. *[14][Hutchinson] The map $T : M(X) \to M(X)$ by*

$$\nu(\cdot) \mapsto \sum_{k=0}^{N-1} \frac{1}{N}\nu(\sigma_k^{-1}(\cdot)),$$

is a Lipschitz contraction in the $(M(X), H)$ metric space, with Lipschitz constant $r := max_{0\leq i\leq N-1}\{r_i\}$.

By applying the Contraction Mapping Theorem to the Lipschitz contraction T, there exists a unique measure, $\mu \in M(X)$, such that $T(\mu) = \mu$. That is

$$\mu(\cdot) = \sum_{k=0}^{N-1} \frac{1}{N}\mu(\sigma_k^{-1}(\cdot)).$$

This unique invariant measure, μ, is called the Hutchinson measure associated to $\mathcal{S}$. Consider the Hilbert space $L^2(X, \mu)$. Define

$$S_i : L^2(X, \mu) \to L^2(X, \mu) \text{ by } \phi \mapsto (\phi \circ \sigma)\sqrt{N}\mathbf{1}_{\sigma_i(X)}$$

for all $i = 0, ..., N-1$, and its adjoint

$$S_i^* : L^2(X, \mu) \to L^2(X, \mu) \text{ by } \phi \mapsto \frac{1}{\sqrt{N}}(\phi \circ \sigma_i)$$

for all $i = 0, ..., N-1$. This leads to the following result due to P. Jorgensen.

Theorem 6.1.2. *[16] [Jorgensen] The maps $\{S_i : 0 \leq i \leq N-1\}$ are isometries, and the maps $\{S_i^* : 0 \leq i \leq N-1\}$ are their adjoints. Moreover, these maps and their adjoints satisfy the Cuntz relations:*

(1) $\sum_{i=0}^{N-1} S_i S_i^* = \mathbf{1}_{\mathcal{H}}$

(2) $S_i^* S_j = \delta_{i,j}\mathbf{1}_{\mathcal{H}}$ *where* $0 \leq i, j \leq N-1$.

Corollary 6.1.3. *[16] [Jorgensen] The Hilbert space $L^2(X,\mu)$ admits a representation of the Cuntz algebra, $\mathcal{O}_N$, on N generators.*

Let $\Gamma_N = \{0, ..., N-1\}$. For $k \in \mathbb{Z}_+$, let $\Gamma_N^k = \Gamma_N \times ... \times \Gamma_N$, where the product is k times. If $a = (a_1, ..., a_k) \in \Gamma_N^k$, where $a_j \in \{0, 1, ..., N-1\}$ for $1 \leq j \leq k$, define

$$A_k(a) = \sigma_{a_1} \circ ... \circ \sigma_{a_k}(X).$$

Using that (6.1) is a disjoint union, we conclude that $\{A_k(a)\}_{a\in\Gamma_N^k}$ partitions X for all $k \in \mathbb{Z}_+$. For $k \in \mathbb{Z}_+$ and $a = (a_1, ..., a_k) \in \Gamma_N^k$ define,

$$P_k(a) = S_a S_a^*,$$

where $S_a = S_{a_1} \circ ... \circ S_{a_k}$. The Cuntz relations suggest that $P_k(a)$ is a projection on the Hilbert space $L^2(X,\mu)$.

We state another result due to Jorgensen.

Theorem 6.1.4. *[17] [18] [Jorgensen] There exists a unique projection-valued measure $E(\cdot)$ defined on the Borel subsets of X, $\mathscr{B}(X)$, taking values in the projections on $L^2(X,\mu)$ such that*

(1) $E(\cdot) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot))S_i^*$, and

(2) $E(A_k(a)) = P_k(a)$ for all $k \in \mathbb{Z}_+$ and $a \in \Gamma_N^k$.

One can calculate that $P_k(a) = M_{\mathbf{1}_{A_k(a)}}$, where $M_{\mathbf{1}_{A_k(a)}}$ is the projection on $L^2(X,\mu)$ given by multiplication by $\mathbf{1}_{A_k(a)}$. Hence, $E(\cdot)$ is the canonical projection valued measure given by multiplication by the indicator function.

In the next two sections, we will provide an alternative proof of this theorem. In particular, we will realize the map

$$F(\cdot) \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot))S_i^*$$

as a Lipschitz contraction on a complete metric space of projection-valued measures from $\mathscr{B}(X)$ into the projections on $L^2(X,\mu)$. The Contraction Mapping Theorem will then guarantee the existence of a unique projection-valued measure E satisfying part (1) of Theorem 6.1.4. Part (2) of Theorem 6.1.4 will follow as a consequence, via induction.

6.2 A Lipschitz Contraction on $(P(X), \rho)$

Suppose that $\mathcal{H} = L^2(X, \mu)$, or more generally, that $\mathcal{H}$ is a Hilbert space which admits a representation of the Cuntz alegra on N generators. Consider the associated complete metric space $(P(X), \rho)$.

Remark 6.2.1. *During the revision process, it was pointed out by Krystal Taylor (University of Minnesota) that my original proof of the below theorem contained an error in the latter half of the proof. The mistake has been corrected in the proof below. The author wants to acknowledge K. Taylor for her important observation. The author's article, see [10], was published before the error was discovered. Hence, there will be an erratum associated to the article.*

Theorem 6.2.2. *[10][Davison] The map $\Phi : P(X) \to P(X)$ given by*

$$E(\cdot) \mapsto \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*$$

is a Lipschitz contraction in the ρ metric.

Proof. We will begin by showing that the map Φ is well defined. That is, we will show that if $E \in P(X)$, then $\Phi(E)$ is a projection-valued measure.

- Let $\Delta_1, \Delta_2 \in \mathscr{B}(X)$. Then,

$$\begin{aligned}
\Phi(E)(\Delta_1 \cap \Delta_2) &= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_1 \cap \Delta_2)) S_i^* \\
&= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_1) \cap \sigma_i^{-1}(\Delta_2)) S_i^* \\
&= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_1)) E(\sigma_i^{-1}(\Delta_2)) S_i^* \\
&= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_1)) S_i^* S_i E(\sigma_i^{-1}(\Delta_2)) S_i^* \\
&= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_1) S_i^* \sum_{j=0}^{N-1} S_j E(\sigma_j^{-1}(\Delta_2)) S_j^* \\
&= \Phi(E)(\Delta_1)\Phi(E)(\Delta_2),
\end{aligned}$$

where the third equality is because E is a projection-valued measure, and the fourth and fifth equalities are because $S_i^* S_j = \delta_{i,j}\mathrm{id}_{\mathcal{H}}$. Next, note that $\Phi(\Delta)$ is self-adjoint, which implies by the above computation that $\Phi(\Delta)$ is a projection.

- $\Phi(E)(\emptyset) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\emptyset))S_i^* = \sum_{i=0}^{N-1} S_i E(\emptyset)S_i^* = 0.$

- $\Phi(E)(X) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(X))S_i^* = \sum_{i=0}^{N-1} S_i E(X)S_i^* = \sum_{i=0}^{N-1} S_i S_i^* = \mathrm{id}_{\mathcal{H}}.$

- Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of disjoint subsets in $\mathscr{B}(X)$. Let $h, k \in \mathcal{H}$. Note that

$$\langle \Phi(E)(\cup_{n=1}^{\infty}\Delta_n)h, k\rangle = \left\langle \left(\sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cup_{n=1}^{\infty}\Delta_n))S_i^*\right) h, k \right\rangle =$$

$$\sum_{i=0}^{N-1} \left\langle S_i E(\sigma_i^{-1}(\cup_{n=1}^{\infty}\Delta_n))S_i^* h, k\right\rangle = \sum_{i=0}^{N-1} \left\langle E(\sigma_i^{-1}(\cup_{n=1}^{\infty}\Delta_n))S_i^* h, S_i^* k\right\rangle. \tag{6.2}$$

Since E is a projection-valued measure, for each $0 \le i \le N-1$ we have that

$$\langle E(\sigma_i^{-1}(\cup_{n=1}^{\infty}\Delta_n))S_i^* h, S_i^* k\rangle = \langle E(\cup_{n=1}^{\infty}\sigma_i^{-1}(\Delta_n))S_i^* h, S_i^* k\rangle =$$

$$\sum_{n=1}^{\infty}\langle E(\sigma_i^{-1}(\Delta_n))S_i^* h, S_i^* k\rangle.$$

Hence, the last expression of equation (6.2) is equal to

$$\sum_{i=0}^{N-1}\sum_{n=1}^{\infty}\langle E(\sigma_i^{-1}(\Delta_n))S_i^* h, S_i^* k\rangle = \sum_{n=1}^{\infty}\sum_{i=0}^{N-1}\langle E(\sigma_i^{-1}(\Delta_n))S_i^* h, S_i^* k\rangle =$$

$$\sum_{n=1}^{\infty}\sum_{i=0}^{N-1}\langle S_i E(\sigma_i^{-1}(\Delta_n))S_i^* h, k\rangle = \sum_{n=1}^{\infty}\langle \Phi(E)(\Delta_n)h, k\rangle.$$

This completes the discussion that Φ is well defined. We will next proceed to the following claim, which will be helpful in showing the Φ is a Lipschitz contraction.

Claim 6.2.3. *Let $h \in \mathcal{H}$. Then,*

$$\Phi(E)_{h,h}(\Delta) = \sum_{i=0}^{N-1} E_{S_i^* h, S_i^* h}(\sigma_i^{-1}(\Delta)),$$

for all $\Delta \in \mathscr{B}(X)$.

Proof of claim: Let $\Delta \in \mathscr{B}(X)$. Then

$$\begin{aligned}
\Phi(E)_{h,h}(\Delta) &= \langle \Phi(E)(\Delta)h, h\rangle \\
&= \left\langle \left(\sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta))S_i^*\right) h, h \right\rangle \\
&= \sum_{i=0}^{N-1} \langle S_i E(\sigma_i^{-1}(\Delta))S_i^* h, h\rangle \\
&= \sum_{i=0}^{N-1} \langle E(\sigma_i^{-1}(\Delta))S_i^* h, S_i^* h\rangle \\
&= \sum_{i=0}^{N-1} E_{S_i^*h,S_i^*h}(\sigma_i^{-1}(\Delta)),
\end{aligned}$$

which completes the proof of the claim.

We will now show that Φ is a Lipschitz contraction in the ρ metric. Accordingly, choose $E, F \in P(X)$. We will show that

$$\rho(\Phi(E), \Phi(F)) \leq r\rho(E, F).$$

Recall that $r = \max_{0\leq i\leq N-1}\{r_i\}$, where r_i is the Lipschitz constant associated to σ_i. Choose $f \in \mathrm{Lip}_1(X)$, and $h \in \mathcal{H}$ with $||h|| = 1$. Then

$$\left|\left\langle \left(\int f d\Phi(E) - \int f d\Phi(F)\right) h, h \right\rangle\right| =$$

$$\left|\left\langle \left(\int f d\Phi(E)\right) h, h \right\rangle - \left\langle \left(\int f d\Phi(F)\right) h, h \right\rangle\right| = \left|\int_X f d\Phi(E)_{h,h} - \int_X f d\Phi(F)_{h,h}\right| =$$

$$\left|\sum_{i=0}^{N-1} \int_X f dE_{S_i^*h,S_i^*h}(\sigma_i^{-1}(\cdot)) - \sum_{i=0}^{N-1} \int_X f dF_{S_i^*h,S_i^*h}(\sigma_i^{-1}(\cdot))\right| =$$

$$\left|\sum_{i=0}^{N-1} \int_X (f\circ\sigma_i) dE_{S_i^*h,S_i^*h} - \sum_{i=0}^{N-1} \int_X (f\circ\sigma_i) dF_{S_i^*h,S_i^*h}\right| =$$

$$\left|\sum_{i=0}^{N-1} \left(\int_X (f\circ\sigma_i) dE_{S_i^*h,S_i^*h} - \int_X (f\circ\sigma_i) dF_{S_i^*h,S_i^*h}\right)\right| =$$

$$\left|\sum_{i=0}^{N-1} r\left(\int_X \left(\frac{f\circ\sigma_i}{r}\right) dE_{S_i^*h,S_i^*h} - \int_X \left(\frac{f\circ\sigma_i}{r}\right) dF_{S_i^*h,S_i^*h}\right)\right| \leq$$

$$r\left(\sum_{i=0}^{N-1}\left|\int_X\left(\frac{f\circ\sigma_i}{r}\right)dE_{S_i^*h,S_i^*h}-\int_X\left(\frac{f\circ\sigma_i}{r}\right)dF_{S_i^*h,S_i^*h}\right|\right)=$$

$$r\left(\sum_{i=0}^{N-1}\left|\left\langle\left(\int\left(\frac{f\circ\sigma_i}{r}\right)dE-\int\left(\frac{f\circ\sigma_i}{r}\right)dF\right)S_i^*h,S_i^*h\right\rangle\right|\right)\leq$$

$$r\left(\sum_{i=0}^{N-1}\left|\left|\int\left(\frac{f\circ\sigma_i}{r}\right)dE-\int\left(\frac{f\circ\sigma_i}{r}\right)dF\right|\right|\,||S_i^*h||^2\right).$$

Note that the function $\frac{f\circ\sigma_i}{r}\in \mathrm{Lip}_1(X)$ for all $0\leq i\leq N-1$. Hence

$$r\left(\sum_{i=0}^{N-1}\left|\left|\int\left(\frac{f\circ\sigma_i}{r}\right)dE-\int\left(\frac{f\circ\sigma_i}{r}\right)dF\right|\right|\,||S_i^*h||^2\right)\leq$$

$$r\rho(E,F)\left(\sum_{i=0}^{N-1}\langle S_i^*h,S_i^*h\rangle\right)=r\rho(E,F)\left(\sum_{i=0}^{N-1}\langle S_iS_i^*h,h\rangle\right)=$$

$$r\rho(E,F)\left\langle\left(\sum_{i=0}^{N-1}S_iS_i^*\right)h,h\right\rangle=r\rho(E,F)\langle h,h\rangle=r\rho(E,F),$$

Therefore

$$\left|\left|\int fd\Phi(E)-\int fd\Phi(F)\right|\right|\leq r\rho(E,F).$$

Since f is an arbitrary element of $\mathrm{Lip}_1(X)$,

$$\rho(\Phi(E),\Phi(F))\leq r\rho(E,F).$$

This proves that Φ is a Lipschitz contraction in the ρ metric on $P(X)$.

□

6.3 An Alternative Proof of Theorem 6.1.4:

By Theorem 3.1.15 and Theorem 6.2.2, we know that Φ is Lipschitz contraction on the complete metric space $(P(X),\rho)$. By the Contraction Mapping Theorem, there exists a unique projection-valued measure $E \in P(X)$ such that

$$E(\cdot) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*. \tag{6.3}$$

It remains to show that $E(A_k(a)) = P_k(a)$ for all $k \in \mathbb{Z}_+$ and $a \in \Gamma_N^k$. This will be done by induction on k. Indeed, suppose that $k = 1$, and consider $A_1(j) = \sigma_j(X)$ for some $j \in \Gamma_N$. Then by equation (6.3),

$$\begin{aligned} E(\sigma_j(X)) &= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\sigma_j(X))) S_i^* \\ &= S_j E(X) S_j^* \\ &= S_j S_j^* \\ &= P_1(j). \end{aligned}$$

This proves the base case. Suppose that $E(A_{n-1}(b)) = P_{n-1}(b)$ for all $b \in \Gamma_N^{n-1}$ where $n \in \mathbb{Z}_+$ with $n > 1$. We will show that $E(A_n(a)) = P_n(a)$. Choose some $a \in \Gamma_N^n$. Suppose that $a = (a_1, ..., a_n)$ and $b = (a_2, ..., a_n)$. Then

$$\begin{aligned} E(A_n(a)) &= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(A_n(a))) S_i^* \\ &= \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\sigma_{a_1}(A_{n-1}(b))) S_i^* \\ &= S_{a_1} E(A_{n-1}(b)) S_{a_1}^* \\ &= S_{a_1} P_{n-1}(b) S_{a_1}^* \\ &= P_n(a). \end{aligned}$$

Hence, an alternative proof of Theorem 6.1.4 is complete.

Remark 6.3.1. *The alternative proof that we have presented depends on the fact that the subsets $A_k(a)$, for $k \in \mathbb{Z}_+$ and $a = (a_1, ..., a_n) \in \Gamma_N^k$, satisfy*

$$A_k(a) = \sigma_{a_1} \circ ... \circ \sigma_{a_k}(X),$$

where $\{\sigma_i\}_{i=0}^{N-1}$ is an iterated function system of Lipschitz contractions on X. The proof of Jorgensen does not require this assumption. Rather, it only requires that for each $k \in \mathbb{Z}_+$, there is a sequence *of subsets* $\{A_k(a)\}_{a\in\Gamma_N^k}$ *which partitions* X *with the following property. Given* $\epsilon > 0$, *there exists a* K *such that for* $k \geq K$

$$diam(A_k(a)) < \epsilon$$

for all $a \in \Gamma_N^k$. *In summary, the result of Jorgensen is more general. Assuming the above decay condition, he shows that the assignment* $A_k(a) \mapsto P_k(a)$ *extends to a projection-valued measure, and the subsets do not have to be constructed from an iterated function system.*

6.4 Multifunctions and IFS

We will take a brief foray away from functional analysis to discuss the notion of a multifunction, and its relationship to iterated functions systems and fixed points. Indeed, let (X, d) be a compact metric space, and let $\mathcal{S} = \{\sigma_0, ..., \sigma_{N-1}\}$ be an iterated function system (IFS) on (X, d), where σ_i has Lipschitz constant $0 < r_i \leq 1$. We adopt the notation from the previous sections in this chapter. That is, let $\Gamma_N = \{0, ..., N-1\}$. If $a = (a_1, ..., a_k) \in \Gamma_N^k$, define

$$A_k(a) = \sigma_{a_1} \circ ... \circ \sigma_{a_k}(X).$$

Let $\mathcal{K}$ denote the collection of all compact subsets of X. As we discussed in Chapter 1, $\mathcal{K}$ can be equipped with the Hausdorff metric, δ, given by

$$\delta(B, C) = \max\{\sup_{b\in B} d(b, C), \sup_{c\in C} d(B, c)\},$$

and the resulting metric space $(\mathcal{K}, \delta)$ is complete.

Definition 6.4.1. *Let* T *be a set. A multifunction with respect to* T *and* X *is a set-valued function* $F : T \to \mathcal{K}$. *That is,* $F(t)$ *is a compact subset of* X *for each* $t \in T$, *and we assume that* $F(\emptyset) = X$.

Let $\mathcal{F}(T, X)$ denote the collection of all multifunctions with respect to T and X. Define a metric d_∞ on $\mathcal{F}(T, X)$ by

$$d_\infty(F, G) = \sup_{t\in T} \delta(F(t), G(t)).$$

We have the following result due to D. Torre, F. Mendivil, and E. Vrscay in [25].

Proposition 6.4.2. *[25] [Torre, Mendivil, Vrscay]The metric space $(\mathcal{F}(T,X), d_\infty)$ is complete.*

We now will discuss a specific application of this theory. For $0 \leq i \leq N-1$, define $\sigma_i^* : \mathcal{K} \to \mathcal{K}$ by $B \mapsto \sigma_i(B)$, which is well defined because the continuous image of a compact set is compact. In addition, σ_i^* is a Lipschitz contraction in the δ metric, with Lipschitz constant r_i . Indeed, if $B, C \in \mathcal{K}$

$$\delta(\sigma_i^*(B), \sigma_i^*(C)) = \delta(\sigma_i(B), \sigma_i(C)) = \max\{\sup_{b\in B}\inf_{c\in C} d(\sigma_i(b), \sigma_i(c)), \sup_{c\in C}\inf_{b\in B} d(\sigma_i(b), \sigma_i(c))\} \leq$$

$$\max\{\sup_{b\in B}\inf_{c\in C} r_i d(b,c), \sup_{c\in C}\inf_{b\in B} r_i d(b,c)\} = r_i \delta(B,C).$$

Next, let $T = \cup_{k\in\mathbb{Z}_+}\Gamma_N^k$. For $0 \leq i \leq N-1$, define $w_i : T \to T$ by $a = (a_1, ..., a_k) \mapsto (i, a_1, ..., a_k)$. Define $U : \mathcal{F}(T,X) \to \mathcal{F}(T,X)$ by

$$U(F)(a) = \sigma_{a_1}^*(F(w_{a_1}^{-1}(a))),$$

where $a = (a_1, ..., a_k) \in T$. Note that if $a = (a_1) \in \Gamma_N$, $U(F)(a) = \sigma_{a_1}(F(\emptyset)) = \sigma_{a_1}(X)$, because we are assuming $F(\emptyset) = X$. The map U is very similar to the union operator presented in [25], and can be shown to be a Lipschitz contraction in the d_∞ metric. To see this, if $F, G \in \mathcal{F}(T,X)$, then

$$d_\infty(U(F), U(G)) = \sup_{a\in T}\delta(U(F)(a), U(G)(a)) = \sup_{a\in T}\delta(\sigma_{a_1}^*(F(w_{a_1}^{-1}(a))), \sigma_{a_1}^*(G(w_{a_1}^{-1}(a)))) \leq$$

$$\sup_{a\in T} r_{a_1}\delta(F(w_{a_1}^{-1}(a)), G(w_{a_1}^{-1}(a))) \leq r \sup_{a\in T}\delta(F(a), G(a)) = r d_\infty(F,G),$$

where $r = \max_{0\leq i\leq N-1} r_i$. Hence, there exists a unique fixed multifunction $E \in \mathcal{F}(T,X)$ that satisfies

$$E(a) = \sigma_{a_1}^*(E(w_{a_1}^{-1}(a)))$$

for all $a \in T$. One can use induction to compute that $E : T \to X$ is given by

$$a \in T \mapsto \sigma_{a_1} \circ ... \circ \sigma_{a_k}(X) = A_k(a).$$

In summary, in this short section we have realized the map $a \mapsto A_k(a)$ as a fixed point of the Lipschitz contraction U.

Chapter 7

Weak Hyperbolic Iterated Function Systems

In this chapter, we will discuss weak hyperbolic iterated functions systems (whIFS). A whIFS is a generalization of an IFS where the members of the whIFS do not have to be Lipschitz contractions; however, there is an assumption that the diameters of the sets $\sigma_{i_1} \circ ... \circ \sigma_{i_n}(X)$ converge to zero as $n \to \infty$. A formal definition is given below. We will begin our discussion of this topic with a motivating example, and then discuss the map $E(\cdot) \mapsto \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*$ in this generalized setting.

7.1 A whIFS Cantor Set

We begin with the definition of a whIFS over a compact metric space, which can be found in [13].

Definition 7.1.1. *[13][Edalat] Let (X, d) be a compact metric space. A weak hyperbolic iterated function system $\mathcal{S} = \{\sigma_0, ..., \sigma_{N-1}\}$ is a family of continuous functions $\sigma_i : X \to X$ for $0 \leq i \leq N-1$ such that*

$$\lim_{n\to\infty} \textit{diam}\ (\sigma_{i_1} \circ ... \circ \sigma_{i_n}(X)) = 0,$$

for any sequence $i_1, ..., i_n, ...$.

Note that if we assume in addition that each σ_i is a non-expansive mapping, meaning that $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$, we can claim the following: For $\epsilon > 0$, there exists a K such that for $k \geq K$

$$\text{diam}\ (\sigma_{i_1} \circ ... \circ \sigma_{i_n}(X)) < \epsilon$$

for all $i_1, ..., i_n$. That is, the diameters converge uniformly to 0. A proof of this follows from Proposition 2.2 of [13], and the fact that each map is non-expansive.

We will now present an interesting example of a non-expansive whIFS, which is inspired by one of the examples presented in [13], and has similarities to the IFS which produces the Cantor set. Consider $f_0(x) = \frac{1}{3}x^3$ and $f_1(x) = \frac{1}{3}x^3 + \frac{2}{3}$ be a family of functions on $[0, 1]$. We note that these function satisfy the following properties:

(1) $f_i'(x) < 1$ for $x \in [0, 1)$ and $f_i(1) = 1$ for $i = 0, 1$.

(2) f_i is increasing and concave up on $[0, 1]$ for $i = 0, 1$.

We can can use property (1) and the mean value theorem to conclude that f_i are non-expansive for $i = 0, 1$. We can can use property (2) and the mean value theorem to conclude the following: If $[a, b]$ and $[c, d]$ are two intervals in $[0, 1]$ such that $b \leq c$, and such that $|b - a| \leq |d - c|$, then $|f_i(b) - f_i(a)| \leq |f_j(d) - f_j(c)|$ for $i, j \in \{0, 1\}$. We call this property $*$.

Let $\Gamma_2 = \{0, 1\}$ and $\Gamma_2^k = \Gamma_2 \times ... \times \Gamma_2$ for $k \in \mathbb{Z}_+$ and the product is k times. One can prove via induction that if $a, b \in \Gamma_2^k$ for some k, and if $a \neq b$, then

$$f_{a_1} \circ ... \circ f_{a_k}([0, 1]) \cap f_{b_1} \circ ... \circ f_{b_k}([0, 1]) = \emptyset.$$

We now have the following claim.

Claim 7.1.2. *Let $a \in \Gamma_2^k$. Then*

$$\textit{diam } (f_{a_1} \circ ... \circ f_{a_k}([0, 1])) \leq \textit{ diam } (f_1^k([0, 1])),$$

where $f_1^k([0, 1]) = f_1 \circ ... \circ f_1([0, 1])$ (k times).

Proof. We first note that $f_{a_1} \circ ... \circ f_{a_k}([0, 1])$ is an interval in $[0, 1]$, and $f_1 \circ ... \circ f_1([0, 1])$ is an interval in $[0, 1]$ of the form $[c, 1]$ for some c. We will prove this claim by induction on k. When $k = 1$

$$\text{diam}(f_{a_1}([0, 1]) = \frac{1}{3} = \text{diam}(f_1([0, 1]).$$

Suppose that the claim is true for $k = l \geq 1$. We need to show it is true for $k = l + 1$. Let $a = (a_1, ..., a_{l+1}) \in \Gamma_2^{l+1}$. By the induction assumption,

$$\text{diam } (f_{a_2} \circ ... \circ f_{a_{l+1}}([0,1])) \leq \text{ diam } (f_1^l([0,1])).$$

We have two cases:

(1) Suppose that $a_2 = 1, ..., a_{l+1} = 1$. Then $f_{a_2} \circ ... \circ f_{a_{l+1}}([0,1]) = f_1^l([0,1])$, which means that

$$\text{diam } (f_{a_1} \circ ... \circ f_{a_{l+1}}([0,1])) = \text{ diam } (f_1^{l+1}([0,1])).$$

(2) Suppose that $a_i \neq 1$ for some $2 \leq i \leq l+1$. Then $f_{a_2} \circ ... \circ f_{a_{l+1}}([0,1]) \cap f_1^l([0,1]) = \emptyset$, and moreover, if $f_{a_2} \circ ... \circ f_{a_{l+1}}([0,1]) := [a,b]$ and $f_1^l([0,1]) := [c,1]$, we must have that $b \leq c$. By invoking property $*$, we can conclude that

$$\text{diam } (f_{a_1} \circ ... \circ f_{a_{l+1}}([0,1])) \leq \text{ diam } (f_1^{l+1}([0,1])).$$

This completes the proof of the claim.

□

We will now show that

$$\lim_{n\to\infty} \text{diam}(f_1^n([0,1])) = 0. \tag{7.1}$$

Note that to show equation (7.1), it is enough to show that $\lim_{n\to\infty} \text{diam}(g^n([0,1])) = 0$, where $g(x) = 1 - f_1(1-x)$. The g function is easier to work with because it has a fixed point at $x = 0$, rather than at $x = 1$ [1] . Consider the function $h(x) = \frac{x}{x+1}$ on $[0,1]$. Notice that

$$h(x) - g(x) = \frac{x^3(2-x)}{3(1+x)} \geq 0$$

for all $x \in [0,1]$, and that $h(0) = g(0)$. It follows by induction that $g^n([0,1]) \subseteq h^n([0,1])$ for all $n = 1, 2, ...$ Additionally, it follows by induction that that $h^n([0,1]) = \left[0, \frac{1}{n+1}\right]$ for all $n = 1, 2, ...$

[1] I would like to acknowledge Darsh Ranjin (UC Berkeley) who provided the idea of considering the function g. This suggestion was made via Math Stack Exchange; see Iterated Function System Question.

Hence, $g^n([0,1]) \subseteq h^n([0,1]) = \left[0, \frac{1}{n+1}\right]$ for all $n = 1, 2, ...$ Since $\lim_{n\to\infty} \operatorname{diam}\left(\left[0, \frac{1}{n+1}\right]\right) = 0$, we get that $\lim_{n\to\infty} \operatorname{diam}(g^n([0,1])) = 0$.

We are now prepared to show that f_0 and f_1 comprise a weak hyperbolic iterated function system. Let $i_1, i_2, ...$ be a sequence in $\Omega := \Gamma_2 \times \Gamma_2 \times ....$ Let $\epsilon > 0$. Choose K such that for $k \geq K$,

$$\operatorname{diam}(f_1^k([0,1]))) \leq \epsilon.$$

By the claim, if $k \geq K$

$$\operatorname{diam}\,(f_{i_1} \circ ... \circ f_{i_k}([0,1])) \leq \operatorname{diam}\,(f_1^k([0,1])) \leq \epsilon.$$

This shows that f_0 and f_1 comprise a weak hyperbolic iterated function system, and moreover, the convergence is uniform in Ω.

Given the non-expansive whIFS, $\mathcal{S} = \{f_0, f_1\}$, the work in [13] shows that there exists a unique compact subset $C \subseteq [0,1]$ such that $C = f_0(C) \cup f_1(C)$. The subset C is a generalized Cantor set. Moreover, if we define $M(C)$ to be the collection of Borel probability measures on C, we can consider the map $T : M(C) \to M(C)$ by $\nu(\cdot) \mapsto \sum_{i=0}^{1} \frac{1}{2}\nu(f_i^{-1}(\cdot))$. Since the maps f_i for $i = 0, 1$ are not Lipschitz contractions, the map T is not a Lipschitz contraction. However, it is still possible to show that T admits a unique fixed measure, which we call μ. The proof of this fact is not included, because a similar kind of proof will be included below for the analogous projection-valued measure fixed point result. Let us now consider the Hilbert space $L^2(C, \mu)$.

Let $\sigma : [0,1] \to [0,1]$ be given by

$$\sigma(x) = \begin{cases} 3^{\frac{1}{3}} x^{\frac{1}{3}} \text{ if } x \in [0, \frac{1}{3}] \\ 0 \text{ if } x \in (\frac{1}{3}, \frac{2}{3}) \\ 3^{\frac{1}{3}}(x - \frac{2}{3})^{\frac{1}{3}} \text{ if } x \in [\frac{2}{3}, 1] \end{cases}$$

One can calculate $\sigma \circ f_0 = \text{id}$ and $\sigma \circ f_1 = \text{id}$, and therefore $\sigma(C) = C$. As we did in the previous chapter, define

$$S_i : L^2(C, \mu) \to L^2(C, \mu) \text{ by } \phi \mapsto (\phi \circ \sigma)\sqrt{2}\mathbf{1}_{f_i(C)}$$

for $i = 0, 1$, and its adjoint

$$S_i^* : L^2(C, \mu) \to L^2(C, \mu) \text{ by } \phi \mapsto \frac{1}{\sqrt{2}}(\phi \circ f_i)$$

for $i = 0, 1$. It can be shown that these maps, S_0 and S_1, and their adjoints satisfy the Cuntz relations, and hence, $L^2(C, \mu)$ admits a representation of the Cuntz algebra on two generators. We now use the notation of the previous chapter. That is for $k \in \mathbb{Z}_+$ and $a \in \Gamma_2^k$, define

$$A_k(a) = f_{a_1} \circ ... \circ f_{a_k}(C).$$

Using that $C = f_0(C) \cup f_1(C)$ is a disjoint union, we conclude that $\{A_k(a)\}_{a \in \Gamma_2^k}$ partitions C for all $k \in \mathbb{Z}_+$. Moreover, the collection of subsets $\{A_k(a)\}_{a \in \Gamma_2^k}$ satisfies the assumptions of Remark 6.3.1, because $\mathcal{S} = \{f_0, f_1\}$ is a non-expansive whIFS. If we define $P(C)$ to be the collection of projection-valued measures with respect to the pair $(C, L^2(C, \mu))$, we can conclude that there exists a unique projection-valued measure $E \in P(C)$ such that

$$E(\cdot) = \sum_{i=0}^{1} S_i E(f_i^{-1}(\cdot)) S_i^*. \tag{7.2}$$

Like we did in Chapter 6, we will now present an alternative proof of this fixed point result by showing that the map $F(\cdot) \mapsto \sum_{i=0}^{1} S_i E(f_i^{-1}(\cdot)) S_i^*$ on $P(C)$ has a unique fixed point, even though it is not a Lipschitz contraction. In particular, let (X, d) be a compact metric space, and let $\mathcal{S} = \{\sigma_0, ..., \sigma_{N-1}\}$ be a non-expansive whIFS on X. Recall that we have the following: for $\epsilon > 0$, there exists a K such that for $k \geq K$, diam $(\sigma_{i_1} \circ ... \circ \sigma_{i_k}(X)) < \epsilon$ for all $i_1, ..., i_k$.

Let $\mathcal{H}$ be a Hilbert space which admits a representation of the Cuntz algebra on N generators; the isometries being $\{S_i\}_{i=0}^{N-1}$. Consider the complete metric space $(P(X), \rho)$ of projection valued measures with respect to the pair $(X, \mathcal{H})$, and the map $\Phi : P(X) \to P(X)$ given by

$$F \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot)) S_i^*.$$

Recall that the topology on $P(X)$ induced by the metric ρ coincides with the weak topology on $P(X)$.

Lemma 7.1.3. *[Davison] The map $\Phi : P(X) \to P(X)$ by $F \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot))S_i^*$ is continuous in the ρ metric.*

Proof. Since the topology on $P(X)$ induced by the metric ρ coincides with the weak topology on $P(X)$, it is enough to show that Φ is continuous in the weak topology. Indeed, suppose that $\{E_n\}_{n=1}^{\infty} \subseteq P(X)$ is a sequence of projection-valued measures that converges weakly to $E \in P(X)$. We need to show that $\Phi(E_n) \to \Phi(E)$ in the weak topology. That is, choose $f \in C_{\mathbb{R}}(X)$ and let $\epsilon > 0$. Since the maps $\{\sigma_i : 0 \leq i \leq N-1\}$ are continuous, we have that $\{f \circ \sigma_i : 0 \leq i \leq N-1\} \subseteq C_{\mathbb{R}}(X)$. Since $E_n \Rightarrow E$, and $\{f \circ \sigma_i : 0 \leq i \leq N-1\}$ is a finite set of maps, there exists an N such that for $n \geq N$

$$\left|\left|\int f \circ \sigma_i dE_n - \int f \circ \sigma_i dE\right|\right| \leq \epsilon.$$

Let $n \geq N$, and choose $h \in \mathcal{H}$ such that $||h|| = 1$. Then

$$\left|\left\langle\left(\int f d\Phi(E_n) - \int f d\Phi(E)\right) h, h\right\rangle\right| \leq$$

$$\sum_{i=0}^{N-1}\left|\left\langle\left(\int f \circ \sigma_i dE_n - \int f \circ \sigma_i d\Phi(E)\right) S_i^* h, S_i^* h\right\rangle\right| \leq$$

$$\sum_{i=0}^{N-1}\left|\left|\int f \circ \sigma_i dE_n - \int f \circ \sigma_i dE\right|\right| ||S_i^* h||^2 \leq$$

$$\epsilon \sum_{i=0}^{N-1} ||S_i^* h||^2 = \epsilon ||h||^2 = \epsilon.$$

Hence, for $n \geq N$

$$\left|\left|\int f d\Phi(E_n) - \int f d\Phi(E)\right|\right| \leq \epsilon,$$

and Φ is continuous in the weak topology, and therefore in the ρ metric.

□

For $x_0 \in X$ define the projection valued measure $E_{x_0} \in P(X)$ as follows:

$$E_{x_0}(\Delta) = \begin{cases} \mathbf{1}_{\mathcal{H}} & \text{if } x_0 \in \Delta \\ 0 & \text{if } x_0 \notin \Delta, \end{cases}$$

where Δ is Borel subset of X.

Lemma 7.1.4. *[Davison] Let $x_0 \in X$. The sequence of projection valued measures $\{\Phi^n(E_{x_0})\}_{n=1}^{\infty}$ is Cauchy in the the ρ metric.*

Proof. It is enough to show that $\{\Phi^n(E_{x_0})\}_{n=1}^{\infty}$ is Cauchy in the weak topology. Let $f \in C_{\mathbb{R}}(X)$ and let $\epsilon > 0$. Note that for $h \in \mathcal{H}$ and $n \in \mathbb{N}$,

$$\left\langle \left(\int f d\Phi^n(E_{x_0}) \right) h, h \right\rangle = \sum_{a \in \Gamma_N^n} \int f \circ \sigma_a(x) dE_{x_0 S_a^* h, S_a^* h} =$$

$$\sum_{a \in \Gamma_N^n} f \circ \sigma_a(x_0) \langle S_a^* h, S_a^* h \rangle,$$

where $a = (a_1, ..., a_n) \in \Gamma_N^n$, and $\sigma_a = \sigma_{a_1} \circ ... \circ \sigma_{a_n}$.

Since $f \in C_{\mathbb{R}}(X)$ and X is compact, f is uniformly continuous. That is, there exists a $\delta > 0$ such that when $x, y \in X$ with $d(x, y) < \delta$, $|f(x) - f(y)| \le \epsilon$. By earlier, we know that there exists an M such that for $k \ge M$,

$$\text{diam}(\sigma_a(X)) \le \delta$$

for all $a \in \Gamma_N^k$.

Suppose that $k, m \ge M$, and without loss of generality that $m \ge k$. Let $b \in \Gamma_N^m$ with $b = (a_1, ..., a_k, b_{k+1}, ..., b_m)$, where $a = (a_1, ..., a_k)$. Note that $\sigma_a(x_0) \in \sigma_a(X)$ and $\sigma_b(x_0) = \sigma_a(\sigma_{b_{k+1}} \circ ... \circ \sigma_{b_m}(x_0)) \in \sigma_a(X)$. Hence $d(\sigma_a(x_0), \sigma_b(x_0)) \le \delta$, which means that $|f(\sigma_a(x_0)) - f(\sigma_b(x_0))| \le \epsilon$.

Let $k, m \ge M$ with $m \ge k$. We will show that

$$\left|\left| \int f d(\Phi^k(E_{x_0})) - \int f d(\Phi^m(E_{x_0})) \right|\right| \le \epsilon.$$

We can assume that $m > k$, because if they are equal, the above quantity is zero. Choose $h \in \mathcal{H}$ with $||h|| = 1$. Then

$$\left| \left\langle \left(\int f d(\Phi^k(E_{x_0})) - \int f d(\Phi^m(E_{x_0})) \right) h, h \right\rangle \right| =$$

$$\left| \sum_{a \in \Gamma_N^k} \langle f \circ \sigma_a(x_0) S_a^* h, S_a^* h \rangle - \sum_{b \in \Gamma_N^m} \langle f \circ \sigma_b(x_0) S_a^* h, S_a^* h \rangle \right|.$$

Now choose $a = (a_1, ..., a_k) \in \Gamma_N^k$. Consider $b \in \Gamma_N^m$ such that $a_1 = b_1, ..., a_k = b_k$, where $b = (b_1, ..., b_k, b_{k+1}, ..., b_m)$. Then

$$
\begin{aligned}
f\circ\sigma_b(x_0)\langle S_b^*h, S_b^*h\rangle &= \langle f\circ\sigma_b(x_0)S_{b_m}^*...S_{b_{k+1}}^*S_{a_k}^*...S_{a_1}^*h, S_{b_m}^*...S_{b_{k+1}}^*S_{a_k}^*...S_{a_1}^*h\rangle \\
&= \langle f\circ\sigma_b(x_0)S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*S_a^*h, S_a^*h\rangle
\end{aligned}
$$

Hence

$$
\left|\sum_{a\in\Gamma_N^k}\langle f\circ\sigma_a(x_0)S_a^*h, S_a^*h\rangle - \sum_{b\in\Gamma_N^m}\langle f\circ\sigma_b(x_0)S_a^*h, S_a^*h\rangle\right| =
$$

$$
\left|\sum_{a\in\Gamma_N^k}\left(\langle f\circ\sigma_a(x_0)S_a^*h, S_a^*h\rangle - \sum_*\langle f\circ\sigma_b(x_0)S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*S_a^*h, S_a^*h\rangle\right)\right| =
$$

(where $*$ denotes all the $b\in\Gamma_N^m$ such that $b_1=a_1,...,b_k=a_k$)

$$
\left|\sum_{a\in\Gamma_N^k}\left\langle\left(f\circ\sigma_a(x_0) - \sum_* f\circ\sigma_b(x_0)S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*\right)S_a^*h, S_a^*h\right\rangle\right| =
$$

$$
\left|\sum_{a\in\Gamma_N^k}\left\langle\left(f\circ\sigma_a(x_0)\hat{\sum} - \sum_* f\circ\sigma_b(x_0)S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*\right)S_a^*h, S_a^*h\right\rangle\right| \leq
$$

(where $\hat{\sum} := \sum_* S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*$ and the above equality is because $\hat{\sum} = \mathbf{1}_{\mathcal{H}}$)

$$
\sum_{a\in\Gamma_N^k}\left(\sum_*|f\circ\sigma_a(x_0) - f\circ\sigma_b(x_0)|\left|\left\langle\left(S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*\right)S_a^*h, S_a^*h\right\rangle\right|\right) \leq
$$

$$
\epsilon\sum_{a\in\Gamma_N^k}\left(\sum_*\left|\langle(S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*)S_a^*h, S_a^*h\rangle\right|\right) =
$$

$$
\epsilon\sum_{a\in\Gamma_N^k}\left(\sum_*\langle(S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*)S_a^*h, S_a^*h\rangle\right) =
$$

(because $S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*$ are positive operators)

$$
\epsilon\sum_{a\in\Gamma_N^k}\left\langle\left(\sum_* S_{b_{k+1}}...S_{b_m}S_{b_m}^*...S_{b_{k+1}}^*\right)S_a^*h, S_a^*h\right\rangle = \epsilon\sum_{a\in\Gamma_N^k}\left\langle\left(\hat{\sum}\right)S_a^*h, S_a^*h\right\rangle =
$$

$$\epsilon \sum_{a\in\Gamma_N^k} \langle S_a^* h, S_a^* h\rangle = \epsilon \left\langle \left(\sum_{a\in\Gamma_N^k} S_a S_a^* \right) h, h \right\rangle = \epsilon ||h||^2 = \epsilon.$$

Hence

$$\left|\left| \int fd(\Phi^k(E_{x_0})) - \int fd(\Phi^m(E_{x_0}))\right|\right| \leq \epsilon$$

for $k, m \geq M$. This completes the proof of the lemma.

□

Since $\{\Phi^n(E_{x_0})\}_{n=1}^{\infty}$ is Cauchy in the the ρ metric, and $(P(X), \rho)$ is a complete metric space, there exists an element $E \in P(X)$ such that $\Phi^n(E_{x_0}) \to E$ in the ρ metric. Since Φ is continuous with respect to the ρ metric, $\Phi(\Phi^n(E_{x_0})) \to \Phi(E)$. But $\Phi(\Phi^n(E_{x_0})) \to E$, since $\{\Phi(\Phi^n(E_{x_0}))\}_{n=1}^{\infty} = \{\Phi^n(E_{x_0})\}_{n=2}^{\infty}$. This implies that $\Phi(E) = E$. We will show that E is the unique fixed point of Φ. We will need the following lemma.

Lemma 7.1.5. *[Davison] Let $F \in P(X)$, and $x_0 \in X$. Given $f \in C_{\mathbb{R}}(X)$ and $\epsilon > 0$, there exists an M such that for $k \geq M$*

$$\left|\left| \int fd\Phi^k(E_{x_0}) - \int fd\Phi^k(F)\right|\right| \leq \epsilon.$$

Proof. Let $f \in C_{\mathbb{R}}(X)$ and $\epsilon > 0$. As before, since f is in $C_{\mathbb{R}}(X)$ and X is compact, f is uniformly continuous. That is, there exists a $\delta > 0$ such that when $x, y \in X$ with $d(x, y) \leq \delta$, $|f(x) - f(y)| \leq \epsilon$. By earlier, we know that there exists an M such that for $k \geq M$, $\text{diam}(\sigma_a(X)) \leq \delta$ for all $a \in \Gamma_N^k$. In particular, if $k \geq M$ and $a \in \Gamma_N^k$, then $d(\sigma_a(x), \sigma_a(x_0)) \leq \delta$ for all $x \in X$, so that $|f \circ \sigma_a(x) - f \circ \sigma_a(x_0)| \leq \epsilon$ for all $x \in X$.

Choose $k \geq M$. Let $h \in \mathcal{H}$ with $||h|| = 1$. Then,

$$\left| \left\langle \left(\int fd\Phi^k(F) - \int fd\Phi^k(E_{x_0}) \right) h, h \right\rangle \right| =$$

$$\left| \sum_{a\in\Gamma_N^k} \left\langle \left(\int f \circ \sigma_a dF - f \circ \sigma_a(x_0) \right) S_a^* h, S_a^* h \right\rangle \right| =$$

$$\left| \sum_{a\in\Gamma_N^k} \left\langle \left(\int f \circ \sigma_a(x) dF - \int f \circ \sigma_a(x_0) dF \right) S_a^* h, S_a^* h \right\rangle \right| \leq$$

(because $\int f \circ \sigma_a(x_0) dF = f \circ \sigma_a(x_0)\mathbf{1}_{\mathcal{H}}$)

$$\sum_{a\in\Gamma_N^k} \left| \int f \circ \sigma_a(x) - f \circ \sigma_a(x_0) dF_{S_a^*h, S_a^*h} \right| \leq$$

$$\sum_{a\in\Gamma_N^k} \int |f \circ \sigma_a(x) - f \circ \sigma_a(x_0)| \, dF_{S_a^*h, S_a^*h} \leq$$

$$\epsilon \sum_{a\in\Gamma_N^k} \int dF_{S_a^*h, S_a^*h} =$$

$$\epsilon \sum_{a\in\Gamma_N^k} \langle F(X) S_a^* h, S_a^* h \rangle =$$

$$\epsilon \left\langle \left(\sum_{a\in\Gamma_N^k} S_a S_a^* \right) h, h \right\rangle =$$

$$\epsilon ||h||^2 = \epsilon.$$

Hence, for $k \geq M$,

$$\left|\left| \int f d\Phi^k(E_{x_0}) - \int f d\Phi^k(F) \right|\right| \leq \epsilon.$$

□

Since $\Phi^n(E_{x_0}) \to E$ in the weak topology, the above lemma implies that $\Phi^n(F) \to E$ in the weak topology, and therefore, in the ρ metric.

Proposition 7.1.6. *[Davison] E is the unique invariant projection-valued measure for Φ.*

Proof. Suppose $F \in P(X)$ such that $\Phi(F) = F$. By the discussion before this proposition, $\Phi^n(F) \to E$ in the ρ metric. But $\Phi^n(F) = F$ for all n, and hence, $\Phi^n(F) \to F$ in the ρ metric. Therefore, $E = F$. □

Chapter 8

Unitary Representations of the Baumslag Solitar Group Associated to the Cantor Set

For $N \in \mathbb{N}$ and $N \geq 2$, the Baumslag-Solitar group, denoted $BS(1, N)$, is the group on two generators a and b that satisfy the relation $a^{-1}ba = b^N$ (or equivalently $aba^{-1} = b^N$). In this chapter, we will be discussing unitary representations of $BS(1, 3)$ on Hilbert spaces. These representations will be related to the Cantor set.

8.1 Background

Let $X =$ Cantor Set $\subseteq [0, 1]$ equipped with the standard metric on $\mathbb{R}$. Recall that $\sigma_0(x) = \frac{1}{3}x$ and $\sigma_1(x) = \frac{1}{3}x + \frac{2}{3}$ comprise the iterated function system which gives rise to X. By Theorem [14], there exists a Borel probability measure μ on X such that

$$\mu(\cdot) = \frac{1}{2}\mu(\sigma_0^{-1}(\cdot)) + \frac{1}{2}\mu(\sigma_1^{-1}(\cdot)). \tag{8.1}$$

Following the work in [12], we define the inflated fractal set $\mathcal{R} \subset \mathbb{R}$ as $\mathcal{R} = \cup_{k,n\in\mathbb{Z}} 3^{-n}(X + k)$. We note that $\mathcal{R}$ satisfies the following properties (see Proposition 2.1 in [12]):

- $\mathcal{R} + \frac{k}{3^n} = \mathcal{R}$ for $k, n \in \mathbb{Z}$, and
- $3^n\mathcal{R} = \mathcal{R}$.

The measure μ can be extended to a Borel measure $\bar{\mu}$ on $\mathcal{R}$ that satisfies the relation $\bar{\mu}(\cdot) = \frac{1}{2}\bar{\mu}(3(\cdot))$. This follows from the fact the μ satisfies the invariance equation (8.1). Moreover, $\bar{\mu}$ is invariant under translation by numbers of the form $\frac{k}{3^n}$ for $k, n \in \mathbb{Z}$. Consider the Hilbert space $L^2(\mathcal{R}, \bar{\mu})$. Define the dilation and translation operators on $L^2(\mathcal{R}, \bar{\mu})$ by

$$Df(x) = \sqrt{2} f(3x),$$

$$Tf(x) = f(x-1),$$

for all $f \in L^2(\mathcal{R}, \bar{\mu})$. The operators D and T are unitary operators on $L^2(\mathcal{R}, \bar{\mu})$. Indeed, the fact that T is a unitary follows from the fact that $\bar{\mu}$ is invariant under translation by the integer 1. The fact that D is a unitary operator follows from the relation $\bar{\mu}(\cdot) = \frac{1}{2}\bar{\mu}(3(\cdot))$. Specifically, if $f \in L^2(\mathcal{R}, \bar{\mu})$,

$$||Df||^2_{L^2(\mathcal{R},\bar{\mu})} = \int_{\mathcal{R}} |Df|^2 d\bar{\mu} = 2\int_{\mathcal{R}} |f(3x)|^2 d\bar{\mu} = 2\int_{\mathcal{R}} |f(3x)|^2 \left(\frac{1}{2}\right) d\bar{\mu}(3(\cdot)) =$$

$$\int_{\mathcal{R}} |f|^2 d\mu = ||f||^2_{L^2(\mathcal{R},\bar{\mu})},$$

which shows that D is an isometry. To show that D is surjective, we note that $D(\frac{1}{\sqrt{2}} f(\frac{x}{3})) = f$ and that

$$\left|\left|\frac{1}{\sqrt{2}} f\left(\frac{x}{3}\right)\right|\right|^2_{L^2(\mathcal{R},\bar{\mu})} = \frac{1}{2}\int \left|f\left(\frac{x}{3}\right)\right|^2 2d\bar{\mu}\left(\frac{1}{3}(\cdot)\right) = ||f||^2_{L^2(\mathcal{R},\bar{\mu})} < \infty.$$

We will now show that $D^{-1}TD = T^3$. This will show that $L^2(\mathcal{R}, \bar{\mu})$ admits a unitary representation of BS(1,3). In particular, if $f \in L^2(\mathcal{R}, \bar{\mu})$

$$TDf = T(\sqrt{2}f(3x)) = \sqrt{2}f(3(x-1)) = \sqrt{2}f(3x-3)$$

and

$$DT^3 f = Df(x-3) = \sqrt{2}f(3x-3).$$

Since T is a unitary operator, the spectrum of T is contained in $\mathbb{T} \subseteq \mathbb{C}$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. By the spectral theorem for normal operators (unitary operators are normal) and the Borel functional calculus, there exists a unique projection valued measure $E : \mathbb{T} \to \mathcal{B}(L^2(\mathcal{R}, \bar{\mu}))$ that satisfies

$$\int \psi(z) dE = \psi(T)$$

for all $\psi \in \mathscr{BB}(\mathbb{T})$, where $\mathscr{BB}(\mathbb{T})$ denotes the bounded Borel functions on $\mathbb{T}$. This allows us to reformulate the relation $D^{-1}TD = T^3$ as

$$D^{-1}\left(\int \psi(z)dE\right)D = \int \psi(z^3)dE \tag{8.2}$$

for all $\psi \in \mathscr{BB}(\mathbb{T})$. Let us now consider the function $\phi = \mathbf{1}_X \in L^2(\mathcal{R}, \bar{\mu})$ and put $V_0 = \overline{\text{span}}\{T^k(\phi) : k \in \mathbb{Z}\}$, which is the closed subspace of $L^2(\mathcal{R}, \bar{\mu})$ spanned by the orthogonal elements $\{T^k(\phi) : k \in \mathbb{Z}\}$. Further, define $V_n = D^n(V_0)$ for all $n \in \mathbb{Z}$. It is shown in [12] that the subspaces $\{V_n : n \in \mathbb{Z}\}$ comprise a multi-resolution analysis of $L^2(\mathcal{R}, \bar{\mu})$. That is,

(1) $V_n \subseteq V_{n+1}$ for all $n \in \mathbb{Z}$

(2) $\cup_{n=-\infty}^{\infty} V_n$ is a dense linear subspace of $L^2(\mathcal{R}, \bar{\mu})$, and

(3) $\cap_{n=-\infty}^{\infty} V_n = \emptyset$.

In particular, $\phi \in V_0 \subseteq V_1 = \overline{\text{span}}\{DT^k(\phi) : k \in \mathbb{Z}\}$. One can calculate that

$$\phi = \frac{1}{\sqrt{2}}D(\phi) + \frac{1}{\sqrt{2}}DT^2(\phi).$$

Equivalently,

$$D^{-1}(\phi) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}T^2\right)(\phi) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\int z^2 dE\right)(\phi) = \left(\int \left(\frac{1+z^2}{\sqrt{2}}\right) dE\right)\phi.$$

We call $h(z) = \frac{1+z^2}{\sqrt{2}} \in \mathscr{BB}(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$ the generating filter corresponding the function ϕ. This generating filter is the starting point for realizing a representation of BS(1,3) on a Hilbert space associated to a compact topological group called the 3-solenoid. One can calculate that this generating function satisfies

$$\sum_{w^3=z} \frac{|h(w)|^2}{3} = 1 \tag{8.3}$$

or equivalently,

$$\sum_{j=0}^{2} \frac{|h(e^{-2\pi i \tau_j(x)})|^2}{3} = 1,$$

where $x \in [0, 1)$ and $\tau_j(x) : [0, 1) \to [0, 1)$ by $x \mapsto \frac{x+j}{3}$ for $0 \leq j \leq 2$.

We now remark that it was shown in [19] that given a generating filter $h \in L^\infty(\mathbb{T})$ that satisfies equation (8.3), there exists a representation of BS(1,3) on a Hilbert space $\mathcal{H}$ (with unitary operators D and T satisfying $D^{-1}TD = T^3$), and a vector $\phi \in \mathcal{H}$ such that

$$D^{-1}\phi = \left(\int h(z)dE \right) \phi, \tag{8.4}$$

where E is the unique projection valued measure associated to the unitary T. Moreover, this representation is unique up to isomorphism. In view of this result, we see that $L^2(\mathcal{R}, \bar{\mu})$ is one realization of this representation. As mentioned above, we will now present another known realization of this representation. Toward this end, we will first introduce needed preliminary material and notation in the below subsections.

8.1.1 The 3-Alphabet Space

Let $\Gamma = \{0, 1, 2\}$, and define $\Omega = \Gamma^{\mathbb{N}}$. Indeed, an element $\omega \in \Omega$ is of the form $\omega = (\omega_1, \omega_2, \omega_3, ...)$, where $\omega_i \in \Gamma$ for all $i = 1, 2, ...$. We call Ω the 3-Alphabet Space. If we put the discrete topology on Γ, then by Tychonoff's Theorem, Ω is compact in the product topology. For $n \in \mathbb{N}$, define

$$A(\omega_1, \omega_2, ..., \omega_n) = \{\omega_1\} \times .... \times \{\omega_n\} \times \Gamma \times \Gamma \times ...,$$

an open subset of Ω. Let $h(z)$ be the generating filter defined above. Define $W(z) = \frac{|h(z)|^2}{3} \in \mathscr{B}\mathscr{B}(\mathbb{T})$. Equation (8.3) becomes

$$\sum_{w^3 = z} W(w) = 1, \tag{8.5}$$

or equivalently,

$$\sum_{j=0}^{2} W(\tau_j(x)) = 1,$$

where we define $W(\tau_j(x)) = W(e^{-2\pi i \tau_j(x)})$. The next result can be found in a textbook of P. Jorgensen ([15] Lemma 2.4.1). We refer the reader to this reference for a proof.

Proposition 8.1.1. *[15][Jorgensen] For each $x \in [0, 1)$, there exists a unique probability measure*

P_x on Ω such that

$$P_x(A(\omega_1, \omega_2, ..., \omega_n)) = W(\tau_{\omega_1}(x)) \cdots W(\tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)),$$

for all open subsets of the form $A(\omega_1, \omega_2, ..., \omega_n)$.

This proposition will become useful when we describe the construction of a measure on the 3-solenoid associated to the generating filter. We make two remarks regarding this proposition.

Remark 8.1.2. *For $x \in [0, 1)$, P_x is a (Borel) probability measure on Ω. To see why this is true, observe that $\Omega = \cup_{\omega_1} A(\omega_1)$, which is a disjoint union. Hence*

$$P_x(\Omega) = \sum_{\omega_1} P_x(A(\omega_1)) = \sum_{\omega_1} W(\tau_{\omega_1}(\xi)) = 1,$$

where the last equality is by equation (8.5).

Remark 8.1.3. *Suppose $f \in C(\omega)$ depends on the first n coordinates. That is, $f(\omega) = f(\omega_1, ..., \omega_n)$. Then*

$$\int f(\omega) dP_x(\omega) = \sum_{\omega_1, ..., \omega_n} W(\tau_{\omega_1}(x)) \cdots W(\tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)) f(\omega_1, ..., \omega_n).$$

8.1.2 The 3-solenoid

As defined above, let $\mathbb{T} \subseteq \mathbb{C}$ be the unit circle. Since $\mathbb{T}$ is compact in the induced topology, the product space $\mathbb{T}^{\mathbb{N} \cup \{0\}}$ is compact in the product topology. The 3-solenoid, denoted $\mathcal{S}_3$, is the subset of $\mathbb{T}^{\mathbb{N} \cup \{0\}}$ described below. An element $(z_n)_{n=0}^{\infty}$ is contained in $\mathcal{S}_3$ if for all $n = 0, ..., \infty$, $z_{n+1}^3 = z_n$.

Proposition 8.1.4. *$\mathcal{S}_3$ is a subgroup of $\mathbb{T}^{\mathbb{N} \cup \{0\}}$.*

Proof. The identity element $(1, 1, 1....) \in \mathcal{S}_3$. If $(z_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ are two elements in $\mathcal{S}_3$, their product

$$(z_n)_{n=0}^{\infty} \cdot (w_n)_{n=0}^{\infty} = (z_n \cdot w_n)_{n=0}^{\infty}$$

is an element on $\mathcal{S}_3$ because for all $n = 0, ..., \infty$, $(z_{n+1} \cdot w_{n+1})^3 = z_{n+1}^3 w_{n+1}^3 = z_n \cdot w_n$. The inverse of an element $(z_n)_{n=0}^{\infty}$ is $(\bar{z}_n)_{n=0}^{\infty}$. Since for all $n = 0, ...\infty$, $z_{n+1}^3 = z_n$, we have that $\bar{z}_{n+1}^3 = \bar{z}_n^2$, which shows that the inverse element is in $\mathcal{S}_3$. Hence, $\mathcal{S}_3$ is a subgroup.

□

We note that $\mathcal{S}_3$ is a closed subset of $\mathbb{T}^{\mathbb{N}\cup\{0\}}$, and hence a compact subset. Moreover, the group operation in $\mathcal{S}_3$ is continuous which makes $\mathcal{S}_3$ a compact abelian group. The Pontryagin dual of the 3-solenoid is the group, $\mathbb{Z}\left[\frac{1}{3}\right]$, defined as follows:

$$\mathbb{Z}\left[\frac{1}{3}\right] = \left\{\frac{l}{3^p} : l \in \mathbb{Z} \text{ and } p \in \mathbb{N} \cup \{0\}\right\},$$

where the group operation is:

$$\frac{l}{3^p} + \frac{m}{3^r} = \frac{l3^r + m3^p}{3^{p+r}}.$$

An element $\frac{l}{3^p} \in \mathbb{Z}\left[\frac{1}{2}\right]$ is a character on $\mathcal{S}_3$ via the map:

$$\left\langle \frac{l}{3^p}, (z_n)_{n=0}^{\infty} \right\rangle = z_p^l \in \mathbb{T}.$$

To see why all the characters are of this form, we will show that the dual of $\mathbb{Z}\left[\frac{1}{3}\right]$ is $\mathcal{S}_3$. By invoking the Pontryagin Duality Theorem, the dual of $\mathcal{S}_3$ will be isomorphic to $\mathbb{Z}\left[\frac{1}{3}\right]$. To this end, choose some λ in the dual of $\mathbb{Z}\left[\frac{1}{3}\right]$. Consider the sequence,

$$(\lambda(1), \lambda\left(\frac{1}{3}\right), \lambda\left(\frac{1}{3^2}\right), ...),$$

which we claim is an element of $\mathcal{S}_3$. First, since λ is a character, each entry in the above sequence is an element of $\mathbb{T}$. Choose some $n = 0, 1, ...$ By the homomorphism property of λ,

$$\lambda\left(\frac{1}{3^n}\right) = \lambda\left(\frac{1}{3^{n+1}}3\right) = \lambda\left(\frac{1}{3^{n+1}}\right)^3,$$

which shows that the above element is in $\mathcal{S}_3$. Moreover, by using the homomorphism property again, for any $l \in \mathbb{Z}$, and $n = 0, 1, ...,$

$$\lambda\left(\frac{l}{3^n}\right) = \lambda\left(\frac{1}{3^n}\right)^l.$$

If $(z_n)_{n=0}^{\infty}$ is an element of $\mathcal{S}_3$, it induces a character on $\mathbb{Z}\left[\frac{1}{3}\right]$ via the map:

$$\frac{l}{3^n} \mapsto z_n^l,$$

where $l \in \mathbb{Z}$, and $n = 0, 1, ...$ The above discussion shows that the dual of $\mathbb{Z}\left[\frac{1}{3}\right]$ is $\mathcal{S}_3$.

We now will relate the 3-solenoid to the 3-alphabet space defined previously. In particular, choose some $x \in [0, 1)$, and $\omega \in \Omega$. Consider the infinite tuple

$$\Phi(x, \omega) := (e^{-2\pi i x}, e^{-2\pi i \tau_{\omega_1}(x)}, ..., e^{-2\pi i \tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)}, ...).$$

Note that $\Phi(x, \omega)$ is an element of $\mathcal{S}_3$. Indeed, for some $n = 0, ..., \infty$,

$$(e^{-2\pi i \tau_{\omega_{n+1}} \circ ... \circ \tau_{\omega_1}(x)})^3 = (e^{-2\pi i \frac{\tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x) + \omega_{n+1}}{3}})^3 =$$

$$e^{-2\pi i \tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)} e^{-2\pi i \omega_{n+1}} = e^{-2\pi i \tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)}.$$

This leads to the following proposition which can be found in a paper of D. Dutkay ([11] Proposition 4.1).

Proposition 8.1.5. *[11][Dutkay] The map $\Phi : [0, 1) \times \Omega \to \mathcal{S}_3$ defined by*

$$\Phi(x, \omega) = (e^{-2\pi i x}, e^{-2\pi i \tau_{\omega_1}(x)}, ..., e^{-2\pi i \tau_{\omega_n} \circ ... \circ \tau_{\omega_1}(x)}, ...),$$

for $x \in [0, 1)$ and $\omega \in \Omega$ is a measurable bijection.

To conclude this subsection, define $S : \mathcal{S}_3 \to \mathcal{S}_3$ by

$$(z_0, z_1, z_2, ...) \mapsto (z_0^3, z_0, z_1, ...),$$

and $S^{-1} : \mathcal{S}_3 \to \mathcal{S}_3$ by

$$(z_0, z_1, z_2, ...) \mapsto (z_1, z_2, ...).$$

8.1.3 Measure on the 3-solenoid

We will now define a measure on $\mathcal{S}_3$ according to [11]. Let $f \in C(\mathcal{S}_3)$ and consider the function $f \circ \Phi : [0, 1) \times \Omega \to \mathbb{C}$, which is measurable by Proposition 8.1.5, and bounded since $\mathcal{S}_3$ is compact. Following the work of D. Dutkay, define $m : C(\mathcal{S}_3) \to \mathbb{C}$ as the iterated integral

$$m(f) := \int_{[0,1)} \int_{\Omega} f \circ \Phi(x, \omega) dP_x(\omega) dx. \tag{8.6}$$

This integral is finite because P_x is a probability measure on Ω. In fact, we have $|m(f)| \leq ||f||_\infty$ for all $f \in C(\mathcal{S}_3)$. Further, observe that m is a positive linear functional on $C(\mathcal{S}_3)$. By the Riesz Representation Theorem, there exists a corresponding measure on $\mathcal{S}_3$ which for ease of notation we also denote by m, and which satisfies the property that for all $f \in C(\mathcal{S}_3)$,

$$\int_{\mathcal{S}_3} f dm = \int_{[0,1)} \int_{\Omega} f \circ \Phi(x, \omega) dP_x(\omega) dx.$$

We will now state some properties of the measure m (see Proposition 4.2 in [11])

Lemma 8.1.6. *[11][Dutkay] m is a probability measure on $\mathcal{S}_3$.*

Proof.

$$m(\mathcal{S}_3) = \int_{\mathcal{S}_3} dm = \int_{[0,1)} \int_{\Omega} dP_x(\omega) dx = 1.$$

□

Theorem 8.1.7. *[11] [Dutkay] Let $f \in C(\mathcal{S}_3)$. Then*

$$\int_{\mathcal{S}_3} f \circ S^{-1} dm = \int_{\mathcal{S}_3} 3W \cdot f dm,$$

where we think of $W \in \mathscr{B}\mathscr{B}(\mathbb{T})$ as a function on $\mathcal{S}_3$ by $W((z_n)_{n=0}^{\infty}) = W(z_0)$.

Remark 8.1.8. *We can reformulate the integral equation in the above theorem as*

$$\int_{\mathcal{S}_3} f d(m \circ S) = \int_{\mathcal{S}_3} 3W \cdot f dm$$

for all $f \in L^1(\mathcal{S}_3, m)$. This implies that $3W = \frac{d(m \circ S)}{dm}$, where $\frac{d(m \circ S)}{dm}$ denotes the Radon-Nikodym derivative. In particular, $m \circ S$ is absolutely continuous with respect to m.

Theorem 8.1.9. *[11] [Dutkay] Let $f \in C(\mathcal{S}_3)$. For $n = 1, ..., \infty$*

$$\int_{\mathcal{S}_3} f \circ S^{-n} dm = \int_{\mathcal{S}_3} 3^n B^{(n)}(\cdot) f dm \tag{8.7}$$

where $B^{(n)}((z_l)_{l=0}^{\infty}) : \mathcal{S}_3 \to \mathbb{C}$ is defined by

$$(z_l)_{l=0}^{\infty} \mapsto W(z_0)...W(z_0^{3^{n-1}}).$$

Proof. This will be proved by induction on n. The case when $n = 1$ is the content of Theorem 8.1.7. Suppose equation (9.2) is true for $n = k$ for $k \geq 1$. We need to show equation (9.2) is true for $n = k + 1$.

$$\int_{\mathcal{S}_3} f \circ S^{-(k+1)} dm = \int_{\mathcal{S}_3} f \circ S^{-k} \circ S^{-1} dm =$$

$$\int_{\mathcal{S}_3} 3W f \circ S^{-k} dm = \int_{\mathcal{S}_3} 3(W \circ S^k) \circ S^{-k} f \circ S^{-k} dm =$$

$$\int_{\mathcal{S}_3} 3 \cdot 3^k B^{(k)}(\cdot)(W \circ S^k) \cdot f dm = \int_{\mathcal{S}_3} 3^{k+1} B^{(k+1)}(\cdot) f dm.$$

□

We now make the observation that the generating filter $h(z) = \frac{1+z^2}{\sqrt{2}}$ satisfies the property that $\nu(\{z \in \mathbb{T} : h(z) = 0\}) = 0$, where ν is the standard Haar measure on $\mathbb{T}$. Following the literature, we say that h is a non-singular generating filter. An important consequence of this observation is the following, which can be found in Theorem 4.3 in [11].

Lemma 8.1.10. *[11] [Dutkay] The measure m on $\mathcal{S}_3$ associated to the non-singular generating function $h = \frac{1+z^2}{\sqrt{2}}$ is absolutely continuous with respect to the measure $m \circ S$ on $\mathcal{S}_3$. Moreover, the Radon-Nikodym derivative is*

$$\frac{dm}{d(m \circ S)} = \frac{1}{3W}.$$

Following the work in [11] (see Theorem 4.3), define the operator $U : L^2(\mathcal{S}_3, m) \to L^2(\mathcal{S}_3, m)$ by

$$Uf((z_n)_{n=0}^{\infty}) = h(z_0) f \circ S((z_n)_{n=0}^{\infty}).$$

The operator U is an isometry. Indeed, if $f \in C(\mathcal{S}_3)$, then f is bounded (because $\mathcal{S}_3$ is compact). Therefore

$$||Uf||^2_{L^2(\mathcal{S}_3,m)} = \int_{\mathcal{S}_3} |Uf|^2 dm = \int_{\mathcal{S}_3} |h(z_0)|^2 |f \circ S|^2 dm = \int_{\mathcal{S}_3} 3W |f \circ S|^2 dm =$$

$$\int_{\mathcal{S}_3} \frac{d(m \circ S)}{dm} |f \circ S|^2 dm = \int_{\mathcal{S}_3} |f \circ S|^2 d(m \circ S) = \int_{\mathcal{S}_3} |f|^2 dm = ||f||^2_{L^2(\mathcal{S}_3,m)}.$$

Since continuous functions are dense in $L^2(\mathcal{S}_3, m)$, U extends to an isometry on $L^2(\mathcal{S}_3, m)$. Moreover, Lemma 8.1.10 allows one to show that U is surjective (see Theorem 4.3 in [11]), which implies that U is a unitary operator on $L^2(\mathcal{S}_3, m)$.

Consider the projection valued measure $F : \mathbb{T} \to \mathcal{B}(L^2(\mathcal{S}_3, m))$ given by

$$\Delta \mapsto M_{\mathbf{1}_\Delta}, \tag{8.8}$$

where Δ denotes an arbitrary Borel subset of $\mathbb{T}$, and where $M_{\mathbf{1}_\Delta} : L^2(\mathcal{S}_3, m) \to L^2(\mathcal{S}_3, m)$ is given by $M_{\mathbf{1}_\Delta} f((z_n)_{n=0}^\infty) = \mathbf{1}_\Delta(z_0) f((z_n)_{n=0}^\infty)$. Note that for all $\psi \in \mathscr{B}\mathscr{B}(\mathbb{T})$

$$\left(\int \psi dF\right) f((z_n)_{n=0}^\infty) = \psi(z_0) f((z_n)_{n=0}^\infty).$$

Define the operator $S = \int z dF \in \mathcal{B}(L^2(\mathcal{S}, m))$. We will show that S is a unitary operator on $L^2(\mathcal{S}, m)$. Let $f \in L^2(\mathcal{S}, m)$. Then

$$||Sf||^2_{L^2(\mathcal{S},m)} = \int_{\mathcal{S}_3} |M_z f|^2 dm = \int_{\mathcal{S}_3} |z_0 f((z_n)_{n=0}^\infty)|^2 dm((z_n)_{n=0}^\infty) = \int_{\mathcal{S}_3} |f|^2 dm = ||f||^2_{L^2(\mathcal{S},m)},$$

where the third equality is because $|z_0|^2 = 1$. This shows that S is an isometry. Observe that $\int z^{-1} dF$ is the inverse of S, which implies that S is a unitary. Let $\phi = 1 \in L^2(\mathcal{S}_3, m)$. Observe that

$$U\phi((z_n)_{n=0}^\infty) = h(z_0)(1 \circ S((z_n)_{n=0}^\infty)) = h(z_0) = \left(\int h(z) dF\right) 1 = \left(\int h(z) dF\right) \phi.$$

We see that U and S satisfy equation (8.4). It remains to show that U and S satisfy the Baumslag-Solitar relation, namely $USU^{-1} = S^3$, or equivalently, $US = S^3 U$. Let $f \in L^2(\mathcal{S}_3, m)$. Observe that

$$USf((z_n)_{n=0}^\infty) = U\left(\int z dF\right) f((z_n)_{n=0}^\infty) = U(z_0 f((z_n)_{n=0}^\infty)) =$$
$$h(z_0)(z_0 \circ S)(f \circ S)((z_n)_{n=0}^\infty) = z_0^3 h(z_0)(f \circ S)((z_n)_{n=0}^\infty).$$

Next note that

$$S^3 U f((z_n)_{n=0}^\infty) = S^3(h(z_0)(f \circ S)((z_n)_{n=0}^\infty)) =$$
$$\left(\int z^3 dF\right)(h(z_0)(f \circ S)((z_n)_{n=0}^\infty)) = z_0^3 h(z_0)(f \circ S)((z_n)_{n=0}^\infty).$$

As we did in equation (8.2), we can reformulate the Baumslag-Solitar relation as

$$U\left(\int \psi(z)dF\right)U^{-1} = \int \psi(z^3)dF \tag{8.9}$$

for all $\psi \in \mathscr{B}\mathscr{B}(\mathbb{T})$, where we recall that $\mathscr{B}\mathscr{B}(\mathbb{T})$ denotes the bounded Borel functions on $\mathbb{T}$. In summary, we have shown that the data h, $L^2(\mathcal{S}_3, m)$, and $\phi = 1 \in L^2(\mathcal{S}_3, m)$ induce a representation of the Baumslag-Solitar group. By work of P. Jorgensen, representations of this form are unique up to isomorphism [19]. The following result in [11] shows that the isomorphism between $L^2(\mathcal{R}, \bar{\mu})$ and $L^2(\mathcal{S}_3, m)$ behaves like a generalized Fourier transform, because under the isomorphism, the translation operator $T \in \mathcal{B}(L^2(\mathcal{R}, \bar{\mu}))$ becomes the multiplication operator $S \in \mathcal{B}(L^2(\mathcal{S}_3, m))$.

Before we state the result, we remark that for $\lambda \in \mathbb{Z}[\frac{1}{3}]$, let $\chi_\lambda : \mathcal{S}_3 \to \mathbb{T}$ be the associated character on $\mathcal{S}_3$ defined earlier in this chapter. Moreover, for $\lambda \in \mathbb{Z}[\frac{1}{3}]$, define the (generalized) translation operator $T_\lambda : L^2(\mathcal{R}, \bar{\mu}) \to L^2(\mathcal{R}, \bar{\mu})$ by $T_\lambda(f)(x) = f(x - \lambda)$. It can be shown that T_λ is a unitary.

Proposition 8.1.11. *[11] [Dutkay] There is a unique isomorphism $\mathcal{F}_3 : L^2(\mathcal{R}, \bar{\mu}) \to L^2(\mathcal{S}_3, m)$ that satisfies:*

(1) $\mathcal{F}_3 T_\lambda \mathcal{F}_3^{-1}(f) = \chi_\lambda f$ for all $\lambda \in \mathbb{Z}[\frac{1}{3}]$, and $f \in L^2(\mathcal{S}_3, m)$ (in particular, $\mathcal{F}_3 T \mathcal{F}_3^{-1}(f) = S(f)$ for $\lambda = 1$).

(2) $\mathcal{F}_3 D^{-1} \mathcal{F}_3^{-1}(f) = U(f)$.

(3) $\mathcal{F}_3(\mathbf{1}_X) = 1$.

8.2 Decomposing the Unitary into a Sum of Partial Isometries

In this section, we will decompose the unitary $U : L^2(\mathcal{S}_3, m) \to L^2(\mathcal{S}_3, m)$ given by

$$Uf((z_n)_{n=0}^{\infty}) = h(z_0) f \circ S((z_n)_{n=0}^{\infty}),$$

for $h(z) = \frac{1+z^2}{\sqrt{2}}$ into a sum of partial isometries $\{T_i\}_{i=0}^{2}$, which will satisfy relations similar to the Cuntz relations.

For $0 \leq i \leq 2$, let $A_i = \{f \in L^2(m) : f \circ \Phi(x,\omega) = 0 \text{ when } \omega_1 \neq i\}$, where $\omega \in \Omega$ is of the form $\omega = (\omega_1, \omega_2, ...)$. These A_i are pairwise orthogonal closed subspaces of $L^2(\mathcal{S}_3, m)$, and moreover

$$L^2(\mathcal{S}_3, m) = \oplus_{i=0}^2 A_i.$$

Indeed, if $f \in L^2(\mathcal{S}_3, m)$

$$f = \sum_{i=0}^{2} f \mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1}(\cdot),$$

where for $0 \leq i \leq 2$, define $\sigma_i : \Omega \to \Omega$ by $(\omega_1, \omega_2, ...) \mapsto (i, \omega_1, \omega_2, ...)$. Accordingly, we let P_i for $0 \leq i \leq 2$ be the orthogonal projection of $L^2(\mathcal{S}_3, m)$ onto A_i. We note that $P_i = M_{\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}\circ\Phi^{-1}(\cdot)}$, where $M_{\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}\circ\Phi^{-1}(\cdot)}$ is multiplication by $\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1}(\cdot)$.

For $0 \leq i \leq 2$, let $B_i = \{f \in L^2(m) : f \circ \Phi(x,\omega) = 0 \text{ when } x \notin \tau_i([0,1))\}$, where $\tau_i : [0,1) \to [0,1)$ is given by $x \mapsto \frac{x+i}{3}$. These B_i are pairwise orthogonal closed subspaces of $L^2(\mathcal{S}_3, m)$, and moreover

$$L^2(\mathcal{S}_3, m) = \oplus_{i=0}^2 B_i.$$

Indeed, if $f \in L^2(\mathcal{S}_3, m)$

$$f = \sum_{i=0}^{2} f \mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}(\cdot).$$

Accordingly, we let Q_i for $0 \leq i \leq 2$ be the orthogonal projection of $L^2(\mathcal{S}_3, m)$ onto B_i. We note that $Q_i = M_{\mathbf{1}_{\tau_i([0,1))\times\Omega}\circ\Phi^{-1}(\cdot)}$. We continue with some notation and definitions:

(1) Define $\tau : [0,1) \to [0,1)$ by $\tau(x) = 3x(\text{mod}1)$. Note that $\tau \circ \tau_i = \text{id}_{[0,1)}$ for $0 \leq i \leq 2$.

(2) Define $\sigma : \Omega \to \Omega$ by $\sigma(\omega_1, \omega_2, ...) = (\omega_2, \omega_3, ...)$. Note that $\sigma \circ \sigma_i = \text{id}_\Omega$ for $0 \leq i \leq 2$.

(3) For $0 \leq i \leq 2$, define $s_i : [0,1) \times \Omega \to [0,1) \times \Omega$ by $(x,\omega) \mapsto (\tau_i(x), \sigma(\omega))$.

(4) For $0 \leq i \leq 2$, define $r_i : [0,1) \times \Omega \to [0,1) \times \Omega$ by $(x,\omega) \mapsto (\tau(x), \sigma_i(\omega))$.

For $0 \leq i \leq 2$, define $T_i : C(\mathcal{S}_3) \to L^2(\mathcal{S}_3, m)$ by

$$f \in C(\mathcal{S}_3) \mapsto (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1})h.$$

We will first show that T_i is a bounded operator on $C(\mathcal{S}_3)$, and then extend T_i uniquely to $L^2(\mathcal{S}_3, m)$ by using a standard density argument. Recall that $|h(z)|^2 = 3W$. Let $f \in C(\mathcal{S}_3)$, and note that f is bounded since $\mathcal{S}_3$ is compact. Moreover

$$||T_i(f)||^2_{L^2(\mathcal{S}_3,m)} = \int_{\mathcal{S}_3} |f \circ \Phi \circ r_i \circ \Phi^{-1}|^2 (\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) 3W dm =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ r_i \circ \Phi^{-1}|^2 (\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) \frac{d(m \circ S)}{dm} dm =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ r_i \circ \Phi^{-1}|^2 (\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) d(m \circ S) =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ r_i \circ \Phi^{-1} \circ S^{-1}|^2 (\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}) dm.$$

Suppose that $(z_n)_{n=0}^{\infty} \in \mathcal{S}_3$ is such that $\Phi^{-1} \circ S^{-1}((z_n)_{n=0}^{\infty}) \in \tau_i([0,1)) \times \Omega$. This implies that $(z_n)_{n=0}^{\infty} = \Phi(x, \omega)$ where $\omega = (i, \omega_2, \omega_3, ...)$. Moreover, for this element $(z_n)_{n=0}^{\infty} \in \mathcal{S}_3$

$$f \circ \Phi \circ r_i \circ \Phi^{-1} \circ S^{-1}((z_n)_{n=0}^{\infty}) = f \circ \Phi \circ r_i(\tau_i(x), \sigma(\omega)) =$$

$$f \circ \Phi(\tau \circ \tau_i(x), \sigma_i \circ \sigma(\omega)) = f \circ \Phi(x, (i, \omega_2, ...)) = f \circ \Phi(x, \omega) = f((z_n)_{n=0}^{\infty}).$$

Hence,

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ r_i \circ \Phi^{-1} \circ S^{-1}|^2 (\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}) dm =$$

$$\int_{S(\Phi(\tau_i([0,1))\times\Omega))} |f|^2 dm \leq \int_{\mathcal{S}_3} |f|^2 dm = ||f||^2_{L^2(\mathcal{S}_3,m)}.$$

This calculation shows that T_i is bounded. It is also clear that T_i is linear. Since continuous functions are dense in $L^2(\mathcal{S}_3, m)$, T_i extends uniquely to a bounded linear operator on $L^2(\mathcal{S}_3, m)$. For $0 \leq i \leq 2$, define $T_i^* : C(\mathcal{S}_3) \to L^2(\mathcal{S}_3, m)$ by

$$f \in C(\mathcal{S}_3) \mapsto (f \circ \Phi \circ s_i \circ \Phi^{-1})(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1}) \frac{1}{h \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}},$$

where $(\tau_i \times \mathrm{id}_\Omega) : [0,1) \times \Omega \to [0,1) \times \Omega$ by $(x, \omega) \mapsto (\tau_i(x), \omega)$. Let $f \in C(\mathcal{S}_3)$. Then

$$||T_i^*(f)||_{L^2(\mathcal{S}_3,m)} = \int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1}|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1}) \frac{1}{3} \frac{1}{W \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}} dm.$$

Now suppose that $(z_n)_{n=0}^{\infty} \in \mathcal{S}_3$ is such that $\Phi^{-1}((z_n)_{n=0}^{\infty}) \in [0,1) \times \sigma_i(\Omega)$. This implies that $(z_n)_{n=0}^{\infty} = \Phi(x, (i, \omega_2, ...))$. Then

$$W \circ S^{-1}((z_n)_{n=0}^{\infty}) = W(e^{-2\pi i \tau_i(x)}).$$

Meanwhile for this $(z_n)_{n=0}^{\infty}$, we also have that

$$W \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}((z_n)_{n=0}^{\infty}) = W \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega)(x, (i, \omega_2, ...)) =$$

$$W \circ \Phi(\tau_i(x), (i, \omega_2, ...)) = W(e^{-2\pi i \tau_i(x)}).$$

These calculations show that

$$(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{3}\frac{1}{W \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}} = (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{3W \circ S^{-1}} =$$

$$(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{dm}{d(m \circ S)} \circ S^{-1}$$

Hence

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1}|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{3}\frac{1}{W \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}} dm =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1}|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{dm}{d(m \circ S)} \circ S^{-1} dm =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1} \circ S \circ S^{-1}|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1} \circ S \circ S^{-1})\frac{dm}{d(m \circ S)} \circ S^{-1} dm =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1} \circ S|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1} \circ S)\frac{dm}{d(m \circ S)} d(m \circ S) =$$

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1} \circ S|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1} \circ S) dm.$$

Suppose that $(z_n)_{n=0}^{\infty} = \Phi(x, (\omega_1, \omega_2, ...))$. Note that

$$\Phi^{-1} \circ S((z_n)_{n=0}^{\infty}) = \Phi^{-1}(e^{-2\pi i \tau(x)}, e^{-2\pi i x}, e^{-2\pi i \tau_{\omega_1}(x)}, ...).$$

Now $\Phi^{-1} \circ S((z_n)_{n=0}^{\infty}) \in [0,1) \times \sigma_i(\Omega)$ means that $x = \tau_i \circ \tau(x)$, so that $\Phi^{-1} \circ S((z_n)_{n=0}^{\infty}) = (\tau(x), (i, \omega_1, \omega_2))$. Then if $\Phi^{-1} \circ S((z_n)_{n=0}^{\infty}) \in [0,1) \times \sigma_i(\Omega)$,

$$f \circ \Phi \circ s_i \circ \Phi^{-1} \circ S((z_n)_{n=0}^{\infty} = f \circ \Phi \circ s_i(\tau(x), (i, \omega_1, \omega_2, ...)) =$$

$$f \circ \Phi(\tau_i \circ \tau(x), \sigma(i, \omega_1, \omega_2, ...)) =$$

$$f \circ \Phi(x, (\omega_1, \omega_2, ...)) = f((z_n)_{n=0}^{\infty}).$$

Hence

$$\int_{\mathcal{S}_3} |f \circ \Phi \circ s_i \circ \Phi^{-1} \circ S|^2 (\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1} \circ S) dm =$$

$$\int_{S^{-1}(\Phi([0,1)\times\sigma_i(\Omega)))} |f|^2 dm \leq \int_{\mathcal{S}_3} |f|^2 dm = ||f||_{L^2(\mathcal{S}_3)}.$$

This calculation shows that T_i^* is bounded. It is also clear that T_i^* is linear. Since continuous functions are dense in $L^2(\mathcal{S}_3, m)$, T_i^* extends uniquely to a bounded linear operator on $L^2(\mathcal{S}_3, m)$.

We will now show that indeed T_i^* is the adjoint of T_i. Let $f, g \in C(\mathcal{S}_3)$. Then

$$\langle T_i f, g\rangle_{L^2(\mathcal{S}_3,m)} = \int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) h\overline{g} dm =$$

$$\int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) \overline{\left(\frac{1}{h} g\right)} h\overline{h} dm =$$

$$\int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) \overline{\left(\frac{1}{h} g\right)} 3W dm =$$

$$\int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) \overline{\left(\frac{1}{h} g\right)} \frac{d(m \circ S)}{dm} dm =$$

$$\int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1}) \overline{\left(\frac{1}{h} g\right)} d(m \circ S) =$$

$$\int_{\mathcal{S}_3} (f \circ \Phi \circ r_i \circ \Phi^{-1} \circ S^{-1}) \overline{(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}) \left(\frac{1}{h \circ S^{-1}}\right) g \circ S^{-1}} dm.$$

We note that the fourth equality is because the integrand in fourth integral is integrable with respect to the measure $m \circ S$. Using similar kinds of computations as above, one can show that

$$(f \circ \Phi \circ r_i \circ \Phi^{-1} \circ S^{-1}) \overline{(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}) \left(\frac{1}{h \circ S^{-1}}\right) g \circ S^{-1}} =$$

$$\overline{f(g \circ \Phi \circ s_i \circ \Phi^{-1})(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{h \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}}} = f\overline{T_i^* g}$$

Hence,

$$\langle T_i f, g\rangle_{L^2(\mathcal{S}_3,m)} = \langle f, T_i^* g\rangle_{L^2(\mathcal{S}_3,m)}$$

Since continuous functions are dense in $L^2(\mathcal{S}_3, m)$, this adjoint relation extends to all of $L^2(\mathcal{S}_3, m)$, and T_i^* is the adjoint operator of T_i.

Proposition 8.2.1. *[Davison] The operators* $\{T_i\}_{i=0}^2$ *satisfy the following:*

$$T_{i_1} \cdots T_{i_n} T_{i_n}^* \cdots T_{i_1}^* = M_{\mathbf{1}_{\tau_{i_1}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega}\circ\Phi^{-1}},$$

where $M_{\mathbf{1}_{\tau_{i_1}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega}\circ\Phi^{-1}}$ *is multiplication by* $\mathbf{1}_{\tau_{i_1}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ \Phi^{-1}$.

Proof. This will be proved by induction. Let $0 \le i \le 2$, and let $f \in L^2(\mathcal{S}_3, m)$. Then

$$T_i T_i^* f = T_i\left(f \circ \Phi \circ s_i \circ \Phi^{-1}(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{h \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}}\right) =$$

$$(f \circ \Phi \circ s_i \circ \Phi^{-1} \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1} \circ \Phi \circ r_i \circ \Phi^{-1})$$
$$\frac{1}{h \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1} \circ \Phi \circ r_i \circ \Phi^{-1}}\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} h.$$

Let $(z_n)_{n=0}^\infty \in \mathcal{S}_3$ be such that $(z_n)_{n=0}^\infty = \Phi(x, \omega)$. Then by the above calculation

$$T_i T_i^* f((z_n)_{n=0}^\infty) = (f \circ \Phi \circ s_i(\tau(x), \sigma_i(\omega)))(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ r_i(x, \omega))$$
$$\frac{1}{h \circ \Phi(\tau_i \circ \tau(x), \sigma_i(\omega))}(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x, \omega))h \circ \Phi(x, \omega) =$$
$$(f \circ \Phi \circ (\tau_i \circ \tau(x), \sigma \circ \sigma_i(\omega)))(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ (\tau(x), \sigma_i(\omega))\frac{1}{h \circ \Phi(\tau_i \circ \tau(x), \sigma_i(\omega))}$$
$$(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x, \omega))h \circ \Phi(x, \omega) =$$
$$(f \circ \Phi \circ (\tau_i \circ \tau(x), \omega))\frac{1}{h \circ \Phi(\tau_i \circ \tau(x), \sigma_i(\omega))}(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x, \omega))h \circ \Phi(x, \omega).$$

Now observe that if $x \in \tau_i([0, 1))$, then $\tau_i \circ \tau(x) = x$. Hence

$$(f \circ \Phi \circ (\tau_i \circ \tau(x), \omega))\frac{1}{h \circ \Phi(\tau_i \circ \tau(x), \sigma_i(\omega))}(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x, \omega))h \circ \Phi(x, \omega) =$$

$$(f \circ \Phi \circ (x,\omega))\frac{1}{h \circ \Phi(x,\sigma_i(\omega))}(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x,\omega))h \circ \Phi(x,\omega) =$$

$$(\mathbf{1}_{\tau_i([0,1))\times\Omega}(x,\omega))(f \circ \Phi \circ (x,\omega)) =$$

$$(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1} \circ \Phi(x,\omega))(f \circ \Phi \circ (x,\omega)) = M_{\mathbf{1}_{\tau_i([0,1))\times\Omega}\circ\Phi^{-1}}f,$$

where the third to last equality is because $h \circ \Phi$ only depends on the variable x. This completes the proof of the base case. Suppose that the proposition is true for $n-1$ with $n \geq 2$. We will show that

$$T_{i_1}\cdots T_{i_n}T_{i_n}^*\cdots T_{i_1}^* = M_{\mathbf{1}_{\tau_{i_1}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega}\circ\Phi^{-1}}.$$

Let $f \in L^2(\mathcal{S}_3, m)$. Then

$$T_{i_1}\cdots T_{i_n}T_{i_n}^*\cdots T_{i_1}^* f =$$

$$T_{i_1}T_{i_2}\cdots T_{i_n}T_{i_n}^*\cdots T_{i_2}^*\left(f \circ \Phi \circ s_{i_1} \circ \Phi^{-1}\mathbf{1}_{[0,1)\times\sigma_{i_1}(\Omega)} \circ \Phi^{-1}\frac{1}{h \circ \Phi \circ (\tau_{i_1} \times \mathrm{id}_\Omega) \circ \Phi^{-1}}\right) =$$

$$T_{i_1}\left(\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ \Phi^{-1} f \circ \Phi \circ s_{i_1} \circ \Phi^{-1}\mathbf{1}_{[0,1)\times\sigma_{i_1}(\Omega)} \circ \Phi^{-1}\frac{1}{h \circ \Phi \circ (\tau_{i_1} \times \mathrm{id}_\Omega) \circ \Phi^{-1}}\right) =$$

$$(\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ r_{i_1} \circ \Phi^{-1})T_{i_1}\left(f \circ \Phi \circ s_{i_1} \circ \Phi^{-1}\mathbf{1}_{[0,1)\times\sigma_{i_1}(\Omega)} \circ \Phi^{-1}\frac{1}{h \circ \Phi \circ (\tau_{i_1} \times \mathrm{id}_\Omega) \circ \Phi^{-1}}\right) =$$

$$(\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ r_{i_1} \circ \Phi^{-1})\mathbf{1}_{\tau_{i_1}([0,1))\times\Omega} \circ \Phi^{-1} f,$$

where the last equality is shown by using the base case. Suppose that $(z_n)_{n=0}^\infty \in \mathcal{S}_3$ is such that $(z_n)_{n=0}^\infty = \Phi(x,\omega)$. Then

$$\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ r_{i_1} \circ \Phi^{-1} \circ \Phi(x,\omega)\mathbf{1}_{\tau_{i_1}([0,1))\times\Omega} \circ \Phi^{-1} \circ \Phi(x,\omega) =$$

$$\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega}(\tau(x),\sigma_{i_1}(\omega))\mathbf{1}_{\tau_{i_1}([0,1))\times\Omega}(x,\omega) =$$

Now $x \in \tau_{i_1}([0,1))$ means that $x = \tau_{i_1}(\gamma)$ for some $\gamma \in [0,1)$. Note that $\tau(x) = \tau \circ \tau_{i_1}(\gamma) = \gamma$. Observe that $\tau(x) \in \tau_{i_2} \circ \ldots \circ \tau_{i_n}([0,1))$ means that $\gamma \in \tau_{i_2} \circ \ldots \circ \tau_{i_n}([0,1))$. Hence $x = \tau_{i_1}(\gamma) \in \tau_{i_1} \circ \ldots \circ \tau_{i_n}([0,1))$. This shows that

$$(\mathbf{1}_{\tau_{i_2}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ r_{i_1} \circ \Phi^{-1})\mathbf{1}_{\tau_{i_1}([0,1))\times\Omega} \circ \Phi^{-1} f = \mathbf{1}_{\tau_{i_1}\circ\ldots\circ\tau_{i_n}([0,1))\times\Omega} \circ \Phi^{-1} f,$$

which completes the proof of the proposition.

□

Proposition 8.2.2. *[Davison] The operators* $\{T_i\}_{i=0}^2$ *satisfy the following:*

$$T_{i_1}^* \cdots T_{i_n}^* T_{i_n} \cdots T_{i_1} = M_{\mathbf{1}_{[0,1)\times\sigma_{i_1}\circ\ldots\circ\sigma_{i_n}(\Omega)}\circ\Phi^{-1}},$$

where $M_{\mathbf{1}_{[0,1)\times\sigma_{i_1}\circ\ldots\circ\sigma_{i_n}(\Omega)}\circ\Phi^{-1}}$ *is multiplication by* $\mathbf{1}_{[0,1)\times\sigma_{i_1}\circ\ldots\circ\sigma_{i_n}(\Omega)}\circ\Phi^{-1}$.

Proof. This will be proved by induction. Let $0 \leq i \leq 2$, and let $f \in L^2(\mathcal{S}_3, m)$. Then

$$T_i^* T_i f = T_i^*((f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ \Phi^{-1})h) =$$

$$(f \circ \Phi \circ r_i \circ s_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega} \circ s_i \circ \Phi^{-1})(h \circ \Phi \circ s_i \circ \Phi^{-1})$$

$$(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1})\frac{1}{h \circ \Phi \circ (\tau_i \times \mathrm{id}_\Omega) \circ \Phi^{-1}}.$$

Suppose that $(z_n)_{n=0}^\infty \in \mathcal{S}_3$ is such that $(z_n)_{n=0}^\infty = \Phi(x, \omega)$. Then

$$T_i^* T_i f((z_n)_{n=0}^\infty) = (f \circ \Phi(\tau \circ \tau_i(x), \sigma_i \circ \sigma(\omega)))(\mathbf{1}_{\tau_i([0,1))\times\Omega}(\tau_i(x), \sigma(\omega)))$$

$$(h \circ \Phi(\tau_i(x), \sigma(\omega)))(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}(x, \omega))\frac{1}{h \circ \Phi(\tau_i(x), \omega)} =$$

$$f \circ \Phi(x, \sigma_i \circ \sigma(\omega))(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}(x, \omega)) = f \circ \Phi(x, \omega)(\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}(x, \omega)) =$$

$$\mathbf{1}_{[0,1)\times\sigma_i(\Omega)} \circ \Phi^{-1}((z_n)_{n=0}^\infty) f((z_n)_{n=0}^\infty).$$

This completes the proof of the base case. Suppose that the proposition is true for $n-1$ with $n \geq 2$. We will show that

$$T_{i_1}^* \cdots T_{i_n}^* T_{i_n} \cdots T_{i_1} = M_{\mathbf{1}_{[0,1)\times\sigma_{i_1}\circ\ldots\circ\sigma_{i_n}(\Omega)}\circ\Phi^{-1}}.$$

Let $f \in L^2(\mathcal{S}_3, m)$. Then

$$T_{i_1}^* \cdots T_{i_n}^* T_{i_n} \cdots T_{i_1} f = T_{i_1}^*(M_{\mathbf{1}_{[0,1)\times\sigma_{i_2}\circ\ldots\circ\sigma_{i_n}(\Omega)}\circ\Phi^{-1}})T_{i_1} f =$$

$$T_{i_1}^*\left[(\mathbf{1}_{[0,1)\times\sigma_{i_2}\circ\ldots\circ\sigma_{i_n}(\Omega)} \circ \Phi^{-1})T_{i_1} f\right] =$$

$$(\mathbf{1}_{[0,1)\times\sigma_{i_2}\circ\ldots\circ\sigma_{i_n}(\Omega)} \circ s_i \circ \Phi^{-1})(\mathbf{1}_{[0,1)\times\sigma_{i_1}(\Omega)} \circ \Phi^{-1})f =$$

$$(\mathbf{1}_{[0,1)\times\sigma_{i_1}\circ\ldots\circ\sigma_{i_n}(\Omega)} \circ \Phi^{-1})f,$$

and this completes the proof of the proposition.

□

We note that by Propositions 8.2.1 and 8.2.2, for $0 \leq i \leq 2$, $T_iT_i^* = Q_i = M_{\mathbf{1}_{\tau_i([0,1))\times\Omega}\circ\Phi^{-1}(\cdot)}$, and $T_i^*T_i = P_i = M_{\mathbf{1}_{[0,1)\times\sigma_i(\Omega)}\circ\Phi^{-1}(\cdot)}$. This leads to the following result.

Proposition 8.2.3. *[Davison] The operators $\{T_i\}_{i=0}^2$ are partial isometries which satisfy the following relations:*

(1) $\sum_{i=0}^2 T_iT_i^* = \sum_{i=0}^2 T_i^*T_i = id_{L^2(\mathcal{S}_3,m)}$.

(2) $T_i^*T_j = T_iT_j^* = 0$ *if* $i \neq j$.

Proof. The fact that T_i are partial isometries follows from the fact that $T_iT_i^*$ and $T_i^*T_i$ are projections. Item (1) is true because $\sum_{i=0}^2 Q_i = \sum_{i=0}^2 P_i = \mathrm{id}_{L^2(\mathcal{S}_3,m)}$. If $i \neq j$, the fact that $T_i^*T_j = 0$ follows from

$$\mathbf{1}_{[0,1)\times\sigma_j(\Omega)} \circ r_i \circ \Phi^{-1} = 0,$$

and the fact that $T_iT_j^* = 0$ follows from

$$\mathbf{1}_{\tau_j([0,1))\times\Omega} \circ s_i \circ \Phi^{-1} = 0.$$

□

Theorem 8.2.4. *[Davison] The unitary $U : L^2(\mathcal{S}_3, m) \to L^2(\mathcal{S}_3, m)$ by*

$$Uf((z_n)_{n=0}^\infty) = h(z_0)f \circ S((z_n)_{n=0}^\infty),$$

satisfies $U = \sum_{i=0}^2 T_i$.

Proof. Let $f \in L^2(\mathcal{S}_3, m)$. Then,

$$\sum_{i=0}^2 T_if = \left(\sum_{i=0}^2 (f \circ \Phi \circ r_i \circ \Phi^{-1})(\mathbf{1}_{\tau_i([0,1))\times\Omega)} \circ \Phi^{-1})\right) h.$$

Choose some $(z_n)_{n=0}^\infty \in L^2(\mathcal{S}_3, m)$. Suppose without loss of generality $(z_n)_{n=0}^\infty = \Phi(x, \omega)$ with $x \in \tau_j([0,1))$. Then

$$\sum_{i=0}^2 T_if((z_n)_{n=0}^\infty) = (f \circ \Phi \circ r_j(x,\omega))(\mathbf{1}_{\tau_j([0,1))\times\Omega)}(x,\omega))h(z_0) =$$

$$(f \circ \Phi(\tau(x), \sigma_j(\omega)))(\mathbf{1}_{\tau_j([0,1))\times\Omega)}(x,\omega))h(z_0) =$$

$$(f \circ S((z_n)_{n=0}^{\infty}))h(z_0) = Uf.$$

□

Consider now the measurable bijection $\psi : [0,1) \to \mathbb{T}$ given by $x \mapsto e^{-2\pi ix}$. For $0 \leq i \leq 2$, define $\rho_i : \mathbb{T} \to \mathbb{T}$ by $\rho_i = \psi \circ \tau_i \circ \psi^{-1}$, and define $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho = \psi \circ \tau \circ \psi^{-1}$. Note that $\rho(z) = z^3$ and $\rho \circ \rho_i = \mathrm{id}_{\mathbb{T}}$ for $0 \leq i \leq 2$.

For $k \in \mathbb{Z}_+$ and $a \in \Gamma_3^k$, let $A_k(a) = \psi \circ \tau_{a_1} \circ ... \circ \tau_{a_k}([0,1))$. Observe that $A_k(a) = \rho_{a_1} \circ ... \circ \rho_{a_k}(\mathbb{T})$. For $a \in \Gamma_3^k$, define the projection $P_k(a) = T_{a_1} \cdots T_{a_k} T_{a_k}^* \cdots T_{a_1}^* = M_{\mathbf{1}_{\tau_{a_1}\circ ... \circ\tau_{a_k}([0,1))\times\Omega}\circ\Phi^{-1}}$ on $L^2(\mathcal{S}_3, m)$. We remark that

- $\mathbb{T} = \cup_{a\in\Gamma_3^k} A_k(a)$ for all $k \in \mathbb{Z}_+$, and the union is disjoint.
- Given $\epsilon > 0$, there exists a $K \in \mathbb{Z}_+$ such that for $k \geq K$, $\mathrm{diam}(A_k(a)) \leq \epsilon$ for all $a \in \Gamma_3^k$.
- $\sum_{a\in\Gamma_3^k} P_k(a) = \mathrm{id}_{L^2(\mathcal{S}_3,m)}$ for all $k \in \mathbb{Z}_+$.
- $P_k(a)P_k(b) = 0$ if $a \neq b$ and $a, b \in \Gamma_3^k$.
- $P_{k+1}(a)$ is contained in some $P_k(b)$ (i.e. $P_k(b)P_{k+1}(a) = P_{k+1}(a)$).

The first two items of the above list imply that $\{A_k(a)\}_{a\in\Gamma_3^k}$ comprise a 3-adic system of partitions of $\mathbb{T}$ (see Definition 3.1 in P. Jorgensen's paper [17]). The last three items of the above list imply that $\{P_k(a)\}_{a\in\Gamma_3^k}$ comprise a 3-adic system of projections on $L^2(\mathcal{S}_3, m)$ (see Definition 3.2 in P. Jorgensen's paper [17]). This allows us to invoke the result of Jorgensen mentioned in Remark 6.3.1 to conclude that there exists a unique projection valued measure G from the Borel subsets of $\mathbb{T}$ to the projections on $L^2(\mathcal{S}_3, m)$ that satisfies

- $G(A_k(a)) = T_{a_1} \cdots T_{a_k} T_{a_k}^* \cdots T_{a_1}^*$, and

- $G(\cdot) = \sum_{i=0}^{2} T_i G(\rho_i^{-1}(\cdot))T_i^*$

Let us now note the following. If $f \in L^2(\mathcal{S}_3, m)$ and $a \in \Gamma_3^k$,

$$T_{a_1} \cdots T_{a_k} T_{a_k}^* \cdots T_{a_1}^* f((z_n)_{n=0}^{\infty}) = M_{\mathbf{1}_{\tau_{a_1} \circ \ldots \circ \tau_{a_k}([0,1)) \times \Omega} \circ \Phi^{-1}(\cdot)} f((z_n)_{n=0}^{\infty}) =$$

$$\mathbf{1}_{\tau_{a_1} \circ \ldots \circ \tau_{a_k}([0,1)) \times \Omega} \circ \Phi^{-1}((z_n)_{n=0}^{\infty}) f((z_n)_{n=0}^{\infty}) = \mathbf{1}_{\psi \circ \tau_{a_1} \circ \ldots \circ \tau_{a_k}([0,1))}(z_0) f((z_n)_{n=0}^{\infty}).$$

$$\mathbf{1}_{A_k(a)}(z_0) f((z_n)_{n=0}^{\infty}).$$

This shows that G is the unique projection valued measure which satisfies

$$G(A_k(a)) f((z_n)_{n=0}^{\infty}) = \mathbf{1}_{A_k(a)}(z_0) f((z_n)_{n=0}^{\infty}) \tag{8.10}$$

for all $f \in L^2(\mathcal{S}_3, m)$. Let us recall that the previously defined projection valued measure F (see equation (8.8)) satisfies the relation of equation (8.10). This proves the following proposition.

Proposition 8.2.5. *[Davison] The projection valued measure F equals the projection valued measure G.*

In summary, what we have shown is that the partial isometries $\{T_i\}_{i=0}^{N-1}$ are the building blocks for constructing the unitary operators U and S on $L^2(\mathcal{S}_3, m)$ which satisfy the Baumslag-Solitar relation $USU^{-1} = S^3$. That is,

- $U = \sum_{i=0}^{N-1} T_i$, and
- $S = \int_{\mathbb{T}} z dF$ where F is the unique projection valued measure satisfying

$$F(\cdot) = \sum_{i=0}^{2} T_i F(\rho_i^{-1}(\cdot))T_i^*.$$

Remark 8.2.6. *The author of this thesis was fortunate to be able to recently discuss this chapter with Palle Jorgensen. He encouraged the author to relate the material in this chapter to research done by M. Marcolli and A.M. Paolucci in [22]. This is something that the author looks forward to doing in future research.*

Chapter 9

Fourier Transform Calculations for Measures on the Solenoid

In this chapter, we will present formulas for the Fourier transforms of several measures on the N-solenoid (for $N \in \mathbb{N}$ with $N \geq 2$). These measures will be derived from generating filters in $\mathscr{B}\mathscr{B}(\mathbb{T})$, the bounded Borel functions on $\mathbb{T}$. The specific filters that we will consider are the Haar filter for dilation by $N = 2$, the (more general) Haar filter for dilation by $N \in \mathbb{N}$ with $N \geq 2$, and the Cantor generating filter discussed in Chapter 8.

9.1 Preliminaries

We first note that the solenoid can be generalized to any natural number $N \in \mathbb{N}$ for $N \geq 2$. Indeed, an element in the the N-solenoid is an infinite tuple $(z_n)_{n=0}^{\infty} \in \mathbb{T}\times\mathbb{T}\times...$ such that $z_{n+1}^N = z_n$ for all $n = 0, 1, ...$ Moreover, an element of the dual of the N-solenoid is of the form $\frac{l}{N^p}$ for $l \in \mathbb{Z}$ and $p = 0, 1, 2, ...$ If $W \in \mathscr{B}\mathscr{B}(\mathbb{T})$ satisfies

$$\sum_{w^N = z} W(w) = 1$$

for all $z \in \mathbb{T}$, one can generalize the techniques discussed in Chapter 8 (to arbitrary $N \in \mathbb{N}$) to show that W induces a Borel probability measure m on the N-solenoid, given by

$$m(f) := \int_{[0,1)} \int_{\Omega} f \circ \Phi(x, \omega) dP_x(\omega) dx \tag{9.1}$$

for all $f \in C(\mathcal{S}_N)$, where Ω, P_x, and Φ are defined according to Chapter 8. This measure satisfies several important integral formulas.

Lemma 9.1.1. *[11] [Dutkay] Let $f \in C(\mathcal{S}_N)$ be such that it depends only on the first coordinate, z_0, of $\mathcal{S}_N$. Then*

$$\int_{\mathcal{S}_N} f dm = \int_{\mathbb{T}} f dz.$$

where we think of f as a function on $\mathbb{T}$ by $f((z_n)_{n=0}^{\infty}) = f(z_0)$.

The next formula is stated in Theorem 8.1.9 for $N = 3$. We restate it in general below.

Theorem 9.1.2. *[11][Dutkay] Let $f \in C(\mathcal{S}_N)$. For $n = 1, ..., \infty$*

$$\int_{\mathcal{S}_N} f \circ S^{-n} dm = \int_{\mathcal{S}_N} N^n B^{(n)}(\cdot) f dm \tag{9.2}$$

where $B^{(n)}((z_l)_{l=0}^{\infty}) : \mathcal{S}_N \to \mathbb{C}$ is defined by

$$(z_l)_{l=0}^{\infty} \mapsto W(z_0)...W(z_0^{N^{n-1}}).$$

9.2 Haar Filter for $N = 2$

Consider the Hilbert space $L^2(\mathbb{R}, \mu)$, where μ is Lebesgue measure. Define dilation and translation operators on $L^2(\mathbb{R}, \mu)$ by

$$Df(x) = \sqrt{2} f(2x),$$

$$Tf(x) = f(x-1),$$

for all $f \in L^2(\mathbb{R}, \mu)$. These operators are unitary operators and they satisfy the relation $D^{-1}TD = T^2$, which is the defining relation for $BS(1,2)$. Moreover, if we let $\phi = \mathbf{1}_{[0,1)}$, one can calculate that

$$\phi = \frac{1}{\sqrt{2}} D(\phi) + \frac{1}{\sqrt{2}} DT(\phi).$$

As before, writing the unitary operator T as $T = \int_{\mathbb{T}} z dE$, for some projection valued measure E, we get that

$$D^{-1}(\phi) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} T \right)(\phi) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \int z dE \right)(\phi) = \left(\int \left(\frac{1+z}{\sqrt{2}} \right) dE \right) \phi.$$

We call $g(z) = \frac{1+z}{\sqrt{2}}$ the Haar generating filter. This is because the dilation and translation operators defined above give rise to a multi-resolution analysis on $L^2(\mathbb{R}, \mu)$. In particular, one can recover the Haar wavelet from this multi-resolution analysis. The Haar wavelet function, ψ, is given by $\psi(t) = 1$ for $0 \leq t < \frac{1}{2}$, $\psi(t) = -1$ for $\frac{1}{2} \leq t < 1$, and $\psi(t) = 0$ otherwise. The doubly indexed family of functions

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

for $j, k \in \mathbb{Z}$ form an orthonormal basis of $L^2(\mathbb{R}, \mu)$. In other words, ψ is a wavelet in $L^2(\mathbb{R}, \mu)$ for dilation by 2. Let us define $W(z) = \frac{|g(z)|^2}{2}$. It can be shown that

$$\sum_{w^2=z} W(z) = 1. \tag{9.3}$$

Therefore W induces a measure m on $\mathcal{S}_2$. We now characterize the Fourier transform of the measure m. In particular, the Fourier transform of m on $\mathcal{S}_2$, denoted $\hat{m}$, is defined by

$$\hat{m}(\lambda) := \int_{\mathcal{S}_2} \chi_\lambda dm,$$

where $\lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$, and χ_λ is the associated character on $\mathcal{S}_2$ given by $\chi_\lambda((z_n)_{n=0}^\infty) = z_p^l$ (assuming $\lambda = \frac{l}{2^p}$)

Lemma 9.2.1. *[11] [Dutkay] Let $\lambda = \frac{l}{2^p}$ for $l \in \mathbb{Z}$ and $p = 1, 2, ...,$. Then*

$$\chi_l \circ S^{-p} = \chi_\lambda$$

Proof. Let $(z_n)_{n=0}^\infty \in \mathcal{S}_2$. Then

$$\chi_l \circ S^{-p}((z_n)_{n=0}^\infty) = z_p^l = \chi_\lambda((z_n)_{n=0}^\infty).$$

□

Proposition 9.2.2. *[11] [Dutkay] For $\lambda = l \in \mathbb{Z}\left[\frac{1}{2}\right]$*

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^l dz.$$

Proof. Since χ_l only depends on the first coordinate, we can apply Lemma 9.1.1 to conclude that

$$\hat{m}(\lambda) = \int_{\mathcal{S}_2} \chi_\lambda dm = \int_{\mathbb{T}} z^l dz.$$

□

Proposition 9.2.3. *[11] [Dutkay] For* $\lambda = \frac{l}{2^p} \in \mathbb{Z}\left[\frac{1}{2}\right]$, *where* $p = 1, 2, ...$

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^l \cdot 2^p B^{(p)}(z) dz.$$

Proof. The proof of this theorem relies on Theorem 9.1.2 and Lemma 9.2.1. That is

$$\hat{m}(\lambda) = \int_{\mathcal{S}_2} \chi_\lambda dm = \int_{\mathcal{S}_2} \chi_l \circ S^{-p} dm =$$

$$\int_{\mathcal{S}_2} 2^p B^{(p)}(\cdot) \chi_l dm.$$

Since $B^{(p)}$ and χ_l only depend on the first coordinate of $\mathcal{S}_2$, we can apply Lemma 9.1.1 to conclude that

$$\int_{\mathcal{S}_2} 2^p B^{(p)} \chi_l dm = \int_{\mathbb{T}} 2^p B^{(p)}(z) z^l dz =$$

$$\int_{\mathbb{T}} z^l 2^p B^{(p)}(z) dz.$$

□

We will investigate the structure of the Laurent polynomial $2^p B^{(p)}(z)$ for $p = 1, 2, ...$ In particular, recall that $g(z) = \frac{1}{\sqrt{2}}(1 + z)$. Hence

$$W(z) = \frac{|g(z)|^2}{2} = \frac{(z + 2 + z^{-1})}{2^2},$$

where we are using the fact that $\overline{z} = z^{-1}$ for $z \in \mathbb{T}$. Therefore

$$2^p B^{(p)}(z) = 2^p W(z) ... W(z^{2^{p-1}}) =$$

$$2\left(\frac{(z + 2 + z^{-1})}{2^2}\right) \cdot ... \cdot 2\left(\frac{(z^{2^{p-1}} + 2 + z^{-2^{p-1}}}{2^2}\right) =$$

$$\left(\frac{(z + 2 + z^{-1})}{2}\right) \cdot ... \cdot \left(\frac{(z^{2^{p-1}} + 2 + z^{-2^{p-1}}}{2}\right) =$$

$$\left(\frac{z}{2}+1+\frac{z^{-1}}{2}\right)\cdot\ldots\cdot\left(\frac{z^{2^{p-1}}}{2}+1+\frac{z^{-2^{p-1}}}{2}\right). \tag{9.4}$$

We will now determine a general formula for expression (9.4).

Proposition 9.2.4. *[Davison] Let $p=1,2,\ldots$. Then*

$$\left(\frac{z}{2}+1+\frac{z^{-1}}{2}\right)\cdot\ldots\cdot\left(\frac{z^{2^{p-1}}}{2}+1+\frac{z^{-2^{p-1}}}{2}\right)=$$

$$\frac{1}{2^p}z^{2^{p-1}+\ldots+2+1}+\frac{2}{2^p}z^{(2^{p-1}+\ldots+2+1)-1}+\ldots+\frac{2^p-1}{2^p}z+1+$$
$$\frac{2^p-1}{2^p}z^{-1}+\ldots+\frac{2}{2^p}z^{-((2^{p-1}+\ldots+2+1)-1)}+\frac{1}{2^p}z^{-(2^{p-1}+\ldots+2+1)}.$$

Proof. We will prove this by induction on p. If $p=1$

$$\left(\frac{z}{2}+1+\frac{z^{-1}}{2}\right)$$

is of the claimed form, and hence the base case is proved. Suppose for $p\geq 1$, the above Laurent polynomial is of the claimed form. We will show the Laurent polynomial is of the claimed form for $p+1$. That is, we will show

$$\left(\frac{z}{2}+1+\frac{z^{-1}}{2}\right)\cdot\ldots\cdot\left(\frac{z^{2^{p}}}{2}+1+\frac{z^{-2^{p}}}{2}\right)=$$

$$\frac{1}{2^{p+1}}z^{2^{p}+\ldots+2+1}+\frac{2}{2^{p+1}}z^{(2^{p}+\ldots+2+1)-1}+\ldots+\frac{2^{p+1}-1}{2^{p+1}}z+1+$$
$$\frac{2^{p+1}-1}{2^{p+1}}z^{-1}+\ldots+\frac{2}{2^{p+1}}z^{-((2^{p}+\ldots+2+1)-1)}+\frac{1}{2^{p+1}}z^{-(2^{p}+\ldots+2+1)}.$$

By the induction assumption, we know that

$$\left(\frac{z}{2}+1+\frac{z^{-1}}{2}\right)\cdot\ldots\cdot\left(\frac{z^{2^{p}}}{2}+1+\frac{z^{-2^{p}}}{2}\right)=$$

$$\left(\frac{z^{2^{p}}}{2}+1+\frac{z^{-2^{p}}}{2}\right)$$
$$\left(\frac{1}{2^p}z^{2^{p-1}+\ldots+2+1}+\ldots+\frac{2^p-1}{2^p}z+1+\frac{2^p-1}{2^p}z^{-1}+\ldots+\frac{1}{2^p}z^{-(2^{p-1}+\ldots+2+1)}\right)=$$

$$\frac{1}{2^{p+1}}z^{2^{p}+\ldots+2+1}+\ldots+\frac{2^p}{2^{p+1}}z^{2^p}+\frac{2^p-1}{2^{p+1}}z^{2^p-1}+\ldots+\frac{1}{2^{p+1}}z^{2^p-(2^{p-1}+\ldots+2+1)}+$$

$$\frac{1}{2^p}z^{2^{p-1}+...+2+1}+...+\frac{2^p-1}{2^p}z+1+\frac{2^p-1}{2^p}z^{-1}+...+\frac{1}{2^p}z^{-(2^{p-1}+...+2+1)}+$$

$$\frac{1}{2^{p+1}}z^{-(2^p-(2^{p-1}+...+2+1))}+...+\frac{2^p-1}{2^{p+1}}z^{-(2^p-1)}+\frac{2^p}{2^{p+1}}z^{-2^p}+...+\frac{1}{2^{p+1}}z^{-(2^p+...+2+1)}=$$

(we will now use that $2^p-1=2^{p-1}+...+2+1$, for $p=1,2,..$)

$$\frac{1}{2^{p+1}}z^{2^p+...+2+1}+...+\frac{2^p}{2^{p+1}}z^{2^p}+\frac{2^p-1}{2^{p+1}}z^{2^p-1}+...+\frac{1}{2^{p+1}}z+$$

$$\frac{1}{2^p}z^{2^p-1}+...+\frac{2^p-1}{2^p}z+1+\frac{2^p-1}{2^p}z^{-1}+...+\frac{1}{2^p}z^{-(2^p-1)}+$$

$$\frac{1}{2^{p+1}}z^{-1}+...+\frac{2^p-1}{2^{p+1}}z^{-(2^p-1)}+\frac{2^p}{2^{p+1}}z^{-(2^p)}+...+\frac{1}{2^{p+1}}z^{-(2^p+...+2+1)}=$$

(by combining like terms we get)

$$\frac{1}{2^{p+1}}z^{2^p+...+2+1}+...+\frac{2^p}{2^{p+1}}z^{2^p}+\frac{2^p+1}{2^{p+1}}z^{2^p-1}+...+\frac{2^{p+1}-1}{2^{p+1}}z+1+$$

$$\frac{2^{p+1}-1}{2^{p+1}}z^{-1}+...+\frac{2^p+1}{2^{p+1}}z^{-(2^p-1)}+\frac{2^p}{2^{p+1}}z^{-(2^p)}+...+\frac{1}{2^{p+1}}z^{-(2^p+...+2+1)}=$$

$$\frac{1}{2^{p+1}}z^{2^p+...+2+1}+...+\frac{2^{p+1}-1}{2^{p+1}}z+1+\frac{2^{p+1}-1}{2^{p+1}}z^{-1}+...+\frac{1}{2^{p+1}}z^{-(2^p+...+2+1)},$$

which is the desired form.

□

Remark 9.2.5. *We note that for $p=1,2,...$, the Laurent polynomial*

$$\frac{1}{2^p}z^{2^{p-1}+...+2+1}+\frac{2}{2^p}z^{(2^{p-1}+...+2+1)-1}+...+\frac{2^p-1}{2^p}z+1+$$

$$\frac{2^p-1}{2^p}z^{-1}+...+\frac{2}{2^p}z^{-((2^{p-1}+...+2+1)-1)}+\frac{1}{2^p}z^{-(2^{p-1}+...+2+1)}$$

has the property that for $1 \leq k \leq 2^{p-1}+...+2+1$, the coefficient of z^k is 2^p-k. Moreover, the coefficient of z^k equals the coefficient of z^{-k}.

We note that the following theorem was first stated in 2008 by J. Packer in [24].

Theorem 9.2.6. *[24][Packer]*

(1) Suppose that $\lambda = l \in \mathbb{Z}\left[\frac{1}{2}\right]$ where $l \in \mathbb{Z}$. Then

$$\hat{m}(\lambda) = \begin{cases} 0 & \text{if } l \neq 0 \\ 1 & \text{if } l = 0 \end{cases}$$

(2) Suppose that $\lambda = \frac{l}{2^p} \in \mathbb{Z}\left[\frac{1}{2}\right]$ where $l \in \mathbb{Z}$ and $p = 1, 2, ...$ Then

$$\hat{m}(\lambda) = \begin{cases} 1-|\lambda| & \text{if } |l| \leq 2^{p-1}+...+2+1 \\ 0 & \text{if } |l| > 2^{p-1}+...+2+1 \end{cases}$$

Proof. We prove (1) first: Recall that for $\lambda = l \in \mathbb{Z}\left[\frac{1}{2}\right]$ where $l \in \mathbb{Z}$, we have by Proposition 9.2.2

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^l dz.$$

This integral is zero for $l \neq 0$ and 1 for $l = 0$, which proves the first part of the theorem.

We now prove (2). Recall that for $\lambda = \frac{l}{2^p} \in \mathbb{Z}\left[\frac{1}{2}\right]$ where $l \in \mathbb{Z}$ and $p = 1, 2, ...$, we have by Proposition 9.2.3, and Proposition 9.2.4,

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^l \cdot 2^p B^{(p)}(z) dz =$$

$$\frac{1}{2^p}\int_{\mathbb{T}} z^l z^{2^{p-1}+...+2+1} dz + ... + \frac{2^p-1}{2^p}\int_{\mathbb{T}} z^l z dz + \int_{\mathbb{T}} z^l dz +$$
$$\frac{2^p-1}{2^p}\int_{\mathbb{T}} z^l z^{-1} dz + ... + \frac{1}{2^p}\int_{\mathbb{T}} z^l z^{-(2^{p-1}+...+2+1)} dz$$

If $l = 0$, we refer to (1), in which case $\hat{m}(\lambda) = 1 = 1 - |0|$. Suppose now that $1 \leq l \leq 2^{p-1} + ... + 2 + 1$. Then the above sum of integrals will reduce to one integral, namely

$$\frac{2^p - l}{2^p} \int_{\mathbb{T}} z^l z^{-l} dz = \frac{2^p - l}{2^p} = 1 - |\lambda|.$$

This is because all other terms will involve an integral of z^n for some non-zero integer n, which will integrate to zero.

Suppose that $-1 \geq l \geq -(2^{p-1} + ... + 2 + 1)$. Similar to above, the above sum of integrals will reduce to one integral, namely

$$\frac{2^p - |l|}{2^p} \int_{\mathbb{T}} z^l z^{-l} dz = \frac{2^p - |l|}{2^p} = 1 - |\lambda|.$$

Suppose that $|l| > 2^{p-1} + ... + 2 + 1$. Then every integral will be zero, and hence, $\hat{m}(\lambda) = 0$. This completes the proof of the theorem.

□

The above theorem can be condensed into the following statement.

Theorem 9.2.7. *[24][Packer] Suppose that* $\lambda = \frac{l}{2^p} \in \mathbb{Z}\left[\frac{1}{2}\right]$ *where* $l \in \mathbb{Z}$ *and* $p = 0, 1, ...$ *Then*

$$\hat{m}(\lambda) = \begin{cases} 1 - |\lambda|, & \text{if } |\lambda| < 1 \\ 0, & \text{if } |\lambda| \geq 1 \end{cases}.$$

9.3 Haar Filter for Arbitrary N

Consider the Hilbert space $L^2(\mathbb{R}, \mu)$, where μ is Lebesgue measure, and choose $N \in \mathbb{N}$ with $N \geq 2$. Define the unitary dilation and translation operators on $L^2(\mathbb{R}, \mu)$ by

$$Df(x) = \sqrt{N} f(Nx),$$

$$Tf(x) = f(x - 1),$$

for all $f \in L^2(\mathbb{R}, \mu)$. Using the same procedure as above, we obtain the Haar generating filter $g(z) = \frac{1+z+...+z^{N-1}}{\sqrt{N}}$. This generating filter induces a measure m_N on the N-solenoid, $\mathcal{S}_N$, which satisfies an analogous result as Theorem 9.2.7. This result, stated below, does require a slightly generalized proof, which we will not include.

Theorem 9.3.1. *Suppose that* $\lambda = \frac{l}{N^p} \in \mathbb{Z}\left[\frac{1}{N}\right]$ *where* $l \in \mathbb{Z}$ *and* $p = 0, 1, \ldots.$ *Then*

$$\hat{m}_N(\lambda) = \begin{cases} 1 - |\lambda|, & \text{if } |\lambda| < 1 \\ 0, & \text{if } |\lambda| \geq 1 \end{cases}.$$

9.4 Cantor Generating Filter

Recall that $h(z) = \frac{1+z^2}{\sqrt{2}}$ is the generating filter associated to the Cantor set. This filter induces a measure, m, on $\mathcal{S}_3$, as described above. In this section, we will determine a formula for the Fourier transform of this measure with respect to the characters on $\mathcal{S}_3$. We note that this formula was first presented by D. Dutkay in [11]; in his proof he relied on Proposition 8.1.11. In our proof we will use an induction argument to find a general formula for the Laurent polynomial $3^p B^{(p)}(z)$, for $p = 1, 2, \ldots.$

Proposition 9.4.1. *[Davison] For* $p = 1, 2, 3, \ldots$*, the Laurent polynomial* $3^p B^{(p)}(z)$ *is of the form*

$$\sum_{j=-M(p)}^{M(p)} c_j z^j$$

where $M(p) := 2 + 2 \cdot 3 + \ldots + 2 \cdot 3^{p-1}$ *and*

$$c_j = \begin{cases} \frac{1}{2^a} & \text{if } j \text{ even} \\ 0 & \text{if } j \text{ odd} \end{cases}$$

for some $a = 0, 1, 2, \ldots$ *We note that this sum has* 3^p *non-zero terms.*

Proof. This will be proven by induction on p. If $p = 1$

$$3B^{(1)}(z) = |h(z)|^2 = \frac{z^2}{2} + 1 + \frac{z^{-2}}{2}.$$

By observation, we see that $3B^{(1)}(z)$ is of the above form, and moreover there are $3^1 = 3$ non-zero terms. Suppose that $3^{p-1}B^{(p-1)}(z)$ is of the above form for $p \geq 2$. We need to show that $3^p B^{(p)}(z)$ is of the above form. Observe that

$$3^p B^{(p)}(z) = \left(1 + \frac{(z^{3^{p-1}})^2}{2} + \frac{(z^{3^{p-1}})^{-2}}{2}\right) 3^{p-1} B^{(p-1)}(z) =$$

$$\left(1 + \frac{(z^{3^{p-1}})^2}{2} + \frac{(z^{3^{p-1}})^{-2}}{2}\right) \left(\sum_{-M(p-1)}^{M(p-1)} c_j z^j\right) =$$

$$\left(1 + \frac{z^{2\cdot 3^{p-1}}}{2} + \frac{z^{-2\cdot 3^{p-1}}}{2}\right) \left(\sum_{j=-M(p-1)}^{M(p-1)} c_j z^j\right)$$

where $M(p-1) = 2 + 2 \cdot 3 + ... + 2 \cdot 3^{p-2}$. At this point, we would like to note the useful facts:

- $-M(p-1) + 2 \cdot 3^{p-1} = M(p-1) + 2$.
- $M(p-1) - 2 \cdot 3^{p-1} = -M(p-1) - 2$.

This allows us to observe that

$$\left(1 + \frac{z^{2\cdot 3^{p-1}}}{2} + \frac{z^{-2\cdot 3^{p-1}}}{2}\right) \left(\sum_{j=-M(p-1)}^{M(p-1)} c_j z^j\right) =$$

$$\sum_{j=-M(p)}^{M(p)} c_j z^j$$

where

- $c_j = 0$ for j odd,
- $c_j = \frac{1}{2} c_{j-2\cdot 3^{p-1}}$ for $M(p-1) < j \leq M(p)$ and j even,
- $c_j = \frac{1}{2} c_{j+2\cdot 3^{p-1}}$ for $-M(p-1) > j \geq -M(p)$ and j even.

Using the induction assumption, we can conclude that $3^p B^{(p)}(z)$ is of the above form, and moreover that there are 3^p non-zero terms.

$\square$

We will now derive a general formula for the coefficients c_j of $3^p B^{(p)})(z)$. We first note that $c_j = c_{-j}$ for all j, which means that we only need to determine the coefficients of c_j for $j \geq 0$. To this end, choose an index j such that $0 \leq j \leq 2 + 2 \cdot 3 + ... + 2 \cdot 3^{p-1}$. If j is odd, the $c_j = 0$. Suppose that j is even. It is possible to express j uniquely as

$$j = d_0 + d_1(3) + ... + d_k(3^k),$$

where $d_i \in \{-2, 0, 2\}$ for all $0 \leq i \leq k$, and $k \leq p-1$. We claim that

$$c_j = \frac{1}{2^{\frac{|d_0|+...+|d_k|}{2}}}.$$

We will prove this by induction on k. If $k = 0$, then $j = d_0$. One can verify that $c_j = \frac{1}{2^{\frac{|d_0|}{2}}}$ for $d_0 \in \{-2, 0, 2\}$. Choose $k \geq 1$, and suppose that the formula is true for all coefficients whose decomposition is of length $k-1$. We will show that it is true for k. That is, suppose $j = d_0 + d_1(3) + ... + d_k(3^k)$, where $d_k = 2$ or $d_k = -2$. Note that we can assume that $d_k \neq 0$, because in this case the decomposition would be of length $k-1$ and we can use the induction assumption.

We will first assume that $d_k = 2$. In this case, $M(k) < j \leq M(k+1)$ which implies that

$$c_j = \frac{1}{2} c_{j-2\cdot 3^k} = \frac{1}{2}\left(\frac{1}{2^{\frac{|d_0|+...|d_{k-1}|}{2}}}\right) = \frac{1}{2^{\frac{|d_0|+...|d_k|}{2}}},$$

where the last equality is by the induction assumption. A similar argument is used to show the formula in the case that $d_k = -2$.

The next result is first due to D. Dutkay in [11]. We will provide another proof of this result that will rely on the previous proposition.

Proposition 9.4.2. *[11][Dutkay] Suppose that λ is a character on $\mathcal{S}_3$, and let m be the measure on $\mathcal{S}_3$ induced from the Cantor generating filter. Then*

$$\hat{m}(\lambda) = \begin{cases} \frac{1}{2^{\frac{|d_1|+...+|d_p|}{2}}} & \text{if } \lambda = \sum_{k=1}^{p} \frac{d_k}{3^k} \\ 0 & \text{else} \end{cases}$$

where $d_k \in \{-2, 0, 2\}$ for all $k = 1, ..., p$.

Proof. If $\lambda = \sum_{k=1}^{p} \frac{d_k}{3^k}$, then $3^p\lambda = d_1 3^{p-1} + ... + d_p$. Note that $3^p\lambda$ is an even integer such that $0 \leq |3^p\lambda| \leq 2 + 2 \cdot 3 + ... + 2 \cdot 3^{p-1}$. Hence, since $\lambda = \frac{3^p\lambda}{3^p}$ where $0 \leq |3^p\lambda| \leq 2 + 2 \cdot 3 + ... + 2 \cdot 3^{p-1}$, we observe that

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^{3^p\lambda} \cdot 3^p B^{(p)}(z) dz = \sum_{j=-M(p)}^{M(p)} c_j \int_{\mathbb{T}} z^{3^p\lambda} z^j dz = c_{|3^p\lambda|} = \frac{1}{2^{\frac{|d_1|+...+|d_p|}{2}}}.$$

Now suppose that λ is not of the form $\sum_{k=1}^{p} \frac{d_k}{3^k}$ for some $p = 1, 2, ...$ Assume that $\lambda = \frac{l}{3^q}$ for some $q = 0, 1, 2, ...$ and some $l \in \mathbb{Z}$. We will consider two cases:

- Suppose that l is even: In this case, we claim that $|l| \geq 2 \cdot 3^q$. This is because if $l < 2 \cdot 3^q$, we could write λ in the above form, which is a contradiction. If $l \geq 2 \cdot 3^q$, $\hat{m}(\lambda) = 0$.
- Suppose that l is odd. In this case, $\hat{m}(\lambda) = 0$, because the coefficients of the odd powers of $3^q B^{(q)}(z)$ are all odd for any $q = 1, 2, ...$

□

We will finish this chapter with a result that was first stated by J. Packer in 2008 in [24]. We can use the above theory to prove it.

Proposition 9.4.3. *[Packer] The quantity $\hat{m}(\lambda) \neq 0$ if and only if $\lambda \in \mathbb{Z}\left[\frac{1}{3}\right]$ with $|\lambda| < 1$ and $|\lambda|$ has an even number of 1's in its standard ternary expansion.*

Proof. We first prove the forwards direction. Suppose that $\hat{m}(\lambda) \neq 0$, and suppose that $\lambda = \frac{l}{3^p}$ for some $p = 0, 1, 2, ...$, and some $l \in \mathbb{Z}$. If $p = 0$, $l = 0$ because $\hat{m}(l) = 0$ for all $l \neq 0$. Hence $|\lambda| = 0 < 1$, and moreover $|\lambda|$ has zero 1's in its standard ternary expansion. If $p = 1, 2, ...$, then l has to be even, and $0 \leq |l| \leq M(p) := 2 + 2(3) + ... + 2(3^{p-1})$ (this is because we are assuming $\hat{m}(l) \neq 0$). Hence, for all $p = 1, 2, ...$,

$$|\lambda| = \left|\frac{l}{3^p}\right| < 1.$$

Since l is even, $|l| = a_0 + a_1(3) + ... + a_{p-1}(3^{p-1})$, where $a_i \in \{0,1,2\}$ for all $0 \leq i \leq p-1$, and where there is an even number of a_i's equal to 1. Observe that

$$|\lambda| = \frac{|l|}{3^p} = \frac{a_0}{3^p} + ... + \frac{a_{p-1}}{3},$$

which shows that $|\lambda|$ has an even number of 1's in its standard ternary expansion.

Conversely, suppose that $|\lambda| < 1$, and that $|\lambda|$ has an even number of 1's in its standard ternary expansion. Suppose that $\lambda = \frac{l}{3^p}$ for some $p = 0, 1, 2, ...$ If $p = 0$, $l = 0$ because $|\lambda| < 1$. Hence, $\hat{m}(\lambda) = \hat{m}(0) \neq 0$. If $p = 1, 2, ...$, $|l| \leq M(p)$, because $|\lambda| < 1$. This means that $|l| = a_0 + a_1(3) + ... + a_{p-1}(3^{p-1})$, where $a_i \in \{0,1,2\}$. Then

$$|\lambda| = \frac{|l|}{3^p} = \frac{a_0}{3^p} + ... + \frac{a_{p-1}}{3}. \tag{9.5}$$

Since the standard ternary expansion of $|\lambda|$ is unique, and since we are assuming that the ternary expansion must have an even number of 1's, we get that there must be an even number of a_i's equal to 1 in expression (9.5). This allows us to claim that $|l|$ is even (and we already know that $|l| \leq M(p)$). Hence, $\hat{m}(\lambda) \neq 0$.

□

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