Teaching and Teacher Education 138 (2024) 104415

Contents lists available at ScienceDirect

Teaching and Teacher Education

journal homepage: www.elsevier.com/locate/tate

Research paper

Examining the role of student-centered versus teacher-centered pedagogical approaches to self-directed learning through teaching

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ARTICLE INFO

Keywords:
Learning through teaching
Mathematics education
Teacher education
Pedagogical content knowledge
Inquiry learning

ABSTRACT

While research has shown that students benefit from student-centered pedagogies, few studies have considered the benefits of this pedagogical approach for educators as they learn through teaching. In response to this need, we analyzed interviews, lesson plans, and video observations from five teachers in elementary schools across the United States who varyingly engaged student-centered and teacher-centered pedagogies. Our analyses revealed that the participating teachers developed a wide breadth of teacher knowledge regardless of their pedagogical approach. However, the teachers who employed student-centered teaching reported more pedagogical content knowledge gains for themselves than the teachers who used direct teaching.

1. Introduction

Within mathematics education research, scholars have repeatedly shown that allowing students to solve well-designed mathematics problems on their own (rather than teachers frontloading solution processes for students) leads to deep and meaningful student learning related to both mathematical content knowledge and socioemotional skills (see Ali et al., 2021). However, less is known about the benefits open ended pedagogical approaches hold for teachers. Thinking specifically through the theoretical lens of learning through teaching (LTT), or the process of teachers developing content and pedagogical knowledge as they lead students through learning experiences, Leikin and Zazkis (2010) contend that studies have repeatedly shown LTT represents an essential component of teacher professional growth and knowledge construction. Leikin (2010) also argues that the design of classroom activities can produce significant influence on what educators learn when teaching. While researchers could build on this assertion by exploring what teachers learn when designing and implementing lessons that position students as agentic problem solvers (in comparison to lessons revolving around direct instruction), studies have yet to explore this particular aspect of LTT. Similarly, extant literature on LTT within mathematics education disproportionately centers teaching in or alongside professional development contexts, highlighting processes of LTT that involve external support or collaboration (e.g., Hart et al., 2011). In other words, LTT research needs to explore what teachers learn when both students and they themselves work independently in learning contexts.

In response, we attend to the following research question: how does the use of student-centered vs teacher-centered pedagogies affect the development of teacher knowledge through LTT on their own? In asking this question, we consider the process of LTT as a multi-step cycle that includes planning, implementing, and reflecting on classroom practices and activities (see Schön, 1987). We also rely on Sengupta-Irving and Enyedy's (2015) definitions of student- and teacher-centered teaching. By either presenting students with open-ended problems to solve and allowing them the freedom to solve them on their own or presenting students with restrictive questions and the specific solution paths to answer them, teachers can potentially construct highly divergent learning environments for not only students but themselves as well. Finally, we specifically name "LTT on their own" to consider the kinds of learning that occur as teachers go about their daily practice, one where educators act in relative isolation compared to those currently enrolled in professional development contexts. Attending to this research question can therefore contribute to ongoing research into LTT, even when teachers have no external professional development support.

To conduct this research, we draw on Simon and Tzur's (1999) accounts of practice methodology and present findings from a broader study into early-career elementary and middle school mathematics teachers in the US. Through a qualitative analysis of teacher interviews, video observations, and lesson plans that track teachers across multiple

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https://doi.org/10.1016/j.tate.2023.104415
Received 16 March 2023; Received in revised form 10 August 2023; Accepted 13 November 2023

implementations of various two-day instructional modules, we show that teachers develop a broad set of skills and knowledge areas that align with Grossman's (1990) model of teacher knowledge and frameworks of mathematical knowledge for teaching (Ball et al., 2008) through LTT regardless of whether educators enable student agency in their classroom activities. However, our analysis also reveals that teachers who relied on student-centered teaching developed a more robust understanding of pedagogical content knowledge (PCK) in comparison to those teachers who took a more prescriptive approach. These findings therefore provide a deeper understanding of LTT, revealing the benefits of engaging well designed, open-ended mathematics for teachers (including those working in relative isolation) and not just students.

2. Theoretical framework

2.1. Teacher knowledge

The knowledge base for teaching specific disciplines has long been of interest to scholars (e.g., Ball et al., 2008; Blömeke et al., 2016; Copur-Gencturk & Tolar, 2022; Grossman, 1990; Shulman, 1986; Tatto et al., 2008). Although researchers differ in their conceptualizations (cf. Blömeke et al., 2016; Fennema & Franke, 1992; Grossman, 1990), they seem to agree that the knowledge needed for teaching has four main domains: subject matter knowledge (SMK), general pedagogical knowledge (GPK), pedagogical content knowledge (PCK), and knowledge of context (KOC) (see Ben-Peretz, 2011; Grossman, 1990; Kereluik, Mishra, Fahnoe, & Terry, 2013; Mishra & Koehler, 2006). Our conceptualization of these four domains was informed by various models and frameworks (see Fig. 1). Focusing on mathematics education, we conceptualized SMK as the robust knowledge of school mathematics that teachers need to know and that is grounded in the National Research Council's (2001) definition of what mathematical proficiency looks like. In particular, mathematical knowledge has four important components: procedural understanding, or knowing and carrying out mathematical rules flexibly and appropriately (National Research Council (NRC), 2001; Rittle-Johnson et al., 2015); conceptual understanding, which includes knowledge of the conceptual underpinnings of mathematical rules and definitions (Copur-Gencturk, 2021; Kilpatrick et al., 2015; Krauss et al., 2008); mathematical reasoning, defined as the logical thinking needed to investigate and evaluate the relationships among mathematical concepts and given situations (NRC, 2001); and word problem-solving skills, which requires translating information given in a word problem into a mathematical expression and being able to solve that problem (Copur-Gencturk & Doleck, 2021). Outside of a math specific context, our conceptualization of GPK was inspired by Grossman's (1990) model of teacher knowledge, which consists of three components: knowledge of learners and learning, which encompasses general knowledge concerning how children learn and how learning occurs; knowledge of classroom management, which entails the general knowledge and skills needed to maintain student engagement and manage the classroom; and knowledge of instruction, or knowledge of the general principles of instruction, such as using wait time.

Combining these two foundational components, PCK represents the third domain of teachers' knowledge. Since its introduction by Shulman (1986), scholars in mathematics education in particular have explored what constitutes PCK by identifying the knowledge that teachers draw on when teaching mathematics (e.g., Ball et al., 2008; Copur-Gencturk & Tolar, 2022; Copur-Gencturk et al., 2019; Tatto et al., 2008). The conceptualization that guided this work has focused on three components that seem to be common elements across various conceptualizations: knowledge of students' mathematical thinking, knowledge of mathematics teaching, and knowledge of mathematics curriculum. The first component, knowledge of students' mathematical thinking, includes teachers' knowledge of how students learn a particular concept, their patterns of learning, and the struggles students demonstrate when learning such concepts (e.g., Ball et al., 2008; Baumert et al., 2010; Copur-Gencturk & Tolar, 2022; Krauss et al., 2008) For instance, grasping students' overgeneralization of the operation rules for translating whole numbers to fractions, such as adding across numerators and denominators, requires knowledge of students' mathematical thinking. The second component, knowledge of mathematics teaching, encompasses knowing how to make the content accessible to students through the use of instructional strategies, mathematical tasks, and representations (Ball et al., 2008; Baumert et al., 2010; Copur-Gencturk & Tolar, 2022; Tatto et al., 2008). As an example, knowing to use fraction rectangles instead of circles when comparing fractions is an indicator of the teacher's knowledge of mathematics teaching because students have difficulties with partitioning equally when using fraction circles. The final component of PCK, knowledge of the mathematics curriculum, encompasses a knowledge of the horizontal and vertical curricula of mathematics in a given curricular framework (Ball et al., 2008; Grossman, 1990). Examples of such knowledge components are how the mathematical concepts that students are expected to learn at a particular grade level are mathematically connected (horizontal knowledge) as well as how these concepts are connected to the mathematical concepts that students learned in prior grades and will learn in later grades (vertical knowledge) (Ball et al., 2008). As numerous studies have shown, the development of PCK represents a crucial part of teacher knowledge development across disciplines that directly relates to student learning (see Baumert et al., 2010; Chang et al., 2020; Ogletree, 2007; Olfos et al., 2014; Purwoko et al., 2019; Thadani et al., 2017), although a need still exists for further analysis in this area (see Jacob et al., 2020). To this end, understanding how teachers develop PCK represents a valuable focus within LTT research and teacher education research more broadly.

In defining the last category of teacher knowledge, knowledge of context (KOC), we center our definition on the specifics of the learning ecology teachers and students find themselves in. KOC involves learning about students (their sociocultural backgrounds, their interests, etc.), the school environment, the local community, and how all of the aspects that compose the learning environment influence (and define) teaching and learning (see Waite & Pratt, 2015). But, as Thomas and Berry (2019) attest, generalizing KOC in a practical or implementable way remains difficult because of the inherent differences that exist between the sociocultural elements of all classrooms. Yet despite the complexity in defining KOC, researchers have employed multiple frameworks that highlight this aspect of teacher knowledge to further develop pedagogical practices rooted in the lives of students. Notions such as funds of knowledge (González et al., 2006), culturally relevant pedagogy (Ladson-Billings, 2021), and culturally responsive teaching (Gay, 2018) all amplify the necessity of developing a thorough understanding of and interaction with the lives of students within effective teaching strategies. This applies to mathematics education, as scholars have shown that understanding the lives of students contributes to effective pedagogy in terms of helping students construct mathematics skills, knowledges, and identities (Lampert, 2001; Ma, 2016; Mukhopadhyay et al., 2009). Yet despite the importance placed on context within these studies, Thomas and Berry (2019) also recognize the tension that teachers often face in navigating mathematical and contextual knowledge, arguing that "more

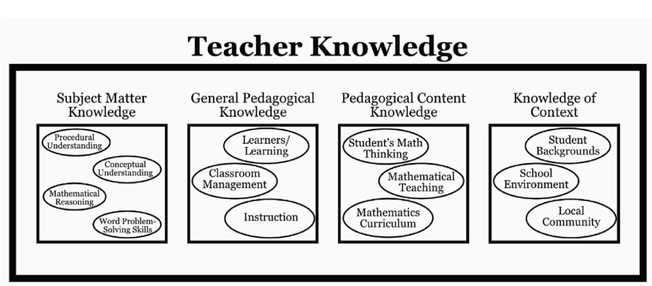

Fig. 1. The component parts of teacher knowledge.

work is needed to understand and unpack the interactions of teachers' KOC and culture with knowledge of mathematics and teaching mathematics" (p. 29).

2.2. Learning through teaching

Within extant literature (and especially mathematics teacher education literature), a subset of scholars have explored and advocated for an approach to developing teacher knowledge via LTT. Beyond merely acknowledging this value, Sherin (2002) contends that LTT exists as a critical and at times intractable aspect of developing as a teacher: as education reforms move through the formal education landscape, teachers often need to adjust their practice to accommodate these shifts without explicit guidance from professional development experiences or teacher educators. Fortunately, previous studies have uncovered a wealth of value within LTT as teachers have developed both discipline specific SMK (see Bausell & Moody, 1974; Elmendorf, 2006; Leikin, 2006; Leikin et al., 2000) and PCK (see Copur-Gencturk & Li, 2023; Cobb & Mcclain, 2001; Dyer, 2016; Lampert et al., 2013; Leikin & Rota, 2006; Perkins et al., 2015; Zazkis & Mamolo, 2018) through LTT. Beyond merely investigating its efficacy, Schön (1987) describes a simple three-part process for the act of teaching that provides a framework for unpacking and further analyzing LTT: planning some sort of lesson or learning experience, implementing that plan with students, and then reflecting on the results. Learning then occurs through the iteration of this process (see Okita & Schwartz, 2013), taking things learned through reflection and applying that knowledge while planning and implementing future lessons. This assertion positions the recursive feedback of seeing students using what teachers taught as a crucial part of developing through LTT.

Building on this framing, extant research has uncovered opportunities for LTT within each phase of the plan/implement/reflect cycle (e.g., Hart et al., 2011). Simon (1997), for instance, describes the planning stage as one in which teachers produce prospective learning trajectories for students that involve learning goals, proposed tasks, and hypotheses as to how the learning process will unfold. Teachers then develop PCK by creating new lessons and anticipating students' responses (Leikin, 2006). These two practices within the planning process enable teachers to deeply engage with content in often unexpected ways. To use Liljedahl's (2007) terminology, planning enables a process of reification wherein teachers pull knowledge from the subconscious into the conscious through the enactment of that knowledge (here in the form of lesson plans or pedagogical tasks). In doing so, teachers also engage a process of negotiation as they adapt existing content knowledge to new contexts (Sherin, 2002). Sherin (2002) expands this point, asserting that teachers also develop new content knowledge through implementation. However, LTT will fail to occur if teachers do not routinely embrace what multiple scholars define as teacher noticing (see Mason, 2002; Sherin et al., 2011; Zazkis & Mamolo, 2018). Within LTT, this involves actively looking for and recognizing student thinking as a valuable part of education, a contention that highlights the fact that teachers learn from students as well (Franke et al., 2001; Jacobs & Empson, 2016). Beyond planning and implementing, many researchers studying teacher education position reflecting on one's practice as the primary means for LTT to occur (see Mason, 2002; Salmon et al., 2020; Santagata et al., 2018; Zaslavsky & Leikin, 2004). According to Tzur (2010), the plan/implement/reflect cycle of teaching provides a "wealth of opportunities to be perturbed, that is, to identify gaps between what they meant their teaching activities to engender and what students actually learned, " (p. 51) even if teachers overlook these opportunities due to feeling threatened by unexpected outcomes or situations.

All told, LTT provides a valuable learning tool for both learners and educators, including novice and veteran teachers (see Landt, 2003), to engage. Yet what remains unknown is the extent to which teachers can learn on their own through their teaching when there is no support or guidance from external sources. This gap exists because the vast majority of research explores (and overemphasizes) LTT within the context of teacher education programs and other professional development communities (McDonald et al., 2014; Salmon et al., 2020; Scanlon et al., 2022). Within math education research in particular, "little is known about the teacher's learning of mathematics in their own classroom" (Leikin & Zazkis, 2010, p. 3), especially when unaccompanied by additional professional development support from instructional coaches, cooperating teachers, university professors, or other professional educators. Still, this body of work holds great value for those creating learning opportunities for pre-service and active teachers. Teacher education curriculum designers, for example, have developed intentional and scaffolded approaches to LTT, including teaching rehearsals (Lampert et al., 2013), co-teaching (Hiebert et al., 2007) and lesson studies (Hart et al., 2011) that engage pre-service teachers and current educators in intentional LTT opportunities. But research that builds on this work by centering the learning of teachers in their day-to-day teaching can help educators more intentionally develop their practice and knowledge when they do not have access to professional development opportunities. For instance, Jackiw and Sinclair (2010) and Leikin (2010) note that not all pedagogical tasks embody the same potential for LTT. Instead, some lessons, especially those that create space for students to respond in new and unexpected ways, create significantly more opportunities for teachers to learn through the act of teaching. According to Lai et al. (2012), "teachers learn about students' thinking when their curricula is conceptually oriented, allows for student creativity, and encourages student contributions" (p. 167). Stated differently, teachers learn through exposure to student thinking and inserting opportunities to come in contact with student-developed approaches to solving mathematics problems creates space for teachers to construct that knowledge, an assertion teachers themselves can employ in their classroom to support their own learning and the learning of students simultaneously.

A greater understanding of LTT outside of professional development opportunities can provide further opportunities for self-directed learning by teachers, especially because not all teachers have access to professional learning communities or initiatives. To this end, we use this paper to explore how LTT occurs for teachers acting in relative isolation (or, at the very least, teaching without having regular contact with or support from teacher educators or conducting formal collaborations with teachers outside of their immediate school context).

2.3. Student-centered pedagogies versus teacher-centered pedagogies

Despite the potential shown by Leikin (2010) in her exploration of pedagogical tasks, extant research has also largely overlooked the influence of pedagogical approaches on teacher knowledge acquisition through LTT on their own. Although an endless number of pedagogical frameworks exist, we rely here on Sengupta-Irving and Enyedy's (2015) definitions of student- and teacher-centered pedagogies. Representing more of a spectrum than a dichotomy, the authors frame the distinction between student- and teacher-centered pedagogies through the lens of student agency in ways that align with Eysink et al.'s (2009) notions of observational learning (guided) and inquiry learning (open). At the core of a teacher-centered approach, the teacher leads students "to and through the math concepts" (Sengupta-Irving and Enyedy, 2015, p. 561) needed to solve a problem chosen by the teacher. While the students have some agency in describing the problem and its solution in their own words or justifying the use of a solution strategy in new problems, the teacher still provides the students with a solution strategy and asks them to replicate that strategy in new settings. A student-centered approach, on the other hand, involves students inventing a unique solution path, discussing and refining this path with other students before justifying both their solutions and their process for arriving at an answer. Importantly, as both Sengupta-Irving and Enyedy (2015) and Hmelo-Silver et al. (2007) attest, student-centered teaching does not imply that students have no guidance at all. Instead, teacher interventions occur less frequently and the nature of these scaffolding moments shifts to focusing on targeted problem solving and questioning strategies rather than correcting student mistakes.

Exploring math education through this lens, existing research has shown that students develop a wide range of mathematical skills and knowledges when engaging well designed (and effectively implemented) open pedagogical tasks. In choosing not to show students how to produce a solution to a given problem and instead asking them to develop their own solution process, student-centered pedagogical approaches create the context necessary for students to develop rich, conceptual understandings of mathematics beyond procedural knowledge (Mackrell & Pratt, 2017; Papert, 1980) and find connections within and across specific concepts (Noss & Hoyles, 1996). In doing so, student-centered teaching creates the context necessary for students to develop creative problem-solving skills (Ali et al., 2021; Jasien & Horn, 2018; Levav-Waynberg & Leikin, 2012) and promote new ways of thinking about mathematics (Bland, 2019).

What teachers learn on their own when employing student-centered teaching in comparison to prescriptive approaches, however, remains largely absent from the literature since prior work has mainly explored LTT in the context of professional development and teacher collaboration. In Leikin's (2010) study of teaching experiments focused on multiple solution tasks (or classroom activities where students provide multiple novel ways for solving a single problem), the author found that teachers intertwined and developed their PCK and SMK, with both aspects of teacher knowledge reinforcing and informing the other. Placing this experience within Schön's (1987) framework for LTT, this knowledge development specifically emerges from teachers' interactions with student problem solving strategies in the implementation phase. Additionally, as Yeh (2016) shows, teachers also bring student problem solving strategies into the reflection phase as they consider not only the choices they made when implementing the lesson but the choices students made as well. In terms of the planning stage, Chapman (2007) recognizes that teachers draw on and subsequently develop an intertwined set of SMK and PCK related to inquiry-based approaches to learning, similar to the kinds of knowledge described by Leikin (2010). Put into conversation, this body of research shows that teachers learn the kinds of SMK and PCK needed to design open ended tasks when engaging students through this pedagogical approach and then reflect on those experiences (when supported by teacher educators or professional learning communities), thus mapping this process onto Schön's (1987) three-part cycle. How this experience compares to LTT when employing teacher-centered pedagogical approaches, however, remains under-explored. While these studies indicate that exposure to student's invented problem-solving methods, or what Sengupta-Irving and Enyedy (2015) describe as the invention of solution paths, represents a unique LTT affordance of open pedagogical tasks that does not exist in more guided approaches, studies directly comparing these two methods have not been conducted. With this oversight in mind, we now turn towards new empirical evidence comparing LTT through student-centered and teacher-centered approaches.

3. Methods

To explore the influence of pedagogical approaches on LTT for math teachers working on their own, we draw on what Simon and Tzur (1999) describe as an "accounts of practice" methodology. In this approach, researchers construct a trajectory of development related to teacher practice, a term that encapsulates both what teachers do and "everything teachers think about, know, and believe about what they do" (Simon & Tzur, 1999, p. 254), thus capturing the development of teacher knowledge. To engage this approach, the locus of research shifts from a teacher's perspective of what they do to the researcher's perspective. This shift addresses the notion that teachers may hold an understanding of their own developing practice that, while highly valuable, does not fully illustrate their learning trajectory due to their embedded perspective.

Broadly speaking, an accounts of practice methodology involves four stages that we adhered to in the following ways. First, researchers develop a conceptual framework to define a teacher's trajectory before data collection begins. As described in our overview of LTT research, we rely on Schön's (1987) model of teaching where educators iteratively learn as they plan, implement, and reflect on classroom activities. Second, researchers collect data that speaks to a teacher's practice over time. To do so, we gathered evidence of teacher learning at all three stages of Schön's (1987) model and through multiple iterations of this process, thus providing temporal insight into the development LTT. Third, researchers develop an account of an educator's professional practice through data analysis. Due to our particular focus in this study, we engaged this aspect of an accounts of practice methodology by tracing the categories of knowledge that our teachers developed throughout our iterative data collection process. Finally, an accounts of practice methodology ends with producing a hypothetical trajectory for that teacher. Diverging somewhat from the original intention of the literature, we reimagined this aspect of Simon and Tzur's (1999) model as a comparison between student-centered and teacher-centered pedagogies, proposing a possible and broad trajectory of LTT within both pedagogical models. With a conceptual argument for the influence of pedagogical approach on LTT established in the previous section, we now turn towards the final three stages of data collection and empirical analysis to produce a deeper understanding of teacher knowledge development within pedagogical interactions.

3.1. Study context

The data utilized for this work comes from a large-scale, longitudinal study designed to investigate teachers' development of content-specific expertise from their own teaching (Copur-Gencturk & Li, 2023). We recruited participating teachers in 2018 from across the United States and then scheduled online, individual meetings with teachers who expressed an interest in participating in the study regarding data collection procedures. In particular, we asked teachers to create lesson plans that documented their plan for instruction over two consecutive days and interviewed them three separate times during each series (before teaching the first lesson, between teaching the first and second lesson, and then after teaching both). This cycle of data collection occurred three times for each teacher whenever a new concept was taught. Our data therefore consists of three series of 2-day lesson plans (6 lessons total) and nine interviews per teacher (see Fig. 2). Thus, if learning occurred from teaching any of these lessons specifically, we were able to capture it through our interviews. In addition, teachers videotaped their instruction while teaching one of these planned lessons. We therefore triangulate the data between the lesson plans, interviews, and video observations to ensure validity (see Denzin, 2012).

As shown in Table 1, five teachers participated in the study. Two self-identified as white, two as multiracial, and one as Black. Three of the teachers self-identified as women and the other two as men. All five teachers held a credential in teaching multiple subjects and had graduated from teacher education programs, with three graduating from a 4-year program and the other two attending a 5-year program. Even though all the teachers were new to the teaching profession, all except

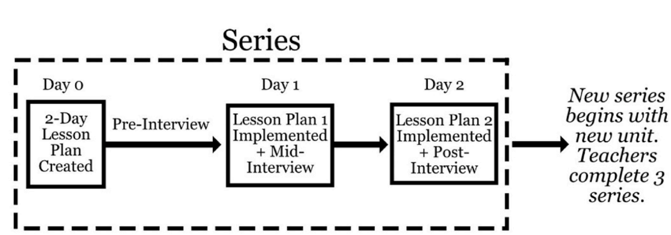

Fig. 2. Illustration of the data collection process.

Table 1
Teachers' personal and educational backgrounds information.

Name	Undergrad major	Gender	Race	Route to the profession	Teaching credential	Grade	Years of teaching experience	Years of math teaching experience	State
Linda	Elementary education	F	White	Traditional 4-year teacher education program	Generalist	5	3	1	AK
Mike	Elementary education	M	White	Traditional 4-year teacher education program	Generalist	4	2	2	IN
Hannah	Education	F	Multi-racial	5-year teacher education program with a master's degree	Generalist	5	0	0	CA
Xavier	Early childhood education	M	Black	Traditional 4-year teacher education program	Generalist	3	2	2	TX
Daniella	Native American studies	F	Multi-racial	5-year teacher education program with a master's degree	Generalist	6	3	2	CA

Note. All names used are pseudonyms.

Hannah had taught mathematics as full-time teachers in K12 settings before the data collection period. Importantly for this study, none of the teachers were currently enrolled in professional development initiatives and this study did not involve any teacher education processes. Instead, we merely asked teachers to share their experience of teaching lessons they would have designed and taught regardless of their involvement in this study.

3.2. Data collection

3.2.1. Lesson plan tasks

We used teachers' lesson plans as a means to gauge the extent to which participants learned from their teaching. Lesson plans provide insights not only into how the teachers teach a lesson, but also into the knowledge and skills they apply to teaching subject matter. Our main interest in this study involved exploring whether and to what extent teachers developed PCK from their teaching. We chose this because of the role that PCK plays in students' disciplinary learning (see Chang et al., 2020; Ogletree, 2007; Olfos et al., 2014; Purwoko et al., 2019; Thadani et al., 2017), its position within our teacher knowledge framework, and its relationship to both GPK and SMK. We therefore focused on the areas of teaching in which participants needed to draw on their PCK. In particular, teachers' task selections and their choices of representations and instructional strategies provided important indicators of their knowledge of mathematics teaching (i.e., their PCK). Similarly, teachers' knowledge of what strategies students would use and what struggles students might have when learning a particular concept represent manifestations of their knowledge of students' mathematical thinking, an element of PCK.

Thus, the lesson plan task we adapted from prior work (Morris & Hiebert, 2017) included specific questions aimed to reveal their PCK. Teachers were asked (1) to provide a learning goal and the key concepts they planned on targeting in each lesson, along with the main task they designed or chose, and (2) to explain why and how this particular task would lead them to accomplish the learning goal. They were also asked questions in the lesson plan document itself about what particular strategies and struggles they anticipated their students having, how their instructional plans addressed these struggles, and their rationale for these plans. The lesson plan included sections to (1) describe the specific things they would be on the lookout for to ensure that their students were making progress toward the learning goal; (2) their organization of solution paths (if predetermined) and their rationale for that particular order in terms of developing students' understanding of the key ideas targeted in the lesson; and (3) the specific questions they would ask students to help them make use of mathematical ideas. In addition, we asked participants to provide another task (a formative assessment or exit ticket) they planned to employ in assessing students' learning at the end of the lesson and their rationale for selecting this problem in terms of why students' work on the problem would help them gain information regarding whether they had achieved their learning goal.

3.2.2. Interviews

The three-part interview series conducted with target teachers represent the second main data source. Initially, we interviewed participants after they had prepared their 2-day lesson plans but before they had taught them to help us understand their lesson plan tasks more accurately. We then interviewed teachers twice thereafter, once immediately after they had taught the first day of their 2-day lesson but before they had taught the second day to capture whether they had learned anything from teaching the first lesson, and once after they had taught the second day of their 2-day lesson. Thus, each teacher was interviewed three times per cycle for a total of nine interviews. We adapted interview questions from prior literature (Smith et al., 2008) and drafted our protocol to explicitly explore teachers' thinking and learning through the lesson. The first interview conducted for each lesson cycle centered on understanding teachers' lesson plans by asking elaborating questions about five key topics: why they chose the main task, why they anticipated the strategies they listed, why they thought students would struggle in particular ways, why they chose the specific items to be on the lookout for, and why they selected a particular task to assess students' mastery of the learning goal. The interviews conducted after the teachers' first and second days of teaching focused on what the teachers learned from teaching in the following areas (that were intentionally similar to the initial interview): the main activity they chose to introduce the concept, the students' strategies they anticipated, the students' struggles they anticipated, their responses to the students' struggles, points they wanted to be on the lookout for to achieve the learning goals of the lesson, their assessment of students' learning (including exit tickets), and any changes they planned to make in their lesson plan based on their implementation of the lesson. In framing the interviews in this way, we position ourselves as outside observers developing an account of teacher practice rather than researchers merely capturing teacher's conceptualization of their own work. All the interviews were conducted online, videotaped, and transcribed verbatim.

3.2.3. Collection procedure

Because we aimed to study the knowledge teachers acquired from teaching, we collected data as frequently as possible to detect potential learning. To do so, the researchers scheduled a one-on-one online meeting with each teacher in the study and decided on three units in the same content area for which teachers created their 2-day lesson plans (i.e., fractions if they were teaching Grades 3–5 and ratios if they were teaching Grades 6–7). Teachers then created a 2-day lesson plan for the first week of each unit in which they were introducing a new concept (see Fig. 2). We focus on this introductory week because teachers would, hypothetically, have more opportunity to learn during the introduction of a new concept as opposed to later in the unit after they had settled into certain pedagogical routines, potentially revealing unexpected outcomes and creating room for them to gain knowledge and skills. The rationale behind focusing on a single content area (i.e., fractions or ratios) was to investigate the extent to which teachers would generalize the PCK of the relevant content area: because PCK is content specific, it is unlikely that students would show similar mathematical struggles and understandings across concepts or content areas.

We then contacted teachers when the time to start a new unit approached and scheduled an online meeting. The teachers were instructed to send their 2-day lesson plans in advance of the interview so that the research team could peruse the plan and ask additional clarifying questions. The first interview of each lesson plan cycle generally occurred a day before teaching the first lesson. The second interview was usually conducted on the same day teachers taught the lesson, and the third interview was conducted either on the same day the lesson was implemented or the day after they taught.

3.3. Data analysis

To analyze the data generated from interviews, lesson plans, and videos, we began by employing an open and iterative approach to what Saldaña (2015) defines as descriptive coding, linking the topics being discussed and choices made by each teacher with a specific code. We generated this set of codes through an emic process, allowing the codes to emerge from the data itself rather than applying a pre-established coding scheme. After both research team members coded the data from each individual teacher, we engaged the consensus building process described by Harry et al. (2005). Rather than determining interrater reliability measures to determine validity, this process involves researchers coding data independently and then comparing all emergent codes and the application of both new codes and established ones from previous analyses of other interviews. From there, we combined our new codes that had significant theoretical overlap and any divergent applications were "debated and clarified until the group agreed on appropriate usage" (Harry et al., 2005, p. 6). In doing so, we eventually came to agreement on all codes and code applications.

After this first round of coding and the subsequent consensus building process, we completed another coding cycle that employed a similarly open and iterative approach to emic, descriptive coding. During this cycle, we centered on moments in the interviews where teachers described what they learned from teaching using their lesson plans (while also continuing to analyze the videos and lesson plans to find evidence or counter-evidence of the teachers' self-reported learning). Beyond this emic approach, however, we also looked for evidence of open and guided approaches to teaching. Drawing on Sengupta-Irving and Enyedy (2015), this part of the analysis distinguished between teachers who provided problems that students would solve on their own or with their colleagues using pre-existing knowledge and problem-solving skills (student-centered teaching) and teachers that not only provided problems but the exact process for solving that problem they expected students to use (teacher-centered). In doing so, this aspect of our coding process specifically focused on Sengupta-Irving and Enyedy's (2015) distinction between the guided seeing ("Students are led to and through the math concepts") and invention ("Students invent the solution path") components of direct and student-centered instruction, respectively (p. 561). We focus on these aspects of teaching because they provide a stark contrast between the two pedagogical approaches while other elements of student- and teacher-centered teaching (such as formalization and best inference) exist in both. Once we completed our second-round coding, we again undertook the same consensus building process to verify our results. With these codes and their applications established, we completed our qualitative analysis by shifting to an etic approach to pattern coding (Saldaña, 2015) that relied on an extant coding scheme. More specifically, we organized our initial descriptive codes that illustrated the breadth of how teachers conceptualized their own LTT into the model of teacher knowledge shown in Fig. 1 (i.e., subject matter knowledge [SMK], general pedagogical knowledge [GPK], pedagogical content knowledge [PCK], and knowledge of context [KOC]). In organizing the coded data in this way, we produced the scope of learning that occurred through LTT by these teachers.

To further analyze our codes and characterize their LTT more holistically, we created a variable to explore the distribution of the different kinds of knowledge teachers reported gaining from their own teaching experience. To do so, we divided the frequency of codes belonging to each knowledge dimension by the total number of code applications for each individual teacher. Thus, a higher percentage score in one knowledge area indicates teachers reported gaining more knowledge in the corresponding knowledge domain. To explore how their teaching style might affect their LTT, we explored the patterns in the development of various kinds of knowledge according to how the teachers structured their mathematics teaching (i.e. through student- vs. teacher-centered pedagogical approaches). We accomplished this by capturing whether they allowed their students to solve a mathematics problem on their own first or whether the teacher solved the problem before giving students a chance to work on it. We coded every instance in which an individual teacher mentioned that the students or the teacher solved the problem first. We then considered which of these two types of instances occurred more often and compared all codes to the specifics of the lesson plans and videos to determine whether the teacher employed a primarily open or guided approach. Finally, we returned to the results from our analysis of teacher learning by looking at these findings in relation to this pedagogical categorization.

4. Findings

Through our analysis of pre, mid, and post interviews, video recordings, and lesson plans, we constructed a broad and encompassing collection of codes that thoroughly documented the breadth of teacher learning that occurred through LTT. These codes connected to all four categories of teacher knowledge but emphasized learning GPK and PCK with only a few codes related to KOC and SMK. In this section, we will more thoroughly discuss these categories and codes before connecting our categories to the pedagogical approaches employed by teachers. In Tables 2 through 4, we list 39 of the 40 codes we developed through our analysis (with a description of the one code related to SMK presented in the main text of the next section), an example of each code taken from one of the teacher interviews, and the total number of lessons where we found evidence of a teacher learning this specific skill or piece of knowledge (both in terms of evidence existing in the interview and in the video observations). The frequency count therefore ranges from 0 (a teacher never mentioned or showed evidence of learning related to this code across the entire study) to 6 (a teacher mentioned and showed evidence of this learning during every single lesson in the study).

4.1. Developing teacher knowledge through teaching

In terms of GPK, our analysis produced eighteen separate skills or ideas related to this category (see Table 2). Within this categorization, we specifically describe moments where the teacher developed a skill or understanding related to teaching that was not specific to mathematics education or mathematical thinking. Broadly speaking, the codes within this category centered on aspects of teaching such as time use and classroom management, reframing classroom activities, or adjusting their overall teaching practice. For instance, some teachers learned that they needed to break up a specific concept from one of their lessons into a multi-day lesson the next time they had to teach that same concept. However, some of this general pedagogical knowledge did not necessarily equate to learning best practices. For instance, some lessons resulted in teachers thinking that they needed to restrict students' ability to creatively explore new concepts or solve problems through invented methods. While this does relate to the development of general pedagogical knowledge (these teachers did develop a new understanding of teaching), it illustrates that this learning may not always equate to an improved teaching practice in certain instances.

Additionally, our analysis uncovered seventeen codes related to the knowledges and skills teachers gained related to PCK (see Table 3). PCK

Table 2
General pedagogical knowledge codes.

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
Activity and Problem Directions	Teacher discusses the need to be clearer or more detailed in reference to the instructions for a problem or activity.	If I were to teach it again, I would figure out a way to give better instructions for the hands-on activity. The instructions kind of confused them, which I think, in turn, made them feel like they were confused about how to find those denominators. Once we kind of cleaned up how to put the chain together, then they realized, "oh, I can do this." But I think that took away from their confidence a little bit at first. So I would come up with better instructions for that activity. (Linda)	1	1	0	2	1
Allowing for Student Exploration and Mistakes	Teacher learns the value of allowing students to try problems on their own and also making and learning from their mistakes.	I've always been scared to let my kids do things the wrong way. I'm afraid that is going to get it stuck in their head and I'll never be able to get them away from doing it the wrong way. But letting them do it their way even though it's wrong, letting them see that doesn't work, it was actually a really good thing for us. (Linda)	0	2	0	0	0
Assessment Practices	Teacher learns something about how they are assessing students/giving feedback.	And then, for the exit tickets, I just gave a graded assignment and they were struggling with it. But when we would go over the practice, I just realized I wasn't fully assessing them. And so that was a big change I made. (Linda)	1	1	0	1	0
Breaking Up Lesson	Teacher discusses the need to break up instruction on a specific concept or an activity over multiple class periods. This is differentiated from the "need for more time" code by considering the structure of the unit as opposed to a specific lesson.	It was successful, but it's just something I realized is a two-day lesson. It's not a one-day lesson. (Linda)	0	1	0	0	0
Clarity of Content Delivery	Teacher discusses the need to more clearly deliver content to students. This can involve wording concepts better, changing what they write on the whiteboard, etc. This code is generic and does not actually connect to the mathematical concept being referenced.	I would be very slow and methodical when I was doing that last example. "This is what I'm thinking as a student, this is what I need to see." That's the biggest one. (Mike)	0	0	0	2	0
Formatting and Communication Strategies	Teacher discusses the need to either support or ask for students to communicate their work or their answers differently. This does not actually refer to teaching content or concepts differently, just writing out the work.	Making sure that they are being very clear about how they should be showing their work so that maybe I won't have to go around to ask them what they did. They can just show me what they did based off of what they wrote. (Hannah)	1	0	0	2	0
Generic Pedagogical Change	Teacher reports plans to change the pedagogical approach in the lesson. This change is not related to the content of the lesson, just the mode of delivery or structure of the student action (i.e. changing a discussion method).	I have to ask questions and not say things, because this is when they start to get really good at reading your body language to know if they have the right answer and repeating exactly what you say so that you think they were paying attention or that they understood what you said. They have those defense mechanisms already, so I can't do those things or I can't get a good assessment on where they are. So, that's the hardest part of teaching in general, it's just figuring out how you can get your students to show you what they know without you giving away the answer without trying to. (Linda)	2	3	1	4	2
Increasing Classroom Management Structure	Teacher discusses the need to increase classroom management structures (i.e. limiting the amount students are able to talk to each other).	It's more of a classroom management problem than an outline problem. I just have to do a better job of making sure I got everybody's attention first before we keep going. Because I feel like I lost some students here and there, and that affected the outcome of the lesson. (Mike)	0	1	0	2	0
Limiting Thinking	Teacher explicitly states they want to limit the range of opportunities the students have to think mathematically or limit the possible choices, values, or operations students can use when solving problems.	I don't feel like it's the best way to teach, but for some of them, they're going to have to have a set strategy. A step one, a step two, a step three that I probably have to sit down and teach them how to do. I don't think that they're going to make that leap on their own. This is the time where I intervene and do some direct teaching and direct modeling rather than letting them figure it out. (Linda)	0	1	0	1	1
Monitoring Student Work	Teacher discusses the need to increasingly monitor what students are doing during class time (i.e. following and checking in on every step of a problem they are working on).	I'm going to give them the main problem and let them start working on it independently, even with all the misconceptions that they have today. But have them show me the first step, show me	0	1	0	0	0

(*continued on next page*)

Table 2 (*continued*)

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
		the second step. Put it like step by step by step. (Linda)					
Need for Differentiation	Teacher discusses the need to differentiate aspects of the lesson in future classes.	I'm going to have three groups. I'm going to have the one that is not regrouping correctly. I'm going to have the one that didn't subtract correctly at all, either problem. And then I do have a few that got the correct answers. So I will probably put together something for them to start estimating and then put the other two separately. So I'll have three different lessons going on tomorrow. (Linda)	0	1	1	0	0
Need for More Assessment	Teacher explains that they want to add more assessments to upcoming lessons or that they would add more assessments to previous lessons when teaching them again.	I know this is part of their homework, but if I had a separate exit ticket that they could give to me, a shorter one that they can just give to me on their way out to recess or something, that might be the only thing I'd change. (Daniella)	0	1	1	2	0
Need for More Time	Teacher discusses the need for more time to teach a concept or do an activity properly	I would try to make it where I have more time before this. I mean, I guess it depends on the class that I have. But if I were to re-teach this same class, I would give myself more time to go over the conversions and the scary percents. (Daniella)	0	0	2	2	1
Need for Practice	Teacher reports learning that future lessons need to involve more opportunities for students to practice skills they have already learned.	I think today, just the idea of how to subtract mixed numbers was achieved. I don't think it was necessarily mastered but I do think they know how. We still need more practice, but the learning part of it has been accomplished. (Linda)	0	1	0	0	0
Need to Reteach	Teacher states that they need to reteach a lesson without indicating the need to change the lesson in any way. This is different than "revisiting concepts" because it is a generic approach to reteaching content. It is not targeted and doesn't include considerations of student knowledge.	I'm just going to go back tomorrow and kind of try it again. Maybe after they've sat with it for a little bit they can see what I'm talking about tomorrow. (Linda)	0	2	1	1	0
Problem Solving Strategies	Teacher learns about the strategies that students already use to solve problems. These strategies are not necessarily related to what is being taught in the class and not necessarily connected to a specific mathematics concept. (This code is different than "Subject Matter Knowledge: Problem Solving Strategies" because the teacher doesn't report learning these strategies themselves. They may already know them, they just didn't expect students to use them.)	I was kind of surprised at some of them and some of [their approaches to] solving the problems, which is really cool. I really liked that there were multiple ways of solving each of these problems. (Daniella)	3	1	1	0	0
Reduced Student Work	Teacher discusses the need to reduce the amount of work students do (i.e. giving less practice problems)	I could've probably made the exit ticket two problems [instead of three]. We seem to have it as a class, so I probably could've left that [extra problem] out. (Mike)	0	0	0	1	0
Revisiting Concepts	Teacher discusses that they should revisit topics already covered in previous lesson in more detail (a.k.a. a refresher). This involves a targeted approach to reteaching concepts that students showed a certain misconception with.	I think that I would just review. We already reviewed the skills we learned the day before, but I think I would increase the review. They needed a little bit more of a review of what we did yesterday before we went into this. (Linda)	0	2	2	3	1

as a category extends beyond GPK by specifically relating to the teaching of mathematics. These codes covered a wide range of knowledges and skills under the PCK umbrella, including how to better write pedagogically valuable problems for students to engage, deeper understandings of their students' curricular knowledge and misconceptions, and techniques to help students connect to mathematical reasoning and previous concepts. Our code "attention to problem details" provides a clear example of what teachers learned related to PCK through teaching. In this code, the teachers recognized specific details within a problem they asked students to solve (such as number selection or wording of the problem) that reduced the complexity of thinking needed to solve the problem or the range of mathematical concepts that students could encounter (i.e., a fraction addition problem would not engage students in finding a common denominator if all of the values had the same denominator).

Lastly, the participants in this study constructed KOC (see Table 4) and SMK. However, the analysis related to these codes proved far less in depth than both GPK and PCK. In terms of KOC, where teachers learned about their individual students and their school environment or learning ecology (the district, etc.), our analysis only produced four different codes: learning how their students think about or relate to learning (not connected to mathematical thinking explicitly), learning about their student's socioemotional knowledge or skills, learning about issues students grappled with outside of the classroom environment, and learning about how to more deeply engage with student data. Regarding their learning related to SMK, our analysis only produced one code. Specifically, Danielle learned new methods for solving a problem she had not considered before, as some of her students framed this learning through a new metaphor (specifically, using money as a means to grapple with proportional reasoning). She describes this moment as follows: "I also learned that they think of such amazing ways to solve problems that I never would have thought of, ever. It was just really cool to see the way that their brains are working. I'm like, 'What, wait, what? I never thought of that!'" Danielle showed evidence of learning SMK

Table 3
Pedagogical content knowledge codes.

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
Attention to Problem Details	Teachers discuss learning how certain details in a problem or activity (i.e. number selection, wording) can either help or hinder student learning.	Another thing I felt like I learned was the way that each part of the problem is so important and can change how the kids solve it. Whether it's the number pairs or how the problem is phrased, changing something small about it or not wording it the right way, you can get different answers from the kids. So I think just being more careful and more thoughtful about the numbers I'm selecting and how I want that to support the learning goal. (Hannah)	4	2	0	1	0
Attention to Vocabulary	Teacher discusses learning how student's use of academic vocabulary leads to helping or hindering mathematical understanding.	The one thing that I did learn was that students seem to respond better to numerator and denominator than being told to multiply the top and the bottom of the fraction. Because the word top and bottom gives them a location automatically where they feel like they can directly apply it. But then when I say numerator or denominator, even though they know where it is, their response to the question is not as urgent. They have to think about it a little bit more. (Mike)	0	0	0	1	0
Breaking Down Problems	Teacher learns that students have a problem with breaking down a problem into different parts and completing those parts in order.	The only thing that I might do differently, depending on the students, is I might give them the main problem just as "how many hot dogs did the vendor sell?" And then after they solved that go, "okay, if I asked you how many were left, could you figure that out?" to make it where they focused on one part at a time rather than giving them both. (Linda)	0	1	1	0	0
Clarity of Mathematical Concepts	Teacher discusses the need to clarify or be more detailed when describing new mathematical concepts to students.	I just need to spend more time on the number line. It's just because we don't have enough time to go into more depth. I would like to extend their learning to go past one. If I had more time, I would have gone a little over one. And maybe showed them one-fourth, and one and one-fourth and one and one-half. Just to get them ready. (Xavier)	0	0	0	1	1
Concept-specific Pedagogical Content Knowledge	Teacher discusses the intersection of specific mathematical concepts and teaching, or how to use or structure mathematical concepts to help students learn a specific mathematical concept (rather than an overarching teaching practice).	So I really didn't know if that would be the way to go, to start with the more complex concept and then focus down into area, because it's a more concrete and easier way to understand multiplying fractions. But it works really, really well, especially for my low-level learners. So I'll probably always do that, use that area relationship until we've gone as far as we can showing that we're trying to find one-half of two-thirds or two-thirds of one-half. (Linda)	3	4	0	1	0
Conceptual Mathematical Knowledge	Teacher learns about students' conceptual knowledge or the development of their conceptual knowledge that they did not expect to see in the lesson.	They didn't need the model, which was really awesome. I did see them multiply by the reciprocal but then I saw a few that realized if I need half of something, that's divided by two. So they just divided the whole number. Because they were like, "it's easier to divide by two than multiply." And I was like, "well, whatever's easier for you." So that was the only strategy I saw that I didn't list. (Linda)	1	1	1	0	1
Connecting to Mathematical Reasoning	Teacher learns that they need to help students connect to mathematical reasoning beyond the practice of solving problems.	So, multiplying decimals by 10, 100, powers of 10: I would draw the number with the decimal and then we would practice moving the decimal. Then I would notice on their work that they could draw that, but then when they wrote their answer they wrote the original number. And I realized they were just making this drawing because that's what I had done without understanding what they were doing. They didn't realize they were moving the place value. So, I had to go back and reteach that, taking away the drawing part, because they were getting so hung up on what I drew on the board. We would go back and do it with our bodies and we would have a ball be the decimal point. And we would practice moving ourselves to move our place value. And then we would say, "Okay,	0	2	0	0	0

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Table 3 (*continued*)

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
		what does that look like?" just making those connections without me giving them something to do. Because if I did that, then they would just focus on that, even if it didn't make sense to them. (Linda)					
Curricular Knowledge	Teacher learns about what students learned in previous years.	I didn't know that my students had some experience with converting measurement last year. I kind of knew it was in 5th grade, but one of my students had his math journal from last year and showed me that they actually did a lot of work with converting measurements. And I just wasn't expecting that. So I feel like they did so well today because of that prior experience that I didn't know they had. (Daniella)	0	1	1	0	0
Knowledge Transfer	Teacher learns about how students use previous mathematical learning to do new tasks or activities, both in terms of their ability and inability to do so.	That's where, even if they know it, they've got to learn it again before they can use it. I didn't realize it would be like reteaching all the parts of fractions. Even though they know what to do and what it means, they just don't hold those connections together. (Linda)	2	2	0		0
Need for Manipulatives	Teacher reports learning that future lessons need to involve more or different manipulatives or that they need to adjust the lesson to fit with manipulatives more fluidly.	I'm going to use the fraction circles and I'm going to have those out. I have them on the side of the room. For this lesson, I'm going to just put them on the tables and say, "Oh, take these, everyone take these. If you need it, you need it. If you don't, just leave it." (Hannah)	1	1	0	0	1
New/Revised Main Activity- Decreased Difficulty	Teacher discusses the need to reduce the difficulty of the main activity or problem.	I felt like, after seeing them do it, I don't think it was necessary to use such a big number. I could have started off with one and one-third and one half instead of eight and one-third. So I feel like those kind of just got in the way. And I think that became confusing, so they weren't able to think about the mixed number and changing it to an improper fraction because it was a little too big of a number. (Hannah)	1	1	0	1	0
New/Revised Main Activity- Increased Difficulty	Teacher discusses the need to increase the difficulty associated with the main problem or activity.	I would make the fraction not one fourth. Because when you hear one fourth, anything multiplied by one is pretty easy to find out. And the multiples of eight are kind of hard to remember for a lot of students. So, I would've picked three eighths. And then I would've had another question on there. It would've been a different scenario where Lina's floor is 20 square feet: "If one fourth of that floor is covered in tile, how many square feet is that floor covering?" Because then they could've multiplied a whole number by a fraction and we could have talked about that. (Mike)	2	1	0	1	0
New/Revised Main Activity- Increased Student Choice	Teacher discusses how they would revise the main activity to include more student choice in responding to the main activity.	I think if I were to teach it again, I like the idea of adding different number sets that they can choose for the fraction. So they can choose what to solve. (Hannah)	1	0	0	0	0
New/Revised Main Activity- Reframed Problem Details	Teacher discusses the need to change the details of a problem to focus on certain mathematical concepts.	I think I'm going to manipulate that main problem so that we can actually use the fraction bars. I only have enough of each fraction to make one whole. So when you regroup and you make that improper fraction, I don't have enough pieces for each group to show that. (Linda)	2	2	0	0	0
Problem Solving Barriers	Teacher learns about the barriers students have when solving problems and the thinking behind that process.	I didn't realize that there would be that big of a misconception with the phrasing. I don't know if it was the phrasing or just having the two denominators where they had to change both of them. I didn't think that the amount would hinder their ability to think about it. (Hannah)	1	0	0	0	0
Students' Mathematical Misconceptions	Teacher learns about specific misconceptions students have related to content.	Regrouping is really hard for 5th graders. I just learned that. I'm still learning how difficult it is for them to connect the same concept across different types of numbers. They can explain regrouping with whole numbers brilliantly. And they can even do that with decimals. But I think just because it's not a set place- like the fractional part- it's not a tenths place, it's not a ones place. It's whatever that denominator says	1	1	1	0	0

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Table 3 (*continued*)

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
		it is. They're just not making that connection. It's not that I didn't know it before, I keep learning it over and over again that they can understand something and just not apply it. (Linda)					
Translating and Understanding Problems	Teacher discusses a student's inability to either read and understand a problem or translate it into mathematical notation.	I guess with fractions and thinking of dividing fractions, it's hard for them to really see a problem and know the equation. That's a little bit more of a struggle for them. They can solve it, but they can't necessarily apply the equation or figure out what the equation is that matches this problem as easily. (Hannah)	1	1	0	0	0

Table 4
Knowledge of context codes.

Code	Definition	Example	Frequency of code applications per teacher				
			Hannah	Linda	Danielle	Mike	Xavier
Learning from Data	Teacher discusses using student data to improve their teaching practice.	I'm really big on student data. I go through everything that they do, to see am I teaching it enough to where they're able to reflect it back to me so I can see that I'm doing my job. And not just with summative assessments but formative assessments as well. I base a lot of stuff on student data in addition to my own reflections of "oh, I could have said that differently." (Danielle)	0	0	1		0
Learning Dispositions	Teacher discusses learning something about how their students think about or conceptualize learning. This code is specifically in reference to their students and not students in general.	I thought for them it might be like a bit scary. I thought they might have a problem if it was a fraction that didn't have a denominator of 100, but they were like all for it. And I was just like, "this is so cool." Like I thought they'd be like a bit intimidated by it, but they weren't at all. They were just totally ready for it. (Danielle)	0	2	2	1	1
Outside Issues	Teacher discusses learning about issues outside of the classroom that affect student learning (i.e. the time of year being a distraction).	I would do it on a day that they didn't have Living History. They were just so hard to keep on task because they were so excited. It's our thematic unit project, so they've been working on it for months. And it's like their exhibition of learning for the whole community. So they were just really anxious and excited. I'd just try to do it on a day when they're not so antsy. (Danielle)	0	1	1		0
Socioemotional Knowledge	Teacher learns something about their students' behavior or socioemotional knowledge. This includes understanding more about students' self-management skills or their classroom behaviors that they feel need managing.	Something I learned was that I need to do more in terms of what an accountable partner looks like. So having more modeling on what partner work should look like. Because sometimes they do great with it and then other times it will get off-task. And so I think having more lessons on what efficient partner work looks like. (Danielle)	0	0	1	1	0

during two separate lessons. This aspect of our analysis therefore shows that teachers engaged KOC and SMK with far less frequency. For instance, only one teacher (Danielle) discussed the code related to student data and only did so during one of her interviews.

4.2. Teacher knowledge development in relation to student-centered versus teacher-centered pedagogical approaches

So far, we have provided a detailed description of what teachers learned while teaching these series of lessons. To better understand the relationship between approaches to teaching and teachers' LTT, we first characterized individual teachers' learning by reporting the frequency of codes applied per teacher knowledge category (PCK, GPK, KOC, or SMK). As shown in Fig. 3, the kinds of knowledge teachers gained from teaching varied from teacher to teacher. For example, Hannah and Linda reported learning more about students' mathematical thinking and what strategies and representations to use to teach mathematics (i.e., PCK). On the other hand, Danielle, Mike, and Xavier gained less PCK from teaching mathematics.

Building on this categorization, we then organized the participants into two categories: (1) those who situated students as problem solvers (student-centered) and (2) those who showed students exactly how to solve problems in their classrooms (teacher-centered). As an example of the first category, students might have needed to use their understanding of how to reduce fractions and add whole numbers to solve an unfamiliar word problem related to adding fractions without receiving instruction in how to do so. Linda exemplifies the open approach when she says, "I like them to see what they know on their own and really investigate before I tell them how to do it. It just kind of lets them make connections on their own." While Linda may eventually show students an efficient way to solve the problem, they still initially solve the problem on their own (and, in this case, share their solutions with their colleagues) without being shown a solution process. As an example of the guided approach, the teacher might instruct students in how to add fractions or solve other problems and then ask students to replicate those steps verbatim in practice problems. Mike strongly aligns his pedagogy with this approach when he says, "The biggest thing is the process. Are they following the process? Are they doing it exactly the way we have it

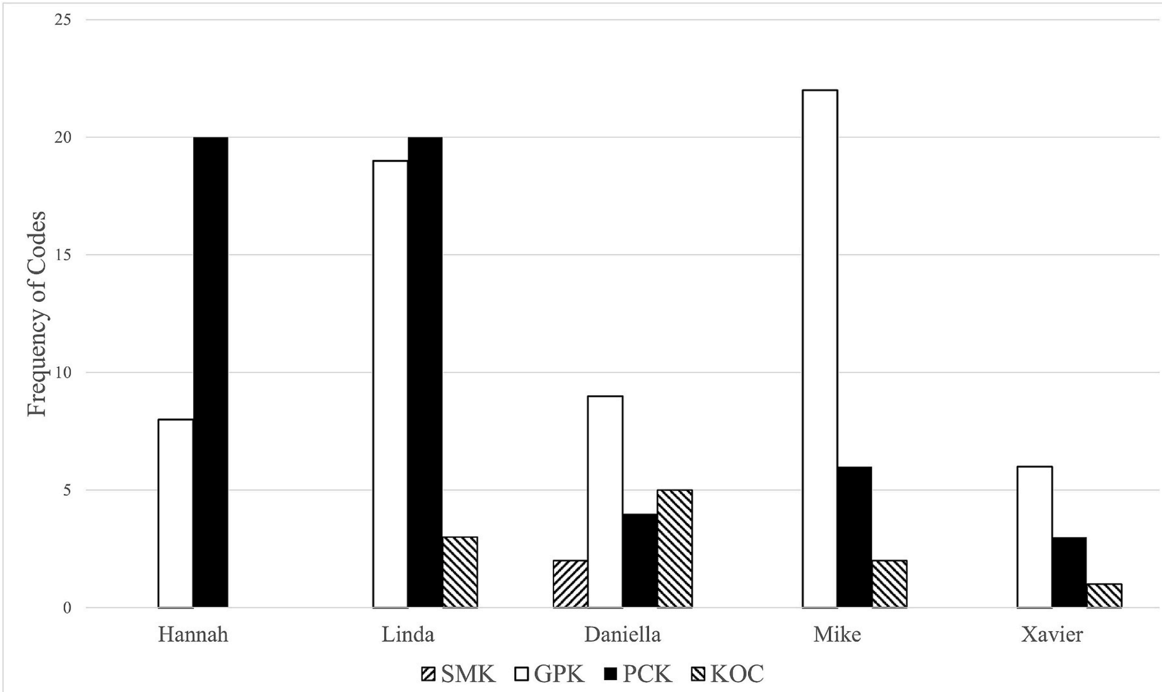

Fig. 3. *Frequency of Codes by Teacher*
Note, SMK = subject matter knowledge, GPK = general pedagogical knowledge, PCK = pedagogical content knowledge, and KOC = knowledge of context.

Provide a brief outline or overview describing how you will introduce students to the Main Problem and facilitate their work on the activity. What would you do first, second, third, etc., and what would your students be doing at each point? (Add as many rows as you need.)

What I will do	During this time, my students will be…
Read the problem to the students. Answer clarifying questions (i.e. not sharing the same burrito, burritos are the same size).	Ask clarifying questions.
Look for common strategies used by students. Look for misconceptions. Sort students by similar strategies.	Students will independently work on problem.
Choose student work to share with the class	Students will be listening and commenting on each other's strategies

What I will do	During this time, my students will be…
Pass 10 skittles to each student in the class The teacher will ask the student how many total skittles they have. The teacher will ask each student to give them half of their skittles Teacher will ask students how they knew that 5 skittles were half of 10 Teacher will introduce the words part and total. Teacher will introduce the word fraction, remember to use the academic language of part and total. Teacher will introduce video on Fractions Teacher will have students discuss with a partner about 3-5 facts they learned from.	Student will observe what materials they have and the material at their desk Students will answer approximately Students will give teacher possibly 5 skittles Students answers will vary Students will remember the terms part a total students will write down all vocabulary in vocabulary book Students will actively watch video on fractions Students will recall facts
Teacher will introduce fraction anchor chart  The teacher will allow students to practice with more skittles with terms such as whole, fourth, half.	Students will copy anchor chart
Teacher will present the main problem as a closer	Students will work alone to label the parts of a fraction

Fig. 4. Sample excerpts from lesson plans.

on our note sheet? When we reference those notes, those notes are going to be laid out in a way that all they have to do is put in their numbers." In this approach, the teacher shows the student exactly how to solve problems and expects students to follow that process without deviation (or input on divergent approaches). We can also see a clear distinction in the lesson plans from the teachers. Fig. 4, for instance, provides a side-by-side comparison between the same section of a lesson plan from Hannah and Xavier. In Hannah's lesson (the table on the left), the teacher clearly demonstrates an open approach to teaching in describing what students will do, stating that "students will independently work on [the] problem" and "students will be listening and commenting on each other's strategies" as opposed to the strategy provided by the teacher. In contrast, Xavier clearly provides students with the solution path they are expected to follow, even going so far as to describe the activity as simply copying curriculum materials for themselves.

In our analysis of the data for each lesson, we found that two of the teachers (Linda and Hannah) employed student-centered teaching every time and structured their lessons pedagogically so that the students would have to invent methods of solving novel problems based on their existing mathematical knowledge. In contrast, three of the teachers (Danielle, Mike, and Xavier) provided students with solution paths before allowing them to find a solution during each lesson and thus applied a direct approach. Our analyses showed a relationship between the approach to teaching employed by each teacher and the kinds of knowledge teachers gained from teaching that lesson. As shown in Fig. 3, Linda and Hannah, the two teachers who designed their instruction so that their students could solve the mathematics problems before discussing how to solve those problems with the teacher, reported more PCK learning than any other category of teacher knowledge. In contrast, the other three teachers, who limited opportunities to hear their students' ideas by not letting them solve mathematics problems through their own processes, did not report gaining as much PCK. On average, the teachers who used student-centered teaching reported an average of 20 PCK codes per teacher, 4.62 times as many as the average PCK codes reported by the three teachers who used direct teaching (1.44). When comparing the average frequencies of the other three categories, a far less distinct comparison emerged. On average, teachers who employed student-centered pedagogies reported and showed evidence of 4.33 GPK codes, 0.5 KOC codes, and 0 SMK codes (i.e., no evidence was shown) while teachers using teacher-centered pedagogies reported and showed evidence of 4.22 GPK codes, 0.89 KOC codes, and 0.22 SMK codes.

Comparing the percentages and frequencies of each category applied to the teacher's total reported knowledge gain illustrates how the decision to employ a student-centered approach over a teacher-centered one played a salient role in teachers' development of PCK. As shown in Fig. 5, the teachers who used student-centered teaching had a much higher percentage of codes related to PCK (60.10%) when compared with the teachers who employed a teacher-centered approach (23.12%). Taken together, these findings suggest that allowing students to agentically work on problems appears to create room for teachers to develop knowledge and skills specific to the work of mathematics teaching. In line with and building on previous research into learning through the application of open approaches to teaching (Leikin, 2010; Yeh, 2016), these findings both reassert that teachers who employ an open pedagogical approach (without the support of professional development initiatives) learn through their encounters with student thinking and reveal that the added exposure to student's novel solution paths during an open approach to teaching provides a significant advantage over direct instruction (in terms of developing PCK).

5. Discussion

Drawing on the importance of reflection in LTT described by multiple scholars (Leikin, 2006; Santagata et al., 2018; Schön, 1987), we build on previous research by recognizing the breadth of knowledge that teachers can recognize and acquire through the act of teaching on their own. In constructing 40 different codes that covered all four areas of teacher knowledge (Grossman, 1990), we provide evidence in this study of the multitude of skills and types of knowledge that teachers can recognize in their own LTT processes. However, it is important to call attention to the fact that this qualitative study is based on the learning patterns of only five teachers. Thus, the generalizability of our findings is limited and requires future work with a nationally representative sample of teachers to explore the replicability of our findings. Our study is explorative in nature in that we collected rich and extensive data from a small group of teachers to identify areas where teacher learning occurred, as opposed to broadly distributed data from a large population that may lack the same depth.

Turning towards the details of our analysis, our findings indicate that

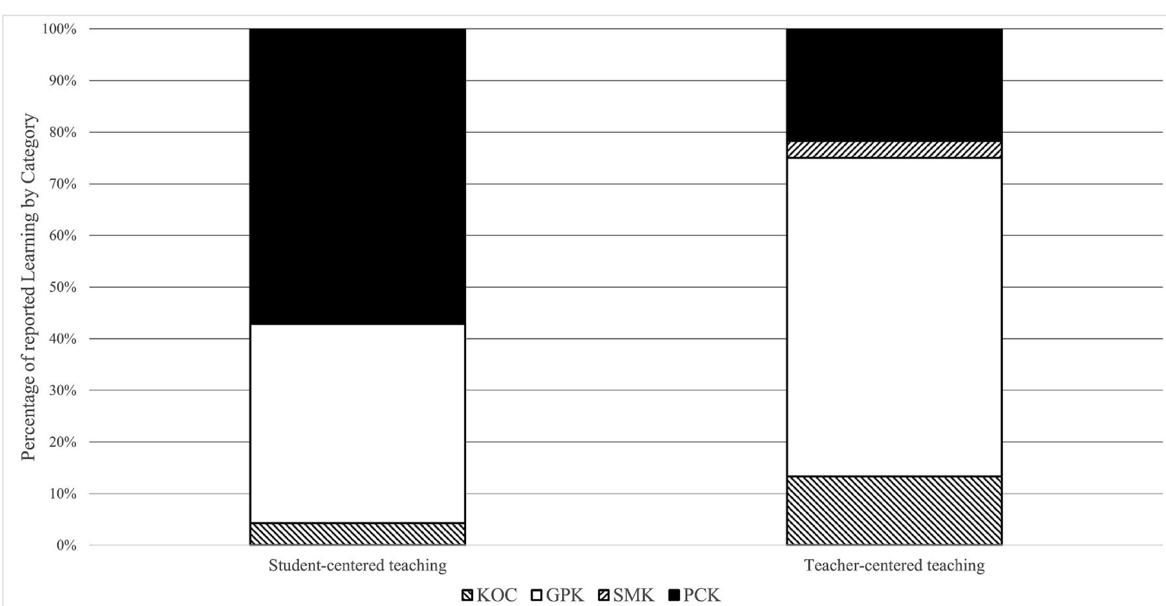

Fig. 5. *Percentage of Code Applications by Pedagogical Approach*
Note: Smk = subject matter knowledge, GPK = general pedagogical knowledge, PCK = pedagogical content knowledge, and KOC = knowledge of context.

participating teachers discussed their learning of GPK and PCK more often than their learning of SMK and KOC. This finding therefore raises questions about the efficacy of LTT in attending to all aspects of teacher knowledge when divorced from the support of teacher educators, particularly SMK and KOC (both of which appeared at an extremely low frequency in the data we collected). While previous studies by Leikin (2006, 2010) and Leikin et al. (2000) have shown that teachers can learn about SMK through LTT processes, these studies have relied on developing interventions for teachers that intentionally and specifically engage teachers in the process of developing SMK. In terms of using LTT to develop KOC, the small number of codes and code applications reiterate the call for researchers to more thoroughly explore multiple approaches to LTT (see Fishman & Davis, 2006; Monte-Sano & Budano, 2013), producing frameworks for researchers and tools for teachers to recognize, analyze, and explore this specific aspect of teacher learning in practice. Left to their own devices (as is the case in this study), teachers may not have opportunities to develop these aspects of their own knowledge or recognize their own development of SMK or KOC. In turn, this study implies the need for continued and targeted interventions with teachers that build various forms of teacher knowledge through LTT. Alternately, future research should consider designing and implementing studies that specifically attend to this form of teacher knowledge (as opposed to the broad and open methodology taken here).

Additionally, our findings indicate that the kinds of knowledge teachers developed were contingent on the structure of the lesson. Teachers who employed a guided approach and centered their lessons on solely communicating prescripted solution strategies to students largely developed GPK, representing 59.73% of code applications for the teachers employing a teacher-centered approach (compared to 23.12% of code applications relating to PCK). In contrast, participants who used a student-centered approach recognized a similar amount of GPK development (with the open approach teachers averaging 4.33 GPK code applications per cycle compared to 4.22 code applications for the guided approach teachers) along with a wider breadth of PCK development as well (6.67 code applications per cycle compared to 1.44 applications). These findings therefore build on Leikin (2010) and Yeh's (2016) assertions that learning experiences where students can respond to problems in unique and novel ways provide a greater opportunity for LTT by creating opportunities for teachers to encounter and reflect on examples of student thinking or problem solving. More than developing a broad set of decontextualized teaching practices (such as classroom management skills or how to structure a lesson), teachers in this study who embraced an open approach to teaching reported developing mathematics-specific teaching knowledge through LTT. Although this data remains correlational, future research with a nationally representative sample of teachers can build on this and previous studies to further explore this alignment between open approaches to teaching and both GPK and PCK.

Moreover, these findings build on the work of scholars that assert the value of student problem solving within mathematics education (see Ainley et al., 2006; Mackrell & Pratt, 2017; Noss & Hoyles, 1996; Papert, 1980) by recognizing and amplifying the value this pedagogical approach holds for teachers as well. By shifting the focus onto teacher knowledge development, our findings show that LTT occurs as teachers join in on this discourse and work with students as they develop solutions to problems themselves. According to Leikin (2010), this process not only engages teachers PCK but their SMK as well, with teachers relying on their mathematical knowledge to explore new mathematical ideas proposed by students. This creates a mutually beneficial process where teachers construct PCK in response to SMK and vice versa. While our findings did not reveal the same depth of SMK knowledge being developed through teaching, the fact that teachers did develop a significant breadth of PCK related knowledges and skills when allowing students to solve problems on their own reinforces the argument made by Leikin (2010) and asserts the value of student-centered teaching for teachers and not just students, as reported by Sengupta-Irving and Enyedy (2015). Future research can build on these findings by exploring the details of interactions between students and teachers within this pedagogical approach and unearthing how these details contribute to LTT.

6. Conclusion

If, as Sherin (2002) argues, learning through teaching represents a vital process through which teachers develop content and pedagogical knowledge, then researchers and educators need to understand both what teachers learn when they teach and the practices through which teachers develop this knowledge outside of professional development contexts and teacher education initiatives. But just as different pedagogical experiences result in different kinds of learning for students, different approaches to teaching also result in teachers learning different kinds of knowledge as well. Stated differently, "expertise grows through personal experience, even if different experiences lead to different levels of expertise" (Leikin & Zazkis, 2010, p. 5). This study contributes to this research initiative by exploring the differences in LTT. In doing so, we argue for the value of student-centered teaching over direct instruction, providing evidence that teachers develop a broader range of pedagogical content knowledge when encountering student's novel problem solving approaches and mathematical thinking within learning contexts. In turn, we not only call on researchers to continue exploring the value of student-centered teaching within LTT but also encourage mathematics educators to structure their pedagogy around allowing students to solve mathematics problems through their own invented solution paths. We do so not only because of the benefits for teachers but because of the previously established benefits for students as well. Although the findings of this study should be interpreted with caution because of the small number of participants, the fact that the majority of these teachers (three out of five) still relied on teacher-centered pedagogical approaches illustrates the importance of not only understanding the value of student-centered pedagogies for teachers but also finding ways to help teachers embrace and engage with this approach to teaching. While future research should explicitly explore how to support the development of teacher's KOC and SMK within these contexts, our findings reveal that student-centered teaching inherently provides a fruitful context for teachers to develop pedagogical content knowledge, creating space for teachers to continue to develop their craft within their classroom while students construct their own knowledge in parallel.

Funding

Work on this paper was supported by the NSF CAREER grant (#1751309) from the National Science Foundation. All opinions and conclusions in this paper are those of the authors and do not necessarily reflect the views of the funding agency.

Declaration of competing interest

The authors whose names are listed immediately below certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data availability

The data that has been used is confidential.

References

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal, 32*(1), 23–38. https://doi.org/10.1080/01411920500401971

Ali, D., Mz, Z. A., Kusnadi, K., & Vebrianto, R. (2021). Literature review: Mathematical creative thinking ability, and students' self regulated learning to use an open ended approach. *Malikussaleh Journal of Mathematics Learning (MJML), 4*(1). https://doi.org/10.29103/mjml.v4i1.3095. Article 1.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.

Bausell, R. B., & Moody, W. B. (1974). Learning through doing in teacher education: A proposal. *Arithmetic Teacher, 21*(5), 436–438. JSTOR.

Ben-Peretz, M. (2011). Teacher knowledge: What is it? How do we uncover it? What are its implications for schooling? *Teaching and Teacher Education, 27*(1), 3–9. https://doi.org/10.1016/j.tate.2010.07.015

Bland, S. (2019). In *Secondary educators use of guided inquiry to increase student engagement in mathematics*. Northcentral University. from https://www.proquest.com/docview/2323920046/abstract/8B9095F0E561472DPQ/1. (Accessed 21 October 2022).

Blömeke, S., Busse, A., Kaiser, G., König, J., & Suhl, U. (2016). The relation between content-specific and general teacher knowledge and skills. *Teaching and Teacher Education, 56*, 35–46.

Chang, S. H., Ward, P., & Goodway, J. D. (2020). The effect of a content knowledge teacher professional workshop on enacted pedagogical content knowledge and student learning in a throwing unit. *Physical Education and Sport Pedagogy, 25*(5), 493–508.

Chapman, O. (2007). Preservice secondary mathematics teachers' knowledge and inquiry teaching approaches. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, 2*, 97–104.

Cobb, P., & Mcclain, K. (2001). An approach for supporting teachers' learning in social context. In F.-L. Lin, & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 207–231). Springer Netherlands. https://doi.org/10.1007/978-94-010-0828-0_10.

Copur-Gencturk, Y. (2021). Teachers' conceptual understanding of fraction operations: Results from a national sample of elementary school teachers. *Educational Studies in Mathematics, 107*, 525–545. https://doi.org/10.1007/s10649-021-10033-4

Copur-Gencturk, Y., & Doleck, T. (2021). Strategic competence for multistep fraction word problems: An overlooked aspect of mathematical knowledge for teaching. *Educational Studies in Mathematics, 107*(1), 49–70.

Copur-Gencturk, Y., & Li, J. (2023). Teaching matters: A longitudinal study of mathematics teachers' knowledge growth. *Teaching and Teacher Education, 121*, Article 103949. https://doi.org/10.1016/j.tate.2022.103949

Copur-Gencturk, Y., & Tolar, T. (2022). Mathematics teaching expertise: A study of the dimensionality of content knowledge, pedagogical content knowledge, and content-specific noticing skills. *Teaching and Teacher Education, 114*, Article 103696. https://doi.org/10.1016/j.tate.2022.103696

Copur-Gencturk, Y., Tolar, T., Jacobson, E., & Fan, W. (2019). An empirical study of the dimensionality of the mathematical knowledge for teaching construct. *Journal of Teacher Education, 70*(5), 485–497. https://doi.org/10.1177/0022487118761860

Denzin, N. K. (2012). Triangulation 2.0. *Journal of Mixed Methods Research, 6*(2), 80–88.

Dyer, E. B. (2016). *Learning through teaching: An exploration of teachers' use of everyday classroom experiences as feedback to develop responsive teaching in mathematics*. Northwestern University.

Elmendorf, H. G. (2006). Learning through teaching: A new perspective on entering a discipline. *Change: The Magazine of Higher Learning, 38*(6), 36–41. https://doi.org/10.3200/CHNG.38.6.36-41.

Eysink, T. H. S., de Jong, T., Berthold, K., Kolloffel, B., Opfermann, M., & Wouters, P. (2009). Learner performance in multimedia learning arrangements: An analysis across instructional approaches. *American Educational Research Journal, 46*(4), 1107–1149. https://doi.org/10.3102/0002831209340235

Fennema, E., & Franke, M. L. (1992). *Teachers' knowledge and its impact*.

Fishman, B. J., & Davis, E. A. (2006). Teacher learning research and the learning sciences. In *The Cambridge handbook of: The learning sciences* (pp. 535–550). Cambridge University Press.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653–689. https://doi.org/10.3102/00028312038003653

Gay, G. (2018). *Culturally responsive teaching: Theory, research, and practice*. teachers college press.

González, N., Moll, L. C., & Amanti, C. (2006). *Funds of knowledge: Theorizing practices in households, communities, and classrooms*. Routledge.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. Teachers College Press, Teachers College, Columbia University.

Harry, B., Sturges, K. M., & Klingner, J. K. (2005). Mapping the process: An exemplar of process and challenge in grounded theory analysis. *Educational Researcher, 34*(2), 3–13.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education*. Springer.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*(1), 47–61. https://doi.org/10.1177/0022487106295726

Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to kirschner, sweller, and clark (2006). *Educational Psychologist, 42*(2), 99–107. https://doi.org/10.1080/00461520701263368

Jackiw, N., & Sinclair, N. (2010). Learning through teaching, when teaching machines: Discursive interaction design in sketchpad. In R. Leikin, & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 153–168). Springer.

Jacob, F., John, S., & Gwany, D. M. (2020). Teachers' pedagogical content knowledge and students' academic achievement: A theoretical overview. *Journal of Global Research in Education and Social Science, 14*(2), 14–44.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM, 48*(1), 185–197. https://doi.org/10.1007/s11858-015-0717-0

Jasien, L., & Horn, I. S. (2018). "Ohhh, now I can do it!": School-age children's spontaneous mathematical sensemaking in construction play. In J. Kay, & R. Luckin (Eds.), *Rethinking learning in the digital age: Making the learning sciences count, 13th international conference of the learning sciences (ICLS) 2018* (Vol. 1, pp. 624–631). International Society of the Learning Sciences. https://repository.isls.org//handle/1/912.

Kereluik, K., Mishra, P., Fahnoe, C., & Terry, L. (2013). What knowledge is of most worth. *Journal of Digital Learning in Teacher Education, 29*(4), 127–140. https://doi.org/10.1080/21532974.2013.10784716

Kilpatrick, J., Blume, G., Heid, M. K., Wilson, J., Wilson, P., & Zbiek, R. M. (2015). Mathematical understanding for secondary teaching: A framework. In K. Heid, P. Wilson, & G. Blume (Eds.), *Mathematical understanding for secondary teaching: A framework and classroom-based situations* (pp. 9–30). Information Age Publishing.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology, 100*(3), 716.

Ladson-Billings, G. (2021). *Culturally relevant pedagogy: Asking a different question*. Teachers College Press.

Lai, Y., Weber, K., & Mejía-Ramos, J. P. (2012). Mathematicians' perspectives on features of a good pedagogical proof. *Cognition and Instruction, 30*(2), 146–169.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. Yale University Press.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226–243. https://doi.org/10.1177/0022487112473837

Landt, S. M. (2003). *Learning through teaching: How teachers use the role of cooperation teacher to improve their own practice*.

Leikin, R. (2006). Learning by teaching: The case of Sieve of Eratosthenes and one elementary school teacher. *Number Theory in Mathematics Education: Perspectives and Prospects*, 115–140.

Leikin, R. (2010). Learning through teaching through the lens of multiple solution tasks. In R. Leikin, & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 69–85). Springer Netherlands. https://doi.org/10.1007/978-90-481-3990-3_4.

Leikin, R., Berman, A., & Zaslavsky, O. (2000). Learning through teaching: The case of symmetry. *Mathematics Education Research Journal, 12*(1), 18–36. https://doi.org/10.1007/BF03217072

Leikin, R., & Rota, S. (2006). Learning through teaching: A case study on the development of a mathematics teacher's proficiency in managing an inquiry-based classroom. *Mathematics Education Research Journal, 18*(3), 44–68. https://doi.org/10.1007/BF03217442

Leikin, R., & Zazkis, R. (Eds.). (2010). *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice*. Springer. https://search.library.wisc.edu/catalog/9910090749502121.

Levav-Waynberg, A., & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *The Journal of Mathematical Behavior, 31*(1), 73–90. https://doi.org/10.1016/j.jmathb.2011.11.001

Liljedahl, P. (2007). TEACHERS'LEARNING reified: The professional growth of INSERVICE teachers through numeracy task design. *Proceedings of the 31st International Conference for the Psychology of Mathematics Education, 1*, 132–141.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences, 25*(3), 335–371. https://doi.org/10.1080/10508406.2016.1180297.

Mackrell, K., & Pratt, D. (2017). Constructionism and the space of reasons. *Mathematics Education Research Journal, 29*(4), 419–435. https://doi.org/10.1007/s13394-017-0194-6

Mason, J. (2002). *Researching your own practice: The discipline of noticing* (1st ed.). Routledge.

McDonald, M., Kazemi, E., Kelley-Petersen, M., Mikolasy, K., Thompson, J., Valencia, S. W., & Windschitl, M. (2014). Practice makes practice: Learning to teach in teacher education. *Peabody Journal of Education, 89*(4), 500–515. https://doi.org/10.1080/0161956X.2014.938997

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017–1054.

Monte-Sano, C., & Budano, C. (2013). Developing and enacting pedagogical content knowledge for teaching history: An exploration of two novice teachers' growth over three years. *Journal of the Learning Sciences, 22*(2), 171–211. https://doi.org/10.1080/10508406.2012.742016.

Morris, A. K., & Hiebert, J. (2017). Effects of teacher preparation courses: Do graduates use what they learned to plan mathematics lessons? *American Educational Research Journal, 54*(3), 524–567.
Mukhopadhyay, S., Powell, A. B., & Frankenstein, M. (2009). An ethnomathematical perspective on culturally responsive mathematics education. In B. Greer, S. Mukhopadhyay, A. B. Powell, & S. Nelson-Barber (Eds.), *Culturally responsive mathematics education* (pp. 65–84). Routledge.
National Research Council. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.
Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers* (Vol. 17). Springer Science & Business Media.
Ogletree, G. L. (2007). *The effect of fifth grade science teachers' pedagogical content knowledge on their decision making and student learning outcomes on the concept of chemical change*. Ph.D., The University of Alabama. https://www.proquest.com/docview/304892963/abstract/ACF446E1982C45F7PQ/1.
Okita, S. Y., & Schwartz, D. L. (2013). Learning by teaching human pupils and teachable agents: The importance of recursive feedback. *The Journal of the Learning Sciences, 22* (3), 375–412. https://doi.org/10.1080/10508406.2013.807263
Olfos, R., Goldrine, T., & Estrella, S. (2014). Teachers' pedagogical content knowledge and its relation with students' understanding. *Revista Brasileira de Educação, 19*, 913–944.
Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Basic Books.
Perkins, R., Aufegger, L., & Williamon, A. (2015). Learning through teaching: Exploring what conservatoire students learn from teaching beginner older adults. *International Journal of Music Education, 33*(1), 80–90. https://doi.org/10.1177/0255761414531544
Purwoko, R. Y., Nugraheni, P., & Instanti, D. (2019). Implementation of pedagogical content knowledge model in mathematics learning for high school. *Journal of Physics: Conference Series, 1254*(1), Article 012079.
Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*, 587–597.
Saldaña, J. (2015). *The coding manual for qualitative researchers* (3rd ed.). Sage Publications Ltd.
Salmon, D., Ko, E. K., Freedman, R., & Zifkin, J. (2020). Adaptive cycles of teaching: Exploring pre-service teacher learning through teaching. *International Perspectives on Knowledge Integration*, 255–289. https://doi.org/10.1163/9789004429499_012
Santagata, R., Yeh, C., & Mercado, J. (2018). Preparing elementary school teachers to learn from teaching: A comparison of two approaches to mathematics methods instruction. *The Journal of the Learning Sciences, 27*(3), 474–516. https://doi.org/10.1080/10508406.2018.1441030
Scanlon, D., MacPhail, A., & Calderón, A. (2022). A rhizomatic exploration of a professional development non-linear approach to learning and teaching: Two teachers' learning journeys in "becoming different.". *Teaching and Teacher Education, 115*, Article 103730. https://doi.org/10.1016/j.tate.2022.103730
Schön, D. A. (1987). *Educating the reflective practitioner: Toward a new design for teaching and learning in the professions* (1st ed.). Jossey-Bass.
Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly guided inquiry. *The Journal of the Learning Sciences, 24*(4), 550–592. https://doi.org/10.1080/10508406.2014.928214
Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119–150. https://doi.org/10.1207/S1532690XCI2002_1
Sherin, M. G., Jacobs, V. R., & Phillip, R. A. (2011). Situating the study of teacher noticing. In *Mathematics teacher noticing* (pp. 3–13). Routledge.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.
Simon, M. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema, & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55–86). Lawrence Erlbaum Associates Publishers.
Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30*(3), 252–264. https://doi.org/10.2307/749835
Smith, M. S., Bill, V., & Hughes, E. K. (2008). Thinking through a lesson: Successfully implementing high-level tasks. *Mathematics Teaching in the Middle School, 14*(3), 132–138.
Tatto, M. T., Ingvarson, L., Schwille, J., Peck, R., Senk, S. L., & Rowley, G. (2008). Teacher education and development study in mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework. In *International association for the evaluation of educational achievement*. International Association for the Evaluation of Educational Achievement. https://eric.ed.gov/?id=ED542390.
Thadani, V., Roth, K. J., Garnier, H. E., Seyarto, M. C., Thompson, J. L., & Froidevaux, N. M. (2017). What can a cognitive coding framework reveal about the effects of professional development on classroom teaching and learning? *The Journal of the Learning Sciences, 27*(4), 517–549. https://doi.org/10.1080/10508406.2017.1396220
Thomas, C. A., & Berry III, R. Q. (2019). A qualitative metasynthesis of culturally relevant pedagogy & culturally responsive teaching: Unpacking mathematics teaching practices. *Journal of Mathematics Education at Teachers College, 10*(1). Article 1 https://doi.org/10.7916/jmetc.v10i1.1668.
Tzur, R. (2010). How and what might teachers learn through teaching mathematics: Contributions to closing an unspoken gap. In R. Leikin, & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 49–68). Springer.
Waite, S., & Pratt, N. (2015). Situated learning (learning in situ). In J. D. Wright (Ed.), *International encyclopedia of the social & behavioral sciences* (2nd ed., pp. 5–12). Elsevier https://doi.org/10.1016/B978-0-08-097086-8.92042-7.
Yeh, C. (2016). *Mathematics, language, and learning: A longitudinal study of elementary teachers and their mathematics teaching practices*. Irvine]: Ph.D., University of California. from https://www.proquest.com/docview/1796371250/abstract/1485736C514A4E2BPQ/1. (Accessed 21 October 2022).
Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education, 7*(1), 5–32. https://doi.org/10.1023/B:JMTE.0000009971.13834.e1
Zazkis, R., & Mamolo, A. (2018). From disturbance to task design, or a story of a rectangular lake. *Journal of Mathematics Teacher Education, 21*(5), 501–516. https://doi.org/10.1007/s10857-016-9361-z