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Research article

The reverse order law for the weighted least square g-inverse of multiple matrix products

Baifeng Qiu and Zhiping Xiong*

School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, Guangdong, China

* **Correspondence:** Email: xzpwhere@163.com.

Abstract: By using the ranks of the generalized Schur complement, the equivalent conditions for reverse order laws of the $\{1, 3M\}-$ and the $\{1, 4N\}-$ inverses of the multiple product of matrices are derived.

Keywords: reverse order law; weighted least square g-inverse; matrix product; generalized Schur complement; maximal and minimal ranks
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1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ matrices in the complex field. For any $A \in \mathbb{C}^{m\times n}$, we denote the conjugate transpose, the rank, the range space and the null space of A by A^*, $r(A)$, $\mathbb{R}(A)$ and $\mathbb{N}(A)$, respectively. In the remainder of this paper, we will adopt

$$\mathbb{A}_i^j = A_i A_{i+1} \cdots A_j, \qquad \mathbb{X}_i^j = X_i^* X_{i+1}^* \cdots X_j^*, \qquad 1 \le i \le j \le n, \tag{1.1}$$

where $A_i \in \mathbb{C}^{m_i\times m_{i+1}}$, $X_i \in \mathbb{C}^{m_{i+1}\times m_i}$ and X_i is called a weighted generalized inverse of A_i, $i = 1, 2, \cdots, n$. In particular, $\mathbb{A}_j^j = A_j$, $\mathbb{A}_1^j = A_1A_2 \cdots A_j$, $\mathbb{X}_j^j = X_j^*$, $\mathbb{X}_1^j = X_1^* X_2^* \cdots X_j^*$, $j = 1, 2, \ldots, n$ and $\mathbb{X}_{n+1}^n = I_{m_{n+1}}$.

For $A \in \mathbb{C}^{m\times n}$, let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be two positive definite Hermitian matrices. We recall that a weighted generalized inverse $X \in \mathbb{C}^{n\times m}$ of A is a matrix that satisfies some of the following equations [1, 13]:

$$(1)\ AXA = A, \ \ (2)\ XAX = X, \ \ (3M)\ (MAX)^* = MAX, \ \ (4N)\ (NXA)^* = NXA. \tag{1.2}$$

We say that $X = A^{(1,3M)}$ is a $\{1, 3M\}$-inverse or a weighted least squares g-inverse of A if X is a common solution of (1) and $(3M)$. Let $A\{1, 3M\}$ denote the set of all $\{1, 3M\}-$inverses of A. We say that

$X = A^{(1,4N)}$ is a $\{1,4N\}$-inverse of A if X is a common solution of (1) and ($4N$). Let $A\{1,4N\}$ denote the set of all $\{1,4N\}-$inverses of A. The unique $\{1,2,3M,4N\}$-inverse of A ia called the weighted Moore-Penrose inverse of A and is denoted by $X = A^{(1,2,3M,4N)} = A^{\dagger}_{M,N}$ [1,23].

The reverse order law for the weighted generalized inverse of the multiple product of matrices has been widely applied in the theoretic research and numerical computations areas (see [1,2,4,5,8,9,14, 23,24]).

For the very time, Greville [6] presented an equivalent condition for the reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Since then, many authors have studied this problem (see [3,4,7,10,11,15–17,19–22,25,26]). It is well known that the core problem concerns with the reverse order law and whether conditions

$$A_n^{(i,j,\cdots,k)} A_{n-1}^{(i,j,\cdots,k)} \cdots A_1^{(i,j,\cdots,k)} = (A_1 A_2 \cdots A_n)^{(i,j,\cdots,k)} \tag{1.3}$$

hold, or whether conditions

$$A_n\{i,j,\cdots,k\}A_{n-1}\{i,j,\cdots,k\}\cdots A_1\{i,j,\cdots,k\} \subseteq (A_1A_2\cdots A_n)\{i,j,\cdots,k\} \tag{1.4}$$

hold, where $(i,j,\cdots,k) \subseteq \{1,2,3M,4N\}$.

The purpose of this paper is to show some equivalent conditions for the following inclusions

$$A_n\{1,3M_n\}A_{n-1}\{1,3M_{n-1}\}\cdots A_1\{1,3M_1\} \subseteq (A_1A_2\cdots A_n)\{1,3M_1\} \tag{1.5}$$

and

$$A_n\{1,4M_{n+1}\}A_{n-1}\{1,4M_n\}\cdots A_1\{1,4M_2\} \subseteq (A_1A_2\cdots A_n)\{1,4M_{n+1}\}, \tag{1.6}$$

where $A_i \in \mathbb{C}^{m_i \times m_{i+1}}$, $i = 1,2,\ldots,n$ and $M_i \in \mathbb{C}^{m_i \times m_i}$, $i = 1,2,\ldots,n+1$ are $n+1$ positive definite Hermitian matrices.

2. Preliminaries

Lemma 2.1. *[23] Let L, M be two complementary subspaces of $\mathbb{C}^m$ and let $P_{L,M}$ be the projector on L along M, then*

$$P_{L,M}A = A \Longleftrightarrow R(A) \subseteq L, \tag{2.1}$$

$$AP_{L,M} = A \Longleftrightarrow N(A) \supseteq M. \tag{2.2}$$

Lemma 2.2. *[1, 23] Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and let M and N be two positive definite Hermitian matrices of order m and n, respectively, then*

$$X \in A\{1,3M\} \Longleftrightarrow A^*MAX = A^*M, \tag{2.3}$$

$$X \in A\{1,4N\} \Longleftrightarrow XAN^{-1}A^* = N^{-1}A^*, \tag{2.4}$$

$$X \in A\{1,4N\} \Longleftrightarrow X^* \in A^*\{1,3N^{-1}\}. \tag{2.5}$$

Lemma 2.3. *[18] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$, and let $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be two positive definite Hermitian matrices, then*

$$\max_{A^{(1,3M)}} r(D - CA^{(1,3M)}B) = \min\left\{ r\begin{pmatrix} A^*MA & A^*MB \\ C & D \end{pmatrix} - r(A),\ r\begin{pmatrix} B \\ D \end{pmatrix} \right\}, \tag{2.6}$$

$$\max_{A^{(1,4N)}} r(D - CA^{(1,4N)}B) = \min\left\{ r\begin{pmatrix} C, & D \end{pmatrix}, r\begin{pmatrix} AN^{-1}A^* & B \\ CN^{-1}A^* & D \end{pmatrix} - r(A) \right\}. \tag{2.7}$$

Lemma 2.4. *[12] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{p\times n}$, then*

$$r\begin{pmatrix} A, & B \end{pmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2.8}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{2.9}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} \le r(A) + r(C), \quad r\begin{pmatrix} A, & B \end{pmatrix} \le r(A) + r(B), \tag{2.10}$$

where the projectors are $E_A = I_m - AA^\dagger$, $E_B = I_m - BB^\dagger$, $F_A = I_n - A^\dagger A$ and $F_C = I_n - C^\dagger C$.

3. Main results

Let

$$\mathbb{A}_i^j = A_i A_{i+1} \cdots A_j, \ \ \mathbb{X}_i^j = X_i^* X_{i+1}^* \cdots X_j^*, \ \ 1 \le i \le j \le n$$

be as given in (1.1), and $M_i \in \mathbb{C}^{m_i\times m_i}$, $i = 1, 2, \ldots, n+1$ are positive definite Hermitian matrices. Then, from (2.3) in Lemma 2.2, we know that (1.5) holds if, and only if,

$$(\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_1^n)^* = (\mathbb{A}_1^n)^* M_1$$

holds for any $X_i \in A_i\{1, 3M_i\}$, $i = 1, 2, \ldots, n$, which is equivalent to:

$$\max_{X_n, X_{n-1}, \cdots, X_1} r((\mathbb{A}_1^n)^* M_1 - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_1^n)^*) = 0. \tag{3.1}$$

Hence, we can present the equivalent conditions for (1.5) if the concrete expressions of the maximal rank involved in (3.1) are derived.

Theorem 3.1. *Let $A_i \in \mathbb{C}^{m_i\times m_{i+1}}$, $X_i = A_i^{(1,3M_i)} \in A_i\{1, 3M_i\}$ and $i = 1, 2, \ldots, n$. Let $M_i \in \mathbb{C}^{m_i\times m_i}$, $i = 1, 2, \ldots, n+1$ be positive definite Hermitian matrices and let $\mathbb{A}_i^j = A_i A_{i+1} \cdots A_j$, $1 \le i \le j \le n$ be given as in* (1.1)*. Then,*

$$
\begin{aligned}
& A_n\{1,3M_n\}A_{n-1}\{1,3M_{n-1}\}\cdots A_1\{1,3M_1\} \subseteq (A_1A_2\cdots A_n)\{1,3M_1\} \\
\Longleftrightarrow\ & r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1} & \cdots & (\mathbb{A}_1^n)^*M_1A_1M_2^{-1} \end{pmatrix} = \sum_{i=2}^{n} r(A_i).
\end{aligned} \tag{3.2}
$$

Proof. From (2.3) in Lemma 2.2 and the definition of the rank of the matrix, we can see that for any $X_i = A_i^{(1,3M_i)} \in A_i\{1,3M_i\}$, $i = 1,2,\ldots,n$, the following three formulas are equivalent:

$$
A_n\{1,3M_n\}A_{n-1}\{1,3M_{n-1}\}\cdots A_1\{1,3M_1\} \subseteq (A_1A_2\cdots A_n)\{1,3M_1\}, \tag{3.3}
$$

$$
(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^* = (\mathbb{A}_1^n)^*M_1 \tag{3.4}
$$

and

$$
\max_{X_n,X_{n-1},\cdots,X_1} r((\mathbb{A}_1^n)^*M_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*) = 0. \tag{3.5}
$$

□

Let $\mathbb{X}_i^j = X_i^*X_{i+1}^*\cdots X_j^*,\ \ 1 \le i \le j \le n$ as in (1.1). Then, from the formula (2.6) in Lemma 2.3 with $A = A_1$, $B = I_{m_1}$, $C = (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^*$ and $D = (\mathbb{A}_1^n)^*M_1$, we have

$$
\begin{aligned}
& \max_{X_1} r((\mathbb{A}_1^n)^*M_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*) \\
= \ & \min\left\{ r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* & (\mathbb{A}_1^n)^*M_1 \end{pmatrix} - r(A_1),\ \ r\begin{pmatrix} I_{m_1} \\ (\mathbb{A}_1^n)^*M_1 \end{pmatrix} \right\} \\
= \ & \min\left\{ r((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* - (\mathbb{A}_1^n)^*M_1A_1),\ \ m_1 \right\} \\
= \ & r((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* - (\mathbb{A}_1^n)^*M_1A_1),
\end{aligned} \tag{3.6}
$$

in which by the row or column elementary block operations from the first equality to the second one, we use the rank identities

$$
r\begin{pmatrix} I_{m_1} \\ (\mathbb{A}_1^n)^*M_1 \end{pmatrix} = m_1,
$$

$$
\begin{aligned}
r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* & (\mathbb{A}_1^n)^*M_1 \end{pmatrix} &= r\begin{pmatrix} O & A_1^*M_1 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* - (\mathbb{A}_1^n)^*M_1A_1 & O \end{pmatrix} \\
&= r((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* - (\mathbb{A}_1^n)^*M_1A_1) + r(A_1)
\end{aligned}
$$

and

$$r((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^* - (\mathbb{A}_1^n)^*M_1A_1) \leq r((\mathbb{A}_1^n)^*) \leq r(A_1) \leq m_1.$$

From (3.6) and again by (2.6) in Lemma 2.3 with $A = A_2$, $B = I_{m_2}$, $C = (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^*$ and $D = (\mathbb{A}_1^n)^*M_1A_1$, we have

$$\begin{aligned}
&\max_{X_2,X_1} r((\mathbb{A}_1^n)^*M_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*) \\
= &\max_{X_2} r((\mathbb{A}_1^n)^*M_1A_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^*) \\
= &\min\left\{ r\begin{pmatrix} A_2^*M_2A_2 & A_2^*M_2 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* & (\mathbb{A}_1^n)^*M_1A_1 \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} I_{m_2} \\ (\mathbb{A}_1^n)^*M_1A_1 \end{pmatrix} \right\} \\
= &\min\left\{ r\begin{pmatrix} O & A_2^*M_2 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2 & (\mathbb{A}_1^n)^*M_1A_1 \end{pmatrix} - r(A_2),\ \ m_2 \right\} \\
= &\min\left\{ r\begin{pmatrix} O & A_2^* \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2 & (\mathbb{A}_1^n)^*M_1A_1M_2^{-1} \end{pmatrix} - r(A_2),\ \ m_2 \right\}.
\end{aligned} \tag{3.7}$$

By (2.9) in Lemma 2.4 we have $\begin{pmatrix} O, & A_2^* \end{pmatrix}^\dagger = \begin{pmatrix} O \\ (A_2^*)^\dagger \end{pmatrix}$, thus

$$\begin{aligned}
&r\begin{pmatrix} O & A_2^* \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2 & (\mathbb{A}_1^n)^*M_1A_1M_2^{-1} \end{pmatrix} \\
= \ & r[\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2,\ \ (\mathbb{A}_1^n)^*M_1A_1M_2^{-1}\right)F_{(O,\ A_2^*)}] + r(A_2) \\
= \ & r[\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2,\ \ (\mathbb{A}_1^n)^*M_1A_1M_2^{-1}\right)(I - (O,\ A_2^*)^\dagger(O,\ A_2^*))] + r(A_2) \\
= \ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2,\ \ (\mathbb{A}_1^n)^*M_1A_1M_2^{-1}F_{A_2^*}\right) + r(A_2) \\
= \ & r[(\mathbb{A}_1^n)^*\left(M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - M_1\mathbb{A}_1^2,\ \ M_1A_1M_2^{-1}F_{A_2^*}\right)] + r(A_2) \\
\leq \ & r((\mathbb{A}_1^n)^*) + r(A_2) \\
\leq \ & r(A_2^*) + r(A_2) \\
\leq \ & m_2 + r(A_2).
\end{aligned} \tag{3.8}$$

Combining (3.6) and (3.7) with (3.8), we have

$$\begin{aligned}
&\max_{X_2,X_1} r((\mathbb{A}_1^n)^*M_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*) \\
= &\max_{X_2} r((\mathbb{A}_1^n)^*M_1A_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_2^n)^*) \\
= \ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_3^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^2,\ \ (\mathbb{A}_1^n)^*M_1A_1M_2^{-1}F_{A_2^*}\right).
\end{aligned} \tag{3.9}$$

Generally, for $2 \leq i \leq n$, we can prove the following fact:

$$\begin{aligned}
&\max_{X_i,X_{i-1},\cdots,X_1} r((\mathbb{A}_1^n)^*M_1 - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*) \\
= \ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_{i+1}^n)^* - (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^i,\ \ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{i-1}M_i^{-1}F_{A_i^*},\ \ \cdots,\ \ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}F_{A_2^*}\right)
\end{aligned}$$

$$= \; r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_{i+1}^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^i, \;\; (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1} M_i^{-1} F_{A_i^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big), \tag{3.10}$$

where $\mathbb{A}_1^1 = A_1$ and $\mathbb{X}_{n+1}^n = I_{m_{n+1}}$.

In fact, (3.10) is true for $i = 2$ (see (3.9)). Now, assume (3.10) is also true for $i-1 (i \geq 3)$, i.e

$$\begin{aligned} & \max_{X_{i-1}, X_{i-2}, \cdots, X_1} r((\mathbb{A}_1^n)^* M_1 - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_1^n)^*) \\ = \; & r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_i^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1}, \;\; (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big). \end{aligned} \tag{3.11}$$

Next, we will prove that (3.10) is also true for i. By formula (3.11) and (2.6) in Lemma 2.3 with $\tilde{B} = \Big(I_{m_i}, \;\; O, \;\; \cdots, \;\; O\Big)$, $\tilde{A} = A_i$, $\tilde{C} = (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_{i+1}^n)^*$,
$\tilde{D} = \Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1}, \;\; -(\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; -(\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big)$ and
$\tilde{E} = (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_i^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1}$, we have

$$\begin{aligned} & \max_{X_i, X_{i-1}, \cdots, X_1} r((\mathbb{A}_1^n)^* M_1 - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_1^n)^*) \\ = \; & \max_{X_i} r\Big(\tilde{E}, \;\; (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big) \\ = \; & \max_{X_i} r(\tilde{D} - \tilde{C} X_i \tilde{B}) \\ = \; & \min\left\{ r\begin{pmatrix} \tilde{A}^* M_i \tilde{A} & \tilde{A}^* M_i \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} - r(\tilde{A}), \;\; r\begin{pmatrix} \tilde{B} \\ \tilde{D} \end{pmatrix} \right\} \\ = \; & r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_{i+1}^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^i, \;\; (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1} M_i^{-1} F_{A_i^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big). \end{aligned} \tag{3.12}$$

By the row or column elementary block operations of formula (2.9) in Lemma 2.4 we have,

$$\begin{aligned} r\begin{pmatrix} \tilde{B} \\ \tilde{D} \end{pmatrix} \; &= \; r\begin{pmatrix} I_{m_i} & O & \cdots & O \\ (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1} & -(\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*} & \cdots & -(\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*} \end{pmatrix} \\ &= \; m_i + r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big) \end{aligned}$$

and

$$\begin{aligned} & r\begin{pmatrix} \tilde{A}^* M_i \tilde{A} & \tilde{A}^* M_i \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = r\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = r\begin{pmatrix} \tau_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \\ = \; & r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_{i+1}^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^i, \;\; \eta_i, \;\; \eta_{i-1}, \;\; \cdots, \;\; \eta_2\Big) + r(A_i^*) \\ \leq \; & r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^n (\mathbb{X}_{i+1}^n)^* - (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^i, \;\; (\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-1} M_i^{-1} F_{A_i^*}\Big) \\ & + r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big) \\ & + r(A_i) \\ \leq \; & r(\mathbb{A}_1^n)^* + r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big) + r(A_i) \\ \leq \; & m_i + r\Big((\mathbb{A}_1^n)^* M_1 \mathbb{A}_1^{i-2} M_{i-1}^{-1} F_{A_{i-1}^*}, \;\; \cdots, \;\; (\mathbb{A}_1^n)^* M_1 A_1 M_2^{-1} F_{A_2^*}\Big) + r(A_i), \end{aligned}$$

where

$$T_{11}=\left(A_i^*M_iA_i,\quad A_i^*M_i\right),\ T_{12}=\left(O,\quad\cdots,\quad O\right),\ T_{21}=\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_{i+1}^n)^*,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{i-1}\right),$$
$$T_{22}=\left(-(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{i-2}M_{i-1}^{-1}F_{A_{i-1}^*},\quad\cdots,\quad-(\mathbb{A}_1^n)^*M_1A_1M_2^{-1}F_{A_2^*}\right)$$
$$\tau_{11}=\left(O,\quad A_i^*\right),$$

and

$$\eta_k=(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{k-1}M_k^{-1}F_{A_k^*},\ \ k=2,3,\ldots,i.$$

In particular, when $i=n$, we get $\mathbb{A}_1^1=A_1$, $\mathbb{X}_{n+1}^n=I_{m_{n+1}}$ and

$$\begin{aligned}&\max_{X_n,X_{n-1},\cdots,X_1} r((\mathbb{A}_1^n)^*M_1-(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*)\\
=\ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_{n+1}^n)^*-(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1}F_{A_n^*},\quad\cdots,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}F_{A_2^*}\right)\\
=\ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n-(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1}F_{A_n^*},\quad\cdots,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}F_{A_2^*}\right)\\
=\ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1}F_{A_n^*},\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1}F_{A_{n-1}^*},\quad\cdots,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}F_{A_2^*}\right).\end{aligned}\tag{3.13}$$

Applying (3.13) with Lemma 2.4, we finally have

$$\begin{aligned}&\max_{X_n,X_{n-1},\cdots,X_1} r((\mathbb{A}_1^n)^*M_1-(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^n(\mathbb{X}_1^n)^*)\\
=\ & r\left((\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1}F_{A_n^*},\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1}F_{A_{n-1}^*},\quad\cdots,\quad(\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}F_{A_2^*}\right)\\
=\ & r\begin{pmatrix}A_n^* & O & \cdots & O\\ O & A_{n-1}^* & \cdots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \cdots & A_2^*\\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1} & \cdots & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1M_2^{-1}\end{pmatrix}-\sum_{i=2}^n r(A_i).\end{aligned}\tag{3.14}$$

According to the formulas (1.1), (3.3)–(3.5) and (3.14), we have

$$\begin{aligned}&A_n\{1,3M_n\}A_{n-1}\{1,3M_{n-1}\}\cdots A_1\{1,3M_1\}\subseteq(A_1A_2\cdots A_n)\{1,3M_1\}\\
\Longleftrightarrow\ & r\begin{pmatrix}A_n^* & O & \cdots & O\\ O & A_{n-1}^* & \cdots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \cdots & A_2^*\\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1} & \cdots & (\mathbb{A}_1^n)^*M_1A_1M_2^{-1}\end{pmatrix}=\sum_{i=2}^n r(A_i).\end{aligned}\tag{3.15}$$

From Lemmas 2.1, 2.4 and Theorem 3.1, we have:

Corollary 3.1. *Let $A_i \in \mathbb{C}^{m_i\times m_{i+1}}$, $X_i = A_i^{(1,3M_i)} \in A_i\{1,3M_i\}$. Let $M_i \in \mathbb{C}^{m_i\times m_i}$ be positive definite Hermitian matrices $i = 1,2,\ldots,n+1$, and let $\mathbb{A}_i^j = A_iA_{i+1}\cdots A_j$, $1 \le i \le j \le n$ be given as in* (1.1). *Then, the following statements are equivalent:*

(1) $A_n\{1,3M_n\}A_{n-1}\{1,3M_{n-1}\}\cdots A_1\{1,3M_1\} \subseteq (A_1A_2\cdots A_n)\{1,3M_1\}$;

(2) $r\begin{pmatrix} A_n^*M_n & O & \cdots & O \\ O & A_{n-1}^*M_{n-1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^*M_2 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2} & \cdots & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1 \end{pmatrix} = \sum_{i=2}^n r(A_i)$;

(3) $\mathbb{R}((\mathbb{A}_1^{i-1})^*M_1\mathbb{A}_1^n) \subseteq \mathbb{R}(M_iA_i)$, $\quad i = 2,3,\ldots,n$;

(4) $A_i(A_i)^\dagger_{M_i,I_{m_{i+1}}}M_i^{-1}(\mathbb{A}_1^{i-1})^*M_1\mathbb{A}_1^n = M_i^{-1}(\mathbb{A}_1^{i-1})^*M_1\mathbb{A}_1^n$, $\quad i = 2,3,\ldots,n$.

Proof. According to Theorem 3.1, we get that (1) and (2) are equivalent since

$$\begin{aligned} & r\left[\begin{pmatrix} A_n^*M_n & O & \cdots & O \\ O & A_{n-1}^*M_{n-1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^*M_2 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2} & \cdots & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^1 \end{pmatrix}\begin{pmatrix} M_n^{-1} & O & \cdots & O \\ O & M_{n-1}^{-1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & M_2^{-1} \end{pmatrix}\right] \\ = & r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1}M_n^{-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2}M_{n-1}^{-1} & \cdots & (\mathbb{A}_1^n)^*M_1A_1M_2^{-1} \end{pmatrix}. \end{aligned} \tag{3.16}$$

Next, we will prove (3) $\Longleftrightarrow$ (2). From (3.16) and (2.8) in Lemma 2.4, we have

$$\begin{aligned} & r\begin{pmatrix} A_n^*M_n & O & \cdots & O \\ O & A_{n-1}^*M_{n-1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^*M_2 \\ (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-1} & (\mathbb{A}_1^n)^*M_1\mathbb{A}_1^{n-2} & \cdots & (\mathbb{A}_1^n)^*M_1A_1 \end{pmatrix} \\ = & r\begin{pmatrix} M_nA_n & O & \cdots & O & (\mathbb{A}_1^{n-1})^*M_1\mathbb{A}_1^n \\ O & M_{n-1}A_{n-1} & \cdots & O & (\mathbb{A}_1^{n-2})^*M_1\mathbb{A}_1^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & M_2A_2 & (\mathbb{A}_1^1)^*M_1\mathbb{A}_1^n \end{pmatrix} \\ = & r\begin{pmatrix} E_{M_nA_n}(\mathbb{A}_1^{n-1})^*M_1\mathbb{A}_1^n \\ E_{M_{n-1}A_{n-1}}(\mathbb{A}_1^{n-2})^*M_1\mathbb{A}_1^n \\ \vdots \\ E_{M_2A_2}(\mathbb{A}_1^1)^*M_1\mathbb{A}_1^n \end{pmatrix} + \sum_{i=2}^n r(M_iA_i). \end{aligned} \tag{3.17}$$

According to (3.17) , we have $r(M_iA_i) = r(A_i)$ and (3) $\Longleftrightarrow$ (2) if, and only if,

$$E_{M_iA_i}(\mathbb{A}_1^{i-1})^* M_1\mathbb{A}_1^n = O, \ \ i = 2, 3, \ldots, n. \tag{3.18}$$

From Lemmas 2.1 and 2.4, we have $E_{M_iA_i} = I_{m_i} - (M_iA_i)(M_iA_i)^\dagger = I_{m_i} - P_{\mathbb{R}(M_iA_i),\mathbb{N}(A_i^*M_i)}$ and (3) $\Longleftrightarrow$ (2), where $i = 2, 3, \ldots, n$.

By using formula (3) and Lemma 2.1, we get (3) $\Longleftrightarrow$ (4) since

$$\begin{aligned} A_i(A_i)^\dagger_{M_i,I_{m_{i+1}}} M_i^{-1}(\mathbb{A}_1^{i-1})^* M_1\mathbb{A}_1^n &= P_{\mathbb{R}(A_i(A_i)^\dagger_{M_i,I_{m_{i+1}}}),\mathbb{N}(A_i(A_i)^\dagger_{M_i,I_{m_{i+1}}})} M_i^{-1}(\mathbb{A}_1^{i-1})^* M_1\mathbb{A}_1^n \\ &= P_{\mathbb{R}(A_i),\mathbb{N}(A_i(A_i)^\dagger_{M_i,I_{m_{i+1}}})} M_i^{-1}(\mathbb{A}_1^{i-1})^* M_1\mathbb{A}_1^n \\ &= M_i^{-1}(\mathbb{A}_1^{i-1})^* M_1\mathbb{A}_1^n, \end{aligned}$$

where $i = 2, 3, \ldots, n$. □ □

From Lemma 2.2, we have $X \in A\{1, 4N\} \Leftrightarrow X^* \in A^*\{1, 3N^{-1}\}$, so from Theorem 3.1 and Corollary 3.1, we have

Theorem 3.2. *Let $A_i \in \mathbb{C}^{m_i\times m_{i+1}}$, $X_i = A_i^{(1,4N_{i+1})} \in A_i\{1, 4N_{i+1}\}$. Let $N_i \in \mathbb{C}^{m_i\times m_i}$ be positive definite Hermitian matrices $i = 1, 2, \ldots, n+1$, and let $\mathbb{A}_i^j = A_iA_{i+1}\cdots A_j$, $1 \le i \le j \le n$ be given as in* (1.1)*. Then,*

$$\begin{aligned} & A_n\{1, 4N_{n+1}\}A_{n-1}\{1, 4N_n\}\cdots A_1\{1, 4N_2\} \subseteq (A_1A_2\cdots A_n)\{1, 4N_{n+1}\} \\ \Longleftrightarrow \ & r\begin{pmatrix} A_1^* & O & \cdots & O & N_2\mathbb{A}_2^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \\ O & A_2^* & \cdots & O & N_3\mathbb{A}_3^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{n-1}^* & N_n\mathbb{A}_n^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \end{pmatrix} = \sum_{i=1}^{n-1} r(A_i). \end{aligned}$$

From Lemmas 2.1, 2.4 and Theorem 3.2, we have

Corollary 3.2. *Let $A_i \in \mathbb{C}^{m_i\times m_{i+1}}$, $X_i = A_i^{(1,4N_{i+1})} \in A_i\{1, 4N_{i+1}\}$. Let $N_i \in \mathbb{C}^{m_i\times m_i}$ be positive definite Hermitian matrices $i = 1, 2, \ldots, n+1$, and let $\mathbb{A}_i^j = A_iA_{i+1}\cdots A_j$, $1 \le i \le j \le n$ be given as in* (1.1)*. Then, the following statements are equivalent:*

(1) $A_n\{1, 4N_{n+1}\}A_{n-1}\{1, 4N_n\}\cdots A_1\{1, 4N_2\} \subseteq (A_1A_2\cdots A_n)\{1, 4N_{n+1}\}$;

(2) $r\begin{pmatrix} N_2^{-1}A_1^* & O & \cdots & O & \mathbb{A}_2^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \\ O & N_3^{-1}A_2^* & \cdots & O & \mathbb{A}_3^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & N_n^{-1}A_{n-1}^* & \mathbb{A}_n^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* \end{pmatrix} = \sum_{i=1}^{n-1} r(A_i)$;

(3) $\mathbb{R}(\mathbb{A}_{i+1}^nN_{n+1}^{-1}(\mathbb{A}_1^n)^*) \subseteq \mathbb{R}(N_{i+1}^{-1}A_i^*)$, $\ i = 1, 2, \ldots, n-1$;

(4) $(A_i)^\dagger_{I_{m_i},N_{i+1}} A_i\mathbb{A}_{i+1}^nN_{n+1}^{-1}(\mathbb{A}_1^n)^* = \mathbb{A}_{i+1}^nN_{n+1}^{-1}(\mathbb{A}_1^n)^*$, $\ i = 1, 2, \ldots, n-1$.

4. Conclusions

The reverse order law for the weighted generalized inverses of the multiple product of matrices has been studied in this article by using the ranks of the generalized Schur complement. The work in this paper was a useful tool in many algorithms for the computation of the weighted least squares technique of matrix equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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