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Poisson derivations of a semiclassical limit of a family of quantum second Weyl algebras

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ABSTRACT

In [16], we studied deformations $A_{\alpha,\beta}$ of the second Weyl algebra and computed their derivations. In the present paper, we identify the semiclassical limits $\mathcal{A}_{\alpha,\beta}$ of these deformations and compute their Poisson derivations. Our results show that the first Hochschild cohomology group $\mathrm{HH}^1(A_{\alpha,\beta})$ is isomorphic to the first Poisson cohomology group $\mathrm{HP}^1(\mathcal{A}_{\alpha,\beta})$.

1. Introduction

Throughout this study, $\mathbb{K}$ will denote a field of characteristic zero.

Given a non-commutative algebra A and its semiclassical limit $\mathcal{A}$, an intriguing question has always been “Do the properties of A always reflect the (Poisson) properties of $\mathcal{A}$?” For example, given the centre, automorphisms, endomorphisms, derivations, and prime ideals of A, can one successfully predict/conjecture the Poisson centre, Poisson automorphisms, Poisson endomorphisms, Poisson derivations and Poisson prime ideals of $\mathcal{A}$? Suppose for instance that the prime spectrum of A reflects the Poisson prime spectrum of $\mathcal{A}$. A follow-on question will be whether they are homeomorphic/isomorphic? The answers to these questions, in some specific cases and for some specific algebras, are affirmative. For example, Goodearl [7] has conjectured that the prime and primitive spectra of the quantized coordinate rings are respectively homeomorphic to the Poisson prime and Poisson primitive spectra of their corresponding semiclassical limits when the base field is algebraically closed and of characteristic zero. This conjecture has been verified for the following quantized coordinate rings: $\mathcal{O}_q(\mathbb{K}^n)$ (see [10, Theorem 4.1]), $\mathcal{O}_q(SL_2(\mathbb{K}))$ (see [7, Example 9.7]), $\mathcal{O}_q(SL_3(\mathbb{K}))$ (see [6, Theorem 5.21 & Corollary 5.22]) and $\mathcal{O}_q(GL_2)$ (see [6, Corollary 5.23]). Moreover, the prime and primitive spectra of the enveloping algebra $U(\mathfrak{g})$ of a solvable finite dimensional complex Lie algebra $\mathfrak{g}$ are respectively homeomorphic to the prime and primitive spectra of its semiclassical limit (see [7, Theorem 8.11, Example 2.6]). From [4], we also have that the Poisson endomorphisms of the Poisson quantum generalized Weyl algebra are precisely the Poisson analogue of the endomorphisms of the quantum generalised Weyl algebra. Belov-Kanel and Kontsevich [1] have also conjectured that the group of automorphisms of an

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n^{th}-Weyl algebra $A_n(\mathbb{K})$ is isomorphic to the group of Poisson automorphisms of the corresponding Poisson Weyl algebra in characteristic zero.

In [16], we studied a family of simple quotients

$$A_{\alpha,\beta} := U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \quad (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$$

of the positive part of the quantized enveloping algebra $U_q(G_2)$, and concluded that the algebra $A_{\alpha,\beta}$ is a q-deformation of a quadratic extension of the second Weyl algebra $A_2(\mathbb{K})$. Since $A_{\alpha,\beta}$ deforms approximately to $A_2(\mathbb{K})$, it is considered as a *quantum second Weyl algebra*. Our goal here is to study a semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$, and compare its Lie algebra of Poisson derivations to the Lie algebra of derivations of $A_{\alpha,\beta}$ studied in [16]. Another property that is also worth investigating is the Belov-Kanel and Kontsevich conjecture [1]. Thus, it is natural to ask if the automorphism group of $A_{\alpha,\beta}$ is isomorphic to the Poisson automorphism group of $\mathcal{A}_{\alpha,\beta}$. We will return to this problem in the near future after we have successfully studied the automorphism group of $A_{\alpha,\beta}$. In the present case and as already mentioned, we only focus on studying the Lie algebra of Poisson derivations of $\mathcal{A}_{\alpha,\beta}$, and comparing them to their non-commutative counterparts in [16].

In the noncommutative world, the knowledge of the derivations of twisted group algebras, studied by Osborn and Passman [20], has helped in studying the derivations of other non-commutative algebras such as the quantum second Weyl algebra (see [16]), quantum matrices (see [14]), generalized Weyl algebras (see [12]) and some specific examples of quantum enveloping algebras (see [15], [21], and [22]). In view of this, we also study the Poisson derivations of the Poisson analogue of the twisted group algebras—called Poisson group algebras—and apply the results to study the Poisson derivations of a semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$. The rest of the paper is organised as follows.

In Section 2, we recall some basics on Poisson algebras and semiclassical limit. We then proceed to study the Poisson derivations of the Poisson group algebras. Similarly to their non-commutative counterparts in [20], every Poisson derivation of a Poisson group algebra is the sum of an inner Poisson derivation and a central/scalar Poisson derivation.

In Section 3, we study a semiclassical limit $\mathcal{A}$ of the quantum algebra $U_q^+(G_2)$ and establish that $\mathcal{A}$ is a Poisson polynomial $\mathbb{K}$-algebra generated by six indeterminates $X_1, \ldots, X_6$. The Poisson algebra $\mathcal{A} = \mathbb{K}[X_1, \ldots, X_6]$ supports the rational action of a torus by Poisson automorphisms, and satisfies the conditions in [13, Hypothesis 1.7]. Hence, we can apply the Poisson deleting derivations algorithm [13] to study its Poisson spectrum, and its Poisson centre, a (commutative) polynomial ring $\mathbb{K}[\Omega_1, \Omega_2]$ in two variables. In Section 4, we study some Poisson $\mathcal{H}$-prime ideals of $\mathcal{A}$ using Goodearl's $\mathcal{H}$-stratification theory [8], and proceed to study a family $(\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle)_{((\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\})}$ of maximal and primitive Poisson prime ideals of $\mathcal{A}$. Consequently, we study their corresponding Poisson simple quotients

$$\mathcal{A}_{\alpha,\beta} := \mathbb{K}[X_1, \ldots, X_6]/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle,$$

and conclude that the Poisson algebra $\mathcal{A}_{\alpha,\beta}$ is a semiclassical limit of the quantum second Weyl algebra $A_{\alpha,\beta}$. Having a complete description of the semiclassical limit $\mathcal{A}_{\alpha,\beta}$ of $A_{\alpha,\beta}$, we proceed to study its Poisson derivations in the final section of this paper, by following procedures similar to its non-commutative counterpart $A_{\alpha,\beta}$ (see [16, §5]). That is, we successively embed $\mathcal{A}_{\alpha,\beta}$ into a suitable Poisson torus $\mathcal{R}_3$ via successive localizations as follows:

$$\mathcal{A}_{\alpha,\beta} = \mathcal{R}_7 \subset \mathcal{R}_6 = \mathcal{R}_7 \Sigma_6^{-1} \subset \mathcal{R}_5 = \mathcal{R}_6 \Sigma_5^{-1} \subset \mathcal{R}_4 = \mathcal{R}_5 \Sigma_4^{-1} \subset \mathcal{R}_3. \tag{1}$$

These embeddings and localizations allow us to extend every Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ successively and uniquely to a Poisson derivation of each of the Poisson algebras $\mathcal{R}_i$ through to the Poisson torus $\mathcal{R}_3$. Since a Poisson torus is an example of a Poisson group algebra, we have that every Poisson derivation of $\mathcal{R}_3$ is the sum of an inner Poisson derivation and a central/scalar Poisson derivation. Given the Poisson derivations of $\mathcal{R}_3$, we backwardly and successively pull the Poisson derivations of $\mathcal{R}_3$ to $\mathcal{A}_{\alpha,\beta}$ using the constraint that our Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ stabilises all Poisson algebras from (1). This gives a complete description of the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. Similarly to their non-commutative counterparts in [16], every Poisson derivation of $\mathcal{A}_{\alpha,\beta}$ is an inner Poisson derivations provided $\alpha\beta \neq 0$, and the sum of inner Poisson and scalar Poisson derivations whenever α or β is zero. More precisely, the first Poisson cohomology group $\mathrm{HP}^1(\mathcal{A}_{\alpha,\beta})$ is a one-dimensional vector space in the case where α or β is zero (but not both).

2. Poisson derivations of Poisson group algebras

This section begins with a reminder about Poisson algebras and semiclassical limits. We will then proceed to introduce Poisson group algebras and, consequently, study their derivations.

2.1. Poisson algebras

A *Poisson algebra* $\mathcal{A}$ is a commutative algebra over $\mathbb{K}$ endowed with a skew-symmetric $\mathbb{K}$-bilinear map $\{-,-\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ which satisfies the Leibniz rule (i.e., $\{x, yz\} = \{x, y\}z + y\{x, z\};\ x, y, z \in \mathcal{A}$) and Jacobi identity (i.e., $\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0;\ x, y, z \in \mathcal{A}$).

The map $\{-,-\}$ is called the *Poisson bracket*. From [17, Prop. 1.7], we have that every Poisson bracket of $\mathcal{A}$ extends uniquely to the localizations of $\mathcal{A}$. A *Poisson ideal* of $\mathcal{A}$ is any ideal I such that $\{\mathcal{A}, I\} \subseteq I$. Given a Poisson ideal I of $\mathcal{A}$, it is well known that the quotient algebra $\mathcal{A}/I$ is a Poisson algebra with an induced Poisson bracket defined by $\{\bar{x}, \bar{y}\} = \overline{\{x, y\}}$, where $\bar{z} := z + I$ for all $z \in \mathcal{A}$. Finally, the subalgebra $Z_P(\mathcal{A}) := \{a \in \mathcal{A} \mid \{a, x\} = 0, \ \forall x \in \mathcal{A}\}$ is called the *Poisson centre* of $\mathcal{A}$.

Remark 2.1. If $\mathcal{A}$ is a Poisson algebra and $\{x_1, \dots, x_n\}$ is a generating set for $\mathcal{A}$ (as an algebra), then

(1) it is always enough to define a Poisson bracket $\{-, -\}$ on $\mathcal{A}$ by defining it on only the generating set.

(2) for all $f, g \in \mathcal{A}$, we have that $\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ (see [7, Example 2.2(a)]).

2.2. Semiclassical limit

Given a non-commutative algebra, one can move from the 'Non-commutative World' to the 'Poisson World' through a process called *semiclassical limit*, and reverse this process through *quantization*. This transformation (semiclassical limit) and its reverse transformation (quantization) have been widely studied (for example, see [5, §§1.1.3], [2, Chapter III.5], and [9, §2]). In line with the presentation in [5, §§1.1.3], we present the following overview of semiclassical limit. Let R be a commutative principal ideal domain containing the field $\mathbb{K}$ and hR be a maximal ideal of R for a fixed $h \in R$. Let A be an algebra which is not necessarily commutative torsion-free R-algebra such that the quotient $\mathcal{A} := A/hA$ is a commutative algebra. For $u, v \in A$; we have that $\bar{u} := u + hA$ and $\bar{v} := v + hA$ are their respective canonical images in $\mathcal{A}$. Since $\bar{u}\bar{v} = \bar{v}\bar{u}$, we have that $[u, v] := uv - vu \in hA$. There exists a unique element $\gamma(u, v)$ of A such that $[u, v] = h\gamma(u, v)$. It follows that

$$\{\bar{u}, \bar{v}\} := \gamma(u, v) + hA = \frac{[u, v]}{h} + hA$$

defines a Poisson bracket on $\mathcal{A}$ (see [5, §§1.1.3] for further details). We say that A is a *quantization* of $\mathcal{A}$, and $\mathcal{A}$ is a *semiclassical limit* of A. Fix $\lambda \in \mathbb{K}$. The algebra $\mathcal{A}_\lambda := A/(h - \lambda)A$ is a *deformation* of the Poisson algebra $\mathcal{A} = \mathcal{A}_0$ if the central element $h - \lambda$ is not invertible in A. We refer the interested reader to [9, §2] for some known examples of semiclassical limits of some families of quantum algebras.

2.3. Introduction to Poisson group algebras

In [20, §1&2], Osborn and Passman studied the derivations of twisted group algebras. In line with their results, we also study the Poisson derivations of Poisson group algebras. The results in this section will be crucial in the final section of this paper where we study the Poisson derivations of a semiclassical limit of the quantum second Weyl algebra $A_{\alpha,\beta}$.

Let G represent a finitely generated abelian group and $\lambda : G \times G \longrightarrow \mathbb{K}$ be a map such that $\lambda(y, x) = -\lambda(x, y)$ and $\lambda(x, yz) = \lambda(x, y) + \lambda(x, z)$. We define a *Poisson group algebra* $\mathbb{K}_P^\lambda[G]$ as a commutative $\mathbb{K}$-algebra which has a copy $\overline{G} := \{\bar{g} \mid g \in G\}$ of G as a basis and define the Poisson bracket via $\{\bar{x}, \bar{y}\} = \lambda(x, y)\bar{x}\bar{y} = \lambda(x, y)\overline{xy}$ for all $x, y \in G$ (note that $\bar{x}\bar{y} = \overline{xy}$). Observe that $\lambda(x, y) = 0$ if and only if $\{\bar{x}, \bar{y}\} = 0$.

Note that $\mathbb{K}_P^\lambda[\mathbb{Z}^n] = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ as a (commutative) $\mathbb{K}$-algebra, where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{Z}^n$ and $X_i := \overline{e_i}$ for all i. Moreover, the Poisson bracket is given by $\{X_i, X_j\} = \lambda(e_i, e_j)X_iX_j$. Conversely, if $M = (\mu_{i,j})$ is a skew-symmetric matrix, we define a Poisson bracket on $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ by setting $\{X_i, X_j\} = \mu_{i,j}X_iX_j$ for all i, j. This is a Poisson group algebra called the *Poisson torus associated to* M.

Let $\gamma \in \mathbb{K}_P^\lambda[G]$. One can write γ as $\gamma = \sum_{g \in G} c_g \bar{g}$, where $c_g \in \mathbb{K}$. Note that $c_g = 0$ for almost all g. The set $\mathrm{supp}(\gamma) := \{g \in G \mid c_g \neq 0 \text{ in } \gamma\}$ is called the *support* of γ.

Given a subset H of G, we set

$$\mathbb{K}_P^\lambda[H] := \{\gamma \in \mathbb{K}_P^\lambda[G] \mid \mathrm{supp}(\gamma) \subseteq H\} \subseteq \mathbb{K}_P^\lambda[G].$$

Remark 2.2.

1. For $x \in G$, we have that $\mathbb{K}_P^\lambda[Hx] = \mathbb{K}_P^\lambda[H]\bar{x}$, where $Hx := \{hx \mid h \in H\}$.
2. We deduce from [20, Sec. 1] that if H is a subsemigroup of G (with the identity e), then $\mathbb{K}_P^\lambda[H]$ is a Poisson subalgebra of $\mathbb{K}_P^\lambda[G]$.

The set $C := \{g \in G \mid \{\bar{g}, \bar{x}\} = 0 \text{ for all } x \in G\}$ and $\Delta(x) := \{g \in G \mid \{\bar{g}, \bar{x}\} = 0\}$ $(x \in G)$ are both subgroups of G. If $\{g_1, \dots, g_n\}$ is a generating set for the group G, then $C = \bigcap_{i=1}^{n} \Delta(g_i)$.

Lemma 2.3. *The Poisson centre $Z_P(\mathbb{K}_P^\lambda[G])$ of $\mathbb{K}_P^\lambda[G]$ is $\mathbb{K}_P^\lambda[C]$.*

Proof. Clearly, $\mathbb{K}_P^\lambda[C] \subseteq Z_P(\mathbb{K}_P^\lambda[G])$. For the reverse inclusion, take $\gamma = \sum_{g\in G} c_g\bar{g} \in Z_P(\mathbb{K}_P^\lambda[G])$. It follows that $0 = \{\gamma, \bar{x}\} = \sum_{g\in G} c_g\{\bar{g}, \bar{x}\} = \sum_{g\in G} c_g\lambda(g,x)\bar{g}\bar{x}$, for any $x \in G$. Consequently, $\lambda(g,x) = 0$, for all $g \in \mathrm{supp}(\gamma)$. This implies that $\mathrm{supp}(\gamma) \subseteq C$, hence $\gamma \in \mathbb{K}_P^\lambda[C]$. □

Remark 2.4. One can easily observe that $Z_P(\mathbb{K}_P^\lambda[G]) = \mathbb{K}$ if and only if $C = \{e\}$, where e is the identity element of G.

2.4. Central and inner Poisson derivations of Poisson group algebras

2.4.1. Central Poisson derivations of Poisson group algebras

Let $\theta : (G, \cdot) \longrightarrow (\mathbb{K}_P^\lambda[C], +)$ be a group homomorphism. That is, $\theta(xy) = \theta(x) + \theta(y)$ for all $x, y \in G$. Define a $\mathbb{K}$-linear operator $\mathcal{D} := \mathcal{D}_\theta$ by

$$\mathcal{D}(\bar{x}) = \theta(x)\bar{x}$$

for all $x \in G$.

Lemma 2.5. *$\mathcal{D}$ is a Poisson derivation of $\mathbb{K}_P^\lambda[G]$.*

Proof. We need to show that $\mathcal{D}(\bar{x}\bar{y}) = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$ and $\mathcal{D}(\{\bar{x}, \bar{y}\}) = \{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\}$ for all $x, y \in G$. Now, $\mathcal{D}(\bar{x}\bar{y}) = \theta(xy)\overline{xy} = \theta(xy)\bar{x}\bar{y} = \theta(x)\bar{x}\bar{y} + \theta(y)\bar{x}\bar{y} = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$. Secondly, $\mathcal{D}(\{\bar{x}, \bar{y}\}) = \lambda(x, y)\mathcal{D}(\bar{x}\bar{y}) = [\theta(x) + \theta(y)]\lambda(x, y)\bar{x}\bar{y} = [\theta(x) + \theta(y)]\{\bar{x}, \bar{y}\} = \theta(x)\{\bar{x}, \bar{y}\} + \theta(y)\{\bar{x}, \bar{y}\} = \{\theta(x)\bar{x}, \bar{y}\} + \{\bar{x}, \theta(y)\bar{y}\} = \{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\}$ (note that $\{\theta(x), \bar{y}\} = \{\bar{x}, \theta(y)\} = 0$, since $\theta(x)$ and $\theta(y)$ are Poisson central elements). □

Similarly to [20], we will refer to $\mathcal{D}$ in Lemma 2.5 as a *central Poisson derivation*. However, when $Z_P(\mathbb{K}_P^\lambda[G]) = \mathbb{K}$, then $\mathcal{D}$ shall be called *scalar Poisson derivation*. Observe that $\mathcal{D}(\bar{x}) = \theta(x)\bar{x} \in \mathbb{K}_P^\lambda[Cx]$ for all $x \in G$.

2.4.2. Inner Poisson derivations of Poisson group algebras

Let $\gamma = \sum_{g\in G} c_g\bar{g} \in \mathbb{K}_P^\lambda[G]$, where $c_g \in \mathbb{K}$, and $\mathrm{ham}_\gamma := \{\gamma, -\}$. It is well known that $\mathrm{ham}_\gamma : \mathbb{K}_P^\lambda[G] \longrightarrow \mathbb{K}_P^\lambda[G]$ is a Poisson derivation called the *hamiltonian derivation associated to* γ. Moreover, $\mathrm{ham}_\gamma(\bar{x}) = \{\gamma, \bar{x}\} = \sum_{g\in G} \lambda(g,x)c_g\bar{g}\bar{x} \in \mathbb{K}_P^\lambda[Gx](= \mathbb{K}_P^\lambda[G]\bar{x})$ for all $x \in G$. Observe that the elements of $C \cap \mathrm{supp}(\gamma)$ do not have any effect on the map ham_γ as $\mathrm{ham}_\gamma = \mathrm{ham}_{\gamma+\mu\bar{t}}$ for all $t \in C \cap \mathrm{supp}(\gamma)$ and $\mu \in \mathbb{K}$. As a result, one can always assume that $C \cap \mathrm{supp}(\gamma) = \emptyset$. Therefore, $\mathrm{ham}_\gamma(\bar{x}) \in \mathbb{K}_P^\lambda[(G \setminus C)x]$ for all $x \in G$. We call the hamiltonian derivation ham_γ an *inner Poisson derivation*.

We can now state our main result in this section in the theorem below.

Theorem 2.6. *Every Poisson derivation of the Poisson group algebra $\mathbb{K}_P^\lambda[G]$ is uniquely the sum of an inner Poisson derivation and a central Poisson derivation.*

Proof. Let $\mathcal{D}$ be a Poisson derivation of $\mathbb{K}_P^\lambda[G]$. Then, for $x \in G$, we have that $\mathcal{D}(\bar{x}) \in \mathbb{K}_P^\lambda[G]$. Hence, $\mathcal{D}(\bar{x}) = \sum_{h\in G} b_h(x)\bar{h} = \sum_{h\in G} b_h(x)\bar{h}\bar{x}^{-1}\bar{x}$. Now, the map $G \to G$ with $h \mapsto hx^{-1}$ is bijective, and so

$$\mathcal{D}(\bar{x}) = \sum_{g\in G} a_g(x)\bar{g}\bar{x},$$

where $g := hx^{-1}$ and $a_g(x) := b_{gx}(x)$. Note that $a_g : G \longrightarrow \mathbb{K}$ and $a_g(x) = 0$ for almost all $x \in G$.

Since $\mathcal{D}$ is a Poisson derivation, we have that $\mathcal{D}(\bar{x}\bar{y}) = \mathcal{D}(\bar{x})\bar{y} + \bar{x}\mathcal{D}(\bar{y})$ for all $x, y \in G$. As a result,

$$\sum_{g\in G} a_g(xy)\bar{g}\bar{x}\bar{y} = \sum_{g\in G} a_g(x)\bar{g}\bar{x}\bar{y} + \sum_{g\in G} a_g(y)\bar{g}\bar{x}\bar{y} = \sum_{g\in G}[a_g(x) + a_g(y)]\bar{g}\bar{x}\bar{y}.$$

Identifying the coefficients in the above equality reveals that

$$a_g(xy) = a_g(x) + a_g(y).$$

Secondly, $\mathcal{D}(\{\bar{x}, \bar{y}\}) = \{\mathcal{D}(\bar{x}), \bar{y}\} + \{\bar{x}, \mathcal{D}(\bar{y})\}$. Now,

$$\mathcal{D}(\{\bar{x}, \bar{y}\}) = \lambda(x, y)\mathcal{D}(\bar{x}\bar{y}) = \sum_{g\in G} \lambda(x, y)a_g(xy)\bar{g}\bar{x}\bar{y}. \tag{2}$$

On the other hand,

$$\begin{aligned}\{\mathcal{D}(\bar{x}),\bar{y}\}+\{\bar{x},\mathcal{D}(\bar{y})\} &= \sum_{g\in G} a_g(x)\{\bar{g}\bar{x},\bar{y}\}+\sum_{g\in G} a_g(y)\{\bar{x},\bar{g}\bar{y}\}\\ &= \sum_{g\in G}[a_g(x)(\lambda(g,y)+\lambda(x,y))+a_g(y)(\lambda(x,g)+\lambda(x,y))]\bar{g}\bar{x}\bar{y}\\ &= \sum_{g\in G}[\lambda(x,y)a_g(xy)+a_g(x)\lambda(g,y)-a_g(y)\lambda(g,x)]\bar{g}\bar{x}\bar{y}. \end{aligned} \tag{3}$$

Since $\mathcal{D}(\{\bar{x},\bar{y}\})=\{\mathcal{D}(\bar{x}),\bar{y}\}+\{\bar{x},\mathcal{D}(\bar{y})\}$, comparing (2) to (3) reveals that

$$\lambda(x,y)a_g(xy)=\lambda(x,y)a_g(xy)+a_g(x)\lambda(g,y)-a_g(y)\lambda(g,x).$$

This implies that

$$a_g(x)\lambda(g,y)=a_g(y)\lambda(g,x). \tag{4}$$

Suppose that $g\in C$. It follows that $\lambda(g,y)=\lambda(g,x)=0$ for all $x,y\in G$. Since $a_g(xy)=a_g(x)+a_g(y)$, the map $\theta:(G,\cdot)\longrightarrow(\mathbb{K}_p^{\lambda}[C],+)$ given by $\theta(x)=\sum_{g\in C}a_g(x)\bar{g}$ is a group homomorphism. Hence, θ defines a central Poisson derivation $\mathcal{D}_\theta$ of $\mathbb{K}_p^{\lambda}[G]$, where

$$\mathcal{D}_\theta(\bar{x})=\sum_{g\in C}a_g(x)\bar{g}\bar{x}. \tag{5}$$

Now, let $g\notin C$. There exists $y\in G$ such that $\lambda(g,y)\neq 0$. Fix y and define

$$c_g:=\frac{a_g(y)}{\lambda(g,y)}.$$

Take any arbitrary element $x\in G$. It follows that

$$c_g\lambda(g,x)=\frac{a_g(y)\lambda(g,x)}{\lambda(g,y)}.$$

From (4), we have that

$$c_g\lambda(g,x)=\frac{a_g(y)\lambda(g,x)}{\lambda(g,y)}=\frac{a_g(x)\lambda(g,y)}{\lambda(g,y)}=a_g(x),$$

for all $x\in G$.

Define $\gamma\in\mathbb{K}_p^{\lambda}[G]$ as $\gamma:=\sum_{g\notin C}c_g\bar{g}$. Then,

$$\mathrm{ham}_\gamma(\bar{x})=\{\gamma,\bar{x}\}=\sum_{g\notin C}c_g\lambda(g,x)\bar{g}\bar{x}=\sum_{g\notin C}a_g(x)\bar{g}\bar{x}. \tag{6}$$

From (5) and (6), one can conclude that every Poisson derivation $\mathcal{D}$ of $\mathbb{K}_p^{\lambda}[G]$ can be written as $\mathcal{D}=\mathcal{D}_\theta+\mathrm{ham}_\gamma$. This decomposition of $\mathcal{D}$ into an inner Poisson derivation (ham_γ) and a central Poisson derivation ($\mathcal{D}_\theta$) is actually unique, because $\mathbb{K}_p^{\lambda}[Gx]=\mathbb{K}_p^{\lambda}[G]$ can be decomposed as $\mathbb{K}_p^{\lambda}[Gx]=\mathbb{K}_p^{\lambda}[Cx]\oplus\mathbb{K}_p^{\lambda}[(G\setminus C)x]$. Now, every central Poisson derivation maps $\bar{x}$ to an element of the subspace $\mathbb{K}_p^{\lambda}[Cx]$, and every inner Poisson derivation maps $\bar{x}$ to an element of the subspace $\mathbb{K}_p^{\lambda}[(G\setminus C)x]$. □

Corollary 2.7. *Suppose that $C=\{e\}$ (equivalently, $Z_P(\mathbb{K}_p^{\lambda}[G])=\mathbb{K}$). Then, every Poisson derivation of $\mathbb{K}_p^{\lambda}[G]$ is uniquely the sum of an inner and a scalar Poisson derivation.*

3. Poisson prime spectrum and Poisson deleting derivations algorithm of a semiclassical limit of $U_q^+(G_2)$

This section aims to study a semiclassical limit of the positive part $U_q^+(G_2)$ of the quantized enveloping algebra of type G_2. Given the semiclassical limit of $U_q^+(G_2)$, we will study its Poisson prime spectrum using Goodearl's $\mathcal{H}$-stratification theory [8], and its Poisson deleting derivations algorithm introduced in [13]. Given the data of the Poisson deleting derivations algorithm, we will study the Poisson centre of the semiclassical limit.

3.1. Semiclassical limit of the algebra $U_q^+(G_2)$

Recall from [16, §2] that $U_q^+(G_2)$ is generated by $E_1, \dots, E_6$, and satisfies the following relations:

$$
\begin{aligned}
&E_2E_1 = q^{-3}E_1E_2 && E_3E_1 = q^{-1}E_1E_3 - (q+q^{-1}+q^{-3})E_2\\
&E_3E_2 = q^{-3}E_2E_3 && E_4E_1 = E_1E_4 + (1-q^2)E_3^2\\
&E_4E_2 = q^{-3}E_2E_4 - \frac{q^4-2q^2+1}{q^4+q^2+1}E_3^3 && E_4E_3 = q^{-3}E_3E_4\\
&E_5E_1 = qE_1E_5 - (1+q^2)E_3 && E_5E_2 = E_2E_5 + (1-q^2)E_3^2\\
&E_5E_3 = q^{-1}E_3E_5 - (q+q^{-1}+q^{-3})E_4 && E_5E_4 = q^{-3}E_4E_5\\
&E_6E_1 = q^3E_1E_6 - q^3E_5 && E_6E_2 = q^3E_2E_6 + (q^4+q^2-1)E_4 +\\
& && \qquad (q^2-q^4)E_3E_5\\
&E_6E_3 = E_3E_6 + (1-q^2)E_5^2 && \\
&E_6E_4 = q^{-3}E_4E_6 - \frac{q^4-2q^2+1}{q^4+q^2+1}E_5^3 && E_6E_5 = q^{-3}E_5E_6.
\end{aligned}
$$

Set $U_i := (q-1)E_i$ for $i = 1, 3, 4, 5$, and $U_i := f(q)(q-1)E_i$ for $i = 2, 6$; where $f(q) = q^4+q^2+1$. Then, $U_q^+(G_2)$ is now generated by $U_1, \dots, U_6$ subject to the relations:

$$
\begin{aligned}
&U_2U_1 = q^{-3}U_1U_2 && U_3U_2 = q^{-3}U_2U_3\\
&U_3U_1 = q^{-1}U_1U_3 - q^{-3}(q-1)U_2 && U_4U_1 = U_1U_4 + (1-q^2)U_3^2\\
&U_4U_2 = q^{-3}U_2U_4 - (q+1)^2(q-1)U_3^3 && U_4U_3 = q^{-3}U_3U_4\\
&U_5U_1 = qU_1U_5 - (1+q^2)(q-1)U_3 && U_5U_2 = U_2U_5 + f(q)(1-q^2)U_3^2\\
&U_5U_3 = q^{-1}U_3U_5 - f(q)(q^{-2}-q^{-3})U_4 && U_5U_4 = q^{-3}U_4U_5\\
&U_6U_1 = q^3U_1U_6 - f(q)(q^4-q^3)U_5 && U_6U_2 = q^3U_2U_6 + f(q)^2(q^2-q^4)U_3U_5 +\\
&U_6U_3 = U_3U_6 + f(q)(1-q^2)U_5^2 && \qquad f(q)^2(q^4+q^2-1)(q-1)U_4\\
&U_6U_4 = q^{-3}U_4U_6 - (q+1)^2(q-1)U_5^3 && U_6U_5 = q^{-3}U_5U_6.
\end{aligned}
$$

We now find a 'new' presentation for $U_q^+(G_2)$ that allows us to introduce a quantisation of $U_q^+(G_2)$. Let $\widehat{U_q^+(G_2)}$ be a $\mathbb{K}[z^{\pm1}]$-algebra generated by $\widehat{U}_1, \dots, \widehat{U}_6$ subject to the relations:

$$
\begin{aligned}
&\widehat{U}_2\widehat{U}_1 = z^{-3}\widehat{U}_1\widehat{U}_2 && \widehat{U}_3\widehat{U}_2 = z^{-3}\widehat{U}_2\widehat{U}_3\\
&\widehat{U}_3\widehat{U}_1 = z^{-1}\widehat{U}_1\widehat{U}_3 - z^{-3}(z-1)\widehat{U}_2 && \widehat{U}_4\widehat{U}_1 = \widehat{U}_1\widehat{U}_4 + (1-z^2)\widehat{U}_3^2\\
&\widehat{U}_4\widehat{U}_2 = z^{-3}\widehat{U}_2\widehat{U}_4 - (z+1)^2(z-1)\widehat{U}_3^3 && \widehat{U}_4\widehat{U}_3 = z^{-3}\widehat{U}_3\widehat{U}_4\\
&\widehat{U}_5\widehat{U}_1 = z\widehat{U}_1\widehat{U}_5 - (1+z^2)(z-1)\widehat{U}_3 && \widehat{U}_5\widehat{U}_2 = \widehat{U}_2\widehat{U}_5 + f(z)(1-z^2)\widehat{U}_3^2\\
&\widehat{U}_5\widehat{U}_3 = z^{-1}\widehat{U}_3\widehat{U}_5 - f(z)(z^{-2}-z^{-3})\widehat{U}_4 && \widehat{U}_5\widehat{U}_4 = z^{-3}\widehat{U}_4\widehat{U}_5\\
&\widehat{U}_6\widehat{U}_1 = z^3\widehat{U}_1\widehat{U}_6 - f(z)(z^4-z^3)\widehat{U}_5 && \widehat{U}_6\widehat{U}_2 = z^3\widehat{U}_2\widehat{U}_6 + f(z)^2(z^2-z^4)\widehat{U}_3\widehat{U}_5 +\\
&\widehat{U}_6\widehat{U}_3 = \widehat{U}_3\widehat{U}_6 + f(z)(1-z^2)\widehat{U}_5^2 && \qquad f(z)^2(z^4+z^2-1)(z-1)\widehat{U}_4\\
&\widehat{U}_6\widehat{U}_4 = z^{-3}\widehat{U}_4\widehat{U}_6 - (z+1)^2(z-1)\widehat{U}_5^3 && \widehat{U}_6\widehat{U}_5 = z^{-3}\widehat{U}_5\widehat{U}_6,
\end{aligned}
$$

where $f(z) = z^4+z^2+1$. Fix $\lambda \in \mathbb{K}^*$. Observe that the element $z-\lambda$ is central and not invertible in $\widehat{U_q^+(G_2)}$, hence we set $\mathcal{A}_\lambda := \widehat{U_q^+(G_2)}/(z-\lambda)\widehat{U_q^+(G_2)}$. Now, $\mathcal{A}_q$ is the non-commutative algebra $U_q^+(G_2)$ and $\mathcal{A}_1 = \mathbb{K}[X_1, \dots, X_6]$ with $X_i := \widehat{U}_i + (z-1)\widehat{U_q^+(G_2)}$ is a Poisson algebra with the Poisson bracket defined as follows:

$$
\begin{aligned}
&\{X_2, X_1\} = -3X_1X_2 && \{X_3, X_1\} = -X_1X_3 - X_2\\
&\{X_3, X_2\} = -3X_2X_3 && \{X_4, X_1\} = -2X_3^2\\
&\{X_4, X_2\} = -3X_2X_4 - 4X_3^3 && \{X_4, X_3\} = -3X_3X_4\\
&\{X_5, X_1\} = X_1X_5 - 2X_3 && \{X_5, X_2\} = -6X_3^2
\end{aligned}
$$

$$\{X_5, X_3\} = -X_3X_5 - 3X_4 \qquad \{X_5, X_4\} = -3X_4X_5$$
$$\{X_6, X_1\} = 3X_1X_6 - 3X_5 \qquad \{X_6, X_2\} = 3X_2X_6 + 9X_4 - 18X_3X_5$$
$$\{X_6, X_3\} = -6X_5^2 \qquad \{X_6, X_4\} = -3X_4X_6 - 4X_5^3$$
$$\{X_6, X_5\} = -3X_5X_6.$$

Therefore, $\mathcal{A}_1$ is a semiclassical limit of the non-commutative algebra $\widehat{U_q^+(G_2)}$, and $\mathcal{A}_q$ is a deformation of the Poisson algebra $\mathcal{A}_1$. For simplicity, we set

$$\mathcal{A} := \mathcal{A}_1 = \mathbb{K}[X_1, \dots, X_6]$$

for the rest of this paper. One can write the Poisson algebra $\mathcal{A}$ as an iterated Poisson-Ore extension over $\mathbb{K}$ (see [18, Theorem 1.1] for the definition of iterated Poisson-Ore extension) as follows:

$$\mathcal{A} = \mathbb{K}[X_1][X_2; \sigma_2]_P[X_3; \sigma_3, \delta_3]_P[X_4; \sigma_4, \delta_4]_P[X_5; \sigma_5, \delta_5]_P[X_6; \sigma_6, \delta_6]_P; \tag{7}$$

where σ_i and δ_i are respectively the Poisson derivations and Poisson σ_i-derivations of

$$\mathbb{K}[X_1][X_2; \sigma_2]_P[X_3; \sigma_3, \delta_3]_P \dots [X_{i-1}; \sigma_{i-1}, \delta_{i-1}]_P$$

($2 \leq i \leq 6$ and $\delta_2 = 0$) defined as follows:

$$\sigma_2(X_1) = -3X_1 \qquad \sigma_3(X_1) = -X_1 \qquad \sigma_3(X_2) = -3X_2 \qquad \sigma_4(X_1) = 0$$
$$\sigma_4(X_2) = -3X_2 \qquad \sigma_4(X_3) = -3X_3 \qquad \sigma_5(X_1) = X_1 \qquad \sigma_5(X_2) = 0$$
$$\sigma_5(X_3) = -X_3 \qquad \sigma_5(X_4) = -3X_4 \qquad \sigma_6(X_1) = 3X_1 \qquad \sigma_6(X_2) = 3X_2$$
$$\sigma_6(X_3) = 0 \qquad \sigma_6(X_4) = -3X_4 \qquad \sigma_6(X_5) = -3X_5,$$

and

$$\delta_3(X_1) = -X_2 \qquad \delta_3(X_2) = 0 \qquad \delta_4(X_1) = -2X_3^2 \qquad \delta_4(X_2) = -4X_3^3$$
$$\delta_4(X_3) = 0 \qquad \delta_5(X_1) = -2X_3 \qquad \delta_5(X_2) = -6X_3^2 \qquad \delta_5(X_3) = -3X_4$$
$$\delta_5(X_4) = 0 \qquad \delta_6(X_1) = -3X_5 \qquad \delta_6(X_2) = 9X_4 - 18X_3X_5 \qquad \delta_6(X_3) = -6X_5^2$$
$$\delta_6(X_4) = -4X_5^3 \qquad \delta_6(X_5) = 0.$$

From [11, Eqn. 5.6], we have that the *rank* of $\mathcal{A}$, denoted by $\text{rank}(\mathcal{A})$, is given by

$$\text{rank}(\mathcal{A}) := |\{j \in \{1, \dots, 6\} \mid \delta_j = 0\}| = 2. \tag{8}$$

Remark 3.1 *([2, Theorem I.1.13] (Hilbert's basis theorem)).* Since $\mathbb{K}$ is a noetherian domain, the Poisson algebra $\mathcal{A} = \mathbb{K}[X_1][X_2; \sigma_2]_P[X_3; \sigma_3, \delta_3]_P[X_4; \sigma_4, \delta_4]_P[X_5; \sigma_5, \delta_5]_P[X_6; \sigma_6, \delta_6]_P$ is a noetherian domain.

3.2. Poisson prime spectrum of the semiclassical limit of the algebra $U_q^+(G_2)$

We study the Poisson prime spectrum of the Poisson algebra $\mathcal{A} = \mathbb{K}[X_1, \dots, X_6]$ in this subsection. Let P be a proper Poisson ideal of a Poisson algebra $\mathcal{A}$. The ideal P is called *Poisson prime ideal* provided that for all Poisson ideals I_1, I_2 of $\mathcal{A}$ such that $P \supseteq I_1I_2$, we have that $P \supseteq I_1$ or $P \supseteq I_2$. Since the Poisson algebra $\mathcal{A}$ is a noetherian domain, every Poisson ideal which is also a prime ideal is a Poisson prime ideal and vice versa (see [8, Lemma 1.1]). The set of all the Poisson prime ideals of $\mathcal{A}$ is called the *Poisson prime spectrum* of $\mathcal{A}$, denoted by $\text{P.Spec}(\mathcal{A})$. The largest Poisson prime ideal contained in a given maximal ideal of $\mathcal{A}$ is called a *Poisson primitive ideal*. The set of all these Poisson primitive ideals is called the *Poisson primitive spectrum* of $\mathcal{A}$, denoted by $\text{P.Prim}(\mathcal{A})$.

One can easily check that the algebraic torus $\mathcal{H} := (\mathbb{K}^*)^2$ acts rationally on $\mathcal{A}$ by Poisson automorphisms via:

$$(\alpha, \beta) \cdot X_1 = \alpha X_1,\ (\alpha, \beta) \cdot X_2 = \alpha^3\beta X_2,\ (\alpha, \beta) \cdot X_3 = \alpha^2\beta X_3,\ (\alpha, \beta) \cdot X_4 = \alpha^3\beta^2 X_4,\ (\alpha, \beta) \cdot X_5 = \alpha\beta X_5,$$
$$\text{and } (\alpha, \beta) \cdot X_6 = \beta X_6 \quad \text{for all } (\alpha, \beta) \in \mathcal{H}.$$

(We refer the reader to [8, §2] for the definition of a rational torus action). A Poisson prime ideal P is $\mathcal{H}$*-invariant* if $h \cdot P = P$ for all $h \in \mathcal{H}$. Moreover, $(P : \mathcal{H}) := \bigcap_{h \in \mathcal{H}} h \cdot P$ is the largest Poisson $\mathcal{H}$-invariant ideal contained in P. Note that $(P : \mathcal{H})$ is a prime ideal.

The set

$$\text{P.Spec}_J(\mathcal{A}) := \{P \in \text{P.Spec}(\mathcal{A}) \mid (P : \mathcal{H}) = J\}$$

is called the *J-stratum* of P.Spec($\mathcal{A}$). The $\mathcal{H}$-strata $\text{P.Spec}_J(\mathcal{A})$ partition P.Spec($\mathcal{A}$) into a disjoint union of strata. Hence,

$$\text{P.Spec}(\mathcal{A}) = \bigsqcup_{J \in \mathcal{H}\text{-P.Spec}(\mathcal{A})} \text{P.Spec}_J(A), \tag{9}$$

where $\mathcal{H}$-P.Spec($\mathcal{A}$) is the collection of all the Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$. This partition is called an $\mathcal{H}$-*stratification* of P.Spec($\mathcal{A}$). In a similar manner, an $\mathcal{H}$-stratification of P.Prim($\mathcal{A}$) is obtained as follows:

$$\text{P.Prim}(\mathcal{A}) = \bigsqcup_{J \in \mathcal{H}\text{-P.Spec}(\mathcal{A})} \text{P.Prim}_J(\mathcal{A}),$$

where $\text{P.Prim}_J(\mathcal{A}) = \text{P.Spec}_J(\mathcal{A}) \cap \text{P.Prim}(\mathcal{A})$.

The iterated-Poisson Ore extension $\mathcal{A}$ has only finitely many Poisson $\mathcal{H}$-primes (see [9, Theorem 1.7]), and so we deduce from [8, Theorem 4.3] the following result.

Proposition 3.2. *Let $P \in \text{P.Spec}_J(\mathcal{A})$. Then, P is a Poisson primitive ideal of $\mathcal{A}$ if and only if P is maximal in $\text{P.Spec}_J(\mathcal{A})$.*

3.3. Poisson deleting derivations algorithm of the semiclassical limit of $U_q^+(G_2)$

In [13], the authors studied a Poisson version of the well-known Cauchon's deleting derivations algorithm (see [3]), called the *Poisson deleting derivations algorithm* (PDDA for short). In this subsection, we study the PDDA of $\mathcal{A} = \mathbb{K}[X_1, \dots, X_6]$. From (7), we have that $\mathcal{A} = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2]_p \dots [X_6; \sigma_6, \delta_6]_P$. One can easily verify that $\mathcal{A}$ satisfies the conditions in the hypothesis below.

Hypothesis 3.3.

(H1) For all $1 \le j < i \le 6$, there exists $\mu_{ij} \in \mathbb{K}$ with $\mu_{ji} := -\mu_{ij}$, such that $\sigma_i(X_j) = \mu_{ij} X_j$.
(H2) The derivations δ_i are all locally nilpotent and $\delta_i\sigma_i - \sigma_i\delta_i = \eta_i\delta_i$ for some non-zero scalar η_i, for all $2 \le i \le 6$.

As a result, the PDDA can be used to study the Poisson prime spectrum of $\mathcal{A}$ (see [13, Hypothesis 1.7]). Let $1 \le i \le 6$ and $2 \le j \le 7$. Using the relation

$$X_{i,j} := \begin{cases} X_{i,j+1} & \text{if } i \ge j \\ \displaystyle\sum_{k=0}^{+\infty} \frac{1}{\eta_j^k k!} \delta_j^k(X_{i,j+1}) X_{j,j+1}^{-k} & \text{if } i < j, \end{cases}$$

(note that since δ_j is locally nilpotent, the summation is finite), one can construct a family $(X_{1,j}, \dots, X_{6,j})$ of elements of the field of fractions Fract($\mathcal{A}$) of $\mathcal{A}$ as follows:

$$\begin{aligned}
X_{i,7} &:= X_i \qquad (i = 1, \dots, 6)\\
X_{1,6} &= X_1 - \frac{1}{2} X_5 X_6^{-1}\\
X_{2,6} &= X_2 + \frac{3}{2} X_4 X_6^{-1} - 3X_3X_5X_6^{-1} + X_5^3 X_6^{-2}\\
X_{3,6} &= X_3 - X_5^2 X_6^{-1}\\
X_{4,6} &= X_4 - \frac{2}{3} X_5^3 X_6^{-1}\\
X_{i,6} &= X_i \qquad (i = 5, 6)\\
X_{1,5} &= X_{1,6} - X_{3,6} X_{5,6}^{-1} + \frac{3}{4} X_{4,6} X_{5,6}^{-2}\\
X_{2,5} &= X_{2,6} - 3X_{3,6}^2 X_{5,6}^{-1} + \frac{9}{2} X_{3,6} X_{4,6} X_{5,6}^{-2} - \frac{9}{4} X_{4,6}^2 X_{5,6}^{-3}\\
X_{3,5} &= X_{3,6} - \frac{3}{2} X_{4,6} X_{5,6}^{-1}\\
X_{i,5} &= X_{i,6} \qquad (i = 4, 5, 6)
\end{aligned}$$

$$\begin{aligned}
X_{1,4} &= X_{1,5} - \frac{1}{3} X_{3,5}^2 X_{4,5}^{-1} \\
X_{2,4} &= X_{2,5} - \frac{2}{3} X_{3,5}^3 X_{4,5}^{-1} \\
X_{i,4} &= X_{i,5} && (i = 3, \dots, 6) \\
X_{1,3} &= X_{1,4} - \frac{1}{2} X_{2,4} X_{3,4}^{-1} \\
X_{i,3} &= X_{i,4} && (i = 2, \dots, 6) \\
T_i &:= X_{i,2} = X_{i,3} && (i = 1, \dots, 6).
\end{aligned}$$

For each $2 \le j \le 7$, the algebra $\mathcal{A}^{(j)}$ represents the subalgebra of Fract($\mathcal{A}$) generated by all the $X_{i,j}$. That is, $\mathcal{A}^{(j)} = \mathbb{K}[X_{1,j}, \dots, X_{6,j}]$. Since $X_{i,7} = X_i$, $1 \le i \le 6$, we have that $\mathcal{A}^{(7)} = \mathcal{A}$. From [13, Prop. 1.11], we have that

$$\mathcal{A}^{(j)} \cong \mathbb{K}[X_1][X_2; \sigma_2, \delta_2]_P \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}]_P [X_j; \tau_j]_P \cdots [X_6; \tau_6]_P,$$

by an isomorphism that maps $X_{i,j}$ to X_i, and $\tau_j, \dots, \tau_6$ denote the Poisson derivations defined by $\tau_l(X_i) = \mu_{li} X_i$ for all $1 \le i < l \le 6$. With a slight abuse of notation, one can identify $\tau_j, \dots, \tau_6$ with $\sigma_j, \dots, \sigma_6$ respectively.

Notation 3.4. $\overline{\mathcal{A}} := \mathcal{A}^{(2)} = \mathbb{K}[T_1, \dots, T_6]$.

One can easily check that $\overline{\mathcal{A}}$ is a Poisson affine space associated to the skew-symmetric matrix

$$M := \begin{bmatrix} 0 & 3 & 1 & 0 & -1 & -3 \\ -3 & 0 & 3 & 3 & 0 & -3 \\ -1 & -3 & 0 & 3 & 1 & 0 \\ 0 & -3 & -3 & 0 & 3 & 3 \\ 1 & 0 & -1 & -3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \end{bmatrix}.$$

That is, $\overline{\mathcal{A}}$ satisfies the relation $\{T_i, T_j\} = \mu_{ij} T_j T_i$ for all $1 \le i, j \le 6$, where μ_{ij} are the entries of M.

3.4. *Canonical embedding*

The set $\Sigma_j := \{X_{j,j+1}^n \mid n \in \mathbb{N}\} = \{X_{j,j}^n \mid n \in \mathbb{N}\}$ is a multiplicative system of regular elements of $\mathcal{A}^{(j)}$ and $\mathcal{A}^{(j+1)}$. Moreover, $\mathcal{A}^{(j)} \Sigma_j^{-1} = \mathcal{A}^{(j+1)} \Sigma_j^{-1}$ [13, Prop. 1.11]. One can use the PDDA to relate P.Spec($\mathcal{A}$) to P.Spec($\overline{\mathcal{A}}$) by constructing an embedding ψ_j: P.Spec($\mathcal{A}^{(j+1)}$) $\hookrightarrow$ P.Spec($\mathcal{A}^{(j)}$) defined by

$$\psi_j(P) := \begin{cases} P\Sigma_j^{-1} \cap \mathcal{A}^{(j)} & \text{if } X_{j,j+1} \notin P, \\ g_j^{-1}(P/\langle X_{j,j+1} \rangle) & \text{if } X_{j,j+1} \in P, \end{cases}$$

for each $2 \le j \le 6$ (see [13, Lemma 2.3]). The map g_j is a surjective homomorphism

$$g_j : \mathcal{A}^{(j)} \to \mathcal{A}^{(j+1)}/\langle X_{j,j+1} \rangle$$

defined by

$$g_j(X_{i,j}) := X_{i,j+1} + \langle X_{j,j+1} \rangle$$

(further details can be found in [13, §2]). From [13, §2.1], there exists an increasing homeomorphism from the topological space

$$\{P \in \text{P.Spec}(\mathcal{A}^{(j+1)}) \mid X_{j,j+1} \notin P\}$$

onto the topological space

$$\{Q \in \text{P.Spec}(\mathcal{A}^{(j)}) \mid X_{j,j} \notin P\}$$

whose inverse is also an increasing homeomorphism (the topology being the Zariski topology). The map ψ_j is injective but not necessarily bijective. However, ψ_j induces a bijection between $\{P \in \text{P.Spec}(\mathcal{A}^{(j+1)}) \mid P \cap \Sigma_j = \emptyset\}$ and $\{Q \in \text{P.Spec}(\mathcal{A}^{(j)}) \mid Q \cap \Sigma_j = \emptyset\}$ [13, Lemma 2.1]. The so-called *canonical embedding*

$$\psi := \psi_2 \circ \cdots \circ \psi_6 : \text{P.Spec}(\mathcal{A}) \hookrightarrow \text{P.Spec}(\overline{\mathcal{A}})$$

is obtained by composing all the ψ_j. This canonical embedding ψ helps to construct a partition, the so-called *canonical partition*, of P.Spec($\mathcal{A}$) into a disjoint union of strata via the notion of Cauchon diagrams that we recall below.

Recall that the torus $\mathcal{H} := (\mathbb{K}^*)^2$ acts rationally on $\mathcal{A}$ by Poisson automorphisms. One can easily check that this induces an action of $\mathcal{H}$ on all $\mathcal{A}^{(j)}$ $(2 \le j \le 7)$ by Poisson automorphisms via:

$$(\alpha,\beta)\cdot X_{1,j} = \alpha X_{1,j},\ (\alpha,\beta)\cdot X_{2,j} = \alpha^3\beta X_{2,j},\ (\alpha,\beta)\cdot X_{3,j} = \alpha^2\beta X_{3,j},\ (\alpha,\beta)\cdot X_{4,j} = \alpha^3\beta^2 X_{4,j},$$
$$(\alpha,\beta)\cdot X_{5,j} = \alpha\beta X_{5,j},\ \text{and}\ (\alpha,\beta)\cdot X_{6,j} = \beta X_{6,j} \quad \text{for all}\ \ (\alpha,\beta)\in\mathcal{H}.$$

One can easily check that the canonical embedding is $\mathcal{H}$-equivariant.

The $\mathcal{H}$-invariant Poisson prime ideals of $\overline{\mathcal{A}}$ have generally been described in [13, §2.2] as follows. For any subset C of $\{1,\dots,6\}$, let K_C denote the Poisson $\mathcal{H}$-invariant prime ideal of $\overline{\mathcal{A}}$ generated by the T_i with $i \in C$. We deduce from [13, §2.2] that

$$K_C = \langle T_i \mid i \in C\rangle,$$

and

$$\mathcal{H}\text{-P.Spec}(\overline{\mathcal{A}}) = \{K_C \mid C \subseteq \{1,\dots,6\}\},$$

so that

$$\psi(\mathcal{H}\text{-P.Spec}(\mathcal{A})) \subseteq \{K_C \mid C \subseteq \{1,\dots,6\}\}.$$

A subset C of $\{1,\dots,6\}$ is called a *Cauchon diagrams* provided $K_C \in \psi(\mathcal{H}\text{-P.Spec}(\mathcal{A}))$.

3.5. Poisson centre of $\mathcal{A}$

The monomials $\Omega_1 := T_1T_3T_5$ and $\Omega_2 := T_2T_4T_6$ are Poisson central elements of the Poisson affine space $\overline{\mathcal{A}}$, since $\{\Omega_i, T_j\} = 0$ for all $i = 1,2$, and $1 \le j \le 6$. We now want to successively pull Ω_1 and Ω_2 from $\overline{\mathcal{A}}$ into $\mathcal{A}$ using the data of the PDDA of $\mathcal{A}$. Through a direct computation, one can confirm that

$$\begin{aligned}
\Omega_1 &= T_1T_3T_5\\
&= X_{1,3}X_{3,3}X_{5,3}\\
&= X_{1,4}X_{3,4}X_{5,4} - \frac{1}{2}X_{2,4}X_{5,4}\\
&= X_{1,5}X_{3,5}X_{5,5} - \frac{1}{2}X_{2,5}X_{5,5}\\
&= X_{1,6}X_{3,6}X_{5,6} - \frac{3}{2}X_{1,6}X_{4,6} - \frac{1}{2}X_{2,6}X_{5,6} + \frac{1}{2}X_{3,6}^2\\
&= X_1X_3X_5 - \frac{3}{2}X_1X_4 - \frac{1}{2}X_2X_5 + \frac{1}{2}X_3^2,
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2 &= T_2T_4T_6\\
&= X_{2,4}X_{4,4}X_{6,4}\\
&= X_{2,5}X_{4,5}X_{6,5} - \frac{2}{3}X_{3,5}^3X_{6,5}\\
&= X_{2,6}X_{4,6}X_{6,6} - \frac{2}{3}X_{3,6}^3X_{6,6}\\
&= X_2X_4X_6 - \frac{2}{3}X_3^3X_6 - \frac{2}{3}X_2X_5^3 + 2X_3^2X_5^2 - 3X_3X_4X_5 + \frac{3}{2}X_4^2.
\end{aligned}$$

We proceed to show that the Poisson centre of $\mathcal{A}^{(j)}$ $(2 \le j \le 7)$ is a polynomial ring in two variables: Ω_1 and Ω_2, for each j. That is, $Z_P(\mathcal{A}^{(j)}) = \mathbb{K}[\Omega_1, \Omega_2]$, for each j. The following discussions will lead us to the proof.

The set $S_j := \{\lambda T_j^{i_j} T_{j+1}^{i_{j+1}} \cdots T_6^{i_6} \mid i_j, \dots, i_6 \in \mathbb{N}\}$ is a multiplicative system of non-zero divisors of $\mathcal{A}^{(j)}$ for each $2 \le j \le 6$. One can therefore localize $\mathcal{A}^{(j)}$ at S_j as follows:

$$\mathfrak{R}_j := \mathcal{A}^{(j)} S_j^{-1}.$$

Recall from [13, Prop. 1.11] that

$$\mathcal{A}^{(j)}\Sigma_j^{-1}=\mathcal{A}^{(j+1)}\Sigma_j^{-1},$$

with $\Sigma_j := \{T_j^n \mid n \in \mathbb{N}\}$. One can easily verify that

$$\mathfrak{R}_j=\mathfrak{R}_{j+1}\Sigma_j^{-1}, \text{ for all } 2\le j\le 6.$$

Moreover, we set

$$\mathfrak{R}_1 := \mathfrak{R}_2[T_1^{-1}].$$

$\mathfrak{R}_1$ is the Poisson torus associated to the Poisson affine space $\overline{\mathcal{A}}$, that is $\mathfrak{R}_1=\mathbb{K}[T_1^{\pm 1},\dots,T_6^{\pm 1}]$, where $\{T_i,T_j\}=\mu_{ij}T_jT_i$ for all $1\le i,j\le 6$. The PDDA helps to construct the following chain of embeddings:

$$\begin{aligned}\mathcal{A}:=\mathfrak{R}_7\subset\mathfrak{R}_6=\mathfrak{R}_7\Sigma_6^{-1}\subset\mathfrak{R}_5=\mathfrak{R}_6\Sigma_5^{-1}\subset\mathfrak{R}_4=\mathfrak{R}_5\Sigma_4^{-1}\\ \subset\mathfrak{R}_3=\mathfrak{R}_4\Sigma_3^{-1}\subset\mathfrak{R}_2=\mathfrak{R}_3\Sigma_2^{-1}\subset\mathfrak{R}_1.\end{aligned}\tag{10}$$

Note that the family $(X_{1,j}^{k_1}\dots X_{6,j}^{k_6})$, where $k_i\in\mathbb{N}$ if $i<j$ and $k_i\in\mathbb{Z}$ otherwise is a PBW-basis of $\mathfrak{R}_j$ for all $2\le j\le 7$. In addition, the family $(T_1^{k_1}\dots T_6^{k_6})_{k_1,\dots,k_6\in\mathbb{Z}}$ is a basis of $\mathfrak{R}_1$.

Lemma 3.5.

1. $Z_P(\mathfrak{R}_1)=\mathbb{K}[\Omega_1^{\pm 1},\Omega_2^{\pm 1}]$.
2. $Z_P(\mathfrak{R}_3)=\mathbb{K}[\Omega_1,\Omega_2]$.
3. $Z_P(\overline{\mathcal{A}})=\mathbb{K}[\Omega_1,\Omega_2]$.
4. $Z_P(\mathcal{A})=Z_P(\mathfrak{R}_6)=Z_P(\mathfrak{R}_5)=Z_P(\mathfrak{R}_4)=\mathbb{K}[\Omega_1,\Omega_2]$.

Proof. First, observe that $\Omega_1=T_1T_3T_5$ and $\Omega_2=T_2T_4T_6$ are algebraically independent. This easily follows from the fact that the monomials in $T_1,\dots,T_6$ are linearly independent.

1. Obviously, $\mathbb{K}[\Omega_1^{\pm 1},\Omega_2^{\pm 1}]\subseteq Z_P(\mathfrak{R}_1)$. For the reverse inclusion, let $y\in Z_P(\mathfrak{R}_1)$. Then, y can be written in terms of the basis of $\mathfrak{R}_1$ as $y=\sum_{(i,\dots,n)\in\mathbb{Z}^6}a_{(i,\dots,n)}T_1^iT_2^jT_3^kT_4^lT_5^mT_6^n$. One can verify that $\{y,T_1\}=(-3j-k+m+3n)yT_1$. Since $y\in Z_P(\mathfrak{R}_1)$, it follows that $-3j-k+m+3n=0$. Following the same pattern for T_2,T_3,T_4,T_5 and T_6, one can confirm that $3i-3k-3l+3n=0$, $i+3j-3l-m=0$, $3j+3k-3m-3n=0$, $-i+k+3l-3n=0$, and $-3i-3j+3l+3m=0$. Solving this system of six equations will reveal that $i=k=m$ and $j=l=n$. One can therefore write

$$y=\sum_{(i,j)\in\mathbb{Z}^2}a_{(i,j)}T_1^iT_2^jT_3^iT_4^jT_5^iT_6^j=\sum_{(i,j)\in\mathbb{Z}^2}a_{(i,j)}T_1^iT_3^iT_5^iT_2^jT_4^jT_6^j=\sum_{(i,j)\in\mathbb{Z}^2}a_{(i,j)}\Omega_1^i\Omega_2^j\in\mathbb{K}[\Omega_1^{\pm 1},\Omega_2^{\pm 1}].$$

2. A similar argument as in (1) will prove the result.
3. Observe that $\mathbb{K}[\Omega_1,\Omega_2]\subseteq Z_P(\overline{\mathcal{A}})\subseteq Z_P(\mathfrak{R}_3)=\mathbb{K}[\Omega_1,\Omega_2]$. Hence, $Z_P(\overline{\mathcal{A}})=\mathbb{K}[\Omega_1,\Omega_2]$.
4. Clearly, $\mathbb{K}[\Omega_1,\Omega_2]\subseteq Z_P(\mathcal{A})$. Since $\mathfrak{R}_i$ is a localization of $\mathfrak{R}_{i+1}$ (see (10)), it follows that $Z_P(\mathfrak{R}_{i+1})\subseteq Z_P(\mathfrak{R}_i)$. Hence, $\mathbb{K}[\Omega_1,\Omega_2]\subseteq Z_P(\mathcal{A})\subseteq Z_P(\mathfrak{R}_6)\subseteq Z_P(\mathfrak{R}_5)\subseteq Z_P(\mathfrak{R}_4)\subseteq Z_P(\mathfrak{R}_3)=\mathbb{K}[\Omega_1,\Omega_2]$. Consequently, $Z_P(\mathcal{A})=Z_P(\mathfrak{R}_6)=Z_P(\mathfrak{R}_5)=Z_P(\mathfrak{R}_4)=\mathbb{K}[\Omega_1,\Omega_2]$. □

Recall that $\mathfrak{R}_j=\mathcal{A}^{(j)}S_j^{-1}$. Since Ω_1 and Ω_2 are both elements of $\mathcal{A}^{(j)}$ and $Z_P(\mathfrak{R}_j)$, we have the following immediate corollary.

Corollary 3.6. $Z_P(\mathcal{A}^{(j)})=\mathbb{K}[\Omega_1,\Omega_2]$ *for each* $3\le j\le 6$.

4. Maximal and primitive Poisson ideals and simple quotients of the semiclassical limit of $U_q^+(G_2)$

This section studies the height one Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}=\mathbb{K}[X_1,\dots,X_6]$. Obviously, $\langle 0\rangle$ is the only height zero Poisson $\mathcal{H}$-invariant prime ideal of $\mathcal{A}$. Now, the Poisson $\mathcal{H}$-invariant prime ideals of at most height one will enable us to study a family of primitive and maximal Poisson ideals of $\mathcal{A}$, and consequently study a family of simple quotients of $\mathcal{A}$.

4.1. Height one Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$

In this subsection, we study the height one Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A} = \mathbb{K}[X_1, \dots, X_6]$. We will begin by showing that $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ are Poisson prime ideals. Note that $\langle \Theta \rangle_R$ will denote the ideal generated by the element Θ in the commutative ring R. Where no doubt arises, we will simply write $\langle \Theta \rangle$.

Recall from Subsection 3.4 that there exists a bijection between $\{P \in \text{P.Spec}(\mathcal{A}^{(j+1)}) \mid P \cap \Sigma_j = \emptyset\}$ and $\{Q \in \text{P.Spec}(\mathcal{A}^{(j)}) \mid Q \cap \Sigma_j = \emptyset\}$. Observe that $\langle T_1 \rangle$ and $\langle T_2 \rangle$ are both elements of $\text{P.Spec}(\overline{\mathcal{A}})$. The following result shows that $\langle T_1 \rangle \in \text{Im}(\psi)$, and that $\langle \Omega_1 \rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle \Omega_1 \rangle) = \langle T_1 \rangle$ (note that we could have used the results of [11] to obtain these results, however we will need some of the intermediate steps here to study the Poisson derivations of the Poisson simple quotients of $\mathcal{A}$).

Lemma 4.1. *$\langle \Omega_1 \rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle \Omega_1 \rangle) = \langle T_1 \rangle$.*

Proof. We will prove this result in several steps by showing that:

1. $\langle T_1 \rangle \in \text{P.Spec}(\mathcal{A}^{(3)})$.
2. $\langle T_1 \rangle [T_3^{-1}] \cap \mathcal{A}^{(4)} = \langle X_{1,4}T_3 - \frac{1}{2}T_2 \rangle$, so that $\langle X_{1,4}T_3 - \frac{1}{2}T_2 \rangle \in \text{P.Spec}(\mathcal{A}^{(4)})$.
3. $\langle X_{1,4}T_3 - \frac{1}{2}T_2 \rangle [T_4^{-1}] \cap \mathcal{A}^{(5)} = \langle X_{1,5}T_3 - \frac{1}{2}X_{2,5} \rangle$, so that $\langle X_{1,5}T_3 - \frac{1}{2}X_{2,5} \rangle \in \text{P.Spec}(\mathcal{A}^{(5)})$.
4. $\langle X_{1,5}T_3 - \frac{1}{2}X_{2,5} \rangle [T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle \Omega_1 \rangle_{\mathcal{A}^{(6)}}$, so that $\langle \Omega_1 \rangle_{\mathcal{A}^{(6)}} \in \text{P.Spec}(\mathcal{A}^{(6)})$.
5. $\langle \Omega_1 \rangle_{\mathcal{A}^{(6)}} [T_6^{-1}] \cap \mathcal{A} = \langle \Omega_1 \rangle_{\mathcal{A}}$, so that $\langle \Omega_1 \rangle_{\mathcal{A}} \in \text{P.Spec}(\mathcal{A})$.

We proceed to prove the above claims.

1. One can easily verify that $\mathcal{A}^{(3)}/\langle T_1 \rangle$ is isomorphic to a Poisson affine space of rank 5. Hence, $\langle T_1 \rangle$ is a Poisson prime ideal in $\mathcal{A}^{(3)}$.

2. Set $I := \langle X_{1,4}T_3 - \frac{1}{2}T_2 \rangle$. One can verify that $\{X_{i,4}, I\} \subseteq I$ for all $i = 1, \dots, 6$. Therefore, I is a Poisson ideal in $\mathcal{A}^{(4)}$. In addition, $\mathcal{A}^{(4)}/I$ is isomorphic to a polynomial ring in 5 variables which is a domain, hence I is a prime ideal. Since I is both Poisson and prime ideal, it is a Poisson prime ideal in $\mathcal{A}^{(4)}$.

3. Similarly to 2.

4. Observe that $\Omega_1' := X_{1,5}T_3 - \frac{1}{2}X_{2,5} = \Omega_1 T_5^{-1}$ in $\mathcal{A}^{(5)}[T_5^{-1}] = \mathcal{A}^{(6)}[T_5^{-1}]$. Since $\langle \Omega_1' \rangle \in \text{P.Spec}(\mathcal{A}^{(5)})$, we want to show that $\langle \Omega_1' \rangle [T_5^{-1}] \cap \mathcal{A}^{(6)} = \langle \Omega_1 \rangle_{\mathcal{A}^{(6)}}$. Observe that $\langle \Omega_1 \rangle_{\mathcal{A}^{(6)}} \subseteq \langle \Omega_1' \rangle [T_5^{-1}] \cap \mathcal{A}^{(6)}$. We establish the reverse inclusion. Let $y \in \langle \Omega_1' \rangle [T_5^{-1}] \cap \mathcal{A}^{(6)}$. Then, $y \in \langle \Omega_1' \rangle [T_5^{-1}]$. There exists $i \in \mathbb{N}$ such that $yT_5^i \in \langle \Omega_1' \rangle$. Hence, $yT_5^i = \Omega_1' v$, for some $v \in \mathcal{A}^{(5)}$. Furthermore, since $\mathcal{A}^{(5)}[T_5^{-1}] = \mathcal{A}^{(6)}[T_5^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_5^j = v'$ for some $v' \in \mathcal{A}^{(6)}$. It follows from $yT_5^i = \Omega_1' v$ that $yT_5^{i+j} = \Omega_1' vT_5^j = \Omega_1' v'$. Hence, $yT_5^\delta = \Omega_1' T_5 v' = \Omega_1 v'$, where $\delta = i + j + 1$ (note that $\Omega_1' T_5 = \Omega_1$ in $\mathcal{A}^{(6)}$). Let $S = \{s \in \mathbb{N} \mid \exists v' \in \mathcal{A}^{(6)} : yT_5^s = \Omega_1 v'\}$. Since $\delta \in S$, we have that $S \neq \emptyset$. Let $s = s_0$ be the minimum element of S such that $yT_5^{s_0} = \Omega_1 v'$ for some $v' \in \mathcal{A}^{(6)}$. We want to show that $s_0 = 0$. Suppose that $s_0 > 0$. Since $T_5 = X_{5,6}$ is irreducible in $\mathcal{A}^{(6)}$, $yT_5^{s_0} = \Omega_1 v'$ implies that T_5 is a factor of Ω_1 or v'. Clearly, T_5 is not a factor of Ω_1 in $\mathcal{A}^{(6)}$. Hence, there exists $v'' \in \mathcal{A}^{(6)}$ such that $v' = T_5 v''$. This implies that $yT_5^{s_0} = \Omega_1 v' = \Omega_1 T_5 v''$, so that $yT_5^{s_0 - 1} = \Omega_1 v''$ and so $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$ and $y = \Omega_1 v' \in \langle \Omega_1 \rangle_{\mathcal{A}^{(6)}}$. Hence, $\langle \Omega_1' \rangle [T_5^{-1}] \cap \mathcal{A}^{(6)} \subseteq \langle \Omega_1 \rangle_{\mathcal{A}^{(6)}}$ as expected.

5. The proof is similar to 4. □

Following similar procedures, one can also prove that $\langle T_2 \rangle \in \text{Im}(\psi)$, and that $\langle \Omega_2 \rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle$ (the interested reader can check out the details of the proof in [19]). We state only the result in the following lemma.

Lemma 4.2. *$\langle \Omega_2 \rangle$ is the Poisson prime ideal of $\mathcal{A}$ such that $\psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle$.*

Observe that $\langle T_1, T_2 \rangle$ and $\langle T_2, T_3 \rangle$ are Poisson prime ideals of $\overline{\mathcal{A}}$. In the next lemma, we will show that $\langle T_1, T_2 \rangle, \langle T_2, T_3 \rangle \in \psi(\text{P.Spec}(\mathcal{A}))$. In other words, we prove that $\{1, 2\}$ and $\{2, 3\}$ are Cauchon diagrams of $\mathcal{A}$.

Lemma 4.3. *$\langle T_1, T_2 \rangle, \langle T_2, T_3 \rangle \in \psi(\text{P.Spec}(\mathcal{A}))$.*

Proof. We only prove that $\langle T_1, T_2 \rangle \in \psi(\text{P.Spec}(\mathcal{A}))$ as the other case is similar.

Set $J_{1,2}^{(3)} := \langle T_1, T_2 \rangle \in \text{P.Spec}(\mathcal{A}^{(3)})$. Observe that $T_3 \notin J_{1,2}^{(3)}$. Therefore, $J_{1,2}^{(4)} := J_{1,2}^{(3)}[T_3^{-1}] \cap \mathcal{A}^{(4)}$ belongs to $\text{P.Spec}(\mathcal{A}^{(4)})$. Suppose that $T_4 \in J_{1,2}^{(4)}$. Then, since $J_{1,2}^{(3)}[T_3^{-1}] = J_{1,2}^{(4)}[T_3^{-1}]$, we have that $T_4 \in J_{1,2}^{(3)}[T_3^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}[T_3^{-1}] \cap \mathcal{A}^{(3)} = J_{1,2}^{(3)}$, a contradiction! Therefore, $T_4 \notin J_{1,2}^{(4)}$. Hence, $J_{1,2}^{(5)} := J_{1,2}^{(4)}[T_4^{-1}] \cap \mathcal{A}^{(5)}$ belongs to $\text{P.Spec}(\mathcal{A}^{(5)})$. Suppose that $T_5 \in J_{1,2}^{(5)}$. Then, $T_5 \in J_{1,2}^{(4)}[T_4^{-1}] \cap \mathcal{A}^{(5)} = J_{1,2}^{(5)}[T_4^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}$, a contradiction! Therefore, $T_5 \notin J_{1,2}^{(5)}$. Hence, $J_{1,2}^{(6)} := J_{1,2}^{(5)}[T_5^{-1}] \cap \mathcal{A}^{(6)}$ belongs to P.Spec($\mathcal{A}^{(6)}$). Similarly, one can show that $T_6 \notin J_{1,2}^{(6)}$. Hence, $J_{1,2} := J_{1,2}^{(6)}[T_6^{-1}] \cap \mathcal{A} = J_{1,2}$ belongs to P.Spec($\mathcal{A}$), and one can easily check that, by construction, $\psi(J_{1,2}) = \langle T_1, T_2\rangle$. □

Recall that $\Omega_1 = T_1T_3T_5$ and $\Omega_2 = T_2T_4T_6$ in $\overline{\mathcal{A}}$. Observe that Ω_1, Ω_2 are both elements of $\langle T_1, T_2\rangle$ and $\langle T_2, T_3\rangle$. From Lemma 4.3, we know that there exist Poisson prime ideals $J_{1,2}$ and $J_{2,3}$ of $\mathcal{A}$ such that $\psi(J_{1,2}) = \langle T_1, T_2\rangle$ and $\psi(J_{2,3}) = \langle T_2, T_3\rangle$. In the next lemma, we show that $J_{1,2}$ and $J_{2,3}$ both contain Ω_1 and Ω_2.

Lemma 4.4. *Ω_1 and Ω_2 are elements of $J_{1,2}$ and $J_{2,3}$.*

Proof. Recall that Ω_1 and Ω_2 are central elements of $\mathcal{A}^{(j)}$ for each $2 \le j \le 7$. We know that Ω_1 and Ω_2 are elements of $J_{1,2}^{(3)} = \langle T_1, T_2\rangle$. With the notation of the proof of Lemma 4.3, observe that this implies that

$$\Omega_1, \Omega_2 \in J_{1,2}^{(3)}[T_j^{-1}] \cap \mathcal{A}^{(4)} = J_{1,2}^{(4)}.$$

By induction on j, we conclude that $\Omega_1, \Omega_2 \in J_{1,2}$.

Similarly, one can also show that $\Omega_1, \Omega_2 \in J_{2,3}$. □

We now want to find the height one Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$, and show that the height two Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$ contain those of height one.

One can easily observe that $\langle\Omega_1\rangle$, $\langle\Omega_2\rangle$, $J_{1,2}$ and $J_{2,3}$ are all $\mathcal{H}$-invariant. From [11, Sec. 5.2], we have that the total number of height 1 Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$ is equal to rank($\mathcal{A}$) = 2 (see (8)). We are therefore certain that $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are the only height 1 Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$.

The next result will allow us to describe height 2 Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$.

Lemma 4.5. *Let $P \in \psi$(P.Spec($\mathcal{A}$)). If $T_j \in P$, then $T_{j-1}, \dots, T_2 \in P$ for all $3 \le j \le 6$.*

Proof. This is an easy consequence of the following observations:

$$T_2 = X_{2,4} = -\{X_{3,4}, X_{1,4}\} - X_{1,4}X_{3,4} \in \langle X_{3,4}\rangle = \langle T_3\rangle_{\mathcal{A}^{(4)}};$$

$$T_3^2 = X_{3,5}^2 = -\frac{1}{2}\{X_{4,5}, X_{1,5}\} \in \langle X_{4,5}\rangle = \langle T_4\rangle_{\mathcal{A}^{(5)}} \ (\Rightarrow T_3 \in \langle T_4\rangle_{\mathcal{A}^{(5)}});$$

$$T_4 = X_{4,6} = -\frac{1}{3}X_{3,6}X_{5,6} - \frac{1}{3}\{X_{5,6}, X_{3,6}\} \in \langle X_{5,6}\rangle = \langle T_5\rangle_{\mathcal{A}^{(6)}};$$

$$T_5 = X_5 = X_1X_6 - \frac{1}{3}\{X_6, X_1\} \in \langle X_6\rangle = \langle T_6\rangle_{\mathcal{A}}. \quad \square$$

Recall from Lemma 4.3 that there exist $J_{1,2}$ and $J_{2,3}$ of P.Spec($\mathcal{A}$) such that $\psi(J_{1,2}) = \langle T_1, T_2\rangle$ and $\psi(J_{2,3}) = \langle T_2, T_3\rangle$. As a consequence of Lemma 4.5, the Poisson ideals $\langle T_1, T_2\rangle$ and $\langle T_2, T_3\rangle$ are the only height two Poisson $\mathcal{H}$-invariant prime ideals of ψ(P.Spec($\mathcal{A}$)). Since ψ preserves Poisson $\mathcal{H}$-invariant prime ideals and the height of a Poisson prime ideal, this implies that $J_{1,2}$ and $J_{2,3}$ are the only height two Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$. We conclude from Lemma 4.4 that the height two Poisson $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$ contain both Ω_1 and Ω_2.

Proposition 4.6. *Every non-zero Poisson $\mathcal{H}$-invariant prime ideal of $\mathcal{A}$ contains Ω_1 or Ω_2. Moreover, every Poisson $\mathcal{H}$-invariant prime ideal of $\mathcal{A}$ of height at least two contains both Ω_1 and Ω_2.*

4.2. Some maximal and primitive Poisson ideals of $\mathcal{A}$

We begin this subsection by finding the $\mathcal{H}$-strata corresponding to $\langle 0\rangle$, $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$. Note that, in this subsection, we write $\langle\Theta_1, \dots, \Theta_d\rangle$ for $\langle\Theta_1, \dots, \Theta_d\rangle_{\mathcal{A}}$, where $\Theta_1, \dots, \Theta_d \in \mathcal{A}$.

Proposition 4.7. *Assume that $\mathbb{K}$ is algebraically closed. Let $\mathcal{P}$ be the set of those monic irreducible polynomials $P(\Omega_1, \Omega_2) \in \mathbb{K}[\Omega_1, \Omega_2]$ with $P(\Omega_1, \Omega_2) \neq \Omega_1$ and $P(\Omega_1, \Omega_2) \neq \Omega_2$. Then* $\text{P.Spec}_{\langle 0\rangle}(\mathcal{A}) = \{\langle 0\rangle\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle\Omega_1 - \alpha, \Omega_2 - \beta\rangle \mid \alpha, \beta \in \mathbb{K}^*\}$.

Proof. Observe first that it follows from the $\mathcal{H}$-stratification of P.Spec($\mathcal{A}$) (see (9)) and Proposition 4.6 that $\text{P.Spec}_{\langle 0\rangle}(\mathcal{A}) = \{Q \in \text{P.Spec}(\mathcal{A}) \mid \Omega_1, \Omega_2 \notin Q\}$.

Set $R := \mathcal{A}[\Omega_1^{-1}, \Omega_2^{-1}]$. The above observation shows that $\phi : Q \mapsto Q[\Omega_1^{-1}, \Omega_2^{-1}]$ is an increasing bijection from $\text{P.Spec}_{\langle 0\rangle}(\mathcal{A})$ onto P.Spec(R).

Before describing P.Spec(R), observe that the action of $\mathcal{H}$ on $\mathcal{A}$ extends to an action of $\mathcal{H}$ on R by Poisson automorphisms since Ω_1 and Ω_2 are $\mathcal{H}$-eigenvectors. The map ϕ is $\mathcal{H}$-equivariant and it follows from Proposition 4.6 that R is Poisson $\mathcal{H}$-simple, that is, $\langle 0\rangle_R$ is the only unique Poisson $\mathcal{H}$-invariant proper ideal of R.

As R is Poisson $\mathcal{H}$-simple, we deduce from [8, Theorem 4.2] that the extension and contraction maps provide mutually inverse bijections between P.Spec(R) and Spec($Z_P(R)$). From Lemma 3.5, $Z_P(\mathcal{A})=\mathbb{K}[\Omega_1,\Omega_2]$, and so $Z_P(R)=\mathbb{K}[\Omega_1^{\pm 1},\Omega_2^{\pm 1}]$. Since $\mathbb{K}$ is algebraically closed, we have that $\text{Spec}(Z_P(R))=\{\langle 0\rangle_{Z_P(R)}\}\cup\{\langle P(\Omega_1,\Omega_2)\rangle_{Z_P(R)}\mid P(\Omega_1,\Omega_2)\in\mathcal{P}\}\cup\{\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_{Z_P(R)}\mid\alpha,\beta\in\mathbb{K}^*\}$. All these show that $\text{P.Spec}(R)=\{\langle 0\rangle_R\}\cup\{\langle P(\Omega_1,\Omega_2)\rangle_R\mid P(\Omega_1,\Omega_2)\in\mathcal{P}\}\cup\{\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\mid\alpha,\beta\in\mathbb{K}^*\}$. It follows that $\text{P.Spec}_{\langle 0\rangle}(\mathcal{A})=\{\langle 0\rangle_R\cap\mathcal{A}\}\cup\{\langle P(\Omega_1,\Omega_2)\rangle_R\cap\mathcal{A}\mid P(\Omega_1,\Omega_2)\in\mathcal{P}\}\cup\{\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}\mid\alpha,\beta\in\mathbb{K}^*\}$.

Undoubtedly, $\langle 0\rangle_R\cap\mathcal{A}=\langle 0\rangle$. We now have to show that $\langle P(\Omega_1,\Omega_2)\rangle_R\cap\mathcal{A}=\langle P(\Omega_1,\Omega_2)\rangle$, $\forall P(\Omega_1,\Omega_2)\in\mathcal{P}$, and $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}=\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$, $\forall\alpha,\beta\in\mathbb{K}^*$, in order to complete the proof.

Fix $P(\Omega_1,\Omega_2)\in\mathcal{P}$. Clearly, $\langle P(\Omega_1,\Omega_2)\rangle\subseteq\langle P(\Omega_1,\Omega_2)\rangle_R\cap\mathcal{A}$. To show the reverse inclusion, let $y\in\langle P(\Omega_1,\Omega_2)\rangle_R\cap\mathcal{A}$. Since $y\in\langle P(\Omega_1,\Omega_2)\rangle_R$, it implies that $y=dP(\Omega_1,\Omega_2)$ for some $d\in R$. Also, $d\in R$ implies that there exists $i,j\in\mathbb{N}$ such that $d=a\Omega_1^{-i}\Omega_2^{-j}$, where $a\in\mathcal{A}$. Therefore, $y=a\Omega_1^{-i}\Omega_2^{-j}P(\Omega_1,\Omega_2)$, which implies that $y\Omega_1^i\Omega_2^j=aP(\Omega_1,\Omega_2)$. Choose $(i,j)\in\mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that $y\Omega_1^i\Omega_2^j\in\langle P(\Omega_1,\Omega_2)\rangle$. Without loss of generality, let us suppose that $i>0$, then $aP(\Omega_1,\Omega_2)\in\langle\Omega_1\rangle$. Since $\langle\Omega_1\rangle$ is a prime ideal, it implies that $a\in\langle\Omega_1\rangle$ or $P(\Omega_1,\Omega_2)\in\langle\Omega_1\rangle$. Given that $P(\Omega_1,\Omega_2)\in\mathcal{P}$, we have that $P(\Omega_1,\Omega_2)\notin\langle\Omega_1\rangle$. Hence, $a\in\langle\Omega_1\rangle$, which implies that $a=t\Omega_1$ for some $t\in\mathcal{A}$. Returning to $y\Omega_1^i\Omega_2^j=aP(\Omega_1,\Omega_2)$, we have that $y\Omega_1^i\Omega_2^j=t\Omega_1P(\Omega_1,\Omega_2)$. Therefore, $y\Omega_1^{i-1}\Omega_2^j=tP(\Omega_1,\Omega_2)\in\langle P(\Omega_1,\Omega_2)\rangle$. This clearly contradicts the minimality of (i,j), hence $(i,j)=(0,0)$. As a result, $y=aP(\Omega_1,\Omega_2)\in\langle P(\Omega_1,\Omega_2)\rangle$. Consequently, $\langle P(\Omega_1,\Omega_2)\rangle_R\cap\mathcal{A}=\langle P(\Omega_1,\Omega_2)\rangle$ for all $P(\Omega_1,\Omega_2)\in\mathcal{P}$ as desired.

Similarly, we show that $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}=\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$; $\forall\alpha,\beta\in\mathbb{K}^*$. Fix $\alpha,\beta\in\mathbb{K}^*$. Obviously, $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle\subseteq\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}$. We now establish the reverse inclusion. Let $y\in\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}$. Since $y\in\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R$, we have that $y=m_0(\Omega_1-\alpha)+n_0(\Omega_2-\beta)$, where $m_0,n_0\in R$. Also, $m_0,n_0\in R$ implies that there exists $i,j\in\mathbb{N}$ such that $m_0=m\Omega_1^{-i}\Omega_2^{-j}$ and $n_0=n\Omega_1^{-i}\Omega_2^{-j}$ for some $m,n\in\mathcal{A}$. Therefore, $y=m\Omega_1^{-i}\Omega_2^{-j}(\Omega_1-\alpha)+n\Omega_1^{-i}\Omega_2^{-j}(\Omega_2-\beta)$, which implies that $y\Omega_1^i\Omega_2^j=m(\Omega_1-\alpha)+n(\Omega_2-\beta)$. Choose $(i,j)\in\mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that $y\Omega_1^i\Omega_2^j\in\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$. Without loss of generality, suppose that $i>0$. Let $f:\mathcal{A}\longrightarrow\mathcal{A}/\langle\Omega_2-\beta\rangle$ be a canonical surjection onto the domain $\mathcal{A}/\langle\Omega_2-\beta\rangle$. We have that $f(y)f(\Omega_1)^if(\Omega_2)^j=f(m)f(\Omega_1-\alpha)$. It follows that $f(m)f(\Omega_1-\alpha)\in\langle f(\Omega_1)\rangle$. This implies that $f(m)\in\langle f(\Omega_1)\rangle$ since $\alpha\neq 0$. Therefore, $\exists\lambda\in\mathcal{A}$ such that $f(m)=f(\lambda)f(\Omega_1)$. Consequently, $f(y)f(\Omega_1)^if(\Omega_2)^j=f(\lambda)f(\Omega_1)f(\Omega_1-\alpha)$. Since $f(\Omega_1)\neq 0$, it implies that $f(y)f(\Omega_1)^{i-1}f(\Omega_2)^j=f(\lambda)f(\Omega_1-\alpha)$. Therefore, there exists $\lambda'\in\mathcal{A}$ such that $y\Omega_1^{i-1}\Omega_2^j=\lambda(\Omega_1-\alpha)+\lambda'(\Omega_2-\beta)\in\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$. This contradicts the minimality of (i,j). Hence, $(i,j)=(0,0)$ and so $y=m(\Omega_1-\alpha)+n(\Omega_2-\beta)\in\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$. In conclusion, $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle_R\cap\mathcal{A}=\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$; $\forall\alpha,\beta\in\mathbb{K}^*$. □

One can also prove the following results in a similar manner. The details of the proof can be deduced from [19, §2.4].

Proposition 4.8. *Assume that $\mathbb{K}$ is algebraically closed.*

1. $\text{P.Spec}_{\langle\Omega_1\rangle}(\mathcal{A})=\{\langle\Omega_1\rangle\}\cup\{\langle\Omega_1,\Omega_2-\beta\rangle\mid\beta\in\mathbb{K}^*\}$.
2. $\text{P.Spec}_{\langle\Omega_2\rangle}(\mathcal{A})=\{\langle\Omega_2\rangle\}\cup\{\langle\Omega_1-\alpha,\Omega_2\rangle\mid\alpha\in\mathbb{K}^*\}$.

Corollary 4.9. *Assume that $\mathbb{K}$ is algebraically closed. The Poisson ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is Poisson primitive in $\mathcal{A}$ for each $(\alpha,\beta)\in\mathbb{K}^2\setminus\{(0,0)\}$.*

Proof. Since the Poisson ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is maximal in its respective strata for each $(\alpha,\beta)\in\mathbb{K}^2\setminus\{(0,0)\}$, it is also Poisson primitive (see Proposition 3.2). □

While we assumed that $\mathbb{K}$ is algebraically closed in Propositions 4.7 and 4.8, Corollary 4.9 is still true when we drop this assumption (with the same proof).

Proposition 4.10. *Let $(\alpha,\beta)\in\mathbb{K}^2\setminus\{(0,0)\}$. The Poisson prime ideal $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is maximal in $\mathcal{A}$.*

Proof. Suppose that there exists a maximal Poisson ideal I of $\mathcal{A}$ such that $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle\subsetneq I\subsetneq\mathcal{A}$. Let J be the Poisson $\mathcal{H}$-invariant prime ideal in $\mathcal{A}$ such that $I\in\text{P.Spec}_J(\mathcal{A})$. By Propositions 4.7 and 4.8, J cannot be $\langle 0\rangle$, $\langle\Omega_1\rangle$ or $\langle\Omega_2\rangle$, since either of these will lead to a contradiction. Every non-zero Poisson $\mathcal{H}$-invariant prime ideal contains only Ω_1 or only Ω_2 or both (Proposition 4.6). Since $J\neq\langle\Omega_1\rangle,\langle\Omega_2\rangle$, this implies that J contains both Ω_1 and Ω_2. Moreover, since $J\subseteq I$, this implies that $\Omega_1,\Omega_2\in I$. Given $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle\subsetneq I$, we have that $\Omega_1-\alpha,\Omega_2-\beta\in I$. It follows that $\alpha,\beta\in I$, hence $I=\mathcal{A}$, a contradiction! This confirms that $\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ is a maximal Poisson ideal in $\mathcal{A}$. □

4.3. Simple quotients of the semiclassical limit of $U_q^+(G_2)$

Given that $\Omega_1-\alpha$ and $\Omega_2-\beta$ generate a maximal Poisson prime ideal of $\mathcal{A}$, the factor algebra

$$\mathcal{A}_{\alpha,\beta}:=\frac{\mathcal{A}}{\langle\Omega_1-\alpha,\Omega_2-\beta\rangle}$$

is a Poisson-simple noetherian domain for all $(\alpha,\beta)\in\mathbb{K}^2\setminus\{(0,0)\}$. Denote the canonical image of X_i by $x_i:=X_i+\langle\Omega_1-\alpha,\Omega_2-\beta\rangle$ for each $1\le i\le 6$. The algebra $\mathcal{A}_{\alpha,\beta}$ is commutative, and satisfies the following two relations:

$$x_1x_3x_5-\frac{3}{2}x_1x_4-\frac{1}{2}x_2x_5+\frac{1}{2}x_3^2=\alpha, \tag{11}$$

$$x_2x_4x_6-\frac{2}{3}x_3^3x_6-\frac{2}{3}x_2x_5^3+2x_3^2x_5^2-3x_3x_4x_5+\frac{3}{2}x_4^2=\beta. \tag{12}$$

We also have the following extra relations in $\mathcal{A}_{\alpha,\beta}$, which can be verified through direct computations.

Lemma 4.11.

(1) $$x_3^2=2\alpha+3x_1x_4+x_2x_5-2x_1x_3x_5.$$

(2) $$\begin{aligned}x_4^2=&\frac{2}{3}\beta+\frac{8}{9}\alpha x_3x_6+\frac{4}{3}x_1x_3x_4x_6+\frac{4}{9}x_2x_3x_5x_6-\frac{16}{9}\alpha x_1x_5x_6-\frac{8}{3}x_1^2x_4x_5x_6\\&+\frac{16}{9}x_1^2x_3x_5^2x_6-\frac{8}{9}x_2x_5^3-\frac{8}{3}\alpha x_5^2-4x_1x_4x_5^2+\frac{8}{3}x_1x_3x_5^3+2x_3x_4x_5-\frac{2}{3}x_2x_4x_6\\&-\frac{8}{9}x_1x_2x_5^2x_6.\end{aligned}$$

(3) $$\begin{aligned}x_3^2x_4=&2\alpha x_4+x_2x_4x_5+2\beta x_1+\frac{8}{3}\alpha x_1x_3x_6+4x_1^2x_3x_4x_6+\frac{4}{3}x_1x_2x_3x_5x_6\\&-8x_1^3x_4x_5x_6-\frac{8}{3}x_1^2x_2x_5^2x_6+\frac{16}{3}x_1^3x_3x_5^2x_6-\frac{8}{3}x_1x_2x_5^3-8\alpha x_1x_5^2-12x_1^2x_4x_5^2\\&+8x_1^2x_3x_5^3+4x_1x_3x_4x_5-2x_1x_2x_4x_6-\frac{16}{3}\alpha x_1^2x_5x_6.\end{aligned}$$

(4) $$\begin{aligned}x_3x_4^2=&\frac{2}{3}\beta x_3+\frac{16}{9}\alpha^2x_6+\frac{16}{3}\alpha x_1x_4x_6+\frac{16}{9}\alpha x_2x_5x_6+\frac{16}{9}\alpha x_1x_3x_5x_6+\frac{4}{9}x_2^2x_5^2x_6\\&+\frac{8}{9}x_1x_2x_3x_5^2x_6-\frac{64}{9}\alpha x_1^3x_5x_6^2-\frac{160}{9}\alpha x_1^2x_5^2x_6-\frac{80}{3}x_1^3x_4x_5^2x_6-\frac{64}{9}x_1^2x_2x_5^3x_6\\&-\frac{8}{9}x_2x_3x_5^3-\frac{8}{3}\alpha x_3x_5^2+4x_1x_3x_4x_5^2+\frac{160}{9}x_1^3x_3x_5^3x_6-16x_1^2x_4x_5^3-\frac{8}{3}x_1x_2x_5^4\\&-\frac{4}{3}x_1x_2x_4x_5x_6+\frac{8}{3}\beta x_1^2x_6+\frac{32}{9}\alpha x_1^2x_3x_6^2+\frac{16}{3}x_1^3x_3x_4x_6^2+\frac{16}{9}x_1^2x_2x_3x_5x_6^2\\&-\frac{32}{3}x_1^4x_4x_5x_6^2-\frac{8}{3}x_1^2x_2x_4x_6^2+4\alpha x_4x_5+2x_2x_4x_5^2+4\beta x_1x_5+\frac{64}{9}x_1^4x_3x_5^2x_6^2\\&-\frac{2}{3}x_2x_3x_4x_6-\frac{32}{3}\alpha x_1x_5^3+\frac{32}{3}x_1^2x_3x_5^4+\frac{32}{3}x_1^2x_3x_4x_5x_6-\frac{32}{9}x_1^3x_2x_5^2x_6^2.\end{aligned}$$

Now, the commutative algebra $\mathcal{A}_{\alpha,\beta}$ is a Poisson $\mathbb{K}$-algebra with the Poisson bracket defined as follows:

$$\begin{array}{lll}\{x_2,x_1\}=-3x_1x_2 & \{x_3,x_1\}=-x_1x_3-x_2 & \{x_3,x_2\}=-3x_2x_3\\ \{x_4,x_1\}=-2x_3^2 & \{x_4,x_2\}=-3x_2x_4-4x_3^3 & \{x_4,x_3\}=-3x_3x_4\\ \{x_5,x_1\}=x_1x_5-2x_3 & \{x_5,x_2\}=-6x_3^2 & \{x_5,x_3\}=-x_3x_5-3x_4\\ \{x_5,x_4\}=-3x_4x_5 & \{x_6,x_1\}=3x_1x_6-3x_5 & \{x_6,x_2\}=3x_2x_6+9x_4-18x_3x_5\\ \{x_6,x_3\}=-6x_5^2 & \{x_6,x_4\}=-3x_4x_6-4x_5^3 & \{x_6,x_5\}=-3x_5x_6.\end{array}$$

Remark 4.12. The Poisson algebra $\mathcal{A}_{\alpha,\beta}$ is the semiclassical limit of the quantum second Weyl algebra $A_{\alpha,\beta}$ studied in [16].

In the remainder of this section, we study a Poisson torus arising from a localization of $\mathcal{A}_{\alpha,\beta}$, and a linear basis of $\mathcal{A}_{\alpha,\beta}$, both of which will be useful in computing the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ in the final section.

4.3.1. A Poisson torus

Let $\alpha, \beta \neq 0$. Recall from Subsection 3.5 that $\Omega_1 = T_1T_3T_5$ and $\Omega_2 = T_2T_4T_6$ in $\overline{\mathcal{A}}$. From [13, Corollaries 3.3 & 3.4], there exists a multiplicative set $S_{\alpha,\beta}$ such that

$$\mathcal{A}_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong \mathscr{P}_{\alpha,\beta} := \frac{\mathfrak{R}_1}{\langle T_1T_3T_5 - \alpha, T_2T_4T_6 - \beta\rangle_{\mathfrak{R}_1}},$$

where $\mathfrak{R}_1 = \mathbb{K}[T_1^{\pm 1}, \dots, T_6^{\pm 1}]$ is the Poisson torus associated to the Poisson affine space $\overline{\mathcal{A}}$ (that is, the Poisson torus associated to the matrix $M = (\mu_{i,j}) \in M_6(\mathbb{K})$ defined after Notation 3.4). Let $t_i := T_i + \langle T_1T_3T_5 - \alpha, T_2T_4T_6 - \beta\rangle_{\mathfrak{R}_1}$ denote the canonical image of T_i in $\mathscr{P}_{\alpha,\beta}$ for each $1 \leq i \leq 6$. The algebra $\mathscr{P}_{\alpha,\beta}$ is generated by $t_1^{\pm 1}, \dots, t_6^{\pm 1}$ subject to the relations:

$$t_1 = \alpha t_3^{-1}t_5^{-1} \text{ and } t_2 = \beta t_4^{-1}t_5^{-1}.$$

One can easily verify that $\mathscr{P}_{\alpha,\beta} \cong \mathbb{K}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$, and that the isomorphism holds whether α or β is zero. As a consequence, $\mathscr{P}_{\alpha,\beta}$ is the Poisson torus associated to the matrix $\overline{M}$ obtained from M by deleting its first two rows and columns.

4.3.2. Linear basis for $\mathcal{A}_{\alpha,\beta}$

Set $\mathcal{A}_\beta := \mathcal{A}/\langle \Omega_2 - \beta\rangle$, $\beta \in \mathbb{K}$. Denote the canonical image of X_i in $\mathcal{A}_\beta$ by $\widehat{x_i} := X_i + \langle \Omega_2 - \beta\rangle$ for each $1 \leq i \leq 6$. It can be verified that $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_\beta/\langle \widehat{\Omega}_1 - \alpha\rangle$. Note that $\mathcal{A}_\beta$ satisfies the relation:

$$\widehat{x_4}^2 = \frac{2}{3}\beta - \frac{2}{3}\widehat{x_2}\widehat{x_4}\widehat{x_6} + \frac{4}{9}\widehat{x_3}^3\widehat{x_6} + \frac{4}{9}\widehat{x_2}\widehat{x_5}^3 - \frac{4}{3}\widehat{x_3}^2\widehat{x_5}^2 + 2\widehat{x_3}\widehat{x_4}\widehat{x_5}. \tag{13}$$

Proposition 4.13. *The set $\mathfrak{F} = \{\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^\xi\widehat{x_5}^l\widehat{x_6}^m \mid (\xi, i, j, k, l, m) \in \{0, 1\} \times \mathbb{N}^5\}$ is a $\mathbb{K}$-basis of $\mathcal{A}_\beta$.*

Proof. Since $(\Pi_{s=1}^6 X_s^{i_s})_{i_s\in\mathbb{N}}$ is a basis of $\mathcal{A}$ over $\mathbb{K}$, we have that $(\Pi_{s=1}^6 \widehat{x_s}^{i_s})_{i_s\in\mathbb{N}}$ is a spanning set of $\mathcal{A}_\beta$ over $\mathbb{K}$. We want to show that $\mathfrak{F}$ is a spanning set of $\mathcal{A}_\beta$. It is sufficient to do that by showing that $\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{i_4}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{F}$ over $\mathbb{K}$ for all $i_1, \dots, i_6 \in \mathbb{N}$. We do this by an induction on i_4. The result is clear when $i_4 = 0$. For $i_4 \geq 0$, suppose that

$$\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{i_4}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^\xi\widehat{x_5}^l\widehat{x_6}^m,$$

where $\underline{v} := (i, j, k, l, m) \in \mathbb{N}^5$, I is a finite subset of $\{0, 1\} \times \mathbb{N}^5$, and the $a_{(\xi,\underline{v})}$ are scalars. It follows that

$$\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{i_4+1}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^{\xi+1}\widehat{x_5}^l\widehat{x_6}^m.$$

We have to show that $\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^{\xi+1}\widehat{x_5}^l\widehat{x_6}^m \in \mathrm{Span}(\mathfrak{F})$ for all $(\xi, \underline{v}) \in I$. The result is obvious when $\xi = 0$. For $\xi = 1$, using (13), one can verify that $\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^2\widehat{x_5}^l\widehat{x_6}^m \in \mathrm{Span}(\mathfrak{F})$. Consequently, $\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{i_4+1}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} \in \mathrm{Span}(\mathfrak{F})$ as desired. Therefore, $\mathfrak{F}$ spans $\mathcal{A}_\beta$.

Before we continue the proof, the following ordering $<_4$ needs to be noted.

♣ Let $(i', j', k', l', m', n'), (i, j, k, l, m, n) \in \mathbb{N}^6$. We say that $(i, j, k, l, m, n) <_4 (i', j', k', l', m', n')$ if $[l < l']$ or $[l = l'$ and $i < i']$ or $[l = l',\ i = i'$ and $j < j']$ or $[l = l',\ i = i',\ j = j'$ and $k < k']$ or $[l = l',\ i = i',\ j = j',\ k = k'$ and $m < m']$ or $[l = l',\ i = i',\ j = j',\ k = k',\ m = m'$ and $n \leq n']$.
Note that the square brackets [ ] are just to differentiate the options.

We proceed to show that $\mathfrak{F}$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}^\xi\widehat{x_5}^l\widehat{x_6}^m = 0.$$

It follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}X_1^iX_2^jX_3^kX_4^\xi X_5^lX_6^m = \nu(\Omega_2 - \beta),$$

where $\nu \in \mathcal{A}$. Write $\nu = \sum_{(i,\dots,n)\in J} b_{(i,\dots,n)}X_1^iX_2^jX_3^kX_4^lX_5^mX_6^n$, where J is a finite subset of $\mathbb{N}^6$, and $b_{(i,\dots,n)}$ are scalars. From Subsection 3.5, we have that

$$\Omega_2 = X_2X_4X_6 - \frac{2}{3}X_3^3X_6 - \frac{2}{3}X_2X_5^3 + 2X_3^2X_5^2 - 3X_3X_4X_5 + \frac{3}{2}X_4^2.$$

It follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} X_1^i X_2^j X_3^k X_4^\xi X_5^l X_6^m = \sum_{(i,\dots,n)\in J} b_{(i,\dots,n)} X_1^i X_2^j X_3^k X_4^l X_5^m X_6^n(\Omega_2-\beta)$$

$$= \sum_{(i,\dots,n)\in J} \frac{3}{2} b_{(i,\dots,n)} X_1^i X_2^j X_3^k X_4^{l+2} X_5^m X_6^n + \mathrm{LT}_{<_4},$$

where $\mathrm{LT}_{<_4}$ contains lower order terms with respect to $<_4$ (see item ♣). Moreover, $\mathrm{LT}_{<_4}$ vanishes when $b_{(i,\dots,n)}=0$ for all $(i,\dots,n)\in J$. One can easily confirm this when the previous line of equality (right hand side) is fully expanded.

Suppose that there exists $(i,j,k,l,m,n)\in J$ such that $b_{(i,j,k,l,m,n)}\neq 0$. Let (i',j',k',l',m',n') be the greatest element of J with respect to $<_4$ such that $b_{(i',j',k',l',m',n')}\neq 0$. Identifying the coefficients of $X_1^{i'}X_2^{j'}X_3^{k'}X_4^{l'+2}X_5^{m'}X_6^{n'}$, we have $\frac{3}{2}b_{(i',j',k',l',m',n')}=0$ (note that the family $(X_1^iX_2^jX_3^kX_4^lX_5^mX_6^n)_{i,\dots,n\in\mathbb{N}}$ is a basis for $\mathcal{A}$ and $\mathrm{LT}_{<_4}$ contains lower order terms). Therefore, $b_{(i',j',k',l',m',n')}=0$, a contradiction! As a result, $b_{(i,j,k,l,m,n)}=0$ for all $(i,j,k,l,m,n)\in J$, and

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} X_1^i X_2^j X_3^k X_4^\xi X_5^l X_6^m = 0.$$

Consequently, $a_{(\xi,i,j,k,l)}=0$ for all $(\xi,i,j,k,l)\in I$. □

We are now ready to find a basis for $\mathcal{A}_{\alpha,\beta}$.

Proposition 4.14. *The set* $\mathfrak{P}=\{x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}x_5^kx_6^l \mid (\epsilon_1,\epsilon_2,i,j,k,l)\in\{0,1\}^2\times\mathbb{N}^4\}$ *is a* $\mathbb{K}$*-basis of* $\mathcal{A}_{\alpha,\beta}$.

Proof. Since the set $\mathfrak{F}=\{\widehat{x_1}^{i_1}\widehat{x_2}^{i_2}\widehat{x_3}^{i_3}\widehat{x_4}^{\xi}\widehat{x_5}^{i_5}\widehat{x_6}^{i_6} \mid (\xi,i_1,i_2,i_3,i_5,i_6)\in\{0,1\}\times\mathbb{N}^5\}$ is a $\mathbb{K}$-basis of $\mathcal{A}_\beta$ (Proposition 4.13) and $\mathcal{A}_{\alpha,\beta}$ is identified with $\mathcal{A}_\beta/\langle\widehat{\Omega}_1-\alpha\rangle$, the family $(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{\xi}x_5^{i_5}x_6^{i_6})_{(\xi,i_1,i_2,i_3,i_5,i_6)\in\{0,1\}\times\mathbb{N}^5}$ is a spanning set of $\mathcal{A}_{\alpha,\beta}$ over $\mathbb{K}$. We want to show that $\mathfrak{P}$ spans $\mathcal{A}_{\alpha,\beta}$ by showing that $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{\xi}x_5^{i_5}x_6^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}$ over $\mathbb{K}$ for all $(\xi,i_1,i_2,i_3,i_5,i_6)\in\{0,1\}\times\mathbb{N}^5$. By Proposition 4.13, it is sufficient to do this by an induction on i_3. The result is obvious when $i_3=0$ or 1. For $i_3\geq 1$, suppose that

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{\xi}x_5^{i_5}x_6^{i_6} = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}x_5^kx_6^l,$$

where $\underline{v}:=(i,j,k,l)\in\mathbb{N}^4$, and the $a_{(\epsilon_1,\epsilon_2,\underline{v})}$ are all scalars. Moreover, I is a finite subset of $\{0,1\}^2\times\mathbb{N}^4$. It follows from the inductive hypothesis that

$$x_1^{i_1}x_2^{i_2}x_3^{i_3+1}x_4^{\xi}x_5^{i_5}x_6^{i_6} = \sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^ix_2^jx_3^{\epsilon_1+1}x_4^{\epsilon_2}x_5^kx_6^l.$$

We need to show that $x_1^ix_2^jx_3^{\epsilon_1+1}x_4^{\epsilon_2}x_5^kx_6^l\in\mathrm{Span}(\mathfrak{P})$ for all $(\epsilon_1,\epsilon_2,\underline{v})\in I$. The result is obvious when $(\epsilon_1,\epsilon_2)=(0,0),(0,1)$. Using Lemma 4.11(1), (3); one can also show that $x_1^ix_2^jx_3^{\epsilon_1+1}x_4^{\epsilon_2}x_5^kx_6^l\in\mathrm{Span}(\mathfrak{P})$ for all $(\epsilon_1,\epsilon_2)=(1,0),(1,1)$; and $i,j,k,l\in\mathbb{N}$. Therefore, $x_1^{i_1}x_2^{i_2}x_3^{i_3+1}x_4^{\xi}x_5^{i_5}x_6^{i_6}\in\mathrm{Span}(\mathfrak{P})$ as expected. As a result, $\mathfrak{P}$ spans $\mathcal{A}_{\alpha,\beta}$.

We proceed to show that $\mathfrak{F}$ is a linearly independent set. Suppose that

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^ix_2^jx_3^{\epsilon_1}x_4^{\epsilon_2}x_5^kx_6^l = 0.$$

Then,

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} \widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{x_5}^k\widehat{x_6}^l = (\widehat{\Omega}_1-\alpha)\nu \tag{14}$$

in $\mathcal{A}_\beta$, where $\nu\in\mathcal{A}_\beta$. Set $\underline{w}:=(i,j,k,l,m)\in\mathbb{N}^5$, and let J_1,J_2 be finite subsets of $\mathbb{N}^5$. One can write ν in terms of the basis $\mathfrak{F}$ of $\mathcal{A}_\beta$ as:

$$\nu = \sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m + \sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m,$$

where $b_{\underline{w}}$ and $c_{\underline{w}}$ are scalars. Note that $\widehat{\Omega}_1=\widehat{x_1}\widehat{x_3}\widehat{x_5}-\frac{3}{2}\widehat{x_1}\widehat{x_4}-\frac{1}{2}\widehat{x_2}\widehat{x_5}+\frac{1}{2}\widehat{x_3}^2$. Given this expression, and the relation (13), one can express (14) as follows:

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{x_5}^k\widehat{x_6}^l = \sum_{\underline{w}\in J_1} b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m(\widehat{\Omega}_1-\alpha)$$
$$+\sum_{\underline{w}\in J_2} c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m(\widehat{\Omega}_1-\alpha)$$
$$=\sum_{\underline{w}\in J_1}\frac{1}{2}b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{k+2}\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m$$
$$-\sum_{\underline{w}\in J_1}\frac{2}{3}b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+3}\widehat{x_5}^l\widehat{x_6}^{m+1}$$
$$-\sum_{\underline{w}\in J_2}\frac{3}{2}c_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m$$
$$+\sum_{\underline{w}\in J_2}\frac{1}{2}c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{k+2}\widehat{x_5}^l\widehat{x_6}^m+\Upsilon,$$

where

$$\Upsilon := \sum_{\underline{w}\in J_1} r_1 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m + \sum_{\underline{w}\in J_1} r_2 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^{1+j}\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^{m+1}$$
$$+\sum_{\underline{w}\in J_1} r_3 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^{j+1}\widehat{x_3}^k\widehat{x_5}^{l+3}\widehat{x_6}^m + \sum_{\underline{w}\in J_1} r_4 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+2}\widehat{x_5}^{l+2}\widehat{x_6}^m$$
$$+\sum_{\underline{w}\in J_1} r_5 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+1}\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^m + \sum_{\underline{w}\in J_1} r_6 b_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^{j+1}\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^m$$
$$+\sum_{\underline{w}\in J_1} r_7 b_{\underline{w}}\beta\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_4}\widehat{x_5}^l\widehat{x_6}^m + \sum_{\underline{w}\in J_1} r_8 b_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+1}\widehat{x_4}\widehat{x_5}^{l+1}\widehat{x_6}^m$$
$$+\sum_{\underline{w}\in J_2} r_9 c_{\underline{w}}\widehat{x_1}^{i+1}\widehat{x_2}^j\widehat{x_3}^{k+1}\widehat{x_5}^{l+1}\widehat{x_6}^m + \sum_{\underline{w}\in J_2} r_{10} c_{\underline{w}}\widehat{x_1}^i\widehat{x_2}^{j+1}\widehat{x_3}^k\widehat{x_5}^{l+1}\widehat{x_6}^m$$
$$+\sum_{\underline{w}\in J_2} r_{11} c_{\underline{w}}\alpha\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^k\widehat{x_5}^l\widehat{x_6}^m.$$

Note that $r_1,\dots,r_{11}$ are some non-zero rational numbers.

Before we continue the proof, the following ordering $<_3$ needs to be noted.

- ♠ Let $(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_5,\vartheta_6)$, $(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5,\varsigma_6)\in\mathbb{N}^5$. We say that $(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5,\varsigma_6) <_3 (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_5,\vartheta_6)$ if $[\vartheta_3>\varsigma_3]$ or $[\vartheta_3=\varsigma_3$ and $\vartheta_1>\varsigma_1]$ or $[\vartheta_3=\varsigma_3,\ \vartheta_1=\varsigma_1$ and $\vartheta_2>\varsigma_2]$ or $[\vartheta_3=\varsigma_3,\ \vartheta_1=\varsigma_1,\ \vartheta_2=\varsigma_2$ and $\vartheta_5>\varsigma_5]$ or $[\vartheta_3=\varsigma_3,\ \vartheta_1=\varsigma_1,\ \vartheta_2=\varsigma_2,\ \vartheta_5=\varsigma_5$ and $\vartheta_6\geq\varsigma_6]$.

Observe that Υ contains lower order terms with respect to $<_3$ in each monomial type (note that there are two different types of monomials in the basis of $\mathcal{A}_\beta$: one with $\widehat{x_4}$ and the other without $\widehat{x_4}$). Now, suppose that there exists $(i,j,k,l,m)\in J_1$ and $(i,j,k,l,m)\in J_2$ such that $b_{(i,j,k,l,m)}\neq 0$ and $c_{(i,j,k,l,m)}\neq 0$. Let (v_1,v_2,v_3,v_5,v_6) and (w_1,w_2,w_3,w_5,w_6) be the greatest elements of J_1 and J_2 respectively with respect to $<_3$ such that $b_{(v_1,v_2,v_3,v_5,v_6)}$ and $c_{(w_1,w_2,w_3,w_5,w_6)}$ are non-zero. Since $\mathfrak{F}$ is a linear basis for $\mathcal{A}_\beta$ and Υ contains lower order terms with respect to $<_3$, we have the following: if $w_3-v_3<2$, then identifying the coefficients of $\widehat{x_1}^{v_1}\widehat{x_2}^{v_2}\widehat{x_3}^{v_3+2}\widehat{x_4}\widehat{x_5}^{v_5}\widehat{x_6}^{v_6}$ implies that $\frac{1}{2}b_{(v_1,v_2,v_3,v_5,v_6)}=0$, a contradiction! Finally, if $w_3-v_3\geq 2$, then identifying the coefficients of $\widehat{x_1}^{w_1}\widehat{x_2}^{w_2}\widehat{x_3}^{w_3+2}\widehat{x_5}^{w_5}\widehat{x_6}^{w_6}$ implies that $\frac{1}{2}c_{(w_1,w_2,w_3,w_5,w_6)}=0$, another contradiction! Therefore, either all $b_{(i,j,k,m,n)}$ or all $c_{(i,j,k,m,n)}$ are zero. Without loss of generality, suppose that there exists $(i,j,k,m,n)\in J_2$ such that $c_{(i,j,k,m,n)}$ is not zero. Then, $b_{(i,j,k,m,n)}$ are all zero. Let (w_1,w_2,w_3,w_5,w_6) be the greatest element of J_2 such that $c_{(w_1,w_2,w_3,w_5,w_6)}\neq 0$. Identifying the coefficients of $\widehat{x_1}^{w_1}\widehat{x_2}^{w_2}\widehat{x_3}^{w_3+2}\widehat{x_5}^{w_5}\widehat{x_6}^{w_6}$ in the above equality implies that $\frac{1}{2}c_{(w_1,w_2,w_3,w_5,w_6)}=0$, a contradiction! We can therefore conclude that $b_{(i,j,k,m,n)}$ and $c_{(i,j,k,m,n)}$ are all zero. Consequently,

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})}\widehat{x_1}^i\widehat{x_2}^j\widehat{x_3}^{\epsilon_1}\widehat{x_4}^{\epsilon_2}\widehat{x_5}^k\widehat{x_6}^l=0.$$

Since $\mathfrak{F}$ is a basis for $\mathcal{A}_\beta$, this implies that $a_{(\epsilon_1,\epsilon_2,\underline{v})}=0$ for all $(\epsilon_1,\epsilon_2,\underline{v})\in I$. Therefore, $\mathfrak{P}$ is a linearly independent set. $\square$

The following corollary will be useful later on.

Corollary 4.15. *Let* $\underline{v} = (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$, *let* I *represent a finite subset of* $\{0,1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2$, *and let* $(a_{(\epsilon_1,\epsilon_2,\underline{v})})_{(\epsilon_1,\epsilon_2,\underline{v})\in I}$ *be a family of scalars. If*

$$\sum_{(\epsilon_1,\epsilon_2,\underline{v})\in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l = 0,$$

then $a_{(\epsilon_1,\epsilon_2,\underline{v})} = 0$ *for all* $(\epsilon_1, \epsilon_2, \underline{v}) \in I$.

Proof. The result is obvious when $k, l \geq 0$ due to Proposition 4.14. When k (resp. l) is negative, then one can multiply the above equality enough times by $x_5 = t_5$ (resp. $x_6 = t_6$) to kill all the negative powers and then apply Proposition 4.14 to complete the proof. □

5. Poisson derivations of the semiclassical limit of the quantum second Weyl algebra $A_{\alpha,\beta}$

This section focuses on studying the Poisson derivations of the Poisson algebra $\mathcal{A}_{\alpha,\beta}$.

5.1. Preliminaries and strategy

Let $2 \leq j \leq 7$ and $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Set

$$\mathcal{A}^{(j)}_{\alpha,\beta} := \frac{\mathcal{A}^{(j)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{\mathcal{A}^{(j)}}},$$

where $\mathcal{A}^{(j)}$ is defined in Subsection 3.3, and Ω_1 and Ω_2 are the generators of the centre of $\mathcal{A}^{(j)}$ for $3 \leq j \leq 7$ (see Lemma 3.5 and Corollary 3.6). Recall that $\mathcal{A}^{(7)} = \mathcal{A} = \mathbb{K}[X_1, \dots, X_6]$ and so $\mathcal{A}^{(7)}_{\alpha,\beta} = \mathcal{A}_{\alpha,\beta}$ is a domain. Actually $\mathcal{A}^{(j)}_{\alpha,\beta}$ is a domain for all $2 \leq j \leq 7$. This is easy to prove using the expressions of Ω_1 and Ω_2 as elements of $\mathcal{A}^{(j)}$, at least when $j \neq 6$. For $j = 6$, this can be done for instance by proving that $\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{\mathcal{A}^{(6)}} = \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{\mathcal{A}}[T_6^{-1}] \cap \mathcal{A}^{(6)}$. In particular, we have that:

$$\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{\mathcal{A}^{(j)}} = \psi_j \circ \cdots \circ \psi_6 \left(\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{\mathcal{A}}\right),$$

where the embeddings ψ_j: P.Spec$(\mathcal{A}^{(j+1)}) \hookrightarrow$ P.Spec$(\mathcal{A}^{(j)})$ have been defined in Section 3.4.

For each $2 \leq j \leq 7$, denote the canonical images of the generators $X_{i,j}$ of $\mathcal{A}^{(j)}$ in $\mathcal{A}^{(j)}_{\alpha,\beta}$ by $x_{i,j}$ for all $1 \leq i \leq 6$. Furthermore, from [13], one can deduce the following data of $\mathcal{A}_{\alpha,\beta}$ from the PDDA of $\mathcal{A}$ (see Subsection 3.3) as follows:

$$\begin{aligned}
x_{i,7} &:= x_i && (i = 1, \dots, 6)\\
x_{1,6} &= x_1 - \frac{1}{2}x_5x_6^{-1}\\
x_{2,6} &= x_2 + \frac{3}{2}x_4x_6^{-1} - 3x_3x_5x_6^{-1} + x_5^3x_6^{-2}\\
x_{3,6} &= x_3 - x_5^2x_6^{-1}\\
x_{4,6} &= x_4 - \frac{2}{3}x_5^3x_6^{-1}\\
x_{i,6} &= x_i && (i = 5, 6)\\
x_{1,5} &= x_{1,6} - x_{3,6}x_{5,6}^{-1} + \frac{3}{4}x_{4,6}x_{5,6}^{-2}\\
x_{2,5} &= x_{2,6} - 3x_{3,6}^2x_{5,6}^{-1} + \frac{9}{2}x_{3,6}x_{4,6}x_{5,6}^{-2} - \frac{9}{4}x_{4,6}^2x_{5,6}^{-3}\\
x_{3,5} &= x_{3,6} - \frac{3}{2}x_{4,6}x_{5,6}^{-1}\\
x_{i,5} &= x_{i,6} && (i = 4, 5, 6)\\
x_{1,4} &= x_{1,5} - \frac{1}{3}x_{3,5}^2x_{4,5}^{-1}\\
x_{2,4} &= x_{2,5} - \frac{2}{3}x_{3,5}^3x_{4,5}^{-1}\\
x_{i,4} &= x_{i,5} && (i = 3, \dots, 6)
\end{aligned}$$

$$x_{1,3} = x_{1,4} - \frac{1}{2} x_{2,4} x_{3,4}^{-1}$$
$$x_{i,3} = x_{i,4} \qquad (i = 2, \dots, 6)$$
$$t_i := x_{i,2} = x_{i,3} \qquad (i = 1, \dots, 6).$$

For simplicity, we will refer to the above data as the PDDA of $\mathcal{A}_{\alpha,\beta}$. Note that the t_i are the canonical images of T_i in $\mathcal{A}_{\alpha,\beta}^{(2)}$ for all $1 \le i \le 6$. For each $2 \le j < 7$, set $S_j := \{\lambda t_j^{i_j} t_{j+1}^{i_{j+1}} \dots t_6^{i_6} \mid i_j, \dots, i_6 \in \mathbb{N},\ \lambda \in \mathbb{K}^*\}$. One can observe that S_j is a multiplicative system of non-zero divisors (or regular elements) of $\mathcal{A}_{\alpha,\beta}^{(j)}$. As a result, the Poisson algebras $\mathcal{A}_{\alpha,\beta}^{(j)}$ can be localised at S_j as follows:

$$\mathcal{R}_j := \mathcal{A}_{\alpha,\beta}^{(j)} S_j^{-1}.$$

Again, the set $\Sigma_j := \{t_j^k \mid k \in \mathbb{N}\}$ is a multiplicative set in both $\mathcal{A}_{\alpha,\beta}^{(j)}$ and $\mathcal{A}_{\alpha,\beta}^{(j+1)}$ for each $2 \le j \le 6$, and one can easily check that

$$\mathcal{A}_{\alpha,\beta}^{(j)} \Sigma_j^{-1} = \mathcal{A}_{\alpha,\beta}^{(j+1)} \Sigma_j^{-1}.$$

It follows that

$$\mathcal{R}_j = \mathcal{R}_{j+1} \Sigma_j^{-1}, \tag{15}$$

for all $2 \le j \le 6$, with the convention that $\mathcal{R}_7 := \mathcal{A}_{\alpha,\beta}$. Similarly to (10), we construct the following embeddings:

$$\mathcal{R}_7 = \mathcal{A}_{\alpha,\beta} \subset \mathcal{R}_6 = \mathcal{R}_7 \Sigma_6^{-1} \subset \mathcal{R}_5 = \mathcal{R}_6 \Sigma_5^{-1} \subset \mathcal{R}_4 = \mathcal{R}_5 \Sigma_4^{-1} \subset \mathcal{R}_3. \tag{16}$$

Observe that $\mathcal{R}_3 = \mathcal{A}_{\alpha,\beta}^{(3)} S_3^{-1} = \mathcal{R}_4 \Sigma_3^{-1}$ is the Poisson torus $\mathscr{P}_{\alpha,\beta} = \mathbb{K}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ from Subsection 4.3.

Strategy to compute the Poisson derivations of $\mathcal{A}_{\alpha,\beta} = \mathcal{R}_7$. Thanks to (16), we can extend the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ to Poisson derivations of $\mathcal{R}_3$. From Corollary 2.7, we know that the Poisson derivations of the Poisson torus $\mathcal{R}_3$ is the sum of an inner and a scalar Poisson derivations. We will then 'pull back' the Poisson derivations of $\mathcal{R}_3$ sequentially through the Poisson algebras in (16) to the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. This will give us a complete description of the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. This process will be carried out in steps, and the linear bases for $\mathcal{R}_i$ will play crucial role. We will therefore compute these bases in the subsequent subsection.

The embeddings in (16) present an opportunity to compute the centre of each of the algebras $\mathcal{R}_i$, which will be helpful in studying the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$.

Lemma 5.1. *As usual, let $Z_P(\mathcal{R}_i)$ denote the Poisson centre of $\mathcal{R}_i$. Then $Z_P(\mathcal{R}_i) = \mathbb{K}$ for each $3 \le i \le 7$.*

Proof. It is easy to verify that $Z_P(\mathcal{R}_3) = \mathbb{K}$ since the matrix associated to this Poisson torus is invertible. Note that $\mathcal{R}_7 = \mathcal{A}_{\alpha,\beta}$. Since $\mathcal{R}_i$ is a localization of $\mathcal{R}_{i+1}$, we have that $\mathbb{K} \subseteq Z_P(\mathcal{R}_7) \subseteq Z_P(\mathcal{R}_6) \subseteq \cdots \subseteq Z_P(\mathcal{R}_3) = \mathbb{K}$. Therefore, $Z_P(\mathcal{R}_7) = Z_P(\mathcal{R}_6) = \cdots = Z_P(\mathcal{R}_3) = \mathbb{K}$. $\square$

5.2. Linear bases for $\mathcal{R}_3$, $\mathcal{R}_4$ and $\mathcal{R}_5$

We aim to find a basis for $\mathcal{R}_j$ for each $j = 3, 4, 5$. Since $\mathcal{R}_3$ is a Poisson torus generated by $t_3^{\pm 1}, \dots, t_6^{\pm 1}$ over $\mathbb{K}$, the set $\{t_3^i t_4^j t_5^k t_6^l \mid i, j, k, l \in \mathbb{Z}\}$ is a basis of $\mathcal{R}_3$.

For simplicity, we set

$$f_1 := x_{1,4} \qquad F_1 := X_{1,4}$$
$$z_1 := x_{1,5} \qquad Z_1 := X_{1,5}$$
$$z_2 := x_{2,5} \qquad Z_2 := X_{2,5}.$$

5.2.1. Basis for $\mathcal{R}_4$

Observe that

$$\mathcal{A}_{\alpha,\beta}^{(4)} = \frac{\mathcal{A}^{(4)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $\Omega_1 = F_1 T_3 T_5 - \frac{1}{2} T_2 T_5$ and $\Omega_2 = T_2 T_4 T_6$ in $\mathcal{A}^{(4)}$ (Subsection 3.3). Set

$$\mathcal{A}_{\beta}^{(4)} S_4^{-1} := \frac{\mathcal{A}^{(4)} S_4^{-1}}{\langle \Omega_2 - \beta \rangle},$$

where $\beta \in \mathbb{K}$. We will denote the canonical images of $X_{i,4}$ (resp. T_i) in $\mathcal{A}_\beta^{(4)}$ by $\widehat{x_{i,4}}$ (resp. $\widehat{t_i}$) for all $1 \le i \le 6$ (resp. $3 \le i \le 6$). Observe that $\widehat{t_2} = \beta \widehat{t_6}^{-1}\widehat{t_4}^{-1}$ in $\mathcal{A}_\beta^{(4)} S_4^{-1}$. As usual, one can identify $\mathcal{R}_4$ with $\mathcal{A}_\beta^{(4)} S_4^{-1}/\langle \widehat{\Omega}_1 - \alpha\rangle$.

Proposition 5.2. *The set* $\mathfrak{P}_4 = \{f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid (i_1, i_4, i_5, i_6) \in \mathbb{N} \times \mathbb{Z}^3\} \cup \{t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid (i_3, i_4, i_5, i_6) \in \mathbb{N} \times \mathbb{Z}^3\}$ *is a* $\mathbb{K}$*-basis of* $\mathcal{R}_4$.

Proof. One can easily verify that $\left(\widehat{f_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,k_4,k_5,k_6)\in \mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $\mathcal{A}_\beta^{(4)} S_4^{-1}$. Since $\mathcal{A}_\beta^{(4)} S_4^{-1} = \mathcal{A}^{(4)} S_4^{-1}/\langle \Omega_2 - \beta\rangle$, the family $(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6})_{(k_1,k_3,k_4,k_5,k_6)\in \mathbb{N}^2\times\mathbb{Z}^3}$ spans $\mathcal{R}_4$. We show that $\mathfrak{P}_4$ is a spanning set of $\mathcal{R}_4$ by showing that $f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}_4$ for all $(k_1, k_3, k_4, k_5, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3$. It is sufficient to do this by an induction on k_1. The result is clear when $k_1 = 0$. Assume that the statement is true for $k_1 \ge 0$. That is,

$$f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = \sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6},$$

where $\underline{i} = (i_1, i_4, i_5, i_6) \in I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $\underline{j} = (i_3, i_4, i_5, i_6) \in I_2 \subset \mathbb{N} \times \mathbb{Z}^3$. Note that the $a_{\underline{i}}$ and $b_{\underline{j}}$ are all scalars.

$$f_1^{k_1+1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = f_1(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}) = \sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6}.$$

Clearly, the monomial $f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \text{Span}(\mathfrak{P}_4)$. We have to also show that $f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \text{Span}(\mathfrak{P}_4)$ for all $i_3 \in \mathbb{N}$ and $i_4, i_5, i_6 \in \mathbb{Z}$. This can easily be achieved by induction on i_3 and the use of the relation $f_1 t_3 = \alpha t_5^{-1} + \frac{1}{2}\beta t_4^{-1} t_6^{-1}$. Therefore, by the principle of mathematical induction, $\mathfrak{P}_4$ is a spanning set of $\mathcal{R}_4$ over $\mathbb{K}$.

We prove that $\mathfrak{P}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0$$

for finite sets $I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $I_2 \subset \mathbb{N} \times \mathbb{Z}^3$, and scalars $a_{\underline{i}}$ ($\underline{i} \in I_1$) and $b_{\underline{j}}$ ($\underline{j} \in I_2$).

It follows that there exists $\nu \in \mathcal{A}_\beta^{(4)} S_4^{-1}$ such that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = \left(\widehat{\Omega}_1 - \alpha\right)\nu.$$

Write $\nu = \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}$, with $\underline{l} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^2 \times \mathbb{Z}^3$ and $c_{\underline{l}} \in \mathbb{K}$. One can easily deduce that $\widehat{\Omega}_1 = \widehat{f_1}\widehat{t_3}\widehat{t_5} - \frac{1}{2}\widehat{t_2}\widehat{t_5} = \widehat{f_1}\widehat{t_3}\widehat{t_5} - \frac{1}{2}\beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}\widehat{t_5}$ (note that $\widehat{t_2} = \beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}$). It follows that

$$\begin{aligned}\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} &= \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6} \\ &\quad - \sum_{\underline{l}\in J} \frac{1}{2}\beta c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4-1}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6-1} \\ &\quad - \sum_{\underline{l}\in J} \alpha c_{\underline{l}} \widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $c_{(i_1,i_3,i_4,i_5,i_6)} \neq 0$. Let $(w_1, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^2 \times \mathbb{Z}^3$) of J such that $c_{(w_1,w_3,w_4,w_5,w_6)} \neq 0$. Since

$$\left(\widehat{f_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,k_4,k_5,k_6)\in \mathbb{N}^2\times\mathbb{Z}^3}$$

is a basis of $\mathcal{A}^{(4)} S_4^{-1}$, we can identify the coefficients of $\widehat{f_1}^{w_1+1}\widehat{t_3}^{w_3+1}\widehat{t_4}^{w_4}\widehat{t_5}^{w_5+1}\widehat{t_6}^{w_6}$ in the above equality to be $c_{(w_1,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Therefore, $c_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in J$. This further implies that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = 0.$$

Consequently, $a_{\underline{i}}$ and $b_{\underline{j}}$ are all zero. In conclusion, $\mathfrak{P}_4$ is a linearly independent set. □

5.2.2. Basis for $\mathcal{R}_5$

We will identify $\mathcal{R}_5$ with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2-\beta\rangle$, where $\mathcal{A}_\alpha^{(5)}S_5^{-1}=\mathcal{A}^{(5)}S_5^{-1}/\langle\Omega_1-\alpha\rangle$. Note that the canonical images of $X_{i,5}$ (resp. T_i) in $\mathcal{A}_\alpha^{(5)}$ will be denoted by $\widehat{x_{i,5}}$ (resp. $\widehat{t_i}$) for all $1\le i\le 6$ (resp. $3\le i\le 6$). We now find a basis for $\mathcal{A}_\alpha^{(5)}S_5^{-1}$. Recall from Subsection 3.3 that $\Omega_1=Z_1T_3T_5-\frac{1}{2}Z_2T_5$ and $\Omega_2=Z_2T_4T_6-\frac{2}{3}T_3^3T_6$ in $\mathcal{A}^{(5)}$ (remember that $Z_1:=X_{1,5}$ and $Z_2:=X_{2,5}$). Since $z_2t_4t_6-\frac{2}{3}t_3^3t_6=\beta$ and $\widehat{z_1}\widehat{t_3}\widehat{t_5}-\frac{1}{2}\widehat{z_2}\widehat{t_5}=\alpha$ in $\mathcal{R}_5$ and $\mathcal{A}_\alpha^{(5)}S_5^{-1}$ respectively, we have the relation $\widehat{z_2}=2\widehat{z_1}\widehat{t_3}-2\alpha\widehat{t_5}^{-1}$ in $A_\alpha^{(5)}S_5^{-1}$ and, in $\mathcal{R}_5$, we have the following two relations:

$$z_2=2(z_1t_3-\alpha t_5^{-1}). \tag{17}$$

$$t_3^3=\frac{3}{2}(z_2t_4-\beta t_6^{-1})=3z_1t_3t_4-\frac{3}{2}\beta t_6^{-1}-3\alpha t_4t_5^{-1}. \tag{18}$$

Proposition 5.3. *The set $\mathfrak{P}_5=\left\{z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6}\mid(\xi,i_1,i_4,i_5,i_6)\in\{0,1,2\}\times\mathbb{N}^2\times\mathbb{Z}^2\right\}$ is a $\mathbb{K}$-basis of $\mathcal{R}_5$.*

Proof. One can easily show that the family $\left(\widehat{z_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,k_4,k_5,k_6)\in\mathbb{N}^3\times\mathbb{Z}^2}$ is a basis of $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2-\beta\rangle$. Since $\mathcal{R}_5$ is identified with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2-\beta\rangle$, we show that $z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{P}_5$ for all $(k_1,k_3,k_4,k_5,k_6)\in\mathbb{N}^3\times\mathbb{Z}^2$. It is sufficient to do this by an induction on k_3. The result is obvious when $k_3=0,1,2$. For $k_3\ge 2$, suppose that

$$z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}=\sum_{(\xi,\underline{i})\in I}a_{(\xi,\underline{i})}z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6},$$

where I is a finite subset of $\{0,1,2\}\times\mathbb{N}^2\times\mathbb{Z}^2$, and the $a_{(\xi,\underline{i})}$ are all scalars. It follows that

$$z_1^{k_1}t_3^{k_3+1}t_4^{k_4}t_5^{k_5}t_6^{k_6}=\left(z_1^{k_1}t_3^{k_3}t_4^{k_4}t_5^{k_5}t_6^{k_6}\right)t_3=\sum_{(\xi,\underline{i})\in I}a_{(\xi,\underline{i})}z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6}.$$

Now, $z_1^{i_1}t_3^{\xi+1}t_4^{i_4}t_5^{i_5}t_6^{i_6}\in\mathrm{Span}(\mathfrak{P}_5)$ when $\xi=0,1$. For $\xi=2$, one can easily verify that $z_1^{i_1}t_3^{3}t_4^{i_4}t_5^{i_5}t_6^{i_6}\in\mathrm{Span}(\mathfrak{P}_5)$ by using the relation in (18). Therefore, by the principle of mathematical induction, $\mathfrak{P}_5$ spans $\mathcal{R}_5$.

We now prove that $\mathfrak{P}_5$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{i})\in I}a_{(\xi,\underline{i})}z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6}=0$$

for a finite subset $I\subset\{0,1,2\}\times\mathbb{N}^2\times\mathbb{Z}^2$ and scalars $a_{(\xi,\underline{i})}$. Since $\mathcal{R}_5$ is identified with $\mathcal{A}_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2-\beta\rangle$, we have that

$$\sum_{(\xi,\underline{i})\in I}a_{(\xi,\underline{i})}\widehat{z_1}^{i_1}\widehat{t_3}^{\xi}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}=\langle\widehat{\Omega}_2-\beta\rangle\nu,$$

where $\nu\in\mathcal{A}_\alpha^{(5)}S_5^{-1}$. Write $\nu=\sum_{\underline{j}\in J}b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}$, with $\underline{j}=(i_1,i_3,i_4,i_5,i_6)\in J\subset\mathbb{N}^3\times\mathbb{Z}^2$ and $b_{\underline{j}}\in\mathbb{K}$. Given $\Omega_2=Z_2T_4T_6-\frac{2}{3}T_3^3T_6$ in $\mathcal{A}^{(5)}$ and the relation (17), one can deduce that

$$\widehat{\Omega}_2=\widehat{z_2}\widehat{t_4}\widehat{t_6}-\frac{2}{3}\widehat{t_3}^3\widehat{t_6}=2\widehat{z_1}\widehat{t_3}\widehat{t_4}\widehat{t_6}-2\alpha\widehat{t_4}\widehat{t_5}^{-1}\widehat{t_6}-\frac{2}{3}\widehat{t_3}^3\widehat{t_6}.$$

Therefore,

$$\begin{aligned}\sum_{(\xi,\underline{i})\in I}a_{(\xi,\underline{i})}\widehat{z_1}^{i_1}\widehat{t_3}^{\xi}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}=&\sum_{\underline{j}\in J}2b_{\underline{j}}\widehat{z_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4+1}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6+1}\\&-\sum_{\underline{j}\in J}\frac{2}{3}b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3+3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6+1}\\&-\sum_{\underline{j}\in J}2\alpha b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4+1}\widehat{t_5}^{i_5-1}\widehat{t_6}^{i_6+1}\\&-\sum_{\underline{j}\in J}\beta b_{\underline{j}}\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1,i_3,i_4,i_5,i_6)\in J$ such that $b_{(i_1,i_3,i_4,i_5,i_6)}\neq 0$. Let $(w_1,w_3,w_4,w_5,w_6)\in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^3\times\mathbb{Z}^2$) of J such that $b_{(w_1,w_3,w_4,w_5,w_6)}\neq 0$. Given that

$$\left(\widehat{z_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,\dots,k_6)\in\mathbb{N}^3\times\mathbb{Z}^2}$$

is a basis of $\mathcal{A}_\alpha^{(5)}S_5^{-1}$, one can identify the coefficients of $\widehat{z_1}^{w_1+1}\widehat{t_3}^{w_3+1}\widehat{t_4}^{w_4+1}\widehat{t_5}^{w_5}\widehat{t_6}^{w_6+1}$ in the above equality as $2b_{(w_1,w_3,w_4,w_5,w_6)}=0$. Hence, $b_{(w_1,w_3,w_4,w_5,w_6)}=0$, a contradiction! Therefore, $b_{(i_1,i_3,i_4,i_5,i_6)}=0$ for all $(i_1,i_3,i_4,i_5,i_6)\in J$. Consequently,

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}\widehat{z_1}^{i_1}\widehat{t_3}^{\xi}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}=0.$$

It follows that $a_{(\xi,\underline{i})}=0$ for all $(\xi,\underline{i})\in I$. As a result, $\mathfrak{P}_5$ is a linearly independent set. $\square$

As an easy consequence of Proposition 5.3 we obtain:

Corollary 5.4. *Let I be a finite subset of $\{0,1,2\}\times\mathbb{N}\times\mathbb{Z}^3$ and let $(a_{(\xi,\underline{i})})_{\underline{i}\in I}$ be a family of scalars. If*

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi,\underline{i})}z_1^{i_1}t_3^{\xi}t_4^{i_4}t_5^{i_5}t_6^{i_6}=0,$$

then $a_{(\xi,\underline{i})}=0$ for all $(\xi,\underline{i})\in I$.

Remark 5.5.

1. Since $\mathcal{R}_7=\mathcal{A}_{\alpha,\beta}$, we already have a basis for $\mathcal{R}_7$ (see Proposition 4.14).
2. We were not successful in finding a basis for $\mathcal{R}_6$. However, this will not prevent us to compute the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ in the next subsection.

5.3. *Poisson derivations of $\mathcal{A}_{\alpha,\beta}$*

We are now (almost) ready to study the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$. We will begin with the case where both α and β are non-zero, and subsequently state our results in the case where either α or β is zero (but not both). Before we proceed, we note the following relations between elements of the various Poisson algebras involved. These relations can easily be obtained through direct computation.

Lemma 5.6.

$$\begin{aligned}
f_1&=t_1+\frac{1}{2}t_2t_3^{-1} & x_{3,6}&=t_3+\frac{3}{2}t_4t_5^{-1}\\
z_1&=f_1+\frac{1}{3}t_3^2t_4^{-1} & x_1&=x_{1,6}+\frac{1}{2}t_5t_6^{-1}\\
z_2&=t_2+\frac{2}{3}t_3^3t_4^{-1} & x_3&=x_{3,6}+t_5^2t_6^{-1}\\
x_{1,6}&=z_1+x_{3,6}t_5^{-1}-\frac{3}{4}t_4t_5^{-2} & x_4&=t_4+\frac{2}{3}t_5^3t_6^{-1}.
\end{aligned}$$

5.3.1. *Poisson derivations of $\mathcal{A}_{\alpha,\beta}$ $(\alpha,\beta\neq 0)$*

Throughout this subsection, we assume that α and β are non-zero. Let $\mathrm{Der}_P(\mathcal{A})$ be the collection of all the Poisson $\mathbb{K}$-derivations of $\mathcal{A}_{\alpha,\beta}$, and $\mathcal{D}\in\mathrm{Der}_P(\mathcal{A})$. Now, $\mathcal{D}$ extends uniquely to a Poisson derivation of each of the algebras in (16) via localization. Hence, $\mathcal{D}$ is a Poisson derivation of the Poisson torus $\mathcal{R}_3=\mathbb{K}[t_3^{\pm1},t_4^{\pm1},t_5^{\pm1},t_6^{\pm1}]$. It follows from Corollary 2.7 that $\mathcal{D}$ can be written as

$$\mathcal{D}=\mathrm{ham}_x+\rho,$$

where ρ is a scalar Poisson derivation of $\mathcal{R}_3$ defined by $\rho(t_i)=\lambda_i t_i,\ \ i=3,4,5,6;$ with $\lambda_i\in Z_P(\mathcal{R}_3)=\mathbb{K}$, and $\mathrm{ham}_x=\{x,-\}:\mathcal{R}_3\to\mathcal{R}_3$ with $x\in\mathcal{R}_3$ (see Corollary 2.7).

We aim to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{A}_{\alpha,\beta}$. We do this in several steps. We first describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{R}_4$.

Lemma 5.7.

1. $x\in\mathcal{R}_4$.
2. $\lambda_5=\lambda_4+\lambda_6$, $\rho(f_1)=-(\lambda_3+\lambda_5)f_1$ *and* $\rho(t_2)=-\lambda_5t_2$.
3. *Set* $\lambda_1:=-(\lambda_3+\lambda_5)$ *and* $\lambda_2:=-\lambda_5$. *Then,* $\mathcal{D}(x_{\kappa,4})=\mathrm{ham}_x(x_{\kappa,4})+\lambda_\kappa x_{\kappa,4}$ *for all* $\kappa\in\{1,\dots,6\}$.

Proof. 1. Recall that $\mathcal{R}_3 = \mathcal{R}_4[t_3^{-1}]$. Observe that $\mathcal{Q} := \mathbb{K}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ is a subalgebra of both $\mathcal{R}_3$ and $\mathcal{R}_4$. One can easily verify that $z := t_4 t_5^{-1} t_6$ is a Poisson central element of $\mathcal{Q}$. Since $\mathcal{R}_3$ is a Poisson torus, it can be presented as a free $\mathcal{Q}$-module with basis $(t_3^j)_{j\in\mathbb{Z}}$. One can therefore write $x \in \mathcal{R}_3$ as $x = \sum_{j\in\mathbb{Z}} b_j t_3^j$, where $b_j \in \mathcal{Q}$. Decompose x as $x = x_- + x_+$, where $x_+ := \sum_{j\geq 0} b_j t_3^j$ and $x_- := \sum_{j<0} b_j t_3^j$. Clearly, $x_+ \in \mathcal{R}_4$. We now want to show that $x_- \in \mathcal{R}_4$. Write $x_- = \sum_{j=-1}^{-m} b_j t_3^j$ for some $m \in \mathbb{N}_{>0}$.

Now, $\mathcal{D}(z) = \mathrm{ham}_x(z) + \rho(z) = \mathrm{ham}_{x_-}(z) + \mathrm{ham}_{x_+}(z) + (\lambda_4 - \lambda_5 + \lambda_6)z \in \mathcal{R}_4$. We have that $\mathrm{ham}_{x_+}(z) + (\lambda_4 - \lambda_5 + \lambda_6)z \in \mathcal{R}_4$, hence $\mathrm{ham}_{x_-}(z) \in \mathcal{R}_4$. Note that $\{t_3, z\} = 2zt_3$, and $\{\gamma, z\} = 0$ for all $\gamma \in \mathcal{Q}$ since z is Poisson central in $\mathcal{Q}$. One can therefore express $\mathrm{ham}_{x_-}(z)$ as follows:

$$\mathrm{ham}_{x_-}(z) = \{x_-, z\} = \sum_{j=-1}^{-m} b_j \{t_3^j, z\} = \sum_{j=-1}^{-m} 2jb_j z t_3^j \in \mathcal{R}_4.$$

Let $n \in \mathbb{N}_{>0}$, and set

$$\ell^{(n)} := \underbrace{\{\{\ldots\{}_{n-\text{times}} x_-, \underbrace{z\}, z\}, \ldots, z\}}_{n-\text{times}} \in \mathcal{R}_4.$$

We claim that

$$\ell^{(n)} = \sum_{j=-1}^{-m} (2j)^n z^n b_j t_3^j,$$

for all $n \in \mathbb{N}_{>0}$. Observe that

$$\ell^{(1)} = \mathrm{ham}_{x_-}(z) = \sum_{j=-1}^{-m} 2jb_j z t_3^j,$$

hence the result is true for $n = 1$. Suppose that the result is true for $n \geq 1$. Then,

$$\ell^{(n+1)} = \{\ell^{(n)}, z\} = \sum_{j=-1}^{-m} (2j)^n z^n b_j \{t_3^j, z\} = \sum_{j=-1}^{-m} (2j)^{n+1} z^{n+1} b_j t_3^j$$

as expected. By the principle of mathematical induction, the claim is proved.

Given that $\ell^{(n)} = \sum_{j=-1}^{-m} (2j)^n z^n b_j t_3^j$, it follows that

$$\mu_n := \ell^{(n)} z^{-n} = \sum_{j=-1}^{-m} (2j)^n b_j t_3^j \in \mathcal{R}_4.$$

The above equality can be written as a matrix equation:

$$\begin{bmatrix} -2 & -4 & -6 & \cdots & -2m \\ (-2)^2 & (-4)^2 & (-6)^2 & \cdots & (-2m)^2 \\ (-2)^3 & (-4)^3 & (-6)^3 & \cdots & (-2m)^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-2)^m & (-4)^m & (-6)^m & \cdots & (-2m)^m \end{bmatrix} \begin{bmatrix} b_{-1}t_3^{-1} \\ b_{-2}t_3^{-2} \\ b_{-3}t_3^{-3} \\ \vdots \\ b_{-m}t_3^{-m} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_m \end{bmatrix}.$$

One can observe that the coefficient matrix

$$\begin{bmatrix} -2 & -4 & -6 & \cdots & -2m \\ (-2)^2 & (-4)^2 & (-6)^2 & \cdots & (-2m)^2 \\ (-2)^3 & (-4)^3 & (-6)^3 & \cdots & (-2m)^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-2)^m & (-4)^m & (-6)^m & \cdots & (-2m)^m \end{bmatrix}$$

is similar to a Vandermonde matrix (since the terms in each column form a geometric sequence) which is well known to be invertible. This therefore implies that each $b_j t_3^j$ is a linear combination of the $\mu_n \in \mathcal{R}_4$. As a result, $b_j t_3^j \in \mathcal{R}_4$ for all $j \in \{-1, \ldots, -m\}$. Consequently, $x_- = \sum_{j=-1}^{-m} b_j t_3^j \in \mathcal{R}_4$ as desired.

2. Recall that $\rho(t_\kappa)=\lambda_\kappa t_\kappa$ for all $\kappa\in\{3,4,5,6\}$ and $\lambda_\kappa\in\mathbb{K}$. From Lemma 5.6, we have that $f_1=t_1+\frac{1}{2}t_2t_3^{-1}$. Again, recall from Subsection 4.3.1 that $t_1=\alpha t_3^{-1}t_5^{-1}$ and $t_2=\beta t_4^{-1}t_6^{-1}$ in $\mathcal{R}_3=\mathscr{P}_{\alpha,\beta}$. As a result, $f_1=\alpha t_3^{-1}t_5^{-1}+\frac{1}{2}\beta t_3^{-1}t_4^{-1}t_6^{-1}$. Therefore,

$$\rho(f_1)=-(\lambda_3+\lambda_5)\alpha t_3^{-1}t_5^{-1}-\frac{1}{2}(\lambda_3+\lambda_4+\lambda_6)\beta t_3^{-1}t_4^{-1}t_6^{-1}. \tag{19}$$

Also, $\rho(f_1)\in\mathcal{R}_4$ implies that $\rho(f_1)$ can be written in terms of the basis $\mathfrak{P}_4$ of $\mathcal{R}_4$ (Proposition 5.2) as

$$\rho(f_1)=\sum_{r>0}a_rf_1^r+\sum_{s\geq 0}b_st_3^s, \tag{20}$$

where a_r and b_s belong to $\mathcal{Q}=\mathbb{K}[t_4^{\pm1},t_5^{\pm1},t_6^{\pm1}]$. Note that

$$\begin{aligned} f_1^r&=\left(\alpha t_3^{-1}t_5^{-1}+\frac{1}{2}\beta t_3^{-1}t_4^{-1}t_6^{-1}\right)^r=\sum_{i=0}^{r}\binom{r}{i}\alpha^i(\beta/2)^{r-1}t_3^{-r}t_4^{i-r}t_5^{-i}t_6^{i-r}\\ &=c_rt_3^{-r}, \end{aligned} \tag{21}$$

where

$$c_r:=\sum_{i=0}^{r}\binom{r}{i}\alpha^i(\beta/2)^{r-i}t_4^{i-r}t_5^{-i}t_6^{i-r}\in\mathcal{Q}\setminus\{0\}. \tag{22}$$

Substitute (21) into (20) to obtain

$$\rho(f_1)=\sum_{r>0}a_rc_rt_3^{-r}+\sum_{s\geq 0}b_st_3^s. \tag{23}$$

One can rewrite (19) as

$$\rho(f_1)=dt_3^{-1}, \tag{24}$$

where $d=-(\lambda_5+\lambda_3)\alpha t_5^{-1}-\frac{1}{2}(\lambda_6+\lambda_4+\lambda_3)\beta t_4^{-1}t_6^{-1}\in\mathcal{Q}$. Comparing (23) to (24) shows that $b_s=0$ for all $s\geq 0$, and $a_rc_r=0$ for all $r\neq 1$. Therefore, $\rho(f_1)=a_1c_1t_3^{-1}$. Moreover, from (22), $c_1=\frac{1}{2}\beta t_4^{-1}t_6^{-1}+\alpha t_5^{-1}$. Hence,

$$\rho(f_1)=a_1c_1t_3^{-1}=a_1\left(\frac{1}{2}\beta t_4^{-1}t_6^{-1}+\alpha t_5^{-1}\right)t_3^{-1}=a_1\alpha t_3^{-1}t_5^{-1}+\frac{1}{2}a_1\beta t_3^{-1}t_4^{-1}t_6^{-1}. \tag{25}$$

Comparing (25) to (19) reveals that $a_1=-(\lambda_5+\lambda_3)=-(\lambda_6+\lambda_4+\lambda_3)$. Consequently, $\lambda_5=\lambda_6+\lambda_4$. Hence, $\rho(f_1)=-(\lambda_5+\lambda_3)\alpha t_3^{-1}t_5^{-1}-\frac{1}{2}(\lambda_5+\lambda_3)\beta t_3^{-1}t_4^{-1}t_6^{-1}=-(\lambda_5+\lambda_3)f_1$. Finally, since $t_2=\beta t_4^{-1}t_6^{-1}$ in $\mathcal{R}_4$, it follows that

$$\rho(t_2)=-(\lambda_6+\lambda_4)\beta t_4^{-1}t_6^{-1}=-(\lambda_6+\lambda_4)t_2=-\lambda_5t_2.$$

3. The result easily follows from the previous points since $f_1=x_{1,4}$ and $t_i=x_{i,4}$ $(2\leq i\leq 6)$. □

We now proceed to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{R}_5$.

Lemma 5.8.

1. *$x\in\mathcal{R}_5$.*
2. *$\lambda_4=3\lambda_3+\lambda_5$, $\lambda_6=-3\lambda_3$, $\rho(z_1)=-(\lambda_3+\lambda_5)z_1$ and $\rho(z_2)=-\lambda_5z_2$.*
3. *$\mathcal{D}(x_{\kappa,5})=\mathrm{ham}_x(x_{\kappa,5})+\lambda_\kappa x_{\kappa,5}$ for all $\kappa\in\{1,\dots,6\}$.*

Proof. In this proof, we denote $\underline{\upsilon}:=(i,j,k,l)\in\mathbb{N}\times\mathbb{Z}^3$.

1. We already know that $x\in\mathcal{R}_4=\mathcal{R}_5[t_4^{-1}]$. Given the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ from Proposition 5.3, x can be written as $x=\sum_{(\xi,\underline{\upsilon})\in I}a_{(\xi,\underline{\upsilon})}z_1^it_3^\xi t_4^jt_5^kt_6^l$, where I is a finite subset of $\{0,1,2\}\times\mathbb{N}\times\mathbb{Z}^3$ and the $a_{(\xi,\underline{\upsilon})}$ are scalars. Write $x=x_-+x_+$, where

$$x_+=\sum_{\substack{(\xi,\underline{\upsilon})\in I\\ j\geq 0}}a_{(\xi,\underline{\upsilon})}z_1^it_3^\xi t_4^jt_5^kt_6^l \text{ and } x_-=\sum_{\substack{(\xi,\underline{\upsilon})\in I\\ j<0}}a_{(\xi,\underline{\upsilon})}z_1^it_3^\xi t_4^jt_5^kt_6^l.$$

Suppose that $x_-\neq 0$. Then, there exists a minimum $j_0<0$ such that $a_{(\xi,i,j_0,k,l)}\neq 0$ for some $(\xi,i,j_0,k,l)\in I$ and $a_{(\xi,i,j,k,l)}=0$ for all $(\xi,i,j_0,k,l)\in I$ with $j<j_0$. Given this assumption, write

$$x_- = \sum_{\substack{(\xi,\underline{\upsilon})\in I\\ j_0\leq j\leq -1}} a_{(\xi,\underline{\upsilon})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

We aim to show that $x_- = 0$. Let $s \in \{3,6\}$. Then, $\mathcal{D}(t_s) = \mathrm{ham}_{x_+}(t_s) + \mathrm{ham}_{x_-}(t_s) + \rho(t_s) \in \mathcal{R}_5$. This implies that $\mathrm{ham}_{x_-}(t_s) \in \mathcal{R}_5$, since $\mathrm{ham}_{x_+}(t_s) + \rho(t_s) = \mathrm{ham}_{x_+}(t_s) + \lambda_s t_s \in \mathcal{R}_5$. Set $\underline{w} := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. One can therefore write $\mathrm{ham}_{x_-}(t_s) \in \mathcal{R}_5$ in terms of the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ as:

$$\mathrm{ham}_{x_-}(t_s) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l, \tag{26}$$

where J is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and the $b_{(\xi,\underline{w})}$ are scalars.

When $s = 6$, we deduce from Remark 2.1(2) that one can also express $\mathrm{ham}_{x_-}(t_6)$ as:

$$\mathrm{ham}_{x_-}(t_6) = \sum_{\substack{(\xi,\underline{\upsilon})\in I\\ j_0\leq j\leq -1}} 3(k+j-i) a_{(\xi,\underline{\upsilon})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1}.$$

Comparing this expression for $\mathrm{ham}_{x_-}(t_6)$ to (26) (when $s = 6$), we have that

$$\sum_{\substack{(\xi,\underline{\upsilon})\in I\\ j_0\leq j\leq -1}} 3(k+j-i) a_{(\xi,\underline{\upsilon})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1} = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

As $\mathfrak{P}_5$ is a basis for $\mathcal{R}_5$ by Proposition 5.3, we deduce from Corollary 5.4 that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};\, j,k,l\in\mathbb{Z};\, \xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Now, at $j = j_0$, denote $\underline{\upsilon} = (i,j,k,l)$ by $\underline{\upsilon}_0 := (i, j_0, k, l)$. Since $\underline{v}_0 \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with $j \geq 0$), it follows from the above equality that, at $\underline{\upsilon}_0$, we must have:

$$3(k + j_0 - i) a_{(\xi,\underline{\upsilon}_0)} = 0.$$

From our initial assumption, we have that $a_{(\xi,\underline{\upsilon}_0)}$ are all not zero. Therefore,

$$k = i - j_0, \tag{27}$$

for some $(\xi, \underline{v}_0) \in I$.

Similarly, when $s = 3$, then using Remark 2.1(2), one can also express $\mathrm{ham}_{x_-}(t_3)$ as:

$$\begin{aligned}\mathrm{ham}_{x_-}(t_3) = &-\sum\left[\frac{3}{2}\beta(3i-k-3j_0)a_{2,i,j_0,k,l+1} + 2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)}\right] z_1^i t_4^{j_0} t_5^k t_6^l\\ &+\sum\left[(3i-k-3j_0)a_{(0,i,j_0,k,l)} - 2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)}\right] z_1^i t_3 t_4^{j_0} t_5^k t_6^l\\ &+\sum\left[(3i-k-3j_0)a_{(1,i,j_0,k,l)} - 2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)}\right] z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K},\end{aligned}$$

where $\mathcal{K} \in \mathrm{Span}\left(\mathfrak{P}_5 \setminus \{z_1^i t_3^\xi t_4^{j_0} t_5^k t_6^l \mid (\xi, i, j_0, k, l) \in \{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^3\}\right)$ (note that one needs the following two expressions $z_2 = 2(z_1 t_3 - \alpha t_5^{-1})$ and $t_3^3 = 3 z_1 t_3 t_4 - 3\alpha t_4 t_5^{-1} - \dfrac{3\beta}{2} t_6^{-1}$ from (17) and (18) to express some of the monomials in terms of the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$). Comparing this expression for $\mathrm{ham}_{x_-}(t_3)$ to (26) (when $s = 3$) reveals that

$$\begin{aligned}&\sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l =\\ &\qquad -\sum\left[\frac{3}{2}\beta(3i-k-3j_0)a_{(2,i,j_0,k,l+1} + 2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)}\right] z_1^i t_4^{j_0} t_5^k t_6^l\\ &\qquad +\sum\left[(3i-k-3j_0)a_{(0,i,j_0,k,l)} - 2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)}\right] z_1^i t_3 t_4^{j_0} t_5^k t_6^l\\ &\qquad +\sum\left[(3i-k-3j_0)a_{(1,i,j_0,k,l)} - 2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)}\right] z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K}.\end{aligned}$$

We have already established that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};\, j,k,l\in\mathbb{Z};\, \xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Since $\underline{v}_0 = (i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with $j \geq 0$), it follows from the above equality that, at $\underline{\upsilon}_0$, we must have:

$$\frac{3}{2}\beta(3i-k-3j_0)a_{(2,i,j_0,k,l+1)}+2(i+1)\alpha a_{(0,i+1,j_0,k+1,l)}=0, \tag{28}$$

$$(3i-k-3j_0)a_{(0,i,j_0,k,l)}-2(i+1)\alpha a_{(1,i+1,j_0,k+1,l)}=0, \tag{29}$$

$$(3i-k-3j_0)a_{(1,i,j_0,k,l)}-2(i+1)\alpha a_{(2,i+1,j_0,k+1,l)}=0. \tag{30}$$

Suppose that there exists $(\xi,i,j_0,k,l)\in I$ such that $(3i-k-3j_0)a_{(\xi,i,j_0,k,l)}=0$. Then, $a_{(\xi,i,j_0,k,l)}=0$. Otherwise, we shall have:

$$k=3(i-j_0). \tag{31}$$

Comparing (31) to (27) clearly shows that $i-j_0=0$ which implies that $i=j_0<0$, a contradiction (note that $i\geq 0$)!

Now, observe that if there exists $\xi\in\{0,1,2\}$ such that $a_{(\xi,i,j_0,k,l)}=0$ for all $(i,j_0,k,l)\in\mathbb{N}\times\mathbb{Z}^3$, then one can easily deduce from equations (28), (29) and (30) that $a_{(\xi,i,j_0,k,l)}=0$ for all $(\xi,i,j_0,k,l)\in I$. This contradicts our initial assumption. Therefore, for each $\xi\in\{0,1,2\}$, there exists some $(i,j_0,k,l)\in\mathbb{N}\times\mathbb{Z}^3$ such that $a_{(\xi,i,j_0,k,l)}\neq 0$. Without loss of generality, let (u,j_0,v,w) be the greatest element in the lexicographic order on $\mathbb{N}\times\mathbb{Z}^3$ such that $a_{(0,u,j_0,v,w)}\neq 0$ and $a_{(0,i,j_0,k,l)}=0$ for all $i>u$.

From (29), at $(i,j_0,k,l)=(u,j_0,v,w)$, we have:

$$(3u-v-3j_0)a_{(0,u,j_0,v,w)}-2(u+1)\alpha a_{(1,u+1,j_0,v+1,w)}=0.$$

From (30), at $(i,j_0,k,l)=(u+1,j_0,v+1,w)$, we have:

$$(3u-v-3j_0)a_{(1,u+1,j_0,v+1,w)}-2(u+1)\alpha a_{(2,u+2,j_0,v+2,w)}=0.$$

Finally, from (28), at $(i,j_0,k,l)=(u+2,j_0,v+2,w-1)$, we have:

$$\frac{3}{2}\beta(3u-v-3j_0)a_{(2,u+2,j_0,v+2,w)}+2(u+1)\alpha a_{(0,u+3,j_0,v+3,w-1)}=0.$$

Since $u+3>u$, it follows from the above three displayed equations (beginning from the last one) that

$$a_{(0,u+3,j_0,v+3,w-1)}=0\Rightarrow a_{(2,u+2,j_0,v+2,w)}=0\Rightarrow a_{(1,u+1,j_0,v+1,w)}=0\Rightarrow a_{(0,u,j_0,v,w)}=0,$$

a contradiction (remember that $(3i-k-3j_0)a_{(\xi,i,j_0,k,l)}=0$ implies that $a_{(\xi,i,j_0,k,l)}=0$)! Hence, $a_{(0,i,j_0,k,l)}=0$ for all $(i,j_0,k,l)\in\mathbb{N}\times\mathbb{Z}^3$. From (28), (29) and (30), one can then conclude that $a_{(\xi,i,j_0,k,l)}=0$ for all $(\xi,i,j_0,k,l)\in I$. This is a contradiction to our assumption, hence $x_-=0$. Consequently, $x=x_+\in\mathcal{R}_5$ as desired.

2. It follows from Lemma 5.6 that $z_2=t_2+\frac{2}{3}t_3^3t_4^{-1}$. Since $\rho(t_\kappa)=\lambda_\kappa t_\kappa,\ \kappa\in\{2,\dots,6\}$, with $\lambda_2=-\lambda_5$ (see Lemma 5.7), it follows that

$$\rho(z_2)=-\lambda_5t_2+\frac{2}{3}(3\lambda_3-\lambda_4)t_3^3t_4^{-1}=-\lambda_5z_2+\frac{2}{3}(3\lambda_3-\lambda_4+\lambda_5)t_3^3t_4^{-1}.$$

Furthermore,

$$\mathcal{D}(z_2)=\mathrm{ham}_x(z_2)+\rho(z_2)=\mathrm{ham}_x(z_2)-\lambda_5z_2+\frac{2}{3}(3\lambda_3-\lambda_4+\lambda_5)t_3^3t_4^{-1}\in\mathcal{R}_5.$$

We have that $(3\lambda_3-\lambda_4+\lambda_5)t_3^3t_4^{-1}\in\mathcal{R}_5$, since $\mathrm{ham}_x(z_2)-\lambda_5z_2\in\mathcal{R}_5$. This implies that $(3\lambda_3-\lambda_4+\lambda_5)t_3^3\in\mathcal{R}_5t_4$. Set $w:=3\lambda_3-\lambda_4+\lambda_5$. Suppose that $w\neq 0$. From (18), we have:

$$t_3^3=3z_1t_3t_4-\frac{3}{2}\beta t_6^{-1}-3\alpha t_4t_5^{-1}.$$

It follows that

$$wt_3^3=3wz_1t_3t_4-3w\alpha t_4t_5^{-1}-\frac{3}{2}w\beta t_6^{-1}\in\mathcal{R}_5t_4.$$

Since $t_3^3,\ t_4t_5^{-1}$ and $z_1t_3t_4$ are all elements of $\mathcal{R}_5t_4$, this implies that $t_6^{-1}\in\mathcal{R}_5t_4$. Hence, $1\in\mathcal{R}_5t_4t_6$, a contradiction (see the basis $\mathfrak{P}_5$ of $\mathcal{R}_5$ (Proposition 5.3)). Therefore, $w=3\lambda_3-\lambda_4+\lambda_5=0$, and so $\lambda_4=3\lambda_3+\lambda_5$. This further implies that $\rho(z_2)=-\lambda_5z_2$ as desired.

Again, from Lemma 5.7, we have that $\rho(f_1)=-(\lambda_3+\lambda_5)f_1$. Recall from Lemma 5.6 that $z_1=f_1+\frac{1}{3}t_3^2t_4^{-1}$. It follows that

$$\begin{aligned}\rho(z_1)&=-(\lambda_3+\lambda_5)f_1+\frac{1}{3}(2\lambda_3-\lambda_4)t_3^2t_4^{-1}=-(\lambda_3+\lambda_5)z_1+\frac{1}{3}(3\lambda_3-\lambda_4+\lambda_5)t_3^2t_4^{-1}\\&=-(\lambda_3+\lambda_5)z_1+\frac{1}{3}(3\lambda_3-(3\lambda_3+\lambda_5)+\lambda_5)t_3^2t_4^{-1}=-(\lambda_3+\lambda_5)z_1.\end{aligned}$$

Finally, we know that $\rho(t_6) = \lambda_6 t_6$, and

$$t_3^3 = 3z_1 t_3 t_4 - 3\alpha t_4 t_5^{-1} - \frac{3\beta}{2} t_6^{-1}.$$

This implies that

$$t_6^{-1} = \frac{2}{3\beta}(3z_1 t_3 t_4 - 3\alpha t_4 t_5^{-1} - t_3^3).$$

Apply ρ to this relation to obtain

$$-\lambda_6 t_6^{-1} = 3\lambda_3 \left(\frac{2}{3\beta} \left(3z_1 t_3 t_4 - 3\alpha t_4 t_5^{-1} - t_3^3 \right) \right).$$

Clearly, $\lambda_6 = -3\lambda_3$ as desired.

3. Recall that $\lambda_1 = -(\lambda_3 + \lambda_5)$ and $\lambda_2 = -\lambda_5$. The result easily follows from the previous points since $z_1 = x_{1,5}$ and $z_2 = x_{2,5}$. □

We are now ready to describe $\mathcal{D}$ as a Poisson derivation of $\mathcal{A}_{\alpha,\beta}$.

Lemma 5.9.

1. $x \in \mathcal{A}_{\alpha,\beta}$.
2. $\rho(x_\kappa) = 0$ *for all* $\kappa \in \{1, \dots, 6\}$.
3. $\mathcal{D} = \mathrm{ham}_x$.

Proof. In this proof, we set $\underline{\upsilon} := (i, j, k, l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. Recall from the PDDA for $\mathcal{A}_{\alpha,\beta}$ that $t_5 = x_5$ and $t_6 = x_6$.

1. Given the basis $\mathfrak{P}$ of $\mathcal{A}_{\alpha,\beta}$ from Proposition 4.14, one can write $x \in \mathcal{R}_5 = \mathcal{A}_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ as

$$x = \sum_{(\epsilon_1,\epsilon_2,\underline{\upsilon}) \in I} a_{(\epsilon_1,\epsilon_2,\underline{\upsilon})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l, \tag{32}$$

where I is a finite subset of $\{0, 1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $a_{(\epsilon_1,\epsilon_2,\underline{\upsilon})}$ are scalars. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{\upsilon}) \in I \\ k,\, l \geq 0}} a_{(\epsilon_1,\epsilon_2,\underline{\upsilon})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l,$$

and

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{\upsilon}) \in I \\ k<0 \text{ or } l<0}} a_{(\epsilon_1,\epsilon_2,\underline{\upsilon})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l.$$

Suppose that $x_- \neq 0$. Then, there exists a minimum negative integer k_0 or l_0 such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} \neq 0$ or $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$ for some $(\epsilon_1, \epsilon_2, i, j, k_0, l), (\epsilon_1, \epsilon_2, i, j, k, l_0) \in I$, and $a_{(\epsilon_1,\epsilon_2,i,j,k,l)} = 0$ whenever $k < k_0$ or $l < l_0$. Write

$$x_- = \sum_{\substack{(\epsilon_1,\epsilon_2,\underline{\upsilon}) \in I \\ k_0 \leq k \leq -1 \text{ or } l_0 \leq l \leq -1}} a_{(\epsilon_1,\epsilon_2,\underline{\upsilon})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l.$$

Now $\mathcal{D}(x_3) = \mathrm{ham}_{x_+}(x_3) + \mathrm{ham}_{x_-}(x_3) + \rho(x_3) \in \mathcal{A}_{\alpha,\beta}$. From Lemma 5.6, we have that $x_3 = x_{3,6} + t_5^2 t_6^{-1}$ and $x_{3,6} = t_3 + \frac{3}{2} t_4 t_5^{-1}$. Putting these two together gives:

$$x_3 = t_3 + \frac{3}{2} t_4 t_5^{-1} + t_5^2 t_6^{-1}.$$

Again, from Lemma 5.6, we also have that $t_4 = x_4 - \frac{2}{3} t_5^3 t_6^{-1}$. Note that $\rho(t_\kappa) = \lambda_\kappa t_\kappa,\ \kappa = 3, 4, 5, 6$.

Now,

$$\begin{aligned} \rho(x_3) &= \lambda_3 t_3 + \frac{3}{2}(\lambda_4 - \lambda_5) t_4 t_5^{-1} + (2\lambda_5 - \lambda_6) t_5^2 t_6^{-1} \\ &= \lambda_3 \left(x_{3,6} - \frac{3}{2} t_4 t_5^{-1} \right) + \frac{3}{2}(\lambda_4 - \lambda_5) t_4 t_5^{-1} + (2\lambda_5 - \lambda_6) t_5^2 t_6^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \lambda_3 x_{3,6} - \frac{3}{2}(\lambda_3 - \lambda_4 + \lambda_5) t_4 t_5^{-1} + (2\lambda_5 - \lambda_6) t_5^2 t_6^{-1} \\
&= \lambda_3 (x_3 - t_5^2 t_6^{-1}) - \frac{3}{2}(\lambda_3 - \lambda_4 + \lambda_5)\left(x_4 - \frac{2}{3} t_5^3 t_6^{-1}\right) t_5^{-1} + (2\lambda_5 - \lambda_6) t_5^2 t_6^{-1} \\
&= \lambda_3 x_3 + \alpha_1 x_4 t_5^{-1} + \alpha_2 t_5^2 t_6^{-1},
\end{aligned} \tag{33}
$$

where $\alpha_1 = \frac{3}{2}(\lambda_4 - \lambda_3 - \lambda_5)$ and $\alpha_2 = (3\lambda_5 - \lambda_4 - \lambda_6)$. Therefore, $\mathcal{D}(x_3) = \mathrm{ham}_{x_+}(x_3) + \mathrm{ham}_{x_-}(x_3) + \lambda_3 x_3 + \alpha_1 x_4 t_5^{-1} + \alpha_2 t_5^2 t_6^{-1} \in \mathcal{A}_{\alpha,\beta}$. It follows that $\mathcal{D}(x_3) t_5 t_6 = \mathrm{ham}_{x_+}(x_3) t_5 t_6 + \mathrm{ham}_{x_-}(x_3) t_5 t_6 + \lambda_3 x_3 t_5 t_6 + \alpha_1 x_4 t_6 + \alpha_2 t_5^3 \in \mathcal{A}_{\alpha,\beta}$. Hence, $\mathrm{ham}_{x_-}(x_3) t_5 t_6 \in \mathcal{A}_{\alpha,\beta}$, since $\mathrm{ham}_{x_+}(x_3) t_5 t_6 + \lambda_3 x_3 t_5 t_6 + \alpha_1 x_4 t_6 + \alpha_2 t_5^3 \in \mathcal{A}_{\alpha,\beta}$.

One can also verify that

$$
\begin{aligned}
\mathrm{ham}_{x_-}(x_3) t_5 t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{v}) \in I} a_{(\epsilon_1,\epsilon_2,\underline{v})} \Big(& (i + 3j - 3\epsilon_2 - k) x_1^i x_2^j x_3^{\epsilon_1+1} x_4^{\epsilon_2} t_5^{k+1} t_6^{l+1} \\
& - 3k x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2+1} t_5^k t_6^{l+1} + i x_1^{i-1} x_2^{j+1} x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^{k+1} t_6^{l+1} \\
& - 6l x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^{k+3} t_6^l \Big).
\end{aligned} \tag{34}
$$

Assume that there exists $l < 0$ such that $a_{(\epsilon_1,\epsilon_2,i,j,k,l)} \neq 0$. It follows from our initial assumption that $a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)} \neq 0$. Now, at $l = l_0$, denote $\underline{v} = (i,j,k,l)$ by $\underline{v}_0 := (i,j,k,l_0)$. From (34), we have that

$$
\mathrm{ham}_{x_-}(x_3) t_5 t_6 = - \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0) \in I} 6 l_0 a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^{k+3} t_6^{l_0} + \mathcal{J}_1,
$$

where $\mathcal{J}_1 \in \mathrm{Span}\left(\mathfrak{P} \setminus \{x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^{l_0} \mid \epsilon_1,\epsilon_2 \in \{0,1\},\ k \in \mathbb{Z} \text{ and } i,j \in \mathbb{N}\}\right)$.

One can also write $\mathrm{ham}_{x_-}(x_3) t_5 t_6 \in \mathcal{A}_{\alpha,\beta}$ in terms of the basis $\mathfrak{P}$ of $\mathcal{A}_{\alpha,\beta}$ from Proposition 4.14 as:

$$
\mathrm{ham}_{x_-}(x_3) t_5 t_6 = \sum_{(\epsilon_1,\epsilon_2,\underline{w}) \in J} b_{(\epsilon_1,\epsilon_2,\underline{w})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l, \tag{35}
$$

where J is a finite subset of $\{0,1\}^2 \times \mathbb{N}^4$ and $b_{(\epsilon_1,\epsilon_2,\underline{w})} \in \mathbb{K}$, with $\underline{w} := (i,j,k,l) \in \mathbb{N}^4$. It follows that

$$
\sum_{(\epsilon_1,\epsilon_2,\underline{w}) \in J} b_{(\epsilon_1,\epsilon_2,\underline{w})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l = - \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0) \in I} 6 l_0 a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^{k+3} t_6^{l_0} + \mathcal{J}_1.
$$

As $\mathfrak{P}$ is a basis for $\mathcal{A}_{\alpha,\beta}$, we deduce from Corollary 4.15 that $\left(x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l\right)_{((\epsilon_1,\epsilon_2,\underline{v}) \in \{0,1\}^2 \times \mathbb{N}^2 \times \mathbb{Z}^2)}$ is a basis for $\mathcal{A}_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$. Since $\underline{v}_0 = (i,j,k,l_0) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with $l_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^4$ (with $l \geq 0$) in the above equality, we must have

$$
6 l_0 a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} = 0.
$$

Note that, since $l_0 \neq 0$, it follows that $a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} = a_{(\epsilon_1,\epsilon_2,i,j,k,l_0)}$ are all zero, a contradiction! Therefore, there is no negative exponent of t_6 appearing in the decomposition of x in (32).

Given that $l \geq 0$, it follows from our initial assumption that there exists $k = k_0 < 0$ such that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} \neq 0$. The rest of the proof will show that this is also impossible.

Set $\underline{v}_0 := (i,j,k_0,l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. From (34), we have that

$$
\mathrm{ham}_{x_-}(x_3) t_5 t_6 = - \sum_{(\epsilon_1,\epsilon_2,\underline{v}_0) \in I} 3k a_{(\epsilon_1,\epsilon_2,\underline{v}_0)} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2+1} t_5^{k_0} t_6^{l+1} + V,
$$

where $V \in \mathcal{J}_2 := \mathrm{Span}\left(\mathfrak{P} \setminus \{x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^{k_0} t_6^l \mid \epsilon_1,\epsilon_2 \in \{0,1\} \text{ and } i,j,l \in \mathbb{N}\}\right)$. It follows that

$$
\begin{aligned}
\mathrm{ham}_{x_-}(x_3) t_5 t_6 = & \\
& - \sum_{(0,0,\underline{v}) \in I} 3k_0 a_{(0,0,\underline{v})} x_1^i x_2^j x_4 t_5^{k_0} t_6^{l+1} - \sum_{(1,0,\underline{v}) \in I} 3k_0 a_{(1,0,\underline{v})} x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^{l+1} \\
& - \sum_{(0,1,\underline{v}) \in I} 3k_0 a_{(0,1,\underline{v})} x_1^i x_2^j x_4^2 t_5^{k_0} t_6^{l+1} - \sum_{(1,1,\underline{v}) \in I} 3k_0 a_{(1,1,\underline{v})} x_1^i x_2^j x_3 x_4^2 t_5^{k_0} t_6^{l+1} + V.
\end{aligned} \tag{36}
$$

Write the relations in Lemma 4.11(2), (4) as follows:

$$x_4^2 = \frac{2}{3}\beta - \frac{2}{3}x_2x_4x_6 + \frac{8}{9}\alpha x_3x_6 + \frac{4}{3}x_1x_3x_4x_6 + L_1, \tag{37}$$

$$\begin{aligned} x_3x_4^2 =& \frac{2}{3}\beta x_3 - \frac{2}{3}x_2x_3x_4x_6 + \frac{16}{9}\alpha^2 x_6 + \frac{16}{3}\alpha x_1x_4x_6 + \frac{8}{3}\beta x_1^2 x_6 \\ & - \frac{8}{3}x_1^2x_2x_4x_6^2 + \frac{32}{9}\alpha x_1^2x_3x_6^2 + \frac{16}{3}x_1^3x_3x_4x_6^2 + L_2, \end{aligned} \tag{38}$$

where L_1 and L_2 are some elements of the left ideal $\mathcal{A}_{\alpha,\beta}t_5 \subseteq \mathcal{J}_2$. Substitute (37) and (38) into (36) and simplify to obtain:

$$\begin{aligned} \mathrm{ham}_{x_-}(e_3)t_5t_6 = & \sum[\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} \\ & + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}]x_1^i x_2^j t_5^{k_0} t_6^l \\ & + \sum[\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} \\ & + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}]x_1^i x_2^j x_3 t_5^{k_0} t_6^l \\ & + \sum[\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} \\ & + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}]x_1^i x_2^j x_4 t_5^{k_0} t_6^l \\ & + \sum[\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} \\ & + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}]x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^l + V', \end{aligned} \tag{39}$$

where $V' \in \mathcal{J}_2$. Also, $\lambda_{s,t} := \lambda_{s,t}(j, k_0, l)$ are some families of scalars which are non-zero for all $s, t \in \{1, 2, 3, 4\}$ and $j, l \in \mathbb{N}$, except $\lambda_{1,4}$ and $\lambda_{2,4}$ which are assumed to be zero since they do not exist in the above expression. Note that although each $\lambda_{s,t}$ depends on j, k_0, l, we have not made this dependency explicit in (39), since the minimum requirement we need to complete the proof is for all the $\lambda_{s,t}$ existing in the above expression to be non-zero, which we already have.

Observe that (39) and (35) are equal, hence

$$\begin{aligned} \sum_{(\epsilon_1,\epsilon_2,\underline{w})\in J} b_{(\epsilon_1,\epsilon_2,\underline{w})} x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l = & \sum[\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} \\ & + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)}]x_1^i x_2^j t_5^{k_0} t_6^l \\ & + \sum[\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} \\ & + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)}]x_1^i x_2^j x_3 t_5^{k_0} t_6^l \\ & + \sum[\lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} \\ & + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)}]x_1^i x_2^j x_4 t_5^{k_0} t_6^l \\ & + \sum[\lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} \\ & + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)}]x_1^i x_2^j x_3 x_4 t_5^{k_0} t_6^l + V'. \end{aligned}$$

We have previously established that $\left(x_1^i x_2^j x_3^{\epsilon_1} x_4^{\epsilon_2} t_5^k t_6^l\right)_{((\epsilon_1,\epsilon_2,\underline{v})\in\{0,1\}^2\times\mathbb{N}^2\times\mathbb{Z}^2)}$ is a basis of $\mathcal{A}_{\alpha,\beta}[t_5^{-1}, t_6^{-1}]$ (remember that $l \geq 0$ in this part of the proof).

Since $\underline{v}_0 = (i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$ (with $k_0 < 0$) and $\underline{w} = (i, j, k, l) \in \mathbb{N}^4$ (with $k \geq 0$) in the above equality, it follows that

$$\lambda_{1,1}\beta a_{(0,1,i,j,k_0,l-1)} + \lambda_{1,2}\alpha^2 a_{(1,1,i,j,k_0,l-2)} + \lambda_{1,3}\beta a_{(1,1,i-2,j,k_0,l-2)} = 0, \tag{40}$$

$$\lambda_{2,1}\alpha a_{(0,1,i,j,k_0,l-2)} + \lambda_{2,2}\beta a_{(1,1,i,j,k_0,l-1)} + \lambda_{2,3}\alpha a_{(1,1,i-2,j,k_0,l-3)} = 0, \tag{41}$$

$$\begin{aligned} & \lambda_{3,1} a_{(0,1,i,j-1,k_0,l-2)} + \lambda_{3,2}\alpha a_{(1,1,i-1,j,k_0,l-2)} + \lambda_{3,3} a_{(1,1,i-2,j-1,k_0,l-3)} \\ & \quad + \lambda_{3,4} a_{(0,0,i,j,k_0,l-1)} = 0, \end{aligned} \tag{42}$$

$$\begin{aligned} & \lambda_{4,1} a_{(0,1,i-1,j,k_0,l-2)} + \lambda_{4,2} a_{(1,1,i,j-1,k_0,l-2)} + \lambda_{4,3} a_{(1,1,i-3,j,k_0,l-3)} \\ & \quad + \lambda_{4,4} a_{(1,0,i,j,k_0,l-1)} = 0. \end{aligned} \tag{43}$$

From (40) and (41), one can easily deduce that

$$a_{(0,1,i,j,k_0,l)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}}a_{(1,1,i,j,k_0,l-1)} - \frac{\lambda_{1,3}}{\lambda_{1,1}}a_{(1,1,i-2,j,k_0,l-1)}, \tag{44}$$

$$a_{(1,1,i,j,k_0,l)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}}a_{(0,1,i,j,k_0,l-1)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}}a_{(1,1,i-2,j,k_0,l-2)}. \tag{45}$$

Note that $a_{(\epsilon_1,\epsilon_2,i,j,k_0,l)} := 0$ whenever $i < 0$, $j < 0$ or $l < 0$ for all $\epsilon_1, \epsilon_2 \in \{0, 1\}$.

Claim. *The coefficients $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero for all $l \geq 0$.*

We justify the claim by an induction on l. From (44) and (45), the result is true when $l = 0$. For $l \geq 0$, assume that $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$. Then, it follows from (44) and (45) that

$$a_{(0,1,i,j,k_0,l+1)} = -\frac{\alpha^2\lambda_{1,2}}{\beta\lambda_{1,1}}a_{(1,1,i,j,k_0,l)} - \frac{\lambda_{1,3}}{\lambda_{1,1}}a_{(1,1,i-2,j,k_0,l)},$$

$$a_{(1,1,i,j,k_0,l+1)} = -\frac{\alpha\lambda_{2,1}}{\beta\lambda_{2,2}}a_{(0,1,i,j,k_0,l)} - \frac{\alpha\lambda_{2,3}}{\beta\lambda_{2,2}}a_{(1,1,i-2,j,k_0,l-1)}.$$

From the inductive hypothesis, $a_{(1,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l)} = a_{(0,1,i,j,k_0,l)} = a_{(1,1,i-2,j,k_0,l-1)} = 0$. Hence, $a_{(1,1,i,j,k_0,l+1)} = a_{(0,1,i,j,k_0,l+1)} = 0$. By the principle of mathematical induction, $a_{(0,1,i,j,k_0,l)} = a_{(1,1,i,j,k_0,l)} = 0$ for all $l \geq 0$ as desired. Given that the families $a_{(0,1,i,j,k_0,l)}$ and $a_{(1,1,i,j,k_0,l)}$ are all zero, it follows from (42) and (43) that $a_{(0,0,i,j,k_0,l)}$ and $a_{(1,0,i,j,k_0,l)}$ are also zero for all $(i, j, k_0, l) \in \mathbb{N}^2 \times \mathbb{Z} \times \mathbb{N}$. This contradicts our assumption. Hence, $x_- = 0$. Consequently, $x = x_+ \in \mathcal{A}_{\alpha,\beta}$ as desired.

2. From Lemma 5.6, we have that $x_4 = x_{4,6} + \frac{2}{3}t_5^3t_6^{-1} = t_4 + \frac{2}{3}t_5^3t_6^{-1}$. Again, from Lemma 5.9, we have that $\lambda_4 = 3\lambda_3 + \lambda_5$ and $\lambda_6 = -3\lambda_3$. Therefore,

$$\begin{aligned}
\rho(x_4) &= \lambda_4 t_4 + \frac{2}{3}(3\lambda_5 - \lambda_6)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)x_{4,6} + 2(\lambda_3 + \lambda_5)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)\left(x_4 - \frac{2}{3}t_5^3t_6^{-1}\right) + 2(\lambda_3 + \lambda_5)t_5^3t_6^{-1} \\
&= (3\lambda_3 + \lambda_5)x_4 + \frac{4}{3}\lambda_5 t_5^3t_6^{-1}.
\end{aligned}$$

Hence,

$$\mathcal{D}(x_4) = \mathrm{ham}_x(x_4) + \rho(x_4) = \mathrm{ham}_x(x_4) + (3\lambda_3 + \lambda_5)x_4 + \frac{4}{3}\lambda_5 t_5^3t_6^{-1} \in \mathcal{A}_{\alpha,\beta}.$$

It follows that $\lambda_5 t_5^3 t_6^{-1} \in \mathcal{A}_{\alpha,\beta}$, since $\mathrm{ham}_x(x_4) + (3\lambda_3 + \lambda_5)x_4 \in \mathcal{A}_{\alpha,\beta}$. Consequently, $\lambda_5 t_5^3 \in \mathcal{A}_{\alpha,\beta}t_6$. Using the basis of $\mathcal{A}_{\alpha,\beta}$ from Proposition 4.14, we easily have that $\lambda_5 = 0$. Therefore, $\rho(x_4) = 3\lambda_3 x_4$ and $\rho(t_5) = 0$. We already know from Lemma 5.9 that $\rho(t_6) = -3\lambda_3 t_6$. From (33), we have $\rho(x_3) = \lambda_3 x_3 + \frac{3}{2}(\lambda_4 - \lambda_3 - \lambda_5)x_4t_5^{-1} + (3\lambda_5 - \lambda_4 - \lambda_6)t_5^2t_6^{-1}$. Given that $\lambda_4 = 3\lambda_3$, $\lambda_5 = 0$ and $\lambda_6 = -3\lambda_3$, we have that $\rho(x_3) = \lambda_3 x_3 + 3\lambda_3 x_4 t_5^{-1}$. Now, $\mathcal{D}(x_3) = \mathrm{ham}_x(x_3) + \rho(x_3) = \mathrm{ham}_x(x_3) + \lambda_3 x_3 + 3\lambda_3 x_4 t_5^{-1} \in \mathcal{A}_{\alpha,\beta}$. Observe that $\mathrm{ham}_x(x_3), \lambda_3 x_3 \in \mathcal{A}_{\alpha,\beta}$. Hence, $\lambda_3 x_4 t_5^{-1} \in \mathcal{A}_{\alpha,\beta}$ implies that $\lambda_3 x_4 \in \mathcal{A}_{\alpha,\beta}t_5$. Similarly, $\lambda_3 = 0$. We now have that $\rho(x_3) = \rho(x_4) = \rho(x_5) = \rho(x_6) = 0$. We finish the proof by showing that $\rho(x_1) = \rho(x_2) = 0$. Recall from (12) that

$$x_2x_4x_6 - \frac{2}{3}x_3^3x_6 - \frac{2}{3}x_2x_5^3 + 2x_3^2x_5^2 - 3x_3x_4x_5 + \frac{3}{2}x_4^2 = \beta.$$

Apply ρ to this relation to obtain $\rho(x_2)x_4x_6 - \frac{2}{3}\rho(x_2)x_5^3 = 0$. This implies that $\rho(x_2)\left(x_4x_6 - \frac{2}{3}x_5^3\right) = 0$. Since $x_4x_6 - \frac{2}{3}x_5^3 \neq 0$, it follows that $\rho(x_2) = 0$. Similarly, from (11), we have that

$$x_1x_3x_5 - \frac{3}{2}x_1x_4 - \frac{1}{2}x_2x_5 + \frac{1}{2}x_3^2 = \alpha.$$

Apply ρ to this relation to obtain $\rho(x_1)\left(x_3x_5 - \frac{3}{2}x_4\right) = 0$. Since $x_3x_5 - \frac{3}{2}x_4 \neq 0$, we must have: $\rho(x_1) = 0$. In conclusion, $\rho(x_\kappa) = 0$ for all $\kappa \in \{1, \ldots, 6\}$.

3. This easily follows from the previous points. □

5.3.2. Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$

Following procedures similar to the previous case (i.e. $\mathcal{A}_{\alpha,\beta}$ with $\alpha\beta \neq 0$), one can also compute the Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$. The computations have been done, however, for the avoidance of redundancy, we are not going to include them here. We only summarize the results here. Before we do that, we compute explicitly some scalar Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$.

Lemma 5.10. *Let $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Suppose that ϑ and $\tilde{\vartheta}$ are linear maps on $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$ respectively, and are defined by:*

$$\vartheta(x_1) = -x_1, \quad \vartheta(x_2) = -x_2, \quad \vartheta(x_3) = 0, \quad \vartheta(x_4) = x_4, \quad \vartheta(x_5) = x_5, \quad \vartheta(x_6) = 2x_6,$$

and

$$\tilde{\vartheta}(x_1) = -2x_1, \quad \tilde{\vartheta}(x_2) = -3x_2, \quad \tilde{\vartheta}(x_3) = -x_3, \quad \tilde{\vartheta}(x_4) = 0, \quad \tilde{\vartheta}(x_5) = x_5, \quad \tilde{\vartheta}(x_6) = 3x_6.$$

Then, ϑ and $\tilde{\vartheta}$ extended to $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$ respectively using the Leibniz rule are $\mathbb{K}$-Poisson derivations of $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{0,\beta}$ respectively.

Proof. The interested reader is referred to [19, Lemma 6.3.6]. □

As usual, we denote by $HP^1(\mathcal{A}_{\alpha,\beta})$ the first Poisson cohomology group of the Poisson algebra $\mathcal{A}_{\alpha,\beta}$. Recall that

$$HP^1(\mathcal{A}_{\alpha,\beta}) := \mathrm{Der}_P(\mathcal{A}_{\alpha,\beta})/\mathrm{InnDer}_P(\mathcal{A}_{\alpha,\beta}),$$

where $\mathrm{Der}_P(\mathcal{A}_{\alpha,\beta})$ is the set of all the Poisson $\mathbb{K}$-derivations of $\mathcal{A}_{\alpha,\beta}$, and $\mathrm{InnDer}_P(\mathcal{A}_{\alpha,\beta}) := \{\mathrm{ham}_x \mid x \in \mathcal{A}_{\alpha,\beta}\} \subseteq \mathrm{Der}_P(\mathcal{A}_{\alpha,\beta})$. For a Poisson derivation d of $\mathcal{A}_{\alpha,\beta}$, we denote by $[d]$ its class modulo $\mathrm{InnDer}_P(\mathcal{A}_{\alpha,\beta})$.

We are now ready to summarize our main results regarding the Poisson derivations of $\mathcal{A}_{\alpha,\beta}$.

Theorem 5.11.

1. *If $\alpha, \beta \neq 0$; then every Poisson derivation $\mathcal{D}$ of $\mathcal{A}_{\alpha,\beta}$ is a Poisson inner derivation. In particular, $HP^1(\mathcal{A}_{\alpha,\beta}) = \{[0]\}$.*
2. *If $\alpha \neq 0$ and $\beta = 0$, then every Poisson derivation $\mathcal{D}$ of $\mathcal{A}_{\alpha,0}$ can uniquely be written as $\mathcal{D} = \mathrm{ham}_x + \lambda\vartheta$, where $\lambda \in \mathbb{K}$ and $x \in \mathcal{A}_{\alpha,0}$. Thus $HP^1(\mathcal{A}_{\alpha,0}) = \mathbb{K}[\vartheta]$.*
3. *If $\alpha = 0$ and $\beta \neq 0$, then every Poisson derivation $\mathcal{D}$ of $\mathcal{A}_{0,\beta}$ can uniquely be written as $\mathcal{D} = \mathrm{ham}_x + \lambda\tilde{\vartheta}$, where $\lambda \in \mathbb{K}$ and $x \in \mathcal{A}_{0,\beta}$. Thus $HP^1(\mathcal{A}_{0,\beta}) = \mathbb{K}[\tilde{\vartheta}]$.*

Let $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. One can easily conclude that the first Poisson cohomology group $\mathrm{HP}^1(\mathcal{A}_{\alpha,\beta})$ is isomorphic to the first Hochschild cohomology group $\mathrm{HH}^1(A_{\alpha,\beta})$ studied in [16, Theorem 5.12].

It is natural to ask whether higher Poisson cohomology groups of $\mathcal{A}_{\alpha,\beta}$ are isomorphic to higher Hochschild cohomology groups of $A_{\alpha,\beta}$, that is, do we have $\mathrm{HP}^i(\mathcal{A}_{\alpha,\beta}) \cong \mathrm{HH}^i(A_{\alpha,\beta})$ for all i?

Data availability

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