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Finitely Generated Simple Graphs

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Abstract

In this paper, Kirchhoff, Hyper-Wiener, Randic, Szeged, Pi index calculations of finitely generated (cyclic) simple graphs on the samples were made and classification of some finitely generated (cyclic) groups was achieved with the help of graph theory.

Keywords: graph theory, identity graphs, simple graphs, group theory

1 Introduction

We will first demonstrate how to express groups as graphs. We will not examine the properties of groups from the structure of graphs. To study a group in graph theory, we make use of the concept of identity in the group and so one says the graph associated with the group the identity graph. We refer authors to read the reference [5] for fundamental definitions. By "the simple graph" we mean that the simple graph of a group G. Assume that G is a group and x, $y \in G$. Then x is to y if and only if $x.y = e$, where e is the identity element of G. We indicate it with a line as of $g^2 = 1$ whenever G = {g, 1 | g^2 = 1}.

Suppose that G is a cyclic group with 7th order. Then we can write $G = \langle h | h^7 = 1 \rangle$ and so the identity graph is shown in the following way:

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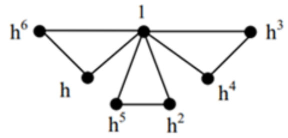
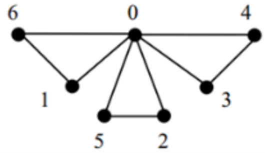


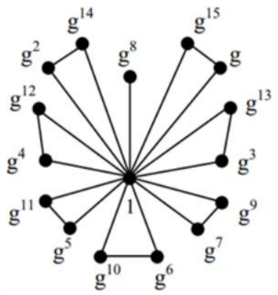
Let the abelian group $\mathbb{Z}_7 = \{0, 1, 2, ..., 6\}$ be defined as a binary operation under addition. The

identity graph of $\mathbb{Z}_7$is shown in the following way:



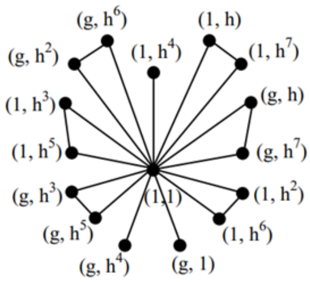
We observe that the identity graphs of $\mathbb{Z}_7$ and G is the same.

Let $G = \mathbb{Z}_{17} \setminus \{0\} = \{1,2,\cdots,16\}$. Then G is an abelian group under the multiplication. Identity graph associated to G, and a cyclic group $G' = \langle g | g^{16} = 1\rangle$. The identity graph is as below.



It follows that these identity graphs are same.

Given $\bar{G} = H \times K = \{1, g | g^2 = 1\} \times \langle h | h^8 = 1\rangle$. Then



We have $|\bar{G}| = 16$ but the identity graph of $\bar{G}$ is distinctive from that of graphs G and G' given in the above.

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2 Research Methodology

In this section some required definitions and theorems are presented. The definitions of index calculations are mentioned. The definitions and theorems in this section are taken from references from [1] to [7].

2.1.Definition

Let G be group. If $X = G$, then the group G is equal to $\langle X \rangle$ and the group G has a generator set. If for $a_1, a_2, \dots, a_n \in G$ such that $G = \langle a_1, a_2, \dots, a_n \rangle$ then G is finitely generated group and $G = \langle \mathrm{a} \rangle$ and if for a $\in G$ such that G is called cyclic group. If for a finite group G to be cyclic $|G|$ = |a| such that a $\in G$ is exist.

2.2. Definition

The binary structure V consisting of a finite non-empty set of V points, whose elements are called points, and a finite set of edges E, its elements are said edges, is said a graph. where e is the set of sides, a set of two-element subsets of v. The $G = (V, E)$ structure, whose elements are called points $V = \{v_1, v_2, \dots, v_n\}$ and edges are called $E = \{e_1, e_2, \dots, e_n\}$.

2.3. Definition

Assume that G is a group and H is a subgroup of G. We say H special identity subgraph of the group G if the identity graph plotted for H. By P, we denote the set of all prime integers.

Theorem 2.1: Let a cyclic group G of order p be given, where $p \in P$. The identity graph created by G has only triangles and the number of these is $(p - 1) / 2$.

Proof: $G = \langle g | g^p = 1 \rangle$. Then we obtain that G doesn't have any proper subgroup. It follows that G does not contain a self inversed element. It means that G can not have 2nd order elements. Hence there is a unique element g^j in G with $g^i . g^j = 1$. It follows from $j = (p - i)$ that the elements $1, g^i, g^{p-i}$ are in the form of a triangle. Hence the identity graph shall not have a line any graphs.

Volume 5, Issue 2, pages 191–200
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Corollary 2.1: Let G be a cyclic group with odd th order. Then G has the identity graph G_i having only by triangles with no lines.

2.4. Definition

A graph that has at most one edge between any two points and does not contain a loop is called a simple graph.

2.5. Definition

A graph with a path between any two points is called a connected graph.

2.6. Definition

Let G be a simple graph with n points and $A(G) = \left(a_{ij}\right)_{nxn}$ be the neighborhood matrix of G. Then the elements is defined by

$$a_{ij} = \begin{cases} 1; & if\ i \sim j \\ 0; & or\ else. \end{cases}$$

2.7. Definition

The Kirchhoff index of G, where G is a simple, connected and n-point graph,

$$Kf(G) = \sum_{i<j} r_{ij}$$

is defined as.

Recall from [4, Lemma 7.1.2] that for n≥2, let G be a connect the point in a graph. In that case,

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\lambda_i}$$

2.8. Definition

For a simple connected graph G, G's Hyper-Wiener index

$$WW(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v) + \frac{1}{2}\sum_{u,v\in V(G)} d^2(u,v)$$

defined as this such that d(u,v) is the shortest distance between u and v.

2.9. Definition

Assume that G is a simple connected graph. d_i is the Randic Index to indicate the degree of the point v_i of G. Thus, G's Randic index

$$R = R(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}}$$

defined as.

2.10. Definition

Assume that G is an simple connected graph and $e = uv$ be a side of the graph G. By $n_u(e)$ (respectively, $n_v(e)$) we denote the number of points closer to the point u than the point v (respectively, the number of points closer to the point v than the point u). By $m_u(e)$ (respectively, $m_v(e)$) we denote the number of edges closer to the point u than the point v (respectively, the number of edges closer to the point v than the point u). Thus the Szeged index of G and the edge Szeged index

$$Sz(G) = \sum_{e\in E} n_u(e)n_v(e)$$

$$Sz_e(G) = \sum_{e\in E} m_u(e)m_v(e)$$

defined as.

2.11. Definition

Assume that G is a simple connected graph and $e = uv$ be a side of the graph G. By $n_u(e)$ (respectively, $n_v(e)$) we denote the number of points closer to the point u than the point v (respectively, the number of points closer to the point v than the point u). By $m_u(e)$ (respectively, $m_v(e)$) we denote the number of edges closer to the point u than the point v (respectively, the number of edges closer to the point v than the point u). Thus, G's Pi index and edge Pi index

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$$PI(G) = \sum_{e \in E} m_u(e) + m_v(e)$$

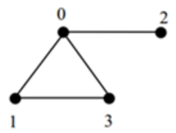
$$PI_v(G) = \sum_{e \in E} n_u(e) + n_v(e)$$

defined as.

3 Results and Discussions

In this section, we have calculated Kirchhoff (Hyper-Wiener, Randic, Szeged, Pi) index on an example of simple graphs of a finitely generated (cyclic) groups.

Identity graph of the group $\mathbb{Z}_4$= {0, 1, 2, 3} is



in the form. According to this, we need to find neighborhood matrix of graph G first.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Eigenvalues of the neighborhood matrix is

$$\begin{bmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 1 & 1 & 0 & -\lambda \end{bmatrix} = 0$$

such that $\det(A - \lambda I_n) = 0$. Then we have

$$\lambda^4 - 4\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda + 1)(\lambda^3 - \lambda^2 - 3\lambda + 1) = 0$$

$$\lambda_1 \cong -1{,}481 \quad , \quad \lambda_2 \cong 0{,}311 \quad , \lambda_3 \cong 2{,}170$$

So we obtain the Kirchhoff index as follows:

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\lambda_i}$$

$$= 4\sum_{i=1}^{3}\frac{1}{\lambda_i}$$

$$= 4\cdot\left(\frac{1}{-1{,}481}+\frac{1}{2{,}170}+\frac{1}{0{,}311}\right)$$

$$= 4.(0{,}460 - 0{,}675 + 3{,}125)$$

$$= 4.3 = 12$$

Let us calculate the Hyper-Wiener index:

$$WW(G) = \frac{1}{2}\sum_{u,v} d(u,v) + \frac{1}{2}\sum_{u,v} d^2(u,v)$$

$WW(G) = \frac{1}{2}\big(d(0{,}1) + d(0{,}2) + d(0{,}3) + d(1{,}2) + d(1{,}3) + d(2{,}3)\big) + \frac{1}{2}(d^2(0{,}1) + d^2(0{,}2) + d^2(0{,}3) + d^2(1{,}2) + d^2(1{,}3) + d^2(2{,}3))=\frac{1}{2}(1+1+1+2+1+2)+\frac{1}{2}(1^2 + 1^2 + 1^2 + 2^2 + 1^2 + 2^2 = \frac{1}{2}(8)+\frac{1}{2}(12) = 4 + 6 = 10$

Let us calculate the Randic index

$$R = R(G) = \sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}}$$

$$= \frac{1}{\sqrt{d_0d_1}}+\frac{1}{\sqrt{d_0d_2}}+\frac{1}{\sqrt{d_0d_3}}+\frac{1}{\sqrt{d_1d_3}}$$

$$= \frac{1}{\sqrt{6}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{4}}$$

$$= \frac{1}{2}+\frac{2}{2{,}449}+\frac{1}{1{,}732}$$

$$= 0{,}5 + 0{,}816 + 0{,}577$$

$$= 1{,}893$$

Let us calculate the Szeged index:

$$e = uv \quad n_u(e) = |\{x \in V : d(x,u) < d(x,v)\}|$$

$$n_u(e) + n_v(e) = |v| = n$$

$e = 01$ için $n_0(e) = |\{x \in v\colon\ d(x,0) < d(x,1)\}| = |\{0{,}2\}| = 2$

$$n_1(e) = |\{x \in v\colon\ d(x,1) < d(x,0)\}| = |\{1\}| = 1$$

$e = 02$ için $n_0(e) = |\{x \in v\colon d(x,0) < d(x,2)\}| = |\{0,1,3\}| = 3$
$n_2(e) = |\{x \in v\colon d(x,2) < d(x,0)\}| = |\{2\}| = 1$
$e = 03$ için $n_0(e) = |\{x \in v\colon d(x,0) < d(x,3)\}| = |\{0,2\}| = 2$
$n_3(e) = |\{x \in v\colon d(x,3) < d(x,0)\}| = |\{3\}| = 1$

$e = 13$ için $n_1(e) = |\{x \in v\colon d(x,1) < d(x,3)\}| = |\{1\}| = 1$
$n_3(e) = |\{x \in v\colon d(x,3) < d(x,1)\}| = |\{3\}| = 1$

$$Sz(G) = \sum_{e \in E} n_u(e) n_v(e)$$

$$\begin{aligned} Sz(G) &= n_0(e)n_1(e) + n_0(e)n_2(e) + n_0(e)n_3(e) \\ &\quad + n_1(e)n_3(e) \\ &= 2.1 + 3.1 + 2.1 + 2.1 + 1.1 \\ &= 10 \end{aligned}$$

Let us calculate the edge Szeged index:

$$Sz_e(G) = \sum_{e \in E} m_u(e) m_v(e)$$

$$\begin{aligned} Sz_e(G) &= m_0(e)m_1(e) + m_0(e)m_2(e) + m_0(e)m_3(e) \\ &\quad + m_1(e)m_3(e) \\ &= 2.1 + 2.0 + 2.1 + 2.1 \\ &= 6 \end{aligned}$$

Finally let us calculate the Pi index:

$$PI(G) = \sum_{e \in E} m_u(e) + m_v(e)$$

$$\begin{aligned} PI(G) &= \big(m_0(e) + m_1(e)\big) + \big(m_0(e) + m_2(e)\big) + \big(m_0(e) + m_3(e)\big) \\ &\quad + \big(m_1(e) + m_3(e)\big) \\ &= (2+1) + (3+1) + (2+1) + (1+1) \\ &= 11 \end{aligned}$$

4 Conclusions

In this study, especially on the problem of representation for finitely generated groups as graphs, an approach has been made with different index types. The main objective of this article is to show that a large number of studies on groups are applicable to graph theory. In this context, finitely generated (cyclic) simple graph, loops, distance between two points and the distance between the two edges of the definitions of graph theory with the help of the theoretical principles on the application of group theory was carried out. In particular, the examples are given supportive of the concept being studied. Therefore, with the help of the basic relationship between the graph structure and the group structure, the Kirchhoff index, Hyper-Wiener index, Randic index, (edge) Szeged index, (edge) Pi index are included in the structure of finitely generated (cyclic) simple graphs, which focus specifically on group properties.

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