ANNIHILATORS OF IRREDUCIBLE REPRESENTATIONS OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON THE SUPERLINE

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Dissertation Prepared for the Degree of

DOCTOR OF PHILOSOPHY

UNIVERSITY OF NORTH TEXAS

May 2023

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Goode, William M. *Annihilators of Irreducible Representations of the Lie Superalgebra of Contact Vector Fields on the Superline*. Doctor of Philosophy (Mathematics), May 2023, 55 pp., 10 numbered references.

The superline has one even and one odd coordinate. We consider the Lie superalgebra of contact vector fields on the superline. Its tensor density modules are a one-parameter family of deformations of the natural action on the ring of polynomials on the superline. They are parameterized by a complex number, and they are irreducible when this parameter is not zero. In this dissertation, we describe the annihilating ideals of these representations in the universal enveloping algebra of this Lie superalgebra by providing their generators. We also describe the intersection of all such ideals: the annihilator of the direct sum of the tensor density modules. The annihilating ideal of an irreducible non-zero left module is called a primitive ideal, and the space of all such ideals in the universal enveloping algebra is its primitive spectrum. The primitive spectrum is endowed with the Jacobson topology, which induces a topology on the annihilators of the tensor density modules. We conclude our discussion with a description of the annihilators as a topological space.

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CHAPTER 1

INTRODUCTION

Let $\mathscr{K}$ be the Lie superalgebra of contact vector fields on the superline $\mathbb{R}^{1|1}$. The topic of this thesis is the two-sided ideals in the universal enveloping algebra of $\mathscr{K}$ annihilating the *tensor density modules*. We describe these annihilators and give generators for them. An important step in the argument is the description of the intersection of all of them.

It is expected that the trivial module and the tensor density modules make up all the irreducible representations of $\mathscr{K}$. Thus their annihilators make up its primitive ideals. The primitive spectrum of a ring is the collection of all primitive ideals, which can be endowed with the Jacobson topology. The annihilators of the tensor density modules then inherit the subspace topology, which we describe: it is equivalent to $\mathbb{C}^\times$ with the co-finite topology.

We follow the approach taken in [3], where the annihilators of the tensor density modules of the Lie algebra $\mathrm{Vec}(\mathbb{R})$ of vector fields on the line were described. $\mathscr{K}$ may be thought of as a square root of $\mathrm{Vec}(\mathbb{R})$: it contains a copy of $\mathrm{Vec}(\mathbb{R})$ as its even part.

The annihilators of the tensor density modules of $\mathrm{Vec}(\mathbb{R})$ and $\mathscr{K}$ are described in terms of the *Casimir element* and other closely related lowest weight elements of the universal enveloping algebra. To be more precise, recall that $\mathrm{Vec}(\mathbb{R})$ contains the infinitesimal linear fractional transformations, which make up a maximal subalgebra, isomorphic to $\mathfrak{sl}_2$. Under this copy of $\mathfrak{sl}_2$, the tensor density modules of $\mathrm{Vec}(\mathbb{R})$ are the duals of the Verma modules. It is well-known that the annihilators of both the Verma modules and their duals are generated by the Casimir element, adjusted by an additive scalar. This result was generalized to arbitrary finite-dimensional complex semisimple Lie algebras by Duflo in [4].

In $\mathscr{K}$, the analog of the copy of $\mathfrak{sl}_2$ in $\mathrm{Vec}(\mathbb{R})$ is a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$. Just as for $\mathrm{Vec}(\mathbb{R})$, under this copy of $\mathfrak{osp}(1|2)$, the tensor density modules of $\mathscr{K}$ are duals of Verma modules. Their $\mathfrak{osp}(1|2)$-annihilators were described by Pinczon in [9]: broadly speaking, they are again generated by the Casimir element adjusted by an additive scalar, but in certain special cases the *ghost*, a square root of the Casimir element, plays a role.

The procedure used in [3] to compute the annihilators of the tensor density modules of $\mathrm{Vec}(\mathbb{R})$ begins with the computation of their intersection with the weight zero degree ≤ 3 subspace of the universal enveloping algebra. This intersection is then shown to generate the ideal. A similary strategy works for the tensor density modules of $\mathscr{K}$, but in general, more lowest weight generators are required. Moreover, in the self-dual case there is an entirely new phenomenon: the ideal is principal and is generated by the $\mathfrak{osp}(1|2)$-ghost.

This dissertation is organized as follows: in Chapter 2, we give basic definitions and results concerning universal enveloping algebras of Lie superalgebras, as well as supersymmetric algebras. In this chapter, we also define the Jacobson topology. In Chapter 3, we define $\mathscr{K}$ and its tensor density modules. In addition, we introduce the universal enveloping algebra $\mathfrak{U}(\mathscr{K})$, define distinguished elements of $\mathfrak{U}(\mathscr{K})$, and discuss differential operators. Chapter 4 contains the statements of our main results. Chapter 5 consists of structural remarks on the universal enveloping algebra, and Chapters 6 and 7 contain the proofs of the main results.

CHAPTER 2

BACKGROUND

In this chapter, we review several terms and results about Lie superalgebras and their representations. Superspaces in this chapter are assumed to be finite-dimensional. Throughout this entire dissertation, the ground field is $\mathbb{C}$. For more information, see [8].

2.1. Lie Superalgebras and Representation Theory

A *superspace* is a $\mathbb{Z}_2$-graded vector space V over a field $\mathbb{F}$. We write $V = V_{\text{even}} \oplus V_{\text{odd}}$. Elements of V_{even} and V_{odd} are said to be *homogeneous*. For a homogeneous element v, we define the *parity* of v to be 0 if $v \in V_{\text{even}}$ and 1 if $v \in V_{\text{odd}}$, and we denote this quantity by $|v|$. The *parity endomorphism* $\epsilon : V \to V$ is the map $\epsilon(v) := (-1)^{|v|}v$. There is a *parity-exchanging functor* Π: the space V^Π is V as a vector space, but $V^\Pi_{\text{odd}} := V_{\text{even}}$ and $V^\Pi_{\text{even}} := V_{\text{odd}}$.

A *Lie superalgebra* $\mathfrak{g}$ is a superspace with a product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the superbracket of $\mathfrak{g}$, that is bilinear, super skew-symmetric, and satisfies the super Jacobi identity:

$$(-1)^{|X||Z|}[X,[Y,Z]] + (-1)^{|Y||X|}[Y,[Z,X]] + (-1)^{|Z||Y|}[Z,[X,Y]] = 0$$

for all homogeneous $X, Y, Z \in \mathfrak{g}$.

If V, W are superspaces over $\mathbb{F}$ then $\mathrm{Hom}(V,W)$ is also naturally a superspace. We have

$$\mathrm{Hom}(V,W)_{\text{even}} = \big\{\phi \in \mathrm{Hom}(V,W) : \phi \text{ preserves parity}\big\},$$

$$\mathrm{Hom}(V,W)_{\text{odd}} = \big\{\phi \in \mathrm{Hom}(V,W) : \phi \text{ exchanges parity}\big\}.$$

$\mathrm{Hom}(V,W)$ is a superalgebra via the bracket given by the supercommutator:

$$[\phi,\psi] := \phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi$$

for any homogeneous $\phi, \psi \in \mathrm{Hom}(V,W)$.

A *representation* (V, π) of a Lie superalgebra $\mathfrak{g}$ is a linear action π of $\mathfrak{g}$ on V that respects brackets. That is, π is an even map from $\mathfrak{g}$ to $\mathrm{End}(V)$ such that

$$\pi([X,Y]) = \pi(X) \circ \pi(Y) - (-1)^{|X||Y|}\pi(Y) \circ \pi(X)$$

for all homogeneous $X, Y \in \mathfrak{g}$. A natural example of a representation is the adjoint action of $\mathfrak{g}$ on itself: $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ with $\mathrm{ad}(X)(Y) = [X, Y]$. We usually abbreviate $\mathrm{ad}(X)$ to ad_X.

If $W \subseteq V$ is a subspace such that $\pi(X)(w) \in W$ for all $X \in \mathfrak{g}$ and $w \in W$, then $(W, \pi|_W)$ is a representation of $\mathfrak{g}$ and we call it a *subrepresentation* of V. The *quotient* representation $(V/W, \overline{\pi})$ is given by $\overline{\pi}(v + W) := \pi(v) + W$. If V has no subrepresentations other than V and $\{0\}$, we say it is *irreducible.*

Fix representations (V, π) and (W, ρ) of $\mathfrak{g}$. If φ is in $\mathrm{Hom}(V, W)$ and has the property

$$\varphi \circ \pi(X) = \rho(X) \circ \varphi$$

for all $X \in \mathfrak{g}$, then we say φ is an *intertwining map* or a $\mathfrak{g}$*-map*. If it is also an isomorphism of vector spaces, we say it is a $\mathfrak{g}$-equivalence.

The direct sum of representations (V, π) and (W, ρ) is denoted $(V \oplus W, \pi \oplus \rho)$ and is defined by

$$(\pi \oplus \rho)(X)(v, w) = (\pi(X)v,\ \rho(X)w)$$

for $X \in \mathfrak{g}$, $v \in V$, and $w \in W$.

2.2. Tensor Products

Given superspaces V and W, the tensor product $V \otimes W$ is also a superspace. The parity function is defined as $|v \otimes w| := |v| + |w|$. If (V, π) and (W, ρ) are representations of a Lie superalgebra $\mathfrak{g}$, then there is an action $\pi \otimes \rho$ of $\mathfrak{g}$ on $V \otimes W$ defined by

$$(\pi \otimes \rho)(X)(v \otimes w) = \pi(X)v \otimes w + (-1)^{|X||v|}\big(v \otimes \rho(X)w\big)$$

for $v \in V$ and $w \in W$. Note that the map $v \otimes w \mapsto w \otimes v$ is not a $\mathfrak{g}$-equivalence in general. Rather, the map $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ is a $\mathfrak{g}$-equivalence between $(V \otimes W, \pi \otimes \rho)$ and $(W \otimes V, \rho \otimes \pi)$.

We write $\otimes V$ for the tensor algebra of V. Given an action π of $\mathfrak{g}$ on V, we write $\otimes\pi$ for the natural action of $\mathfrak{g}$ on $\otimes V$ by superderivations. When the meaning is clear from the context, we sometimes abbreviate $\otimes\pi$ to π. In particular, we write ad for the action $\otimes$ ad of $\mathfrak{g}$ on $\otimes\mathfrak{g}$. For a non-negative integer r, we denote the r^{th} tensor power of V by $\otimes^r V$. Fix bases $\mathcal{B}_{\text{even}} = \{v_1, \dots, v_n\}$ of V_{even} and $\mathcal{B}_{\text{odd}} = \{w_1, \dots, w_m\}$ of V_{odd}. It will be convenient to establish alternate notation:

$$u_1 = v_1, \ \dots, \ u_n = v_n; \qquad u_{n+1} = w_1, \ \dots, \ u_{n+m} = w_m.$$

Thus, $\mathcal{B}_V = \mathcal{B}_{\text{even}} \cup \mathcal{B}_{\text{odd}} = \{u_1, \dots, u_{n+m}\}$. We say that

$$\mathcal{B}_{\otimes^r V} := \{u_{i_1} \otimes \cdots \otimes u_{i_r} : u_{i_1}, \dots, u_{i_r} \in \mathcal{B}_V\}$$

is the basis for $\otimes^r V$ induced by $\mathcal{B}_V$.

We also put $\otimes_r V := \bigoplus_{j=0}^{r} \otimes^j V$. For any non-zero Θ in $\otimes V$, the smallest integer d for which $\Theta \in \otimes_d V$ is called the *degree* of Θ and is denoted $\deg(\Theta)$. We say $\otimes^r V$ is the space of homogeneous tensors of degree r. For any subspace A of $\otimes V$, define

$$A^r := A \cap \otimes^r V, \qquad A_r := A \cap \otimes_r V.$$

Lemma 2.1. *If V is a representation of $\mathfrak{g}$, then $\otimes^r V$ and $\otimes_r V$ are subrepresentations of $\otimes V$.*

Definition 2.2. Let $1 \leq j \leq r$, and let X be of homogeneous parity in $\mathfrak{g}$. We define

$$\pi(X)_j : \otimes^r V \to \otimes^r V, \qquad \pi(X)_j := (\otimes^{j-1} \epsilon^{|X|}) \otimes \pi(X) \otimes (\otimes^{r-j} 1).$$

Thus, the action of $\otimes\pi$ on $\otimes^r V$ may be expressed as

$$\pi(X) = \sum_{j=1}^{r} \pi(X)_j.$$

When context makes the meaning unambiguous, we will sometimes write π_j for the map $X \mapsto \pi(X)_j$ which yields $\pi = \sum_j \pi_j$.

2.3. The Supersymmetric Algebra

This section reviews the supersymmetric algebra and some of its properties. It is the super-analog of the usual symmetric algebra of a vector space. Let $V = V_{\text{even}} \oplus V_{\text{odd}}$ be any superspace.

DEFINITION 2.3. Let I be the two-sided ideal of $\otimes V$ generated by

$$\{v \otimes w - (-1)^{|v||w|} w \otimes v : v, w \in V\}.$$

LEMMA 2.4. *The ideal I is homogeneous with respect to degree: $I = \bigoplus_{r=0}^{\infty} I^r$.*

Let S_r be the symmetric group on r letters. For $1 \leq j \leq r-1$, let s_j be the transposition $(j, j+1)$. We recall the *Coxeter presentation* of S_r:

$$S_r = \langle s_1, \ldots, s_{r-1} : s_j^2 = 1,\ s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1},\ s_i s_j = s_j s_i \text{ for } |i-j| > 1 \rangle.$$

The relation $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ is the so-called *braid relation.*

LEMMA 2.5. *Let $v_1, \ldots, v_r$ be any vectors of homogeneous parity in V, and let $\sigma \in S_r$. Then there exists an $\varepsilon \in \mathbb{Z}_2$ depending only on σ and the parities $|v_1|, \ldots, |v_r|$ such that*

$$v_1 \otimes \cdots \otimes v_r - (-1)^{\varepsilon} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)} \in I^r.$$

PROOF. We will use the fact that S_r is generated by adjacent transpositions. That is, we may write any $\sigma \in S_r$ as $\sigma = (j_k, j_k+1)(j_{k-1}, j_{k-1}+1) \cdots (j_1, j_1+1)$ for some positive integer k and $j_i \in \{1, \ldots, r-1\}$ for each $1 \leq i \leq k$. We must induct on the length of the product k. When $k = 1$, verify that $\varepsilon = |v_{j_1}||v_{j_1+1}|$ works. As an inductive hypothesis, we assume the statement holds for products of $k-1$ adjacent transpositions. For an arbitrary $\sigma = (j_k, j_k+1)(j_{k-1}, j_{k-1}+1) \cdots (j_1, j_1+1)$ in S_r, let us denote the $k-1^{\text{st}}$ partial product $(j_{k-1}, j_{k-1}+1) \cdots (j_1, j_1+1)$ by σ_{k-1}. Let ε be the integer provided by the inductive hypothesis such that $v_1 \otimes \cdots \otimes v_r - (-1)^{\varepsilon} v_{\sigma_{k-1}^{-1}(1)} \otimes \cdots \otimes v_{\sigma_{k-1}^{-1}(r)} \in I^r$. We have

$$\begin{aligned} v_1 \otimes \cdots \otimes v_r &- (-1)^{|v_{\sigma(j_k)}||v_{\sigma(j_k)+1}|+\varepsilon} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)} \\ &= v_1 \otimes \cdots \otimes v_r - (-1)^{\varepsilon} v_{\sigma_{k-1}^{-1}(1)} \otimes \cdots \otimes v_{\sigma_{k-1}^{-1}(r)} \end{aligned}$$

$$+ (-1)^{\varepsilon}\big(v_{\sigma_{k-1}^{-1}(1)} \otimes \cdots \otimes v_{\sigma_{k-1}^{-1}(r)} - (-1)^{|v_{\sigma(j_k)}||v_{\sigma(j_k)+1}|} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}\big).$$

The first summand on the right-hand side of the above equation is in I^r by the inductive hypothesis. The second directly satisfies the definition of elements of I^r. □

Recall the bases $\mathcal{B}_{\text{even}} = \{v_1, \ldots, v_n\}$ of V_{even}, $\mathcal{B}_{\text{odd}} = \{w_1, \ldots, w_m\}$ of V_{odd}, their union $\mathcal{B}_V = \{u_1, \ldots, u_{n+m}\}$, and the basis $\mathcal{B}_{\otimes^r V}$ of $\otimes^r V$ induced by $\mathcal{B}_V$.

Definition 2.6. Let $u_{i_1}, \ldots, u_{i_r} \in \mathcal{B}_V$ and $\sigma \in S_r$. Let ε be the integer provided by Lemma 2.5 for σ and the parities $|u_{i_1}|, \ldots, |u_{i_r}|$. We define the map $\widehat{\sigma} : \mathcal{B}_{\otimes^r V} \to \otimes^r V$ by

$$\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) := (-1)^{\varepsilon} u_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes u_{i_{\sigma^{-1}(r)}}.$$

Since $\dim(V) = n + m$, we have $\dim(\otimes^r V) = (n+m)^r$. Let $\{f_1, \ldots, f_{(n+m)^r}\}$ be an enumeration of $\mathcal{B}_{\otimes^r V}$. Then for $1 \le i \le (n+m)^r$, each f_i is a pure tensor of homogeneous basis elements of V. Thus, f_i is of homogeneous parity for all i. Recall that for homogeneous $v, w \in V$ we have $|v \otimes w| = |v| + |w|$.

Lemma 2.7. *For each $\sigma \in S_r$, the map $\widehat{\sigma} : \mathcal{B}_{\otimes^r V} \to \otimes^r V$ may be extended linearly to an endomorphism $\widehat{\sigma} : \otimes^r V \to \otimes^r V$. This defines a representation of S_r on $\otimes^r V$.*

Proof. Let $\sigma \in S_r$ be arbitrary. Given a basis element f_i of $\otimes^r V$, we may write $f_i = u_{j_1} \otimes \cdots \otimes u_{j_r}$ for some $u_{j_1}, \ldots, u_{j_r} \in \mathcal{B}_V$. For simplicity, we will denote the expression $u_{j_{\sigma^{-1}(1)}} \otimes \cdots \otimes u_{j_{\sigma^{-1}(r)}}$ by f_i^{σ}. Recall that the integer ε provided by Lemma 2.5 depends only on σ and the parities $|u_{j_1}|, \ldots, |u_{j_r}|$. Therefore, we may denote the ε corresponding to σ and $|u_{j_1}|, \ldots, |u_{j_r}|$ as $\varepsilon(f_i, \sigma)$ without ambiguity. This verifies that the extension will be well-defined.

To prove that $\sigma \mapsto \widehat{\sigma}$ is a representation of S_r, it is enough to verify that the action satisfies the relations provided in the Coxeter presentation of S_r. The proofs for the first and third relations are straightforward. We will check the braid relation. Let $1 \le j < r - 1$ and write s_j for the transposition $(j, j+1)$. Let $x_1, \ldots, x_r$ be of homogeneous parity in V.

Then

$$\begin{aligned}\widehat{s}_j\widehat{s}_{j+1}\widehat{s}_j(x_1\otimes\cdots\otimes x_r) &= (-1)^{|x_j||x_{j+1}|}\widehat{s}_j\widehat{s}_{j+1}(x_1\otimes\cdots\otimes x_{j+1}\otimes x_j\otimes\cdots\otimes x_r)\\ &= (-1)^{|x_{j+1}||x_{j+2}|}\widehat{s}_j(x_1\otimes\cdots\otimes x_{j+1}\otimes x_{j+2}\otimes x_j\otimes\cdots\otimes x_r)\\ &= x_1\otimes\cdots\otimes x_{j+2}\otimes x_{j+1}\otimes x_j\otimes\cdots\otimes x_r,\end{aligned}$$

and on the other hand

$$\begin{aligned}\widehat{s}_{j+1}\widehat{s}_j\widehat{s}_{j+1}(x_1\otimes\cdots\otimes x_r) &= (-1)^{|x_{j+1}||x_{j+2}|}\widehat{s}_{j+1}\widehat{s}_j(x_1\otimes\cdots\otimes x_{j+2}\otimes x_{j+1}\otimes\cdots\otimes x_r)\\ &= (-1)^{|x_j||x_{j+1}|}\widehat{s}_{j+1}(x_1\otimes\cdots\otimes x_{j+2}\otimes x_j\otimes x_{j+1}\otimes\cdots\otimes x_r)\\ &= x_1\otimes\cdots\otimes x_{j+2}\otimes x_{j+1}\otimes x_j\otimes\cdots\otimes x_r\end{aligned}$$

as desired. □

The next lemma is immediate from the basis for $\otimes^r V$ induced by $\mathcal{B}_V$.

LEMMA 2.8. *The set*

$$\left\{u_{i_1}\otimes\cdots\otimes(u_{i_j}\otimes u_{i_{j+1}}-(-1)^{|u_{i_j}||u_{i_{j+1}}|}u_{i_{j+1}}\otimes u_{i_j})\otimes\cdots\otimes u_{i_r} : 1\le j\le r-1, u_{i_1},\ldots,u_{i_r}\in\mathcal{B}_V\right\}$$

is a (not necessarily linearly independent) spanning set for I^r.

PROPOSITION 2.9. *I^r is an S_r-subrepresentation of $\otimes^r V$, and the action of S_r on $\otimes^r V/I^r$ is trivial.*

PROOF. To see that I^r is an S_r-subrepresentation, fix $u_{i_1},\ldots,u_{i_r}\in\mathcal{B}_V$. For brevity, let us write f to denote the element $u_{i_1}\otimes\cdots\otimes u_{i_r}$. Then given any adjacent transposition $\rho\in S_r$, we have $f-\widehat{\rho}(f)\in I^r$ by Lemma 2.5. Let $\sigma\in S_r$ be arbitrary. In light of Lemma 2.8, we may prove that $\widehat{\sigma}\big(f-\widehat{\rho}(f)\big)\in I^r$, and then use the linearity of $\widehat{\sigma}$ to complete the proof. By Lemma 2.5 we have

$$\widehat{\sigma}\big(f-\widehat{\rho}(f)\big)=\widehat{\sigma}(f)-(\widehat{\sigma}\widehat{\rho}\widehat{\sigma}^{-1})\widehat{\sigma}(f)\in I^r$$

as desired. The fact that S_r acts trivially on the quotient $\otimes^r V/I^r$ is immediate from Lemma 2.5 and Definition 2.6. □

Given a basis element f_i of $\otimes^r V$, we have its stabilizer:

$$\mathrm{Stab}_{S_r}(f_i) := \{\sigma \in S_r : \widehat{\sigma}(f_i) = f_i\}.$$

Consider $S_r/\mathrm{Stab}_{S_r}(f_i) - \{\mathrm{Stab}_{S_r}(f_i)\}$, the set of non-identity left cosets of the stabilizer. For each of these non-identity cosets, select one representative and denote the set of selected representatives as $G(f_i)$.

Definition 2.10. Let a and b be non-negative integers with $a+b=r$. Define $\mathcal{B}_{I^r}$ to be the set of elements of the form

$$v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes w_{j_1} \otimes \cdots \otimes w_{j_b} - \widehat{\sigma}(v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes w_{j_1} \otimes \cdots \otimes w_{j_b})$$

where $1 \le i_1 \le \ldots \le i_a \le n$, $1 \le j_1 \le \ldots \le j_b \le m$, and $\sigma \in G(v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes w_{j_1} \otimes \cdots \otimes w_{j_b})$.

For brevity, we will say a basis element of $\otimes^r V$ is *properly ordered* if it is of the form $v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes w_{j_1} \otimes \cdots \otimes w_{j_b}$ where $1 \le i_1 \le \ldots \le i_a \le n$ and $1 \le j_1 \le \ldots \le j_b \le m$.

Lemma 2.11. *$\mathcal{B}_{I^r}$ is a basis of I^r.*

Proof. Independence of elements of $\mathcal{B}_{I^r}$ is guaranteed by the facts that σ runs over all non-identity coset representatives and that elements of $\mathcal{B}_{I^r}$ are linear combinations of basis elements of $\otimes^r V$. We will prove that the elements of $\mathcal{B}_{I^r}$ span I^r. It is enough to prove that $\mathcal{B}_{I^r}$ spans the spanning set of I^r provided in Lemma 2.8.

To begin, let $u_{i_1}, \ldots, u_{i_r} \in \mathcal{B}_V$ be arbitrary and let $1 \le j \le r-1$. Recall that we write s_j for the transposition $(j, j+1)$. There exists a $\sigma \in S_r$ such that $\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r})$ is properly ordered. We have

$$\begin{aligned}
&u_{i_1} \otimes \cdots \otimes \left(u_{i_j} \otimes u_{i_{j+1}} - (-1)^{|u_{i_j}||u_{i_{j+1}}|} u_{i_{j+1}} \otimes u_{i_j}\right) \otimes \cdots \otimes u_{i_r} \\
&= u_{i_1} \otimes \cdots \otimes u_{i_r} - (-1)^{|u_{i_j}||u_{i_{j+1}}|} u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r} \\
&= u_{i_1} \otimes \cdots \otimes u_{i_r} - (-1)^{|u_{i_j}||u_{i_{j+1}}|} u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r} \\
&\qquad - \widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) + \widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) \\
&= u_{i_1} \otimes \cdots \otimes u_{i_r} - (-1)^{|u_{i_j}||u_{i_{j+1}}|} u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r}
\end{aligned}$$

$$- \widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) + (-1)^{|u_{i_j}||u_{i_{j+1}}|} \widehat{\sigma}\widehat{s}_j(u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r})$$

$$= (-1)^{|u_{i_j}||u_{i_{j+1}}|}\big(\widehat{\sigma}\widehat{s}_j(u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r}) - u_{i_1} \otimes \cdots \otimes u_{i_{j+1}} \otimes u_{i_j} \otimes \cdots \otimes u_{i_r}\big)$$

$$- \big(\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) - u_{i_1} \otimes \cdots \otimes u_{i_r}\big)\big),$$

which completes the proof. □

DEFINITION 2.12. For each $r \geq 0$, define an endomorphism $\int_{S_r} : \otimes^r V \to \otimes^r V$ by the rule

$$\int_{S_r} \Theta = \frac{1}{r!} \sum_{\sigma \in S_r} \widehat{\sigma}(\Theta).$$

The following lemma is immediate.

LEMMA 2.13. *$\int_{S_r}$ is the unique S_r-invariant projection operator from $\otimes^r V$ to its S_r-invariant subspace: for all $\rho \in S_r$, $\widehat{\rho} \circ \int_{S_r} = \int_{S_r} \circ \widehat{\rho} = \int_{S_r}$, and $\int_{S_r}^2 = \int_{S_r}$.*

PROPOSITION 2.14. $\ker(\int_{S_r}) = I^r$.

PROOF. To prove that $I^r \subseteq \ker(\int_{S_r})$, apply Lemma 2.11. To prove the other direction of containment, let $\Theta \in \otimes^r V$, and assume that $\int_{S_r} \Theta = 0$. Recall that the action of S_r satisfies $\Theta - \widehat{\sigma}(\Theta) \in I^r$ for any $\sigma \in S_r$. Thus,

$$\Theta = \frac{1}{r!}\Big(r!\Theta - \sum_{\sigma \in S_r} \widehat{\sigma}(\Theta)\Big) = \frac{1}{r!}\Big(\sum_{\sigma \in S_r} \Theta - \widehat{\sigma}(\Theta)\Big) \in I^r,$$

completing the proof. □

COROLLARY 2.15. $\otimes^r V = I^r \oplus \int_{S_r} \otimes^r V$.

DEFINITION 2.16. Let $\mathcal{B}_{\mathcal{S}^r}$ be the set of all properly ordered basis elements of $\otimes^r V$ without any repeated odd terms:

$$\mathcal{B}_{\mathcal{S}^r} = \big\{v_{i_1} \otimes \cdots v_{i_a} \otimes w_{j_1} \otimes \cdots \otimes w_{j_b} : i_1 \leq \ldots \leq i_a,\ j_1 < \ldots < j_b,\ a + b = r\big\}.$$

LEMMA 2.17. *$\mathcal{B}_{\mathcal{S}^r} \cap \mathcal{B}_{I^r} = \varnothing$ and $\mathcal{B}_{\mathcal{S}^r} \cup \mathcal{B}_{I^r}$ is a basis for $\otimes^r V$.*

PROOF. It is clear that $\mathcal{B}_{\mathcal{S}^r}$ and $\mathcal{B}_{I^r}$ are disjoint. Moreover, the definition of $\mathcal{B}_{I^r}$ and the independence of elements of $\mathcal{B}_{\otimes^r V}$ make it clear that elements of $\mathcal{B}_{\mathcal{S}^r} \cup \mathcal{B}_{I^r}$ must be independent. We will prove that $\mathcal{B}_{\mathcal{S}^r} \cup \mathcal{B}_{I^r}$ spans $\otimes^r V$. Let $u_{i_1}, \ldots, u_{i_r} \in \mathcal{B}_V$ be arbitrary. Either $u_{i_1} \otimes \cdots \otimes u_{i_r}$ contains a repeated odd term or it does not.

Assume that it contains a repeated odd term. That is, assume there are integers j and k with $j \neq k$ such that $u_{i_j} = u_{i_k}$ is odd. Then

$$\frac{1}{2}\big(u_{i_1} \otimes \cdots \otimes u_{i_r} - \widehat{(j,k)}(u_{i_1} \otimes \cdots \otimes u_{i_r})\big) = u_{i_1} \otimes \cdots \otimes u_{i_r} \in I^r.$$

Since $u_{i_1} \otimes \cdots \otimes u_{i_r} \in I^r$, it is a linear combination of elements of $\mathcal{B}_{I^r}$ by Lemma 2.11.

Now assume that $u_{i_1} \otimes \cdots \otimes u_{i_r}$ does not contain a repeated odd term. Let $\rho \in S_r$ such that $\widehat{\rho}(u_{i_1} \otimes \cdots \otimes u_{i_r})$ is properly ordered. If $\rho \in \mathrm{Stab}_{S_r}(u_{i_1} \otimes \cdots \otimes u_{i_r})$, then $u_{i_1} \otimes \cdots \otimes u_{i_r} \in \mathcal{B}_{\mathcal{S}^r}$ and we are done. So assume that ρ does not stabilize $u_{i_1} \otimes \cdots \otimes u_{i_r}$. Let $\sigma \in G(u_{i_1} \otimes \cdots \otimes u_{i_r})$ be the left-coset representative selected for $\rho\mathrm{Stab}_{S_r}(u_{i_1} \otimes \cdots \otimes u_{i_r})$. We have

$$u_{i_1} \otimes \cdots \otimes u_{i_r} = \widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) - \big(\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) - u_{i_1} \otimes \cdots \otimes u_{i_r}\big).$$

Then $\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r})$ is an element of $\mathcal{B}_{\mathcal{S}^r}$ up to sign, and $\widehat{\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}) - u_{i_1} \otimes \cdots \otimes u_{i_r}$ is an element of $\mathcal{B}_{I^r}$. This completes the proof. □

COROLLARY 2.18. *$\int_{S_r} \mathcal{B}_{\mathcal{S}^r}$ is a basis for $\int_{S_r} \otimes^r V$ and hence $\otimes^r V = \mathrm{span}_{\mathbb{C}}\mathcal{B}_{\mathcal{S}^r} \oplus \mathrm{span}_{\mathbb{C}}\mathcal{B}_{I^r}$.*

PROOF. Apply Proposition 2.14 and Lemma 2.17. □

DEFINITION 2.19. The *supersymmetric algebra* $\mathcal{S}(V)$ is $\otimes V/I$. We denote projection to $\mathcal{S}(V)$ along I by

$$\mathrm{proj}_{\mathcal{S}} : \otimes V \twoheadrightarrow \mathcal{S}(V).$$

We denote the r^{th} *symmetric power* of V by $\mathcal{S}^r(V) := \otimes^r V/I^r$. The map $\mathrm{proj}_{\mathcal{S}}$ restricts to a projection $\mathrm{proj}_{\mathcal{S}}|_{\otimes^r(V)} : \otimes^r V \twoheadrightarrow \mathcal{S}^r(V)$, which has kernel I^r. This restriction will be denoted $\mathrm{proj}_{\mathcal{S}^r}$.

We drop the tensor symbol and denote multiplication in $\mathcal{S}(V)$ by concatenation, writing vw for the image of $v \otimes w$ under $\mathrm{proj}_{\mathcal{S}}$. We also have the *degree* of elements of $\mathcal{S}(V)$, induced by the degree on $\otimes V$. Elements of $\mathcal{S}^r(V)$ are said to be homogeneous of degree r. Note that if $V_{\text{odd}} = 0$, then the supersymmetric algebra is simply the usual symmetric algebra. The following proposition is immediate from Corollary 2.18.

Proposition 2.20. *Elements of the form*

$$v_{i_1} \cdots v_{i_a} w_{j_1} \cdots w_{j_b}, \qquad i_1 \leq \ldots \leq i_a,\ j_1 < \ldots < j_b$$

are a basis for $\mathcal{S}(V)$. Moreover, elements of this form with $a + b = r$ are a basis for $\mathcal{S}^r(V)$.

Lemma 2.21. $\mathcal{S}(V) = \bigoplus_{r=0}^{\infty} \mathcal{S}^r(V)$

Corollary 2.22. *We have*

$$\mathcal{S}^r(V_{\text{even}} \oplus V_{\text{odd}}) = \bigoplus_{j=0}^{r} \left(\mathcal{S}^j(V_{\text{even}}) \otimes \bigwedge\nolimits^{r-j}(V_{\text{odd}}) \right)$$

where $\mathcal{S}^j(V_{\text{even}})$ and $\bigwedge^{r-j}(V_{\text{odd}})$ are the usual j^{th} symmetric and $(r-j)^{th}$ exterior powers, respectively.

One then concludes that $\mathcal{S}(V) = \mathcal{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$.

We now discuss representations. For the remainder of this subsection $\mathfrak{g} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}$ will denote a Lie superalgebra, and (V, π) is representation of $\mathfrak{g}$.

Lemma 2.23. $\int_{S_r}$ *is a $\mathfrak{g}$-map.*

Proof. By Definition 2.2, $\otimes\pi = \sum_{j=1}^{r} \pi_j$. Furthermore, we have $\sigma \circ \pi_j = \pi_{\sigma(j)} \circ \sigma$. Thus, σ is a $\mathfrak{g}$-map for each $\sigma \in S_r$, which implies the result. $\square$

Corollary 2.24. $\int_{S_r} \otimes^r \mathfrak{g}$ *is a subrepresentation of $\otimes^r V$. Moreover, I is a subrepresentation of $\otimes V$ and I^r is a subrepresentation of $\otimes^r V$.*

It follows from this corollary that $\text{proj}_{\mathcal{S}}$ is a $\mathfrak{g}$-map, and that any representation of $\mathfrak{g}$ on V defines a representation of $\mathfrak{g}$ on $\mathcal{S}(V)$ and $\mathcal{S}^r(V)$.

Proposition 2.25. *The restriction* $\text{proj}_{\mathcal{S}^r} : \int_{S_r} \otimes^r V \to \mathcal{S}^r(V)$ *is a $\mathfrak{g}$-equivalence.*

PROOF. By Lemma 2.23, $\mathrm{proj}_{\mathcal{S}^r}$ is a $\mathfrak{g}$-map. By Proposition 2.14, it is also a bijection when restricted to $\int_{S_r} \otimes^r V$. $\square$

2.4. The Universal Enveloping Algebra $\mathfrak{U}$

Let $\mathfrak{g} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}$ be a Lie superalgebra. We now establish the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ and provide several results about $\mathfrak{U}(\mathfrak{g})$ with proof. Throughout, the reader is encouraged to consult the diagram given in Proposition 2.42 on page 17.

DEFINITION 2.26. Let J be the two-sided ideal of $\otimes\mathfrak{g}$ generated by

$$\big\{X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y] : X, Y \in \mathfrak{g}\big\}.$$

LEMMA 2.27. *J is filtered by the subspaces J_r for $r \geq 0$. Moreover, we have $J_r + \otimes_{r-1}\mathfrak{g} = I^r \oplus \otimes_{r-1}\mathfrak{g}$.*

PROOF. For the first sentence, use the fact that $\otimes\mathfrak{g}$ is graded by the subspaces $\otimes^r\mathfrak{g}$ for $r \geq 0$. We will now prove the second sentence. We begin by showing $J_r + \otimes_{r-1}\mathfrak{g} \subseteq I^r \oplus \otimes_{r-1}\mathfrak{g}$. Let X and Y be of homogeneous parity in $\mathfrak{g}$, and let Θ_1, Θ_2 be in $\otimes\mathfrak{g}$ with $\deg(\Theta_1) + \deg(\Theta_2) \leq r - 2$. Then

$$\Theta_1 \otimes \big(X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y]\big) \otimes \Theta_2$$

is in J_r. If $\deg(\Theta_1) + \deg(\Theta_2) < r - 2$, then this expression is contained in $J_{r-1} \subset \otimes_{r-1}\mathfrak{g}$ and we are done. So assume that $\deg(\Theta_1) + \deg(\Theta_2) = r - 2$. Then $\Theta_1 \otimes [X, Y] \otimes \Theta_2$ is in $\otimes_{r-1}\mathfrak{g}$ and $\Theta_1 \otimes \big(X \otimes Y - (-1)^{|X||Y|} Y \otimes X\big) \otimes \Theta_2$ is in I^r, finishing this direction of containment.

For the other direction of containment, consider $\Theta_1 \otimes \big(X \otimes Y - (-1)^{|X||Y|} Y \otimes X\big) \otimes \Theta_2 \in I^r$. We again use the fact that $\Theta_1 \otimes [X, Y] \otimes \Theta_2$ is in $\otimes_{r-1}\mathfrak{g}$ to write it as

$$\Theta_1 \otimes \big(X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y]\big) \otimes \Theta_2 + \Theta_1 \otimes [X, Y] \otimes \Theta_2$$

which is in $J_r + \otimes_{r-1}\mathfrak{g}$, as desired. $\square$

DEFINITION 2.28. The *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ is $\otimes\mathfrak{g}/J$. We denote projection to $\mathfrak{U}(\mathfrak{g})$ along J by

$$\mathrm{proj}_{\mathfrak{U}} : \otimes\mathfrak{g} \twoheadrightarrow \mathfrak{U}(\mathfrak{g}).$$

We denote the r^{th} *filtration* of $\mathfrak{U}(\mathfrak{g})$ by $\mathfrak{U}_r(\mathfrak{g}) := \otimes_r \mathfrak{g}/J_r$. The map $\text{proj}_{\mathfrak{U}}$ restricts to a projection $\text{proj}_{\mathfrak{U}}|_{\otimes_r \mathfrak{g}} : \otimes_r \mathfrak{g} \twoheadrightarrow \mathfrak{U}_r(\mathfrak{g})$, which has kernel J_r. This restriction will be denoted $\text{proj}_{\mathfrak{U}_r}$.

Again, we drop the tensor symbol and denote multiplication in $\mathfrak{U}(\mathfrak{g})$ by concatenation, writing XY for the image of $X \otimes Y$ under $\text{proj}_{\mathfrak{U}}$. The action of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ is by superderivations:

$$\text{ad}_X(\Theta_1\Theta_2) = \text{ad}_X(\Theta_1)\Theta_2 + (-1)^{|X||\Theta_1|}\Theta_1 \, \text{ad}_X(\Theta_2)$$

for $X \in \mathfrak{g}$ and $\Theta_1, \Theta_2 \in \mathfrak{U}(\mathfrak{g})$.

Proposition 2.29. *Under the adjoint action, J is a subrepresentation of $\otimes \mathfrak{g}$ and J_r is a subrepresentation of $\otimes_r \mathfrak{g}$.*

It follows from this proposition that $\text{proj}_{\mathfrak{U}}$ is a $\mathfrak{g}$-map, and that the adjoint action of $\mathfrak{g}$ on itself defines a representation of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}_r(\mathfrak{g})$.

Let V be a superspace. It is a fact that any representation π of $\mathfrak{g}$ on V extends to an associative algebra representation of $\mathfrak{U}(\mathfrak{g})$ on V via the assignment

$$\pi(X_1 X_2 \cdots X_p) = \pi(X_1) \circ \pi(X_2) \circ \cdots \circ \pi(X_p).$$

In particular, we can extend the adjoint representation of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ to a representation of $\mathfrak{U}(\mathfrak{g})$ on $\mathfrak{U}(\mathfrak{g})$.

Definition 2.30. Put $\mathcal{S}_r(\mathfrak{g}) := \bigoplus_{j=0}^{r} \mathcal{S}^j(\mathfrak{g})$ and $\text{proj}_{\mathcal{S}_r} := \bigoplus_{j=0}^{r} \text{proj}_{\mathcal{S}^j} : \otimes_r \mathfrak{g} \twoheadrightarrow \mathcal{S}_r(\mathfrak{g})$.

It is clear from Definition 2.30 that $\mathcal{S}_r/\mathcal{S}_{r-1}$ and $\mathcal{S}^r$ are naturally isomorphic as representations of $\mathfrak{g}$.

Definition 2.31. The canonical projection from $\mathfrak{U}_r(\mathfrak{g})$ to $\mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g})$ is denoted by ρ_r. The canonical projection from $\mathcal{S}_r(\mathfrak{g})$ to $\mathcal{S}_r(\mathfrak{g})/\mathcal{S}_{r-1}(\mathfrak{g})$ is denoted by ϕ_r.

Definition 2.32. For each non-zero $\Theta \in \mathfrak{U}(\mathfrak{g})$, the smallest integer d for which $\Theta \in \mathfrak{U}_d(\mathfrak{g})$ is called the *degree* of Θ and is denoted $\deg(\Theta)$. The image of Θ under $\rho_{\deg(\Theta)}$ is called the *symbol* of Θ.

Lemma 2.33. *We have* $\ker(\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r}) = J_r + \otimes_{r-1}\mathfrak{g}$ *and* $\ker(\phi_r \circ \mathrm{proj}_{\mathcal{S}_r}) = I^r \oplus \otimes_{r-1}\mathfrak{g}$. *Thus,* $\ker(\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r}) = \ker(\phi_r \circ \mathrm{proj}_{\mathcal{S}_r})$.

Proof. For the first equation, use the fact that $\ker(\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r}) = \mathrm{proj}_{\mathfrak{U}_r}^{-1}\big(\mathfrak{U}_{r-1}(\mathfrak{g})\big)$. For the second, use the analogous fact that $\ker(\phi_r \circ \mathrm{proj}_{\mathcal{S}_r}) = \mathrm{proj}_{\mathcal{S}_r}^{-1}\big(\mathcal{S}_{r-1}(\mathfrak{g})\big)$, combined with the fact that I is homogeneous. To prove the second sentence of the lemma, use the second sentence and Lemma 2.27. □

Proposition 2.34. $\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r} \circ \int_{S_r} : \otimes^r\mathfrak{g} \to \mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g})$ *is a* $\mathfrak{g}$*-equivalence.*

Proof. Recall that Lemma 2.23 shows that $\int_{S_r}$ is a $\mathfrak{g}$-map. As previously stated, ρ_r and $\mathrm{proj}_{\mathfrak{U}_r}$ are also $\mathfrak{g}$-maps. So it is enough to prove that $\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r} \circ \int_{S_r}$ is a bijection. Use Lemma 2.14 to write $\otimes^r\mathfrak{g} = \int_{S_r} \otimes^r\mathfrak{g} \oplus I^r$. Since $\int_{S_r} \otimes^r\mathfrak{g} \cap \otimes_{r-1}\mathfrak{g} = 0$, Lemma 2.33 implies that $\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r} \circ \int_{S_r}$ is an injection. To see that it is also a surjection, note that $\otimes_r\mathfrak{g} = \otimes^r\mathfrak{g} \oplus \otimes_{r-1}\mathfrak{g}$. We conclude from Lemma 2.33 that $\rho_r \circ \mathrm{proj}_{\mathfrak{U}_r}(\otimes_r\mathfrak{g}) = \rho_r \circ \mathrm{proj}_{\mathfrak{U}_r} \circ \int_{S_r} \otimes^r\mathfrak{g}$, which finishes the claim. □

Definition 2.35. Put $\mathfrak{U}^r(\mathfrak{g}) := \mathrm{proj}_{\mathfrak{U}_r}\big(\int_{S_r} \otimes^r\mathfrak{g}\big)$.

Corollary 2.36. *The restriction* $\rho_r : \mathfrak{U}^r(\mathfrak{g}) \to \mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g})$ *is a* $\mathfrak{g}$*-equivalence.*

We will sometimes write $\mathfrak{U}^r$ for $\mathfrak{U}^r(\mathfrak{g})$. The next result follows directly from Proposition 2.34.

Corollary 2.37. $\mathfrak{U}_r(\mathfrak{g}) = \mathfrak{U}^r(\mathfrak{g}) \oplus \mathfrak{U}_{r-1}(\mathfrak{g})$ *as representations of* $\mathfrak{g}$.

Proposition 2.25 states that $\mathrm{proj}_{\mathcal{S}^r} : \int_{S_r} \otimes^r\mathfrak{g} \to \mathcal{S}^r(\mathfrak{g})$ is a $\mathfrak{g}$-equivalence. Thus, $\mathrm{proj}_{\mathcal{S}^r}^{-1} : \mathcal{S}^r(\mathfrak{g}) \to \int_{S_r} \otimes^r\mathfrak{g}$ is a well-defined $\mathfrak{g}$-equivalence.

Definition 2.38. The r^{th} *symmetrizer map* $\mathrm{sym}_r : \mathcal{S}^r(\mathfrak{g}) \to \mathfrak{U}^r(\mathfrak{g})$ is defined as

$$\mathrm{sym}_r = \mathrm{proj}_{\mathfrak{U}_r} \circ \ {\mathrm{proj}_{\mathcal{S}^r}}^{-1}.$$

We further define the *symmetrizer map* $\mathrm{sym} : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ to be $\mathrm{sym} := \oplus_{r\geq 0}\mathrm{sym}_r$.

It is clear that sym_r is a bijective map. Fix bases $X_1, \ldots, X_n$ of $\mathfrak{g}_{\text{even}}$ and $Y_1, \ldots, Y_m$ of $\mathfrak{g}_{\text{odd}}$.

Lemma 2.39. *Let $X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}$ be an arbitrary basis element of $\mathcal{S}^r(\mathfrak{g})$ as described in* Proposition 2.20. *Then $\rho_r \circ \text{sym}_r(X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}) = X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}$.*

Proof. Let $\sigma \in S_r$ be arbitrary. Then $X_{i_1} \otimes \cdots \otimes X_{i_a} \otimes Y_{j_1} \otimes \cdots \otimes Y_{j_b} - \widehat{\sigma}(X_{i_1} \otimes \cdots \otimes X_{i_a} \otimes Y_{j_1} \otimes \cdots \otimes Y_{j_b}) \in I^r$. So by Lemma 2.27, we have

$$\rho_r \circ \text{proj}_{\mathfrak{U}_r}(X_{i_1} \otimes \cdots \otimes X_{i_a} \otimes Y_{j_1} \otimes \cdots \otimes Y_{j_b}) = \rho_r \circ \text{proj}_{\mathfrak{U}_r} \circ \widehat{\sigma}(X_{i_1} \otimes \cdots \otimes X_{i_a} \otimes Y_{j_1} \otimes \cdots \otimes Y_{j_b}).$$

Now, write

$$\begin{aligned} &\rho_r \circ \text{proj}_{\mathfrak{U}_r} \circ {\text{proj}_{\mathcal{S}^r}}^{-1}(X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}) \\ &\qquad = \rho_r \circ \text{proj}_{\mathfrak{U}_r}\Big(\frac{1}{r!} \sum_{\sigma \in S_r} \widehat{\sigma}(X_{i_1} \otimes \cdots \otimes X_{i_a} \otimes Y_{j_1} \otimes \cdots \otimes Y_{j_b})\Big). \end{aligned}$$

Combining the linearity of ρ_r and $\text{proj}_{\mathfrak{U}_r}$ with the above completes the proof. □

Theorem 2.40 (Poincaré-Birkhoff-Witt). *Fix bases $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}_{even}$ and $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{g}_{odd}$. Then elements of the form*

$$X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b} \qquad i_1 \leq \ldots \leq i_a,\ j_1 < \ldots < j_b$$

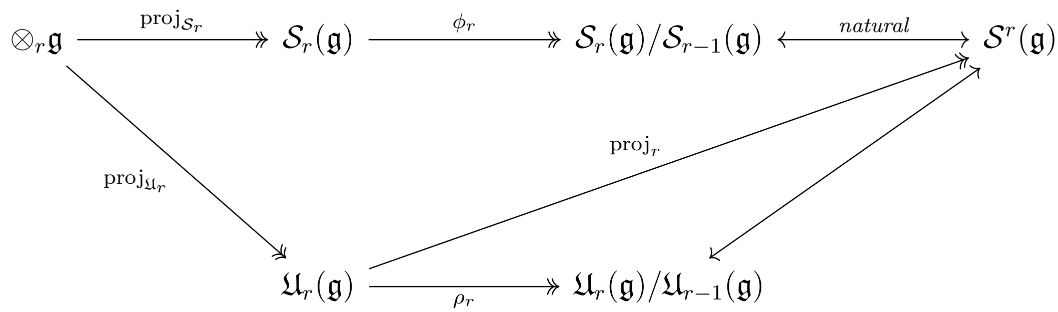
are a basis for $\mathfrak{U}(\mathfrak{g})$. Furthermore, elements of this form with $a+b \leq r$ are a basis for $\mathfrak{U}_r(\mathfrak{g})$.

Proof. The second sentence of the theorem implies the first. For brevity, let us say that elements of the form $X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}$ with $i_1 \leq \ldots \leq i_a$, $j_1 < \ldots < j_b$ and $a + b = r$ are *PBW monomials of degree* r. It follows from Corollary 2.37 that $\mathfrak{U}_r(\mathfrak{g}) = \oplus_{j=0}^{r} \mathfrak{U}^j(\mathfrak{g})$. Lemma 2.39 and Proposition 2.20 imply that $\mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g})$ has a basis of PBW monomials of degree r. Now, Corollary 2.36 states that $\rho_r^{-1} : \mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g}) \to \mathfrak{U}^r(\mathfrak{g})$ is a $\mathfrak{g}$-equivalence. By Lemma 2.39, ρ_r^{-1} must satisfy $\rho_r^{-1}(X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}) = X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}$. Hence, $\mathfrak{U}^r(\mathfrak{g})$ has a basis of PBW monomials of degree r. Thus, the second sentence of the theorem is proven. □

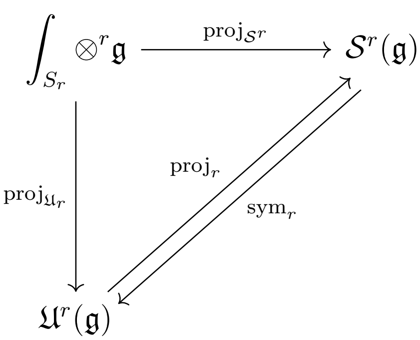
Definition 2.41. Define $\text{proj}_r : \mathfrak{U}_r(\mathfrak{g}) \to \mathcal{S}^r(\mathfrak{g})$ by the rule

$$\text{proj}_r(X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b}) = \begin{cases} X_{i_1} \cdots X_{i_a} Y_{j_1} \cdots Y_{j_b} & a+b=r \\ 0 & a+b<r \end{cases}$$

Proposition 2.42. proj_r *is a surjective* $\mathfrak{g}$*-map with kernel* $\mathfrak{U}_{r-1}(\mathfrak{g})$*. It induces a* $\mathfrak{g}$*-equivalence from* $\mathfrak{U}_r(\mathfrak{g})/\mathfrak{U}_{r-1}(\mathfrak{g})$ *to* $\mathcal{S}^r(\mathfrak{g})$ *such that the following diagram commutes.*



Corollary 2.43. *The restriction* $\text{proj}_r : \mathfrak{U}^r(\mathfrak{g}) \to \mathcal{S}^r(\mathfrak{g})$ *is a* $\mathfrak{g}$*-equivalence. Its inverse is* sym_r*, and we have the following commutative diagram. Each map below is a* $\mathfrak{g}$*-equivalence.*



Definition 2.44. Let (V, π) be a representation of a Lie superalgebra $\mathfrak{g}$. The *annihilator* of V is defined as

$$\text{Ann}_{\mathfrak{g}}(V) := \ker(\pi|_{\mathfrak{U}(\mathfrak{g})}).$$

Given any representation (V, π), the annihilator $\text{Ann}_{\mathfrak{g}}(V)$ is a two-sided ideal in the universal enveloping algebra. In particular, $\text{Ann}_{\mathfrak{g}}(V)$ is invariant under the adjoint action of $\mathfrak{U}(\mathfrak{g})$.

Definition 2.45. Let A be a unital ring. A proper ideal I is called a *primitive ideal* if I is the annihilator of some non-zero irreducible left A-module.

Within the context of representations of Lie superalgebras, if (V, π) is a non-trivial irreducible representation of $\mathfrak{g}$, then $\mathrm{Ann}_{\mathfrak{g}}(V)$ is a primitive ideal. For example, we have the following:

DEFINITION 2.46. Let $\mathfrak{U}^+(\mathfrak{g})$ be the two-sided ideal of $\mathfrak{U}(\mathfrak{g})$ generated by $\mathfrak{g}$.

Then $\mathfrak{U}(\mathfrak{g}) = \mathbb{C}1 \oplus \mathfrak{U}^+(\mathfrak{g})$, and $\mathfrak{U}^+(\mathfrak{g}) = \mathrm{Ann}_{\mathfrak{g}}(\mathbb{C})$. Thus, $\mathfrak{U}^+(\mathfrak{g})$ is primitive.

The set of primitive ideals of A is denoted by $\mathrm{Prim}(A)$. Let S be a non-empty subset of $\mathrm{Prim}(A)$. Put $I(S) := \bigcap_{J \in S} J$. Then $I(S)$ is a two-sided ideal of A. The *closure operator* on $\mathrm{Prim}(A)$ is defined to be

$$\overline{S} := \{J \in \mathrm{Prim}(A) : J \supseteq I(S)\}.$$

One further defines $\overline{\varnothing} := \varnothing$. The reader may check that closure defines a topology on $\mathrm{Prim}(A)$.

DEFINITION 2.47. The *Jacobson topology* on $\mathrm{Prim}(A)$ is the topology defined by closure.

The Jacobson topology first appears at this level of generality in [7]. Note that in general, points are not closed in the Jacobson topology. If I is a primitive ideal of A, the set $\{I\}$ is closed if and only if I is not contained in any other primitive ideal of A.

CHAPTER 3

THE LIE SUPERALGEBRA $\mathscr{K}$

In this chapter, we introduce the Lie superalgebra $\mathscr{K}$ of contact vector fields on $\mathbb{R}^{1|1}$ and its tensor density modules. Then we discuss differential operators. We further describe the universal enveloping algebra $\mathfrak{U}(\mathscr{K})$ and identify distinguished elements of this algebra.

3.1. The Superline

Here we discuss the supermanifold with one even variable and one odd variable as well as its associated polynomial vector fields. Let $\mathbb{R}^{1|1}$ be the superline, with even coordinate x and odd coordinate ξ. Here $\xi^2 = 0$, so $\mathbb{C}[x, \xi]$ has a basis of $\{1, \xi\}$ over $\mathbb{C}[x]$. The space of polynomial vector fields on $\mathbb{R}^{1|1}$ is

$$\mathrm{Vec}(\mathbb{R}^{1|1}) := \mathrm{span}_{\mathbb{C}[x,\xi]}\{\partial_x, \partial_\xi\}.$$

It is a Lie superalgebra acting by superderivations on $\mathbb{C}[x, \xi]$: for $X, Y \in \mathrm{Vec}(\mathbb{R}^{1|1})$ and $F, G \in \mathbb{C}[x, \xi]$, we have

$$[X, Y] = XY - (-1)^{|X||Y|}YX, \quad X(FG) = X(F)G + (-1)^{|F||X|}FX(G).$$

Obviously, $\mathbb{C}[x, \xi]$ is a two-sided $\mathbb{C}[x, \xi]$-module via the usual polynomial multiplication. We may compose the actions of $\mathrm{Vec}(\mathbb{R}^{1|1})$ and $\mathbb{C}[x, \xi]$: given $X \in \mathrm{Vec}(\mathbb{R}^{1|1})$ and $F \in \mathbb{C}[x, \xi]$, the expression XF acts on $\mathbb{C}[x, \xi]$ via the assignment $XF(G) := X(FG)$. Hence as elements of $\mathrm{End}(\mathbb{C}[x, \xi])$, one has $[X, F] = X(F)$. Define elements D, $\overline{D}$ and ϵ of $\mathrm{Vec}(\mathbb{R}^{1|1})$ by

$$D := \partial_\xi + \xi\partial_x, \qquad \overline{D} := \partial_\xi - \xi\partial_x, \qquad \epsilon := 1 - 2\xi\partial_\xi.$$

The operator ϵ is the parity operator: it acts by 1 on $\mathbb{C}[x]$ and by -1 on $\mathbb{C}[x]\xi$.

Proposition 3.1. *The operators ∂_ξ, D and $\overline{D}$ are odd and satisfy the following formulae:*

(1) $\partial_\xi^2 = 0$

(2) $[D, D] = 2D^2 = 2\partial_x$

(3) $[\overline{D}, \overline{D}] = 2\overline{D}^2 = -2\partial_x$

(4) $[\overline{D}, D] = 0$

(5) $\xi\partial_\xi = \xi D = \xi\overline{D}$

(6) $D = \overline{D} - 2\xi\overline{D}^2 = \epsilon\overline{D} = -\overline{D}\epsilon$

(7) $fD = Df - \xi f'$ *for any* $f(x) \in \mathbb{C}[x]$

(8) $f\overline{D} = \overline{D}f + \xi f'$ *for any* $f(x) \in \mathbb{C}[x]$

3.2. Contact Vector Fields

This section is devoted to discussing the subspace of $\mathrm{Vec}(\mathbb{R}^{1|1})$ of contact vector fields: those vector fields that preserve the contact structure induced by the contact form $\omega := dx + \xi d\xi$. The Lie superalgebra $\mathscr{K}$ of *contact vector fields* on $\mathbb{R}^{1|1}$ is the image of the even linear injection $\mathbb{X} : \mathbb{C}[x, \xi] \to \mathrm{Vec}(\mathbb{R}^{1|1})$ defined by

$$\mathbb{X}(f) = f\partial_x + \tfrac{1}{2}f'\xi\partial_\xi, \qquad \mathbb{X}(f\xi) = \tfrac{1}{2}fD,$$

where $f \in \mathbb{C}[x]$. It has brackets

$$[\mathbb{X}(f), \mathbb{X}(g)] = \mathbb{X}(fg' - f'g),$$

$$[\mathbb{X}(f), \mathbb{X}(g\xi)] = \mathbb{X}\left(fg' - \tfrac{1}{2}f'g\xi\right),$$

$$[\mathbb{X}(f\xi), \mathbb{X}(g\xi)] = \mathbb{X}\left(\tfrac{1}{2}fg\right).$$

Given $X \in \mathscr{K}$, the polynomial $\mathbb{X}^{-1}(X) \in \mathbb{C}[x, \xi]$ is called the *contact Hamiltonian* of X. We have the following basis for $\mathscr{K}$:

$$\left\{e_{n-1} := \mathbb{X}(x^n),\ e_{n-1/2} := 2\mathbb{X}(\xi x^n) : n \in \mathbb{N}\right\}.$$

These basis elements satisfy

$$[e_n, e_m] = (m - n)e_{n+m} \qquad \text{if } n, m \in \mathbb{N} - 1,$$

$$[e_n, e_m] = (m - n/2)e_{n+m} \qquad \text{if } n \in \mathbb{N} - 1, m \in \mathbb{N} - 1/2,$$

$$[e_n, e_m] = 2e_{n+m} \qquad \text{if } n, m \in \mathbb{N} - 1/2.$$

Note that $\mathscr{K}_{\text{even}} = \mathrm{span}_{\mathbb{C}}\{e_n : n \in \mathbb{N} - 1\}$ and $\mathscr{K}_{\text{odd}} = \mathrm{span}_{\mathbb{C}}\{e_n : n \in \mathbb{N} - 1/2\}$.

LEMMA 3.2. *$\mathscr{K}_{\mathrm{odd}} = \mathbb{C}[x,\xi]D$ and $\mathscr{K}$ is generated by $\mathscr{K}_{\mathrm{odd}}$.*

Now $\mathscr{K}$ contains a maximal subalgebra

$$\mathfrak{s} := \mathrm{span}_{\mathbb{C}}\big\{e_{-1},\ e_{-1/2},\ e_0,\ e_{1/2},\ e_1\big\},$$

which is isomorphic to $\mathfrak{osp}(1|2)$. Its even part is $\mathfrak{a} := \mathrm{span}_{\mathbb{C}}\{e_{-1},\ e_0,\ e_1\}$, which is isomorphic to $\mathfrak{sl}_2$. We will also need the *affine subalgebra* $\mathfrak{t}$ of $\mathfrak{s}$, defined as

$$\mathfrak{t} := \mathrm{span}_{\mathbb{C}}\big\{e_{-1},\ e_{-1/2},\ e_0\big\}.$$

If (V,π) is any representation of $\mathfrak{t}$, then the eigenvalues and eigenspaces of $\pi(e_0)$ are called *weights* and *weightspaces*, respectively. We write V_λ for the λ-weightspace of V. We denote the kernels of $\mathrm{ad}(e_{-1/2})$ and $\mathrm{ad}(e_{1/2})$ by $V^{e_{-1/2}}$ and $V^{e_{1/2}}$, respectively.

LEMMA 3.3. *Under* ad*, e_n is a vector of weight n. In any representation (V,π) of $\mathscr{K}$, $\pi(e_n)$ maps V_λ to $V_{\lambda+n}$. If W is another representation, then $V_\lambda \otimes W_\mu \subseteq (V\otimes W)_{\lambda+\mu}$.*

3.3. Tensor Density Modules

From its definition, we see that $\mathscr{K}$ has a natural action on $\mathbb{C}[x,\xi]$. The tensor density modules are a one-parameter family of deformations of this action. For $\lambda \in \mathbb{C}$, the *tensor density module* (TDM) of degree λ is the vector space $\mathbb{C}[x,\xi]$, with the Lie superalgebra representation π_λ of $\mathscr{K}$ defined by

$$\pi_\lambda\big(\mathbb{X}(F)\big) := \mathbb{X}(F) + \lambda F',$$

where $F' := \partial_x(F)$. Applying this to $G \in \mathbb{C}[x,\xi]$ yields $\pi_\lambda\big(\mathbb{X}(F)\big)(G) = \mathbb{X}(F)G + \lambda F'G$. Thus the natural action of $\mathscr{K}$ on $\mathbb{C}[x,\xi]$ is the TDM of degree 0. On the basis of $\mathscr{K}$ in the previous section, we have

$$\pi_\lambda(e_{n-1}) = x^n\partial_x + nx^{n-1}(\tfrac{1}{2}\xi\partial_\xi + \lambda), \quad \pi_\lambda(e_{n-1/2}) = x^nD + 2n\lambda\xi x^{n-1}. \tag{1}$$

We will write $\mathbb{F}_\lambda$ to denote $\mathbb{C}[x,\xi]$ with this action. For reference, we provide the image of $\mathfrak{s}$ under π_λ:

$$\pi_\lambda(e_{-1}) = \partial_x, \quad \pi_\lambda(e_{-1/2}) = D, \quad \pi_\lambda(e_0) = x\partial_x + \tfrac{1}{2}\xi\partial_\xi + \lambda,$$

$$\pi_\lambda(e_{1/2}) = xD + 2\lambda\xi, \quad \pi_\lambda(e_1) = x^2\partial_x + x(\xi\partial_\xi + 2\lambda).$$

Note that ∂_x has weight -1, x has weight 1, ξ has weight 1/2, and ∂_ξ, D, and $\overline{D}$ all have weight $-1/2$.

PROPOSITION 3.4.

(1) *As an $\mathfrak{s}$-module, $\mathbb{F}_\lambda$ is irreducible unless $\lambda \in -\frac{1}{2}\mathbb{N}$, when it contains a unique $\mathfrak{s}$-subrepresentation*

$$\mathbb{L}_\lambda := \mathrm{span}_{\mathbb{C}}\big\{1, \xi, x, \xi x, x^2, \ldots, x^{-2\lambda-1}, \xi x^{-2\lambda-1}, x^{-2\lambda}\big\}$$

of dimension $-4\lambda+1$. Note that $x^{-2\lambda}$ is of weight $-\lambda$ in $\mathbb{F}_\lambda$, and so the weights of $\mathbb{L}_\lambda$ are evenly spaced about zero.

(2) *As a $\mathscr{K}$-module, $\mathbb{F}_\lambda$ is irreducible unless $\lambda = 0$ when its unique $\mathscr{K}$-subrepresentation is $\mathbb{L}_0$.*

(3) *The quotient $\mathbb{F}_\lambda/\mathbb{L}_\lambda$ is $\mathfrak{s}$-equivalent to $\big(\mathbb{F}_{-\lambda+1/2}\big)^\Pi$.*

(4) $(\mathscr{K}, \mathrm{ad})$ *is equivalent to $\mathbb{F}_{-1}$ as a $\mathscr{K}$-module, and $\mathfrak{s}$ corresponds to $\mathbb{L}_{-1}$. The map $F \mapsto \mathbb{X}(F)$ is a $\mathscr{K}$-equivalence.*

3.4. Differential Operators

Consider the space of polynomial differential operators on $\mathbb{C}[x, \xi]$:

$$\mathrm{Diff}(\mathbb{R}^{1|1}) := \mathrm{span}_{\mathbb{C}[x,\xi]}\left\{\partial_x^i,\ \partial_x^i\partial_\xi : i \in \mathbb{N}\right\}.$$

It is a superalgebra under composition. Clearly, it contains $\mathscr{K}$ and thus is naturally a $\mathscr{K}$-module under the adjoint action. In light of Proposition 3.1, we have the following claim.

LEMMA 3.5. $\mathrm{Diff}(\mathbb{R}^{1|1}) = \mathrm{span}_{\mathbb{C}}\big\{F\overline{D}^i : F \in \mathbb{C}[x, \xi],\ i \in \mathbb{N}\big\}$.

This observation leads us to define the *fine filtration* $\mathrm{Diff}(\mathbb{R}^{1|1})$. It is described in [5].

DEFINITION 3.6. For $k \in \frac{1}{2}\mathbb{N}$, set

$$\mathrm{Diff}^k(\mathbb{R}^{1|1}) := \mathrm{span}_{\mathbb{C}}\big\{F\overline{D}^i : F \in \mathbb{C}[x, \xi],\ i \leq 2k\big\}.$$

Given $\sum_{i=0}^{\infty} F_i\overline{D}^i$ in $\mathrm{Diff}(\mathbb{R}^{1|1})$, the largest half-integer d for which $F_{2d} \neq 0$ is called the *$\overline{D}$-degree* of $\sum_{i=0}^{\infty} F_i\overline{D}^i$.

Proposition 3.7. *For $k \in \frac{1}{2}\mathbb{N}$, $\mathrm{Diff}^k(\mathbb{R}^{1|1})$ is invariant under the adjoint action of $\mathscr{K}$.*

Proof. First, one proves by induction that for each half-integral $k \geq 1$ and $F \in \mathbb{C}[x,\xi]$, the $\overline{D}$-degree of $[\overline{D}^{2k}, F]$ is $k-1$. The base case is the fact that $[\partial_x, F] = F'$ and $[\overline{D}, F] = \overline{D}(F)$ and the inductive step amounts to an application of the superderivation property. Next, let $G \in \mathbb{C}[x,\xi]$, and assume that F and G are of homogeneous parity. Then

$$[FD, G\overline{D}^i] = F[D,G]\overline{D}^i - (-1)^{|G|(|F|+1)}G[\overline{D}^i, FD].$$

Then $[D,G] = D(G)$ and $[\overline{D}^i, FD] = [\overline{D}^i, F]D$, which completes the proof. ☐

3.5. $\mathfrak{U}(\mathscr{K})$

Let us now describe the universal enveloping algebra of $\mathscr{K}$ and identify some distinguished elements in it. By Theorem 2.40,

$$\{e_{i_1}e_{i_2}\cdots e_{i_a}e_{j_1}e_{j_2}\cdots e_{j_b} : i_1 \leq \ldots \leq i_a \in \mathbb{N}-1,\ j_1 < \ldots < j_b \in \mathbb{N} - \tfrac{1}{2}\ \} \tag{2}$$

is a basis of the universal enveloping algebra of $\mathfrak{U}(\mathscr{K})$. Recall the degree filtration on $\mathfrak{U}_r(\mathscr{K})$. It is spanned by elements of the basis (2) satisfying $a+b \leq r$. Note that in general, degree is sub-additive rather than strictly additive in universal enveloping algebras of Lie superalgebras. For example, $e_{-1/2}^2 = e_{-1}$, so $\deg(e_{-1/2}^2) = \deg(e_{-1/2})$.

The weight of an element of the basis (2) is the sum of the indices. The space $\mathfrak{U}(\mathscr{K})^{e_{-1/2}}$, the kernel of $\mathrm{ad}(e_{-1/2})$, will be particularly important for us. The following lemma will be useful in describing it.

Lemma 3.8. *For $r > 0$, $\mathrm{ad}(e_{-1/2})$ acts surjectively on $\otimes^r\mathscr{K}$, $\mathfrak{U}^r(\mathscr{K})$, and $\mathcal{S}^r(\mathscr{K})$.*

Proof. We will prove via induction that $\mathrm{ad}(e_{-1/2})$ acts surjectively on the tensor powers of $\mathscr{K}$. It is clear that $\mathrm{ad}(e_{-1/2})$ acts surjectively on $\mathscr{K}$. For $n \in \frac{1}{2}\mathbb{N} - 1$, let us write a_n for the scalar satisfying $\mathrm{ad}(e_{-1/2})e_{n+1/2} = a_n e_n$. For the inductive hypothesis, assume that

$\mathrm{ad}(e_{-1/2})$ acts surjectively on $\otimes^{r-1}\mathscr{K}$. Let $m \in \frac{1}{2}\mathbb{N}-1$, and define the following subspaces of $\mathscr{K}$:

$$V_m := \mathrm{span}_{\mathbb{C}}\{e_n : n \leq m\}.$$

For $m < -1$, put $V_m := 0$.

We must prove that $V_m \otimes (\otimes^{r-1}\mathscr{K}) \subset \mathrm{ad}(e_{-1/2})\,(\otimes^r\mathscr{K})$ for every m. We will do this by inducting on m. First, we consider $m = -1$ when $V_{-1} = \mathbb{C}e_{-1}$. Let $\Theta \in \otimes^{r-1}\mathscr{K}$ be arbitrary. The original inductive hypothesis assumes that $\mathrm{ad}(e_{-1/2})$ acts surjectively on $\otimes^{r-1}\mathscr{K}$, so there exists an $\Omega \in \otimes^{r-1}\mathscr{K}$ for which $\mathrm{ad}(e_{-1/2})\Omega = \Theta$. Using the superderivation property of the adjoint action and the fact that $\mathrm{ad}(e_{-1/2})(e_{-1}) = 0$, we write

$$\mathrm{ad}(e_{-1/2})(e_{-1} \otimes \Omega) = e_{-1} \otimes \Theta.$$

Thus, $V_{-1} \otimes (\otimes^{r-1}\mathscr{K}) \subset \mathrm{ad}(e_{-1/2})(\otimes^r\mathscr{K})$.

Now we may proceed with the induction on m. As a secondary inductive hypothesis, assume that $V_m \otimes (\otimes^{r-1}\mathscr{K}) \subset \mathrm{ad}(e_{-1/2})(\otimes^r\mathscr{K})$. Again, let Θ be arbitrary in $\otimes^{r-1}\mathscr{K}$. Since $V_{m+1/2} = \mathbb{C}e_{m+1/2} \oplus V_m$, it is sufficient to prove that $e_{m+1/2} \otimes \Theta \in \mathrm{ad}(e_{-1/2})(\otimes^r\mathscr{K})$. By the original inductive hypothesis, there is an $\Omega \in \otimes^{r-1}\mathscr{K}$ with $\mathrm{ad}(e_{-1/2})\Omega = \Theta$. As in the base case, we have

$$\mathrm{ad}(e_{-1/2})(e_{m+1/2} \otimes \Omega) = a_m e_m \otimes \Omega + (-1)^{2m+1} e_{m+1/2} \otimes \Theta.$$

Subtracting $a_m e_m \otimes \Omega$ from both sides and applying the secondary inductive hypothesis completes the proof of the fact that $V_m \otimes (\otimes^{r-1}\mathscr{K}) \subset \mathrm{ad}(e_{-1/2})(\otimes^r\mathscr{K})$. Thus, $\mathrm{ad}(e_{-1/2})$ is a surjective endomorphism of $\otimes^r\mathscr{K}$. To complete the proof of the lemma, use the facts that $\mathrm{proj}_{\mathcal{S}_r} : \otimes^r\mathscr{K} \twoheadrightarrow \mathcal{S}^r(\mathscr{K})$ is a surjective $\mathscr{K}$-map, and that $\mathcal{S}^r(\mathscr{K})$ is $\mathscr{K}$-equivalent to $\mathfrak{U}^r(\mathscr{K})$. □

We write $\mathfrak{U}_r(\mathscr{K})_m$ for the m-weightspace of the r^{th} filtration. If the context is clear, we write $\mathfrak{U}$ for $\mathfrak{U}(\mathscr{K})$ and $(\mathfrak{U}_r)_m$ for $\mathfrak{U}_r(\mathscr{K})_m$. As stated before, given any ideal A of $\mathfrak{U}(\mathscr{K})$, we write A_r for $A \cap \mathfrak{U}_r(\mathscr{K})$. We use $(A_r)_m$ for the m-weightspace of A_r.

For each $\lambda \in \mathbb{C}$, we extend π_λ to a representation of $\mathfrak{U}(\mathscr{K})$ on $\mathbb{F}_\lambda$. For the remainder of this dissertation, the symbol π_λ will mean this extension. Then π_λ is an associative algebra homomorphism mapping $\mathfrak{U}(\mathscr{K})$ into $\mathrm{Diff}(\mathbb{R}^{1|1})$. The next proposition states that the degree filtrations on $\mathfrak{U}(\mathscr{K})$ and $\mathrm{Diff}(\mathbb{R}^{1|1})$ are compatible.

PROPOSITION 3.9. *For $r \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, we have $\pi_\lambda\big(\mathfrak{U}_r(\mathscr{K})\big) \subseteq \mathrm{Diff}^r(\mathbb{R}^{1|1})$.*

Let us now introduce several key elements of $\mathfrak{U}(\mathscr{K})$ and give their images under π_λ. At this point, we only define what is necessary to state our main results. For a more thorough discussion, see Chapter 5.

DEFINITION 3.10. The *Casimir operator* $Q_{\mathfrak{s}}$ of $\mathfrak{U}(\mathfrak{s})$ and the *Scasimir operator* $T_{\mathfrak{s}}$ of $\mathfrak{U}(\mathfrak{s})$ are defined as

$$Q_{\mathfrak{s}} := e_0^2 + \tfrac{1}{2}e_0 + \tfrac{1}{2}e_{-1/2}e_{1/2} - e_{-1}e_1, \qquad T_{\mathfrak{s}} := e_0 - e_{1/2}e_{-1/2} - \tfrac{1}{4}.$$

PROPOSITION 3.11.

(1) *$Q_{\mathfrak{s}}$ is central in $\mathfrak{U}(\mathfrak{s})$ and the center of $\mathfrak{U}(\mathfrak{s})$ is $\mathbb{C}[Q_{\mathfrak{s}}]$.*

(2) *$T_{\mathfrak{s}}$ is not central in $\mathfrak{U}(\mathfrak{s})$; it commutes with $\mathfrak{s}_{even}$ and skew-commutes with $\mathfrak{s}_{odd}$.*

(3) *$Q_{\mathfrak{s}} = T_{\mathfrak{s}}^2 - \frac{1}{16}$.*

(4) *$T_{\mathfrak{s}}$ is not a LWV, but $\mathrm{ad}(e_{-1/2})T_{\mathfrak{s}}$ is.*

We remark that the subspace $\mathbb{C}[T_{\mathfrak{s}}]$ of $\mathfrak{U}(\mathfrak{s})$ is called the *ghost center* of $\mathfrak{U}(\mathfrak{s})$. See for example [6]. The following lemma describes the kernel of $\mathrm{ad}(D)$, a subalgebra of $\mathrm{Diff}(\mathbb{R}^{1|1})$.

LEMMA 3.12. $\mathrm{Diff}(\mathbb{R}^{1|1})^{e_{-1/2}} = \mathbb{C}[\overline{D}]$.

PROOF. Use Definition 3.6 to write any element of $\mathrm{Diff}(\mathbb{R}^{1|1})$ as $\sum_{i\in\mathbb{N}} F_i\overline{D}^{2i}$. By Proposition 3.1, D and $\overline{D}$ commute. Thus,

$$\mathrm{ad}(D)\Big(\sum_{i\in\mathbb{N}} F_i\overline{D}^i\Big) = \sum_{i\in\mathbb{N}} \mathrm{ad}(D)(F_i)\overline{D}^i,$$

which is zero if and only if $D(F_i) = 0$ for all $i \in \mathbb{N}$. That is, if and only if F_i is a scalar for all i. □

For the remainder of this section, results are proven via direct computation. These have been left to the reader.

LEMMA 3.13. *Recall that* $\epsilon := 1 - 2\xi\partial_\xi$. *The Casimir and Scasimir operators have the following images under* π_λ :

$$\pi_\lambda(Q_{\mathfrak{s}}) = \lambda^2 - \tfrac{1}{2}\lambda, \qquad \pi_\lambda(T_{\mathfrak{s}}) = (\lambda - \tfrac{1}{4})\epsilon.$$

DEFINITION 3.14.

$$\begin{aligned}
Z_{1/2} &:= \tfrac{1}{4}\left((2e_0+1)e_{1/2} - e_{-1/2}e_1 - e_{-1}e_{3/2}\right)\\
Z_1 &:= \mathrm{ad}(e_{1/2})Z_{1/2}\\
Y_0 &:= Q_{\mathfrak{s}}(e_0 - \tfrac{1}{4}) - \tfrac{1}{2}Z_{1/2}e_{-1/2} - Z_1 e_{-1}\\
\widehat{T} &:= -\tfrac{1}{4}\left(Y_0 + \mathrm{ad}(e_{1/2})(Z_{1/2}e_{-1} - \tfrac{1}{2}Q_{\mathfrak{s}}e_{-1/2})\right)
\end{aligned}$$

LEMMA 3.15.

(1) $Z_{1/2}$ *is of weight* $\frac{1}{2}$. *We have* $\mathrm{ad}(e_{-1/2})Z_{1/2} = Q_{\mathfrak{s}}$ *and* $\pi_\lambda(Z_{1/2}) = (\lambda^2 - \frac{1}{2}\lambda)\xi$.

(2) Z_1 *is of weight* 1. *We have* $\mathrm{ad}(e_{-1/2})Z_1 = Z_{1/2}$ *and* $\pi_\lambda(Z_1) = (\lambda^2 - \frac{1}{2}\lambda)x$.

(3) Y_0 *is a LWV of weight 0. We have* $\pi_\lambda(Y_0) = (\lambda - \frac{1}{4})(\lambda^2 - \frac{1}{2}\lambda)$.

(4) $\widehat{T}$ *is of weight 0, but is not a LWV. We have* $\pi_\lambda(\widehat{T}) = (\lambda^2 - \frac{1}{2}\lambda)\epsilon$.

COROLLARY 3.16. *For* $\lambda \in \mathbb{C}$, *the following elements are in* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$:

(1) $Q_{\mathfrak{s}} - \lambda^2 + \frac{1}{2}\lambda$

(2) $Y_0 - (\lambda - \frac{1}{4})(\lambda^2 - \frac{1}{2}\lambda)$

(3) $(\lambda - \frac{1}{4})\widehat{T} - (\lambda^2 - \frac{1}{2}\lambda)T$

On the other hand, $T_{\mathfrak{s}}$ *is in* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ *only when* $\lambda = \frac{1}{4}$. *Similarly,* $Z_{1/2}$ *is in* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ *only when* $\lambda = 0$ *or* $\frac{1}{2}$.

CHAPTER 4

MAIN RESULTS

In this chapter, we state our main results.

DEFINITION 4.1. Define polynomials

$$p_1(\lambda) := \lambda - \tfrac{1}{4}, \qquad p_2(\lambda) := \lambda^2 - \tfrac{1}{2}\lambda, \qquad p_3(\lambda) := p_1(\lambda)p_2(\lambda).$$

THEOREM 4.2. *For $\lambda \neq 0, 1/4$, or $1/2$, the ideals $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ are all distinct. Each of them is generated by its intersection with $\mathfrak{U}_3(\mathscr{K})_0$, the subspace of weight 0 of degree ≤ 3. This intersection is 4-dimensional and spanned by*

$$Q_{\mathfrak{s}} - p_2(\lambda), \qquad Y_0 - p_3(\lambda), \qquad \big(Q_{\mathfrak{s}} - p_2(\lambda)\big)e_0, \qquad p_1(\lambda)\widehat{T} - p_2(\lambda)T.$$

Therefore,

$$\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = \big\langle Q_{\mathfrak{s}} - p_2(\lambda),\ Y_0 - p_3(\lambda),\ p_1(\lambda)\widehat{T} - p_2(\lambda)T \big\rangle_{\mathscr{K}}.$$

Note that $p_1(\lambda)\widehat{T} - p_2(\lambda)T$ may be replaced by $\mathrm{ad}(e_{-1/2})\big(p_1(\lambda)\widehat{T} - p_2(\lambda)T\big)$ so as to have all generators be lowest weight vectors.

THEOREM 4.3. *We have $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0) = \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2}) = \langle Z_{1/2} \rangle_{\mathscr{K}}$.*

THEOREM 4.4. *We have $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/4}) = \langle T \rangle_{\mathscr{K}}$.*

THEOREM 4.5. *The ideal $\bigcap_{\lambda \in \mathbb{C}} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ contains all lowest weight vectors of positive weight. Its intersection with $\mathfrak{U}(\mathfrak{s})$ is zero. It is not generated by any single lowest weight vector. It is generated by a single element of weight 2. We have*

$$\bigcap_{\lambda \in \mathbb{C}} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = \big\langle\, \mathrm{ad}(e_2)T \big\rangle_{\mathscr{K}}.$$

Now, let $S \subset \mathbb{C}$ and put $\mathcal{A}(S) := \big\{\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) : \lambda \in S\big\}$. As $\mathcal{A}(S)$ is a subspace of $\mathrm{Prim}\big(\mathfrak{U}(\mathscr{K})\big)$, it can be equipped with the subspace topology. The final theorem describes $\mathcal{A}(\mathbb{C})$ as a topological space.

Theorem 4.6. *The space $\mathcal{A}(\mathbb{C})$ is topologically equivalent to $\mathbb{C}^\times$ equipped with the co-finite topology. For $\lambda \in \mathbb{C}^\times$, the map $\operatorname{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) \mapsto \lambda$ is a homeomorphism.*

CHAPTER 5

THE $\mathfrak{s}$-STRUCTURE OF $\mathfrak{U}(\mathscr{K})$

In this section, we give decompositions of $\mathcal{S}^2(\mathscr{K})$ and $\mathcal{S}^3(\mathscr{K})$ under the adjoint action of $\mathfrak{s}$ and $\mathfrak{t}$. We further describe subspaces of these decompositions by explicitly defining their weight vectors and stating how those elements behave under the adjoint actions of $\mathfrak{s}$ and $\mathfrak{t}$. Given any subalgebra $\mathfrak{a}$ of $\mathscr{K}$, we write $\stackrel{\mathfrak{a}}{\cong}$ to mean $\mathfrak{a}$-equivalence. For elements $\Omega \in \mathfrak{U}(\mathscr{K})$ and $X \in \mathscr{K}$, we will sometimes use the abbreviation

$$\Omega^X := \mathrm{ad}(X)(\Omega).$$

One finds the following results stated in [2].

Proposition 5.1. *Suppose that V is a representation of $\mathfrak{s}$ containing a $\mathbb{Z}_2$-homogeneous lowest weight vector v of weight λ. If $\lambda \notin -\mathbb{N}/2$, then v generates a copy of $\mathbb{F}_\lambda^{|v|\Pi}$. If $\dim V < \infty$, then $\lambda \in -\mathbb{N}/2$ and v generates a copy of $\mathbb{L}_\lambda^{|v|\Pi}$.*

Proposition 5.2. *We have the following decompositions for $\mathcal{S}^2(\mathscr{K})$ and $\mathcal{S}^3(\mathscr{K})$:*

$$\mathcal{S}^2(\mathscr{K}) \stackrel{\mathfrak{s}}{\cong} \bigoplus_{j\in\mathbb{N}} \mathbb{F}_{2j-2} \oplus \mathbb{F}^{\Pi}_{2j-1/2}, \quad \mathcal{S}^3(\mathscr{K}) \stackrel{\mathfrak{t}}{\cong} \bigoplus_{\substack{i,j\in\mathbb{N},\\ b\in\{0,\, 3/2,\, 5/2,\, 4\}}} \mathbb{F}^{2b\Pi}_{b+2j+3(i-1)}.$$

It follows from this proposition that in $\mathcal{S}^2(\mathscr{K})$, there is a LWV of weight 2 that is unique up to scalar. We will denote its image under sym_2 as R. It is given explicitly in Lemma 6.26. Below, we give $\mathfrak{s}$-decompositions for $\mathfrak{U}_2(\mathscr{K})$ and $\mathfrak{U}_3(\mathscr{K})$. There are subspaces of $\mathfrak{U}_3(\mathscr{K})$ that are indecomposable as $\mathfrak{s}$-modules, but decomposable as $\mathfrak{t}$-modules. To signify this for subspaces A and B, we write $A \oplus_{\mathfrak{t}} B$.

Corollary 5.3. *$\mathfrak{U}_2(\mathscr{K})$ and $\mathfrak{U}_3(\mathscr{K})$ have the following $\mathfrak{s}$-decompositions. The lowest weight vectors of the first few summands are written beneath them.*

$$\begin{array}{cccccccccc} \mathfrak{U}_2(\mathscr{K}) \stackrel{\mathfrak{s}}{\cong} & \mathbb{C} & \oplus\, \mathbb{F}_{-2} & \oplus\, \mathbb{F}_{-1} & \oplus\, \mathbb{F}^{\Pi}_{-1/2} & \oplus\, \mathbb{F}_0 & \oplus\, \mathbb{F}^{\Pi}_{3/2} & \oplus\, \mathbb{F}_2 & \oplus\ \cdots, \\ & 1 & e_{-1}^2 & e_{-1} & T^{e_{-1/2}} & Q_{\mathfrak{s}} & Q_{\mathfrak{s}}^{e_{3/2}} & R & \end{array}$$

$$\mathfrak{U}_3(\mathscr{K}) \overset{\mathfrak{s}}{\cong} \mathfrak{U}_2(\mathscr{K}) \oplus \underset{e_{-1}^3}{\mathbb{F}_{-3}} \oplus \underset{T^{e_{-1/2}}e_{-1}}{\mathbb{F}_{-3/2}} \oplus \underset{Q_{\mathfrak{s}}e_{-1}}{\mathbb{F}_{-1}} \oplus \big(\underset{\widehat{T}^{e_{-1/2}}}{\mathbb{F}_{-1/2}} \oplus_{\mathfrak{t}} \mathbb{F}_1\big) \oplus \big(\underset{Y_0}{\mathbb{F}_0} \oplus_{\mathfrak{t}} \underset{Q_{\mathfrak{s}}^{e_{3/2}}e_{-1}}{\mathbb{F}_{1/2}}\big) \oplus \underset{Re_{-1}}{\mathbb{F}_1} \oplus \cdots$$

Additionally, we have $\mathrm{ad}(e_{1/2})Q_{\mathfrak{s}}^{3/2} = 2Q_{\mathfrak{s}}^{e_2}$ *and* $T^{e_2} = \frac{2}{3}(R - Q_{\mathfrak{s}}^{e_2})$.

Proof. To see the decomposition, we recall Corollary 2.37: we have $\mathfrak{U}_r(\mathscr{K}) = \bigoplus_{j=0}^r \mathfrak{U}^j(\mathscr{K})$. Now $\mathfrak{U}^j(\mathscr{K})$ is $\mathscr{K}$-equivalent to $\mathcal{S}^j(\mathscr{K})$, and hence applying Proposition 5.2 verifies the decomposition. The LWVs are results of Lemma 3.15 and direct computation. For the last sentence, note that Proposition 3.11 implies $\mathrm{ad}(e_{-1})T = \mathrm{ad}(e_1)T = 0$. Thus $\mathrm{ad}(e_{-1}e_2)T = 0$, but $\mathrm{ad}(e_{-1/2}e_2)T \neq 0$. Hence $\mathrm{ad}(e_2)T$ is an element of the subspace $\mathbb{F}_{3/2}^{\Pi} \oplus \mathbb{F}_2 \subset \mathcal{S}^2(\mathscr{K})$. This is an easy way to see that T^{e_2} is a linear combination of R and Q^{e_2}. To find its exact value, use direct computation. □

Proposition 5.4. *Let* $n \in \frac{1}{2}\mathbb{N}$ *with* $n > 0$. *Suppose* $\Omega \in \mathfrak{U}(\mathscr{K})$ *is of weight* n *and satisfies* $\mathrm{ad}(e_{-1/2})^{2n}\Omega = 0$. *Then* $\pi_\lambda(\Omega) = 0$.

Proof. Toward a contradiction, assume that $\pi_\lambda(\Omega) \neq 0$. Let k be the largest integer for which $\pi_\lambda\big(\mathrm{ad}(e_{-1/2})^k\Omega)\big) \neq 0$. Then $\mathrm{ad}(e_{-1/2})^k\Omega$ is a LWV of weight $n - \frac{k}{2} > 0$. Now since π_λ is a $\mathscr{K}$-map, $\pi_\lambda\big(\mathrm{ad}(e_{-1/2})^k\Omega\big)$ is zero or a LWV of weight $n - \frac{k}{2}$. Thus, by Lemma 3.12 we have $\pi_\lambda(\Omega) \in \mathbb{C}[\overline{D}]$. However, $\overline{D}$ is of weight $-\frac{1}{2}$, so $\mathbb{C}[\overline{D}]$ contains only elements of non-positive weight. Therefore, $\pi_\lambda\big(\mathrm{ad}(e_{-1/2})^k\Omega\big) = 0$, which is a contradiction. □

The following facts describe the quadratic portions of the individual annihilators, as well as of the intersection over all annihilators: the annihilator of the direct sum of all the $\mathbb{F}_\lambda$. For the remainder of this section, we use the abbreviation

$$I := \bigcap_{\lambda\in\mathbb{C}} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda),$$

and as previously stated we write I_2 for $I \cap \mathfrak{U}_2(\mathscr{K})$.

Lemma 5.5. *For all* $\lambda \in \mathbb{C}$, $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ *contains every* $\mathfrak{s}$*-submodule of* $\mathfrak{U}(\mathscr{K})$ *equivalent to* $\mathbb{F}_{n/2}$ *for any* $n \geq 1$. *Furthermore, we have* $I_2 \overset{\mathfrak{s}}{\cong} \bigoplus_{j\geq 0} \mathbb{F}_{2j+2} \oplus \mathbb{F}_{2j+3/2}^{\Pi}$.

PROOF. The first sentence of the lemma follows immediately from Propositon 5.4. Corollary 5.3 verifies that $\bigoplus_{j\geq 0} \mathbb{F}_{2j+2} \oplus \mathbb{F}^{\Pi}_{2j+3/2} \subseteq I_2$. To show equality, use Corollary 3.16 with Corollary 5.3 to deduce that the TDMs in $\mathfrak{U}_2(\mathscr{K})$ with non-positive weights do not annihilate for every λ. Specifically, $Q_{\mathfrak{s}}$ only annihilates for $\lambda = 0, \frac{1}{2}$, and $T^{e_{-1/2}}$ only annihilates for $\lambda = \frac{1}{4}$. Moreover, it is clear that e_{-1} and e_{-1}^2 are not contained $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ for any λ. □

PROPOSITION 5.6. *The ideals* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)$ *and* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2})$ *are equal and contained in* $\mathfrak{U}^+(\mathscr{K})$. *Conversely, if* $\lambda \neq 0, 1/2$, *then* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ *is not equal to* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\mu)$ *for any* $\lambda \neq \mu$, *and* $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ *is not contained in* $\mathfrak{U}^+(\mathscr{K})$.

Let j *be an integer. For* $\lambda \neq 0, 1/4$ *or* $1/2$, *we have the following* $\mathfrak{s}$*-decomposition:*

$$\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)_2 \overset{\mathfrak{s}}{\cong} \mathbb{C} \oplus \Big(\bigoplus_{j\geq 0} \mathbb{F}_{2j+2} \oplus \mathbb{F}^{\Pi}_{2j+3/2} \Big).$$

When $\lambda = 0$ *or* $1/2$ *we have the* $\mathfrak{s}$*-decomposition*

$$\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)_2 = \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2})_2 \overset{\mathfrak{s}}{\cong} \mathbb{F}_0 \oplus \Big(\bigoplus_{j\geq 0} \mathbb{F}_{2j+2} \oplus \mathbb{F}^{\Pi}_{2j+3/2} \Big).$$

Finally, for $\lambda = 1/4$ *we arrive at the* $\mathfrak{s}$*-decomposition*

$$\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/4})_2 \overset{\mathfrak{s}}{\cong} \mathbb{C} \oplus \mathbb{F}_{-1/2} \oplus \Big(\bigoplus_{j\geq 0} \mathbb{F}_{2j+2} \oplus \mathbb{F}^{\Pi}_{2j+3/2} \Big).$$

PROOF. To prove the first sentence, note that $\mathbb{F}_0/\mathbb{C}$ is $\mathscr{K}$-equivalent to $\mathbb{F}_{1/2}$. Thus we have $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0) \subseteq \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2})$. To show equality, assume that Ω is in $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2})$. Consider the differential operator $\pi_0(\Omega)$. Since $\mathbb{F}_0/\mathbb{C}$ is $\mathscr{K}$-equivalent to $\mathbb{F}_{1/2}$, we have $\pi_0(\Omega) : \mathbb{F}_0 \to \mathbb{C}$. Hence, $\pi_0(\Omega)$ sends infinitely many weight spaces to zero. In light of the filtration on $\mathrm{Diff}(\mathbb{R}^{1|1})$, we may write $\pi_0(\Omega) = \sum_{i=0}^{N} F_i \overline{D}^i$ for $F_i \in \mathbb{C}[x, \xi]$. Thus, either $\pi_0(\Omega)$ only annihilates finitely many weight spaces, which is a contradiction, or we have that F_i is zero for all $0 \leq i \leq N$. So we conclude that $\pi_0(\Omega) = 0$ and hence Ω is in $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)$, as desired. To see that $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)$ is contained in $\mathfrak{U}^+(\mathscr{K})$, consider the trivial submodule of scalars at the bottom of $\mathbb{F}_0$: it is annihilated by every member of $\mathscr{K}$ and thus by $\mathfrak{U}^+(\mathscr{K})$. On the other hand, it is not annihilated by the action of the non-zero scalars from $\mathfrak{U}(\mathscr{K})$, completing the proof of the first sentence.

Now we prove the second sentence. Fix $\lambda, \mu \neq 0$ or $\frac{1}{2}$. Suppose that $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\mu)$, and consider the operators $Q_{\mathfrak{s}} - (\lambda^2 - \frac{1}{2}\lambda)$ and $Y_0 - (\lambda - \frac{1}{4})(\lambda^2 - \frac{1}{2}\lambda)$. Their images under π_μ are zero by hypothesis. Thus, $\mu^2 - \frac{1}{2}\mu = \lambda^2 - \frac{1}{2}\lambda$ and $(\mu - \frac{1}{4})(\mu^2 - \frac{1}{2}\mu) = (\lambda - \frac{1}{4})(\lambda^2 - \frac{1}{2}\lambda)$, so $\lambda = \mu$. Moreover, the fact that $Q_{\mathfrak{s}} - (\lambda^2 - \frac{1}{2}\lambda)$ is contained in $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ proves that $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ is not contained in $\mathfrak{U}^+(\mathscr{K})$.

Lastly we prove the equivalences as $\mathfrak{s}$-modules, recalling the decomposition provided for $\mathfrak{U}_2(\mathscr{K})$ in Corollary 5.3. It is clear that $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)_2$ contains I_2 for every $\lambda \in \mathbb{C}$, so we must prove that adding the additional subspaces yield equivalences. As stated in Corollary 5.3, e_{-1} and e_{-1}^2 do not annihilate for any λ: they have images ∂_x and ∂_x^2, respectively. As these are the unique LWVs in $\mathfrak{U}_2(\mathscr{K})$ of weight -1 and -2, respectively, no annihilator's second filtration contains a copy of $\mathbb{F}_{-1}$ or $\mathbb{F}_{-2}$. Clearly, $Q_{\mathfrak{s}} - (\lambda^2 - \frac{1}{2}\lambda)$ annihilates for any $\lambda \in \mathbb{C}$, so $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ contains a copy of $\mathbb{C}$ for every λ. In the case of $\lambda = 0$ or $\frac{1}{2}$, this copy of $\mathbb{C}$ is $\mathbb{C}Q_{\mathfrak{s}}$ and is contained in the $\mathbb{F}_0$. Lemma 3.15 states that $\pi_\lambda(Z_{1/2}) = (\lambda^2 - \frac{1}{2}\lambda)\xi$. For $\lambda = 0$ or $\frac{1}{2}$, use the irreducibility of $\mathbb{F}_0/\mathbb{C}$ under $\mathfrak{s}$ to deduce that all of the positive weight spaces of the $\mathbb{F}_0$ annihilate. Finally, a direct computation reveals $\pi_\lambda(T^{e_{-1/2}}) = 2(\lambda - \frac{1}{4})\overline{D}$, which annihilates only for $\lambda = \frac{1}{4}$. The fact that each LWV in $\mathfrak{U}_2(\mathscr{K})$ is of unique weight completes the proof. □

Definition 5.7. Put $Z_0 := Q_{\mathfrak{s}}$. In Definition 3.14, we defined

$$Z_{1/2} := \tfrac{1}{4}\left((2e_0 + 1)e_{1/2} - e_{-1/2}e_1 - e_{-1}e_{3/2}\right), \qquad Z_1 := \mathrm{ad}(e_{1/2})Z_{1/2}.$$

Recall from Lemma 3.15 that $Z_{1/2}$ is the unique element of weight $\frac{1}{2}$ satisfying $\mathrm{ad}(e_{-1/2})Z_{1/2} = Q_{\mathfrak{s}}$. For $n > 1$, we recursively define

$$Z_{n+1/2} := \tfrac{1}{n}\,\mathrm{ad}(e_{1/2})Z_n, \qquad Z_{n+1} := \mathrm{ad}(e_{1/2})Z_{n+1/2}.$$

We recall again from Definition 3.14 and Lemma 3.15 that

$$Y_0 := Z_0\left(e_0 - \tfrac{1}{4}\right) - \tfrac{1}{2}Z_{1/2}e_{-1/2} - Z_1e_{-1}$$

is up to scalar and symbol the unique cubic LWV of weight zero. Put

$$Y_{1/2} := Z_{1/2}\left(e_0 - \tfrac{1}{4}\right) - Z_{3/2}e_{-1}, \qquad X_{1/2} := \mathrm{ad}(e_{1/2})Y_0, \qquad X_1 := \mathrm{ad}(e_{1/2})X_{1/2}.$$

so that for $n > 1$ we may define

$$\begin{aligned} X_{n+1/2} &:= \tfrac{1}{n+1}\,\mathrm{ad}(e_{1/2})X_n, \qquad & Y_{n+1/2} &:= \tfrac{1}{n}\,\mathrm{ad}(e_{1/2})Y_n + X_{n+1/2} \\ X_{n+1} &:= \mathrm{ad}(e_{1/2})X_{n+1/2}, \qquad & Y_{n+1} &:= \mathrm{ad}(e_{1/2})Y_{n+1/2} - X_{n+1} \end{aligned}$$

For simplicity, we take $X_0 := 0$.

Note that above, the subscript of each element is its weight. The next proposition computes the images of all these elements under π_λ and shows that they make up several key subspaces of $\mathfrak{U}_3(\mathscr{K})$: for $m \in \frac{1}{2}\mathbb{N}$ the Z_m make up the copy of $\mathbb{F}_0$ in $\mathcal{S}^2(\mathscr{K})$. The copy of $\mathbb{F}_0$ in $\mathfrak{U}^3(\mathscr{K})$ consists of the Y_m, and the copy of $\mathbb{F}_{1/2}$ in $\mathfrak{U}^3(\mathscr{K})$ consists of the X_m. Together, the span of the X_m and Y_m is the $\mathfrak{s}$-indecomposable $\mathbb{F}_0 \oplus_{\mathfrak{t}} \mathbb{F}^{\Pi}_{1/2}$ in $\mathfrak{U}_3(\mathscr{K})$.

Proposition 5.8. *Let $n \in \mathbb{N}$. We have*

$$\begin{aligned} &\pi_\lambda(Z_n) = p_2(\lambda)x^n, \qquad &&\pi_\lambda(Y_n) = p_3(\lambda)x^n, \qquad &&\pi_\lambda(X_n) = 0 \\ &\pi_\lambda(Z_{n+1/2}) = p_2(\lambda)x^n\xi, \qquad &&\pi_\lambda(Y_{n+1/2}) = p_3(\lambda)x^n\xi, \qquad &&\pi_\lambda(X_{n+1/2}) = 0. \end{aligned}$$

The ad*-action of $e_{-1/2}$ on these elements is*

$$\begin{aligned} \mathrm{ad}(e_{-1/2})Z_{n+1} &= (n+1)Z_{n+1/2}, \qquad & \mathrm{ad}(e_{-1/2})Z_{n+1/2} &= Z_n, \\ \mathrm{ad}(e_{-1/2})Y_{n+1} &= (n+1)Y_{n+1/2}, \qquad & \mathrm{ad}(e_{-1/2})Y_{n+1/2} &= Y_{n+1/2}, \\ \mathrm{ad}(e_{-1/2})X_{n+1} &= nX_{n+1/2}, \qquad & \mathrm{ad}(e_{-1/2})X_{n+1/2} &= X_n. \end{aligned}$$

Proof. It is easy to use the equality $e_{-1/2}e_{1/2} = -e_{1/2}e_{-1/2} + 2e_0$ to check that $X_{1/2}$ is a LWV of weight $\frac{1}{2}$. By Lemma 5.5, its image under π_λ must be zero. It follows immediately that $\pi_\lambda(X_m) = 0$ for all $m \in \frac{1}{2}\mathbb{N}$. Next, for $m = 0$ and $\frac{1}{2}$, verify the formulae for the images of $\pi_\lambda(Z_m)$ and $\pi_\lambda(Y_m)$ by direct computation. For $m > \frac{1}{2}$, use the definition of Z_m and Y_m

combined with the fact that π_λ is a $\mathscr{K}$-map to check the formulae for $\pi_\lambda(Z_m)$ and $\pi_\lambda(Y_m)$. For $n \in \mathbb{N}$, we have

$$[xD, x^n] = xD(x^n) = nx^n\xi, \qquad [xD, x^n\xi] = xD(x^n\xi) = x^{n+1}.$$

For the action of $e_{-1/2}$ on $Z_{n/2}$ and $Y_{n/2}$, again use the fact that π_λ is a $\mathscr{K}$-map to compute the action in $\mathrm{Diff}(\mathbb{R}^{1|1})$. Write

$$[D, x^n] = D(x^n) = nx^{n-1}\xi, \qquad [D, x^n\xi] = D(x^n\xi) = x^n.$$

To prove the formulae for the ad-actions of $e_{-1/2}$ on the X_m, use direct computation for $m = \frac{1}{2}$ and 1. A straightforward induction completes the proof. □

CHAPTER 6

PROOF OF THEOREM 4.5

In this chapter, we describe the intersection of the annihilators of the tensor density modules. We show that it is a principal ideal generated by the image of the ghost under $\mathrm{ad}(e_2)$. In this chapter and the next, we will use the abbreviations

$$I := \bigcap_{\lambda\in\mathbb{C}} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda), \qquad T^{e_2} := \mathrm{ad}(e_2)T, \qquad \mathfrak{U} := \mathfrak{U}(\mathscr{K}), \qquad \mathfrak{U}^+ := \mathfrak{U}^+(\mathscr{K}).$$

We also need to introduce a few key items.

Definition 6.1. Define a subspace J of $\mathfrak{U}$ by

$$J := \mathrm{span}_{\mathbb{C}} \left\{ e_0^i e_{-1/2}^j,\ e_{n/2} e_0^i e_{-1/2}^j : i, j \geq 0 \text{ and } n \geq 1 \right\}.$$

The strategy to prove that $I = \langle T^{e_2} \rangle_{\mathscr{K}}$ is as follows: first, we show that J is complementary to I so that $\mathfrak{U} = I \oplus J$. Then, we will show that $\mathfrak{U} = \langle I_2 \rangle_{\mathscr{K}} + J$. An immediate corollary is $I = \langle I_2 \rangle_{\mathscr{K}}$. Lastly, we verify that $I_2 \subset \langle T^{e_2} \rangle_{\mathscr{K}}$, which implies the desired result.

Recall that $\mathbb{X}$ is the function defined in Section 3.2 that bijectively associates to each polynomial of $\mathbb{C}[x, \xi]$ a vector field in $\mathrm{Vec}(\mathbb{R}^{1|1})$. Continuing the set-up, we define the following $\mathscr{K}$-module:

Definition 6.2. Put $\mathbb{F}_\Lambda := \mathbb{C}[x, \xi, \Lambda]$ where Λ is an indeterminate. Define a representation π of $\mathscr{K}$ on $\mathbb{F}_\Lambda$ by

$$\pi\big(\mathbb{X}(F)\big)(G) = \mathbb{X}(F)G + \Lambda F' G.$$

Let us also fix some useful polynomials. We define $F_{-1} := 1$ and $F_{-1/2} := 2\xi$. Then for each $n \in \mathbb{N}$, we set

$$F_n := x^{n+1}, \qquad F_{n+1/2} := 2x^{n+1}\xi, \qquad G_n := \tfrac{1}{n!} x^n, \qquad G_{n+1/2} := \tfrac{1}{n!} x^n \xi.$$

The module $\mathbb{F}_\Lambda$ has precisely the same $\mathscr{K}$-action as the tensor density module of degree λ, except the scalar λ has been replaced with the indeterminate Λ. Since π is a representation of $\mathscr{K}$, it is also a representation of $\mathfrak{U}$ in the usual way. For $m \in \frac{1}{2}\mathbb{N}$, note that F_m and G_m are chosen so that $\mathbb{X}(F_m) = e_m$ and $D^{2m}(G_m) = 1$.

DEFINITION 6.3. Let $\mathrm{eval}_\lambda : \mathbb{F}_\Lambda \to \mathbb{F}_\lambda$ be defined as

$$\mathrm{eval}_\lambda\big(f(x,\Lambda) + g(x,\Lambda)\xi\big) = f(x,\lambda) + g(x,\lambda)\xi$$

for $f, g \in \mathbb{C}[x,\Lambda]$.

Next, we define an algebra of differential operators on $\mathbb{F}_\Lambda$.

DEFINITION 6.4. Put $\mathrm{Diff}(\Lambda) := \mathbb{C}[x,\xi,\overline{D}][\Lambda]$ where Λ is central.

PROPOSITION 6.5. *Extend* eval_λ *to* $\mathrm{eval}_\lambda : \mathrm{Diff}(\Lambda) \to \mathrm{Diff}(\mathbb{R}^{1|1})$ *via*

$$\mathrm{eval}_\lambda\big(F(x,\xi,\Lambda)\overline{D}\big) := F(x,\xi,\lambda)\overline{D}.$$

Then eval_λ *intertwines* π *and* π_λ.

Recall the fine filtration on $\mathrm{Diff}(\mathbb{R}^{1|1})$ from Definition 3.6. We now give an analogous filtration on $\mathrm{Diff}(\Lambda)$ that accounts for Λ.

DEFINITION 6.6. Let $k \in \frac{1}{2}\mathbb{N}$. Then the space $\mathrm{Diff}(\Lambda)$ has k^{th}-filtration

$$\mathrm{Diff}^k(\Lambda) := \mathrm{span}_{\mathbb{C}}\big\{x^n \xi^\delta \Lambda^j\, \overline{D}^i : \delta = 0 \text{ or } 1,\ n \geq 0,\ 2j + i \leq 2k\big\}.$$

In particular, $\mathrm{Diff}^0(\Lambda) = \mathbb{C}[x,\xi]$.

PROPOSITION 6.7. *For each* $k \in \frac{1}{2}\mathbb{N}$, $\mathrm{Diff}^k(\Lambda)$ *is invariant under the adjoint action of* $\mathscr{K}$.

The fact that $\mathrm{Diff}^k(\Lambda)$ is invariant under the adjoint action of $\mathscr{K}$ follows from Proposition 3.7 and the fact that Λ is central. In the event that $k < 0$, we take $\mathrm{Diff}^k(\Lambda) := 0$. Given any subspace V of $\mathrm{Diff}(\Lambda)$, we write V_k for $V \cap \mathrm{Diff}^k(\Lambda)$. Similarly to the fine filtration on $\mathrm{Diff}(\mathbb{R}^{1|1})$, the $\frac{1}{2}\mathbb{N}$-filtration on $\mathrm{Diff}(\Lambda)$ is compatible with the $\mathbb{N}$-filtration on $\mathfrak{U}$:

PROPOSITION 6.8. *For* $r \in \mathbb{N}$, $\pi(\mathfrak{U}_r) \subset \mathrm{Diff}^r(\Lambda)$.

Now we may begin describing the kernel and image of π.

LEMMA 6.9. $\ker(\pi) = I$.

Proof. To show that $\ker(\pi) \subseteq I$, we let $\lambda \in \mathbb{C}$ and $\Omega \in \ker(\pi)$ both be arbitrary. Then $\pi_\lambda(\Omega) = \mathrm{eval}_\lambda\big(\pi(\Omega)\big) = \mathrm{eval}_\lambda(0)$. Since λ was arbitrary, $\Omega \in \bigcap_{\lambda\in\mathbb{C}} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = I$ as desired.

For the other direction of containment, we let $\Omega \in I$ be arbitrary. We must prove that $\pi(\Omega) = 0$. We may write

$$\pi(\Omega) = \sum_{n=0}^{\infty} p_n(x,\xi,\Lambda)\overline{D}^n$$

for some $p_0, p_1, p_2, \ldots \in \mathbb{C}[x,\xi,\Lambda]$. If $p_n = 0$ for every n, then we are done. Toward a contradiction, assume that not all p_n are zero. Let N be the smallest integer for which p_N is not zero. Recall the polynomial $G_{N/2}$ defined in Definition 6.2 which satisfies $D^N(G_{N/2}) = 1$. We also have $\overline{D}^N(G_{N/2}) = +1$ or -1, depending on the value of N. Hence $\pi(\Omega)G_{N/2} = \pm p_N(x,\xi,\Lambda)$. Therefore,

$$\pm p_N(x,\xi,\lambda) = \pi_\lambda(\Omega)G_{N/2} = \mathrm{eval}_\lambda\big(\pi(\Omega)\big)G_{N/2} = 0$$

for every $\lambda \in \mathbb{C}$. In other words, $\mathrm{eval}_\lambda\big(p_N(x,\xi,\Lambda)\big) = 0$ for every $\lambda \in \mathbb{C}$. But the polynomial division algorithm implies that either $\Lambda - \lambda$ divides $p_N(x,\xi,\Lambda)$ for every $\lambda \in \mathbb{C}$, or that $p_N = 0$. In either case, we have a contradiction. □

Corollary 6.10. $J \cap I = 0$.

Proof. From the previous lemma, $I = \ker(\pi)$. So it is enough to show that $\pi|_J$ has kernel zero. Without loss of generality, we may restrict to a fixed weight $n \in \frac{1}{2}\mathbb{Z}$. Let $m \in \frac{1}{2}\mathbb{Z}$. Any non-zero element Ω of J of weight n may be written as

$$\Omega = p_{-n}(e_0)e_{-1/2}^{-2n} + \sum_{m>-n} e_{n+m}p_m(e_0)e_{-1/2}^{2m},$$

where $p_m \in \mathbb{C}[e_0]$ and $p_m = 0$ for $m < 0$. Note that there are only finitely many m such that p_m is not identically zero.

Recall the polynomials F_m from Definition 6.2: $\mathbb{X}(F_m) = e_m$. Since π is an associative algebra homomorphism,

$$\pi(\Omega) = p_{-n}\big(\mathbb{X}(F_0) + \Lambda\big)D^{-2n} + \sum_{m>-n} \big(\mathbb{X}(F_{n+m}) + \Lambda F'_{n+m}\big)p_m\big(\mathbb{X}(F_0) + \Lambda\big)D^{2m}.$$

We must show that $\pi(\Omega)$ is non-zero in $\mathrm{Diff}(\Lambda)$. To do this, we examine its action on $\mathbb{F}_\Lambda$. We will prove that there exists a polynomial G in $\mathbb{F}_\Lambda$ with $\pi(\Omega)G \neq 0$.

Let M be the smallest half-integer such that $p_M \neq 0$. In the notation of Definition 6.2, we consider the polynomial $G_M \in \mathbb{F}_\Lambda$. For each $m \in \frac{1}{2}\mathbb{N}$, $D^{2m}(G_m) = 1$. First, consider the case where $M = -n$. Then $-2n$ is the minimal power of D appearing in the expression $\pi(\Omega)$. We have $\pi(\Omega)G_{-n} = p_{-n}(\Lambda)$, which is non-zero by assumption. On the other hand, we have the case where $M > -n$. We have $\pi(\Omega)G_M = \Lambda F'_{n+M} p_M(\Lambda)$, which is zero if and only if $F'_{n+M} = 0$. That is, $\pi(\Omega)G_M = 0$ if and only if F_{n+M} is a constant. However, this would imply that $M = -n$, violating the assumption that $M > -n$. Thus, $\pi(\Omega)G_M$ is not zero. $\square$

So far, we have described the kernel of π. We now seek to describe its image: it is the direct sum of two distinguished subspaces of $\mathrm{Diff}^r(\Lambda)$. We now define one of these subspaces. Recall that Proposition 3.1 (6) yields $D = -\overline{D}\epsilon$.

Definition 6.11. For a positive integer r, we set

$$\Delta_r^0 := \mathrm{span}_{\mathbb{C}}\big\{\big(\mathbb{X}(F_n) + \Lambda F'_n\big)\overline{D}^{2i} : i \in \mathbb{N},\ 0 \leq i \leq r-1,\ n \in \tfrac{1}{2}\mathbb{N} - 1\big\}.$$

Then we define $\Delta_1^1 := 0$, and for $r \geq 2$

$$\Delta_r^1 := \mathrm{span}_{\mathbb{C}}\big\{\big(\mathbb{X}(F_n) + \Lambda F'_n\big)\overline{D}^{2i+1}\epsilon : i \in \mathbb{N},\ 0 \leq i \leq r-2,\ n \in \tfrac{1}{2}\mathbb{N}\big\}.$$

We put $\Delta_r := \Delta_r^0 \oplus \Delta_r^1$ and finally $\Delta := \bigcup_{r \geq 1} \Delta_r$.

The next lemma checks that the sum in the definition of Δ_r is indeed a direct sum. First let us remark that $\Delta_r = \Delta \cap \mathrm{Diff}^r(\Lambda)$. In other words, the indices used in the definitions of Δ_r^0 and Δ_r^1 are compatible with the filtration on $\mathrm{Diff}(\Lambda)$.

Lemma 6.12. $\Delta_r^0 \cap \Delta_r^1 = 0$.

Proof. We induct on r; the claim is clear for $r = 1$. So assume that $\Delta_{r-1}^0 \cap \Delta_{r-1}^1 = 0$. This allows us to work modulo $\mathrm{Diff}^{r-1}(\Lambda)$. We may also restrict to a fixed weight $n \in \frac{1}{2}\mathbb{Z}$, as elements of different weights are independent. The minimal weight of Δ_r^0 is $-r$, and the minimal weight of Δ_r^1 is $-r+\frac{3}{2}$, so it is sufficient to verify the claim for $n \geq -r+\frac{3}{2}$.

We will prove that for each $n \geq -r+\frac{3}{2}$, both $\Delta_r^0/\Delta_{r-1}^0$ and $\Delta_r^1/\Delta_{r-1}^1$ have up to scalar a unique element of weight n, and that these elements are not equivalent modulo $\mathrm{Diff}^{r-1}(\Lambda)$. We remind the reader that for $m \in \frac{1}{2}\mathbb{Z}$, the polynomials F_m are of weight $m+1$. Hence $\mathbb{X}(F_m)$ is of weight m. Thus, it is clear that

$$\big(\mathbb{X}(F_{n+r-1})+\Lambda F'_{n+r-1}\big)\overline{D}^{2r-2} \in \Delta_r^0/\Delta_{r-1}^0, \qquad \big(\mathbb{X}(F_{n+r-3/2})+\Lambda F'_{n+r-3/2}\big)\overline{D}^{2r-3}\epsilon \in \Delta_r^1/\Delta_{r-1}^1,$$

are, up to scalars, the only elements of weight n in their respective subspaces. To see that they are not equivalent at the level of symbol, it is enough to prove that for any scalar $\alpha \in \mathbb{C}$ we have

$$\Lambda F'_{n+r-1}\overline{D}^{2r-2} - \alpha\Lambda F'_{n+r-3/2}\overline{D}^{2r-3}\epsilon \notin \mathrm{Diff}^{r-1}(\Lambda).$$

The equation $D = -\overline{D}\epsilon$ and the fact that D commutes with $\overline{D}$ allows us to rewrite the above expression as $\Lambda(F'_{n+r-1}\overline{D}^2 + \alpha F'_{n+r-3/2}D)\overline{D}^{2r-4}$. Additionally, we have from Proposition 3.1 (6) that $D = \overline{D} - 2\xi\overline{D}^2$. So we obtain

$$\Lambda(F'_{n+r-1}\overline{D}^2 + \alpha F'_{n+r-3/2}\overline{D} - \alpha F'_{n+r-3/2}\xi\overline{D}^2)\overline{D}^{2r-4}.$$

This expression has a Λ-degree of 1 and a $\overline{D}$-degree of at least $2r-3$, regardless of the value of α. So it is not contained in $\mathrm{Diff}^{r-1}(\Lambda)$, which completes the proof. ☐

As previously stated, the space Δ_r is one of the subspaces that comprise $\pi(\mathfrak{U}_r^+)$. It will be shown in Lemma 6.16 that the subspace $p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$ is complementary to Δ_r in $\pi(\mathfrak{U}_r^+)$. However, we must first make additional definitions and gather results. For $n \in \frac{1}{2}\mathbb{N}$, recall the elements Z_n and Y_n of $\mathfrak{U}$ as given in Definition 5.7.

Definition 6.13. Let $n \in \frac{1}{2}\mathbb{N}$. Define subspaces

$$\mathcal{Z} := \mathrm{span}_{\mathbb{C}}\{Z_n : n \in \tfrac{1}{2}\mathbb{N}\}, \qquad \mathcal{Y} := \mathrm{span}_{\mathbb{C}}\{Y_n : n \in \tfrac{1}{2}\mathbb{N}\}.$$

Further define for each $m \in \mathbb{N}$:

$$Z'_m := Z_m e_{-1/2} + 2Z_{m+1/2}e_{-1}, \qquad Z'_{m+1/2} := Z_m e_{-1/2}, \qquad \mathcal{Z}' := \mathrm{span}_{\mathbb{C}}\{Z'_n : n \in \tfrac{1}{2}\mathbb{N}\},$$

$$Y'_m := Y_m e_{-1/2} + 2Y_{m+1/2}e_{-1}, \qquad Y'_{m+1/2} := Y_m e_{-1/2}, \qquad \mathcal{Y}' := \mathrm{span}_{\mathbb{C}}\{Y'_n : n \in \tfrac{1}{2}\mathbb{N}\}.$$

The following lemma can be verified via computation using the fact that $p_3(\Lambda) + \frac{1}{4}p_2(\Lambda) = \Lambda p_2(\Lambda)$ and Proposition 5.8.

Lemma 6.14. *For $n \in \frac{1}{2}\mathbb{N}$, we have*

$$\pi(Z_n) = p_2(\Lambda)F_{n-1}, \qquad \pi(Y_n) = p_3(\Lambda)F_{n-1},$$

$$\pi(Z'_n) = p_2(\Lambda)F_{n-1}\overline{D}, \quad \pi(Y'_n) = p_3(\Lambda)F_{n-1}\overline{D}.$$

Hence

$$\pi(\mathcal{Z} \oplus \mathcal{Y}) = \Lambda p_2(\Lambda)\mathbb{C}[x,\xi] \oplus p_2(\Lambda)\mathbb{C}[x,\xi], \quad \pi(\mathcal{Z}' \oplus \mathcal{Y}') = \Lambda p_2(\Lambda)\mathbb{C}[x,\xi]\overline{D} \oplus p_2(\Lambda)\mathbb{C}[x,\xi]\overline{D}.$$

Definition 6.15. For brevity, we put

$$\mathcal{W} := \mathcal{Z} \oplus \mathcal{Y} \oplus \mathcal{Z}' \oplus \mathcal{Y}', \qquad \mathcal{D} := \mathrm{span}_{\mathbb{C}}\{e_{-1}^i : i \in \mathbb{N}\}.$$

We finally arrive at the description of $\pi(\mathfrak{U}_r^+)$.

Lemma 6.16. *For all $r \geq 1$, $\pi(\mathfrak{U}_r^+) = \Delta_r \oplus p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$.*

Proof. First we will show that the right-hand side (RHS) is contained in the left-hand side (LHS). We use the fact that π is an associative algebra homomorphism to write

$$\pi(e_n e_{-1}^i) = (-1)^i\big(\mathbb{X}(F_n) + \Lambda F'_n\big)\overline{D}^{2i} \tag{3}$$

for $n \in \frac{1}{2}\mathbb{N} - 1$ and $0 \leq i \leq r-1$. Now for $n \in \frac{1}{2}\mathbb{N}$ and $0 \leq j \leq r-2$ we have

$$\pi(e_n e_{-1/2} e_{-1}^j) = (-1)^j\big(\mathbb{X}(F_n) + \Lambda F'_n\big)D\overline{D}^{2j} \tag{4}$$

Using the fact that D commutes with $\overline{D}$ and applying Proposition 3.1 (6) to (4) proves that Δ_r is contained in the LHS.

Now we will prove that $p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$ is contained in the LHS. Recall that $\pi(Q_{\mathfrak{s}}) = p_2(\Lambda)$. Consider the subspace $\mathbb{C}[Q_{\mathfrak{s}}]\mathcal{W}\mathcal{D}$ of $\mathfrak{U}^+$. We will show

$$\pi(\mathbb{C}[Q_{\mathfrak{s}}]\mathcal{W}\mathcal{D} \cap \mathfrak{U}_r^+) = p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$$

by inducting on r. When $r=1$, the claim is trivial since $\mathbb{C}[Q_{\mathfrak{s}}]\mathcal{W}\mathcal{D}\cap\mathfrak{U}_1^+=0$ and we have $\mathrm{Diff}^{-1}(\Lambda)=0$ by definition. On the other hand, when $r=2$ we have

$$\mathbb{C}[Q_{\mathfrak{s}}]\mathcal{W}\mathcal{D}\cap\mathfrak{U}_2^+=\mathscr{Z},\qquad \pi(\mathscr{Z})=p_2(\Lambda)\mathbb{C}[x,\xi]$$

by Lemma 6.14. So we may assume the claim holds for $r-1$. The inductive hypothesis allows us to prove the claim at the level of symbol. That is, it is sufficient to check that

$$\pi(\mathfrak{U}_r/\mathfrak{U}_{r-1})=p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)/p_2(\Lambda)\mathrm{Diff}^{r-3}(\Lambda).$$

Let

$$p_2(\Lambda)\Lambda^j F_{n-1}\overline{D}^i\in p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$$

with $2r-6\leq 2j+i\leq 2r-4$. The table below gives the construction of the element in $\mathfrak{U}_r$ whose image has symbol $p_2(\Lambda)\Lambda^j F_{n-1}\overline{D}^i$ under π. It is left to the reader to apply Lemma 6.14 and find that each element has the desired image.

	i even	i odd
j even	$Q_{\mathfrak{s}}^{\frac{j}{2}}Z_n e_{-1}^{\frac{i}{2}}$	$Q_{\mathfrak{s}}^{\frac{j}{2}}Z'_n e_{-1}^{\lfloor\frac{i}{2}\rfloor}$
j odd	$Q_{\mathfrak{s}}^{\lfloor\frac{j}{2}\rfloor}Y_n e_{-1}^{\frac{i}{2}}$	$Q_{\mathfrak{s}}^{\lfloor\frac{j}{2}\rfloor}Y'_n e_{-1}^{\lfloor\frac{i}{2}\rfloor}$

This completes the proof of the fact that $\Delta_r\oplus p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)\subset\pi(\mathfrak{U}_r^+)$.

Now, we will argue that the LHS is contained in the RHS via induction on r. When $r=1$, the claim is clear, as $\mathfrak{U}_1^+=\mathscr{K}$ and $\pi(\mathscr{K})=\Delta_1^0$. Assume the claim holds for $r-1$. We must prove that $\pi(e_{i_1}\cdots e_{i_r})$ is in $\Delta_r\oplus p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$ for arbitrary $i_1,\ldots,i_r\in\frac{1}{2}\mathbb{N}-1$. For convenience, put $i:=i_1+\cdots+i_r$. Consider $\mathrm{Diff}^r(\Lambda)$ modulo $\pi(\mathfrak{U}_{r-1}^+)$. Given elements Ω and Θ of $\mathrm{Diff}^r(\Lambda)$, write $\Omega\equiv\Theta$ whenever $\Omega-\Theta$ is in $\pi(\mathfrak{U}_{r-1}^+)$. For natural numbers n and m, lengthy but straightforward calculations yield the following:

$$\pi\Big(e_ne_m-e_{n+m+1}e_{-1}-\frac{(n+1)(m+1)}{2(n+m+1)}e_{n+m+1/2}e_{-1/2}-\frac{(m+1)(n+2m+1)}{2(n+m+1)}e_{n+m}\Big)$$
$$=(n+1)(m+1)p_2(\Lambda)x^{n+m},$$
$$\pi\Big(e_ne_{m+1/2}-\frac{m+1}{n+m+2}e_{n+m+3/2}e_{-1}-\frac{n+1}{n+m+2}e_{n+m+1}e_{-1/2}-(m+1)e_{n+m+1/2}\Big)$$
$$=2(n+1)(m+1)p_2(\Lambda)x^{n+m}\xi,$$

$$\pi\Big(e_{n+1/2}e_{m+1/2} - \frac{n-m}{n+m+2}e_{n+m+3/2}e_{-1/2} - \frac{2(m+1)}{n+m+2}e_{n+m+1}\Big) = 0.$$

Repeated applications of these facts show that for some $\alpha, \beta \in \mathbb{C}$ and some $\Omega \in \mathrm{Diff}^{r-2}(\Lambda)$, one has

$$\pi(e_{i_1} \cdots e_{i_r}) \equiv p_2(\Lambda)\Omega + \pi\left(\alpha e_{i+r-1}e_{-1}^{r-1} + \beta e_{i+r-3/2}e_{-1}^{r-2}e_{-1/2}\right).$$

It is clear from (3) and (4) that the RHS of this equivalence is in $\Delta_r \oplus p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$, which completes the proof. □

Corollary 6.17. *For each $r \in \mathbb{N}$, $\pi(\mathfrak{U}_r) = \pi(\mathfrak{U})_r$. Furthermore, we have*

$$\pi(\mathfrak{U}) = \mathbb{C}1 \oplus \big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \oplus \Delta.$$

Now that we have described both the image and kernel of π, we aim to show that $\pi(J) = \pi(\mathfrak{U})$: the proof of this fact will be Proposition 6.20. It is known from Lemma 6.10 that $\pi|_J$ is an injection, so we must show it is a surjection. To do this, we will prove in the following lemma that there is a correspondence between the weight spaces of $\pi(J)$ and $\pi(\mathfrak{U})$.

Lemma 6.18. *Let $n \in \mathbb{Z}$, and let r be a positive integer. Then*

$$\dim\big((J_r/J_{r-1})_{n/2}\big) = \dim\big((\pi(\mathfrak{U}_r)/\pi(\mathfrak{U}_{r-1}))_{n/2}\big) = \begin{cases} 0 & \text{if } \frac{n}{2} < -r \\ 1 & \text{if } -r \le \frac{n}{2} \le -r+1 \\ 2r-1+n & \text{if } -r+1 < \frac{n}{2} < 0 \\ 2r-1 & \text{if } 0 \le \frac{n}{2} \end{cases}$$

Proof. We will address each case individually and count the number of elements that comprise a basis for each space. The first case is clear: in both spaces, the minimal weight for the r^{th} filtration is $-r$.

For the second case, we have the following table:

weight	element in $(J_r/J_{r-1})_{n/2}$	element in $\big(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U})_{r-1}\big)_{n/2}$
$-r$	e_{-1}^r	$\overline{D}^{2r}$
$-r+\frac{1}{2}$	$e_{-1}^{r-1}e_{-1/2}$	$\overline{D}^{2r-1}\epsilon$
$-r+1$	$e_0e_{-1}^{r-1}$	$(-1)^{r-1}\big(\mathbb{X}(x)+\Lambda\big)\overline{D}^{2r-2}$

An examination of the bases for $\mathfrak{U}$ and Diff(Λ) quickly reveals there can be no other elements with these weights.

The fourth case will be completed before the third case, as it will then be used in the third case. So suppose that $n \geq 0$. When $r = 1$, the claim is clear, as $\mathscr{K}$ has 1 basis element of each nonnegative half-integral weight. For simplicity, we will handle the subcases where n is even or odd separately, with even first.

Write $n = 2m$ for some $m \in \mathbb{N}$, so that the weight equals m. For $r \geq 2$, Lemma 6.16 states that $\big(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U}_{r-1})\big)_{n/2}$ is spanned by elements of the following form:

$$p_2(\Lambda)\Lambda^{r-i-2}x^{m+i}\overline{D}^{2i} \text{ for } 0 \leq i \leq r-2, \tag{5}$$

$$p_2(\Lambda)\Lambda^{r-j-2}x^{m+j}\xi\overline{D}^{2j+1} \text{ for } 0 \leq j \leq r-3, \tag{6}$$

$$\big(\mathbb{X}(x^{m+r}) + (m+r)\Lambda x^{m+r-1}\big)\overline{D}^{2r-2}, \tag{7}$$

$$\big(\mathbb{X}(x^{m+r-1}\xi) + (m+r-1)\Lambda x^{m+r-2}\xi\big)\overline{D}^{2r-3}\epsilon. \tag{8}$$

The first two forms correspond to the symbol of the $p_2(\Lambda)\mathrm{Diff}^{r-2}(\Lambda)$-component of the image, and the last two elements correspond to the symbol of the Δ_r-component of the image. In particular, we note that (8) is a basis element of Δ_r, since $r \geq 2$ and $m \geq 0$ imply that $m+r-1 \geq 1$. We see that in total there are $2r-1$ elements. Now $(J_r/J_{r-1})_{n/2}$ is also $2r-1$ dimensional. Recall that $e_{-1/2}^2 = e_{-1}$. Thus $(J_r/J_{r-1})_{n/2}$ has a basis consisting of cosets of the form $e_{m+i}e_0^{r-i-1}e_{-1}^i$ with $0 \leq i \leq r-1$, and $e_{m+j+1/2}e_0^{r-j-2}e_{-1}^j e_{-1/2}$ with $0 \leq j \leq r-2$.

We may now proceed with the odd case. Write $n = 2m+1$ for $m \in \mathbb{N}$, so that the weight is $m + \frac{1}{2}$. For $r \geq 2$, Lemma 6.16 states that $\big(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U}_{r-1})\big)_{n/2}$ is spanned by elements of the form

$$p_2(\Lambda)\Lambda^{r-i-2}x^{m+i}\xi\overline{D}^{2i} \text{ for } 0 \leq i \leq r-2, \tag{9}$$

$$p_2(\Lambda)\Lambda^{r-j-2}x^{m+j+1}\overline{D}^{2j+1} \text{ for } 0 \leq j \leq r-3, \tag{10}$$

$$\big(\mathbb{X}(x^{m+r}\xi) + (m+r)\Lambda x^{m+r-1}\xi\big)\overline{D}^{2r-2}, \tag{11}$$

$$\big(\mathbb{X}(x^{m+r}) + (m+r)\Lambda x^{m+r-1}\big)\overline{D}^{2r-3}\epsilon. \tag{12}$$

Again, there are a total of $2r-1$ elements. To complete this sub-case, note that $(J_r/J_{r-1})_{n/2}$

has a basis of elements $e_{m+i+1/2}e_0^{r-i-1}e_{-1}^i$ with $0 \leq i \leq r-1$, and $e_{m+j+1}e_0^{r-j-2}e_{-1}^j e_{-1/2}$ with $0 \leq j \leq r-2$.

The third case is similar to the fourth, except as the weight decreases the size of the basis is reduced accordingly. Assume that $-r+1 < \frac{n}{2} < 0$. We will again treat the case of $n = 2m$ first. Here, the weight is m. Then (5) and (6) in the first list of the fourth case are only permissible elements when $i, j \geq -m$, while (7) and (8) are always valid expressions since $-r+1 < m$ implies that $m+r$ and $m+r-1$ are both positive. So in sum, $-2m$ elements of the above list are impermissible. Hence there are $2r-1+n$ elements in the basis for $\left(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U})_{r-1}\right)_{n/2}$. Now we consider $(J_r/J_{r-1})_{n/2}$. This weight space is spanned by elements of the form $e_{m+i}e_0^{r-i-1}e_{-1}^i$ for $-m \leq i \leq r-1$, and $e_{m+j+1/2}e_0^{r-j-2}e_{-1}^j e_{-1/2}$ for $-m \leq j \leq r-2$. As in the third case, there are $2r-1+n$ total basis elements. To prove the case where $n = 2m+1$, we consult (9 – 12) in the fourth case. Again, notice that we necessarily have $i, j \geq -m$ in (9) and (10). Furthermore, (11) and (12) are still permissible basis elements of Δ_r, since $1 < m+r$ by hypothesis. Thus, the dimension of $\left(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U})_{r-1}\right)_{n/2}$ is again $2r-1+n$. Finally, we have a basis of $(J_r/J_{r-1})_{n/2}$ consisting of elements of the form $e_{m+i}e_0^{r-i-1}e_{-1}^i$ with $-m \leq i \leq r-1$, and $e_{m+j+1/2}e_0^{r-j-2}e_{-1}^j e_{-1/2}$ with $-m \leq j \leq r-2$. There are $2r-1+n$ of these, as desired. □

Lemma 6.19. $\mathrm{Diff}(\Lambda)^{e_{-1/2}} = \mathbb{C}[\Lambda, \overline{D}]$.

Proof. This is an immediate consequence of Lemma 3.12 and the fact that Λ is central in $\mathrm{Diff}(\Lambda)$. □

Proposition 6.20. *$\pi : J_r \to \pi(\mathfrak{U}_r)$ is bijective for all r.*

Proof. We will induct on r. The claim is true when $r = 1$, as $J_1 = \mathfrak{U}_1$. Assume it holds for $r-1$. By induction, it is sufficient to show that π is a bijection between J_r/J_{r-1} and $\pi(\mathfrak{U}_r)/\pi(\mathfrak{U}_{r-1})$. Recall that π preserves weights. Therefore, we may restrict to a fixed weight $n/2$. Lemma 6.9 and Corollary 6.10 give $J \cap \ker(\pi) = 0$, so π must be an injection. Lemma 6.18 shows that $(J_r/J_{r-1})_{n/2}$ and $\left(\pi(\mathfrak{U}_r)/\pi(\mathfrak{U}_{r-1})\right)_{n/2}$ are both finite dimensional with equal dimensions. As the kernel is trivial, π must be a surjection and hence a bijection. □

Corollary 6.21. $\mathfrak{U} = I \oplus J$.

Proof. Apply Lemma 6.9, Corollary 6.10 and Proposition 6.20. $\square$

This concludes the first phase of showing that $I = \langle T^{e_2} \rangle_{\mathscr{K}}$. Next, we prove that I is generated by its quadratic part. That is, $I = \langle I_2 \rangle_{\mathscr{K}}$. To this end, we define the following subspace of $\mathfrak{U}$:

Definition 6.22. Set $B^2 := \operatorname{span}_{\mathbb{C}}\{e_n e_{m-1/2} : n, m \in \mathbb{Z}^+\}$.

Lemma 6.23. *$\mathfrak{U}_2 = B^2 \oplus J_2$ and $B^2 \equiv I_2$ mod J_2.*

Proof. The first equation is clear from the standard basis for $\mathfrak{U}$. For the second equation, apply Proposition 6.20. $\square$

Given any subspaces V, W of $\mathfrak{U}$, we write VW for the set $\{vw : v \in V, w \in W\}$. Consider the left ideal $\mathfrak{U}B^2$ of $\mathfrak{U}$.

Lemma 6.24. *For all $r \geq 2$, $\mathfrak{U}_r = \mathfrak{U}_{r-2}B^2 \oplus J_r$.*

Proof. The claim is true for $r = 2$ by Lemma 6.23, so we proceed with an induction on r. Assume that $\mathfrak{U}_{r-1} = \mathfrak{U}_{r-3}B^2 \oplus J_{r-1}$. Evidently, $\mathfrak{U}_{r-2}B^2 \cap J_r = 0$, so we will indeed have a direct sum. We only need to prove that $\mathfrak{U}_r \subseteq \mathfrak{U}_{r-2}B^2 \oplus J_r$. The inductive hypothesis allows us to work modulo $\mathfrak{U}_{r-1}$ and restrict our attention to homogeneous degree r elements.

Let $e_{i_1} \cdots e_{i_r} \in \mathfrak{U}_r$ with $i_1, \ldots, i_r \in \frac{1}{2}\mathbb{N} - 1$. Suppose there exist integers t and s with $1 \leq s \leq t \leq r$ such that i_t and i_s are both positive. For a pure tensor product $v \otimes w$, let $\widehat{v} \otimes w := w$. That is, a hat indicates that term has been removed. Then for some $\varepsilon \in \{0, 1\}$,

$$e_{i_1} \cdots e_{i_r} \equiv (-1)^{\varepsilon} e_{i_1} \cdots \widehat{e_{i_s}} \cdots \widehat{e_{i_t}} \cdots e_{i_r} e_{i_s} e_{i_t} \mod \mathfrak{U}_{r-1}.$$

Consequently, $e_{i_1} \cdots e_{i_r} \in \mathfrak{U}_{r-2}B^2$.

On the other hand, if there is exactly one index t with $i_t \geq \frac{1}{2}$, then we necessarily have $i_s \leq 0$ for all $1 \leq s \leq r$ with $s \neq t$. In other words, for some $\varepsilon \in \{0, 1\}$ and some integers α, β we have

$$e_{i_1} \cdots e_{i_r} \equiv (-1)^{\varepsilon} e_{i_t} e_0^{\alpha} e_{-1/2}^{\beta} \mod \mathfrak{U}_{r-1}.$$

If there are no indices t for which $i_t \geq \frac{1}{2}$, then

$$e_{i_1} \cdots e_{i_r} \equiv (-1)^{\varepsilon} e_0^{\alpha} e_{-1/2}^{\beta} \mod \mathfrak{U}_{r-1}$$

again for some integers α and β and some $\varepsilon \in \{0,1\}$. In either case, the representative on the right hand side of the equivalence is in J_r. □

PROPOSITION 6.25. $I = \langle I_2 \rangle_{\mathscr{K}}$.

PROOF. Throughout this proof, we write $\langle I_2 \rangle$ for $\langle I_2 \rangle_{\mathscr{K}}$. We will prove that $\mathfrak{U} = \langle I_2 \rangle + J$. It is enough to show that $\mathfrak{U}_r = \langle I_2 \rangle_r + J_r$ for every positive integer r. The claim is clear for $r = 1$, and Corollary 6.21 proves the claim for $r = 2$, as $\langle I_2 \rangle_2 = I_2$. So we may proceed with the induction. Let $\Omega \in \mathfrak{U}_r$ be arbitrary. By Lemma 6.24, there exists $\Theta \in \mathfrak{U}_{r-2}$, $b \in B^2$ and $X \in J_r$ for which $\Omega = \Theta b + X$. Also by Lemma 6.24, there exists a $v \in I_2$ such that $b - v \in J_2$. So then

$$\Omega = \Theta b + X = \Theta b + X - \Theta v + \Theta v = \Theta(b - v) + X + \Theta v.$$

Now we apply the inductive hypothesis to write $\Theta = w + Y$, where $w \in \langle I_2 \rangle_{r-2}$ and $Y \in J_{r-2}$. Thus

$$\Omega = \big(w(b-v) + \Theta v\big) + \big(Y(b-v) + X\big) \in \langle I_2 \rangle_r + J_r,$$

whence $Y(b - v) \in J_r$. This completes the proof of $\mathfrak{U} = \langle I_2 \rangle + J$. Then Corollary 6.21 and the fact that $\langle I_2 \rangle \subseteq I$ yield $I = \langle I_2 \rangle$. □

This concludes phase two of proving that $I = \langle T^{e_2} \rangle_{\mathscr{K}}$. The following lemma is the third and final phase.

LEMMA 6.26. $I_2 \subset \langle T^{e_2} \rangle_{\mathscr{K}}$.

PROOF. As in the previous proof, we write $\langle T^{e_2} \rangle$ for $\langle T^{e_2} \rangle_{\mathscr{K}}$. We will begin by showing that T^{e_2} generates all LWVs of positive weight in $\mathfrak{U}_2$. To this end, define the following elements of $\mathfrak{U}_2$:

$$S_{3/2} := (e_0 - 1)e_{3/2} - e_1 e_{1/2}, \quad S_2 := (4e_0 - 2)e_2 - 3e_1^2 - 3e_{1/2}e_{3/2}.$$

They are elements of the so-called *step algebra*, and their images under ad are called *step operators*. For reference, see [10]. One checks that $e_{-1/2}S_{3/2}$ and $e_{-1/2}S_2$ are in $\mathfrak{U}e_{-1/2}$, which implies that the ad-actions of $S_{3/2}$ and S_2 preserve LWVs. That is, if v is a LWV of weight λ, then $\mathrm{ad}(S_{3/2})(v)$ is either zero or a LWV of weight $\lambda + \frac{3}{2}$. Similarly, $\mathrm{ad}(S_2)(v)$ is either zero or a LWV of weight $\lambda + 2$. Recall that R is the unique LWV of weight 2 in $\mathfrak{U}_2$ up to scalar. Explicitly, we have

$$R := e_{-1}e_3 - e_{-1/2}e_{5/2} - 4e_0e_2 + 3e_{1/2}e_{3/2} + 3e_1^2.$$

As previously mentioned, T^{e_2} is not a LWV: one may write $T^{e_2} = \frac{2}{3}(R - Q_{\mathfrak{s}}^{e_2})$, and so $\mathrm{ad}(e_{-1/2})T^{e_2} = -Q_{\mathfrak{s}}^{e_{3/2}} \neq 0$. However, $Q_{\mathfrak{s}}^{e_{3/2}}$ is a LWV, and $\mathrm{ad}(e_{1/2})Q_{\mathfrak{s}}^{e_{3/2}} = 2Q_{\mathfrak{s}}^{e_2}$. Consequently, $\langle T^{e_2}\rangle_{\mathscr{K}}$ contains both the LWV of weight 3/2 and the LWV of weight 2. Our next goal is to prove that all quadratic LWVs of positive weight are in the image of $Q_{\mathfrak{s}}^{e_{3/2}}$ and R under repeated applications of $\mathrm{ad}(S_{3/2})$ and $\mathrm{ad}(S_2)$.

We only need to track the coefficient of $e_{-1}e_{n/2}$ to obtain conditions for when $S_{3/2}$ and S_2 kill the LWVs. Since $Q_{\mathfrak{s}}^{e_{3/2}}$ and R involve terms $e_{-1}e_{5/2}$ and $e_{-1}e_3$ respectively, it is sufficient to do this for $n \geq 5$. One finds that $\mathrm{ad}(S_{3/2})$ annihilates all LWVs of half-integral weight and does not annihilate any LWVs of integral weight. On the other hand, the coefficient of the $e_{-1}e_{2+n/2}$ term in $\mathrm{ad}(S_2)(e_{-1}e_{n/2})$ is zero for $n = -1, 2, 10, 11$. The values $n = -1, 2, 11$ are inconsequential: -1 and 2 are less than 5 and Corollary 5.3 shows there is no LWV in $\mathfrak{U}_2$ of weight 9/2. However, $n = 10$ is relevant. We conclude that $\mathrm{ad}(S_2)$ annihilates the LWV of weight 4 in $\mathfrak{U}_2$. Again, there is no LWV of weight 9/2 in $\mathfrak{U}_2$, so it is impossible to reach the LWV of weight 6 with $S_{3/2}$ as well. In light of this, we define one more element of the step algebra:

$$S_{5/2} := (2e_0 - 3)\big((2e_0 - 1)e_{5/2} - 2e_{1/2}e_2 - 3e_1e_{3/2}\big) - 6e_1^2e_{1/2}.$$

A calculation shows that the coefficient of the $e_{-1}e_7$ term in $\mathrm{ad}(S_{5/2})(e_{-1}e_{9/2})$ is -146. Since this is not zero, $\mathrm{ad}(S_{5/2})$ carries the LWV of weight 7/2 to the LWV of weight 6.

Hence $\langle T^{e_2}\rangle$ contains every quadratic LWV of positive weight in $\mathfrak{U}_2$. Moreover, since $\mathrm{ad}(e_{1/2})\langle T^{e_2}\rangle_2 \subset \langle T^{e_2}\rangle_2$, it follows that $\langle T^{e_2}\rangle_2$ contains a subspace U that is $\mathfrak{s}$-isomorphic to

$\bigoplus_{j=0}^{\infty} \mathbb{F}_{2+2j} \oplus \mathbb{F}_{3/2+2j}$. From Corollary 5.5, we also have $I_2 \overset{\mathfrak{s}}{\cong} \bigoplus_{j=0}^{\infty} \mathbb{F}_{2+2j} \oplus \mathbb{F}_{3/2+2j}$. But by Corollary 5.3, there is exactly one LWV of weight $2 + 2j$ and exactly one LWV of weight $3/2 + 2j$ for each $j \geq 0$. So $I_2^{e_{-1/2}} = \langle T^{e_2} \rangle_2^{e_{-1/2}}$. As their lowest weight spaces are equal, they generate equal spaces under repeated applications of $\mathrm{ad}(e_{1/2})$. Therefore, $I_2 \subseteq \langle T^{e_2} \rangle_{\mathscr{K}}$. $\square$

CHAPTER 7

PROOFS OF THEOREMS 4.2, 4.3, 4.4, AND 4.6

In this chapter, we prove the remaining main results. Theorems 4.2, 4.3, and 4.4 are consequences of Theorem 4.5.

LEMMA 7.1. *For all* $\lambda \in \mathbb{C}$, $\ker(\mathrm{eval}_\lambda) = \langle \Lambda - \lambda \rangle_{\mathrm{Diff}(\Lambda)}$.

PROOF. It is clear that $\langle \Lambda - \lambda \rangle_{\mathrm{Diff}(\Lambda)} \subseteq \ker(\mathrm{eval}_\lambda)$, so we will prove the other direction of containment. Let $\Omega \in \ker(\mathrm{eval}_\lambda)$. In light of the filtration on Diff(Λ), we have

$$\Omega = \sum_{i \in \mathbb{N}} F_i(x, \xi, \Lambda) \overline{D}^i$$

for some $F_0, F_1, F_2, \ldots \in \mathbb{C}[x, \xi, \Lambda]$. Since $\Omega \in \ker(\mathrm{eval}_\lambda)$, we have $\mathrm{eval}_\lambda(F_i) = 0$ for each $i \in \mathbb{N}$. Now the polynomial division algorithm implies that $\Lambda - \lambda$ divides F_i for every $i \in \mathbb{N}$, which finishes the proof. □

LEMMA 7.2. $\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big) = \ker(\mathrm{eval}_\lambda|_{\pi(\mathfrak{U})}) = \langle \Lambda - \lambda \rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U})$

PROOF. As stated in Chapter 6, eval_λ intertwines π and π_λ. We have $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = \ker(\pi_\lambda)$. The previous lemma gives us $\ker(\mathrm{eval}_\lambda) = \langle \Lambda - \lambda \rangle_{\mathrm{Diff}(\Lambda)}$. Therefore,

$$\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big) = \pi\big(\ker(\mathrm{eval}_\lambda \circ \pi)\big) = \ker(\mathrm{eval}_\lambda|_{\pi(\mathfrak{U})}) = \ker(\mathrm{eval}_\lambda) \cap \pi(\mathfrak{U}) = \langle \Lambda - \lambda \rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U})$$

as desired. □

LEMMA 7.3. *Let H be any two-sided ideal in $\mathfrak{U}$. Then $H = \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ if and only if $I \subset H$ and $\pi(H) = \pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big)$.*

PROOF. The forward direction of implication is obvious, so we will assume that H is some two-sided ideal in $\mathfrak{U}$ satisfying $I \subset H$ and $\pi(H) = \pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big)$. We prove $H = \mathrm{Ann}(\mathbb{F}_\lambda)$ via double containment. Let $\Omega \in H$ be arbitrary. Recall that Lemma 6.9 yields $I = \ker(\pi)$. Since $\pi(H) = \pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big)$, there exists a $Y \in \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$ such that $\Omega - \Theta \in I \subset \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$. Therefore, $\Omega \in \mathrm{Ann}(\mathbb{F}_\lambda)$ and hence $H \subseteq \mathrm{Ann}(\mathbb{F}_\lambda)$. Now assume that $\Omega \in \mathrm{Ann}(\mathbb{F}_\lambda)$. Since

$\pi(H) = \pi\big(\mathrm{Ann}(\mathbb{F}_\lambda)\big)$, there exists a $\Theta \in H$ such that $\Omega - \Theta \in I \subset H$. So $\Omega \in H$, which completes the proof. $\square$

The following lemma is immediate from the fact that π is an associative algebra homomorphism.

LEMMA 7.4. *Let G be any subset of $\mathfrak{U}$. Then* $\pi\big(\langle G\rangle_{\mathfrak{U}}\big) = \big\langle \pi(G)\big\rangle_{\pi(\mathfrak{U})}$.

Proof of Theorem 4.2

Fix $\lambda \in \mathbb{C}$ with $\lambda \neq 0, 1/4$, or $1/2$. For brevity, we make the assignments

$$Z_0(\lambda) := Z_0 - p_2(\lambda), \qquad T_0(\lambda) := p_1(\lambda)\widehat{T} - p_2(\lambda)T,$$

$$Y_0(\lambda) := Y_0 - p_1(\lambda)Z_0, \qquad T_{-1/2}(\lambda) := \mathrm{ad}(e_{-1/2})T_0(\lambda),$$

$$I(\lambda) := \big\langle\ Z_0(\lambda), Y_0(\lambda), T_0(\lambda)\ \big\rangle_{\mathscr{K}}.$$

We seek to apply Lemma 7.3, so we must show that $I \subset I(\lambda)$ and $\pi\big(I(\lambda)\big) = \pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big)$. From Theorem 4.5, it is sufficient to show $T^{e_2} \in I(\lambda)$ to prove $I \subset I(\lambda)$. To this end, recall the step operators S_2 and $S_{5/2}$, first defined in Lemma 6.26. The reader may verify that

$$18T^{e_2} = 3\,\mathrm{ad}(S_{5/2})T_{-1/2}(\lambda) + 4\,\mathrm{ad}(S_2)Y_0(\lambda) + 12\,\mathrm{ad}(e_2)Z_0(\lambda),$$

and hence $I \subset I(\lambda)$. Next, use Proposition 3.15 to verify

$$\pi\big(Z_0(\lambda)\big) = (\Lambda - \lambda)\big(p_1(\Lambda) + p_1(\lambda)\big),$$

$$\pi\big(Y_0(\lambda)\big) = (\Lambda - \lambda)p_2(\Lambda),$$

$$\pi\big(T_0(\lambda)\big) = (\Lambda - \lambda)p_1(\Lambda)p_1(\lambda)\epsilon.$$

From Lemma 7.2, we have $\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)\big) = \langle \Lambda - \lambda\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U})$. Therefore, the statement of Theorem 4.2 is reduced to proving the following equality:

$$\text{(13)}\quad \langle \Lambda - \lambda\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U}) = \Big\langle (\Lambda-\lambda)\big(p_1(\Lambda)+p_1(\lambda)\big),\ (\Lambda-\lambda)p_2(\Lambda),\ (\Lambda-\lambda)p_1(\Lambda)p_1(\lambda)\epsilon \Big\rangle_{\pi(\mathfrak{U})}.$$

It is clear that the LHS contains the RHS, so we must show the other direction of containment. To this end, we will now give a more convenient form of the LHS.

Corollary 6.17 gives $\pi(\mathfrak{U}) = \mathbb{C}1 \oplus \big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \oplus \Delta$. Additionally, one finds that $\mathrm{Diff}(\Lambda) = \mathbb{C}[x, \xi, \overline{D}] \oplus \langle \Lambda - \lambda\rangle_{\mathrm{Diff}(\Lambda)}$. From these facts, we deduce

$$\pi(\mathfrak{U}) = p_2(\Lambda)\mathbb{C}[x, \xi, \overline{D}] \oplus \big\langle (\Lambda - \lambda)p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \oplus \big(p_2(\Lambda) - p_2(\lambda)\big)(\mathbb{C}1 \oplus \Delta).$$

Since λ is not 0 or $\frac{1}{2}$, we have $\mathrm{eval}_\lambda\big(p_2(\Lambda)\big) \neq 0$. Therefore, eval_λ carries $p_2(\Lambda)\mathbb{C}[x, \xi, \overline{D}]$ bijectively into $\mathrm{Diff}(\mathbb{R}^{1|1})$. The other summands in the above expression for $\pi(\mathfrak{U})$ are killed by evaluation. Thus,

$$\langle \Lambda - \lambda\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U}) = \big\langle (\Lambda - \lambda)p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \oplus \big(p_2(\Lambda) - p_2(\lambda)\big)(\mathbb{C}1 \oplus \Delta). \tag{14}$$

To finish the proof of the theorem, it is sufficient to show that the RHS of (13) contains the RHS of (14). This will be handled one summand at a time.

To see that the RHS of (13) contains $\big(p_2(\Lambda) - p_2(\lambda)\big)(\mathbb{C}1 \oplus \Delta)$, note that

$$(\Lambda - \lambda)\big(p_1(\Lambda) + p_1(\lambda)\big) = p_2(\Lambda) - p_2(\lambda)$$

and $\mathbb{C}1 \oplus \Delta$ is contained in $\pi(\mathfrak{U})$. For the first summand, we note that Corollary 6.17 implies that the RHS of (13) contains both $\big(p_2(\Lambda) - p_2(\lambda)\big)p_2(\Lambda)\mathrm{Diff}(\Lambda)$ and $(\Lambda - \lambda)p_2^2(\Lambda)\mathrm{Diff}(\Lambda)$. Then, we again rely on the fact that $p_2(\lambda)$ is not zero to write

$$\tfrac{1}{p_2(\lambda)}\Big((\Lambda - \lambda)p_2^2(\Lambda) - (\Lambda - \lambda)\big(p_2(\Lambda) - p_2(\lambda)\big)p_2(\Lambda)\Big) = (\Lambda - \lambda)p_2(\Lambda).$$

In other words, $\big\langle (\Lambda - \lambda)p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)}$ is contained in the ideal generated by the images of $Z_0(\lambda)$ and $Y_0(\lambda)$, which completes the proof. □

Proof of Theorem 4.3

Recall from Proposition 5.6 that $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0) = \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/2})$. It is straightforward to check that

$$\mathrm{ad}(S_{3/2} - e_1e_{1/2})Z_{1/2} = -8T^{e_2}.$$

Therefore, $Z_{1/2}$ generates T^{e_2} and hence $I \subset \langle Z_{1/2}\rangle_{\mathscr{K}}$ by Theorem 4.5. We aim to prove that $\pi\big(\langle Z_{1/2}\rangle_{\mathscr{K}}\big) = \pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)\big)$ so that we may apply Lemma 7.3.

Now, Lemma 5.8 yields $\pi(Z_{1/2}) = p_2(\Lambda)\xi$, and hence Lemma 7.4 implies $\pi\big(\langle Z_{1/2}\rangle_{\mathscr{K}}\big) = \langle p_2(\Lambda)\xi\rangle_{\pi(\mathfrak{U})}$. By Lemma 7.2, $\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_0)\big) = \langle\Lambda\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U})$, and Corollary 6.17 states that

$\pi(\mathfrak{U}) = \mathbb{C}1 \oplus \big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \oplus \Delta$. Note that since Λ divides $p_2(\Lambda)$, we have $\big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} \subset \langle\Lambda\rangle_{\mathrm{Diff}(\Lambda)}$. It follows that $\langle\Lambda\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U}) = \big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)}$. Since Λ is central, we have $\big\langle p_2(\Lambda)\big\rangle_{\mathrm{Diff}(\Lambda)} = p_2(\Lambda)\mathrm{Diff}(\Lambda)$. Moreover, it is not hard to verify that

$$\mathrm{Diff}(\Lambda) = \mathbb{C}[x, \xi, \overline{D}] \oplus \Lambda\mathbb{C}[x, \xi, \overline{D}] \oplus p_2(\Lambda)\mathrm{Diff}(\Lambda).$$

Putting this all together means that the proof of Theorem 4.3 amounts to verifying the following:

$$\big\langle p_2(\Lambda)\xi\big\rangle_{\pi(\mathfrak{U})} = p_2(\Lambda)\big(\mathbb{C}[x, \xi, \overline{D}] \oplus \Lambda\mathbb{C}[x, \xi, \overline{D}] \oplus p_2(\Lambda)\mathrm{Diff}(\Lambda)\big). \tag{15}$$

The LHS is clearly contained in the RHS, so we must prove the other direction of containment. Again, we check containment one summand at a time.

Recall the subspaces $\mathcal{W}$ and $\mathcal{D}$ of $\mathfrak{U}$ from Definition 6.15. Note that $Y_0 \in \langle Z_{1/2}\rangle_{\mathscr{K}}$. Combining this fact with Definition 6.13 and Definition 5.7 yields $\mathcal{W}\mathcal{D} \subset \langle Z_{1/2}\rangle_{\mathscr{K}}$. Using Lemma 6.14, we find

$$\pi(\mathcal{W}\mathcal{D}) = p_2(\Lambda)\mathbb{C}[x, \xi, \overline{D}] \oplus p_2(\Lambda)\Lambda\mathbb{C}[x, \xi, \overline{D}] \subset \big\langle p_2(\Lambda)\xi\big\rangle_{\pi(\mathfrak{U})}.$$

Thus, we are only left to check $p_2^2(\Lambda)\mathrm{Diff}(\Lambda) \subset \big\langle p_2(\Lambda)\xi\big\rangle_{\pi(\mathfrak{U})}$. By Corollary 6.17, we have $p_2(\Lambda)\mathrm{Diff}(\Lambda) \subset \pi(\mathfrak{U})$. The observation $\pi\big(\mathrm{ad}(e_{-1/2})Z_{1/2}\big) = p_2(\Lambda)$ completes the proof. □

Proof of Theorem 4.4

Again, we must apply Lemma 7.3. Obviously, $T^{e_2} \in \langle T\rangle_{\mathscr{K}}$, so Theorem 4.5 gives $I \subset \langle T\rangle_{\mathscr{K}}$. By Lemma 7.4, it is sufficient to prove $\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/4})\big) = \big\langle \pi(T)\big\rangle_{\pi(\mathfrak{U})}$. By Lemma 3.13, we have $\pi(T) = (\Lambda - \frac{1}{4})\epsilon$ where $\epsilon = 1 - 2\xi\partial_\xi \in \mathrm{Diff}(\Lambda)$. Note that since ϵ acts by 1 on even elements and -1 on odd elements, we have $\epsilon^2 = 1$. Now, by Lemma 7.2 we have $\pi\big(\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_{1/4})\big) = \langle\Lambda - \frac{1}{4}\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U})$. Since $\epsilon^2 = 1$, we have $\langle\Lambda - \frac{1}{4}\rangle_{\mathrm{Diff}(\Lambda)} = \big\langle(\Lambda - \frac{1}{4})\epsilon\big\rangle_{\mathrm{Diff}(\Lambda)}$. To summarize, the claim will be verified if we show the following:

$$\big\langle(\Lambda - \tfrac{1}{4})\epsilon\big\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U}) = \big\langle(\Lambda - \tfrac{1}{4})\epsilon\big\rangle_{\pi(\mathfrak{U})}. \tag{16}$$

It is clear that the RHS is contained in the LHS, so we will prove the other direction of containment.

We start by finding a better description of the LHS. Corollary 6.17 states $\pi(\mathfrak{U}) = \mathbb{C}1 \oplus p_2(\Lambda)\mathrm{Diff}(\Lambda) \oplus \Delta$. Use the fact that $\mathrm{Diff}(\Lambda) = \mathbb{C}[x, \xi, \overline{D}] \oplus (\Lambda - \frac{1}{4})\mathrm{Diff}(\Lambda)$ to write

$$\pi(\mathfrak{U}) = p_2(\Lambda)\mathbb{C}[x, \xi, \overline{D}] \oplus (\Lambda - \tfrac{1}{4})p_2(\Lambda)\mathrm{Diff}(\Lambda) \oplus \big(p_2(\Lambda) + \tfrac{1}{16}\big)(\mathbb{C}1 \oplus \Delta).$$

Now again, $p_2(\frac{1}{4}) = -\frac{1}{16} \neq 0$, so $\mathrm{eval}_{1/4}$ carries $p_2(\Lambda)\mathbb{C}[x, \xi, \overline{D}]$ bijectively into $\mathrm{Diff}(\mathbb{R}^{1|1})$. Also, we have $p_2(\Lambda) + \frac{1}{16} = (\Lambda - \frac{1}{4})^2$, and hence

$$\big\langle(\Lambda - \tfrac{1}{4})\big\rangle_{\mathrm{Diff}(\Lambda)} \cap \pi(\mathfrak{U}) = (\Lambda - \tfrac{1}{4})p_2(\Lambda)\mathrm{Diff}(\Lambda) \oplus (\Lambda - \tfrac{1}{4})^2(\mathbb{C}1 \oplus \Delta). \tag{17}$$

We must prove the RHS of (17) is contained in the RHS of (16). By Corollary 6.17, $p_2(\Lambda)\mathrm{Diff}(\Lambda)$ is contained in $\pi(\mathfrak{U})$. Thus, the first summand of the RHS of (17) is contained in $\big\langle(\Lambda - \frac{1}{4})\epsilon\big\rangle_{\pi(\mathfrak{U})}$. For the second summand, note that $(\Lambda - \frac{1}{4})^2 = \big((\Lambda - \frac{1}{4})\epsilon\big)^2$. As $(\Lambda - \frac{1}{4})\epsilon$ is an element of $\pi(\mathfrak{U})$ and $\mathbb{C}1 \oplus \Delta$ is contained in $\pi(\mathfrak{U})$, the proof is complete. □

Now we move on to proving Theorem 4.6. Given any non-empty subset S of $\mathbb{C}$, put $\mathcal{A}(S) := \{\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) : \lambda \in S\}$. Note that Proposition 5.6 yields $\mathcal{A}(\mathbb{C}) = \mathcal{A}(\mathbb{C}^\times)$.

Lemma 7.5. *Points are closed in $\mathcal{A}(\mathbb{C}^\times)$.*

Proof. Let $S \subset \mathrm{Prim}(\mathfrak{U})$. Recall that Definition 2.47 defines the Jacobson topology: put $I(S) := \bigcap_{J \in S} J$. Then the closure of S in the Jacobson topology is

$$\overline{S} := \{J \in \mathrm{Prim}(\mathfrak{U}) : J \supseteq I(S)\}.$$

Thus if $S \subseteq \mathcal{A}(\mathbb{C}^\times)$, the subspace topology inherited by $\mathcal{A}(\mathbb{C}^\times)$ satisfies

$$\overline{S} = \big\{J \in \mathcal{A}(\mathbb{C}^\times) : J \supseteq I(S)\big\}.$$

Fix $\lambda \in \mathbb{C}$ and consider the singleton set $\mathcal{A}(\lambda) := \big\{\mathrm{Ann}(\mathbb{F}_\lambda)\big\} \subset \mathrm{Prim}(\mathfrak{U})$. Then $I\big(\mathcal{A}(\lambda)\big) = \mathcal{A}(\lambda)$, so

$$\overline{\mathcal{A}(\lambda)} = \big\{J \in \mathcal{A}(\mathbb{C}^\times) : J \supseteq \mathcal{A}(\lambda)\big\}.$$

It follows from Theorems 4.2, 4.3 and 4.4 that $\overline{\mathcal{A}(\lambda)} = \mathcal{A}(\lambda)$. □

Corollary 7.6. *If F is any finite subset of $\mathbb{C}^\times$, then $\mathcal{A}(F)$ is closed.*

Proposition 7.7. *Let S be an infinite subset of $\mathbb{C}^{\times}$. Then* $\bigcap_{\lambda \in S} \mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) = I$.

Proof. It is clear that $I \subseteq \cap_{\lambda\in S}\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda)$, so we will prove $\cap_{\lambda\in S}\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) \subseteq I$. For $\Omega \in \cap_{\lambda\in S}\mathrm{Ann}(\mathbb{F}_\lambda)$, $(\Lambda - \lambda)$ divides $\pi(\Omega)$ for every $\lambda \in S$, so $\pi(\Omega) = 0$. Therefore by Lemma 6.9, $\Omega \in I$. □

Proof of Theorem 4.6

In light of Theorems 4.2, 4.3, and 4.4, the map $\mathrm{Ann}_{\mathscr{K}}(\mathbb{F}_\lambda) \mapsto \lambda$ is a bijection between $\mathcal{A}(\mathbb{C}^{\times})$ and $\mathbb{C}^{\times}$. Proposition 7.7 shows that if $\mathbb{C}^{\times}$ is equipped with the co-finite topology, then this map is a homeomorphism. □

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