Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **12** (2023) 154–158
DOI: 10.47443/dml.2023.131

Research Article

Some results about ID-path-factor critical graphs

Zhiren Sun[1], Sizhong Zhou[2,*]

[1]*School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China*
[2]*School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*

(Received: 27 August 2023. Received in revised form: 25 October 2023. Accepted: 2 November 2023. Published online: 13 November 2023.)

Abstract

Let G be a graph of order n. A spanning subgraph F of G is said to be a $P_{\geq k}$-factor of G if every component of F is a path with at least k vertices, where $k \geq 2$. In this paper, we introduce the concept of an ID-$P_{\geq k}$-factor critical graph; a graph G is said to be an ID-$P_{\geq k}$-factor critical graph if for any independent set I of G, $G - I$ admits a $P_{\geq k}$-factor. We prove that a graph G of a given order is an ID-$P_{\geq 2}$-factor critical graph if its binding number is at least 2. We also prove that a graph G of a fixed order is an ID-$P_{\geq 3}$-factor critical graph if its binding number is at least $\frac{9}{4}$. Furthermore, we show that the obtained results are the best possible in some sense.

Keywords: graph; independent set; binding number; $P_{\geq k}$-factor; ID-$P_{\geq k}$-factor critical graph.

2020 Mathematics Subject Classification: 05C70, 05C38.

1. Introduction

In this work, we consider only undirected finite graphs which have no multiple edges and no loops. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For every $x \in V(G)$, $d_G(x)$ denotes the degree of x in G and $N_G(x)$ denotes the neighborhood of x in G. We define $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For any $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X, and we define $N_G(X) = \cup_{x \in X} N_G(x)$ and $G - X = G[V(G) \setminus X]$. A set $Y \subseteq V(G)$ is independent if $G[Y]$ has no edges. We denote by $I(G)$ the set of isolated vertices of G and by $i(G)$ the number of isolated vertices of G. We use K_n to denote the complete graph of order n, and use $K_{n,m}$ to denote the complete bipartite graph with partite sets A and B, with $|A| = n$, $|B| = m$, and $A \cup B = V(K_{n,m})$. Let G_1 and G_2 be two graphs. The graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$ is denoted by $G_1 \vee G_2$.

The binding number of a graph was first introduced by Woodall [15], and is defined as follows:

$$bind(G) = min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Lemma 1.1 (see [15]). *Let G be a graph of order n and let β be a positive real. If $bind(G) \geq \beta$, then $\delta(G) \geq n - \frac{n-1}{\beta}$.*

A spanning subgraph F of G is said to be a path factor of G if all components of F are paths. We denote by P_k the path with k vertices, and write $P_{\geq k} = \{P_i | i \geq k\}$, where k is a positive integer. Therefore, a $P_{\geq k}$-factor means a path factor, each component of which is a path with at least k vertices. It is easy to see that a perfect matching is a $P_{\geq 2}$-factor with each component being P_2. A graph G is called an ID-$P_{\geq k}$-factor critical graph if for any independent set I of G, $G - I$ admits a $P_{\geq k}$-factor.

Let R be a graph. If $R - u$ has a perfect matching for every vertex u of R, then R is called factor-critical. A graph H is called a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with at least three vertices, i.e., H is obtained from R by adding a new vertex $w = w(u)$ together with a new edge uw for any $u \in V(R)$. A sun with at least six vertices is called a big sun. We denote the number of sun components of a graph G by $sun(G)$.

Vergnas [10] posed a criterion for a graph having a $P_{\geq 2}$-factor. Matsubara, Matsumura, Tsugaki and Yamashita [9] provided a degree sum condition for the existence of path factors in bipartite graphs. Kelmans [8] obtained some results about path factors in claw-free graphs. Zhang and Zhou [17] gave two necessary and sufficient conditions for graphs to have $P_{\geq k}$-factors containing any given edge e, where $k = 2, 3$. Zhou, Wu and Xu [29] presented two results on the existence of path factors in graphs with prescribed properties. In [13, 16, 20, 21, 26, 27] several results on the graphs admitting path

*Corresponding author (zsz_cumt@163.com).

factors were derived. The relationships between binding number and graph factors can be found in [1, 7, 11, 18]. For more results on graph factors, we refer the reader to [2, 6, 12, 14, 19, 22–25, 28, 30].

Vergnas [10] proved a necessary and sufficient condition for a graph having a $P_{\geq 2}$-factor.

Theorem 1.1 (see [10]). *A graph G has a $P_{\geq 2}$-factor if and only if $i(G-X) \leq 2|X|$ for any $X \subseteq V(G)$.*

Kaneko [3] obtained a necessary and sufficient condition for a graph having a $P_{\geq 3}$-factor, which is very useful in the proof of our main theorems; Kano, Katona and Király [5] provided a simpler proof for the mentioned result of Kaneko.

Theorem 1.2 (see [3, 5]). *A graph G has a $P_{\geq 3}$-factor if and only if $sun(G-X) \leq 2|X|$ for any $X \subseteq V(G)$.*

A claw is a graph isomorphic to $K_{1,3}$; namely, the graph with four vertices and three edges having a common end-vertex. A graph is called claw-free if it contains no induced subgraph isomorphic to a claw. Kaneko, Kelmans and Nishimura [4] proved the following result on the existence of a P_3-factor in a claw-free graph.

Theorem 1.3 (see [4]). *Suppose that G is a 2-connected claw-free graph of order n. If $n \equiv 1$ (mod 3), then $G - \{x\}$ has a P_3-factor for some $x \in V(G)$.*

Kelmans [8] proved that if we replace 2-connected by 3-connected in Theorem 1.3, then a stronger claim holds.

Theorem 1.4 (see [8]). *Suppose that G is a 3-connected claw-free graph of order n. If $n \equiv 1$ (mod 3), then $G - \{x\}$ has a P_3-factor for any $x \in V(G)$.*

Theorem 1.5 (see [8]). *Suppose that G is a 3-connected claw-free graph of order n. If $n \equiv 2$ (mod 3), then $G - \{x, y\}$ has a P_3-factor for every edge xy in G.*

Note that $\{x\}$ is an independent set of G for any $x \in V(G)$. Motivated by Theorems 1.3 and 1.4, we consider the following more general question: For any independent set I of a graph G, does $G - I$ have a path factor? In other words, is a graph G an ID-path factor critical graph?

In this paper, we provide two binding-number conditions for graphs to be ID-$P_{\geq k}$-factors critical graphs when $k = 2, 3$; the results about these conditions are proved in Sections 2 and 3, respectively. Our main results imply that the answer to the above question is positive.

2. Binding numbers and ID-$P_{\geq 2}$-factor critical graphs

Theorem 2.1. *A graph G of order n is an ID-$P_{\geq 2}$-factor critical graph if its binding number $bind(G) \geq 2$.*

Remark 2.1. We show that the binding number condition $bind(G) \geq 2 = \frac{4}{2}$ in Theorem 2.1 cannot be replaced by $bind(G) \geq \frac{4}{3}$. In order to demonstrate this, we construct a graph $G = (3K_1 \vee K_1) \vee 3K_1$. It is obvious that $bind(G) = \frac{4}{3}$. Set $I = V(3K_1)$ and $H = G - I = 3K_1 \vee K_1$. Let $X = V(K_1)$. Thus, we obtain

$$i(H - X) = i(3K_1) = 3 > 2 = 2|X|.$$

In light of Theorem 1.1, $H = G - I$ has no $P_{\geq 2}$-factor. Hence, G is not an ID-$P_{\geq 2}$-factor critical graph.

Proof of Theorem 2.1. For an independent set I of G, take $H = G - I$. Suppose that the result is not true. Then by Theorem 1.1, there exists a vertex subset X of H satisfying

$$i(H - X) > 2|X|. \tag{1}$$

It follows from (1) that $i(H - X) \geq 1$, which implies $N_G(V(G) \setminus (I \cup X)) = N_G(V(H) \setminus X) \neq V(G)$. Combining this with the definition of $bind(G)$ and the hypothesis of Theorem 2.1, we have

$$2 \leq bind(G) \leq \frac{|N_G(V(G) \setminus (I \cup X))|}{|V(G) \setminus (I \cup X)|} \leq \frac{|V(G)| - i(G - I - X)}{|V(G)| - |I| - |X|} = \frac{n - i(H - X)}{n - |I| - |X|},$$

that is,

$$i(H - X) \leq 2|I| + 2|X| - n. \tag{2}$$

In order to complete the proof of Theorem 2.1, we first prove the following claim.

Claim 2.1. $|I| \leq \frac{n-1}{2}$.

Proof of Claim 2.1. Since I is an independent set of G and $d_G(x) \geq \delta(G)$ for every $x \in I$, we have

$$n \geq d_G(x) + |I| \geq \delta(G) + |I|. \quad (3)$$

In terms of (3) and Lemma 1.1, we get

$$n \geq \delta(G) + |I| \geq n - \frac{n-1}{2} + |I|,$$

which implies

$$|I| \leq \frac{n-1}{2}.$$

Thus, Claim 2.1 is verified. ■

It follows from (2) and Claim 2.1 that

$$i(H - X) \leq 2|I| + 2|X| - n \leq n - 1 + 2|X| - n = 2|X| - 1,$$

which contradicts (1). Therefore, Theorem 2.1 is verified. □

3. Binding numbers and ID-$P_{\geq 3}$-factor critical graphs

Theorem 3.1. *A graph G of order n is an ID-$P_{\geq 3}$-factor critical graph if its binding number $bind(G) \geq \frac{9}{4}$.*

Remark 3.1. We show that the condition $bind(G) \geq \frac{9}{4}$ in Theorem 3.1 cannot be replaced by $bind(G) \geq \frac{9}{5}$. In order to demonstrate this, we construct a graph $G = (3K_1 \vee K_1) \vee 3K_2$. It is easy to see that $bind(G) = \frac{9}{5}$. Let $I = V(3K_1)$ and $H = G - I$. For $X = V(K_1)$, we have

$$sun(H - X) = sun(3K_2) = 3 > 2 = 2|X|.$$

In terms of Theorem 1.2, $H = G - I$ has no $P_{\geq 3}$-factor. Therefore, G is not an ID-$P_{\geq 3}$-factor critical graph.

Proof of Theorem 3.1. For an independent set I of G, set $H = G - I$. Assume that the result is not true. Then by Theorem 1.2, there exists a vertex subset X of H such that

$$sun(H - X) > 2|X|. \quad (4)$$

In order to complete the proof of Theorem 3.1, we first prove the following claim.

Claim 3.1. $|I| \leq \frac{4}{9}(n - 1)$.

Proof of Claim 3.1. Since $d_G(x) \geq \delta(G)$ for every $x \in I$, where I is an independent set of G, we conclude

$$n \geq d_G(x) + |I| \geq \delta(G) + |I|. \quad (5)$$

According to (5) and Lemma 1.1, we have

$$|I| \leq n - \delta(G) \leq n - (n - \frac{4}{9}(n-1)) = \frac{4}{9}(n-1).$$

This completes the proof of Claim 3.1. □

Assume that there exist "a" isolated vertices, bK_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, in $H - X$. Obviously, we obtain

$$sun(H - X) = a + b + c. \quad (6)$$

Using (4) and (6), we get

$$a + b + c = sun(H - X) \geq 2|X| + 1 \geq 1. \quad (7)$$

In the following, we consider two cases depending on whether $a = 0$ or not.

Case 1. $a \geq 1$.
It is obvious that $i(G - (I \cup X)) = i(H - X) = a \geq 1$. By the definition of $bind(G)$ and the hypothesis of Theorem 3.1, we have

$$\frac{9}{4} \leq bind(G) \leq \frac{|N_G(V(G) \setminus (I \cup X))|}{|V(G) \setminus (I \cup X)|} \leq \frac{n - a}{n - |I| - |X|},$$

that is,

$$0 \geq \frac{5}{4}n - \frac{9}{4}|I| - \frac{9}{4}|X| + a. \quad (8)$$

Note that $n \geq |I| + |X| + a + 2b + 6c$. In view of (4), (7), (8) and Claim 3.1, we obtain

$$\begin{aligned}
0 &\geq \frac{5}{4}n - \frac{9}{4}|I| - \frac{9}{4}|X| + a \\
&= \frac{5}{4}n - \frac{9}{5}|I| - \frac{9}{20}|I| - \frac{9}{4}|X| + a \\
&\geq \frac{5}{4}n - \frac{9}{5} \cdot \frac{4}{9}(n-1) - \frac{9}{20}|I| - \frac{9}{4}|X| + a \\
&= \frac{9}{20}n - \frac{9}{20}|I| - \frac{9}{4}|X| + a + \frac{4}{5} \\
&\geq \frac{9}{20}(|I| + |X| + a + 2b + 6c) - \frac{9}{20}|I| - \frac{9}{4}|X| + a + \frac{4}{5} \\
&= \frac{9}{20}(a + 2b + 6c) - \frac{9}{5}|X| + a + \frac{4}{5} \\
&> \frac{9}{20}(2a + 2b + 2c) - \frac{9}{5}|X| + \frac{4}{5} \\
&> \frac{9}{20}(2sun(H-X)) - \frac{9}{5} \cdot \frac{sun(H-X)}{2} + \frac{4}{5} \\
&= \frac{4}{5},
\end{aligned}$$

which is a contradiction.

Case 2. $a = 0$.
By (7), we get $b + c \geq 1$. Hence, there exist two vertices x, y of $H' = bK_2 \cup H_1 \cup H_2 \cup \cdots \cup H_c$ such that $d_{H'}(x) = 1$ and $xy \in E(H')$. Thus, we have

$$i(G - (I \cup X \cup \{y\})) = i(H - (X \cup \{y\})) = 1.$$

Combining this with the definition of $bind(G)$ and the hypothesis of Theorem 3.1, we obtain

$$\frac{9}{4} \leq bind(G) \leq \frac{|N_G(V(G) \setminus (I \cup X \cup \{y\}))|}{|V(G) \setminus (I \cup X \cup \{y\})|} \leq \frac{n-1}{n - |I| - |X| - 1},$$

which implies

$$0 \geq \frac{5}{4}n - \frac{9}{4}|I| - \frac{9}{4}|X| - \frac{5}{4}. \tag{9}$$

Note that $a = 0$, and so $n \geq |I| + |X| + 2b + 6c$. It follows from (7), (9) and Claim 3.1 that

$$\begin{aligned}
0 &\geq \frac{5}{4}n - \frac{9}{4}|I| - \frac{9}{4}|X| - \frac{5}{4} \\
&= \frac{5}{4}n - \frac{9}{5}|I| - \frac{9}{20}|I| - \frac{9}{4}|X| - \frac{5}{4} \\
&\geq \frac{5}{4}n - \frac{9}{5} \cdot \frac{4}{9}(n-1) - \frac{9}{20}|I| - \frac{9}{4}|X| - \frac{5}{4} \\
&= \frac{9}{20}n - \frac{9}{20}|I| - \frac{9}{4}|X| - \frac{9}{20} \\
&\geq \frac{9}{20}(|I| + |X| + 2b + 6c) - \frac{9}{20}|I| - \frac{9}{4}|X| - \frac{9}{20} \\
&= \frac{9}{10}(b + 3c) - \frac{9}{5}|X| - \frac{9}{20} \\
&\geq \frac{9}{10}(b + c) - \frac{9}{5}|X| - \frac{9}{20} \\
&\geq \frac{9}{10}sun(H-X) - \frac{9}{5} \cdot \frac{sun(H-X) - 1}{2} - \frac{9}{20} \\
&= \frac{9}{20},
\end{aligned}$$

which is a contradiction. Thus, Theorem 3.1 holds. □

Acknowledgment

The authors would like to express their gratitude to the anonymous reviewers for their helpful comments and valuable suggestions, which improved the quality of this paper.

References

[1] W. Gao, W. Wang, Y. Chen, Tight bounds for the existence of path factors in network vulnerability parameter settings, *Int. J. Intel. Syst.* **36** (2021) 1133–1158.
[2] W. Gao, W. Wang, J. Guirao, The extension degree conditions for fractional factor, *Acta Math. Sin.* (*Engl. Ser.*) **36** (2020) 305–317.
[3] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, *J. Combin. Theory Ser. B* **88** (2003) 195–218.
[4] A. Kaneko, A. Kelmans, T. Nishimura, On packing 3-vertex paths in a graph, *J. Graph Theory* **36** (2001) 175–197.
[5] M. Kano, G. Katona, Z. Király, Packing paths of length at least two, *Discrete Math.* **283** (2004) 129–135.
[6] M. Kano, A. Saito, Star-factors with large component, *Discrete Math.* **312** (2012) 2005–2008.
[7] M. Kano, N. Tokushige, Binding numbers and f-factors of graphs, *J. Combin. Theory Ser. B* **54** (1992) 213–221.
[8] A. Kelmans, Packing 3-vertex paths in claw-free graphs and related topics, *Discrete Appl. Math.* **159** (2011) 112–127.
[9] R. Matsubara, H. Matsumura, M. Tsugaki, T. Yamashita, Degree sum conditions for path-factors with specified end vertices in bipartite graphs, *Discrete Math.* **340** (2017) 87–95.
[10] M. L. Vergnas, An extension of Tutte's 1-factor theorem, *Discrete Math.* **23** (1978) 241–255.
[11] S. Wang, W. Zhang, Research on fractional critical covered graphs, *Probl. Inf. Transm.* **56** (2020) 270–277.
[12] S. Wang, W. Zhang, On k-orthogonal factorizations in networks, *RAIRO Oper. Res.* **55** (2021) 969–977.
[13] S. Wang, W. Zhang, Isolated toughness for path factors in networks, *RAIRO Oper. Res.* **56** (2022) 2613–2619.
[14] S. Wang, W. Zhang, Degree conditions for the existence of a $\{P_2, P_5\}$-factor in a graph, *RAIRO Oper. Res.* **57** (2023) 2231–2237.
[15] D. Woodall, The binding number of a graph and its Anderson number, *J. Combin. Theory Ser. B* **15** (1973) 225–255.
[16] J. Wu, Path-factor critical covered graphs and path-factor uniform graphs, *RAIRO Oper. Res.* **56** (2022) 4317–4325.
[17] H. Zhang, S. Zhou, Characterizations for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, *Discrete Math.* **309** (2009) 2067–2076.
[18] S. Zhou, A note of generalization of fractional ID-factor-critical graphs, *Fund. Inform.* **187** (2022) 61–69.
[19] S. Zhou, A neighborhood union condition for fractional (a, b, k)-critical covered graphs, *Discrete Appl. Math.* **323** (2022) 343–348.
[20] S. Zhou, Some results on path-factor critical avoidable graphs, *Discuss. Math. Graph Theory* **43** (2023) 233–244.
[21] S. Zhou, Degree conditions and path factors with inclusion or exclusion properties, *Bull. Math. Soc. Sci. Math. Roumanie* **66** (2023) 3–14.
[22] S. Zhou, Remarks on restricted fractional (g, f)-factors in graphs, *Discrete Appl. Math.*, DOI: 10.1016/j.dam.2022.07.020, In press.
[23] S. Zhou, H. Liu, Characterizing an odd $[1, b]$-factor on the distance signless Laplacian spectral radius, *RAIRO Oper. Res.* **57** (2023) 1343–1351.
[24] S. Zhou, H. Liu, Two sufficient conditions for odd $[1, b]$-factors in graphs, *Linear Algebra Appl.* **661** (2023) 149–162.
[25] S. Zhou, H. Liu, Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs, *Discrete Appl. Math.* **319** (2022) 511–516.
[26] S. Zhou, Z. Sun, H. Liu, Some sufficient conditions for path-factor uniform graphs, *Aequationes Math.* **97** (2023) 489–500.
[27] S. Zhou, J. Wu, Q. Bian, On path-factor critical deleted (or covered) graphs, *Aequationes Math.* **96** (2022) 795–802.
[28] S. Zhou, J. Wu, H. Liu, Independence number and connectivity for fractional (a, b, k)-critical covered graphs, *RAIRO Oper. Res.* **56** (2022) 2535–2542.
[29] S. Zhou, J. Wu, Y. Xu, Toughness, isolated toughness and path factors in graphs, *Bull. Aust. Math. Soc.* **106** (2022) 195–202.
[30] S. Zhou, Y. Zhang, Sufficient conditions for graphs to have strong parity factors, *RAIRO Oper. Res.* **57** (2023) 2465–2471.