DEFINABLE STRUCTURES ON THE SPACE OF FUNCTIONS FROM TUPLES OF INTEGERS INTO 2

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We give some background on the free part of the action of tuples of integers into 2. We will construct specific structures on this space, and then show that certain other structures cannot exist.

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CHAPTER 1

INTRODUCTION

Countable Borel equivalence relations occur naturally in different contexts in which an equivalence relation occurs as a result of an action by some group. Some of these contexts include ergodic theory and the theory of operator algebras, but there is a natural connection into descriptive set theory. The theory of definable equivalence relations serves as a base for studying classification and complexity problems, which can often be represented as equivalence relations which are definable subsets of some Polish space; thus, it is natural to try to develop a theory on these objects themselves. The study of countable equivalence relations in a purely descriptive set theoretic context began in the mid 1990's, with the papers [9] and [2] being two of the earliest. Kechris gives an extensive overview of the topic of countable equivalence relations in [12].

A theorem by Feldman and Moore states that any countable Borel equivalence relation E of some Polish space X can be viewed as being induced by some countable group. Thus, studying Borel equivalence relations by the groups that induced them became standard practice. For example, it is now known that the relation induced by any abelian group is hyperfinite. To prove this, sets known as marker sets with certain properties on each equivalence class had to be defined. One of the first arugments using marker sets was given by Slaman and Steel. Marker sets have now been used in various different spaces to construct explicit sets, functions, and graphs that have certain properties on each equivalence class.

Marker sets are an excellent tool for explicitly constructing various sets, but there is no clear way to use them to prove the nonexistence of sets with specific properties. To do this, special points called hyperaperiodic points were constructed for specific spaces. The closure of the orbits of these points are compact, which means any open structure defined on them is subject to incredibly strict, often impossible conditions. In [6], Gao, Jackson, Krohne, and Seward constructed hyper-aperiodic elements via forcing. These new elements when used cleverly can impose strong conditions even on Borel sets, allowing us to prove the nonexistence of Borel structures.

In this paper, we will focus on a specific collection of spaces. We will consider the sets $2^{\mathbb{Z}^n}$, where the action of $\mathbb{Z}^n$ is the *shift action*, $g \cdot x(h) = x(-g+h)$. The equivalence classes of these spaces are the points which are shifts of each other, meaning each equivalence class looks like a copy of $\mathbb{Z}^n$. This means there is some interesting geometry to consider when trying to define structures such as linings or treeings on each class.

The goal of this thesis is to analyze which structures can be defined on the specific space $F(2^{\mathbb{Z}^n})$ and which cannot. We will use marker sets to construct some sets explicitly, but we will also define a new set of marker sets. It is known that there is no sequence of clopen marker sets $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ which have empty intersection. In this paper, we define an open sequence of marker sets of $F(2^{\mathbb{Z}^n})$ that have this property, and discuss the limitations of these sets.

We have also taken results about clopen structures and generalized them to open structures. Occasionally, changing clopen to open will allow some objects to exist that couldn't previously, such as in the case of our marker sets above. Conversely, there are nonexistence results about clopen structures which can be generalized to open. We have developed techniques that allow us to generalize clopen structures with one component to open structures that can have multiple components. An interesting phenomena occurs where changing "exactly" to "at most" can change whether or not a certain type of structure can exist. The following results give such an example.

Theorem 1.1. *For any $n \in \omega$, There is no open treeing T of $F(2^{\mathbb{Z}^2})$ which has exactly n components on each equivalence class.*

Theorem 1.2. *There is an open treeing T of $F(2^{\mathbb{Z}^2})$ which has at most 4 components on each equivalence class.*

The discrepancy between these two theorems is perhaps a little surprising. Forcing a treeing to have exactly the same number of components on each class is too restrictive, but by giving up a little ground and letting the number of components vary, we can prove the existence of the same structure for a fairly low number of components. If we only care that the resulting structure is Borel instead of clopen, we can define structures with stronger properties. We will also provide an alternative proof of the following theorem using a method that may generalize to other spaces.

THEOREM 1.3. *There is a Borel lining L of $F(2^{\mathbb{Z}^2})$ which has exactly n components on each equivalence class.*

CHAPTER 2

BOREL EQUIVALENCE RELATIONS

In this chapter, we will give some background on the building blocks of descriptive set theory, the Borel sets, and build up the basic theory and notation we use in the subject of countable Borel equivalence relations. We will also discuss the notion of hyperfiniteness and pose a few of the major open questions in the subject.

2.1. Borel Sets and Standard Borel Spaces

A *Polish space* is a separable completely-metrizable topological space. The collection of Borel sets of a topological space X is the smallest σ-algebra containing the open sets of X. We say a function $f : X \to Y$ is a *Borel function* if the inverse image of any open subset of Y is Borel in X. Since inverse images are closed under unions, intersections and complements, it's not hard to see that an equivalent definition would be to say that the inverse image of any Borel set of Y is Borel in X. Two spaces X, Y are *Borel isomorphic* if there is a bijection f between them such that f and f^{-1} are Borel functions. A fundamental theorem of the subject of descriptive set theory is the *Borel isomorphism theorem* (A proof for which can be found in [11]).

THEOREM 2.1. (Borel Isomorphism Theorem) *If X, Y are two uncountable Polish spaces, then X and Y are Borel isopmorphic.*

This theorem simplifies studying the Borel structure of Polish spaces since the Borel sets (as a whole) of any two uncountable Polish spaces are essentially the same. Thus, it makes sense to prove theorems for spaces which are structured like Polish spaces. A *standard Borel space* is a set equipped with a σ-algebra which is Borel isomorphic to the σ-algebra of the Borel sets of some Polish space. Many theorems involving Borel sets of Polish spaces are proven for a carefully chosen Polish space, and then extended, or "transferred", to other spaces with help from the above theorem.

2.2. Borel Equivalence Relations

Many classifications problems can be viewed as equivalence relations, and a standard question of the subject is to ask about the complexity of certain objects. A *Borel equivalence relation* E on a Polish space X is an equivalence relation that is a Borel subset of X^2. For each Borel $Y \subseteq X$, we let $E \upharpoonright Y = E \cap Y^2$. E is said to be *countable* if each equivalence class of E is countable, and similarly, E is *finite* if each equivalence class of E is finite. The definitions below give us a few natural ways to try and categorize the complexity of Borel equivalence relations.

Let (X, E), and (Y, F) be two Borel equivalence relations. Then we say that E is *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. Such a function f induces an injection which maps the equivalence classes of E into the equivalence classes of F. The intuition of this definition is that deciding E-equivalence is "simpler" than deciding F-equivalence, i.e., if we can decide F-equivalence, we can, in a definable way, decide E equivalence.

E is *Borel embeddable* into F, denoted $E \sqsubseteq F$, if E is reducible to F by an injective Borel map. An equivalent formulation is that E is embeddable into F iff there is some $Z \subseteq Y$ such that E is Borel isomorphic to $F \upharpoonright Z$. E is Borel invariantly embeddable into F if E is Borel isomorphic to $F \upharpoonright Z$, where $Z \subseteq Y$ is a Borel subset of Y which is invariant under F, i.e. $z \in Z, xFz \Rightarrow y \in Z$.

The simplest Borel equivalence relations are the ones for which there is Borel function that can pick out an element from each equivalence class, i.e., a Borel selector. These equivalence relations are the *smooth* relations, and they are generally too simple to be interesting. A countable Borel equivalence relation E is *hyperfinite* if there is an increasing union of finite equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$ such that $E = \cup_{n\in\omega} F_n$. The study of hyperfinite equivalence relations is an incredibly active area of the subject, and there are many open questions about hyperfiniteness. A fundamental example of a hyperfinite equivalence relation is the eventually equal relation E_0.

EXAMPLE 2.2. (E_0) Let 2^ω denote the set of all functions $f : \omega \to 2$. Then E_0 is the *eventually equal* equivalence relation defined by

$$xE_0y \Leftrightarrow \exists N\, \forall n > N,\ x(n) = y(n).$$

We can view E_0 as the first nonsmooth equivalence relation via the following dichotomy theorem.

THEOREM 2.3. (Harrington-Kechris-Louveau, [14]) *Let E be a Borel equivalence relation on a Polish space X. Then exactly one of the following hold.*

(1) E is smooth

(2) $E_0 \sqsubseteq E$ via a continuous function.

Thus, we can think of hyperfinite equivalence relations as being the simplest nontrivial relations. A common tactic for proving a given equivalence relation is hyperfinite is to reduce it to E_0, but before we explain how these problems are being approached, we must first introduce how groups play into the study of Borel equivalence relations.

Let G be a countable group and X a standard Borel space. A *Borel action* is an action $(g, x) \to g \cdot x$ of G on X satisfying $1 \cdot X = x$, $gh \cdot x = g \cdot (h \cdot x)$, and for each g, the action $g(x) := g \cdot x$ is Borel. Given a Borel action of G on X, we denote by E_G the induced orbit equivalence relation

$$xE_Gy \Leftrightarrow \exists g \in G(y = g \cdot x).$$

A theorem proven by Feldman and Moore in [3] shows that any countable Borel equivalence relation of a Polish space X occurs as the Borel action of some countable group G on X.

THEOREM 2.4 (Feldman-Moore). *If E is a countable Borel equivalence relation on a standard Borel space X, then there is a countable group G and a Borel action of G on X such that $E = E_G$*

By making use of the Feldman-Moore theorem, we are able to assume that an arbitrary countable Borel equivalence relation is given by the action of some countable group. Thus, we can determine characteristics of an equivalence relation based on which group induces its equivalence classes. In [15], Slaman and Steel give an example of a group which induces a non-hyperfinite equivalence relation, so not all groups induce hyperfinite relations. On the other hand, Gao and Jackson proved in [4] that the orbit equivalence relation generated by any countable abelian group is hyperfinite. It is currently an open question if the equivalence relation generated by any amenable group is hyperfinite, and progress has been made on this question for amenable groups with specific conditions. Another big open problem about hyperfinite equivalence relations is the following; if $E = \cup_{n\in\omega} F_n$, where $F_n \subseteq F_{n+1}$, and F_n is hyperfinite, is E hyperfinite?

The following result from [2] gives a few equivalencies for hyperfinite.

THEOREM 2.5. *Let E be a countable Borel equivalence relation. Then the following are equivalent.*

(1) E is hyperfinite.

(2) $E = \cup_{n\in\omega} F_n$, where F_n are finite Borel equivalence relations, $F_n \subseteq F_{n+1}$, and each F_n-equivalence class has cardinality at most n.

(3) $E = \cup_{n\in\omega} F_n$, where F_n are smooth Borel equivalence relations, $F_n \subseteq F_{n+1}$.

(4) $E = E_{\mathbb{Z}}$, i.e. there is a Borel automorphism T of X with $xEy \Leftrightarrow \exists n \in \mathbb{Z}(T^n(x) = y)$;

(5) There is a Borel assignment $C \mapsto <_C$ giving for each E-equivalence class C a linear order $<_C$ of C of order type finite or $\mathbb{Z}$.

We give the proof for (5) $\Rightarrow$ (1) as that particular argument produces one of the earliest instances of a marker set, a concept which has become fundamental to the study of Borel equivalence relations. The S_n^C constructed in the proof are known as *Slaman-Steel* markers. In chapter 3, we will construct more elaborate marker structures.

PROOF. Assume without loss of generality (by the Borel Isomorphism theorem) that $X = 2^\omega$ and each E-equivalence class is inifinite, and hence ordered by $<_C$ in order type $\mathbb{Z}$. For each E-equivalence class C, let x_C be the lexicographically-least element of the closure of C. The map $y \mapsto x_{[y]_E}$ is Borel. Let

$$S_n^C = \{x \in C : x \upharpoonright n = x_c \upharpoonright n\}$$

.

Define now the equivalence relations $E_n \subseteq E$ as follows: if $(x, y) \in E$ with $[x]_E = [y]_E = C$ are such that $x_C \in C$, then let

$$xE_ny :\Leftrightarrow [x = y \vee \text{ the distance of } x_C \text{ from } x, y \text{ in } <_c \text{ is at most } n].$$

If $x_c \notin C$ but there is an m such that S_m^C is bounded below in $<_C$, let m_0 be the least such m and z_C the $<_C$-least element of $S_{m_0}^C$ and let

$$xE_ny :\Leftrightarrow [x - y \vee \text{ the distance of } z_C \text{ from } x, y \text{ in } <_C \text{ is at most } n]$$

If $x_C \notin C$, and S_m^C is not bounded below in C, but is bounded above, we may define E_n similarly to how we did above.

Finally, if $x_C \notin C$, and S_m^C is not bounded above or below in C, then $\{S_n^C\}$ form a decreasing sequence of subsets of C with S_n^C unbounded in both directions in $<_C$ and $\cap_{n\in\omega} S_n^C = \emptyset$, then let

$$xE_ny \Leftrightarrow \exists a \exists b[a, b \text{ are consecutive members of } S_n^C \text{ and } a \leq_c x < b \text{ and } a \leq_c y <_c b].$$

It is clear that the relations E_n are increasing finite Borel equivalence relations with $\cup_{n\in\omega} E_n = E$, so E is hyperfinite. $\square$

If X is a standard Borel space and G is a countable group, we denote by X^G the set of maps from G into X with the standard product Borel structure. Then there is a natural action of G on X^G, namely, $g \cdot x(h) = x(g^{-1}h)$ for $x \in X^G$ and $g, h \in G$. We then denote $E(G, X)$ as the corresponding equivalence relation, and for any x, we let $[x]$ denote the orbit of x

$$[x] = \{y \in X^G : \exists g \in G \, y = g \cdot x\}$$

An interesting fact, proven in [2], is that there is a universal countable Borel equivalence relation $E(F_2, 2)$. A *universal* equivalence relation for a class $\mathcal{C}$ of equivalence relations is some relation $F \in \mathcal{C}$ such that for any $E \in \mathcal{C}$, E is Borel-reducible to F. Thus, we may think of a universal equivalence relation for a class as the most complicated one in that class. For the following proposition, we recall that F_2 is the free group with 2 generators.

PROPOSITION 2.6. *Let E be a countable Borel equivalence relation. Then $E \sqsubseteq E(F_2, 2)$.*

We have discussed three different "levels" of countable Borel equivalence relations: The smooth ones, which are the simplest, the hyperfinite ones, which are the next step up, and a universal equivalence relation. These are far from the only ones; in fact, there is a massive collection of equivalence relations having the property that $E_0 <_B E <_B E(F_2, 2)$.

THEOREM 2.7. (Adams-Kechris, [1]) *There exist uncountably many countable Borel equivalence relations up to Borel bireducibility.*

For this paper, we will specifically work with the groups $G = \mathbb{Z}^n$, in which case the equivalence relations we are interested in are $E(2^{\mathbb{Z}^n}, \mathbb{Z}^n)$. We establish some terminology that will be used throughout the remainder of the paper. When n is fixed we let $e_1, e_2, \cdots, e_n$ be the standard *generators* of $\mathbb{Z}^n$.

$$e_1 = (1, 0, \ldots, 0),$$

$$e_2 = (0, 1, \ldots, 0),$$

$$\ldots$$

$$e_n = (0, 0, \ldots, 1)$$

.

Any $g \in \mathbb{Z}^n$ can be uniquely expressed as $(g_1, g_2, \ldots, g_n)$ or $g_1e_1 + g_2e_2 + \cdots + g_ne_n$ for $g_1, g_2, \ldots, g_n \in \mathbb{Z}$. We define the l_∞ norm on $\mathbb{Z}^n$ by

$$||g|| = ||(g_1, g_2, \ldots, g_n)|| = \max\{|g_1|, |g_2|, \ldots, |g_n|\}.$$

This norm will be useful in its own right, but it also gives rise to a natural distance function which is well-defined on $F(2^{\mathbb{Z}^n})$.

$$\rho(x,y) = \begin{cases} ||g||, & \text{if } g \cdot x = y \\ \infty, & \text{if } (x,y) \notin E(G). \end{cases}$$

It is easy to check that ρ is a pseudometric, and corresponds exactly to the taxicab metric on each class of $F(2^{\mathbb{Z}^n})$.

By the definition of the product topology, if $D \subseteq G$ is a finite, then a function $s : D \to 2$ determines a basic clopen set

$$N_s = \{x \in 2^G : \forall g \in D,\ \ x(g) = s(g)\}$$

Via a relatively easy proof, we may assume without loss of generality that the domain of each s is an n dimensional rectangle, so that $\text{dom}(s) = [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$, so that we may work with a base with a nicer geometric structure.

For a subequivalence relation R of $F(2^{\mathbb{Z}^n})$, we can say R is "relatively" clopen if for each $g \in \mathbb{Z}^n$, the set

$$\{x \in F(2^{\mathbb{Z}^n}) : (x, g \cdot x) \in R\}$$

is relatively clopen in $F(2^{\mathbb{Z}^n}) \times \mathbb{Z}^n$. We may replace the word "clopen" with "Borel", to get an analagous definition. We can now define structures with topological properties, and try to prove whether or not they exist. To prove a certain structure does exist, we generally use marker regions. To prove that they cannot exist, we generally use hyper-aperiodic elements. We introduce and discuss these concepts in later chapters.

CHAPTER 3

MARKER REGIONS

One of the main difficulties working in $F(2^{\mathbb{Z}^2})$ is that we cannot pick a point of each class in a Borel way. The smooth equivalence relations are considered simple for exactly this reason, as if there is a structure one wants to construct on a smooth relation, they can usually just select a point and use it as an origin to perform a relatively straightforward construction.

Without the ability to pick out a single point from each class, we must rely on marker sets. A *marker set* is any Borel complete section of $F(2^{\mathbb{Z}^n})$. The exact definitions of these sets vary, but the main idea is that they are points which occur somewhat regularly in the space. These points then induce subequivalence relations which partition the space into finite regions, allowing us to outline algorithms in these regions to construct the structures we want. For example, in chapter 2, we encountered the Slaman-Steel markers for $F(2^{\mathbb{Z}})$ which allowed us to $\mathbb{Z}$-order the classes of a hyperfinite equivalence relation. In this chapter, we discuss various marker sets, their constructions, and their applications.

3.1. General Marker Regions

Most constructions of intricate marker sets begin by starting with "basic" marker sets having the property that the marker points of M, to some degree, are regularly spaced.

LEMMA 3.1. (Gao-Jackson, [4]) *Let d be a positive integer. Then there is a relatively clopen set $S \subseteq F(2^{\mathbb{Z}^n})$ such that*

(1) if $x, y \in S$ are distinct, then $\rho(x, y) > d$.

(2) for any $x \in F(2^{\mathbb{Z}^n})$, $\rho(x, S) \leq d$.

We will refer to a set satisfying (1) and (2) as a *basic clopen marker set* for the marker distance d. We will use these sets to build new marker sets that give rise to subequivalence relations with various geometric properties. Before we do that, we give an example of how even these basic marker sets can be used to prove theorems. The definitions of Cayley graph and Schreier graph are necessary for our results anyway, so we are not going out of our way to define them here.

If G is a countable group having a generating set S, the *Cayley graph* $\Gamma = \Gamma(G, S)$ is a labeled directed graph with the vertex set $V(\Gamma) = G$ and the edge set defined by $(u, v) \in E(\Gamma)$ iff $\exists g \in S$ such that $g \cdot u = v$. If G acts freely on a Polish space X, then we define the *Schreier graph* $\Gamma_G(X)$ on X by $V(\Gamma_G(X)) = X$, and $(x, y) \in E(\Gamma_G(X))$ iff $\exists g \in S$ with $g \cdot x = y$. We note that every orbit of X is a connected component of $\Gamma_G(X)$, and since G acts freely on X, each component is isomorphic to the Cayley graph of G.

For a graph Γ and a set K of colors, a *proper* ($K-$)*coloring* is a map $\kappa : V(\Gamma) \to K$ such that if $(x, y) \in E(\Gamma)$, then $\kappa(x) \neq \kappa(y)$. The *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the least cardinality of a set K such that there exists a proper K-coloring for Γ. When the graph Γ is a topological graph, we may consider *continuous chromatic number* and *Borel chromatic number*, where the function k must be continuous or Borel respectively. Using our basic clopen marker regions, we can prove the following exercise

Example 3.2. The continuous chromatic number of $F(2^{\mathbb{Z}})$ is at most 3.

Proof. Let M be a marker set given by Lemma 3.1 for distance $d > 2$ and note that M partitions each class of $F(2^{\mathbb{Z}})$ into finite intervals of length at most $2d$. We now define κ as follows. For each $y \in F(2^{\mathbb{Z}})$, let $x_y \in M$ be such that $y = n \cdot x_y$ for $n \geq 0$, and if $0 < m < n$, $m \cdot x_y \notin M$. Since M is clopen and the distance between any two points of M is at most $2d$, the map $y \to x_y$ is continuous.

Define the coloring $\kappa(y)$ by

$$\kappa(y) = \begin{cases} 0 & \text{if } y = n \cdot x_y, \text{ where } n \geq 0 \text{ is even and } 1 \cdot y \notin M \\ 1 & \text{if } y = n \cdot x_y, \text{ where } n > 0 \text{ is odd} \\ 2 & \text{otherwise} \end{cases}$$

In other words, let the leftmost point of any interval be 0, and then alternate colors, with the exception that the vertex which is to the left of the next marker point can be colored

2. κ is continuous, since all searches are bounded, and is also clearly a proper 3-coloring. $\square$

The above proof is an elementary example of how we can use regions defined by marker sets to construct various structures on $F(2^{\mathbb{Z}^n})$. The regions defined by the marker sets allow us to give algorithms which can be applied uniformly in each region, bypassing the need to use a function which can select a point in each class.

We commonly make use of sequences of marker sets with growing marker distance. For example, consider the Slaman-Steel markers we constructed in chapter 2. As $n \to \infty$, the spaces between the marker sets became larger and larger, which forced more and more points to be put into the same region, which was a crucial component of the argument we used them for. These marker sets had a few other nice properties. If we let $S_0, S_1, \ldots$ denote the sets we constructed at each stage, we note that $S_0 \supseteq S_1 \supseteq \ldots$, and that $\cap_{n \in \omega} S_n = \emptyset$. Those sets are Borel, so any construction that makes use of them can at best be Borel (as opposed to simply clopen). It would be natural then to ask if we can construct a clopen sequence of marker sets satisfying the "vanishing" property that the Slaman-Steel markers have. Gao, Jackson and Seward proved the following.

THEOREM 3.3. (Gao, Jackson, Seward, [7]) *Let $F(G)$ be the free part of the shift action on 2^G by G. Then there is no infinite sequence of closed complete sections*

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

such that $\cap_n S_n = \emptyset$.

For $F(2^{\mathbb{Z}})$, a consequence of this is that if $S_0 \supseteq S_1 \cdots \supseteq S_n \supseteq \cdots$ is a sequence of clopen marker sets, there will necessarily be classes on which the intersection of the S_n is a single point. In this case, the limit of the regions is two infinite intervals. This theorem has a corollary which forbids the existence of an increasing sequence of open marker regions for which any two points of $F(2^{\mathbb{Z}^n})$ are eventually contained in the same marker region, meaning that constructions of continuous structures of $F(2^{\mathbb{Z}^n})$ must either forego containment of the marker regions or the marker sets having full union.

Corollary 3.4. *There is no increasing sequence of nondegenerate, relatively open subequivalence relations of* $F(2^{\mathbb{Z}^n})$

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_k \subseteq \cdots$$

such that $\cup_{n\in\omega} R_k = F(2^{\mathbb{Z}^n})$.

3.2. Rectangular Marker Regions

In general, we want regions that have more specific properties that align with the problems that we want to solve. One useful such construction is that of the rectangular marker regions, which are points that, in a clopen way, induce subequivalence relations of $F(2^{\mathbb{Z}^n})$ which are n-dimensional rectangles. When we say that a marker set R^n_d is clopen, we mean that $\{(x,g) \in F(\mathbb{Z}^n) \times \mathbb{Z}^n : g \cdot x R^n_d x\}$ is a clopen subset of $F(\mathbb{Z}^n) \times \mathbb{Z}^n$. The theorems below, all proven in §3 of [4], require some notation and definitions that we provide here for the reader's convenience.

A *rectangular polyhedron* is a finite union of rectangles in $\mathbb{Z}^n$. We define a *face* F of a rectangular polyhedron P to be a set $F \subseteq P$ such that for some $1 \le i \le n$ we have that F is a maximal subset of P satisfying the following:

(1) for any $x, y \in F$ and g such that $g \cdot x = y$, the ith coordinate of g is zero, and
(2) either $e_i \cdot F \cap P = \emptyset$ or $-e_i \cdot F \cap P = \emptyset$.

We refer to such a face F as being *perpendicular* to e_i or an i-face. We note that i-faces need not be "connected", i.e. there does not need to be a sequence

$$g_1, \dots, g_m \in \{\pm e_1, \dots, \pm e_n\}$$

such that $y = (g_1 + \cdots + g_m) \cdot x$ and $(g_1 + \cdots + g_l) \cdot x \in F$ for all $1 \le l \le m$. We say two faces are *parallel* if they are perpendicular to the same e_i, i.e., they are both i-faces for some $1 \le i \le n$. If F_1 and F_2 are both i-faces, then their *perpendicular distance* is the absolute value of the unique integer a_i whenever there are $a_j \in \mathbb{Z}$ for all $1 \le j \le n$ with

$$(a_1 e_1 + \cdots + a_i e_i + \cdots + a_n e_n) \cdot F_1 \cap F_2 \neq \emptyset$$

Theorem 3.5. (Gao-Jackson [4]) *Let $d > 0$ be an integer. Then there is a subequivalence relation R^n_d of $F(\mathbb{Z}^n)$ such that R^n_d is relatively clopen and the R^n_d-marker regions are n-dimensional rectangles with edge lengths either d or $d+1$.*

This theorem is proven in multiple steps using what is referred to as the big-marker-little-marker method. In essence, the authors build rectangular regions that are very large compared to the rectangular regions they actually want to construct, and then subdivide those regions into rectangular regions with length d or $d+1$.

Lemma 3.6 is used to divide an arbitrary marker region whose perpendicular faces are far apart into marker regions which are n-dimensional rectangles with edge length at least the distance between parallel faces.

Lemma 3.6. *Let $D > 0$ be an integer. Let R_0 be a subequivalence relation of $F(\mathbb{Z}^n)$ so that the R_0-marker regions are n-dimensional polyhedra with faces perpendicular to the coordinate axes. Suppose that for each R_0-marker region every pair of parallel faces has a perpendicular distance greater than D. Then there is a subequivalence relation $R_1 \subseteq R_0$ so that every R_0-marker region is partitioned into R_1-marker regions, which are n-dimensional rectangles with edge lengths greater than D. Moreover, if R_0 is clopen and there is $\Delta > D$ so that each R_0-marker region is contained in an n-dimensional cube of edge lengths Δ, then R_1 can also be clopen.*

The proof can be summarized as follows. If P is a finite-polyhedral region P in $\mathbb{Z}^n$ satisfying the hypotheses of the theorem, then any face F_j of P partitions P into at most two parts. The first part is the collection of points which are on one "side" of P with respect to the e_i that F_j is perpendicular to, denoted F_j^+, but still in P. The other part is simply the rest of P. The classes of the defined subequivalence relation are simply the sets of points which lie in exactly the same F_j^+.

Lemma 3.7 establishes the subdivision algorithm given that a marker set that induces sufficiently large n-dimensional rectangular regions has already been constructed. Furthermore, it asserts that if these large regions are clopen and there is a universal bound on the size of them, then the resulting subdivided regions are also clopen.

Lemma 3.7. *Let $d > 0$ and $D > d^2$ be integers. Let R_D be a subequivalence relation of $F(\mathbb{Z}^n)$ so that the R_D-marker regions are n-dimensional rectangles with edge lengths greater than D. Then there is a subequivalence relation $R_d \subseteq R_D$ so that every R_D-marker region is partioned into R_d-marker regions which are n-dimensional rectangles with edge lengths either d or $d+1$. Moreover, if R_D is clopen and there is $\Delta > D$ so that each R_D-maerker region has edge lengths $\leq \Delta$, then R_d can also be clopen.*

The important fact that makes the proof possible is that any integer $D > d^2$ can be written as a linear combination of d and $d+1$ with nonnegative coefficients, allowing a rectangle with side length at least to be partitioned into smaller rectangles with side lengths d or $d+1$.

Thus, by Lemma 3.6, if one can construct polyhedral marker regions where the faces have large perpendicular distance, they can subdivide it into large n-dimensional rectangular regions. By Lemma 3.7, these large regions can be subdivided further into rectangles with edge lengths which are almost regular, so the only step that remains is to construct the polyhedral marker regions which is, unsurprisingly, a difficult and technical proof.

The rectangular regions defined by Theorem 3.5 allow for constructions in regions with uniform shape and with edge lengths that are fairly regular. The fact that the regions are clopen and have a bounded size make it possible to give constructions that are continuous. For example, these regions were used in [4] to prove the upcoming theorem. These regions are nice to work with since the regions have a very nice geometry. We note Theorem 3.5 provides a tiling of $F(2^{\mathbb{Z}^2})$ by tiles with dimensions $d \times d, (d+1) \times d, d \times (d+1)$, and $(d+1) \times (d+1)$, however the following proposition, stated in [4], implies that there is no tiling of $F(2^{\mathbb{Z}^2})$ in which exactly one of these tiles is used. It is currently unclear if all four different types of tiles are needed, or if it is possible to tile $F(2^{\mathbb{Z}^2})$ in a clopen way with two or three types of tiles. A few specific combinations have been ruled out, and Gao and Jackson conjecture in [4] that 4 is the optimal number.

Theorem 3.8. *There is no Borel marker set $M \subseteq F(\mathbb{Z}^n)$ such that for any $x \in M$ there is a proper subgroup G of $\mathbb{Z}^n$ such that $[x] \cap M \subseteq G \cdot x$*

An application of the rectangular marker regions is the following.

THEOREM 3.9. *For each $n > 1$, there is a continuous 4-coloring of $F(2^{\mathbb{Z}^n})$.*

In [5], Gao, Jackson, Krohne and Seward proved that there is no continuous 3-coloring of $F(2^{\mathbb{Z}^n})$ for $n > 1$. This illustrates an interesting phenomena, as we showed that there is a continuous 3-coloring of $F(2^{\mathbb{Z}})$, implying that the answers to questions about $F(2^{\mathbb{Z}^n})$ can change depending on the value of n, with $n = 1$ being an outlier case. One reason for why this can happen is the different possible hyper-aperiodic points (which we discuss in detail in chapter 4) that can be constructed for these spaces.

3.3. Orthogonal Marker Regions

In this section we will discuss a special sequence of marker regions which don't cohere in a very strong sense. We will focus our attention to $\mathbb{Z}^2$, where the sets are easier to visualize, but the analogs for $\mathbb{Z}^n$ are natural, and the theorems and constructions still hold given changes to a few constants. In fact, in [4], Gao and Jackson constructed regions of $F(2^{\mathbb{Z}^{<\omega}})$ and used them to define an embedding from F_ω to $E_0(\omega^\omega)$. We will provide a special case of the main theorem they used.

For the rest of this paper, $d_j \gg d_{j-1}$ is defined to mean that

$$\frac{1}{9000^j 16^{j^2}(j+1)^2} d_j > 24(d_1 + d_2 + \cdots + d_{j-1})$$

LEMMA 3.10. (S. Gao, S. Jackson [4]) *For any pair of positive integers i, j with $j < i$. Let d_{j+1}, d_j be positive integers with $d_{j+1} \gg d_j$, then there is a clopen subequivalence relation $R^i_j \subseteq F(\mathbb{Z}^2)$ satisfying the following:*

(1) R^i_i induces rectangular regions with side lengths d_i or $d_i + 1$.

(2) On each class of $F(2^{\mathbb{Z}^2})$, for each region of $\mathcal{R}^i_j$, there is a region R' induced by $\mathcal{R}^i_{j+1}$ such that each face of R is within $12d_j$ of a face of R'.

(3) $\mathcal{R}^i_{j+1}$ induces a partition into polyhedral regions R each of which is a union of rectangles with edge lengths between $9d_j$ and $12d_j$.

(4) *In any ball* B *of radius* $100,000 \cdot 16^2 d_j$ *contained in a class of* $F(2^{\mathbb{Z}^2})$*, there are at most* 2 *values of* k *with* $j < k \leq i$ *such that some region induced by the restriction of* R_j^k *has a face intersecting* B.

(5) *For any* $j < k_1 < k_2 \leq i$ *and regions* R_1*,* R_2 *contained in a class of* $F(2^{\mathbb{Z}^2})$ *induced by the restrictions of* $R_j^{k_1}$*,* $R_j^{k_2}$ *respectively, if* $\mathcal{F}_1$*,* $\mathcal{F}_2$ *are parallel faces of* R_1*,* R_2*, then* $\rho(\mathcal{F}_1, \mathcal{F}_2) > \frac{1}{9000^j 16^{j^2}(j+1)} d_j$

If for each n we let $\mathcal{R}_n = \mathcal{R}_1^n$, The marker sets $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_n, \ldots$ satisfy an incredibly important property. For any two $x, y \in F(2^{\mathbb{Z}^2})$ there is an $N \in \omega$ such that for all $n > N$, x and y will fall into the same region of $\mathcal{R}_n$. This is proven in [4], but the intuition is fairly simple. If x and y fell into different regions infinitely often, then one could consider a shortest path $x = x_0, x_1, \ldots, x_N = y$ from x to y. Since this path is finite, some fixed x_l will fall on the boundary of infinitely many $\mathcal{R}_n$, but this would imply at least 3 higher-level region have faces close to each other, contradicting the fact that parallel faces of higher-level regions must be far apart. This is the benefit of the orthogonal marker regions, and is the main reason to use them over the rectangular regions defined in the previous section. When using these marker sets, there is the concern that two points x and y might lie in the same region in one stage, and then be placed into different regions at a subsequent step. Property (3) of Theorem 3.10 affirms that this can only happen a twice. Thus, if we can work around a finite number of interruptions, we are free to devise constructions which rely on any two points of $F(2^{\mathbb{Z}^2})$ eventually lying in the same region.

CHAPTER 4

HYPER-APERIODIC ELEMENTS

In this chapter, we will discuss hyper-aperiodic elements of 2^G for countable groups G. These points often require structures on 2^G to have certain properties, making them a very useful tool for showing certain structures cannot exist on these spaces.

DEFINITION 4.1. Let G be a countable group. A point $x \in 2^G$ is *hyper-aperiodic* if the closure of its orbit is contained in the free part, i.e., if $\overline{[x]} \subset F(2^G)$.

These elements are significant because the closures of their orbits are compact, giving us a greater depth of topological arguments for analyzing continuous structures of $F(2^G)$. Proving that a point is hyper-aperiodic by the definition given above is often cumbersome and tedious; instead, we usually test hyper-aperiodicity by a combinatorial condition using the following lemma. (1) $\iff$ (2) was proven in [8], and (2) $\iff$ (3) was proven in [5].

LEMMA 4.2. *Let G act on 2^G by right-shifts. Then the following are equivalent.*

(1) $x \in 2^G$ is hyper-aperiodic

(2) For any $s \neq 1_g$ in G there is a finite set $T \subseteq G$ such that

$$\forall g \in G \ \exists t \in T \ \ x(tg) \neq x(tsg)$$

.

(3) For any $s \neq 1_g$ in G there is a finite $T \subseteq G$ such that

$$\forall g \in G \ \ [(\exists t_1 \in T x(t_1 g) \neq x(t_1 sg)) \wedge (\exists t_2 \in T x(t_2 g) = x(t_2 sg))].$$

We will refer to (2) as the combinatorial condition for hyper-aperiodicity and we will use it almost exclusively to prove theorems about hyper-aperiodic elements. Hyper-aperiodic elements were originally refereed to as *2-colorings*, but the current name became more widely used as the concept extended beyond Bernoulli subflows.

In [8], it was proven that for any G, there is a hyper-aperiodic element of 2^G. Since we will be working extensively with $F(2^{\mathbb{Z}^n})$, we will explicitly construct an element of $F(2^{\mathbb{Z}})$. When we say a periodic point witnesses hyper-aperiodicity for a specific shift s_0, we mean that it satisfies the combinatorial condition above for that particular shift. We note that for any shift s, there is some periodic point which will witness hyper-aperiodicity for s. In particular, a point having a period with s zeros followed by a one will work.

Example 4.3. Let $s_0, s_1, \ldots$ enumerate $\mathbb{Z}$. Let B_0 be a periodic element of $2^{\mathbb{Z}}$ which witnesses hyper-aperiodicity for s_0, i.e., there is a finite set T such that for all $z \in \mathbb{Z}$, $\exists t \in T$ such that $x(z+t) \neq x(z+s+t)$. We define $\overline{B_0}(x) = 1 - B_0(x)$.

Suppose inductively that for all $k < n$, B_k is a periodic point of $2^{\mathbb{Z}}$ which has been defined so that B_k is a tiling of blocks of the form B_{k-1} or $\overline{B_{k-1}}$ and B_k witnesses the combinatorial condition of hyper-aperiodicity for s_k. If we let N_n denote the length of each period of B_n, then we additionally assume that $N_k > N_{k-1}$ for all $k < n$, and that for all $x \in [-N_{k-1}, N_{k-1})$, $B_k(x) = B_{k-1}(x)$. Let A_n be a periodic point of $2^{\mathbb{Z}}$ which witnesses hyper-aperiodicity for s_n. We may without loss of generality assume $A_n(0) = 0$, or else we can use $\overline{A_n}$. For all $m \in \mathbb{Z}$. We define B_n to be a tiling where if $B_n(z) = 0$, we insert a copy of the period of B_{n-1}. Otherwise, we insert a copy of $\overline{B_{n-1}}$. More precisely,

$$B_n \upharpoonright [m \cdot N_{n-1}, (m+1) \cdot N_{n-1}) = \begin{cases} B_{n-1} \upharpoonright [0, N_{n-1}) & \text{if } A_n(m) = 0 \\ \overline{B_{n-1}} \upharpoonright [0, N_{n-1}) & \text{if } A_n(m) = 1 \end{cases}$$

B_n is a tiling of blocks of B_{n-1} and $\overline{B_{n-1}}$ by construction, which implies $N_n > N_{n-1}$. Since $A_n(0) = 0$, $B_n \upharpoonright [0, N_{n-1}) = B_{n-1} \upharpoonright [0, N_{n-1})$. We next show that B_n witnesses the hyper-aperiodicity for s_n. Let T' be the set witnessing that hyper-aperiodicity for A_n and let $T = N_n \cdot T'$. Then if $z \in \mathbb{Z}$, there is $t \in T'$ such that if $y = \lfloor \frac{z}{N_n} \rfloor$, $A_n(y+t) \neq A_n(y+s_n+t)$. Thus, $B_n(z + N_n \cdot t) \neq (z + s_n + N_n \cdot t)$.

To define the hyper-aperiodic element, let $x(z) = \lim_{n\to\infty} B_n(|z|)$. x is well-defined since for all n, $B_n \upharpoonright [0, N_{n-1}) = B_{n-1} \upharpoonright [0, N_{n-1})$ and $N_n > N_{n-1}$. x is hyper-aperiodic as for any shift s and $z \in \mathbb{Z}$, $x(z)$ lies in some block of either B_n or $\overline{B_n}$ which will witness hyper-aperiodicity for x.

We now cite a general lemma that will let us use this hyper-aperiodic element of $2^{\mathbb{Z}}$ to construct hyper-aperiodic elements of $2^{\mathbb{Z}^n}$. If G, H are countable groups and $x \in 2^G$, $y \in 2^H$, let $x \oplus y \in 2^{G \times H}$ be given by $(x \oplus y)(g, h) = x(g) + y(h) \mod 2$. By iteratively applying the following lemma, we may construct a hyper-aperiodic point of $F(2^{\mathbb{Z}^n})$ for any n.

Lemma 4.4. (Gao, Jackson, Krohne, Seward, [5]) *Let $x \in 2^G$ and $y \in 2^H$ be hyper-aperiodic elements. Then $x \oplus y \in 2^{G \times H}$ is a hyper-aperiodic element.*

In the case $n = 1$, forcing a point to be hyper-aperiodic is a significant commitment, and it is often impossible to construct an element of $F(2^{\mathbb{Z}})$ with additional useful properties. For $n > 1$, hyper-aperiodicity is a much less restrictive condition. In fact, there are hyper-aperiodic elements x for which any vertical "slice" of x is periodic. The definition of this element uses points which are *orthogonal*. If G is a countable group, two points $x, y \in 2^G$ are *orthogonal*, denoted $x \perp y$, if there exists a finite $T \subseteq G$ such that for any $g, h \in G$ there is $t \in T$ with $x(tg) \neq y(th)$.

Lemma 4.5. (Gao, Jackson, Krohne, Seward, [5]) *Let x, y_0, y_1 be hyper-aperiodic elements of $2^{\mathbb{Z}}$ with $y_0 \perp y_1$. Let Λ be an infinite set of prime numbers and $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function satisfying the following conditions:*

(1) for all $u \in \mathbb{Z}$ there are $p \in \Lambda$ and $n \in \mathbb{Z}^+$ such that $f(u) = p^n$;

(2) for all $p \in \Lambda$ and $m \in \omega$, there are $a \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$ such that $f(i) = p^k$ for all $i \in [a, a+m]$;

(3) $f(u)$ is monotone increasing for $u > 0$, monotone decreasing for $u < 0$, and $f(u) \to \infty$ as $|u| \to \infty$

Then the element $z \in F(2^{\mathbb{Z}^2})$ defined by

$$z(u, v) = y_{x(u)}(v \mod f(u))$$

is hyper-aperiodic

By using the hyper-aperiodic point constructed in Lemma 4.5, the authors of [5] were able to prove that there is no continuous proper 3-coloring of $F(2^{\mathbb{Z}^2})$. The next theorem doesn't require any special condition on the hyper-aperiodic point used. In chapter 6, we will generalize this result using a hyper-aperiodic element constructed using forcing.

THEOREM 4.6. (Gao, Jackson, Krohne, Seward, [5]) *There does not exist a complete clopen lining L of $F(2^{\mathbb{Z}^2})$.*

Where by clopen lining, we mean a symmetric Borel relation S which is a subset of the Schreier graph for which the degree of every vertex is exactly 2. The proof heavily leverages the compactness of $K := \overline{[x]}$ to require that any point of K is a bounded distance from the line L. It then uses compactness again to bound the length of the line segment connecting any two points which are a bounded distance apart. Applying both of these properties, the line is forced to stay inside a "tube" with bounded height on K, contradicting the fact that every point of K is a bounded distance from L. We noticed that, by slightly altering the argument, we could prove an even stronger result. Instead of bounding L in a tube of fixed height, we could instead force L to have a nontrivial cycle by combining four sufficiently long tubes in the shape of a torus, implying that it is impossible to construct a clopen treeing of $F(2^{\mathbb{Z}^2})$.

CHAPTER 5

FORCING NOTIONS

In this chapter, we define forcing posets that correspond to elements x_G of $F(2Y\mathbb{Z}^n)$. These elements will force certain formulas on $[x]_G$. The books written by Jech [10], and Kunen [13] both provide a good basics for readers unfamiliar with the basics of forcing.

We will need a version of *Shoenfield's Absoluteness*. We note that a formula φ being *absolute* between two models M and N means that M models φ if and only if N models φ.

LEMMA 5.1. *If $M \subseteq N$ are transitive models of enough of ZF and $\omega_1 \subseteq M$, then $\mathbf{\Sigma}^2_1$ statements are absolute between M and N.*

Σ^2_1 is a pointclass which contains the Borel sets, so in particular, any Borel statement between any two "reasonable" models is absolute.

5.1. Minimal Two-Coloring and Grid-Periodicity Forcing

We now discuss generics for specific posets that naturally correspond to elements x_G of $F(2^{\mathbb{Z}^2})$. The elements of the generic will force various formulas, and, in particular, will force them via a clopen set. Due to the absoluteness of Borel formulas, these formulas will hold in $[x_G]$ as well, as the Forcing Theorem asserts some condition will force the formula.

THEOREM 5.2. *(The Forcing Theorem) Let $(P, <)$ be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over M,*

$$M[G] \models \sigma \textit{ if and only if } (\exists p \in G)\ p \Vdash \sigma.$$

If we have structures which are not clopen, or would like to do define formulas that aren't clopen, there will be a clopen neighborhood in the generic extension that will force the statement.

We start by defining the minimal 2-coloring forcing.

DEFINITION 5.3. The *minimal 2-coloring forcing* $\mathbb{P}_{mt}$ on $\mathbb{Z}^2$ is defined by the conditions

$$\mathfrak{p} = (p, n, t_1, \ldots, t_n, T_1, \ldots, T_n, m, f_1, \ldots, f_m, F_1, \ldots, F_m)$$

where $m, n \in \mathbb{N}, p \in 2^{<\mathbb{Z}^2}$ with $\text{dom}(p) = [a, b] \times [c, d]$ for some $a, b, c, d \in \mathbb{Z}$, $t_1, \ldots, t_n \in \mathbb{Z}^2 - \{(0, 0)\}, f_1, \ldots, f_m \in 2^{<\mathbb{Z}^2}$, and $T_1, \ldots, T_n, F_1, \ldots, F_m$ are finite subsets of $\mathbb{Z}^2$ such that the following conditions are satisfied:

(1) For any $1 \leq i \leq n$ and $g \in \text{dom}(p)$ there is $\tau \in T_i$ such that $g+\tau, g+t_i+\tau \in \text{dom}(p)$ and $p(g + \tau) \neq p(g + t_i + \tau)$;

(2) For any $1 \leq j \leq m$ and $g \in \text{dom}(p)$ there is $\sigma \in F_j$ such that $g + \sigma + \text{dom}(f_j) \subseteq \text{dom}(p)$ and for all $u \in \text{dom}(f_j), p(g + \sigma + u) = f_j(u)$.

(3) For any $1 \leq j \leq m$ and $g \in \text{dom}(p)$ there is $\sigma \in F_j$ such that $g + \sigma + \text{dom}(f_j) \subseteq \text{dom}(p)$ and for all $u \in \text{dom}(f_j), p(g + \sigma + u) = 1 - f_j(u)$.

If $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}_{mt}$, then $\mathfrak{q} \leq \mathfrak{p}$ iff $q \supseteq p$, $n(\mathfrak{q}) \geq n(\mathfrak{p})$, $m(\mathfrak{q}) \geq m(\mathfrak{p})$, $t_i(\mathfrak{q}) = t_i(\mathfrak{p})$, and $T_i(\mathfrak{q}) = T_i(\mathfrak{p})$ for all $1 \leq i \leq n(\mathfrak{p})$, and $f_j(\mathfrak{q}) = f_j(\mathfrak{p})$, $F_j(\mathfrak{q}) = F_j(\mathfrak{p})$ for all $1 \leq j \leq m(\mathfrak{p})$

Properties (1) and (2) will ensure that the generic x_G (the component corresponding to the finite functions of $\mathfrak{p}$) is hyper-aperiodic and minimal respectively, while (3) will be helpful for certain arguments.

We now provide a few lemmas which will show that $\mathbb{P}_{mt}$ does add a minimal 2-coloring in $2^{\mathbb{Z}^2}$. The following lemmas, proven in [6] will imply that x_G is as desired.

Lemma 5.4. *For any $g \in \mathbb{Z}^2$, the set $D_g = \{\mathfrak{q} \in \mathbb{P}_{mt} : g \in dom(q)\}$ is dense in $\mathbb{P}_{mt}$.*

This lemma will show x_G is an element of $2^{\mathbb{Z}^2}$. The next will show that if $s \in \mathbb{Z}^2$ is a nontrivial shift, then there is some condition $\mathfrak{p}$ which contains s. Thus, x_G will be a hyper-aperiodic element.

Lemma 5.5. *For any $t \in \mathbb{Z}^2 - \{(0, 0)\}$ the set*

$$E_t = \{\mathfrak{q} \in \mathbb{P}_{mt} : \exists 1 \leq i \leq n(\mathfrak{q}) t_i(\mathfrak{q}) = t\}$$

is dense in $\mathbb{P}_{mt}$

The next lemma will require that we consider neighborhoods of arbitrarily large sizes.

Lemma 5.6. *For any finite set $A \subseteq \mathbb{Z}^2$, the set*

$$D_A = \{\mathfrak{q} \in \mathbb{P}_{mt} : \exists 1 \leq j \leq m(\mathfrak{q}) A \subseteq dom(f_j(\mathfrak{q}))\}$$

is dense in $\mathbb{P}_{mt}$

Using the above lemma, and the trick of putting a copy of p and $\bar{p}$ adjacent to each other, we get the last lemma that we need, which says that if $p \in \mathfrak{p}$, then p will be contained in one of the f_j of some $\mathfrak{q} \leq \mathfrak{p}$. This is vital, as if we say $\mathfrak{p}$ forces some formula φ, then p will occur regularly. But then in the generic extension, we will see p, and know there is some $\mathfrak{q} < \mathfrak{p}$ which has $f_j \supseteq p$ as a component. Thus, every time we see an occurrence of p, φ will be forced.

Lemma 5.7. *For any $\mathfrak{p} \in \mathbb{P}_{mt}$, the set*

$$D_p = \{q \in \mathbb{P}_{mt} : \exists 1 \leq j \leq m(\mathfrak{q}) p \subseteq f_j(\mathfrak{q})$$

is dense below $\mathfrak{p}$ in $\mathbb{P}_{mt}$.

So we get the following lemma.

Lemma 5.8. *If x_G is generic for $\mathbb{P}_{mt}$, then x_G is minimal and hyper-aperiodic.*

We now introduce the *grid-periodicity forcing*. This generic x_G for this forcing will exhibit a very regular grid-like structure.

Definition 5.9. Let n be a positive integer. The *grid-periodicity* forcing $\mathbb{P}_{gp}(n)$ is defined as follows. A condition $p \in \mathbb{P}_{gp}(n)$ is a function $p : R \setminus \{u\} \to \{0, 1\}$ where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{Z}^2$ with $w = b - a + 1$, $l = d - c + 1$ both powers of n and $u \in R$. We write $R(p), w(p), h(p), u(p)$ for the corresponding objects and parameters.

We define $q \leq p$ iff $R(q)$ is obtained by a rectangular tiling by copies of $R(p)$ and if $c \in R(q)$ is in th ecopy $R(p) + t$ and $c - t \neq u(p)$, then $q(c) = p(c - t)$. Also, $u(q)$ must be equal to one of the copied translates of $u(p)$.

Lemma 5.10. *Let x_G be generic for $\mathbb{P}_{gp}$. Then x_G is a minimal and hyper-aperiodic.*

Not only is x_G a minimal hyper-aperiodic element. It also satisfies a weak form of periodicity, as proven in [6]

Lemma 5.11. *Let x_G be generic for $\mathbb{P}_{gp}(n)$.*

(1) For any vertical or horizontal line l in $\mathbb{Z}^2$, $x_G \restriction l$ is periodic with period a power of n.

(2) For any finite $A \subseteq \mathbb{Z}^2$, ther eis a lattice $L = (w\mathbb{Z}) \times (h\mathbb{Z})$, with both w and h powers of n, and there is a $u \in \mathbb{Z}^2 \setminus (A + L)$ such that x_G is constant on $k + L$ whenever $k + L \neq u + L$.

Having a hyper-aperiodic element which also corresponds to a generic for a forcing poset is incredibly powerful. For example, the following theorem asserts that any Borel complete structure must contain a lattice on at least one class.

Theorem 5.12. [6] *Let $B \subseteq F(2^{\mathbb{Z}^2})$ be a Borel complete section. Then there is an $x \in F(2^{\mathbb{Z}^2})$ and a lattice $L = k + \{(iw, jh) : (i, j) \in \mathbb{Z}^2\}$ such that $L \cdot x \subseteq B$.*

CHAPTER 6

OPEN STRUCTURES

In this chapter we will define a decreasing sequence of open sets that vanishes, and show that there is no open treeing of $F(2^{\mathbb{Z}^2})$ having exactly n-components on any class, while there is an open treeing that has at most 4 components on any class.

6.1. Marker Sets

Despite the fact that we cannot construct a sequence of descending clopen sets with empty intersection, [4] does provide an algorithm for constructing a sequence for which the intersection of the M_n contains at most one point. Currently, we have nice regularity results for $F(2^{\mathbb{Z}^2})$, but not for $F(2^{\mathbb{Z}^n})$ in general.

THEOREM 6.1. (Gao, Jackson [4]) *For any value of $0 < \epsilon < 1$, there is a clopen sequence of d_n-marker sets $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ of $F(2^{\mathbb{Z}^n})$ satisfying*

(1) For all $x \in F(2^{\mathbb{Z}^2})$, $\rho(x, M_n) < (1+\epsilon)d_n$.

(2) If $x, y \in M_n$ and $x \in [y]$, then $\rho(x, y) > (1-\epsilon)d_n$

(3) For all $z \in F(2^{\mathbb{Z}^n})$, $|\cap_n M_n \cap [z]| \leq 1$.

Starting with these sets as a base, we can construct a sequence of decreasing open marker sets which have somewhat regular spacing and empty intersection.

THEOREM 6.2. *For any sequence of integers $d_0 < d_1 < \cdots < d_n < \ldots$, there is sequence of open d_n-marker sets $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ of $F(2^{\mathbb{Z}^n})$ satisfying*

(1) For all $x \in F(2^{\mathbb{Z}^2})$, $\rho(x, M_n) < 2(1+\epsilon)d_n$.

(2) If $x, y \in M_n$ and $x \in [y]$, then $\rho(x, y) > (1-\epsilon)d_n$.

(3) $\cap_{n\in\omega} M_n = \emptyset$

PROOF. Let $M_0', M_1', \ldots$ be given by Theorem 6.1. Define M_n by

$$x \in M_n \Leftrightarrow x \in M_n' \wedge \exists m\, x \notin M_m$$

M_n is open since each M'_m is clopen; $\cap_n M_n = \emptyset$ since for each $x \in M_n$, there is an m with $x \notin M'_m$. $M_n \subseteq M'_n$, so for all x, y in the same class of $F(2^{\mathbb{Z}^n})$, $\rho(x,y) > (1-\epsilon)d_n$. However, each class might be missing exactly one point. Suppose z is such a point, and suppose x and y were points which were within $(1+\epsilon)d_n$ of z. Then

$$\rho(x,y) < \rho(x,z) + \rho(y,z) < 2(1+\epsilon)d_n$$

□

The marker sets constructed in the above proof have almost exactly the same structure as the ones constructed in 6.1. The only difference is that each class might have exactly one point missing. Thus, two points of M_n could be as close as roughly d_n, or as far apart as roughly $2d_n$. We know that the $2(1+\epsilon)d_n$ in the statement of 6.2 is not optimal in certain cases. For example, we can get an analogous theorem for $F(2^{\mathbb{Z}^2})$ with the bounds unchanged.

THEOREM 6.3. *For any sequence of integers $d_0 < d_1 < \cdots < d_n < \ldots$, there is sequence of open d_n-marker sets $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ of $F(2^{\mathbb{Z}^2})$ satisfying*

(1) For all $x \in F(2^{\mathbb{Z}^2})$, $\rho(x, M_n) < (1+\epsilon)d_n$.

(2) If $x, y \in M_n$ and $x \in [y]$, then $\rho(x,y) > (1-\epsilon)d_n$.

(3) $\cap_{n\in\omega} M_n = \emptyset$

To prove this, we will tile each class somewhat regularly by diamonds.

DEFINITION 6.4. A *diamond D* around a point x is an l_1 ball centered at x.

Once the space is tiled this way, showing the hypotheses holds will be relatively simple. This tiling is given by the following theorem, where the term diagonal axes refers to the lines of the form $f_0 = \{(x,x) : x \in \mathbb{Z}\}$ and $f_1 = \{x, -x\} : x \in \mathbb{Z}\}$. The construction of the diamond marker sets of free follows exactly like the construction of clopen rectangular marker sets in the specific case of $F(2^{\mathbb{Z}^2})$. This construction does not naturally extend to $F(2^{\mathbb{Z}^n})$, since in general, l_1 balls can't tile $\mathbb{Z}^n$. If $D_0, D_1, \ldots, D_n, \ldots$ are the set of center points of the diamond regions, by possibly adjusting the marker points (not the regions), we can get a sequence of marker sets $D'_0 \supseteq D'_1 \supseteq \cdots \supseteq D'_n \supseteq \dots$ which induce the same diamonds.

THEOREM 6.5. *Let $d > 0$ be an integer. Then there is a subequivalence relation D^n_d of $F(\mathbb{Z}^n)$ such that D^n_d is relatively clopen and the D^n_d-marker regions are diamonds with diagonal edge lengths either d or $d+1$.*

We now present the proof of Theorem 6.3

PROOF. Let $D'_0 \supseteq D'_1, \dots$ be a sequence of clopen diamond marker regions with edge lengths on the scale of $\frac{d_n}{2}$. Define D_n by

$$x \in D_n \Leftrightarrow x \in D'_n \wedge \exists m > n \; x \notin D'_m$$

Then each D_n is easily open. To see that each x is at most $(1+\epsilon)$ away from a point of D_n, we note that each x is in a l_1 neighborhood around some point y of D'_n. If y is in D_n, then we are done. Otherwise, since at most one point of each class was thrown out when constructing D_n, x is less than $\frac{(1+\epsilon)d_n}{2}$ away from the boundary of the diamond around y. Thus, x is $\frac{1+\epsilon d_n}{2}$ away from a different l_1 neighborhood around some point z of D_n, and this proves the claim. □

We might question how regular we can make the points of these marker sets. If we would like a descending sequence which has empty intersection, then it is necessary that the sets not be closed. The next theorem says that if we try to place the points of a marker set of $F(2^{\mathbb{Z}})$ too close together in an open way, the set will end up being clopen.

THEOREM 6.6. *IfM is an open marker set of $F(2^{\mathbb{Z}^n})$ satisfying the following conditions, then M is clopen*

(1) For each $x \in F(2^{\mathbb{Z}^n})$, $\rho(x, M) < d_n$

(2) For each $x, y \in M$, $\rho(x, y) > d_n$

PROOF. The claim holds since $F(2^{\mathbb{Z}^n}) \setminus M$ is open, which follows from the formula

$$x \notin M_n \Leftrightarrow \exists m \left(0 < ||m|| < \frac{d}{2} \wedge m \cdot x \in M_n \right).$$

□

It follows from this theorem and Theorem 3.3, that an sequence of open decreasing marker sets with empty intersection must have some amount of irregularity in the placement of the points.

6.2. Treeings

Before proceeding, we will make a distinction in terminology. We will say a formula is *forced* by a neighborhood if there is some element of a forcing poset having that neighborhood which forces it. We will say a formula is *determined* by a neighborhood if the neighborhood witnesses the formula in the usual way (i.e. if M is a clopen lining, then for each x, there is some clopen neighborhood U which determines that $x \in V(M)$.)

THEOREM 6.7. *There is no open exact n-treeing of $F(2^{\mathbb{Z}^2})$.*

PROOF. Suppose that T is an open exact n-treeing of $F(2^{\mathbb{Z}^2})$. Let x_G be generic for P_{gp} and set $K = \overline{[x_G]}$, so $K \subseteq F(2^{\mathbb{Z}^2})$ is compact. $[x_G]$ has k components, so we fix $\mathfrak{q} \in G$ that forces the tree structure of T in a square with some side length d_0 satisfying the following.

- There are n points $x_0, x_1, \ldots, x_{n-1} \in V(T)$
- There is an $N \in \omega$ such that for all $0 \leq a \leq n$, $x_a \restriction [-N, N]^2$ determines that $x_a \in V(T)$.
- x_a and x_b are not connected if $a \neq b$.

Let U_q be the open set corresponding to $\mathfrak{q}$,and $U_a = x_a \restriction [-N, N]^2$. If $\pi_{(i,j)}(g)$ is defined to be the translation map $g + (iw(q), jh(q))$, then $\pi_{(i,j)}$ induces an automorphism of $\mathbb{P}_{gp}$, so $\pi_{(i,j)}(U_q)$ forces the formulas above when $\dot{x_a}$ is replaced with $\pi_{(i,j)}(\dot{x_a})$. Furthermore, $\pi_{(i,j)}(x_a) \restriction [-N, N]^2$ will force that $\pi_{(i,j)}(x_a) \in V(T)$. Let $d_1 = \max\{w(q), h(q)\}$.

For each $x \in K$, and for each $z_0, z_1, \ldots, z_{n-1}, z_n \in K$ within $3d_1$ of x satisfying $z_a \restriction [-N, N]^2 = U_a$, there is some path connecting two of the points $z_0, z_1, \ldots, z_n, z$. Since these sets are clopen, K is compact, and T is complete, there is a maximal such path length d_2.

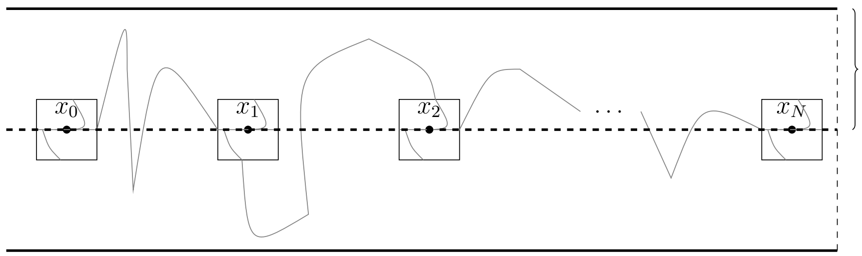

FIGURE 6.1. The subtree L_0. Some element of G forces all of the points of T within d_0 of each z_0^k.

Let $z_0, z_1, \dots z_N$ be defined by $z_k = (3kd_1, 0) \cdot x_0$, where $N > 3d_2$. For each k, $\pi_{3k,0}(U_q)$ will force the tree structure within d_0 of z_k; furthermore, for each $a \in \{0, 1, \dots, n-1\}$, $\pi_{(i,0)}(U_a)$, will appear within d_0 of z_k and determine points $x_a^k \in V(T)$. Since there are exactly n components of T on each class, for each k, two of $x_0^k, \dots x_n^k$ and x_0^{k+1} are connected by a line segment with length at most d_2. Thus, some x_a^k must connect to x_0^{k+1}, and by possibly renaming vertices, we may assume that $x_a^k = x_0^k$. We can repeat this argument, and assume without loss of generality that for each a, x_a^k is connected to x_a^{k+1}, meaning there is a line segment s_k of T with length at most d_2 which connects x_0^k and x_0^{k+1}. Some open neighborhood will force that there is some path p_k connecting x_0^k to x_0^k. We let

$$L_0 = \bigcup_{k \in \{0, \dots, n-1\}} s_k \cup p_k$$

L_0 is a subtree of T since each s_k and p_k are paths. L_0 contains only one component since s_k intersects p_k at the vertex x_0^k, while s_k and p_{k+1} will intersect at the vertex x_0^{k+1}. We note L_0 is contained in a rectangle with length at most $5d_2$ and height at most $2d_2$ (As pictured in Figure 6.1).

Repeat the construction above by taking a sequence of vertically aligned points $z_N = z'_0, \dots, z'_N$ defined similarly to the z_i sequence to construct a line segment L_1 which stays within a rectangle of width at most $2d_2$ and height $5d_2$. L_0 and L_1 both have the point z_n in common, so the two subtrees are connected. We note that $L_0 \cap L_1$ is contained in a square with side length at most d_2, i.e. L_1 can only "backtrack" a small distance into L_0.

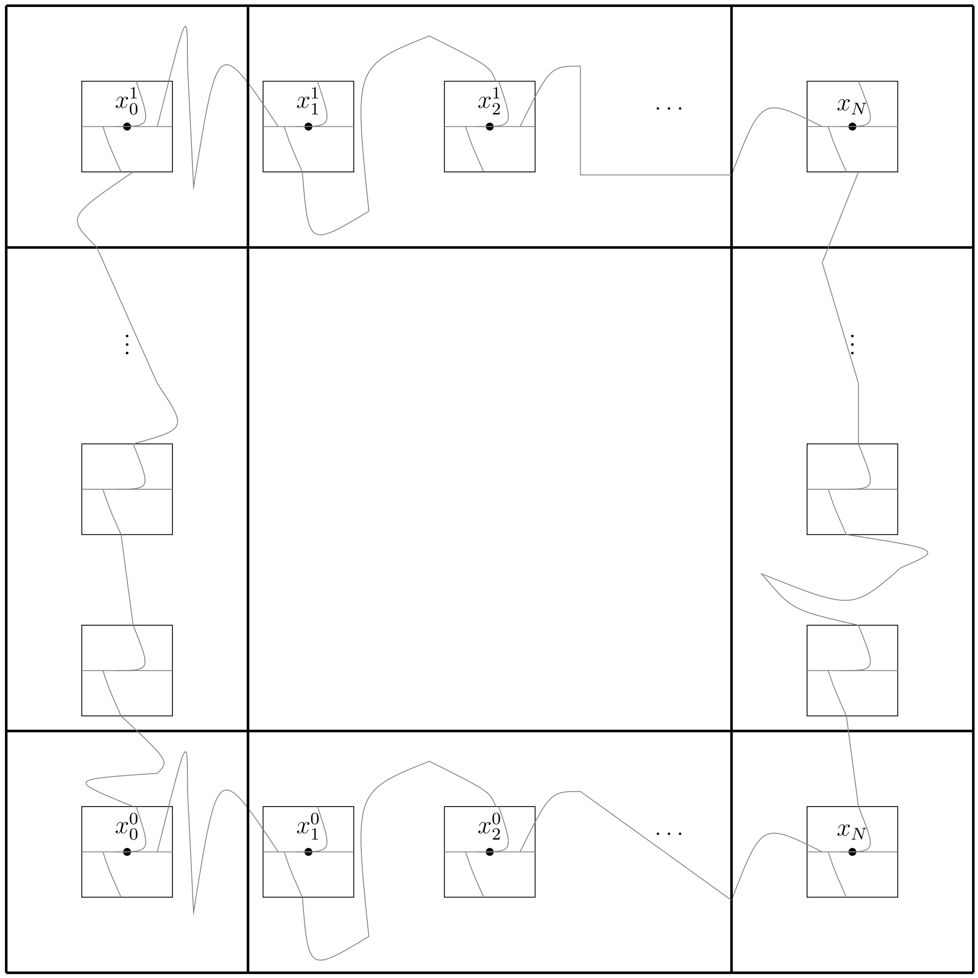

FIGURE 6.2. The L_i are all connected in tubes, and the forcing neighborhoods connect the L_i, forming a cycle of T.

Now repeat the construction again by moving left from z'_n to construct L_2, and then down to construct L_3. L_2 connects to L_1 via a segment contained in a square with side length at most d_1, and similarly for the pairs L_2, L_3, and L_3, L_0. Let $L = L_0 \cup L_1 \cup L_2 \cup L_3$.

We claim that L contains a nontrivial cycle in T. There is a path P from x_0^0 to $(0, 3d_2) \cdot x_0^0$ which is contained in $L_0 \cup L_1 \cup L_2$. There is also a path P' from x_0^0 to $(0, 3d_2) \cdot x_0^0$ contained in L_3. We claim that $P' {}^\frown P$ forms a cycle in T. $P' {}^\frown P$ is easily seen to be a path in T, but $P' \cap P$ is contained in the union of the two squares of size d_2 around x_0^0 and $(0, 3d_2) \cdot x_0^0$.

□

So for any specified n, there is no open treeing with exactly n components on each class; however, if we allow each class to have n or fewer components on each class, then the next lemma states that an n-treeing does exist. For $F(2^{\mathbb{Z}^2})$, the minimal known n is 4. It is currently not known if this bound can be lowered.

THEOREM 6.8. *There is an open* 4*-treeing of* $F(2^{\mathbb{Z}^2})$.

PROOF. For each i, Let $\mathcal{R}_i$ be an equivalence relation of $F(2^{\mathbb{Z}^2})$ which gives rise to rectangular regions with side lengths d_i or $d_i + 1$. We put an order $<$ on the regions of each class using the lexicographic order on the bottom left corner of each region. We now define the open treeing T inductively.

Let T_0 be the set of bottom left corner points of $\mathcal{R}_0$. Inductively, suppose that T_i has been defined for all $i < k$ such that each component of T_i is a tree contained in a square with side length at most $2d_i$, and for all i, each component of T_{i-1} is a proper subtree of some component of T_i. For each region R of $\mathcal{R}_k$, we define

$$C_R = \{t : t \text{ is a component of } T_{k-1} \text{ and } R \text{ is } <\text{-least such that } t \cap R \neq \emptyset\}.$$

C_R is well-defined since each component of T_{k-1} is finite; furthermore, if t is a component of C_R, then t can intersect at most four regions of $\mathcal{R}_k$, as $d_{k-1} \ll d_k$. We define an algorithm to connect all of the trees in C_R using the notion that when we say there is a path connecting two trees, that path must be contained in R. Let t_0 be an arbitrary component of C_R and let $T'_0 = t_0$. For each $t \in C_R \setminus \{t_0\}$, there is some shortest path connecting t_0 to t, as any trees on the boundary of $\mathcal{R}_k$ extend at most $2d_{k-1}$ into $\mathcal{R}_k$. Thus, we may connect t_0 to the tree $t_1 \in C_R$ which has the least such path p_0. It is easy to see that $T'_1 = t_0 \cup p_0 \cup t_1$ is still a tree, as if p_0 introduced a cycle, it would not be a shortest path. We now iterate this construction to define T'_n until it is connected to each component of C, and define T_k to be union of all of the T'_n. T_k is clopen, and $T_{k-1} \subseteq T_k$. We let

$$T = \bigcup_{k \in \omega} T_k$$

T is open, and all of the components of T are infinite since each component of T_{k-1} is contained in a strictly larger component of T_k. Each component of T is a tree, as otherwise,

T would have a cycle on some class, but that would imply that some T_k has a cycle on that class, which is false by the induction assumption.

We now show T has at most four components on each class. Towards a contradiction, suppose there are five different components of T on the same class $[x]$, and that x_0, x_1, x_2, x_3, x_4 are elements of the different components. Then there is a $k \in \omega$ such that the five points are contained in a square with side length d_k. But this square can intersect at most four regions of $\mathcal{R}_k$, so two of the x_i must be in the same region. This is a contradiction, as the trees containing the two points would have been connected by stage k.

□

CHAPTER 7

A BOREL N-LINING

In this chapter, we use the orthogonal marker sets $\mathcal{R}_0, \mathcal{R}_1, \dots \mathcal{R}_n, \dots$ from Theorem 3.10 to construct a Borel lining of $F(2^{\mathbb{Z}^2})$ which has exactly n components on each class. The intuition of the construction is that we will first define finite linings in each region R of $\mathcal{R}_0$, then by induction, we will connect the linings defined in the regions of $\mathcal{R}_{k-1}$ that are contained inside the same region of $\mathcal{R}_k$. The first problem we encounter is that the linings we defined in the previous step might overlap the boundary of the new region. To fix this, we would like to delete any points of L_k which get "too close" to the boundary of a region, but this will damage the lining. We will first prove Lemma 7.6, which we will refer to as the rewiring lemma. This will be what allows us to repair the linings of regions that are intersected by higher level regions. By the orthogonal marker construction, we can guarantee that the limit of the linings will be eventually constant on any fixed square region of $F(2^{\mathbb{Z}^2})$.

We now introduce the terminology we will be using through the rest of the chapter. The definitions and lemmas below will use a horizontal line, but we note they can be made analogously using a vertical line (or in fact, any line that partitions a region into two smaller regions). For this chapter, when we say $L = (L_0, L_1, \dots, L_{n-1})$ is an n-lining, we mean L is the union of n disjoint linings of $F(2^{\mathbb{Z}^2})$.

Given a lining L of $F(2^{\mathbb{Z}^2})$ and a horizontal line H, a *loop* l of L is any connected component of L for which either all points of l are above H, or all points of l are below H. A loop l is *embedded* in a different loop l' if every point of l is inside the region created by l' and H. A *loop structure* is a finite collection of loops $l_0, \dots, l_n$ such that for all $k \neq 0$, l_k is embedded inside of l_0. We refer to l_0 as the *outer loop* of the loop structure. The *rank* of a single loop is 0; inductively, the rank of a loop structure L is defined to be $k + 1$, where k is the highest rank of any loop structure inside the region created by L and H.

Definition 7.1. Two lines L_0 and L_1 are *adjacent* if for all $x \in L_i$, there is a $y \in L_{1-i}$ such that $\rho(x, y) = 1$. An n-lining is *tight* if it consists of lines $L_0, L_1, \ldots, L_{n-1}$ such that for all i, L_i is adjacent to L_{i-1} and L_{i+1}.

In other words, n lines are tight if they functionally act as one line. We can define loops, loop structures, and ranks for tight n-linings similarly to how we defined them for 1-linings. An *n-loop* is a collection of n loops $l_0, \ldots, l_{n-1}$ satisfying the tightness property above. An *n-loop structure* is a loop structure consisting only of n-loops.

Given a horizontal line H, a *cut* of an n-lining $L = (L_0, L_1, \ldots, L_{n-1})$ is a sequence of n-adjacent vertices $x_0, \ldots x_{n-1}$ of L such that each point intersects H and $x_0 \in V(L_0), \ldots, x_{n-1} \in V(L_{n-1})$. We will write a cut as $x = (x_a, x_b, \ldots)$, where $x_a \in V(L_a)$ is the leftmost vertex, followed by x_b and so on. For vertical cuts, x_a will be the lowest point, x_b the next, and then so on. We think of cuts as the equivalent of endpoints for structures that have n loops.

Definition 7.2. An n-loop structure is *good* if whenever l is an n-loop with left cut $x = (x_0, x_1, \ldots, x_{n-1})$, k is an n-loop embedded in l, and there are no loops in the region created by k and l, then the left cut y of k is ordered $(y_{n-1}, \ldots, y_0)$.

In other words, if L is a good loop structure, then however its outer loops are ordered, any n-loops below will be in the reverse order. Furthermore, if L is a good loop structure, then it's cut points are either in the order $c_0, c_1, \ldots, c_{n-1}$ or $c_{n-1}, \ldots, c_0$, and these orders alternate as we move from left to right along H.

Suppose that L is a tight n-lining and H is a horizontal line. We can convert the resulting loop structure into a good one by letting $H' = (0, -n) \cdot H$, and deleting any n-loops that do not intersect the original region.

Lemma 7.3. *Suppose S is a loop structure created by a tight n-lining L and a horizontal line H, then there is an n-loop structure $S \subseteq S' \subseteq V(L)$ in the region created by $H' = (0, -n) \cdot H$.*

Proof. Let S be given by the hypotheses, and let S'' be the loop structure induced by H' and L. Let $s(l)$ be the set of loops k of S'' such that there are loops $l_0, \ldots l_N$ of S'' such that

$l = l_i$ for some i and for all j, l_j and l_{j+1} are adjacent. Define S' by $l \subset S' \iff s(l) \cap S \neq \emptyset$.

$S \subset S'$ since we only deleted lines that didn't start in S; furthermore, S' only consists of n-loops via the tightness of L. The order flips as a corollary to the following claim, which is proven easily via induction.

<u>Claim:</u> If L is a single line intersecting a horizontal line H, and $x_0, x_1, \ldots, x_n$ are the vertices of $H \cap V(L)$ ordered left-to-right. Then if x_n and x_m are the endpoints of some loop, $n - m$ is odd. □

For any loop structure L, we will let x_L and y_L denote the left and right cuts of the outer loop of L. Whenever we say there are loop structures $l_1, l_2, \ldots, l_n$ under some loop structure L, we will assume the cuts of l_i are to the left of l_j if $i < j$. We are now ready to prove lemma 7.5. Intuitively the lemma traces out the outer loop of a loop structure L, then crosses over the outer loops of all the structures beneath L . It then commits to connecting all points under the first structure and moves to the next loop structure. For the following definition and proof, we will just say loop instead of n-loop.

Definition 7.4. Let $x = (x_0, \ldots, x_{n-1})$ and $y = (y_{n-1}, \ldots, y_0)$ be two cuts of some n-lining L. The *cable* h connecting x and y is defined as follows.

- If x and y are horizontal cuts with x to the left of y, then let a_i be the line segment that starts at x_i, extends down $i+1$ units, to the right until it is under y_i, and then up to connect to y_i. We let h be the union of the a_i.
- If $x = (x_0, \ldots, x_{n-1})$ is a horizontal cut and $y = (y_0, \ldots, y_{n-1})$ is a vertical cut such that $y_{n-1} = (-m, -n) \cdot x_{n-1}$, where $m, n > 0$, let a_i be the line segment that starts at x_i, extends down until it is to the right of y_i, and then left until it connects to y_i. We let h be the union of the a_i.

Lemma 7.5. *Given a good n-loop structure L contained inside the intersection of a finite region $R \subseteq F(2^{\mathbb{Z}^2})$ and a horizontal line H, there is a good n-loop structure $\mathcal{L}$ satisfying:*

(1) $L \subseteq \mathcal{L}$.

(2) If s is a loop and x, y are points of $V(\mathcal{L})$ in the region S created by s and H, then the part of the path connecting x and y above H is contained in S.

(3) Any point of $\mathcal{L}$ is at most $2n$ below H.

(4) $\mathcal{L}$ shares the cut $x_L = (x_0, \ldots, x_{n-1})$ with L and its other "endpoint" is the vertical cut $y_{\mathcal{L}} = (y'_0 = (0, -2n) \cdot y_{n-1}, \ldots y'_{n-1} = (0, -n-1) \cdot y_{n-1})$, where y_{n-1} is the leftmost point of y_L.

Moreover, if R is induced by a Borel equivalence relation and L is Borel, then $\mathcal{L}$ is Borel.

Proof. We prove the claim by induction based on the rank of the loop structure L. If L has rank 0, then L is exactly one loop, so we just let $\mathcal{L} = L$. If L has rank 1, then L has an outer loop l with left cut $x = (x_0, \cdots, x_{n-1})$, and right cut $y_L = (y_{n-1}, \ldots, y_0)$. l also has some number of embedded loops $l_0, l_1, \ldots, l_m$ whose left cuts are in the opposite order of the left cuts of l and similarly for the right cuts. For all $i < m$, let h_i be the cable connecting y_{l_i} and $x_{l_{i+1}}$, and let h_m be the cable connecting y_{l_m} and y_l. We define

$$L' = l \cup \bigcup_{0 \leq i \leq m} (h_i \cup l_i),$$

Let $\mathcal{L} = L' \cup v \cup h_{n+1}$, where v is the set of vertical lines v_i with endpoints x_{l_1}, and $(0, -n) \cdot x_{l_i}$, and h_{n+1} is the cable connecting $(0, -n) \cdot x_{L_1}$, and $(0, -n) \cdot y_{n-1}$. Intuitively, $\mathcal{L}$ traces out the outer loops moving left to right, and then all of the inner loops moving right to left. It is easily checked that (1)-(4) are satisfied.

Now suppose that the claim holds for any good loop structure of rank k or less; we show that the claim holds for a loop structure of rank $k+2$. Suppose L is a loop structure of rank $k+2$. Then L has an outer loop l, with some number of loop structures of rank at most $k+1$ below it, say $l_0, l_1, \ldots, l_m$. Under the outer loop of each l_i are loop structures $l_0^i, l_1^i, \ldots l_{m_i}^i$ with rank at most k. For each $0 \leq i \leq m$ and $0 \leq j \leq m_i$, Let K_j^i be the n-lining which satisfies the induction hypotheses for l_j^i. By induction hypothesis K_j^i has a horizontal cut x_j^i and vertical cut y_j^i. Define

$$\mathcal{L} = L'_0 \cup h \cup \bigcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m_n}} \left(K_j^i \cup h_j^i \cup v_j^i \right)$$

$\mathcal{L}$ has left cut x_l as that was the left cut of L_0, and has a vertical right cut $(0,-2n) \cdot y_l$, so $\mathcal{L}$ satisfies (4). Similarly, all of the horizontal cables connected adjacent endpoints, so those went at most n below H. The cables connecting the K_j^i extend at most n below the other cables, so $\mathcal{L}$ satisfies (3).

We next check (1). Let $x, y \in V(L)$. x is a point of one of l, l_i or some K_j^i, so it suffices to show l, l_i, and K_j^i are all connected. l and all of the l_i were connected in L_0. L connected L_0 to K_0^0, and then connected all of the K_i^j to each other. Thus, any two points of L are connected in $\mathcal{L}$. (2) holds by construction.

□

Lemma 7.5 allows us to make a tight n-lining out of an arbitrary good loop-structure which extends a bounded distance below the loop structure. for the purposes of proving theorem 7.10, we will need a slight strengthening. We will leave room under the lining L so that we can "encase" the lining at a later stage.

Corollary 7.6. (The Rewiring Lemma) *Suppose L is a good n-loop structure inside the intersection of a finite region $R \subseteq F(2^{\mathbb{Z}^2})$, and a horizontal line H. Then there is a lining $\mathcal{L}$ satisfying the hypotheses of Lemma 7.5 and for any substructure A of $\mathcal{L}$ having an outer loop l which has distance $k > 50n$ from any other point of A, and any vertical line V intersecting $A \setminus l$, there are cuts a_1,a_2,b_1,b_2 of $V(L) \cap V(H)$ satisfying:*

(1) The cuts are ordered left to right by a_1,a_2,b_1,b_2, and every point of one of these cuts is either a point of l, or is in the region created by A and H.

(2) a_1 and a_2 are to the left of V, and b_1 and b_2 are to the right of V.

(3) a_1 and a_2 are the cuts of loops l_{a_1} and l_{a_2} which are at least $4n$ units away from each other and there are no points of L in the region created by the two loops and H. Similarly b_1 and b_2 are the cuts of loops l_{b_1} and l_{b_2} which are at least $4n$ away from each other, and there is no point of L in the region created by the two loops and H.

(4) l_{b_1} is embedded in all of the other loops above, and the distance from l_{b_1} to any point

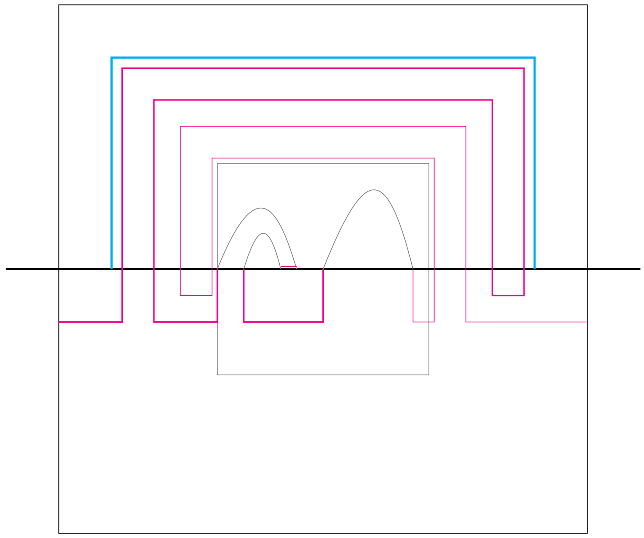

FIGURE 7.1. The rewiring arcs over the inner loops in order to leave space for a different set to come in later. The blue loop represents the original outer loop of the lining.

of $A \setminus l_{b_1}$ *is at least* $k - 25n$*.*

PROOF. In the proof of 7.5, whenever we encounter an outer loop l satisfying the hypotheses, we can alter the algorithm as follows. If we would construct a cable h_i connecting x_l and x_{l_1}, then we replace it with $S = h_0 \cup l_{a_1} \cup h_1 \cup l_{a_2} \cup h_2$, where each piece is defined below.

- h_0 is the cable below H connecting $x_l = (x_0, \ldots x_{n-1})$ and the horizontal cut $a_1 := ((2, 0) \cdot x_{n-1}, \ldots, (2 + n - 1) \cdot x_{n-1})$.
- l_{a_1} is an n-loop with cuts a_1 and $(-2, -2n) \cdot y_1$ for which every point of the outer loop has distance either 1 or 2 from l.
- h_1 is the set of horizontal lines starting at $(-2, -2n) \cdot y_l$ and ending at $(-7n, -2n) \cdot y_l$.
- l_{a_2} is an n-loop starting at $(-7n, -2n)$ that stays roughly $6n$ away from l_{a_1} and ends at the cut $a_2 = (7n, 0) \cdot x_l$.
- h_2 is the cable starting at a_2 and ending at x_{l_1}.

,

If the algorithm does not connect x_l and x_{l_1} with a cable, then we can let $a_1 = x_l$ and $a_2 = x_{l_1}$, and the hypotheses are easily satisfied.

If we would draw a cable connecting $(0,-2n)\cdot y_{l_n}$ and $(0,-2n)\cdot y_l$, we instead replace it with the line segment $S = h_0 \cup l_{b_1} \cup h_1 \cup l_{b_2} \cup h_2$, where each set is defined below.

- h_0 is the cable of length 2 starting at $(0,-2n)\cdot y_l$ and ending at $b_1 = (-21n,-2n)\cdot y_{l_n}$.
- l_{b_1} is a loop starting at $(-21n,-2n)\cdot y_{l_n}$ and ending at $(21n,-2n)\cdot x_l$ where every point of the outer loop of l_{b_1} has a distance roughly $14n$ from l_{a_2}.
- h_1 is is the cable connecting $(21n,-2n)\cdot x_l$ and $(14n,-2n)\cdot x_l$
- l_{b_2} is a loop starting at $(14n,-2n)\cdot x_l$ and ending at $b_2 = (-14n,-2n)\cdot y_l$
- h_2 is the cable connecting b_2 to $(0,-4n)\cdot y_l$.

Both of these linings are well-defined since there were no points of $\mathcal{L}$ within $50n$ units of l. Any point of l_{b_1} is at most $22n$ away from the outer loop by construction, so $\rho(l_{b_1}, A\setminus l) > 25n$. □

We now work towards an algorithm which will connect the disjoint linings in each of the regions. For the rest of the section, we will assume we have a sequence of integers $d_0 << d_1, \dots$, and a sequence of orthogonal marker sets $\mathcal{R}_0, \mathcal{R}_1, \dots$ as defined in Theorem 3.10. Furthermore, we will assume that there is a fixed $N > 0$ such that each region R of $\mathcal{R}_k$ intersects at most N regions of $\mathcal{R}_{k-1}$. We also assume b is a number much greater than $3^{(2N)^2}$. We will now define a few notions that will be used in the constructions of the connection algorithm and the Borel n-lining.

Definition 7.7. For a region R of $\mathcal{R}_k$, the set of *boundary points*, denoted $\mathrm{bd}(R)$ is defined by

$$x \in \mathrm{bd}(R) \Leftrightarrow x \in F(2^{\mathbb{Z}^2}) \text{ and } \exists g \in \mathbb{Z}^2 \ (||g|| = 1 \text{ and } (g\cdot x, x) \notin R).$$

Let $I_R = \{x \in R : \rho(x, \mathrm{bd}(R)) \geq b\}$ be the *interior* of R. Let $B_R = \{x \in R : \rho(x, bd(R)) < b\}$ be the *buffer* of R.

The rewiring lemma connects any components that a higher level region disconnects, and we showed that this reconnection can happen in a fixed amount of space. Since the linings are originally contained in the interiors of the regions, this reconnecting will extend a fixed distance into the buffer. We will use the remaining space in the buffer to connect the interior linings at each step. To do this, we require a topological lemma for $\mathbb{R}^2$. Whenever we use the term distance with respect to $\mathbb{R}^2$, we mean distance using the l_∞ metric. We view n as a fixed number for the lemma below, but will allow it to vary in Lemma 7.9.

LEMMA 7.8. *Let $R \subseteq \mathbb{R}^2$ be a square with side lengths at least $d_0 >> 3^{2Nn}$, where $n \in \mathbb{Z}^+$ is arbitrary. Let L be a line segment satisfying the following properties for some $k > 2$.*

(1) For each $x \in V(L)$, $B_{l_\infty}(x, 3^{nk})$ contains exactly one component of L.

(2) If $x, y \in V(L)$ are such that $x = (r, 0) + y$ or $x = (0, r) + y$, and x and y are not connected by a vertical/horizontal line, then $r > 3^{nk}$,

(3) For each $x \in V(L)$, $l_\infty(x, bd(R)) > 3^{nk}$.

Then if z is a point which is at least 3^{nk} away from L, there is a line segment $\mathcal{L}$ such that $L \subset \mathcal{L}$, z is an endpoint of $\mathcal{L}$, and $\mathcal{L}$ satisfies (1)-(3) if every instance of k is replaced with $k - 2$.

PROOF. We will say that any line segment satisfying (1) and (2) together is *essentially diagonal*, and note that for all $x \in L$, $V(L) \cap B(x, 3^{nk})$ is essentially diagonal by hypothesis. Define

$$T = \{x \in R : l_\infty(x, L) \leq 3^{n(k-1)}\},$$

and let $S = R \setminus T$. We note that T is closed. We will show that S is connected, which will imply it is path connected as $S \subseteq \mathbb{R}^2$ is open. Let x_0 be an endpoint of L, and define

$$s_0 = \{z \in V(L) : l_\infty(x_0, z) \leq 3^{nk}\}, \text{ and}$$

$$t_0 = \{z \in R : l_\infty(z, s_0) \leq 3^{n(k-1)}\}.$$

It is clear that s_0 is essentially diagonal and $R \setminus t_0$ is connected.

Inductively suppose essentially diagonal line segments $s_0, \ldots, s_{m-1} \subseteq L$, and vertices $x_0, \ldots, x_m \in V(L)$ have been defined such that $s_i \subseteq L$ has endpoints x_i and x_{i+1} and for all $i < m - 1$:

a) $x_i \in V(L)$, $l_\infty(x_i, x_{i+1}) = 3^{nk}$, and $l_\infty(x_i, x_j) > 3^{nk}$ if $|i - j| \neq 1$

b) $\bar{B}(x_i, 3^{nk}) \cap L = s_{i-1} \cup s_i$.

c) $s_i \cap s_{i+1} = x_{i+1}$.

d) $s_i \cap s_j = \emptyset$ if $|j-i| \neq 1$.

e) if $t_i = \{z \in R : l_\infty(z, s_i) \leq 3^{n(k-1)}\}$, then $R \setminus (t_0 \cup \cdots \cup t_{m-1})$ is connected.

Let s'_m be the line segment $\bar{B}(x_m, 3^{nk}) \cap L$, and $s_m = x_m \cup (s'_m \setminus s_{m-1})$. One of the endpoints of s_m is x_m and we denote the other one as x_{m+1}. By construction $s_0, \ldots, s_m$ satisfy (b) and (c). (a) and (d) follow from the triangle inequality and the fact that each s_i is essentially diagonal. We define $t_m = \{z \in R : l_\infty(z, s_m) \leq 3^{n(k-1)}\}$ and show $R \setminus (t_0 \cup \cdots \cup t_m)$ is connected.

Towards a contradiction, suppose not. It is easy to check that $R \setminus (t_{m-1} \cup t_m)$ is connected, as $s_{m-1} \cup s_m$ is the component of L in $\bar{B}(x_m, 3^{nk})$. Therefore, $t_m \cap t_i \neq \emptyset$ for some $i < m-1$, so there is a $z_i \in s_i$ such that $l_\infty(z_i, s_m) < 2 \cdot 3^{n(k-1)}$. But then $\bar{B}(z_i, 3^{nk}) \cap s_m \neq \emptyset$, so in fact the line segment connecting s_i and s_m is contained in $\bar{B}(z_i, 3^{nk})$, which is a contradiction, as $\bar{B}(z_i, 3^{n(k-1)}) \cap L \subseteq s_{i-1} \cup s_i \cup s_{i+1}$, which s_m does not intersect. Thus, $R \setminus (t_0 \cup \cdots \cup t_m)$ is connected.

Our recursive definition eventually halts since L is finite, meaning that $L = s_0 \cup \cdots \cup s_M$ for some integer M; therefore, $T = \cup_n t_n$, so $R \setminus T$ is path connected. We now define the components that will union to be $\mathcal{L}$.

$\bar{B}(x_0, 3^{n(k-1)})$ consists of an essentially diagonal line, so there is a vertical line segment $l_0 \subseteq T$ which has one endpoint x_0, and another endpoint y on the boundary of T which satisfies that for all $x \in l_0$, and $a \in s_0$, $x \neq (0, r) + a$, and similarly for $(r, 0)$. If there is a horizontal or vertical line that connects z (The point we would like to connect to L) to some point y' on the boundary of T, then let l_1 be that line. Otherwise, let v be the shortest vertical line, such that there is a horizontal line h with $h \cup v$ connecting x to a point y' of T, and let $l_1 = h \cup v$.

Let l'_2 be a line segment that travels along the boundary of T with endpoints y and y'. l'_2 could move clockwise or counterclockwise, and one of these orientations will guarantee no point of l'_2 is $r \cdot e_i + x$ for some $x \in l_0$, where $r < 3^{n(k-1)}$. l'_2 might not implicitly satisfy (2) if k is replaced with $k-2$, so we can edit it in the following way. If $a, b \in l'_2$ would fail to satisfy

(2) using $k-1$, then replace the segment connecting a and b with a vertical/horizontal line segment. We argue this segment is at least $3^{n(k-2)}$ away from L since a and b were $3^{n(k-1)}$ away from L. Towards a contradiction, suppose there is a point c of the segment and a point z of S such that $l_\infty(c,z) < 3^{n(k-2)}$. Without loss of generality, $l_\infty(a,c) < \frac{3^{k(n-1)}}{2}$ (Otherwise, it is at least this close to b). But then

$$l_\infty(a,z) < l_\infty(a,c) + l_\infty(z,c) < \frac{3^{k(n-1)}}{2} + 3^{k(n-2)} < 3^{k(n-1)}$$

contradicting that a was a point of $l'_2 \subseteq T$. Finally, we let

$$\mathcal{L} = L \cap l_0 \cup l_1 \cup l_2.$$

$\mathcal{L}$ connects z to L by definition. We now show it satisfies (1) and (2) if k is replaced by $k-2$ (so now when we say (1) or (2) holds, we mean with respect to $k-2$).

Suppose $x \in \mathcal{L}$ we show $B := \bar{B}(x, 3^{n(k-2)}) \cap \mathcal{L}$ is essentially diagonal, which will imply (1) and (2). If B intersects exactly one of the sets L, l_0, l_1 or l_2, then B is essentially diagonal, so it suffices to check that if B intersects a union, then B is essentially diagonal. The pairs $(l_0,l_1),(L,l_0),(L,l_2)$ are too far apart for a ball to intersect them both. Also, the pairs $(l_0,l_2),(l_1,l_2),(L,l_1)$ were specifically defined so that any such ball would be essentially diagonal. Thus, the claim holds. □

LEMMA 7.9. *Suppose R is a region of $\mathcal{R}_k$ such that $R = R_0 \cup R_1 \cup \cdots \cup R_N$, where each R_i is a region of $\mathcal{R}_{k-1}$. Suppose that for all $0 \le i \le N$, L_i is a tight n-lining of each R_i such that the following hold:*

(1) L_i is contained in I_{R_i}

(2) L_i has cuts x_i and y_i on the boundary of I_{R_i}.

Then there is a single tight n-lining $\mathcal{L}$ of R such that for all $i,j < N$, $\mathcal{L} \restriction I_i = L_i$, and if l_1 is a component of L_1, then for each any k, there is a path connecting l_1 to some component of L_k.

PROOF. Before we construct the lining, we build a function to convert R into a region $R' \subseteq \mathbb{R}$. Let $z \in \mathrm{bd}(R)$ be arbitrary and let $f(z) = (0,0)$. For all $y \in R$, if $y = (n,m) \cdot z$, let

$f(y) = (n, m)$. If $y \in \mathrm{bd}(R)$, let $f(y) \in R''$. If $x, y \in bd(R)$, and $y = \pm e_i \cdot x$, then connect $f(x)$ and $f(y)$ via a straight line in $\mathbb{R}$. Then R'' forms the boundary of a region R' which is nearly a square with side length d_n. This space is connected, and changing the initial choice of z would yield a translation of R, which would be homeomorphic to R.

If M is a tight line segment in R, then we can convert M into a line segment M' of R' using the following algorithm. Let $(x_0, x_1, \ldots, x_n)$ order the vertices of the zeroth segment of M. Let M_i be the straight-line with endpoints $f(x_i)$ and $f(x_{i+1})$ and let M' be the union of the M_i. Given any line segment that is a union of straight lines that consist of at least one integer coordinate, we can convert that into a lining in R in a Borel way.

We next define tight segments p_k with the properties below and let $\mathcal{L}$ be the union of the p_k and L_k. Property (1) holding for all k will imply $\mathcal{L}$ satisfies the conclusion of the theorem.

(1) p_k connects L_k to L_{k+1}
(2) For all $m < k$, $\rho(p_m, p_k) \geq 3^{(2N-2k)(2N-2k)}$
(3) p_k has distance at least $3^{(2N-2k)(2N-2k)}$ from $\mathrm{bd}(R)$
(4) p_k has distance at least $3^{(2N-2k)(2N-2k)}$ from the interior of any region other than I_{R_k} and $I_{R_{k+1}}$
(5) For any $x \in V(p_k)$, $B(x, 3^{(2N-2k)(2N-2k)})$ is essentially diagonal.

It is easy to see that a path p_0 satisfying (1)-(5) exists, so let p_0 be a shortest such path. Now suppose that p_m has been defined for all $m < k$ satisfying (1)-(5), and let p'_m in R' be analogous to p_m in R using the algorithm given above. Then the paths p_m satisfy (1)-(5) with value m if ρ is replaced with l_∞. Properties (3) and (5) imply that $p' = \cup_{m<k}\ p'_m$ satisfy the hypotheses of Lemma 7.8. Let S be the line segment given by the lemma for the point y_k and $n = (2N - 2k)$ and define $s' = S \setminus P'$.

Next, let s be the line segment of R analogous to s'. Then s will easily satisfy $(1)_k$ and $(5)_k$. s will also satisfy $(2)_k$, and $(3)_k$ for the larger value $3^{(2N-2k+2)(2N-2k)}$, but will not necessarily satisfy $(4)_k$. Thus, we must adjust the segment so that if $i \notin \{k-1, k\}$, s avoids the boundary of I_{R_i}. For each i where $(4)_k$ does not hold, fix an orientation of s and let

x be the least point of s with $\rho(x, I_{R_i}) = 3^{(2N-2k+1)(2N-2k)}$, and let y be the greatest such point. Since p_m satisfies $(4)_m$ for each $m < k$, $\{z \in R_i : \rho(z, \mathrm{bd}(I_{R_i})) \leq 3^{(2N-2k+1)(2N-2k)}\} \cap V(\cup_{m<k} p_m) = \emptyset$. Define l_i to be a shortest line segment in $\{z \in R_i : \rho(z, \mathrm{bd}(A_i)) \leq 3^{(2N-2k+1)(2N-2k)}\}$ that connects x to y, and replace the original path connecting x and y with l_i.

We therefore have a line segment s that connects the zeroth point of x_{k-1} to y_k and satisfies $(1)_k - (5)_k$, and satisfies $(2)_k - (4)_k$ if $3^{(2N-2k)^2}$ is replaced by $3^{(2N-2k+1)(2N-2k)}$. With this extra space, we can add in $n-1$ lines adjacent to s to form the full tight lining p_k satisfying $(1)_k - (5)_k$. □

We now give a proof of the existence of a Borel n-lining of $F(2^{\mathbb{Z}^2})$. At each step, we will use the buffer to draw "spirals" around the interior, and leave some room to connect two different regions at the next level.

THEOREM 7.10. *There is a tight Borel n-lining of $F(2^{\mathbb{Z}^2})$*

PROOF. We define the L_k inductively so that for each k, L_k will satisfy the three clauses below. For each region R, of $\mathcal{R}_i$, t_R is the number of times that R has been intersected by a region R' of $\mathcal{R}_j$ for $j > i$. We note that by the orthogonal marker construction, t_R can be at most 2.

(1) If x, y are in the interior of some $m-$region R' and there is no j with $m < j \leq k$ such that there are different j-regions R_1 and R_2 with $x \in R_1, y \in R_2$, then there is a path connecting x and y that is contained in $I_{R'}$.
(2) The endpoints of $L_k \restriction R$ lie on the boundary of $\{x \in R : \rho(x, I_R) \leq 300n\}$.
(3) Any horizontal line H intersecting R induces loops l_1 and l_2 such that l_1 is contained in I_R^t above H, every $x \in V(L_k) \cap I_R$ is in the region created by l_1 and H, and l_1 has distance at least $100n - 25nt_R$ from any other loop. l_2 is defined similarly to l_1, but replacing "above" with "below". Furthermore, a similar claim holds for any vertical line intersecting R.

Let R be a region of $\mathcal{R}_0$. Then R is a rectangle with side lengths d_0 or $d_0 + 1$. Let

L'_0 in R be n adjacent vertical lines with endpoints on I_R. The zeroth line can be picked to go through a well-defined center point so that L'_0 is Borel. We define the "spiral" part of L_0 below, giving an algorithm that yields the structure given in Figure 7.2. Let S_0 be a tight line segment which shares an endpoint with the top part of L'_0 which spirals around I_R in the following sense. S_0 moves up $10n$, right until it extends $100n$ past the left edge of I_R, down until it extends $100n$ below I_R, right until it is $200n$ to the left of I_R, up until it is $200n$ above I_R, left until it is $300n$ to the right of I_R, and finally down to the boundary of I^0_R. Define S_1 similarly, and let $L_0 = S_0 \cup L'_0 \cup S_1$. $(1)_0$ is trivially satisfied, while $(2)_0$ and $(3)_0$ are true by construction.

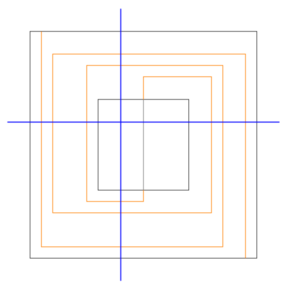

FIGURE 7.2. We define a lining in the interior of R, and then let S spiral around so that L_0 satisfies $(3)_0$

.

Now suppose that L_{k-1} has been defined, let R be an arbitrary region of $\mathcal{R}_k$, and let $L'_{k-1} = \{(x,y) \in L_{k-1} \restriction R : x, y \in I_R\}$. $V(L'_{k-1})$ is the collection of points of L_{k-1} that are not in the buffer of R. If S is a region of $\mathcal{R}_m$ intersecting R, where $0 \leq m < k$ then we let $S' = \{(x,y) \in S : x, y \in I_R\}$. and define $I_{S'}$, and $B_{S'}$, similarly. We will call S' a *perimeter region* of R. When we perform a construction involving a perimeter region, we will use S' in place of S, so that $S' \subseteq R$.

Let $P_0, P_1, \ldots P_m$ enumerate without repetition all of the perimeter $(k-1)$-regions of R such that for each $i < m$, P_i and P_{i+1} are adjacent. Inductively, suppose that for all $s \in \omega^{<\omega}$ with $lh(s) = j$, $P_{\bar{s}}$ is a perimeter $k - lh(s)$ region, and let $P_{\bar{s}^\frown 0}, P_{\bar{s}^\frown 1} \ldots P_{\bar{s}^\frown n_{\bar{s}}}$ be an enumeration of perimeter regions contained in $P_{\bar{s}}$ such that $P_{\bar{s}^\frown 0}$ is the closest region to the boundary of $P_{\bar{s}}$ and for all i, $P_{\bar{s}^\frown i}$ and $P_{\bar{s}^\frown (i+1)}$ are adjacent regions. By $(3)_{j-1}$, $P_{\bar{s}}$ has a loop $l_{\bar{s}}$ in $A_{\bar{s}}$ which is at least $100 - 25t_{\bar{s}}$ away from $I_{P_{\bar{s}}}$

For each $i \leq m$, we may assume P_i (a $k-1$-region) has a good loop structure by possibly extending the boundary of I_{P_i} by n using Lemma 7.3. Thus, there is a lining L_i of P_i satisfying the hypothesis of lemma 7.1. In particular, for each perimeter region $P_{\bar{s}}$, L_i induces four loops $l^{\bar{s}}_{a_1}$, $l^{\bar{s}}_{a_2}$, $l^{\bar{s}}_{b_1}$ and $l^{\bar{s}}_{b_2}$ in $A_{\bar{s}}$ which are at least $4n$ units apart, and there are no points in the area created by any two loops and H. We now use the $P_{\bar{s}}$ to define the spiral set for R. Let $H^{\bar{s}}_1 = l_1 \cup h \cup l_2$, where each set is defined as follows:

- l_1 is a cable whose outer loop is distance 1 from l_{a_1}, whose left cut is $150n$ units below $I_{P_{\bar{s}}}$, and whose right cut extends $4n$ below $I_{P_{\bar{s}}}$.
- l_2 is a loop below l_1 whose outer loop is distance 1 from l_1 with cuts that are both 1 unit away from the cuts of l_2.
- h is a cable connecting the right cuts of l_1 and l_2.

$H^{\bar{s}}_1$ does not intersect L'_{n-1}, l_{a_1}, or l_{a_2}. Define $H^{\bar{s}}_2$ similarly using b_1 and b_2 for $P_{\bar{s}}$. Let $S_{\bar{s}} = H^{\bar{s}}_1 \cup h \cup H^{\bar{s}}_2$, where h is the horizontal cable connecting the right cut of $H^{\bar{s}}_1$ to the left cut of $H^{\bar{s}}_2$. For each $\bar{s}$, let $\bar{t}$ be least such that $\bar{t} \geq_{lex} \bar{s}$, and define $C_{\bar{s}}$ to be a cable that connects the right cut of $S_{\bar{s}}$ to $S_{\bar{t}}$ which stays $300n$ away from the interiors of the two regions. For the maximal $\bar{t}$, have $C_{\bar{t}}$, connect a cut of the lining in $P_{\bar{t}}$ to $S_{\bar{t}}$. We now define S by

$$S = \bigcup_{\bar{s} \in \omega^{<\omega}} S_{\bar{s}} \cup C_{\bar{s}}$$

Let $P_0 = R_0, R_1, \ldots R_N$ be an enumeration of the $k-1$-regions of R. Apply lemma 7.9 to these regions. This will produce a lining L'_n. We extend L'_n by adding two line segments S^k_0 and S^k_1 that are defined similarly to S_0 and S_1 from the base case using the free cuts of S and P_0.

We now claim the lining L_n satisfies $(1)_k - (3)_k$. $(3)_k$ holds for R itself since S spirals around A_R. Let T be an m-region for $m < k$. If T is not a perimeter region of R, then we only added points to B_T, so $(3)_k$ holds because it held for T at the previous step. If T is a perimeter region of R, then without loss of generality, suppose R intersects T horizontally

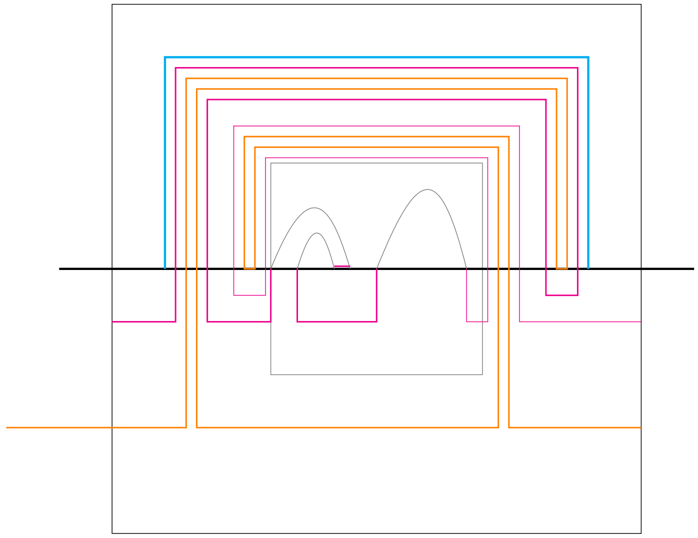

FIGURE 7.3. S (The orange line) enters in the free spaces between the loops set up by lemma 7.5

and T lies above the boundary of R. Let H be a horizontal line intersecting I_T. Then S induces loops above and below H in I_T. The argument is similar for any vertical line.

We next show that $(1)_k$ holds. Let $m \leq k$, and suppose x and y are points of L_k in a region R_m of $\mathcal{R}_m$ satisfying the hypothesis of $(1)_k$. If $m = k$, we are done, as we connected the linings in all of the lower level regions. If $m < k$, and R_m is not a perimeter region of R, then the lining inside of I_{R_m} was unchanged, so $(1)_k$ holds because it was true at the previous step. Finally, if R_m is a perimeter region, then the lining connecting x and y stayed within $3n$ of I_{R_m} by lemma 7.6. $(2)_k$ holds from the definition of S.

We define $\mathcal{L}$ as follows:

$$\mathcal{L} = \bigcup_{m \in \omega} \bigcap_{n > m} L_n,$$

so $\mathcal{L}$ is Borel. For any $x \in F(2^{\mathbb{Z}^2})$ and $m \in \omega$, the lining within a rectangle of side length m containing x will be eventually constant, as this rectangle can only be intersected by a finite number of $\mathcal{R}_i$ regions. We now show that in each class, $\mathcal{L}$ has exactly n-lines. It is enough to show that there is exactly one tight n-line. Suppose $x, y \in V(\mathcal{L})$ are in the same equivalence class of $F(2^{\mathbb{Z}^2})$. By Theorem 7.1, there is some least N such that for all $m \geq N$, x and y are in the same region of $\mathcal{R}_m$, and suppose R is the region containing x and y. Then the n-lining containing x and y remained in I_R. If at some stage m, R was intersected by a higher level region S, then by lemma 7.6, and $(3)_m$, the rewiring stayed inside of $\{x : \rho(x, I_R) < b\}$. If the lining was intersected again, then the rewiring must still stay inside of the previous set. Thus, if x and y were connected, the path connecting them is eventually constant, so x and y are connected in $\mathcal{L}$ by that path. If x and y were not connected then they would necessarily stay disconnected at later stages. $\square$

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