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ゲージ場理論の数学的構造

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ゲージ場理論の数学的構造

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Mathematical Structure of Gauge Field Theories

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1 前書き

場の理論 - Field Theory - は，時空の各点で定義された力学変数のなす無限自由度の力学系の理論である．もとより，極微の世界の素粒子現象を記述する素粒子論の基本的な言語であるが，さらに，統計力系やニューラルネットワークにも広い応用を持つ．特にゲージ場が導入され局所ゲージ変換に対する対称性を持ったゲージ場理論は数学的にも大変美しい幾何構造を持つ．我々がどのようなコンテクストで共同研究したか，を示すために場の理論のこの様な特性を少し述べてみよう．

局所的ゲージ対称性とは，次のような概念である．素粒子論では，基本粒子としてクォークとレプトンを考える． 時空の各点ごとに，クォーク場とかレプトン場はそれらが生きている内部空間で値を持つ．もっとも簡単な電磁相互作用の場合では，場は複素空間絶対値と位相で特定される値を持つ．このとき，位相を測る基準は時空の各点ごとに本来，全く自由に考えて良いはずである．従って物質場のある一つの時空上の配位に対して，時空の各点ごとに，位相を与え直す変換 - 局所的ゲージ変換 - を施したもう一つの場の配位も全く同じ物理現象を記述するべきである． このことを理論は局所的ゲージ変換に対して対称性を持つという．このような局所的ゲージ対称性を持った場の理論を構成するには，クォーク場とかレプトン場の様な物質場に加えて，各点ごとの内部空間の関係を与える役割を持つ場を理論に組み込まなければならない．これが，ゲージ場であって，電磁相互作用の場合には，電磁場がそれである．重要なことは，局所的ゲージ対称性の要請から電磁場がなければならないことが導かれることであり，ローレンツ変換に対する対称性と局所的ゲージ対称性の両者を理論に要請すると，物質場と電磁場との相互作用の形は殆ど完全に決まってしまうことである．今考えたゲージ変換は，複素空間での位相の変換であった．これは，二つの実数を成分とする２次元ベクトルに２行２列の行列をかけてベクトルを回転させる変換と見なすことができる．この考えを押し進めてもっと高次の内部空間でのベクトル場に対する行列のかけ算に対する対称性を要請すると，クォークとレプトンの持つ様々な相互作用の大統一理論が出来る．これは宇宙の初期のクォークとレプトンの振る舞いを記述している理論である．そして宇宙が冷えてくるに伴って，物質場のなかで特定のもの(ヒッグス場)が，自分たちの相互作用のために，特定の場の配位を自発的にとるようになると元々の高い対称性が，一部壊れて，現在の強い相互作用，電磁相互作用，弱い相互作用に相互作用が分化した世界になった，と考えられる．

以上は，物理屋の立場から，ゲージ理論を概述したものであるが，数学者は，幾何学の立場からゲージ理論をファイバー束の理論として捉えている．４次元の時空がファイバー束のベース多様体であり，時空の各点ごとの内部空間が，ファイバー空間である．そしてゲージ場は，２つのファイバー同士の平行移動の対応をあたえる接続場に他ならない．有名な話であるが，ゲージ場理論の創始者の一人であるC.N.Yangが，数学者のS. Chernに向かって，嘗て，こう言ったという．"This is both thrilling and puzzling, since you matihematicians dreamed up these concepts out of nowhere. " すなわち物理屋にとっては，電磁相互作用の理論を吟味していけば，局所的ゲージ対称性を保証する理論構造が見えてくるわけだが，いったい数学者は，どこからファイバー束の理論というものを夢見たのだろうか，と． これに対するChenの答えは，"No, no, These concepts were not dreamed up. They were natural and real. " ，数学者にとっては，むしろ，ファイバー束こそ実在なのですよ，と．

このように，物理学と数学とそれぞれの世界で生まれ育った一つの理論は，共通の土壌として，それぞれの世界に強い刺激を与え続けている．特に，幾何学的不変量を理論の基本的作用に持つ場の理論は，無限次元の幾何学などにも新しい息吹きを与えている．我々の共同研究についても，物理の立場からゲージ場理論を専攻する研究者の一人(島田) にとっては，和泉や神田の共同研究室で，夜更け，サンドウィッチをかじりながら，数学者であるゲージ場理論の研究者のもう一人(林)と，ゲージ場理論の新しい側面を勉強したことは，刺激に満ちたかけがえのない時間であった．

場の理論の研究の土壌としての豊かさは，これだけにはとどまらない．本研究のなかで，発展させられた有限温度の場の理論は，非平衡系の量子統計力学の新しい定式化を与えるものである．また，多数の力学変

数の相互作用により系が新たな様相を示すことを記述する理論として, 場の理論を考えてみるとネットワークの物理学は, まさに場の理論の土壌のもとで育つべきものであり, 実際, 本研究の中のネットワークの理論では, ヒッグス場が凝縮をおこすのに対応するカオス素子たちの集団モードの発生が見いだされ, その機構が解明されている. ネットワーク理論の心理学への適用も考察された. 大統一理論よりも, もっと高いエネルギースケールでは, 粒子間の相互作用ではなく, 例えば弦などの広がりを持ち, もっと高い対称性を自然に表現できるものの相互作用が重要であったかもしれない. その意味では, 場の理論は, 究極の理論に対する現在のエネルギースケールでの有効的理論 (Effective Field theory) と見なすべきである. この観点にもとづいて, 重力場の理論を弦理論から導くことも, 本研究において達成されている.

我々の研究は, 次のように分類される.

A ゲージ場理論の新しい摂動展開法

B ゲージ場理論の幾何学的量子化

C 有限温度の場の理論

D 離散的時空間における非線形素子場のカオスダイナミックス

E 結合写像系, 及び結合カオス系における自己構造発生

以下, 分野ごとに簡単に解説を与えた後, 発表論文を採録する. 場の理論という柔軟な枠組みの中で, それぞれの考察が深く結び付いていることを読み取って頂ければ幸いである.

A: ゲージ場理論の新しい摂動展開法

弦理論の散乱振巾は, 径路積分法の立場では, 弦の様々な軌跡 (世界面) の寄与の足し上げで与えられる. 弦の生きている時空中 (例えば２６次元) で, 最小面積の面のまわりの面の揺らぎの足し上げを計算すれば弦の散乱振巾が与えられることになる. これは, 幾何学的に美しい問題で計算は容易である.

一方, 点粒子の高次の散乱振巾は, 非常に多数のファインマン・ダイアグラムを総計して初めて計算される. 特に, 真空中から粒子・反粒子の対が生成し再びその対が消滅する過程を含む振巾を 1-loop 振巾と呼ぶ. 特定の場の理論の本質的に量子論的な特質を調べるには, この 1-loop 振巾を計算しなければならない. ところが, もっとも基本的な重力理論の場合, 従前の Feynman diagram による摂動計算法を使って重力子の 1-loop 振巾を計算しようとすると, 結合因子 (vertex) も伝搬因子 (propagator) も非常に複雑なテンソルなので,, Feynman パラメーター積分を要する項がおよそ 10^8 個も現れてしまう. 大型計算機による代数処理を駆使しても重力理論の 1-loop 計算は, その重要性にも関わらず実行不可能だった. 実は, このような大量の項の殆どすべてが, 適当に組み合わせるとキャンセルしあって消えてしまうことを筆者 (島田) が在外研究中に, 弦理論における場の再定義の研究で見いだしたのがこの研究の起こりである. 本当に知りたい結果を得る前に, 山ほどの"ゴミ"を算出しなければならない— これは Feynman diagram による摂動計算の宿命的欠点である. .

我々は, 弦の張力が無限大になり, 弦が点粒子に縮む極限を考えて, 超弦理論の散乱振巾から点粒子の散乱振巾を導く一般的なスキームを開発した.

具体的には, トポロジー的にトーラスの寄与を考えた超弦理論の振巾を作り, その張力無限大の極限をとれば, 重力子の 1-loop レベルの散乱振巾が得られる. 我々の方法では, Feynman 図形の方法では, 10^8 個の項の計算を要するところを, わずか数個の弦振巾の計算に置き換えてしまう. これまで計算不可能だった重力子の４点の 1-loop helicity 振巾に, わずか１行であらわされる結果を与えた. 特に興味深いのは, 幾何的にトーラスが annulus ２枚に分解できるために, 重力の理論の 1-loop 振巾が, 非可換ゲージ場理論での 1-loop 振巾の積であらわされることが見いだされたことである. (文献 [A1]) かって, 数十ページに及ぶ Klein-仁

科のコンプトン散乱の計算がファインマン・ダイヤグラムによって教室の黒板でできるようになったように, 現代の高速大型計算機による代数処理を駆使し漸く行っていた量子色力学や重力のファインマン流の高次計算を簡単に誰でもできるように構成するのが我々の目標であったが, その到達点が見えてきたようである.

B: ゲージ場理論の幾何学的量子化

場の理論，とくに共形不変性を持つ共形場理論と呼ばれるものが，最近になって，物理学のみならず数学の中でも興味深い対象とみなされ，活発な研究が行なわれている．

共形不変性とは，一般には n 次元空間上の変換で，長さは変えてもよいが，角度は保つような変換（これを共形変換という）による不変性であるが，いま考えるのは，２次元で，とくにミンコフスキー計量で，空間１次元，時間１次元の時空間における共形不変性である．このことを，１＋１次元時空間における共形場理論という．

実際にわれわれの住んでいる時空間は，３＋１次元であるから，その意味でこれは架空の世界の理論であるが，物理学で，粒子は弦からできていると考える弦理論は，１＋１次元の場の理論で，場の行き先が，われわれの住んでいる時空間であると考えるものである．文献 [B1] では, この共形不変性と,(可換な) ゲージ対称性を併せて持つ理論として, 閉じた自由ボソン弦の理論を考察し, その symplectic 構造を定式化した. 特に, Noether の定理がどのように効くかを明確にし, ゲージ対称性については古典論のレベルでも, Noether 電荷の間のポアッソン括弧に, cocycle 項が現れることを示した.

引き続き, Wess-Zumino-Witten 模型での Noether の定理を考察し, この模型の charged system $(Q;\rho,u,\beta)$ としての定式化を得た.[B2]. Wess-Zumino-Witten 模型は, その古典論をラグランジュ形式で記述しようとするとき、最小作用の原理で運動方程式を導くときに基本となるラグランジュ関数が配位空間の接束上、グローバルには与えられないという特徴がある。配位空間 Q としてはコンパクトリー群 G に対するループ群

$$LG = \{\gamma : S^1 \longrightarrow G\}$$

をとる。この上に電磁力学における磁場 $\boldsymbol{B}$ に対応する２次微分形式 β を考える。β は閉形式ではあるが、完全形式ではない。すなわち、Q 上の１次微分形式 α で, $\beta = d\alpha$ とすることはできない。この α は、$\boldsymbol{B}$ のベクトルポテンシャル $\boldsymbol{A}$ にあたるもので, ラグランジュ関数は、この α を用いて書かれるべきものなので, グローバルには存在できないものである。そこで、この系の古典論は相空間としては通常の余接束 T^*Q をとるが, シンプレクティック形式としては, 通常の ω_0 に加えて β をあわせて $\omega_\beta = \omega_0 + \beta$ を考える。

さらに, このシンプレクティック多様体 (T^*Q, ω_β) に対して、幾何的量子化をほどこした [B3]. すなわち、T^*Q 上, ω_β を第１チャーン類とする複素線束を考え, その切断全体のつくる線形空間上に系を表現するのである。

この定式化では，状態の入っているヒルベルト空間や，関数とそのヒルベルト空間上の作用素との対応が，具体的に与えられるので，いままで（少なくとも数学的には）はっきりしなかった WZW 模型のヒルベルト空間がどのようなものであるかが明確にされた．

WZW 模型の配位空間は，ループ群 $LG = Map(S^1, G)$ である．幾何学的量子化の結果によると，模型の量子状態の空間は, $C^\infty_K(\tilde{L}G)$ であることが主張される [B3].

ここで $\tilde{L}G$ は, $\mathbf{T} = S^1$ による LG の中心拡大であり，また，$\psi \in C^\infty_K(\tilde{L}G)$ とは $\psi : \tilde{L}G \to \mathbf{C}$ 但し ,$\psi(\tilde{g}t) = t^{-K}\psi(\tilde{g})$ $(t \in \mathbf{T})$ である事を意味する． さらに，この考察は，なぜ，中心拡大 $\tilde{L}G$ と, そのレベル K の表現が現れるか, その理由を数学的に明確にしている [B3].

C: 有限温度の場の理論

有限温度の系をあつかう場の量子論の通常の定式では，有限温度の状態は密度行列演算子で表され，任意の物理量の熱平均は，その物理量を表す演算子と密度行列演算子の積のトレースで与えられる.しかし，自由度が無限大である場の理論では，演算子の表現は一意的ではなく本質的に異なる多くの表現が可能（ユニタリ非同値な表現の存在）であり密度行列を用いる定式も数学的には必ずしも明確ではない.

温度を含まない場の量子論では対称性の自発的な破れや相転移などの非同値表現にかかわる多くの事柄が研究されている.これらの成果を取り入れるためには，温度を含まない場合とできるだけ似た形に，有限温度の場の理論を定式化することが望まれる.この観点に立って，梅沢と高橋は有限温度の場の理論の新しい定式化を提唱した. Thermo Field Dynamics（以下では TFD と略称する）と呼ばれるこの定式では，有限温度の状態は，密度行列ではなくひとつの状態ベクトルで表され,，物理量の熱平均はこの状態ベクトルによる期待値で与えられる.混合状態を純粋状態に置き換えることは，状態空間を二重化することで可能になる.そこでは，密度行列にかわる状態ベクトルは二重化された空間を混ぜ合わせるある種のボゴリュウボフ変換を真空に作用させたものとして定義され，状態の温度は変換の回転角と関連するパラメータの役割をする.　このようにして定式化された TFD は，平衡系の理論としては，すでに確立された理論として受け入れられており，他の定式との同等性も証明されている（H.Umezawa *Advanced Field Theory* (American Institute of Physics, New york 1993) を参照）.　TFD を非平衡系をあつかう理論に拡張することも，梅沢を中心としたグループによって精力的に進められてきた.この一般化は，ボゴリュウボフ変換のパラメータを時間に依存させるという方法でなされた.しかし，その段階では取り扱われる系は空間的に均一なものに限られていた.多くの非可逆現象が空間的に不均一な系で起こることを考えると，この拡張はまだ不十分であり，空間的に不均一な系にも適用できる形への拡張が望まれた.

文献［C1］は，この拡張のひとつの定式を提唱したものである.そこでは，上記のボゴリュウボフ変換は，空間座標に依存した積分核をもつ積分変換に一般化される.空間の不均一性に対応して，ボゴリュウボフ変換が局所化されたことになる.

変換の局所化は，この変換のゲージ化の可能性を示唆する.そこで，本研究の一部として，まず熱的ボゴリュウボフ変換のゲージ化を行った.ゲージ理論の一般論に従って，積分変換に拡張されたボゴリュウボフ変換から共変微分を定義し，それに対応する接続としてゲージ場を導入した.この結果は 1992 年に Perugia(Italy) で開催された International Workshop on Field Theory and Collective Phenomena で発表した［C2］.

TFD に特有の状態空間の二重性は，それ自体で，ゲージ不変性に関して興味ある性質を持っている.二重化された空間のある部分空間では，一方の空間のゲージ場を他方の空間の電流と等価だと見なすことができる.文献［C3］で，SU(1) ゲージの場合にこのことを示した.［C2］で導入した熱的ボゴリュウボフ変換のゲージ理論に対して，この議論を展開することも可能である.ただし，その場合には状態空間を四重にする必要がある［C4］.これらの議論はいずれも形式的なものであってその物理的な意味は必ずしも明白ではない.その意味を明らかにするのは今後の課題である.

上に述べたように，平衡系の TFD では，状態を特徴づけるパラメータである温度はボゴリュウボフ変換の回転角を与える.非平衡系の TFD では，粒子の数密度をこのパラメータにすることができる.すなわち，粒子数密度分布関数の汎関数としてボゴリュウボフ行列が与えられる.そして，系の熱的状態の時間変化は，状態を定義するボゴリュウボフ変換の時間変化，すなわち，粒子数密度分布関数の時間発展として表現される.

この粒子数密度分布関数の時間発展を規定する運動方程式が，熱的な状態での一粒子の自己エネルギーに対する繰り込み条件から導出されるという興味深い結果を文献［C5］で示した.ただし，この論文では，空間的には均一な場合のみを考察した.

この結果を，空間的に不均一な場合に拡張することは，文献［nakamura 1］の定式を用いて一粒子の

自己エネルギーを計算し，適当な繰り込み条件を課せばよい.しかし，空間的に不均一な系では，運動量がよい量子数でないため，繰り込み条件を設定する際の on-mass-shel 条件が必ずしも自明ではないという問題が生じる.文献［C6］では，ひとつの on-mass-shell 条件の定義を提案し，その条件の下でボルツマン方程式に似た輸送方程式が得られることを示した.この方程式は量子効果を含むボルツマン方程式の一般化になっている.さらに，この方程式からエントロピー密度の時間変化を記述する方程式を導くことができ，それを用いてエントロピーの増加則を証明することができる.この結果は，1993 年に Banff(Canada) で開催された 3rd International Workshop on Thermal Field Theories and Their Applications で報告した［C7].

本研究の成果をもとに，研究期間の終了後に二つの結果を得ることができたので併せて報告する.1）文献［C6］で導いた輸送方程式から出発して，適当な近似の下に，エネルギー密度に対する拡散方程式を導出した.2）上に述べたように，空間的に不均一な系では，疑似粒子を特徴づける量子数として運動量をとることは適当ではない.そこで，ウィグナー分布関数を対角化するモードを疑似粒子と考え，文献［C1］の定式を書き換えることを試みた.いずれの結果も，1995 年に大連（中国）で行われた 4th International Workshop on Thermal Theories and Their Applications で報告した.

D:非線形素子場のカオスダイナミックスとその心理学への適用

近年，ソリトンやカオスなどの研究の急速な進展によって，非線形ダイナミクスに対する理解が深まりつつある．また，コンピュータテクノロジーの進歩によってニューラルネットワークの工学的成功がもたらされ，非線形情報処理素子を空間的に相互結合した非線形システムの研究が盛んである．これらの進歩によって，従来は数理モデルによる解析が困難と思われていた人間心理のダイナミクスを解析できる可能性さえ生じている．離散的時空間における非線形素子場を対象に，主としてカオス的挙動に注目し，システムのダイナミックスを総合的に研究した.

D.1 二つのアトラクタ間の遷移を表わす非線形数理モデル

代表的なコネクショニストモデルとしてラメルハートたちのスキーマモデルを取り上げ，これが示すカオス的挙動を解明して，あいまい図形の反転現象を説明するカオスモデルを提案し，その応用を考察した.スキーマモデルでは，ネットワーク素子の非線形性がニューロンモデルよりも弱く，単体では，せいぜい発散振動を示すだけである．したがって，これまで，PDP スキーマモデルは，カオスとは無関係と思われていた．われわれは，大規模複雑系としてのスキーマモデルのダイナミクスを明らかにするため，素子のスキーマ表示とスキーマ内での平均場近似を導入した．その結果，この系のダイナミクスは，本質的にはたった 1 自由度で記述できることを発見し，1 次元の写像関数を導出することに成功した [D1]．この写像関数は，1 次元ロジスティック写像を点対称関数へと拡張し，さらに非対称性を付け加えたものに相当している．したがって，1 次元ロジスティック写像よりも，はるかに多様なカオスダイナミックスを引き起こす．例えば，この 1 次元写像関数は，ヒステリシスをもつ新しいタイプのカオス分岐として山口・阪井が 1983 年に発見した乗り換え危機に対するもっとも簡単なモデルを与えるものである．カオス領域においては，2 つのカオスアトラクタ間を往来するカオス的バーストが見出された．これは，He の混合液を用いた実験で昨年はじめて観測されたカオス的バーストのよいモデルを与える．実験で見出された奇妙なスケール則も，本モデルにより，ヒューマンファクタによる仮想的な効果であることが明らかになった．さらに，ヒステリシスを引き起こす分岐パラメータに，準静的なダイナミクスを導入して 2 次元写像へ拡張することにより，ネッカーの立方体に代表される多義図形の解釈の反転現象を説明することができた [D2-4].

このモデルをもとにバイアスに対しての断熱的なダイナミクスを導入することにより，認知心理学におけるあいまい図形の反転現象を説明可能になった．これを検証するため,96 人の学生による認知心理学的実験を行ない，その結果が上記のカオス理論によって解釈されることを検証した [D2-4]．これらの成果は，過去半世紀にわたって受け入れられてきたあいまい多義図形反転現象の疲労効果による解釈に対して変更を迫るものである.

D.2 多アトラクタの共存を示す論理モデル

多アトラクタの共存を示す論理モデルとして、自己再帰型ファジィ推論が示すカオスについて考察した．すなわち，ファジィ推論を自己再帰的に行なうことによって、ファジィ分割した実数区間の写像力学を解析した [D5-6]．われわれは，ファジィ分割数をシステマティックに増やして非線形ファジィルールを生成し，極限において1次元ロジスティック写像関数に漸近する方法を提案した．えられた写像関数は，区間ごとに非線形関数を継ぎ合せた非線形継ぎ合せ型となり，ファジィ推論の非線形性が区間非線形性に反映されている．

このファジィカオス系では，(1) 3ルールでカオスが発生，　(2) 4ルール以上では一般に複数のアトラクタが共存，　(3) 分岐パラメータ線上でヒステリシスをもつ—— などの結果が見いだされた [D6]．

この種の自己再帰型ファジィ制御系は，制御対象として線形応答を示す系だけではない．自己再帰型ファジィ制御系を，ファジィ制御系と非線形対象システムの合成とみなした場合，ある種のクラスの非線形性を含んでいる [D7]．原理的に実効的な自己再帰型ファジィ制御系とみなせない対象システムの非線形性のクラスも明らかにした．さらに $2n$ 個のルールによって n 個の共存アトラクタを構築できることを示し，この性質を利用することにより，多値論理素子モデルとしての実現性が高いことを指摘した [D7]．

前述のニューロカオスと比べると，ニューロカオスの写像関数が極大値と極小値の両方をもつに対して，ファジィカオスでは1つの極大値しかもたないところが全く異なっている．それにもかかわらず，両者ともに複数アトラクターが共存し，不安定点近傍でベイスンが集積している点が共通しており興味深い．

ファジィカオスにおいては，揺らぎがもたらす秩序を見つけることができた．すなわち，揺らぎを加えることによりカオスから逆に1周期への転移が起こるのである．さらに強い揺らぎによってカオスと1周期との相互乗り入れの現象も見つけられている．

D.3 多アトラクタ間の遷移を表わす区間線形モデル

脳は生理学的に実現された生体系のニューラルネットワークである．この系のダイナミクスを調べる方法として，頭蓋骨外から電極をあてて電位差の変動を脳波として捉える方法が使われている．われわれは脳波の時系列解析を行って，情緒や感情が脳波によってどのように判定可能かを調べた．その結果，これらの知見をもとに，特に次の2つの研究 (1) 脳波から感情要素を自動的に識別するエキスパートシステムの構築 [D8]，(2) 1次元写像関数によって擬似的な異常脳波を生成するカオスモデルの構築 [D9] —— を行った．(いずれも (株) 脳機能研究所の武者利光氏と共同で行った感情脳波の測定と解析の成果がもとになっている.) 感情脳波の識別は，感情を自在にコントロールできるイメージトレーニングの達人を対象にすることで可能になった．

(1) は感情脳波の時系列をウェーブレット解析し，その係数をファジィメンバーシップ関数とみなしてファジィ推論と同様の方法でパターンマッチングする方法を用いたものである．脳波による感情の識別方法については，武者氏たちによるニューラルネット的な感情解析システムによって10電極間の相関係数を用いれば可能であることが示されていた．われわれはウェーブレット解析を用いた時系列解析により1電極だけからの脳波の場合に感情識別の可能性を指摘した [D8]

これらの感情解析の過程で，怒りの感情などにはいわゆる異常とされる脳波が含まれることを発見した．この異常脳波はリラックス時の α 波などのような明確な周期性をもたず，むしろ，そのパワースペクトルが $1/f$ 的な特性をもつことに特徴がある．われわれは，この時系列をアトラクタ間のバースト的な遷移によって実現する1次元写像によるカオスモデルを構築した [D9]．

したがってモデルパラメータは，カオスアトラクタ間の遷移確率から得た平均的な量によって与えられる．本モデルは，カオスアトラクタの遷移を決定する2つのパラメータによって記述され、異常脳波の定量的な再現に成功している．また、2つのモデルパラメータが表わす位相空間では、低周波成分の増大による異常脳波のおよそすべての波形を再現できることが判明した [D9]．

本モデルは，従来，簡単なモデルが不可能と考えられていた脳波生成の非常に単純なモデルである．すでにわれわれは，指先脈波の時系列を用いて情緒や感情の識別をする手法も，カオス解析を応用すること

によって確立している [D10-11]. この手法を応用することで, さらに詳細な感情脳波を識別できる可能性がある.

非線形ダイナミクスの重要性は，自然現象に限らず心理現象や社会現象においてこそ本領を発揮するものである. 実際, われわれのモデルは，多義図形の解釈の反転現象 [D1-4] にとどまらず，発見・発明などの創造的飛躍のメカニズムとしてカオスが有効であることを具体的に示している．さらに脳波の異常度解析 [D9] をはじめとして，幅広い心理・社会現象のダイナミクスを解明するモデルとして発展が期待される. 1次元写像関数には相当大きい可能性と潜在能力としてのポテンシャルがあり，モデルが簡単な分だけ多分野への応用が容易といえよう. 将来の研究においては，1次元写像がもつこれらの利点を生かし，物理学や認知心理学のみならず社会学や経済学などの社会科学系諸分野へ応用を試みて行きたいと考えている.

E: 結合写像系, 及び結合カオス系における自己構造発生

簡単な非線形写像（例えば $f(x) = 1 - ax^2$）の結合写像系では，その結合が大域的であり, 各々の写像素子が全写像素子の平均場を通して相互作用して非線形発展をする場合，個々の写像素子は単純な振る舞いをするにも関わらず, その集団は, 各素子の非線形性と, 平均操作による秩序化の微妙なバランスのもとで驚くほど多様性に富んだ振る舞いをする.

この基本的な知見に対して, 一方では, ニューラル・ネットワークでの認識の問題に適用し, もう一方では, 場の理論からの基本的な理解を得るためには, 次の２点の解析が必須である.

(1) 写像素子のような極端な変化を各ステップごとに行う素子でなく, 連続的な微分方程式で時間発展が記述される流れ素子の結合系ではどのような集団運動が起こるか.

(2) 大域的な結合素子模型では, 各素子が平均場に対して全く民主的に, 同等な寄与をするので素子間の距離というものが全く考えられない．いわば, 大域的な結合素子模型は零次元の模型である．相互作用が, 近いもの同士の間では強く, 遠いもの同士の間では弱い様な結合写像素子模型を作った場合, どのような相が観測されるか.

本研究では, この (1),(2) の点を平行して研究した．流れ素子の軌道を一定の時間ごとにスナップショットして考えてみると, このいわゆるポアンカレ切断面上では, 軌道の切断された点が, フローケ行列による繰り返し写像のもとで運動していることになる．従って, 写像素子模型と流れ素子模型との間には一定の対応があり, 結合写像素子模型で得られた新知見は, 流れ写像素子の結合系へ新しい刺激を与え, また逆のこともしばしば起こりこの非線形相乗効果により, 始め予想していたよりも遥かに興味深い結果が得られた.

(1) について，２つの流れ素子の非線形パラメーターが全く異なっていて, 例えば単独では, 一方がカオス的な振る舞いを持ち, もう一方は周期的な振る舞いをする場合でも平均場を通して両者を結合させると, 驚くべき事に, 解の間の引き込みが起こり, 新しいアトラクターが構成されることを見いだした．このアトラクターでは，２つの素子の軌道は大きさや位置は相空間のなかで異なるが, 位相は全く同期をしている．このことに基づいて, 我々は, この新現象を位相同期 (phase synchronization) と名付けた．ほぼ同時に, Max-Planck 研のグループでも, 同様な発見がなされた．但し, このグループの解析は, 元々，２つの周期性を持つ流れ素子に非線形性のほぼ近い摂動を与えたときの位相同期に限られているので, 我々の解析の方が, 一般性を持っているといえる．脳の記憶の機構において重要なのは, 各ニューロンの保持する電圧の絶対値ではなく, その変化の位相であるから, 我々の得た新知見は, 記憶・認識機構に新しい手がかりを与えるものである [E1,E2]. さらに, この位相同期現象は, 多数の流れ素子の結合系でも, 集団的な位相同期運動の発現として観測される．これは, 超伝導体やヒッグス場の場合の秩序相の発現ときわめて類似しており, 場の理論の非線形素子ネットワークへの新しい応用を示唆している．[E2].

(2) については, 素子間の距離に相互作用の強さが逆比例するような結合写像模型を構成し, その行動を解析した．大域的模型は素子間の距離の概念がないために空間的なパターンの形成を議論出来ないという大きな欠陥があるが, そのかわり, 場の理論的にいうと, 模型のこの零次元性 (スケール不変性) のお陰で素子間の長距離相関が保持される．我々の模型では相互作用が（指数型でなく）べき乗の法則に従っているために, ２次元模型であるにも関わらず, 余計な距離のスケールは導入されない．そこで，もっとも単純化された極限である興味深い大域的模型の様相を保ちつつ, しかも空間的パターンの形成を観察できる．空間構造を持ち, しかも有効的には大域的な相互作用を持つ写像素子系が存在することを示したのは, おそらく本研究がはじめてであり， その数値シミュレーション解析の完璧版を目指したので, 解析は, 本研究のまとめ期間のほぼ全部を使って, ようやく完了した．我々の模型は, 簡単な大域的写像素子模型と同じ相図を持ち, 特に, 系が相転移を遂げる過程では, あたかも密度揺らぎからの銀河形成に類似したクラスター発生が観測されている． この解析と同時に, 大域的結合写像素子模型で謎とされていた Posi-nega switch 現象も解明できたので, 併せて報文を準備中である [E4,E5].

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PHYSICS LETTERS B

String-based methods in perturbative gravity

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String theory implies a relatively modest growth in computational complexity for perturbative gravity calculations as compared to gauge theory calculations, contrary to field theory expectations. An explicit string-based calculation, which would be extremely difficult using conventional techniques, is presented to illustrate this.

1. Introduction

Perturbative computations in gravity are notorious for their algebraic complexity, being many orders of magnitude more complicated than the corresponding gauge theory computations. For example, a brute force computation of the one-loop four-graviton scattering amplitude using conventional Feynman diagram techniques [1] involves $\sim 10^8$ terms. Even with the background field method [2] in a brute-force computation one would encounter $\sim 10^6$ terms. The size of these intermediate expressions may be compared to the final results which are quite compact; indeed the amplitude for one minus and three plus helicities fits on a line.

Recently, a string-based technique significantly more efficient than conventional Feynman diagram techniques was developed for the computation of one-loop n gluon amplitudes [3–7]. Obvious questions are whether this technique can be extended to other cases and whether string theory provides additional non-trivial guidance for these extensions. In this letter, we address these questions by extending these string-based techniques to perturbative gravity. At tree level, Berends, Giele and Kuijf [8] have already used string theory [9] to give compact expressions for a class of tree-level gravity amplitudes using known Yang–Mills tree amplitudes [10]. For $N=8$ supergravity, Green, Schwarz and Brink have used the Green–Schwarz formulation of string theory to give compact results for the four-graviton one-loop amplitude [11].

The string-based technique was originally developed to compute one-loop gluon matrix elements that are formidable to compute but which are required for current and future experiments. This technology was a key ingredient in the first calculation of the one-loop five-gluon amplitude (which will enter into the analysis of three-jet events at hadron colliders) [6]. This technique has been summarized in terms of systematic rules [4,5]

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for the one-loop n-gluon amplitude which require no knowledge of string theory and bypass much of the algebra associated with Feynman diagram calculations.

To convert the rules to the case of one-loop graviton scattering amplitudes we alter the details of the string construction to recover gravity amplitudes rather than gauge theory amplitudes in the infinite string tension limit [12]. Since the string-based rules for gauge theories are already computationally efficient one expects considerable advantage in using string-based rules for gravity.

Calculations of one-loop gravity amplitudes have never been performed using traditional Feynman diagram methods. With the string-based method [4,5] we exhibit a four-point graviton-by-graviton scattering calculation for a particular helicity configuration and arbitrary particle content. This computation would be exceedingly difficult by traditional Feynman diagram techniques but is very simple with string-based techniques.

Given the conventional field theory understanding of the efficiency of the string-based methods when performing a one-loop Yang–Mills calculation [13], one might think to apply this knowledge to other cases without further appeals to string theory [14]. However, for the case of one-loop gravity the field theory understanding of Yang–Mills is insufficient to obtain the full benefit of the string-based methods. In particular, the structure of the string integrand is

$$(\text{Closed String}) \sim (\text{Open String})^2 .$$

Since closed strings contain gravity and open strings contain gauge theory there should be a formulation of gravity with the property that the integrands of the diagrams satisfy

$$(\text{Gravity}) \sim (\text{Yang–Mills})^2 . \tag{1}$$

In string theory this relationship cn be made precise.

Given that string theory has this property one can attempt to reorganize field theory to mimic this. To do so non-trivial field redefinitions and gauge choices are required. In this way one can attempt to mimic the string simplicity of the amplitude, but in a conventional field theory approach there is no guiding principle. (The field theory first-quantized formalism [15,16] could be used for studying the effective action.)

2. Field theory structure

We will now examine the properties that a reorganization of conventional field theory must satisfy to mimic the string-based structure (1). The starting point in field theory is the Einstein–Hilbert action

$$S[h] = \frac{2}{\kappa^2} \int d^4x \sqrt{-g} R .$$

Our conventions are chosen so that the kinetic term has the correct canonical normalization. The metric is expanded as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ where $h_{\mu\nu}$ is the graviton field. The first step in finding a conventional field theory formulation which mimics string theory is to find a suitable propagator for $h_{\mu\nu}$. In string theory the propagator $(L_0 + \tilde{L}_0 - 2)^{-1}$, where L_0 and $\tilde{L}_0$ are left- and right-mover world-sheet Hamiltonians, does not contain Lorentz indices. This indicates that the required field theory propagator should have a trivial Lorentz structure and therefore be proportional to the unit tensor

$$I_{\mu\nu,\rho\sigma} = \tfrac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) .$$

The unit tensor is a symmetrization of a product of $\eta_{\mu\nu}$'s which is the tensor in the propagator of Feynman-like gauges in Yang–Mills. The commonly used de Donder gauge gravity propagator [1] is

$$P_{\mu\nu;\rho\sigma}=\mathrm{i}\left(\frac{I_{\mu\nu;\rho\sigma}}{p^2+\mathrm{i}\varepsilon}-\frac{1}{D-2}\frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{p^2+\mathrm{i}\varepsilon}\right)$$

where the signature of $q_{\mu\nu}$ is $(+,-,-,-)$. This propagator is not of the desired form, since there is an extra trace piece and so the de Donder gauge is not an appropriate candidate to mimic the string organization. Although it is not possible to obtain a propagator with only a unit tensor within the class of standard gauges, since the de Donder gauge propagator is close to the desired form one might suspect that there exists a modification of the theory with the desired field theory propagator. String theory suggests a natural way of accomplishing this.

In string theory there is always an additional field associated with the graviton – the dilaton φ. This suggests that one can add a dilaton to the theory in order to produce a simple propagator to aid in calculations. At the end of a calculation one would subtract out the dilaton contribution, which is quite simple because it is a scalar. In a string-based calculation one also needs to subtract the dilaton contribution. In string theory there is in addition an antisymmetric tensor which must be subtracted; in four dimensions, this is effectively another scalar.

From the field theory understanding of the gauge theory rules [13], the background field method [2] is needed to mimic the loop part of the string-based rules. Consider the one-loop effective action of gravity coupled to a dilaton and carry out a background field expansion $g_{\mu\nu}=\bar{g}_{\mu\nu}+\kappa h_{\mu\nu}$. With the background field de Donder gauge choice, the part of the action quadratic in the quantum fields is

$$S=\int \mathrm{d}^4x\sqrt{-\bar{g}}\,(-\tfrac{1}{2}h_{\mu\nu}\mathrm{D}^2h^{\mu\nu}+h^{\mu\nu}R_{\mu\rho\nu\sigma}h^{\rho\sigma}+\tfrac{1}{4}h^{\mu}_{\mu}\mathrm{D}^2h^{\mu}_{\mu}-\tfrac{1}{2}\varphi\mathrm{D}^2\varphi-\bar{\chi}^{\mu}\mathrm{D}^2\chi_{\mu})\,,$$

where we have used the on-shell conditions on the background field and have included the ghosts χ_μ. The curvature and covariant derivatives are with respect to the background field. Consider the field redefinition

$$h_{\mu\nu}=\tilde{h}_{\mu\nu}+\frac{\eta_{\mu\nu}}{\sqrt{D-2}}\tilde{\varphi}\,,\quad \varphi=\frac{1}{\sqrt{2}}\tilde{h}^{\mu}_{\mu}+\sqrt{\frac{D-2}{2}}\,\tilde{\varphi}\,. \tag{2}$$

This has no effect on the value of the effective action since it is only a change of variables for the internal quantum field. (There is a trivial Jacobian in the path integral which is unity in dimensional regularization.) Performing the field redefinition yields

$$S=\int \mathrm{d}^4x\sqrt{-\bar{g}}\,(-\tfrac{1}{2}\tilde{h}_{\mu\nu}\mathrm{D}^2\tilde{h}^{\mu\nu}+\tilde{h}^{\mu\nu}R_{\mu\rho\nu\sigma}\tilde{h}^{\rho\sigma}+\tfrac{1}{2}\tilde{\varphi}\mathrm{D}^2\tilde{\varphi}-\bar{\chi}^{\mu}\mathrm{D}^2\chi_{\mu})\,,$$

where again we have dropped terms that vanish after imposing the equation of motion on the background field $\bar{g}$. In this action the "graviton" propagator is proportional to the unit tensor and is thus of the required form to mimic the string organization. Furthermore, the background field graviton three-vertex G_3 derived from this action is

$$G^{\kappa\lambda\delta}_{3\mu\nu\rho}(k,p,q)=-\tfrac{1}{8}\mathrm{i}\kappa[\,V_{3\mu\nu\rho}(k,p,q)\times V_3^{\kappa\lambda\delta}(k,p,q)+\{\mu\leftrightarrow\kappa\},\{\nu\leftrightarrow\lambda\},\{\rho\leftrightarrow\delta\}\,]\,,$$

where V_3 is proportional to the kinematic part of the Feynman gauge background field Yang–Mills three-vertex,

$$V_{3\mu\nu\rho}(k,p,q)=\eta_{\nu\rho}\frac{(p-q)_\mu}{2}-\eta_{\mu\rho}k_\nu+\eta_{\mu\nu}k_\rho\,,$$

where k is the momentum of the on-shell background field line and p and q are the momenta of the internal lines. The vertex G_3 is therefore of the desired form to mimic string theory since it is composed of products of Yang–Mills vertices.

In background field method one would sew tree diagrams in some other gauge onto the one-particle-irreduc-

ible diagrams [2]. For the tree parts of diagrams, the relevant gauge choices and field redefinitions necessary to mimic the string form are more complicated but are similar (although not identical) to the choices made by van de Ven [17] in his computation of the two-loop infinity of gravity. A field redefinition is also needed to remove the trace term in the tree-level graviton propagator.

One can expect that this process of reformulating field theory to mimic the string-based structure can be continued, but the process becomes increasingly obscure. For example, Yang–Mills only has three- and four-point vertices while gravity has infinitely many vertices [#1]. A simpler approach to carry out calculations is to proceed directly using string theory. The procedure for obtaining field theory rules from string theory has been described in refs. [4,18].

3. One-loop rules for gravity

The one-loop string-based rules for gravity are similar to those for gauge theory [4] so we only outline the differences between the two sets of rules. We use the bosonic string form of the rules [18,19] since the kinematic expression is simpler than the heterotic form originally used [4,5] although it contains identical information. (The heterotic string was used in the original derivation of the rules because of its full consistency.)

The starting point of these rules are labeled ϕ^3 diagrams (excluding tadpoles). Considering that gravity has an infinite set of higher point Feynman vertices, a description in terms of ϕ^3 diagrams may seem surprising, but the contributions from all such higher vertices are implicitly included. There is no need to consider diagrams with loops isolated on external legs as these vanish in dimensional regularization.

The external legs of the diagrams should be labeled in the same way as ordinary Feynman diagrams with all orderings included. This is unlike the gauge theory case where the legs were color ordered. The inner lines of a tree attached to a loop are labeled according to the rule that as one moves from the outer lines to the inner lines, one labels the inner line by the label of the most clockwise of the two outer lines. (See refs. [4,5,19] for further details.) According to the rules, each labeled n-point ϕ^3-like diagram evaluates to

$$\mathcal{I}=\mathrm{i}\frac{(-\kappa)^n}{(4\pi)^{2-\epsilon/2}}\Gamma(n_l-2+\tfrac{1}{2}\epsilon)\int_0^1 \mathrm{d}x_{i_{n_l-1}}\int_0^{x_{i_{n_l-1}}}\mathrm{d}x_{i_{n_l-2}}\cdots\int_0^{x_{i_3}}\mathrm{d}x_{i_2}\int_0^{x_{i_2}}\mathrm{d}x_{i_1}\frac{K_{\mathrm{red}}}{[\sum_{l<m}^{n_l}P_{i_l}\cdot P_{i_m}x_{i_{m_l}}(1-x_{i_{m_l}})]^{n_l-2+\epsilon/2}} \tag{3}$$

where the ordering of the loop parameter integrals corresponds to the ordering of the n_l lines attached to the loop, $x_{ij}\equiv x_i-x_j$, and K_{red} is the reduced kinematic factor. The string-based rules efficiently yield K_{red} in a compact form. The lines attached to the loop carry momenta P_i which will be off-shell if there is a tree attached to that line. The dimensional regularization parameter $\epsilon=4-D$ handles all ultraviolet and infrared divergences. The x_{i_m} are related to ordinary Feynman parameters by $x_{i_m}=\sum_{j=1}^{m}a_j$. The amplitude is then given by summing over all diagrams.

The starting point for evaluating K_{red} for any diagram is the graviton kinematic expression

$$\mathcal{K}=\int\prod_{i=1}^{n}x_i\mathrm{d}\bar{x}_i\prod_{i<j}^{n}\exp(k_i\cdot k_jG_{\mathrm{B}}^{ij})\exp[(k_i\cdot\varepsilon_j-k_j\cdot\varepsilon_i)\hat{G}_{\mathrm{B}}^{ij}-\varepsilon_i\cdot\varepsilon_j\dot{G}_{\mathrm{B}}^{ij}]\exp[(k_i\cdot\bar{\varepsilon}_j-k_j\cdot\bar{\varepsilon}_i)\tilde{G}_{\mathrm{B}}^{ij}-\bar{\varepsilon}_i\cdot\bar{\varepsilon}_j\ddot{G}_{\mathrm{B}}^{ij}]\,|_{\text{multi-linear}} \tag{4}$$

where the "multi-linear" indicates that only the terms linear in all ε_i and $\bar{\varepsilon}_i$ are included. The graviton polarization tensor is reconstructed by taking $\varepsilon_i^{\mu}\bar{\varepsilon}_i^{\nu}\to\varepsilon_i^{\mu\nu}$. This kinematic expression is obtained from a bosonic string and contains the same information as that obtained from a superstring [18]. The structure of this kinematic expression is that the polarization factors are a product of two gauge theory factors [4,5,18], corresponding to

[#1] In the field theory limit of string theory higher point vertices appear from a combination of δ-functions in the Schwinger proper time and by cancellation of kinematic poles agains factors in the kinematic expression.

the left- and right-movers of the underlying closed string theory. In string theory the G_B are Green functions on the world sheet, but in the field theory limit these become "Feynman parameter functions". From a conventional Feynman diagram point of view, the existence of a universal kinematic function is surprising as there is apparently no simple relationship between the various Feynman diagrams contributing to a given process.

In the form of the rules presented in refs. [4,18], one integrates by parts to remove all $\dot{G}_B$. In the case of gravity the integration by parts on the left and right are not independent and certain cross-terms where a left-mover derivative hits right-mover terms must be taken into account. We will discuss the cross-terms elsewhere since the integration by parts is not necessary for the calculation in the next section. It is this integration by parts step which reduces gravity to a ϕ^3 structure. (This step is not an essential part of the string-based method, which can be formulated without integration by parts [7].)

Given the integrated by parts kinematic expression, for a particular diagram with a two-point tree with lines labeled by i and j, with i appearing before j in the clockwise ordering, the tree rules tell one to replace a $(\dot{G}_B^{ij})^n(\bar{G}_B^{ij})^m$ in each term by a factor of $\delta_{n,1}\delta_{m,1}(-2k_i\cdot k_j)^{-1}$. One moves from the outside inward iteratively, replacing the functions as described. These tree rules do not depend on what particles circulate in the loop and are similar to those in refs. [4,5,19].

After the tree rules are applied to a given diagram one then applies loop substitution rules. These are essentially identical rules as for Yang–Mills applied independently to both the left- and right-mover parts of the kinematic expression. This provides an explicit diagram-by-diagram relationship between the one-loop gravity amplitude and the corresponding gauge theory amplitude. For gravitons (and the associated ghosts) circulating in the loop, every term generates two types of contributions.

The first contribution for left-movers is obtained by multiplying the kinematic expression by an overall factor of $(2-\epsilon\delta_R)$ and substituting

$$\dot{G}_B^{ij} \to \tfrac{1}{2}[-\mathrm{sign}(x_{ij})+2x_{ij}]\,, \tag{5}$$

and exactly the same substitution for the right-mover $\bar{\dot{G}}_B^{ij}$. The parameter δ_R depends on the precise form of the regularization scheme used [4]. When this first type of term occurs for both left- and right-movers instead of a factor of $(2-\epsilon\delta_R)^2$, the correct factor is $\frac{1}{2}(4-\epsilon\delta_R)(1-\epsilon\delta_R)$, which is the number of graviton degrees of freedom. More generally for a theory of gravity containing various particle types, the factor $(2-\epsilon\delta_R)^2$ is replaced by

$$N_s = N_b - N_f\,,$$

where N_b is the number of bosonic states (including any modifications due to dimensional regularization) and N_f is the number of fermionic states which circulate in the loop.

The second type of contribution for gravitons arises if a particular term contains a cycle of $\dot{G}_B$'s [18,19]. The rules for cycle contributions are essentially the same as for gauge theory that now there are both left and right contributions. For the graviton in the loop one simply takes the gauge theory vector rules on the right and on the left.

For other particles in the loop one applies rules appropriate for the particle under consideration. For example, a contribution from a gravitino in the loop can be obtained by using gauge theory vector loop rules on the left and fermion loop rules on the right. In this way the contribution of gravitons, gravitinos, vectors, fermions or scalars to the one-loop gravity amplitudes can be obtained by independently choosing gauge theory scalars, fermions, or vector loop rules given in refs. [18,19] for the left and right pieces.

Modifications to include masses for the internal fermions or scalars is simple; the only change is in the denominator in eq. (3) where the massless Feynman denominator is replaced with one corresponding to massive states circulating in the loop.

To illustrate the gravity rules we now turn to an explicit example.

4. Sample calculation

We now calculate the $\mathscr{A}(1^-, 2^+, 3^+, 4^+)$ four-graviton helicity amplitude. From a conventional Feynman diagram point of view, this computation requires a total of 12 distinct diagrams or 54 diagrams including permutations of external legs. Since gravity vertices contain many terms this would be an extremely difficult calculation with conventional Feynman diagram techniques; using string-based rules we show that this calculation is in fact very easy.

The first step is to insert spinor helicity simplifications into the kinematic expression (4). The spinor helicity method for gravitons [8,20] is related to that for vectors [21] by

$$\varepsilon^{++}=\varepsilon^{+}\bar{\varepsilon}^{+}\,,\quad \varepsilon^{--}=\varepsilon^{-}\bar{\varepsilon}^{-}\,,$$

where $\varepsilon^{\pm\pm}$ are the graviton helicity polarizations and $\varepsilon^{\pm}$ are the vector helicity polarizations defined by Xu, Zhang and Chang. We use the notation for spinor inner products $\langle k_1^- | k_2^+ \rangle = \langle 1\,2 \rangle$ and $\langle k_1^+ | k_2^- \rangle = [1\,2]$. Using the same choice of spinor helicity reference momenta as in the Yang–Mills computation of ref. [5] simplifies the kinematic coefficient to

$$K=S(\dot{G}_B^{13}-\dot{G}_B^{12})(\dot{G}_B^{24}-\dot{G}_B^{23})(\dot{G}_B^{34}+\dot{G}_B^{23})(\dot{G}_B^{34}-\dot{G}_B^{24})(\dot{\bar{G}}_B^{13}-\dot{\bar{G}}_B^{12})(\dot{\bar{G}}_B^{24}-\dot{\bar{G}}_B^{23})(\dot{\bar{G}}_B^{34}+\dot{\bar{G}}_B^{23})(\dot{\bar{G}}_B^{34}-\dot{\bar{G}}_B^{24})$$

where

$$S=\left(\frac{s^2t}{4}\right)^2\left(\frac{[2\,4]^2}{[1\,2]\langle 2\,3\rangle\langle 3\,4\rangle[4\,1]}\right)^2$$

and the Mandelstam variables are $s=2k_1\cdot k_2$, $t=2k_1\cdot k_4$ and $u=-s-t$. Due to the special helicity configuration, $\ddot{G}_B$'s do not appear and there is therefore no need to integrate by parts.

The next step is to determine which diagrams vanish trivially by the tree rules. There are a total of twelve ϕ^3-like diagrams. Of these, seven vanish by the tree rules. For example, a diagram containing a 1–4 tree vanishes because there are no $\dot{G}_B^{14}$ Green functions. Other diagrams which contain a 2–3 tree vanish because the remaining factors vanish after setting the labels of the two pinched legs together; in this case $(\dot{G}_B^{34}-\dot{G}_B^{24})\to 0$ for $2\to 3$. The only non-vanishing diagrams are the five shown in figs. 1a–1e.

First consider diagram 1a. This diagram has no trees so we immediately apply the loop rules. It is not difficult to check that all cycle contributions of the loop cancel amongst themselves whether fermion or vector rules are applied to the right- or left-movers. Thus the reduced kinematic expression can be obtained by applying the substitution rule (5) and multiplying by the number of states N_s yielding the Feynman parameter polynomial

$$N_s S x_2^2(1-x_3)^2(x_3-x_2)^4\,.$$

Up to an overall constant this is *precisely* the square of the Yang–Mills Feynman parameter polynomial for the corresponding diagram derived in refs. [5,19]. Inserting this into the loop integral yields

$$D_a=\frac{i\kappa^4}{(4\pi)^2}N_s S\int_0^1 dx_3\int_0^{x_3}dx_2\int_0^{x_2}dx_1\,\frac{x_2^2(1-x_3)^2(x_3-x_2)^4}{[sx_1(x_3-x_2)+t(x_2-x_1)(1-x_3)]^2}\,.$$

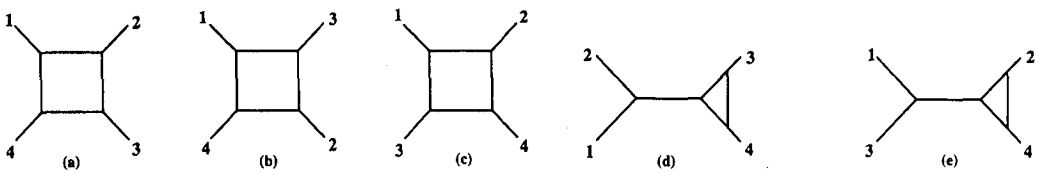

Fig. 1. The diagrams which do not vanish after applying the tree rules.

Since this and all following integrals are finite we have set the dimensional regularization parameter ϵ to zero. This integral is easy to evaluate as the numerator cancels the denominator after performing the x_1 integral. Diagrams 1b and 1c are just as easy to evaluate. The three contributions are

$$D_a = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s S}{840st}, \quad D_b = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s S}{840ut}, \quad D_c = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s S}{252su}.$$

This takes care of the box diagrams.

Now we evaluate the two triangle diagrams. First consider diagram 1d. Applying the rules for a 1–2 tree reduces the kinematic coefficient (4) to

$$K = -\frac{1}{s} S(\dot{G}_B^{24} - \dot{G}_B^{23})(\dot{G}_B^{34} + \dot{G}_B^{23})(\dot{G}_B^{34} - \dot{G}_B^{24})(\dot{\bar{G}}_B^{24} - \dot{\bar{G}}_B^{23})(\dot{\bar{G}}_B^{34} + \dot{\bar{G}}_B^{23})(\dot{\bar{G}}_B^{34} - \dot{\bar{G}}_B^{24}).$$

Applying the loop substitution rule yields the loop integral

$$D_d = -i\frac{\kappa^4}{(4\pi)^2}\frac{N_s S}{s}\int_0^1 dx_3 \int_0^{x_3} dx_2 \frac{(1-x_3)^2 x_2^2 (x_3 - x_2)^2}{s x_2 (x_2 - x_2)},$$

which is a trivial integral since the denominator cancels against the numerator. The last non-zero diagram 1e is similar to evaluate and the two diagrams are

$$D_d = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s S}{360s^2}, \quad D_e = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s S}{360u^2}.$$

Summing over all diagrams we have the four-graviton amplitude in a theory with any particle content as

$$\mathscr{A}(1^-, 2^+, 3^+, 4^+) = \frac{i\kappa^4}{(4\pi)^2}\frac{N_s}{5760}\frac{s^2t^2}{u^2}(u^2 - st)\left(\frac{[2\,4]^2}{[1\,2]\langle 2\,3\rangle\langle 3\,4\rangle[4\,1]}\right)^2.$$

For pure gravity $N_s = 2$ because the graviton has two helicity states. It is easy to verify that this amplitude has the required crossing symmetry under the interchange of legs 2, 3 and 4. In a supergravity theory with equal numbers of bosonic and fermionic states $N_s = 0$ so the amplitude vanishes in agreement with the supersymmetry identities [22]. Note that in this string-based calculation this identity holds at the level of the integrand.

The helicity conserving process $\mathscr{A}(1^-, 2^-, 3^+, 4^+)$ is more difficult to compute since cycle contributions no longer vanish and it is infrared divergent. However, even this is relatively easy to compute using the string-based methods.

These calculations may be compared to the corresponding QED calculation of light-by-light scattering. One has a few more diagrams and more complicated Feynman parameter polynomials to integrate, but the extra complication is very slight when compared to the traditional field theory expectation that gravity computations are exceedingly more complicated than QED ones.

In conclusion, gravity provides a further example of how string-based methods can be used to obtain results which would be extremely difficult to obtain using field theory methods. We expect new methods based on string theory to have further non-trivial applications to field theory calculations.

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Symplectic Structure of CFT of Free Bosons

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Synopsis: We consider a system of free bosons (bosonic closed string model) which has the conformal and (abelian) gauge symmetry.
In this note we state Noether's Theorem explicitly, emphasizing that there is a possibility of the appearance of a cocycle term in the relation of Poisson brackets of Noether charges already in the classical level, and apply it to the system. The cocycle term appears in the case of the gauge symmetry, but does not in the conformal symmetry one. The reason of the difference is not yet understood.

§1 Introduction

Conformal Field Theory (CFT) is governed by the Virasoro algebra

(1.1a) $$\mathrm{Vir} = \bigoplus_{n\in\mathbf{Z}} \mathbf{C}\hat{L}_n \oplus \mathbf{C}\hat{c}$$

with $(m, n \in \mathbf{Z})$

(1.1b) $$[\hat{L}_m, \hat{L}_n] = (m-n)\,\hat{L}_{m+n} + \frac{\hat{c}}{12}\,(m^3 - m)\,\delta_{m+n,0}\ ,$$

(1.1c) $$[\hat{L}_m,\ \hat{c}\,] = 0\ .$$

A representation of Vir is constructed from the following Heisenberg algebra

(1.2a) $$\mathrm{Heis} = \bigoplus_{n\in\mathbf{Z}} \mathbf{C}\hat{a}_n \oplus \mathbf{C}\hat{\hbar}$$

with $(m, n \in \mathbf{Z})$

(1.2b) $$[\hat{a}_m, \hat{a}_n] = m\,\delta_{m+n,0}\,\hat{\hbar}\,,$$

(1.2c) $$[\hat{a}_m, \hat{\hbar}] = 0\,,$$

as follows. See Lecture 2 in [4] for detail. We first represent Heis on the space B of polynomials in infinitely many variables :

(1.3) $$B = \mathbf{C}[t_1,\ t_2,\ \cdots]$$

as

(1.4) $$\hat{a}_n = \frac{\partial}{\partial t_n}\,,\quad \hat{a}_{-n} = n\,t_n \quad (\,n \in \mathbf{Z}_{>0}\,)\,,\quad \hat{a}_0 = \mu\,I\,,\quad \hat{\hbar} = I$$

where $I : B \longrightarrow B$ is the identity mapping and $\mu \in \mathbf{R}$. This is the highest weight representation for a weight

(1.5) $$\hat{a}_0 \mapsto \mu,\quad \hat{\hbar} \mapsto 1\,,$$

considering $\mathbf{C}\hat{a}_0 \oplus \mathbf{C}\hat{\hbar}$ as a Cartan subalgebra, whose highest weight vector is 1 as a polynomial. The space B is regarded as the Fock space of a system of free bosons, whose vacuum is the 1.

We put (Sugawara construction, (2.9) in [4])

(1.6) $$\hat{L}_n = \frac{1}{2}\sum_{j\in\mathbf{Z}} :\hat{a}_{-j}\,\hat{a}_{j+n}:\,,\qquad n \in \mathbf{Z}$$

where $:\ :$ means the normal ordering defined by

(1.7) $$:\hat{a}_i\,\hat{a}_j: = \begin{cases} \hat{a}_i\,\hat{a}_j & \text{if} \quad i \le j\,, \\ \hat{a}_j\,\hat{a}_i & \text{if} \quad i > j\,. \end{cases}$$

Then we have (Prop. 2.3 in [4])

(1.8) $$[\hat{L}_m, \hat{L}_n] = (m-n)\,\hat{L}_{m+n} + \frac{1}{12}\,(m^3 - m)\,\delta_{m+n,0}\,.$$

This means $\hat{L}_n$'s given by (1.6) provide a representation of Vir in B with central charge $c = 1$ (c is a number for which $\hat{c}$ is represented as $\hat{c} = c\,I$). We also have (Lemma 2.2 in [4])

$$[\hat{L}_m, \hat{a}_n] \;=\; -n\,\hat{a}_{m+n}\,, \quad m, n \;\in\; \mathbf{Z}\,. \tag{1.9}$$

The relation (1.6) is equivalent to the following one between field operators:

$$\hat{T}(\theta) = \frac{1}{2} : \hat{J}(\theta)\,\hat{J}(\theta) :, \quad \theta \;\in\; S^1 = \mathbf{R}/2\pi\mathbf{Z} \tag{1.10}$$

where

$$\hat{T}(\theta) = \sum_{n\in\mathbf{Z}} e^{-in\theta}\,\hat{L}_n\,, \tag{1.11}$$

$$\hat{J}(\theta) = \sum_{n\in\mathbf{Z}} e^{-in\theta}\,\hat{a}_n\,. \tag{1.12}$$

We call (1.10) the *Sugawara relation.* (We do not care the convergence problem. The above expressions (1.11) and (1.12) are regarded as formal ones.)

We renumber the relations (1.8), (1.2b) with $\hat{\hbar} = I$ and (1.9) for later convenience : ($m, n \;\in\; \mathbf{Z}$)

$$[\hat{L}_m, \hat{L}_n] \;=\; (m-n)\,\hat{L}_{m+n} + \frac{1}{12}\,(m^3 - m)\,\delta_{m+n,0}\,, \tag{1.13a}$$

$$[\,\hat{a}_m, \hat{a}_n\,] \;=\; m\,\delta_{m+n,0}\,, \tag{1.13b}$$

$$[\hat{L}_m, \hat{a}_n\,] \;=\; -n\,\hat{a}_{m+n}\,. \tag{1.13c}$$

The group theoretical meaning of (1.13 a – c) is as follows. We consider the two groups

$$\mathcal{D} = \mathrm{Diff}^+ S^1 = \{\phi : S^1 \to S^1,\ \text{orientation preserving diffeomorphism}\} \tag{1.14}$$

whose group operation is given by the composition and

$$\mathcal{N} = S^1\mathbf{R}^N = \{h : S^1 \to \mathbf{R}^N,\ C^\infty\text{-map}\} \tag{1.15}$$

with the operation of the pointwise addition in $\mathbf{R}^N$ (hereafter we take $N = 1$ for simplicity, but retain N to distinguish $\mathbf{R}^N$ from $\mathbf{R}$ as the space of τ, θ or x^+).

We let $\mathcal{D}$ and $\mathcal{N}$ act on the space

(1.16) $$S^1\mathbf{R}^N = \{X : S^1 \to \mathbf{R}^N,\ C^\infty\text{-map}\}$$

by

(1.17) $$(\phi \cdot X)(\theta) = X(\phi^{-1}(\theta)) \quad \text{for} \quad \phi \in \mathcal{D},\ X \in S^1\mathbf{R}^N\ ,$$

(1.18) $$(h \cdot X)(\theta) = X(\theta) + h(\theta) \quad \text{for} \quad h \in \mathcal{N},\ X \in S^1\mathbf{R}^N\ .$$

The action of the semi-direct product $\mathcal{D} \ltimes \mathcal{N}$ is given by

(1.19) $$((\phi, h) \cdot X)(\theta) = X(\phi^{-1}(\theta)) + h(\phi^{-1}(\theta))\ ,$$

then, in the operation form in $\mathcal{D} \ltimes \mathcal{N}$, we have

(1.20) $$(\phi_1, h_1) \cdot (\phi_2, h_2) = (\phi_1 \cdot \phi_2, h_1 \circ \phi_2 + h_2)\ .$$

The Lie algebra of $\mathcal{D}$ is

(1.21) $$\underline{\mathcal{D}} = S^1\mathbf{R} = \{\xi : S^1 \to \mathbf{R}\}\ ,$$

whose Lie bracket is given by

(1.22a) $$[\xi, \eta] = \xi'\eta - \xi\eta'\ , \quad \xi, \eta \in \underline{\mathcal{D}}\ , \qquad ' = \frac{d}{d\theta}\ .$$

Remark that the sign is different from the usual one. We give the bracket through the action (1.17), so ξ means $-\xi(\theta)\dfrac{d}{d\theta}$ in the usual sense.

We identify $\mathcal{N}$ and its Lie algebra $\underline{\mathcal{N}}$: $\underline{\mathcal{N}} = \mathcal{N}$. Since $\mathcal{N}$ is abelian, we have

(1.22b) $$[h_1, h_2] = 0 \quad \text{for} \quad h_1, h_2 \in \underline{\mathcal{N}}\ .$$

We also have, in $\underline{\mathcal{D} \ltimes \mathcal{N}}$

(1.22c) $$[\xi, h] = -\xi h' \quad \text{for} \quad \xi \in \underline{\mathcal{D}}\ ,\ h \in \underline{\mathcal{N}}\ .$$

where the right hand side of (1.22c) is considered as an element of $\underline{\mathcal{N}}$.

In the complexifications $\underline{\mathcal{D}} \otimes \mathbf{C}$ and $\underline{\mathcal{N}} \otimes \mathbf{C}$, we take the bases

(1.23) $$\ell_n = \ell_n(\theta) = e^{in\theta} \in \underline{\mathcal{D}} \otimes \mathbf{C}, \quad n \in \mathbf{Z},$$

(1.24) $$j_n = j_n(\theta) = e^{in\theta} \in \underline{\mathcal{N}} \otimes \mathbf{C}, \quad n \in \mathbf{Z}.$$

We remark that, although the functional forms of $\ell_n(\theta)$ and $j_n(\theta)$ are the same, the meanings as actions on $S^1\mathbf{R}^N$ (more precisely its complexification $(S^1\mathbf{R}^N)^{\mathbf{C}} = S^1\mathbf{C}^N$) are different. Then, from (1.22 a – c), we have

$$[\ell_m, \ell_n] = i(m-n)\,\ell_{m+n}\,, \tag{1.25a}$$

$$[j_m, j_n] = 0\,, \tag{1.25b}$$

$$[\ell_m, j_n] = -in\,j_{m+n}\,. \tag{1.25c}$$

Thus the relations (1.13 a – c) are considered as a central extension of (1.25 a – c) under the correspondence $\hat{L}_n \leftrightarrow \ell_n$, $\hat{a}_n \leftrightarrow j_n$ and

(1.26) $[\ ,\ ]$ in $\underline{\mathcal{D} \ltimes \mathcal{N}}$ $\leftrightarrow$ $i[\ ,\ ]$ in the operator algebra of the Hilbert space

where the Hilbert space is in our case B.

Through Noether's Theorem, the bracket $[\ ,\ ]$ in the Lie algebra of the symmetry group is transformed into the Poisson bracket $\{\ ,\ \}$ of Noether charges with a possibility of cocycle terms ((iii) of Theorem 1 in §2). We also have the correspondence under the canonical quantization:

(1.27) $\{\ ,\ \}$ $\leftrightarrow$ $i[\ ,\ ]$ in the operator algebra of the Hilbert space

with a possibility of the appearance of cocycle terms, in this case by the socalled *anomaly*. These correspondences give the group theoretical meaning of (1.13 a – c).

The above situation is represented by a classical system of free bosons with a symmetry governed by the group $\mathcal{D} \ltimes \mathcal{N}$ (the history is a converse one of course). Noether's theorem yields conserved quantities (Noether charges) whose Poisson brackets represent (with a possible cocycle term) the Lie algebra of the symmetry group.

In this note, we attempt to clarify the following points:

a) On what space and how does the symmetry group $\mathcal{D} \times \mathcal{N}$ act?

b) What is the explicit statement of Noether's theorem which we apply to the situation?

c) What is the meaning of the Sugawara relation (1.10) in the classical level?

In §2, we give an answer to b), Theorem 1. The proof is given in §5, where the proof of a key formula is postponed to the appendix. Theorem 1 is considered as a case which satisfies the assumption of Theorem 4.2.8 in [1]. Namely, after linearization,

$$J : TM \to \mathbf{g}^* ; w \mapsto \left[\xi \mapsto I_\xi{}^{(t)}(w)\right] \tag{1.28}$$

is a momentum mapping (for fixed $t \in \mathbf{R}$). Usually, in the literature (for example, Chap. 4 in [1], especially statements above Theorem 4.2.8, Chap. 3 in [7] or Appendix 5 in [2]), the case, where the symmetry group acts on the configuration space M or TM (or T^*M), is considered. So there is a novelty in Theorem 1, where we let the symmetry group act on the space of all paths on M. (In field theory, such a situation is often considered, but it seems that such a relation as (2.27) has not been stated explicitly.)

In §3, we consider the classical system of free bosons with the conformal symmetry corresponding to $\mathcal{D}$. In this case, the cocycle term in (iii) does not appear, hence the cocycle term in the right hand side of (1.13a) is a quantum one.

In §4, the (abelian) gauge symmetry corresponding to $\mathcal{N}$ is considered. In this case, there appears the cocycle term of (iii), thus the cocycle term in the right hand side of (1.13b) is a classical one.

We answer to a) in §3 and §4. An answer to c) is given in §4.

We apply Noether's theorem of finete degrees of freedom in §2 to the system in §3 and §4 which is of infinitely degrees of freedom. So the application is formal.

In §5, we also briefly explain the symplectic geometry, emphasizing the point of view of Lie algebras and homomorphisms between them. It clarifies the reason why, in the relation of the Poisson bracket of Noether charges ((iii) in Theorem 1), the Lie bracket of the Lie algebra of the symmetry group and the cocycle term appear.

We hope the above setting is a starting point to understand the symplectic structure of the Wess–Zumino–Witten model [6]. There are some ambiguities

such as: What are the configuration space and the Lagrangian function(al)? What is the meaning of the calculations of Poisson brackets in [6]? To understand such points is our forthcoming problem.

§2 Noether's Theorem

Let M be a configuration space (C^∞ manifold, but we use notations as if manifolds are Euclidian spaces for simplicity) of a classical system of N-degrees of freedom:

$$q = (q^i) = (q^1, q^2, \cdots, q^N) \in M . \tag{2.1}$$

Although this section is written for finite dimensional cases, the results for infinite dimensional cases are also valid.

The system is governed by a Lagrangian function

$$L = L(q, v) : TM \to \mathbf{R} \tag{2.2}$$

where $(q, v) = (q^1, \cdots, q^N; v^1, \cdots, v^N)$ is a point of the tangent bundle TM (we have $v^i = \dot{q}^i = \frac{d}{dt} q^i$ in mind). The motion is determined by the principle of "least" action:

$$\delta \int dt\, L(q(t), \dot{q}(t)) = 0 , \quad \dot{} = \frac{d}{dt} \tag{2.3}$$

whose Euler–Lagrange equation is

$$\frac{d}{dt} \frac{\partial L}{\partial v^i} (q(t), \dot{q}(t)) = \frac{\partial L}{\partial q^i} (q(t), \dot{q}(t)) . \tag{2.4}$$

For simplicity we assume $L(q, v)$ has the form

$$L(q, v) = \frac{1}{2} a_{ij} v^i v^j - U(q) \tag{2.5}$$

where $U : M \to \mathbf{R}$ is a potential and

(2.6)

(a_{ij}) is symmetric, positive definite $N \times N$ matrix, independent of q and v

(repeated upper–lower indexes are summed over $i, j = 1$ to N). Then we have

$$\frac{\partial L}{\partial v^i}(q, v) = a_{ij} v^j \tag{2.7}$$

and the equation of motion (2.4) becomes

$$\ddot{q}^i = -U^i(q) \tag{2.8}$$

where

$$U^i(q) = a^{ij}U_j(q), \quad U_j(q) = \frac{\partial}{\partial q^j}U(q), \quad (a^{ij}) = (a_{ij})^{-1} . \tag{2.9}$$

We consider a Lie group G which acts on the space of paths on M:

$$\mathrm{Path}\, M = \mathrm{Map}(\mathbf{R}, M) \ni q(\cdot) \tag{2.10}$$

where $\cdot$ in $q(\cdot)$ is the dummy of $t \in \mathbf{R}$. We consider a situation under which

$$\left\{ \begin{array}{l} \text{if } q(\cdot) \in \mathrm{Path}\, M \text{ is a solution of (2.4),} \\ \text{then so is } \gamma \cdot ((q(\cdot)) \text{ for any } \gamma \in G. \end{array} \right. \tag{2.11}$$

This is the case if the following condition is satisfied (we denote by $\mathbf{g}$ the Lie algebra of G):

$$\left\{ \begin{array}{l} \text{for } \forall \xi \in \mathbf{g}, \text{ there exists } \Lambda_\xi = \Lambda_\xi(t, q, v) \text{ such that} \\ \text{for } \forall q(\cdot) \in \mathrm{Path}\, M, \text{ putting } q^\epsilon(\cdot) = (\exp - \epsilon\xi) \cdot (q(\cdot)) \text{ , we have} \\ \qquad \dfrac{d}{d\epsilon} L(q^\epsilon(t), \dot{q}^\epsilon(t))\Big|_{\epsilon=0} = \dfrac{d}{dt} \Lambda_\xi(t, q(t), \dot{q}(t)) \ . \end{array} \right. \tag{2.12}$$

Under the condition (2.12), if $q(\cdot)$ is a solution of (2.4), then we have

$$\frac{d}{dt} \left[\frac{\partial L}{\partial v^i}(q(t), \dot{q}(t)) \left(\frac{d}{d\epsilon} q^{\epsilon,i}(t)\Big|_{\epsilon=0} \right) - \Lambda_\xi(t, q(t), \dot{q}(t)) \right] = 0 \ . \tag{2.13}$$

In fact, putting

$$A^i(t) = \frac{d}{d\epsilon} q^{\epsilon,i}(t)\Big|_{\epsilon=0} \tag{2.14}$$

and remarking $\dfrac{d}{d\epsilon} \dot{q}^\epsilon(t)\Big|_{\epsilon=0} = \dfrac{\partial}{\partial t}\dfrac{\partial}{\partial \epsilon} q^\epsilon(t)\Big|_{\epsilon=0} = \dot{A}(t)$, we have for solution $q(t)$ of (2.4)

$$(2.15)\qquad \begin{aligned} \frac{d}{d\epsilon}L(q^\epsilon(t),\dot{q}^\epsilon(t))\Big|_{\epsilon=0} &= \frac{\partial L}{\partial q^i}A^i + \frac{\partial L}{\partial v^i}\dot{A}^i \\ &= \left(\frac{d}{dt}\frac{\partial L}{\partial v^i}\right)A^i + \frac{\partial L}{\partial v^i}\dot{A}^i \qquad \text{(by (2.4))} \\ &= \frac{d}{dt}\left[\frac{\partial L}{\partial v^i}A^i\right] , \end{aligned}$$

on the other hand this is equal to $\frac{d}{dt}\Lambda_\xi$ by (2.12), giving (2.13). This is the fundamental idea of Noether's Theorem.

To obtain a function $I_\xi = I_\xi(t,q,v) \in C^\infty(\mathbf{R}\times TM)$ for which we have

$$(2.16)\qquad \text{if } q(\cdot) \text{ is a solution of (2.4), then } \frac{d}{dt}I_\xi(t,q(t),\dot{q}(t)) = 0 ,$$

we entail the following condition for the G–action on Path M:

$$(2.17)\qquad \left[\begin{array}{l} \text{for } \forall\xi\in\mathrm{g}, \text{ there exists } A_\xi = A_\xi(t,q,v) \\ \text{such that for } \forall q(\cdot)\in \text{Path}\, M, \text{ we have} \\ \frac{d}{d\epsilon}q^\epsilon(t)\Big|_{\epsilon=0} = A_\xi(t,q(t),\dot{q}(t)) , \text{ where } q^\epsilon(\cdot) = (\exp-\epsilon\xi)\cdot(q(\cdot)). \end{array}\right.$$

This condition ensures a "locality"of the action. Under the condition (2.17), putting

$$(2.18)\qquad I_\xi(t,q,v) = \frac{\partial L}{\partial v^i}(q,v)\,A^i_\xi(t,q,v) - \Lambda_\xi(t,q,v) ,$$

we have (2.16) by (2.13) (assuming (2.12) of course). This is the first part (i) of Noether's Theorem. We call I_ξ the *Noether charge* (a naming having an application to field theories in mind) for $\xi\in\mathrm{g}$.

To yield the relation on Poisson brackets of Noether charges, we finally entail

$$(2.19)\qquad A_\xi(t,q,v) \text{ , appeared in (2.17), is linear in } v$$

(see (A.1) for the exact meaning).

Notation: for a function $f = f(t, q, v)$ of t, q and v, we denote by $f^{(t)}$ the function of q and v given by $f^{(t)}(q, v) = f(t, q, v)$. Under such situations, we have

Theorem 1. (Noether's theorem)

Let M be a configuration space and consider a Lagrangian $L = L(q, v)$ of the form (2.5) with (2.6). We assume a Lie group G acts on Path M so as to satisfy (2.12). We also assume that the G-action satisfies (2.17) and (2.19). Then, for ${I_\xi}^{(t)} \in C^\infty(TM)$, $\xi \in \mathbf{g}$, given by (2.18), the followings are valid.

(i) *If $q = q(t)$ is a solution of (2.4), then we have*

$$\frac{d}{dt} I_\xi(t, q(t), \dot{q}(t)) = 0 \quad \text{for } \forall \xi \in \mathbf{g} .$$

(ii) *The Noether charge ${I_\xi}^{(t)} \in C^\infty(TM)$ generates the transformation given by $\exp - \epsilon\xi$ through the Poisson bracket (see (5.15) or (5.16)) in the following sense:*

$$\left\{ {I_\xi}^{(t)}, q^i \right\} = {A_\xi}^{(t)i} , \tag{2.20a}$$

$$\left\{ {I_\xi}^{(t)}, v^i \right\} = {B_\xi}^{(t)i} , \tag{2.20b}$$

where

$$B_\xi(t, q, v) = \frac{\partial}{\partial t} A_\xi(t, q, v) + \frac{\partial A_\xi}{\partial q^i}(t, q, v)\, v^i - \frac{\partial A_\xi}{\partial v^i}(t, q, v)\, U^i(q) . \tag{2.21}$$

(iii) *For $\forall \xi, \eta \in \mathbf{g}$, we have*

$$\left\{ {I_\xi}^{(t)}, {I_\eta}^{(t)} \right\} = I^{(t)}_{[\xi,\eta]} \qquad \text{(up to cocycle)} .$$

The proof shall be given in §5.

Remarks:

a) For $f = f(q, v)$, $g = g(q, v) \in C^\infty(TM)$, the Poisson bracket $\{f, g\} \in C^\infty(TM)$ is given by

$$\{f, g\} = a^{ij} \left(\frac{\partial f}{\partial v^i} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial v^j} \right) , \tag{2.22}$$

where $(a^{ij}) = (a_{ij})^{-1}$ and (a_{ij}) is the one in (2.5). This means we pull-back the canonical Poisson bracket (5.18) for T^*M, the cotangent bundle of M, by

the isomorphism

$$(2.23) \qquad \partial_2 L \ : \ TM \ \longrightarrow \ T^*M \ ; \ (q,v) \ \mapsto \ (q,p), \ p_i = \frac{\partial L}{\partial v^i}(q,v) = a_{ij}v^j \ ,$$

where $\partial_2 L$ means the fibre derivative of L, denoted as FL in Difinition 3.5.2 in [1]. See §5 for a detailed discussion.

b) The definition (2.21) of B_ξ means we have

$$(2.24) \qquad B_\xi(t,q(t),\dot{q}(t)) = \frac{d}{dt}A_\xi(t,q(t),\dot{q}(t))$$

for any solution $q(t)$ of (2.4) (or (2.8) for our Lagrangian (2.5)). In the sense we may write as $B_\xi(t,q,v) = \dfrac{d}{dt}A_\xi(t,q,v)$.

c) The precise meaning of (iii) of the theorem is as follows. In general, an antisymmetric bilinear form $\alpha : \mathrm{g} \times \mathrm{g} \longrightarrow \mathbf{R}$ is called a *cocycle* if it satisfies

$$(2.25) \qquad \alpha([\xi,\eta],\zeta) + \alpha([\eta,\zeta],\xi) + \alpha([\zeta,\xi],\eta) = 0 \ , \quad \text{for } \forall \xi,\eta \text{ and } \zeta \ \in \ \mathrm{g} \ .$$

Then (iii) means that, adding a constant to each Λ_ξ if necessary, the correspondence (for each t)

$$(2.26) \qquad \mathrm{g} \ \longrightarrow \ C^\infty(TM) \ ; \xi \ \mapsto \ I_\xi{}^{(t)}$$

is linear and there exists a cocycle $\alpha^{(t)} : \mathrm{g} \times \mathrm{g} \longrightarrow \mathbf{R}$ for which we have

$$(2.27) \qquad \left\{ I_\xi{}^{(t)}, I_\eta{}^{(t)} \right\} = I^{(t)}_{[\xi,\eta]} + \alpha^{(t)}(\xi,\eta) \ , \qquad \text{for} \quad \xi,\eta \ \in \ \mathrm{g} \ .$$

§3 Conformal symmetry of free bosons

We consider a system of free bosons (bosonic closed string model) described by the Lagrangian density (Chap. 2 in [3])

$$(3.1) \qquad \mathcal{L}(X) = \frac{1}{2}\partial^\mu X \partial_\mu X = \frac{1}{2}\left\{(\partial_0 X)^2 - (\partial_1 X)^2\right\} \ ,$$

where $x = (x^\mu) = (x^0, x^1) \ \in \ \mathbf{R}^2$, $X = X(x) \ \in \ \mathbf{R}^N$ and $\partial_\mu = \partial/\partial x^\mu$, $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, $(\eta_{\mu\nu}) = \mathrm{diag}[1,-1]$, $(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1}$. We also take $\tau = x^0$ and $\sigma = x^1$, then (3.1) becomes

$$(3.2) \qquad \mathcal{L}(X) = \frac{1}{2}\{\dot{X}^2 - X'^2\} \ , \quad \dot{} = \frac{\partial}{\partial\tau} \ , \quad {}' = \frac{\partial}{\partial\sigma} \ .$$

The action functional is given by

$$S[X] = \int d\tau d\sigma \, \mathcal{L}(X) \tag{3.3}$$

and the equation of motion (Euler–Lagrange equation of $\delta S[X] = 0$) is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu X} = \frac{\partial \mathcal{L}}{\partial X} \, . \tag{3.4}$$

For our case we have $\partial\mathcal{L}/\partial\partial_\mu X = \partial^\mu X$, hence (3.4) becomes

$$\partial_\mu \partial^\mu X = 0 \qquad \text{i.e.,} \quad \partial_0\partial_0 X - \partial_1\partial_1 X = 0 \, . \tag{3.5}$$

We consider the closed string case: $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$, and regarding τ as time (written as t in §2), we take the configuration space

$$M = S^1 \, \mathbf{R}^N = \{ \; X = X(\sigma) \; \in \; \mathbf{R}^N \text{ with } X(\sigma + 2\pi) = X(\sigma) \; \} \, . \tag{3.6}$$

Since $S^1 \, \mathbf{R}^N$ is a linear space, the tangent bundle is

$$TM = S^1 \, \mathbf{R}^N \times S^1 \, \mathbf{R}^N \quad \ni \quad (X(\square), Y(\square)) \, , \tag{3.7}$$

where $\square$ is the dummy of σ and $Y(\sigma) = \dot{X}(\sigma)$ in mind. The Lagrangian L is given by

$$L \; : \; TM \; \rightarrow \; \mathbf{R} \; ; \; (X(\square), Y(\square)) \; \mapsto \; \frac{1}{2}\int_0^{2\pi} d\sigma \, \{Y(\sigma)^2 - X'(\sigma)^2\} \tag{3.8}$$

giving $S[X]$ of (3.3) after further τ integration.

To give the conformal (and later gauge) symmetry, we take the light–cone coordinate:

$$x^+ = x^0 + x^1 \, , \quad x^- = x^0 - x^1 \tag{3.9}$$

and consider an orientation preserving diffeo of S^1 (considered as a space of x^+) :

$$\phi : S^1 \rightarrow S^1 \text{ (or } \mathbf{R} \rightarrow \mathbf{R} \text{ with } \phi(x^+ + 2\pi) = \phi(x^+) + 2\pi) \, . \tag{3.10}$$

Then the diffeo given by

$$\tilde{\phi} \; : \mathbf{R} \times S^1 \; \rightarrow \; \mathbf{R} \times S^1 \; ; \; (x^+, x^-) \; \mapsto \; (\phi(x^+), x^-) \, , \tag{3.11}$$

or returning to the original coordinates

$$(3.12)\quad \tilde{\phi}\ :\ (x^0, x^1) \mapsto \left(\tfrac{1}{2}\{\phi(x^0+x^1)+x^0-x^1\}, \tfrac{1}{2}\{\phi(x^0+x^1)-x^0+x^1\}\right)$$

is a conformal mapping with respect to $(\eta_{\mu\nu})$, that is

$$(3.13)\qquad (\tilde{\phi}^*\eta)_x = \Omega(x)\eta \quad \text{for some function } \Omega = \Omega(x) > 0\ ,$$

where $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ is the Lorenztian metric on $\mathbf{R} \times S^1$.

Considering infinitesimally, we take

$$(3.14)\qquad \xi\ :\ S^1 \to \mathbf{R} \quad (\text{or } \mathbf{R} \to \mathbf{R} \text{ with } \xi(x^+ + 2\pi) = \xi(x^+))$$

and for $\epsilon \in \mathbf{R}$, $|\epsilon| \ll 1$,

$$(3.15)\qquad \phi(x^+) = x^+ + \epsilon\xi(x^+)\ .$$

(Hereafter we omit the higher order terms of ϵ for simplicity.) Then $\tilde{\phi}$ written with the coordinate (τ, σ) is

$$(3.16)\qquad (\tau, \sigma) \mapsto (\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma))\ .$$

Through this conformal transformation of $\tau\sigma$–space, we consider the transformation of field $X(\tau, \sigma)$:

$$(3.17)\qquad X(\tau, \sigma) \mapsto X^\epsilon(\tau, \sigma) = X(\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma))\ .$$

or in the light–cone coordinate

$$(3.18)\qquad X(x^+, x^-) \mapsto X^\epsilon(x^+, x^-) = X(x^+ + \epsilon\xi(x^+), x^-)\ .$$

The correspondence to the notations in §2 is as follows. As the symmetry group G, we take $\mathcal{D}$ of (1.14) with Lie algebra $\underline{\mathcal{D}}$ ($\theta \leftrightarrow x^+$ in mind). The path space is in this case

$$(3.19)\quad \mathrm{Path}\, M = \mathrm{Map}(\mathbf{R}, \mathrm{Map}(S^1, \mathbf{R}^N)) = \mathrm{Map}(\mathbf{R} \times S^1, \mathbf{R}^N) \ni X(\cdot, \square)\ .$$

The action of $\mathcal{D}$ on $\mathrm{Path}\, M$ is

(3.20)

$$(\phi \cdot X)(\tau, \sigma) = X(\tilde{\phi}^{-1}(\tau, \sigma))\ , \quad \text{for} \quad \phi \in \mathcal{D},\ X = X(\tau, \sigma) \in \mathrm{Path}\, M\ ,$$

hence, infinitesimally for $\xi \in \underline{\mathcal{D}}$, if we put

$$X^\epsilon(\cdot, \square) = (\exp - \epsilon\xi) \cdot (X(\cdot, \square)) \ , \tag{3.21}$$

we have

$$X^\epsilon(\tau, \sigma) = X(\tau + \tfrac{1}{2}\epsilon\xi(\tau+\sigma), \sigma + \tfrac{1}{2}\epsilon\xi(\tau+\sigma)) \ , \tag{3.22}$$

giving (3.17).

For this $\mathcal{D}$–action, we can show the existence of Λ_ξ of (2.12) as follows. Remark that, in light–cone coordinates, we have

$$\mathcal{L}(X) = 2\partial_+ X \partial_- X \ . \tag{3.23}$$

For $\xi \in \underline{\mathcal{D}}$, taking $X^\epsilon = X^\epsilon(\tau, \sigma)$ as (3.22) or

$$X^\epsilon(x^+, x^-) = X(x^+ + \epsilon\xi(x^+), x^-) = X(x^+, x^-) + \epsilon\xi(x^+)\partial_+ X(x^+, x^-) \ , \tag{3.24}$$

we have

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}(X^\epsilon)\bigg|_{\epsilon=0} &= 2\left(\frac{d}{d\epsilon}\partial_+ X^\epsilon\bigg|_{\epsilon=0}\right)\partial_- X + 2\partial_+ X\left(\frac{d}{d\epsilon}\partial_- X^\epsilon\bigg|_{\epsilon=0}\right) \\
&= 2\{\xi'(x^+)\partial_+ X + \xi(x^+)\partial_+\partial_+ X\}\partial_- X \\
&\qquad + 2\partial_+ X \xi(x^+)\partial_-\partial_+ X \\
&= 2\partial_+\{\xi(x^+)\partial_+ X \partial_- X\} \\
&= \frac{1}{2}(\partial_0 + \partial_1)[\xi(\tau+\sigma)\mathcal{L}] \\
&= \partial_\mu\left[\frac{1}{2}(\delta_0^\mu + \delta_1^\mu)\xi(\tau+\sigma)\mathcal{L}\right] \ ,
\end{aligned} \tag{3.25}$$

where δ_ν^μ is Kronecker's delta. Thus taking the σ–integral for $\mu = 0$ component of [ ] of the final formula,

$$\Lambda_\xi(\tau, X(\square), Y(\square)) = \frac{1}{2}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma) \cdot \frac{1}{2}\{Y(\sigma)^2 - X'(\sigma)^2\} \tag{3.26}$$

gives the desired function(al):

$$\frac{d}{d\epsilon} L(X^\epsilon(\tau, \square), \dot{X}^\epsilon(\tau, \square))\bigg|_{\epsilon=0} = \frac{d}{d\tau}\Lambda_\xi(\tau, X(\tau, \square), \dot{X}(\tau, \square)) \ . \tag{3.27}$$

Therefore all assumptions of Noether's theorem are satisfied (for the linearity (2.19) in v, in our case $Y = \partial_0 X$, see (3.31)).

The corresponding Noether charge

$$I_\xi = I_\xi(\tau, X(\square), Y(\square)) \in C^\infty(\mathbf{R} \times TM) \tag{3.28}$$

is given as follows. To obtain the first term of the right hand side of (2.18):

$$\frac{\partial L}{\partial v^i} A^i_\xi(t, q, v) \ , \tag{3.29}$$

we remark that

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu X} = \partial^\mu X \tag{3.30}$$

and

$$\frac{d}{d\epsilon} X^\epsilon \bigg|_{\epsilon=0} = \xi(x^+)\partial_+ X = \frac{1}{2}\xi(\tau+\sigma)(\partial_0 X + \partial_1 X) \ . \tag{3.31}$$

Thus (3.29) corresponds to the σ–integral of the $\mu = 0$ component of

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu X}\left(\frac{d}{d\epsilon} X^\epsilon \bigg|_{\epsilon=0}\right) = \frac{1}{2}\,\partial^\mu X\,[\xi(\tau+\sigma)(\partial_0 X + \partial_1 X)] \ . \tag{3.32}$$

Hence we have

$$\begin{aligned} &I_\xi(\tau, X(\square), Y(\square)) \\ &= \frac{1}{2}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)\left[Y(\sigma)(Y(\sigma)+X'(\sigma)) - \tfrac{1}{2}(Y(\sigma)^2 - X'(\sigma)^2)\right] \\ &= \frac{1}{4}\int_0^{2\pi} d\sigma\, \xi(\tau+\sigma)\,[Y(\sigma)+X'(\sigma)]^2 \ . \end{aligned} \tag{3.33}$$

For these Noether charges $I_\xi^{(\tau)}$, one can verify (i) and (ii) of Noether's Theorem directly.

For (iii), Noether's Theorem does not determine the cocycle term $\alpha^{(t)}(\xi, \eta)$ in (2.27). So we compute the Poisson brackets actually. In this case, we have $\alpha^{(t)} = 0$ as follows.

For $\xi \in \underline{\mathcal{D}}$, we have

$$\text{(3.34a)} \qquad \frac{\delta I_\xi{}^{(\tau)}}{\delta X(\sigma)} = -\frac{1}{2}[\xi'(\tau+\sigma)(Y(\sigma)+X'(\sigma)) + \xi(\tau+\sigma)(Y'(\sigma)+X''(\sigma))] ,$$

$$\text{(3.34b)} \qquad \frac{\delta I_\xi{}^{(\tau)}}{\delta Y(\sigma)} = \frac{1}{2}\xi(\tau+\sigma)[Y(\sigma)+X'(\sigma)] .$$

Thus for ξ, $\eta \in \mathcal{D}$, we have

$$\text{(3.35)} \quad \{I_\xi{}^{(\tau)}, I_\eta{}^{(\tau)}\}(X(\square), Y(\square))$$

$$= \int_0^{2\pi} d\sigma \left[\frac{\delta I_\xi{}^{(\tau)}}{\delta Y(\sigma)}\frac{\delta I_\eta{}^{(\tau)}}{\delta X(\sigma)} - \frac{\delta I_\xi{}^{(\tau)}}{\delta X(\sigma)}\frac{\delta I_\eta{}^{(\tau)}}{\delta Y(\sigma)}\right]$$

$$= \frac{1}{4}\int_0^{2\pi} d\sigma\, [\xi'(\tau+\sigma)\eta(\tau+\sigma) - \xi(\tau+\sigma)\eta'(\tau+\sigma)][Y(\sigma)+X'(\sigma)]^2$$

$$= I_{[\xi,\eta]}^{(\tau)}(X(\square), Y(\square)) ,$$

here we recall (1.22a). This means $\alpha^{(t)} = 0$. The above normalization of Poisson bracket $\{\ ,\ \}$ on $C^\infty(TM)$ is given as follows. If we give $S^1\mathbf{R}^N$ the "inner product" $<Y_1, Y_2>$ by $\int_0^{2\pi} d\sigma\, Y_1(\sigma)\, Y_2(\sigma)$, then the form of the Lagrangian (3.8) means the finite dimensional corresponding a_{ij} in (2.5) is Kronecker's delta. So (2.22) gives the normalization.

Using the energy–momentum tensor

$$\text{(3.36)} \qquad T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu X}\partial_\nu X - \delta^\mu_\nu\mathcal{L} = \partial^\mu X\partial_\nu X - \delta^\mu_\nu\mathcal{L} ,$$

the Noether charge $I_\xi{}^{(\tau)}$ is written as follows. From (3.25) and (3.32), we see that the Noether charge $I_\xi{}^{(\tau)}$ is the σ–integral of the $\mu = 0$ component of

(3.37)

$$\frac{1}{2}\xi(\tau+\sigma)\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu X}(\partial_0 X + \partial_1 X) - (\delta^\mu_0 + \delta^\mu_1)\mathcal{L}\right] = \frac{1}{2}\xi(\tau+\sigma)[T^\mu{}_0 + T^\mu{}_1] .$$

In the light–cone coordinate, we have (see (2.1.40) in [3])

$$\text{(3.38)} \qquad T_{++} = \frac{1}{2}(T_{00} + T_{01}) = (\partial_+ X)^2 = \frac{1}{4}(\partial_0 X + \partial_1 X)^2$$

where

$$T_{\mu\nu} \equiv \eta_{\mu\rho} T^{\rho}{}_{\nu} = \partial_\mu \partial_\nu X - \eta_{\mu\nu} \mathcal{L} \ . \tag{3.39}$$

Thus we have

$$I_\xi{}^{(\tau)}(X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, \xi(\tau+\sigma) T_{++} \ , \tag{3.40}$$

where, in the right hand side, $\partial_0 X$ in T_{++} is replaced by $Y(\sigma)$.

So far we consider the real Lie algebra $\underline{\mathcal{D}} \ni \xi$ and the real valued functions $C^\infty(TM) \ni I_\xi{}^{(\tau)}$. For fixed $\tau \in \mathbf{R}$, we extend the correspondence $\xi \mapsto I_\xi{}^{(\tau)}$ to the complexified one:

$$\underline{\mathcal{D}} \otimes \mathbf{C} \to C^\infty(TM)^{\mathbf{C}} \equiv \mathrm{Map}(TM, \mathbf{C})$$

by the obvious fashion. We define $L_n{}^{(\tau)} \in C^\infty(TM)^{\mathbf{C}}$, $n \in \mathbf{Z}$, by

$$L_n{}^{(\tau)}(X(\square), Y(\square)) \equiv I_{\ell_n}{}^{(\tau)}(X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, e^{in(\tau+\sigma)} T_{++} \ , \tag{3.41}$$

where $\ell_n \in \underline{\mathcal{D}} \otimes \mathbf{C}$ is the one in (1.23) with $\theta \leftrightarrow x^+$. Then we have, from (3.35)

$$\begin{aligned} \{L_m{}^{(\tau)}, L_n{}^{(\tau)}\}(X(\square), Y(\square)) &= i(m-n) \int_0^{2\pi} d\sigma\, e^{i(m+n)(\tau+\sigma)} T_{++} \\ &= i(m-n) L_{m+n}^{(\tau)}(X(\square), Y(\square)) \ . \end{aligned} \tag{3.42}$$

This is the relation in the classical level corresponding to (1.13a) without cocycle term.

§4 Gauge symmetry of free bosons and Sugawara relation

The system of free bosons, whose configuration space is (3.6), with the Lagrangian (3.8) has also the following (abelian) gauge symmetry.

Let the group $\mathcal{N}$ of (1.15) act on Path M (3.19) as (recall the correspondence $\theta \leftrightarrow x^+$)

$$X^\epsilon(x^+, x^-) = X(x^+, x^-) + \epsilon h(x^+) \ , \qquad \text{for } h \in \mathcal{N} \tag{4.1}$$

or

$$X^\epsilon(\tau,\sigma) = X(\tau,\sigma) + \epsilon h(\tau+\sigma)\ , \qquad \text{for } h \in \mathcal{N} \tag{4.2}$$

here we have written the final form corresponding to (3.22) for the case of the conformal symmetry. (To obtain this, the action (1.18) must be $(h\cdot X)(\theta) = X(\theta) - h(\theta)$. This changes j_n to $-j_n$, retaining the relation (1.25 a – c).) In this case, we have, using (3.23),

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}(X^\epsilon)\Big|_{\epsilon=0} &= 2\left(\frac{d}{d\epsilon}\partial_+ X^\epsilon\Big|_{\epsilon=0}\right)\partial_- X + 2\partial_+ X\left(\frac{d}{d\epsilon}\partial_- X^\epsilon\Big|_{\epsilon=0}\right) \\
&= 2\,h'(x^+)\,\partial_- X \\
&= h'(\tau+\sigma)\,(\partial_0 X - \partial_1 X) \\
&= \partial_0\{-h(\tau+\sigma)\,\partial_1 X\} + \partial_1\{h(\tau+\sigma)\,\partial_0 X\} \\
&= \partial_\mu\left[-\epsilon^{\mu\nu}\,h(\tau+\sigma)\,\partial_\nu X\right]\ ,
\end{aligned} \tag{4.3}$$

here $\epsilon^{\mu\nu}$ is the antisymmetric tensor with $\epsilon^{01} = 1$. Thus $\Lambda_h(\tau, X(\square), Y(\square))$ is obtained as the σ–integral of the $\mu = 0$ component of [ ] of the final formula of (4.3):

$$\Lambda_h(\tau, X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\ [-h(\tau+\sigma)\,X'(\sigma)]\ . \tag{4.4}$$

Hence all assumptions of Theorem 1 are satisfied.

The Noether charge $I_h(\tau, X(\square), (\square))$ is the σ–integral of the $\mu = 0$ component of

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu X}\left(\frac{d}{d\epsilon}X^\epsilon\Big|_{\epsilon=0}\right) - [-\epsilon^{\mu\nu}\,h(\tau+\sigma)\,\partial_\nu X] = h(\tau+\sigma)\ [\partial^\mu X + \epsilon^{\mu\nu}\partial_\nu X]\ , \tag{4.5}$$

that is,

$$I_h(\tau, X(\square), Y(\square)) = \int_0^{2\pi} d\sigma\, h(\tau+\sigma)\,[Y(\sigma) + X'(\sigma)]\ . \tag{4.6}$$

For this Noether charge, we have

$$\frac{\delta I_h{}^{(\tau)}}{\delta X(\sigma)} = -h'(\tau+\sigma)\ , \tag{4.7a}$$

$$\frac{\delta I_h^{(\tau)}}{\delta Y(\sigma)} = h(\tau+\sigma) \ . \tag{4.7b}$$

Hence, for $h_1, h_2 \in \mathcal{N}$, we have

$$\begin{aligned} \left\{I_{h_1}^{(\tau)}, I_{h_2}^{(\tau)}\right\} &= \int_0^{2\pi} d\sigma \, [h_1'(\tau+\sigma)\, h_2(\tau+\sigma) - h_1(\tau+\sigma)\, h_2'(\tau+\sigma)] \\ &= \int_0^{2\pi} d\sigma \, [h_1'(\sigma)\, h_2(\sigma) - h_1(\sigma)\, h_2'(\sigma)] \ . \end{aligned} \tag{4.8}$$

We remark that the right hand side is a constant as a function of $(X(\square), Y(\square))$ and defines a cocycle (since $\mathcal{N}$ is abelian, only the anti–symmetricity is required).

After complexification, we define $J_n^{(\tau)} \in C^\infty(TM)^{\mathbf{C}}$ as $I_h^{(\tau)}$ for $h = h(\theta) = e^{in\theta}/\sqrt{4\pi} \in \mathcal{N} \otimes \mathbf{C}$:

$$J_n^{(\tau)}(X(\square), Y(\square)) = \frac{1}{\sqrt{4\pi}} \int_0^{2\pi} d\sigma \, e^{in(\tau+\sigma)} \, [Y(\sigma) + X'(\sigma)] \ . \tag{4.9}$$

Then we have from (4.8)

$$\left\{J_m^{(\tau)}, J_n^{(\tau)}\right\} = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \, i(m-n) e^{i(m+n)\sigma} = im\delta_{m+n,0} \ . \tag{4.10}$$

Thus the correspondence

$$J_n^{(\tau)} \longleftrightarrow \hat{a}_n \tag{4.11}$$

with (1.27) gives the relation (1.13b).

To see the correspondence to the Sugawara relation (1.10), we reverse the relation of $\partial_+ X = \frac{1}{2}[Y(\sigma) + X'(\sigma)]$ and $J_n^{(\tau)}$ in (4.9) as follows. First we expand $\partial_+ X$ as

$$\frac{1}{2}[Y(\sigma) + X'(\sigma)] = \sum_{n \in \mathbf{Z}} c_n^{(\tau)}(X(\square), Y(\square)) \, e^{in(\tau+\sigma)} \ , \tag{4.12}$$

here $c_n^{(\tau)}(X(\square), Y(\square))$ is the Fourier coefficient showing the dependence on $\tau, X(\square), Y(\square)$ explicitly, and substitute (4.12) to (4.9), to obtain

$$J_n^{(\tau)} = 2\sqrt{\pi} c_{-n}^{(\tau)} \ . \tag{4.13}$$

Thus we have

$$\partial_+ X(\sigma) = \frac{1}{2\sqrt{\pi}} \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} J_n^{(\tau)} , \tag{4.14}$$

where the left hand side is considered as the function(al)

$$(X(\square), Y(\square)) \in TM = S^1\mathbf{R}^N \times S^1\mathbf{R}^N \mapsto \frac{1}{2}[Y(\sigma) + X'(\sigma)] . \tag{4.15}$$

After (1.12), considering the correspondence (4.11), we put

$$J^{(\tau)}(\sigma) = \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} J_n^{(\tau)} , \tag{4.16}$$

which is a function(al) on TM. Then (4.14) becomes

$$\partial_+ X(\sigma) = \frac{1}{2\sqrt{\pi}} J^{(\tau)}(\sigma) . \tag{4.17}$$

Both sides of (4.17) are considered as function(al) on TM.

We apply the same method to (3.41) to obtain

$$T_{++}(\sigma) = [\partial_+ X(\sigma)]^2 = \frac{1}{2\pi} T^{(\tau)}(\sigma) \tag{4.18}$$

where, after (1.11), we put

$$T^{(\tau)}(\sigma) = \sum_{n\in\mathbf{Z}} e^{-in(\tau+\sigma)} L_n^{(\tau)} . \tag{4.19}$$

In (4.18), $T_{++}(\sigma)$ means the function(al)

$$(X(\square), Y(\square)) \in TM \mapsto \frac{1}{4}[Y(\sigma) + X'(\sigma)]^2 . \tag{4.20}$$

Thus, comparing (4.17) and (4.18), we have

$$T^{(\tau)}(\sigma) = \frac{1}{2}\left[J^{(\tau)}(\sigma)\right]^2 . \tag{4.21}$$

This is the clas ' ' relation corresponding to the Sugawara relation (1.10).

Remark that the equality (1.10) is the one as operators on the Hilbert space, but the equality (4.21) is the one as function(al)s on $TM = S^1\mathbf{R}^N \times S^1\mathbf{R}^N$.

§5 Symplectic geometry and proof of Noether's Theorem

To see a general aspect of Noether's theorem, we first point out that we have three Lie algebras. First the Lie algebra $\mathbf{g}$ of the Lie group G, the second is the space of all real valued C^∞ functions $C^\infty(TM)$ with the Poisson bracket (2.22), and as the third, the space of vector fields on a manifold with the usual bracket [,] of vector fields.

There are some cases in which we have a homomorphism of these Lie algebras. For example we assume that the Lie group G acts on a manifold Z. For $\xi \in \mathbf{g}$, we define the vector field $V_\xi \in V(Z)$ by (we denote by $V(Z)$ the space of all vector field on Z)

$$V_\xi(z) = \frac{d}{d\epsilon}\left[(\exp - \epsilon\xi)\cdot z\right]\Big|_{\epsilon=0} \qquad \text{for} \quad z \in Z\,. \tag{5.1}$$

Then it is known that (Prop. 4.1.26 in [1]. Remark that our sign convention is different. Ours are chosen so as to make mappings between Lie algebras homomorphism.)

Proposition 5.1 *The correspondence*

$$\xi \in \mathbf{g} \quad \mapsto \quad V_\xi \in V(Z) \tag{5.2}$$

is a Lie algebra homomorphism, that is, it is linear and

$$[V_\xi, V_\eta] = V_{[\xi,\eta]} \qquad \textit{in} \quad V(Z) \quad \textit{for} \quad \forall \xi, \eta \in \mathbf{g}\,. \tag{5.3}$$

As a next example, we consider a symplectic manifold W. This means W is a C^∞ manifold with a 2–form ω satisfying

$$\omega \text{ is closed, i.e., } \quad d\omega = 0 \quad , \tag{5.4a}$$

$$\omega \text{ is nondegenerate} \quad , \tag{5.4b}$$

in the sense that for $V \in V(W)$

$$\omega(V, \cdot) = 0 \qquad \text{if and only if} \qquad V = 0\,, \tag{5.5}$$

where $\omega(V,\cdot)$, which is also written as $i(V)\cdot\omega$, means the 1–form

(5.6) $$V_1 \in V(W) \quad \mapsto \quad \omega(V,V_1)\ .$$

The 2–form ω is called a *symplectic 2–form.* The condition (5.5) is equivalent to

(5.7) for $\forall$ 1–form α on W, $\exists$ the unique $V \in V(W)$, for which $\alpha = \omega(V,\cdot)$.

For any C^∞ manifold M, its cotangent bundle

(5.8) $$T^*M \ni (q,p) = (q^1,q^2,\cdots,q^N;p_1,p_2,\cdots,p_N)$$

has the canonical symplectic 2–form

(5.9) $$\omega = dp_i \wedge dq^i \qquad \text{(summed over } i = 1 \text{ to } N)\ .$$

We have

(5.10) $$\omega = d\theta$$

where θ is the 1–form on T^*M

(5.11) $$\theta = p_i \wedge dq^i\ ,$$

called the *canonical 1–form.* This ω is exact, although we require only closedness for general cases.

Returning to a general symplectic manifold (W,ω), we define, for $f \in C^\infty(W)$, the vector field $V_f \in V(W)$ by

(5.12) $$-df = \omega(V_f,\cdot) \qquad \text{as 1–forms on } W\ .$$

This V_f is uniquely determined from f by virtue of (5.7). It is seen that the correspondence

(5.13) $$f \in C^\infty(W) \quad \mapsto \quad V_f \in V(W)$$

is a Lie algebra homomorphism :

(5.14) $$[V_f,V_g] = V_{\{f,g\}} \qquad \text{for} \quad f,g \in C^\infty(W)\ ,$$

where $\{f,g\} \in C^\infty(W)$ is the Poisson bracket defined by

(5.15) $$\{f,g\} = \omega(V_f,V_g)\ .$$

(see Coro. 3.3.18 of [1], remarking the sign convention.) Remark that the definition (5.15) is also written as

$$\{f, g\} = V_f \cdot g \ . \tag{5.16}$$

For a cotangent bundle $W = T^*M$ with the symplectic structure (5.9), we have

$$V_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \ , \qquad \text{for} \quad f = f(q,p) \ \in \ C^\infty(T^*M) \ , \tag{5.17}$$

hence, by (5.16)

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \ . \tag{5.18}$$

In general, a vector field $V \in V(W)$ is called *Hamiltonian vector field* if $V = V_f$ for some $f \in C^\infty(W)$. A vector field $V \in V(W)$ is called *locally Hamiltonian vector field* if for any $w \in W$, we can find a C^∞ function f defined on a neighborhood of w for which $-df = \omega(V, \cdot)$ on the neighborhood. We denote by $HV(W)$ (resp. $LHV(W)$) the space of all Hamiltonian (resp. locally Hamiltonian) vector fields on W. In the words of forms, we can say that for $V \in V(W)$,

$$V \ \in \ HV(W) \quad \Longleftrightarrow \quad \omega(V, \cdot) \ : \ \text{exact} \ , \tag{5.19}$$

$$V \ \in \ LHV(W) \quad \Longleftrightarrow \quad \omega(V, \cdot) \ : \ \text{closed} \ . \tag{5.20}$$

It is seen that

$$V \ \in \ LHV(W) \quad \Longleftrightarrow \quad \mathcal{L}_V\omega = 0 \ , \tag{5.21}$$

where $\mathcal{L}_V\omega$ is the Lie derivative (see Chap. 2 in [1], especially p.121), so locally Hamiltonian vector fields are also called *symplectic vector fields.* It is easy to see that $HV(W)$ and $LHV(W)$ are closed under [,], hence we have three Lie algebras:

$$HV(W) \quad \subset \quad LHV(W) \quad \subset \quad V(W) \ .$$

We have from (5.19),(5.20) and (5.7)

$$LHV(W)/HV(W) \quad \cong \quad \mathrm{H}^1(W, \mathbf{R}) \ , \text{de Rham cohomology} \ . \tag{5.22}$$

(Proof of Theorem 1)

(i) is already given in §2.

Let M be the configuration space in the theorem. We pull–back the symplectic structure (5.9) on T^*M to the one on TM by (2.23) :

$$\omega = d\theta = dv_i \wedge dq^i = a_{ij}\, dv^i \wedge dq^j \ , \tag{5.23}$$

$$\theta = v_i dq^i = a_{ij} v^i dq^j \ , \tag{5.24}$$

where we use the same notation ω and θ on TM for simplicity and use (a_{ij}), which has the inverse $(a_{ij})^{-1} = (a^{ij})$ by (2.6), to let the indices $i, j, \cdots$ up or down:

$$v_i = a_{ij} v^j \ , \quad A_{\xi\, i}^{(t)}(q, v) = a_{ij} A_{\xi}^{(t)\, j}(q, v) \ , \ \cdots, \text{ etc.} \tag{5.25}$$

Under the assumption of the theorem, for any fixed $t_0 \in \mathbf{R}$, we can obtain a G–action on TM as follows. For $\gamma \in G$ and $(q_0, v_0) \in TM$, let $q = q(t) \in \mathrm{Path}\, M$ be the solution of (2.4) with

$$q(t_0) = q_0, \quad \dot{q}(t_0) = v_0 \tag{5.26}$$

and put

$$q^\gamma(\cdot) = \gamma \cdot (q(\cdot)) \tag{5.27}$$

(we assume all solutions of (2.4) with initial condition as (5.26) can be defined for all $t \in \mathbf{R}$).

Then defining as

$$\gamma \cdot (q_0, v_0) = (q^\gamma(t_0), \dot{q}^\gamma(t_0)) \ , \tag{5.28}$$

we have

Proposition 5.2 *For any fixed $t_0 \in \mathbf{R}$, the above construction (5.28) gives a G–action on TM.*

(proof) For given $(q_0, v_0) \in TM$, let $q_1(\cdot)$ be the solution of (2.4) with $q_1(t_0) = q_0, \dot{q}_1(t_0) = v_0$. Then for γ_1 and $\gamma_2 \in G$, $q_2(\cdot) \equiv \gamma_1 \cdot (q_1(\cdot))$ and $q_3(\cdot) \equiv \gamma_2 \cdot (q_2(\cdot)) = (\gamma_2\gamma_1) \cdot (q_1(\cdot))$ are also solutions of (2.4) by (2.11). Here we use the fact that G acts on $\mathrm{Path}\, M$. Then $\gamma_1 \cdot (q_0, v_0) = (q_2(t_0), \dot{q}_2(t_0))$, $\gamma_2 \cdot (q_2(t_0), \dot{q}_2(t_0)) = (q_3(t_0), \dot{q}_3(t_0)) = (\gamma_2\gamma_1) \cdot (q_0, v_0)$, hence

$$\gamma_2 \cdot (\gamma_1(q_0, v_0)) = (\gamma_2\gamma_1) \cdot (q_0, v_0) \ .$$

This means (5.28) gives a group action. Q.E.D.

For fixed $t \in \mathbf{R}$, using A_ξ in (2.17) and B_ξ in (2.21), we define the vector field on TM by

$$V_\xi{}^{(t)} = A_\xi{}^{(t)i}(q,v)\frac{\partial}{\partial q^i} + B_\xi{}^{(t)i}(q,v)\frac{\partial}{\partial v^i} . \tag{5.29}$$

Then the definition of $A_\xi{}^{(t)}$ and $B_\xi{}^{(t)}$ just means that the vector field $V_\xi{}^{(t)}$ is the one given by (5.1) for the G–action (5.28) on TM for $t(=t_0)$. See remark b) after Theorem 1. Thus we have by Proposition 5.1

$$\left[V_\xi{}^{(t)}, V_\eta{}^{(t)}\right] = V_{[\xi,\eta]}{}^{(t)} , \qquad \text{for} \quad \xi,\eta \in \mathbf{g} , \quad t \in \mathbf{R} . \tag{5.30}$$

The key step to prove the theorem is the following

Proposition 5.3 *For $t \in \mathbf{R}$ and $\xi \in \mathrm{g}$, let $V_\xi{}^{(t)} \in V(TM)$ given by (5.29) and $\Lambda_\xi{}^{(t)} \in C^\infty(TM)$ be the function in (2.12). Then we have*

$$\mathcal{L}_{V_\xi{}^{(t)}}\theta = d\Lambda_\xi{}^{(t)} . \tag{5.31}$$

The proof shall be given in the Appendix. From (5.31), we see immediately

$$\mathcal{L}_{V_\xi{}^{(t)}}\omega = 0 , \quad \text{i.e.,} \quad V_\xi{}^{(t)} \in LHV(TM) , \tag{5.32}$$

because $\mathcal{L}_{V_\xi{}^{(t)}}\omega = \mathcal{L}_{V_\xi{}^{(t)}}(d\theta) = d\mathcal{L}_{V_\xi{}^{(t)}}\theta = dd\Lambda_\xi{}^{(t)} = 0$.

We point out here that $I_\xi{}^{(t)} \in C^\infty(TM)$ of (2.18) is written as

$$I_\xi{}^{(t)} = \left\langle \theta, V_\xi{}^{(t)}\right\rangle - \Lambda_\xi{}^{(t)} , \tag{5.33}$$

because of $\left\langle \theta, V_\xi{}^{(t)}\right\rangle = \left\langle v_i dq^i, A_\xi{}^{(t)}\frac{\partial}{\partial q} + B_\xi{}^{(t)}\frac{\partial}{\partial v}\right\rangle = v_i A_\xi{}^{(t)i}$ and (2.7). Compare this to the formula in Theorem 4.2.10 in [1]. We assume that M is connected but do not assume the vanishing of $\mathrm{H}^1(M,\mathbf{R}) \cong \mathrm{H}^1(TM,\mathbf{R})$. So (5.32) does not imply $V_\xi{}^{(t)} \in HV(TM)$ (see(5.22)). Nevertheless, in our situation, we have

Proposition 5.4 *Let $V_\xi{}^{(t)} \in V(TM)$ as above and $I_\xi{}^{(t)} \in C^\infty(TM)$ be the function given by (2.18). Then we have*

$$-dI_\xi{}^{(t)} = \omega(V_\xi{}^{(t)}, \cdot) . \tag{5.34}$$

(proof) Putting $V = V_\xi{}^{(t)}$, we have

$$\begin{aligned}\omega(V,\cdot) &= i(V)\cdot\omega = i(V)\cdot d\theta = \{\mathcal{L}_V - d\cdot i(V)\}\,\theta \\ &\overset{(5.31)}{=} d\Lambda_\xi{}^{(t)} - d\langle\theta, V\rangle \\ &\overset{(5.33)}{=} -dI_\xi{}^{(t)}\,. \qquad \text{Q.E.D.}\end{aligned}$$

This proposition means

(5.35) $\quad V_\xi{}^{(t)}$ given by (5.29) $= V_{I_\xi{}^{(t)}}$ in the sense of (5.12) ,

hence $V_\xi{}^{(t)} \in HV(TM)$ and, by (5.30), we have a Lie algebra homomorphism

(5.36) $$\xi \in \mathrm{g} \quad \mapsto \quad V_\xi{}^{(t)} \in HV(TM)\,.$$

We can now prove (ii) in the theorem. In fact we have

$$\left\{I_\xi{}^{(t)}, q^i\right\} = V_{I_\xi{}^{(t)}}\cdot q^i \overset{(5.35)}{=} V_\xi{}^{(t)}\cdot q^i = \left(A_\xi{}^{(t)j}\frac{\partial}{\partial q^j} + B_\xi{}^{(t)j}\frac{\partial}{\partial v^j}\right)\cdot q^i = A_\xi{}^{(t)i}$$

and (2.20b) similarly, proving (ii).

To prove (iii), we choose a basis of g (as a vector space)

(5.37) $$e_1, e_2, \cdots \in \mathrm{g}$$

and, for each e_k, we fix $\Lambda_{e_k}^{(t)} \in C^\infty(TM)$. Then for genaral

(5.38) $$\xi = \sum_k c_k e_k \in \mathrm{g}\,, \qquad c_k \in \mathbf{R}\,,$$

we put

(5.39) $$\tilde{\Lambda}_\xi{}^{(t)} = \sum_k c_k \Lambda_{e_k}^{(t)}\,.$$

It can be seen that (assuming M is connected) the difference between $\tilde{\Lambda}_\xi{}^{(t)}$ and the original $\Lambda_\xi{}^{(t)}$ is constant. We remark that the addition of a constant to $\Lambda_\xi{}^{(t)}$ is admissible, because, in the condition (2.12), $\Lambda_\xi{}^{(t)}$ is concerned only through its derivative. Using this new $\tilde{\Lambda}_\xi{}^{(t)}$ as $\Lambda_\xi{}^{(t)}$ from the first, we have the linear mapping

(5.40) $$\xi \in \mathrm{g} \quad \mapsto \quad I_\xi{}^{(t)} \in C^\infty(TM)\,.$$

Thus, for fixed $t \in \mathbf{R}$, we have the commutative diagram:

$$\begin{tikzcd} \xi \in \mathbf{g} \arrow[r, "(5.40)"] \arrow[dr, "(5.36)"'] & C^\infty(TM) \ni I_\xi^{(t)} \arrow[d, "(5.13)"] \arrow[dl, phantom, "\circlearrowright"] \\ & HV(TM) \ni V_\xi^{(t)} . \end{tikzcd} \tag{5.41}$$

In the diagram (5.41), the mappings (5.13) and (5.36) are Lie algebra homomorphism but (5.40) is only a linear mapping in general. This is the origin of the appearance of the cocycle term $\alpha^{(t)}(\xi, \eta)$ in (2.27) as shown below.

In this situation, we have

Proposition 5.5 *We fix $t \in \mathbf{R}$. For $\xi, \eta \in \mathbf{g}$, the function on TM*

$$\left\{ I_\xi^{(t)}, I_\eta^{(t)} \right\} - I_{[\xi,\eta]}^{(t)} \tag{5.42}$$

is constant. Furthermore the bilinear form $\alpha^{(t)} : \mathbf{g} \times \mathbf{g} \to \mathbf{R}$ defined by

$$\alpha^{(t)}(\xi, \eta) = \left\{ I_\xi^{(t)}, I_\eta^{(t)} \right\} - I_{[\xi,\eta]}^{(t)} \tag{5.43}$$

is a cocycle in the sense of (2.25).

(*proof*) Through the homomorphism (5.13), we have, omitting $^{(t)}$,

$$\{I_\xi, I_\eta\} \overset{(5.13)}{\longmapsto} V_{\{I_\xi, I_\eta\}} \overset{(5.14)}{=} \left[V_{I_\xi}, V_{I_\eta}\right] \overset{(5.35)}{=} [V_\xi, V_\eta] \overset{(5.30)}{=} V_{[\xi,\eta]} ,$$

$$I_{[\xi,\eta]} \overset{(5.13)}{\longmapsto} V_{I_{[\xi,\eta]}} \overset{(5.35)}{=} V_{[\xi,\eta]} ,$$

hence $V_{(5.42)} = 0$, giving $d(5.42) \overset{(5.12)}{=} -\omega(V_{(5.42)}, \cdot) = 0$. Since we have assumed M (hence TM) is connected, we see that (5.42) is a constant.

The argument in p. 44 of [7] gives that (5.43) is a cocycle. Q.E.D.

This proposition yields (iii), proving the whole theorem. Q.E.D.

In the point of view of Theorem 4.2.8 of [1], the relation (5.34) and the linearity of (5.40) mean that the mapping (1.28) is a momentum mapping. A more explicit formula for $\alpha^{(t)}$ of (5.43) is given in Theorem 4.2.8 of [1].

Appendix : Proof of Proposition 5.3

In this appendix, we fix $\xi \in \mathfrak{g}$ and omit it in notations for simplicity, so $A^{(t)}, \Lambda^{(t)}$ means $A_\xi^{(t)}, \Lambda_\xi^{(t)}$ etc. A similar calculation is done in Appendix of [5], where the proof of the fact corresponding to (ii) of our theorem is given.

The condition (2.19) means

$$A^{(t)i}(q,v) = D^{(t)i}(q) + E^{(t)i}{}_j\, v^j \ , \tag{A.1}$$

which corresponds to (A.2) in [5]. For $\forall q(\cdot) \in \mathrm{Path}\, M$, putting $q^\epsilon(\cdot) = (\exp - \epsilon\xi)\cdot(q(\cdot))$, we have from (2.17)

$$\frac{d}{d\epsilon}\, q^{\epsilon,i}(t)\Big|_{\epsilon=0} = A^{(t)i}(q(t), \dot q(t)) \ . \tag{A.2}$$

Thus, for the Lagrangian $L = L(q,v)$ of the form (2.5), we obtain

$$\begin{aligned}
(\mathrm{A.3})\quad & \frac{d}{d\epsilon} L\left(q^\epsilon(t), \dot q^\epsilon(t)\right)\Big|_{\epsilon=0} \\
&= \frac{\partial L}{\partial q^i}(q(t),\dot q(t)) \left\{ \frac{d}{d\epsilon} q^{\epsilon,i}(t)\Big|_{\epsilon=0} \right\} + \frac{\partial L}{\partial v^i}(q(t),\dot q(t)) \left\{ \frac{d}{d\epsilon}\frac{d}{dt}\, q^{\epsilon,i}(t)\Big|_{\epsilon=0} \right\} \\
&= -U_i(q(t))\, A^{(t)i}(q(t),\dot q(t)) + \dot q_i(t)\, \frac{d}{dt} A^{(t)i}(q(t),\dot q(t)) \\
&= -U_i(q)\, D^{(t)i}(q) + \left\{ \dot D^{(t)i}(q) - U_j(q)\, E^{(t)ji} \right\} \dot q_i \\
&\quad + \left\{ \dot E^{(t)}{}_{ij} + D^{(t)}{}_{i;j}(q) \right\} \dot q^i\, \dot q^j + E^{(t)}{}_{ij}\, \dot q^i\, \ddot q^j \ ,
\end{aligned}$$

where $\dot D^{(t)}(q) = \frac{\partial}{\partial t} D(t,q)$, $D^{(t)}{}_{i;j} = \frac{\partial}{\partial q^j} D^{(t)}{}_i$, and we let $i, j, \cdots$ up and down using (a_{ij}) or (a^{ij}) :

$$E_{ij} \equiv a_{ik} E^k{}_j \ , \ E^{ij} \equiv E^i{}_k\, a^{kj} \ , \ \cdots, \ \text{etc.} \tag{A.4}$$

Now we put

$$\Lambda^{(t)}(q,v) = \alpha^{(t)}(q) + \beta^{(t)}_i(q)\, v^i + \frac{1}{2}\gamma^{(t)}_{ij}(q)\, v^i\, v^j + [\text{higher terms in } v] \ . \tag{A.5a}$$

We can assume $\left(\gamma^{(t)}{}_{ij}(q)\right)$ is symmetric :

(A.5b) $$\gamma^{(t)}{}_{ji}(q) = \gamma^{(t)}{}_{ij}(q) \ .$$

Then we have

(A.6) $$\begin{aligned} &\frac{d}{dt}\Lambda^{(t)}(q(t), \dot{q}(t)) \\ &= \dot{\alpha}^{(t)}(q) + \left\{\alpha^{(t)}{}_{;i}(q) + \dot{\beta}^{(t)}{}_{i}(q)\right\} \dot{q}^i + \beta^{(t)}{}_{i}(q)\, \ddot{q}^i \\ &\quad + \left\{\beta^{(t)}{}_{i;j}(q) + \frac{1}{2}\dot{\gamma}^{(t)}{}_{ij}(q)\right\} \dot{q}^i \dot{q}^j + \gamma^{(t)}{}_{ij}(q)\, \dot{q}^i \ddot{q}^j \\ &\quad + \frac{1}{2}\gamma^{(t)}{}_{ij;k}(q)\, \dot{q}^i \dot{q}^j \dot{q}^k + [\,\text{terms of } \dot{q},\ \ddot{q} \text{ of order } \geq 3\,] \ . \end{aligned}$$

We remark that the condition (2.12) must be satisfied for any $q(\cdot) \in$ Path M, hence the coefficients for $\dot{q}$, $\ddot{q}$ in (A.3) and (A.6) must be equal independently. So [higher terms in v] of (A.5a) is absent and we have $\gamma^{(t)}{}_{ij;k}(q) = 0$ and $\beta^{(t)}{}_{i}(q) = 0$. Thus we have

(A.7) $$\Lambda^{(t)}(q, v) = \alpha^{(t)}(q) + \frac{1}{2}\gamma^{(t)}{}_{ij}\, v^i v^j \ , \qquad \gamma^{(t)}{}_{ji} = \gamma^{(t)}{}_{ij} \ .$$

We also have

(A.8a) $$-U_i(q) D^{(t)i}(q) = \dot{\alpha}^{(t)}(q) \ ,$$

(A.8b) $$\dot{D}^{(t)}{}_{i}(q) - U_j(q)\, E^{(t)j}{}_{i} = \alpha^{(t)}{}_{;i}(q) \ ,$$

(A.8c) $$\left\{\dot{E}^{(t)}{}_{ij} + D^{(t)}{}_{i;j}(q) - \frac{1}{2}\dot{\gamma}^{(t)}{}_{ij}\right\} \dot{q}^i \dot{q}^j = 0 \ ,$$

(A.8d) $$E^{(t)}{}_{ij} = \gamma^{(t)}{}_{ij} \ .$$

We see, from (A.8d) and (A.5b), that $E^{(t)}{}_{ij}$ is symmetric:

(A.9) $$E^{(t)}{}_{ji} = E^{(t)}{}_{ij} \ .$$

Since (A.8c) is satisfied for all $\dot{q}$, the symmetrization of i, j in $\{\quad\}$ must vanish, thus

(A.10) $$D^{(t)}{}_{i;j}(q) + D^{(t)}{}_{j;i}(q) + \dot{E}^{(t)}{}_{ij} = 0 \ ,$$

here we used (A.8d). Using these relations we have

$$
\begin{aligned}
\text{(A.11)}\quad B^{(t)}{}_i(q,v) &= \frac{d}{dt} A^{(t)}{}_i(q,v) \\
&= \frac{d}{dt}\left[D^{(t)}{}_i(q) + E^{(t)}{}_{ij}\, v^j\right] \\
&= \dot{D}^{(t)}{}_i(q) + D^{(t)}{}_{i;j}(q)\, v^j + \dot{E}^{(t)}{}_{ij}\, v^j + E^{(t)}{}_{ij}\,(-U^j(q)) \\
&= \alpha^{(t)}{}_{;i}(q) - D^{(t)}{}_{j;i}(q)\, v^j \qquad \text{(by (A.8b) and (A.10))}
\end{aligned}
$$

Now we can prove (5.31) as follows. We put

$$
V = V^{(t)} = V_\xi{}^{(t)} = A^{(t)i}\,\frac{\partial}{\partial q^i} + B^{(t)i}\,\frac{\partial}{\partial v^i}
$$

and omit $^{(t)}$. Then

$$
\begin{aligned}
\text{(A.12)}\qquad \mathcal{L}_V\,\theta &= \mathcal{L}_V\,(v_i\, dq^i) \\
&= (\mathcal{L}_V\, v_i)\, dq^i + v_i\, \mathcal{L}_V\,(dq^i) \\
&= (V\cdot v_i)\, dq^i + v_i\, d(V\cdot q^i) \\
&= B_i\, dq^i + v_i\, dA^i \\
&= \left[\alpha_{;j}(q) - D_{k;j}\, v^k\right] dq^j \qquad \text{(by (A.11))} \\
&\quad + v_i\,\left[D^i{}_{;j}(q)\, dq^j + E^i{}_j\, dv^j\right] \\
&= \alpha_{;j}(q)\, dq^j + E_{ij}\, v^i\, dv^j \\
&= d\Lambda\,, \qquad \text{(by (A.7) and (A.8d))}
\end{aligned}
$$

yielding the proposition. Q.E.D.

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Wess–Zumino–Witten Model and Noether's Theorem

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Synopsis : Wess–Zumino–Witten model based on a compact Lie group G has gauge and conformal symmetry. The gauge symmetry means that we have an $LG \equiv \{\gamma : S^1 \to G\}$ action on the space of G–valued fields, under which a Lagrangian density "$\mathcal{L}$" is invariant. The Poisson bracket of the Noether charges for the LG–action gives the Kac–Moody relation.

As the double quotations in "$\mathcal{L}$" suggest, the density is not defined globally. In this note, we give a formalism (charged symplectic structure) suitable to such a situation. We also have Noether's theorem with a formula, (3.26), yielding the Kac–Moody cocycle term only knowing the form of the LG–action on the space of fields.

Key words : WZW model, Noether's theorem, charged system.

§1 Introduction

Let G be a compact Lie group, which we assume to be simple, connected and simply connected for simplicity. For a given map $g : S^2 \to G$, $S^2 \equiv \{y \in \mathbf{R}^3; |y| = 1\}$ (map(ping)s are considered as of C^∞–class unless otherwise

specified), we have an extension $\tilde{g} : B^3 \to G$ of g, i.e., $\tilde{g}_{|S^2} = g$, to $B^3 \equiv \{y \in \mathbf{R}^3; |y| \leq 1\}$, $\partial B^3 = S^2$, because the second homotopy group $\pi_2(G) = \{0\}$. We define the functional S_{WZ}, called the Wess–Zumino term [8], by

$$S_{\mathrm{WZ}}[g] = -\sqrt{-1}\,\frac{K}{12\pi}\int_{B^3} d^3y\,\epsilon^{ijk}\,\mathrm{tr}\left(\tilde{g}^{-1}\partial_i\tilde{g}\,\tilde{g}^{-1}\partial_j\tilde{g}\,\tilde{g}^{-1}\partial_k\tilde{g}\right)\ , \tag{1.1}$$

where ϵ^{ijk} is antisymmetric with $\epsilon^{123} = 1$ and tr means the trace in the defining representation. The upper and lower indices are summed over. This functional is determined mod. $2\pi\sqrt{-1}\,K\mathbf{Z}$. (This is the case for $G = SU(N)$. Otherwise it may be necessary to renormalize tr.)

The action functional of the Wess–Zumino–Witten (WZW) model [10] is given by

$$S[g] = S_0[g] + S_{\mathrm{WZ}}[g] \tag{1.2}$$

with

$$S_0[g] = -\frac{1}{4\lambda^2}\int_{S^2} d^2x\,\sqrt{\rho_{S^2}}\,\rho_{S^2}^{ij}\,\mathrm{tr}\left(g^{-1}\partial_i g\,g^{-1}\partial_j g\right)\ , \tag{1.3}$$

where $\rho_{S^2} = \rho_{S^2,ij}(x)dx^i\otimes dx^j$ is a Riemannian metric on S^2, $\left(\rho_{S^2}^{ij}\right) = (\rho_{S^2,ij})^{-1}$ and $\sqrt{\rho_{S^2}} = \sqrt{\det(\rho_{S^2,ij})}$.

The functional (1.2) is the one in the Euclidean regime. The corresponding formalism in the Minkowskian regime is given as follows. We consider the 3–form Ω_G on G :

$$\Omega_{G,g}(\xi,\eta,\zeta) = \frac{1}{4\pi}\left\langle g^{-1}\xi\,,\,\left[g^{-1}\eta\,,\,g^{-1}\zeta\right]\right\rangle, \quad \text{for } \xi,\eta,\zeta \in T_gG\ , \tag{1.4}$$

where $\langle\ ,\ \rangle$ is the (positive) Killing form given by

$$\langle\xi,\eta\rangle \;=\; -\mathrm{tr}(\xi\eta)\ , \quad \text{for } \xi,\,\eta \in \mathrm{g}\ , \tag{1.5}$$

here $\mathrm{g} \equiv T_1G$, the Lie algebra of G (1 is the identity of G).

Then the Wess–Zumino term (1.1) is written as

$$S_{\mathrm{WZ}}[g] = \sqrt{-1}\,K\int_{B^3}\tilde{g}^*\,\Omega_G\ , \tag{1.6}$$

where $\tilde{g}^*\,\Omega_G$ is the pull–back of Ω_G by $\tilde{g} : B^3 \to G$. The 3–form Ω_G is closed and its de Rham cohomology class is a non–zero element of $\mathrm{H}^3(G;\mathbf{R}) \cong \mathbf{R}$

(the coefficient is arranged so as to satisfy $\int_{SU(2)} \Omega_G = 2\pi$, where $SU(2)$ is a subgroup of G), hence *not exact.*

If Ω_G *were* exact, there *would be* a 2–form "ω_G" with d"ω_G" $= \Omega_G$. Then the term (1.6) would be (see Appendix in [10]), by Stokes' theorem,

"(1.7)"

$$\int_{B^3} \tilde{g}^* \Omega_G = \int_{B^3} \tilde{g}^*(d\text{``}\omega_G\text{''}) = \int_{B^3} d(\tilde{g}^* \text{``}\omega_G\text{''}) = \int_{\partial B^3} g^* \text{``}\omega_G\text{''} = \int_{S^2} g^* \text{``}\omega_G\text{''} .$$

This integral (times $\sqrt{-1}$) is considered as the Euclidefication ($x^0 \to \sqrt{-1}\, x^0$) of

$$\text{(1.8a)} \qquad \int dx^0 \int_0^{2\pi} dx^1 \text{``}\omega_{ij}(\phi)\text{''} \partial_0\phi^i \partial_1\phi^j ,$$

where $\phi = (\phi^i)$ is in a coordinate system on G, $\partial_\mu = \partial/\partial x^\mu$, $x = (x^\mu) = (x^0, x^1)$, and

$$\text{(1.8b)} \qquad \text{``}\omega_G\text{''} = \tfrac{1}{2} \text{``}\omega_{ij}(\phi)\text{''} d\phi^i \wedge d\phi^j , \ \text{``}\omega_{ij}(\phi)\text{''} = \text{``}\omega_{G,\phi}\text{''} \left(\tfrac{\partial}{\partial\phi^i}, \tfrac{\partial}{\partial\phi^j}\right) .$$

Thus, compactifying the x^1–space to $S^1 \equiv \mathbf{R}/2\pi\mathbf{Z}$, that is, $x^1 \equiv x^1 + 2\pi$, we have the virtual Lagrangian density

$$\text{(1.9a)} \qquad \text{``}\mathcal{L}\text{''}(\phi) = \mathcal{L}_0(\phi) + \text{``}\mathcal{L}\text{''}_{\mathrm{WZ}}(\phi)$$

with

$$\text{(1.9b)} \quad \mathcal{L}_0(\phi) = \tfrac{1}{2} \eta^{\mu\nu} \rho_{G,ij}(\phi) \partial_\mu\phi^i \partial_\nu\phi^j = \tfrac{1}{2} \rho_{G,ij}(\phi) \left[\partial_0\phi^i \partial_0\phi^j - \partial_1\phi^i \partial_1\phi^j\right] ,$$

$$\text{(1.9c)} \quad \text{``}\mathcal{L}\text{''}_{\mathrm{WZ}}(\phi) = \tfrac{1}{2} \cdot 2\lambda^2 K \epsilon^{\mu\nu} \text{``}\omega_{ij}(\phi)\text{''} \partial_\mu\phi^i \partial_\nu\phi^j = 2\lambda^2 K \text{``}\omega_{ij}(\phi)\text{''} \partial_0\phi^i \partial_1\phi^j ,$$

where $(\eta^{\mu\nu}) = \mathrm{diag}[1, -1]$, $\epsilon^{\mu\nu}$: antisymmetric with $\epsilon^{01} = 1$, $\rho_G = \rho_{G,ij}(\phi) d\phi^i \otimes d\phi^j$ is the Riemannian metric on G given by $\langle\ ,\ \rangle$ of (1.5), that is,

$$\text{(1.10)} \qquad \rho_{G,g}(\xi, \eta) = \left\langle g^{-1}\xi\, ,\, g^{-1}\eta \right\rangle , \quad \xi, \eta \in T_g G .$$

The whole (1.2) is considered as the Euclidefication of the Minkowskian "functional"

$$\text{(1.11)} \qquad \frac{1}{2\lambda^2} \int dx^0 \int_0^{2\pi} dx^1 \text{``}\mathcal{L}\text{''}(\phi) .$$

In the coordinate system, the 3–form Ω_G is written as

$$\text{(1.12)} \quad \Omega_G = \tfrac{1}{6} \Omega_{ijk}(\phi)\, d\phi^i \wedge d\phi^j \wedge d\phi^k , \quad \Omega_{ijk}(\phi) = \Omega_{G,\phi} \left(\tfrac{\partial}{\partial\phi^i}, \tfrac{\partial}{\partial\phi^j}, \tfrac{\partial}{\partial\phi^k}\right) ,$$

and $d``\omega_G" = \Omega_G$ means

$$\Omega_{ijk}(\phi) = \partial_i``\omega_{jk}(\phi)" + \partial_j``\omega_{ki}(\phi)" + \partial_k``\omega_{ij}(\phi)" \ . \tag{1.13}$$

In many cases, there appears only the combination of the right hand side of (1.13), giving a global meaning.

A mathematically more clear formalism is given in §2 and §4. We only give here the Euler–Lagrange equation for $\delta(1.11) = 0$:

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial``\mathcal{L}"}{\partial\partial_\mu\phi^i} - \frac{\partial``\mathcal{L}"}{\partial\phi^i} \\ &= \rho_{G,ij}(\phi)\eta^{\mu\nu}\partial_\mu\partial_\nu\phi^j \\ &\qquad + \left[\Gamma_{G,ijk}(\phi)\eta^{\mu\nu} - \tfrac{1}{2}\cdot 2\lambda^2 K\Omega_{ijk}(\phi)\epsilon^{\mu\nu}\right]\partial_\mu\phi^j\partial_\nu\phi^k \ , \end{aligned} \tag{1.14a}$$

where $\Gamma_{G,ijk}(\phi)$ is the Christoffel symbol for the metric (1.10) :

$$\Gamma_{G,ijk} = \tfrac{1}{2}\{\, \partial_j\rho_{G,ik} + \partial_k\rho_{G,ij} - \partial_i\rho_{G,jk} \,\} \ , \tag{1.14b}$$

and we used (1.13). Remark that, although the density (1.9a) does not have global meaning, the equation (1.14a) does.

The equation of motion (1.14a) is equivalent to (see eq. (15) of [10])

$$\left(\frac{1}{2\lambda^2} + \frac{K}{4\pi}\right)\partial_-(g^{-1}\partial_+ g) + \left(\frac{1}{2\lambda^2} - \frac{K}{4\pi}\right)\partial_+(g^{-1}\partial_- g) = 0 \ , \tag{1.15}$$

where we have introduced the light–cone coordinates

$$x^\pm = x^0 \pm x^1 \ , \quad \partial_\pm = \frac{\partial}{\partial x^\pm} = \tfrac{1}{2}(\partial_0 \pm \partial_1) \ . \tag{1.16}$$

For the case

$$2\lambda^2 K = 4\pi \ , \tag{1.17}$$

the equation of motion becomes

$$\partial_-(g^{-1}\partial_+ g) = 0 \ , \tag{1.18}$$

whose solution is of the form

$$g(x) = g(x^+, x^-) = C_-(x^-)C_+(x^+) \ , \quad C_\pm(x^\pm) \in G \ . \tag{1.19}$$

As the form of (1.19) suggests, the model has the following symmetry if (1.17) is satisfied. Let $LG = \mathrm{Map}(S^1, G) = \{\gamma : S^1 \to G\}$ be the loop group [7], and consider the action of $LG \times LG$ on the space of fields $\mathrm{Map}(\mathbf{R} \times S^1, G)$ given by

$$(1.20) \qquad g(x^+, x^-) \to \gamma_-(x^-) g(x^+, x^-) \gamma_+(x^+)^{-1}, \quad (\gamma_-, \gamma_+) \in LG \times LG ,$$

where for $g = g(x^0, x^1) \in \mathrm{Map}(\mathbf{R} \times S^1, G)$, the variables x^0, x^1 are changed to the light–cone ones : $x^\pm = x^0 \pm x^1$. The would–be Lagrangian density (1.9a) is invariant under the transformation (1.20). The precise meaning of this statement is clarified in §3 and §4.

The symmetry leads to the existence of Noether charges. For our case, considering only the half of the transformation (1.20) for simplicity, we take an element $\xi = \xi(x^+)$ of the Lie algebra $\underline{LG} = L\mathbf{g}$ of LG, for which the transformation is (here and hereafter, we only write terms of necessary order)

$$(1.21) \qquad g(x^+, x^-) \to g(x^+, x^-) + \epsilon g(x^+, x^-) \xi_+(x^+), \quad |\epsilon| \ll 1 ,$$

that is, we took $\gamma_- = i_d$, the identity map, and $\gamma_+(x^+) = \exp\{-\epsilon\xi(x^+)\} = 1 - \epsilon\xi(x^+)$ in (1.20). The Noether charge corresponding to the transformation (1.21) is given by (see (4.20))

$$(1.22) \qquad \fint \left\langle 2g^{-1}(x^+, x^-)\partial_+ g(x^+, x^-),\ \xi(x^+) \right\rangle , \quad \fint \equiv \frac{1}{2\lambda^2}\int_0^{2\pi} dx^1 .$$

This is x^0–independent if $g = g(x)$ is a solution of (1.18).

In our formalism developed in §3, the Noether charge is considered as a function (for fixed $x^0 = \tau$) on the tangent bundle TQ of the configuration space of the model :

$$(1.23) \qquad Q = LG \equiv \{g : S^1 \to G\} .$$

Remark that we have three LG's, all of which are the same as sets, but S^1's are considered as the space of x^- , x^+ and $x^1 = \sigma$ respectively.

We write an element of $Q = LG$ as $g(\square)$, where $\square$ is the dummy of σ, and an element of TQ as $(g(\square), h(\square))$, which means $h(\square) \in T_{g(\square)}Q$, that is, $h(\sigma) \in T_{g(\sigma)}G$ for $\forall\sigma \in S^1$. Then, for fixed $\tau \in \mathbf{R}$, the charge (1.22) is condidered as a function on TQ :

(1.24)

$$I_\xi^{(\tau)} : TQ \to \mathbf{R} ;\ (g(\square), h(\square)) \mapsto \fint \left\langle g^{-1}(\sigma)\{h(\sigma) + g'(\sigma)\} ,\ \xi(\tau + \sigma) \right\rangle ,$$

where $g'(\sigma)$ means $\frac{d}{d\sigma}g(\sigma)$ (recall that $2\partial_+ = \partial_0 + \partial_1$ and we have $h = \partial_0 g$ in mind).

Since we have a natural symplectic 2–form ω_{TQ}, (2.26), on TQ, we can consider the Poisson brackets, (B.8), of functions on TQ. We have (see (4.24) or (D.9))

$$(1.25) \qquad \left\{ I_\xi^{(\tau)},\ I_\eta^{(\tau)} \right\} \ = \ I_{[\xi,\eta]}^{(\tau)} + c_K(\xi,\eta)\ , \quad \text{for } \xi,\eta \in L\mathbf{g}\ ,$$

where $c_K : L\mathbf{g} \times L\mathbf{g} \to \mathbf{R}$ is the cocycle term given by (cf. (4.2.2) of [7])

$$(1.26) \qquad c_K(\xi,\eta) = \frac{K}{2\pi}\int_0^{2\pi} d\theta\, \langle\, \xi'(\theta)\,,\ \eta(\theta)\,\rangle\ , \qquad \xi'(\theta) = \tfrac{d}{d\theta}\,\xi(\theta)\ .$$

This is the relation of the affine Kac–Moody Lie algebra of level K. The relation was originally obtained in [10] using the light–cone coordinate, but the symplectic structure was not so explicitly written. We shall give a formula in Theorem 3.1, which yields the cocycle term knowing only the action of the symmetry group, without exact calculation using the symplectic 2–form ω_{TQ}. A direct calculation is given in Appendix D. A similar result for Poisson brackets is obtained in §9.5 in [6], with the symplectic manifold being LG/T, where T is a maximal torus of G.

We also have the conformal symmetry, whose symmetry group is Diff $S^1 \times$ Diff S^1, where Diff S^1 is the group of all orientation preserving diffeomorphisms of S^1. The transformation is given by

$$(1.27) \quad g(x^+,x^-) \to g(D_+^{-1}(x^+), D_-^{-1}(x^-))\,, \quad (D_+, D_-) \in \text{Diff}\, S^1 \times \text{Diff}\, S^1\ .$$

Remark that $(x^+,x^-) \to (D_+(x^+), D_-(x^-))$ is a conformal transformation with respect to the Minkowskian metric $\eta_{\mu\nu}dx^\mu \otimes dx^\nu = d\tau \otimes d\tau - d\sigma \otimes d\sigma$ on $\mathbf{R} \times S^1$.

For the case $D_- = i_d$, $D_+^{-1}(x^+) = x^+ + \epsilon E(x^+)$, where

$$(1.28) \qquad E \in \underline{\text{Diff}\, S^1}\ , \text{ i.e., } E\ :\ \mathbf{R} \to \mathbf{R} \ \text{ with } \ E(x^+ + 2\pi) = E(x^+)\ ,$$

the transformation (1.27) becomes

$$(1.29) \quad g(x^+,x^-) \to g(x^+ + \epsilon E(x^+), x^-) \ = \ g(x^+,x^-) + \epsilon E(x^+)\partial_+ g(x^+,x^-)\ .$$

The corresponding Noether charge is given by (see (4.32) with (4.33))

$$\fint E(\tau+\sigma)\left\langle g^{-1}\partial_+ g\,,\ g^{-1}\partial_+ g\right\rangle\ . \tag{1.30}$$

Comparing this to (1.22), we see here the origin of the Sugawara construction.

As a function on TQ, (1.30) is regarded as

$$\begin{aligned} I_E^{(\tau)}\ :\ & (g(\square), h(\square)) \\ & \mapsto \fint \tfrac{1}{4}E(\tau+\sigma)\left\langle g^{-1}(\sigma)\left\{h(\sigma)+g'(\sigma)\right\}\ ,\ g^{-1}(\sigma)\left\{h(\sigma)+g'(\sigma)\right\}\right\rangle\ . \end{aligned} \tag{1.31}$$

In this case, for $E_1,\ E_2 \in \underline{\mathrm{Diff}\, S^1}$, we have ((4.34) and the remark below it)

$$\left\{I_{E_1}^{(\tau)}\ ,\ I_{E_2}^{(\tau)}\right\} \ =\ I_{[E_1\,,\,E_2]}^{(\tau)}\ . \tag{1.32}$$

This means there are no cocycle terms for the conformal symmetry. Thus the Virasoro anomaly is a quantum one. The cocycle term (which is 0) in (1.32) is also derived only seeing the action (1.29) of the Lie algebra $\underline{\mathrm{Diff}\, S^1}$.

We organize this paper as follows.

In §2, we give a formalism suitable for systems with Lagrangian functions defined only locally. §3 gives Noether's theorem for such systems. There is a novelty in the formula (3.26), which gives the cocycle term in the relation concerning Poisson brackets of Noether charges, only knowing the form of the action of the symmetry group on the space of fields. Supplementary arguments are given in Appendix A. We give a generality of symplectic actions used to prove Theorem 3.1 in Appendix B.

Although the configuration space of the WZW model is infinite dimensional, we state the necessary formalism in the finite dimensional form in §2 and §3. It is straightforward to extend it to the infinite dimensional form.

In §4, we apply the formalism developed in §2 and §3 to the WZW model. We calculate the Kac–Moody cocycle term using the formula (3.26).

In Appendix C, we give calculations leading Noether currents discussed in §1 and §4, and in Appendix D, the Kac–Moody relation (1.25) is obtained directly, containing the case $2\lambda^2 K \neq 4\pi$.

§2 Charged Symplectic Structure

Let Q be a configuration space (C^∞–manifold) of a classical system. We denote by $q = (q^i) = (q^1, q^2, \cdots, q^{\dim Q})$ an element of Q, by $(q, v) = (q^i, v^i)$ for TQ, the tangent bundle, and by $(q, p) = (q^i, p_i)$ for T^*Q, the cotangent bundle.

On T^*Q, we have the canonical symplectic 2–form

$$\omega_0 = dp_i \wedge dq^i \ , \tag{2.1}$$

which is exact : $\omega_0 = d\theta_0$, where

$$\theta_0 = p_i dq^i \tag{2.2}$$

is the canonical 1–form globally defined on T^*Q. See [1], [3] or Chap. 2 of [11] for a generality of the symplectic geometry.

A novel feature of our formalism is that we take a *closed* 2–form on Q :

$$\beta = \tfrac{1}{2}\,\beta_{ij}(q)\,dq^i \wedge dq^j \ , \quad \beta_{ij} = \beta\left(\tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}\right) \ , \tag{2.3}$$

and define the closed 2–form on T^*Q :

$$\omega_\beta \equiv \omega_0 + \beta = dp_i \wedge dq^i + \tfrac{1}{2}\,\beta_{ij}(q)\,dq^i \wedge dq^j \ , \tag{2.4}$$

which is always nondegenerate, hence is a symplectic 2–form on T^*Q (the form β in (2.4) may have to be written as $\pi^*\beta$, where $\pi : T^*Q \to Q$ is the projection, but we drop π^* for simplicity). We call ω_β the *charged symplectic structure* (or 2–form) and β a *magnetic 2–form* . (The names are originated from the system of charged particle q in a submanifold Q of $\mathbf{R}^3$ under a magnetic field $\mathbf{B}$ on Q, which is described by ω_β with $\beta = \Omega_{\mathbf{R}^3}(\mathbf{B}, \cdot, \cdot)$, where $\Omega_{\mathbf{R}^3}$ is the volume element of $\mathbf{R}^3$. Remark that $d\beta = 0 \ \Leftrightarrow \ \operatorname{div}\mathbf{B} = 0$. See §2.6 of [11] and references therein. The Dirac monopole is such a field $\mathbf{B}$ on $Q = \mathbf{R}^3 \setminus \{0\}$, for which β is not exact, that is, there are no globally defined vector potentials. A relation between the Dirac monopole and WZW model is discussed in [9].)

The system is governed by the Hamiltonian function H on T^*Q :

$$H(q, p) = \tfrac{1}{2}\,\rho^{ij}(q)\,p_i\,p_j + u(q) \ , \tag{2.5}$$

where $(\rho^{ij}) = (\rho_{ij})^{-1}$ with

(2.6)

$$\rho = \rho_{ij}(q) dq^i \otimes dq^j \ , \quad \rho_{ji} = \rho_{ij} \ : \text{ a (positive) Riemannian metric on } Q \ ,$$

and

(2.7) $$u \ : \ Q \to \mathbf{R} \ , \quad \text{a potential } (C^\infty\text{-function}) \ .$$

This Hamiltonian function H is the usual one derived from the Lagrangian function $L_0 = L_0(q, v)$ on TQ :

(2.8) $$L_0(q, v) = \tfrac{1}{2}\, \rho_{ij}(q) v^i v^j - u(q)$$

as

(2.9a) $$H(q, p) = p_i\, v^i(q, p) - L_0(q, v(q, p)) \ ,$$

where $v = v(q, p)$ is determined by solving

(2.9b) $$p_i = \frac{\partial L_0}{\partial v^i} = \rho_{ij}(q) v^j \ .$$

Although the Hamiltonian function H is usual, the symplectic 2–form ω_β is not. The Poisson bracket $\{\ ,\ \}$ with respect to ω_β for functions on T^*Q, that is,

(2.10) $$\{f_1, f_2\} \equiv \omega_\beta(V_{f_1}, V_{f_2}) = V_{f_1} \cdot f_2 \quad \text{for } f_1,\ f_2 \in C^\infty(T^*Q) \ ,$$

where for $f \in C^\infty(T^*Q)$, the vector field V_f on T^*Q is determined by

(2.11) $$-d\, f = \omega_\beta(V_f, \cdot) \ , \qquad \omega_\beta(V_f, \cdot) \ : \ V \mapsto \omega_\beta(V_f, V) \ ,$$

is given as follows. Since $\omega_\beta(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) = \beta_{ij}$, $\omega_\beta(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_j}) = -\delta_i^j$, $\omega_\beta(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0$, we have

(2.12) $$\omega_\beta(\tfrac{\partial}{\partial q^i}, \cdot) = -dp_i + \beta_{ij}\, dq^j \ , \quad \omega_\beta(\tfrac{\partial}{\partial p_i}, \cdot) = dq^i \ ,$$

hence, for $f = f(q, p) \in C^\infty(T^*Q)$, the vector field V_f is written as

(2.13) $$V_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left[\frac{\partial f}{\partial q^i} + \beta_{ji}\frac{\partial f}{\partial p_j}\right]\frac{\partial}{\partial p_i} \ .$$

Thus for $f_1,\ f_2 \in C^\infty(T^*Q)$, we have

(2.14) $$\{f_1, f_2\} = \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q^i} - \frac{\partial f_1}{\partial q^i}\frac{\partial f_2}{\partial p_i} - \beta_{ij}(q)\frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial p_j} \ .$$

The equation of motion is given by

$$(2.15a) \qquad \dot{q}^i = \{H, q^i\} = \frac{\partial H}{\partial p_i} = \rho^{ij}(q)\, p_j \ ,$$

$$(2.15b) \qquad \dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q^i} + \beta_{ij}(q)\, \frac{\partial H}{\partial p_j}$$

$$= -\frac{\partial u}{\partial q^i} - \tfrac{1}{2}\, \frac{\partial \rho^{jk}}{\partial q^i}\, p_j\, p_k + \beta_{ij}\, \rho^{jk}\, p_k \ ,$$

which is equivalent to the flow on T^*Q derived from the vector field V_H on T^*Q.

If we transform the motion onto TQ by the isomorphism (see (2.9b))

$$(2.16) \qquad \dot{\partial} L_0 \ : \ TQ \to T^*Q \ ; \ (q, v) \mapsto (q, p), \ p_i = \frac{\partial L_0}{\partial v^i} = \rho_{ij}(q) v^j \ ,$$

the equation (2.15ab) becomes (remark the sign : $\partial_i \rho^{jk} \rho_{k\ell} = -\rho^{jk} \partial_i \rho_{k\ell}$, $\partial_i \equiv \frac{\partial}{\partial q^i}$)

$$(2.17a) \qquad \dot{q}^i = v^i \ ,$$

$$(2.17b) \qquad \tfrac{d}{dt}\left[\rho_{ij}(q) v^j\right] = -\partial_i u(q) + \tfrac{1}{2}\, \partial_i \rho_{jk}(q) v^j v^k + \beta_{ij}(q) v^j$$

which is equivalent to

$$(2.18a) \qquad \rho_{ij}(q) \ddot{q}^j = -\Gamma_{ijk}(q) \dot{q}^j \dot{q}^k - \partial_i u(q) + \beta_{ij}(q) \dot{q}^j \ ,$$

where $\Gamma_{ijk}(q)$ is the Christoffel symbol for the Riemannian metric ρ on Q :

$$(2.18b) \qquad \Gamma_{ijk} = \tfrac{1}{2}\left\{\partial_j \rho_{ik} + \partial_k \rho_{ij} - \partial_i \rho_{jk}\right\} \ , \quad \Gamma^i{}_{jk} = \rho^{i\ell}\, \Gamma_{\ell jk} \ .$$

The Lagrangian formalism is a somewhat awkward one, but it is suitable for Noether's theorem, because the invariance under a group action is stated using the Lagrangian functions. Since β is a closed 2–form on Q, we have an open covering

$$(2.19) \qquad \{U, V, \ldots\}$$

of Q such that on each U, there is a 1–form $\alpha^U = \alpha^U_i(q) dq^i$ with

$$(2.20) \qquad d\alpha^U = \beta_{|U} \ , \quad \text{i.e.,} \quad \partial_i \alpha^U_j - \partial_j \alpha^U_i = \beta_{ij} \ .$$

Using it, we define the Lagrangian function on TU (not on the whole TQ) by
(2.21)
$$L^U \ : \ TU \to \mathbf{R} \ ; \ (q,v) \mapsto L_0(q,v) + \langle \alpha^U(q), v \rangle = \tfrac{1}{2}\rho_{ij}(q)v^i v^j - u(q) + \alpha^U_i(q)v^i \ ,$$

where $\langle \ , \ \rangle$ is the pairing of 1–forms and vectors. Although the Lagrangian function L^U depends on U, the equation of motion derived from the Principle of Least Action : $\delta \int dt \ L^U = 0$, i.e.,

$$\frac{d}{dt}\frac{\partial L^U}{\partial v^i}(q(t), \dot{q}(t)) = \frac{\partial L^U}{\partial q^i}(q(t), \dot{q}(t)) \ , \tag{2.22}$$

which is equivalent to (2.18a), has a global meaning (because it depends only on β).

The charged symplectic structure ω_β and the Hamiltonian function H on T^*Q are derived from these L^U's as follows. We pull–back the forms θ_0 and ω_0 on T^*Q by the isomorphism

$$\dot{\partial} L^U \ : \ TU \to T^*U \ ; \ (q,v) \mapsto (q,p), \ p_i = \frac{\partial L^U}{\partial v^i} = \rho_{ij}(q)v^j + \alpha^U_i(q) \ , \tag{2.23}$$

to obtain the forms on TU :

$$\theta_{TU} \ \equiv \ \left(\partial_{\mathrm{F}} L^U\right)^* (\theta_{0|T^*U}) \ = \ \frac{\partial L^U}{\partial v^i} dq^i \ = \ \frac{\partial L_0}{\partial v^i} dq^i + \alpha^U_i dq^i \ , \tag{2.24}$$

$$\omega_{TU} \ \equiv \ \left(\partial_{\mathrm{F}} L^U\right)^* (\omega_{0|T^*U}) \ = \ d\theta_{TU} \ = \ d\left[\frac{\partial L_0}{\partial v^i} dq^i\right] + \beta \ . \tag{2.25}$$

As these formulas show, although θ_{TU}'s do not have a global meaning, ω_{TU}'s do and give the globally defined 2–form on TQ :
(2.26)
$$\omega_{TQ} = d\left[\frac{\partial L_0}{\partial v^i} dq^i\right] + \beta = \rho_{ij}(q) dv^i \wedge dq^j + \left[\partial_i \rho_{jk}(q) v^k + \tfrac{1}{2}\beta_{ij}(q)\right] dq^i \wedge dq^j \ .$$

This 2–form is closed and nondegenerate, hence a symplectic 2–form on TQ, corresponding to ω_β on T^*Q through the isomorphism (2.16). The Hamiltonian function H_{T^*U} on T^*U derived by the usual procedure from L^U is given by
(2.27)
$$H_{T^*U}(q,p) = p_i v^i - L^U(q,v) = \tfrac{1}{2}\rho^{ij}(q)\left(p_i - \alpha^U_i(q)\right)\left(p_j - \alpha^U_j(q)\right) + u^U(q) \ ,$$

where $v = v(q,p)$ is determined by solving

$$p_i = \frac{\partial L^U}{\partial v^i}(q,v) = \rho_{ij}(q)v^j + \alpha^U_i(q) \ . \tag{2.28}$$

Although H_{T^*U} depends on α^U, the function $H_{TU} \equiv \left(\partial_{\mathrm{F}} L^U\right)^* H_{T^*U}$ on TU does not and gives a globally defined function H_{TQ} on TQ :

$$H_{TQ}(q,v) = \tfrac{1}{2}\,\rho_{ij}(q)v^i v^j + u(q) \ , \tag{2.29}$$

which is the pull–back of H of (2.5) by the isomorphism (2.16) (not by (2.23)). Thus we have the two Hamiltonian systems (T^*Q, ω_β, H) and $(TQ, \omega_{TQ}, H_{TQ})$, which are isomorphic under (2.16).

We also have another symplectic manifold. We assume that

(2.30) [Assumption of completeness of solutions] :

for any $(q_0, v_0) \in TQ$, the solution $q(t)$ of (2.18a) with

$$q(t_0) = q_0, \quad \dot{q}(t_0) = v_0 \tag{2.31}$$

(for fixed $t_0 \in \mathbf{R}$) can be extended to $-\infty < t < \infty$.

We denote by $\mathrm{Sol}\,Q$ the set of all solutions of (2.18a), then $\mathrm{Sol}\,Q \subset \mathrm{Path}\,Q$, where $\mathrm{Path}\,Q$ is the space of all paths on Q :

$$\mathrm{Path}\,Q \equiv \mathrm{Map}(\mathbf{R}, Q) \ . \tag{2.32}$$

For fixed t_0, we have the bijection

$$TQ \to \mathrm{Sol}\,Q \ ; \ (q_0, v_0) \mapsto [\text{ the solution of (2.18a) with (2.31) }] \ , \tag{2.33}$$

through which we can consider $\mathrm{Sol}\,Q$ as a symplectic manifold. The explicit form of the symplectic 2–form on $\mathrm{Sol}\,Q$ is given in §2.3 of [11].

§3 Noether's Theorem

We give here Noether's theorem for a system with a charged symplectic structure explained in §2. Supplementary arguments are given in Appendix A and B. We assume that the configuration space Q is connected and simply connected. We have on Q a Riemannian metric ρ, a potential u and a magnetic 2–form β. In the sense, we write the *charged system* as $(Q;\rho,u,\beta)$. We also impose the assumption (2.30).

Let $\mathbf{G}$ be a connected Lie group acting on $\mathrm{Path}\,Q$: (2.32). We write an element of $\mathrm{Path}\,Q$ as $q(\cdot)$, where $\cdot$ is the dummy of $t \in \mathbf{R}$, and write the action as $\gamma \cdot (q(\cdot)) \in \mathrm{Path}\,Q$ for $\gamma \in \mathbf{G}$. We denote by $\underline{\mathbf{G}}$ the Lie algebra of $\mathbf{G}$. We assume :

[Assumption of locality]

for any $\xi \in \underline{\mathbf{G}}$, there are C^∞–functions

$$A^i_\xi = A^i_\xi(t,q,v)\ , \quad (t,q,v) \in \mathbf{R} \times TQ\ , \tag{3.1}$$

for which we have

$$\tfrac{d}{d\epsilon}\, q^{\epsilon,i}(t)\ \Big|_{\epsilon=0} = A^i_\xi(t,q(t),\dot{q}(t))\ , \tag{3.2}$$

where $q(\cdot)$ is an arbitrary element of $\mathrm{Path}\,Q$ and

$$q^\epsilon(\cdot) \equiv (\exp - \epsilon\xi)\cdot(q(\cdot))\ . \tag{3.3}$$

As the formula (3.2) shows, for fixed $t \in \mathbf{R}$,

$$A^{(t)}_\xi \equiv A^i_\xi(t,q,v)\tfrac{\partial}{\partial q^i} \tag{3.4}$$

is a vector field on TQ, which is "horizontal" in the sense that there are no $\frac{\partial}{\partial v^i}$ terms.

The invariance of the system under the $\mathbf{G}$–action is stated as follows. Let $\{U,V,\cdots\}$ be the open covering (2.19) of Q and $L^U : TU \to \mathbf{R}$ the Lagrangian function (2.21) in §2. For each U of the covering and $\xi \in \underline{\mathbf{G}}$, we have a function

$$R^U_\xi = R^U_\xi(t,q,v)\ , \quad (t,q,v) \in \mathbf{R} \times TU\ , \tag{3.5}$$

such that for any $q(\cdot) \in \mathrm{Path}\, Q$, defining $q^\epsilon(\cdot)$ by (3.3), we have

$$\frac{d}{d\epsilon} L^U(q^\epsilon(t), \dot{q}^\epsilon(t)) \Big|_{\epsilon=0} = \frac{d}{dt} R^U_\xi(t, q(t), \dot{q}(t)) \tag{3.6}$$

as long as $q(t) \in U$. Then, as the usual Noether's theorem says, we have

$$\gamma \in \mathbf{G},\ q(\cdot) \in \mathrm{Sol}\, Q \ \Rightarrow\ \gamma \cdot (q(\cdot)) \in \mathrm{Sol}\, Q\ , \tag{3.7}$$

that is, $\mathrm{Sol}\, Q$ is an invariant subset of $\mathrm{Path}\, Q$. So, for fixed $t_0 \in \mathbf{R}$, we have an action of $\mathbf{G}$ on TQ through the bijection (2.33). We also write the action as

$$\gamma \cdot z \in TQ \quad \text{for } \gamma \in \mathbf{G},\ z \in TQ\ , \tag{3.8}$$

not showing t_0 explicitly. The vector field $V^{(t_0)}_\xi$ on TQ given by the action (3.8) :

$$V^{(t_0)}_\xi(z) \equiv \frac{d}{d\epsilon} (\exp - \epsilon\xi) \cdot z \Big|_{\epsilon=0}\ , \quad z \in TQ \tag{3.9}$$

is nothing but

$$V^{(t)}_\xi = A^i_\xi(t, q, v) \frac{\partial}{\partial q^i} + B^i_\xi(t, q, v) \frac{\partial}{\partial v^i}\ , \tag{3.10}$$

where

$$\begin{aligned} B^i_\xi(t, q, v) &\equiv \frac{d}{dt} A^i_\xi(t, q, v) \\ &\equiv \frac{\partial}{\partial t} A^i_\xi(t, q, v) + \frac{\partial}{\partial q^j} A^i_\xi(t, q, v)\, v^j + \frac{\partial}{\partial v^j} A^i_\xi(t, q, v)\, \dot{v}^j(q, v) \end{aligned} \tag{3.11}$$

with

$$\dot{v}^i(q, v) \equiv \rho^{ij}(q) \left[-\Gamma_{jk\ell}(q) v^k v^\ell - \partial_j u(q) + \beta_{jk}(q) v^k \right] \tag{3.12}$$

(see (2.18)). By the general argument of a group action on manifolds, (B.2), we have

$$\left[V^{(t)}_\xi,\ V^{(t)}_\eta \right] = V^{(t)}_{[\xi, \eta]}\ , \quad \text{for } \xi, \eta \in \underline{\mathbf{G}}\ . \tag{3.13}$$

We assume further that $A^i_\xi(t, q, v)$ in (3.1) is at most linear in v :

(3.14)

$$A^i_\xi(t, q, v) = D^i_\xi(t, q) + E_\xi{}^i{}_j(t, q) v^j \qquad \text{for some } D^i_\xi,\ E_\xi{}^i{}_j : \mathbf{R} \times Q \to \mathbf{R}\ .$$

Then we can show that, for fixed t (earlier written as t_0), the action (3.8) on TQ is symplectic with respect to ω_{TQ}, hence $V_\xi^{(t)}$ is a symplectic vector field on TQ (see (A.3)). Since we have assumed Q, hence TQ, is simply connected (see the statement above (B.4)), $V_\xi^{(t)}$ is globally Hamiltonian, that is, there is a function $I_\xi^{(t)} \in C^\infty(TQ)$ for which

$$-dI_\xi^{(t)} = \omega_{TQ}(V_\xi^{(t)}, \cdot) \ . \tag{3.15}$$

This function $I_\xi^{(t)}$ is the same as the one derived by the usual Noether's procedure :

$$\tfrac{\partial L^U}{\partial v^i}(q,v)\, A_\xi^i(t,q,v) - R_\xi^U(t,q,v) \ , \tag{3.16}$$

as long as $q \in U$ (up to constant, remark that we can add a constant to R_ξ^U), because $d(3.16) = d(\mathrm{A}.5) \overset{(\mathrm{A}.6)}{=\!=} -\omega_{TQ}(V_\xi^{(t)}, \cdot) \overset{(3.15)}{=\!=} dI_\xi^{(t)}$. Hence, by the usual argument of Noether's theorem, $I_\xi^{(t)}$ is constant along solutions, that is,

$$\tfrac{d}{dt} I_\xi^{(t)}(q(t), \dot{q}(t)) = 0 \quad \text{if } q(\cdot) \text{ is a solution of (2.18a)} \ . \tag{3.17}$$

By Prop. B.4, we have in our case, choosing $z = (q,v) \in TQ$ arbitrarily and adding constants to $I_\xi^{(t)}$'s if necessary,

$$\left\{ I_\xi^{(t)} \ , \ I_\eta^{(t)} \right\} = I_{[\xi,\eta]} + c_z(\xi,\eta) \ , \tag{3.18}$$

where $c_z : \underline{\mathbf{G}} \times \underline{\mathbf{G}} \to \mathbf{R}$ is the cocycle given by

$$c_z \ : \ (\xi,\eta) \mapsto \omega_{TQ}(V_\xi^{(t)}, V_\eta^{(t)}) \quad \text{at } z \ . \tag{3.19}$$

We can always impose that (see the statement below (B.10))

$$\text{the correspondence} \quad \xi \in \underline{\mathbf{G}} \ \mapsto \ I_\xi^{(t)} \in C^\infty(TQ) \quad \text{is linear} \ . \tag{3.20}$$

When our potential $u : Q \to \mathbf{R}$ has a critical point $q_0 \in Q$, that is,

$$\partial_i u(q_0) = 0 \quad \text{for all } i \ , \tag{3.21}$$

we can obtain a simpler formula for the cocycle term in (3.18) as follows. At $q = q_0$, $v = 0$, by (3.11) and (3.12), the vector field $B_\xi^{(t)} \equiv B_\xi^i(t,q,v)\frac{\partial}{\partial v^i}$ has the simpler form :

$$B_\xi^{(t)}(q_0,0) = \tfrac{\partial}{\partial t} A_\xi^i(t,q_0,0)\, \tfrac{\partial}{\partial v^i} \ . \tag{3.22}$$

On the other hand, the 2–form ω_{TQ} of (2.26) has the shape at $(q,0)$:

$$\omega_{TQ} \text{ at } (q,0) \;=\; \rho_{ij}(q)dv^i \wedge dq^j + \beta_q \;, \tag{3.23}$$

where β_q means β at q. Hence the 2–cocycle (3.19) at $z_0 = (q_0, 0)$ is given as

$$\begin{aligned} c_{z_0}(\xi,\eta) &= \omega_{TQ,z_0}(V_\xi^{(t)}, V_\eta^{(t)}) \\ &= \rho_{ij}(q_0)\left[B_\xi^i(t,q_0,0)A_\eta^j(t,q_0,0) - A_\xi^i(t,q_0,0)B_\eta^j(t,q_0,0)\right] \\ &\quad + \beta_{q_0}\left(A_\xi^{(t)}(q_0,0), A_\eta^{(t)}(q_0,0)\right) \;. \end{aligned} \tag{3.24}$$

Thus we have proved

Theorem 3.1. *Let Q be a connected and simply connected configuration space, on which we have a Riemannian metric ρ, a potential u and a magnetic 2–form β. We assume* (2.30). *We also have a connected Lie group* $\mathbf{G}$ *acting on* Path Q, *having a local property* (3.2) *with* (3.14). *We assume that the system $(Q;\rho,u,\beta)$ is invariant under the* $\mathbf{G}$*–action in the sense that, for each U of an open covering* (2.19) *of Q and each $\xi \in \underline{\mathbf{G}}$, we have $R_\xi^U : \mathbf{R} \times TU \to \mathbf{R}$, for which* (3.6) *is satisfied.*

Then, for all $\xi \in \underline{\mathbf{G}}$ and $t \in R$, we have $I_\xi^{(t)} : TQ \to \mathbf{R}$, which is constant along solutions of (2.18a) *and gives the vector field $V_\xi^{(t)}$ on TQ expressing the* $\mathbf{G}$*–action* (3.8) *in the sense of* (3.9), *through* (3.15). *The explicit form of $I_\xi^{(t)}$ is given by* (3.16) *on each U up to constant. Adding constants to $I_\xi^{(t)}$'s if necessary, the correspondence $\xi \in \underline{\mathbf{G}} \;\mapsto\; I_\xi^{(t)} \in C^\infty(TQ)$ is linear and their Poisson brackets satisfy the relation* (3.18).

When the potential $u : Q \to \mathbf{R}$ has a critical point $q_0 \in Q$, the relation becomes

$$\left\{I_\xi^{(t)},\, I_\eta^{(t)}\right\} = I_{[\xi,\eta]}^{(t)} + c_{q_0}(\xi,\eta) \;, \quad \text{for } \xi,\eta \in \underline{\mathbf{G}} \;, \tag{3.25}$$

where $c_{q_0} : \underline{\mathbf{G}} \times \underline{\mathbf{G}} \to \mathbf{R}$ is the cocycle given by

(3.26)

$$c_{q_0}(\xi,\eta) = \left\langle \dot{A}_\xi^{(t)},\, A_\eta^{(t)} \right\rangle - \left\langle A_\xi^{(t)},\, \dot{A}_\eta^{(t)} \right\rangle + \beta(A_\xi^{(t)}, A_\eta^{(t)}) \quad \text{at } q = q_0, v = 0 \;.$$

Here $\langle\ ,\ \rangle$ is the Riemannian metrin ρ on Q, $A_\xi = A_\xi(t,q,v)$ is the one in (3.1), *and $\dot{A}_\xi^{(t)} = \frac{\partial}{\partial t} A_\xi(t,q,v)$.*

§4 Wess–Zumino–Witten Model

We now here apply the formalism developed in §2 and §3 to the Wess–Zumino–Witten model. We take the loop group LG, which is connected and simply connected (G is assumed to be simply connected), as the configuration space Q. As the magnetic 2–form β on $Q = LG$, we take

$$(4.1) \quad \beta_{g(\square)} : \quad T_{g(\square)}Q \times T_{g(\square)}Q \rightarrow \mathbf{R} ,$$

$$(h_1(\square), h_2(\square)) \mapsto 2\lambda^2 K \fint \Omega_{g(\sigma)}(g'(\sigma), h_1(\sigma), h_2(\sigma))$$

where $\Omega = \Omega_G$ is the 3–form (1.4) on G. The Riemannian metric ρ on Q is given by

$$(4.2) \quad \rho_{g(\square)} : T_{g(\square)}Q \times T_{g(\square)}Q \rightarrow \mathbf{R} ; \ (h_1(\square), h_2(\square)) \mapsto \fint \langle h_1(\sigma), h_2(\sigma)\rangle_{g(\sigma)} ,$$

and we define the potential u by

$$(4.3) \qquad u : Q \rightarrow \mathbf{R} ; \ g(\square) \mapsto \tfrac{1}{2} \fint \langle g'(\sigma), g'(\sigma) \rangle_{g(\sigma)} ,$$

where $\langle \ , \ \rangle_g = \rho_{G,g}$ is the Riemannian metric on G given by (1.10).

This charged formalism, $(Q; \rho, u, \beta)$, is equivalent to the system described by the would–be Lagrangian density "$\mathcal{L}$" of (1.9) in the following sense. The function L_0 of (2.8) for our ρ of (4.2) and u of (4.3) corresponds to the σ–integral of the term $\mathcal{L}_0$ of "$\mathcal{L}$" with $\partial_0\phi \leftrightarrow h(\sigma)$, $\partial_1\phi \leftrightarrow g'(\sigma)$ in mind. To treat the term "$\mathcal{L}$"$_{\mathrm{WZ}}$, we take an open covering

$$(4.4) \qquad \{U', V', \cdots\}$$

of G, on each open set U' of which we have a 1–form $\omega^{U'}$ on U' satisfying $d\omega^{U'} = \Omega_{|U'}$. This can be done because Ω is closed. As the open subset U of Q, we take

$$(4.5) \qquad U = \{g(\square) \in Q \ ; \ g(S^1) \subset U'\} ,$$

and as the 1–form α^U on U, we take

$$(4.6) \qquad \alpha^U_{g(\square)}(h(\square)) = 2\lambda^2 K \fint \omega^{U'}_{g(\sigma)}(h(\sigma), g'(\sigma)) , \quad (g(\square), h(\square)) \in TU .$$

Then $\alpha^U_{g(\square)}(h(\square))$, whose finite dimensional shape is $\alpha^U_i(q)v^i$, gives the σ–integral of "$\mathcal{L}$"$_{\rm WZ}$. Hence the Lagrangian function L^U of the form (2.21) is the σ–integral of the whole "$\mathcal{L}$".

The 2–form β of (4.1) is obtained as $d\alpha^U$ on each U. In fact, in the coordinate system $g \leftrightarrow \phi = (\phi^i)$, $h \leftrightarrow \psi = (\psi^i)$, we have

$$\alpha_{\phi(\square)}(\psi(\square)) = 2\lambda^2 K \fint \omega_{ij(\phi(\sigma))}\psi^i(\sigma)\phi^{j\prime}(\sigma) \ , \tag{4.7}$$

where $\alpha^U \leftrightarrow \alpha_{\phi(\square)}$ and $\omega_{ij}(\phi)$ is the component of $\omega^{U'}$, and

$$\begin{aligned} d\alpha_{\phi(\square)}&(\psi_1(\square), \psi_2(\square)) \\ &= \alpha_{\phi(\square)}(\psi_1(\square)) + \alpha_{\phi(\square)+\psi_1(\square)}(\psi_2(\square)) \\ &\qquad -\alpha_{\phi(\square)}(\psi_2(\square)) - \alpha_{\phi(\square)+\psi_2(\square)}(\psi_1(\square)) \\ &= 2\lambda^2 K \fint \Big[\{\partial_j\omega_{ki(\phi(\sigma))} - \partial_k\omega_{ji(\phi(\sigma))}\}\, \phi^{i\prime}(\sigma)\psi_1^j(\sigma)\psi_2^k(\sigma) \\ &\qquad\qquad -\omega_{jk(\phi(\sigma))}\tfrac{d}{d\sigma}\left\{\psi_1^j(\sigma)\psi_2^k(\sigma)\right\} \Big] \\ &\overset{(1.13)}{=} 2\lambda^2 K \fint \Omega_{ijk(\phi(\sigma))}\phi^{i\prime}(\sigma)\psi_1^j(\sigma)\psi_2^k(\sigma) \ , \end{aligned} \tag{4.8}$$

which has a global meaning on Q, giving (4.1). This also shows that β is closed. The derivation of β is the original one. We reversed the order of the construction in §2.

We now apply Noether's theorem of §3 to our system. In this case,

$$\mathrm{Path}\, Q = \mathrm{Map}(\mathbf{R}, \mathrm{Map}(S^1, G)) = \mathrm{Map}(\mathbf{R} \times S^1, G) \ . \tag{4.9}$$

We see, by the form (1.19) of solutions, that the assumption (2.30) is satisfied.

We first consider the half of the loop group symmetry, that is, $\gamma_- = i_d$ in (1.20). As (1.21) shows, for $\xi = \xi(x^+) \in Lg$, the vector field $A^{(\tau)}_\xi$ on TQ (we use here τ in stead of t) of (3.4) is in our case given by

$$A^{(\tau)}_\xi \text{ at } (g(\square), h(\square)) \ = \ g(\square)\xi(\tau + \square) \ , \tag{4.10}$$

which does not depend on $h(\square)$, hence satisfies the condition (3.14) with $E_\xi{}^i{}_j = 0$.

To prove the invariance in the sense of (3.6), we introduce a coordinate system. Let U' be an open subset of G in the covering (4.4), which is assumed to be a coordinate neighborhood, having an (onto) diffeomorphism for some open subset U'' of $\mathbf{R}^{\dim G}$:

$$\chi \ : \ U'' \to U' \ . \tag{4.11}$$

The coordinates $\phi = (\phi^i)$ used earlier in (1.8) or (4.7) is considered as a point of U''. We take an orthonormal basis t_a of $\mathbf{g}$; $\langle t_a, t_b \rangle = \delta_{ab}$, and put

$$L_{ia}(\phi) = \left\langle \chi^{-1}(\phi)\partial_i\chi(\phi) \, , \, t_a \right\rangle \ , \quad \phi \in U'' \ , \tag{4.12}$$

where $\partial_i\chi = \partial\chi/\partial\phi^i$, hence $\partial_i\chi(\phi) \in T_{\chi(\phi)}G$, and $\chi^{-1}(\phi)$ means the inverse element of $\chi(\phi)$ in G (not the inverse function $U' \to U''$). We write the transformation (1.21) in the coordinate system as

$$\phi(x) \to \phi^\epsilon(x) = \phi(x) + \epsilon\Phi(x) \ , \tag{4.13}$$

which means

$$g(x) = \chi(\phi(x)) \ , \quad g^\epsilon(x) \equiv g(x) + \epsilon g(x)\xi(x^+) = \chi(\phi^\epsilon(x)) \ , \tag{4.14}$$

then we have

$$\Phi^i(x) = L^i{}_a(\phi(x))\xi^a(x^+) \ , \tag{4.15}$$

where $L^i{}_a = \rho^{ij}L_{ja}$, $(\rho^{ij}) = (\rho_{ij})^{-1}$,

$$\rho_{ij}(\phi) \equiv \left\langle \chi^{-1}(\phi)\partial_i\chi(\phi) \, , \, \chi^{-1}(\phi)\partial_j\chi(\phi) \right\rangle \ , \tag{4.16}$$

written as $\rho_{G,ij}(\phi)$ in (1.9b), and $\xi(x^+) = \xi^a(x^+)t_a$.

The change of $\mathcal{L}(\phi)$, where we drop " " for simplicity, under the transformation (1.21) is given by (see Appendix C)

$$\frac{d}{d\epsilon}\mathcal{L}(\phi^\epsilon)\Big|_{\epsilon=0} = \partial_\mu R^\mu_\xi \ , \tag{4.17}$$

where

$$R^\mu_\xi = -\epsilon^{\mu\nu}\left[\rho_{ij}(\phi) - \tilde{\omega}_{ij}(\phi)\right] L^i{}_a(\phi)\partial_\nu\phi^j\xi^a(x^+) \ , \tag{4.18}$$

with $\tilde{\omega}_{ij} = 2\lambda^2 K\omega_{ij}$. The Noether current is then

$$I^\mu_\xi = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\Phi^i - R^\mu_\xi = (\eta^{\mu\nu} + \epsilon^{\mu\nu})\, L_{ia}(\phi)\, \xi^a(x^+)\, \partial_\nu\phi^i \ , \tag{4.19}$$

which has a global meaning (the dependence on ω_{ij} has disappeared) as it should be, and is conserved : $\partial_\mu I^\mu_\xi = 0$ if a solution ϕ is substituted.

The Noether charge is the σ–integral of the $\mu = 0$ component of I^μ_ξ :

$$(4.20)\quad I^{(\tau)}_\xi = \fint I^0_\xi = \fint L_{ia}(\phi)\,\xi^a(x^+)\,(\partial_0+\partial_1)\phi^i = \fint \left\langle 2g^{-1}\partial_+ g, \xi(x^+)\right\rangle ,$$

giving (1.22) (τ of $I^{(\tau)}_\xi$ means x^0 of $x^+ = x^0 + x^1$ in $\xi(x^+)$).

Now we can obtain the cocycle term of (1.25) by Theorem 3.1. Let $g_0(\square) \in Q = LG$ be the constant loop at the identity 1_G : $g_0(\sigma) \equiv 1_G$. It is easily seen that, for the potential $u : Q \to \mathbf{R}$ of (4.3), and the magnetic 2–form β of (4.1),

(4.21) $\quad g_0$ is a critical point of u,

(4.22) $\quad \beta = 0$ at g_0.

We also have from (4.10)

$$(4.23)\quad A^{(\tau)}_\xi = \xi(\tau+\square)\ , \quad \dot{A}^{(\tau)}_\xi = \xi'(\tau+\square) \quad \text{at } g(\square) = g_0 \text{ and } h(\square) = 0\ ,$$

where $A^{(\tau)}_\xi$ corresponds to $A^{(t)}_\xi$ in Theorem 3.1 and ξ' means $\frac{d}{dx^+}\xi$.

Hence, when we assume (1.17) : $2\lambda^2 K = 4\pi$, the formula (3.26) gives

$$\begin{aligned}(4.24)\quad c_{g_0}(\xi,\eta) &= \langle \dot{A}^{(\tau)}_\xi,\ A^{(\tau)}_\eta\rangle - \langle A^{(\tau)}_\xi,\ \dot{A}^{(\tau)}_\eta\rangle \\ &= \fint [\,\langle \xi'(\tau+\sigma),\ \eta(\tau+\sigma)\rangle - \langle \xi(\tau+\sigma),\ \eta'(\tau+\sigma)\rangle\,] \\ &= \frac{1}{2\lambda^2}\cdot 2\int_0^{2\pi} d\sigma\,\langle \xi'(\sigma),\ \eta(\sigma)\rangle \\ &= \frac{K}{2\pi}\int_0^{2\pi} d\sigma\,\langle \xi'(\sigma),\ \eta(\sigma)\rangle\ ,\end{aligned}$$

yielding (1.25) with (1.26).

The first impression is that the cocycle term of (1.25) originates from the β–term, which contains K, in (3.26). However, by (4.22), it is 0 and the level K appears through the relation $2\lambda^2 K = 4\pi$.

Next we consider the conformal symmetry (1.27) or (1.29). (In this case we may not assume $2\lambda^2 K = 4\pi$.) The vector field $A^{(\tau)}_E$ on TQ of (3.4) for $E \in \underline{\mathrm{Diff}\,S^1}$ is given, from (1.29), as

$$(4.25)\qquad A^{(\tau)}_E \text{ at } (g(\square), h(\square)) = \tfrac{1}{2}\,E(\tau+\square)\,(h(\square)+g'(\square))\ ,$$

which is also of the form (3.14). The change of $\mathcal{L}$ under (1.29) is given by (see (C.10) and (C.11))

(4.26) $$\frac{d}{d\epsilon}\mathcal{L}(\phi^\epsilon)\Big|_{\epsilon=0} = \partial_\mu R^\mu_E \ ,$$

with

(4.27) $$R^\mu_E = \tfrac{1}{2}(\delta^\mu_0 + \delta^\mu_1)\, E(x^+)\, \mathcal{L}$$

($\phi^\epsilon = \phi(x) + \epsilon E(x^+)\partial_+\phi(x)$, see the statement above (C.9)). Hence the Noether current is

(4.28) $$\begin{aligned} I^\mu_E &\equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\left(\frac{d}{d\epsilon}\phi^{\epsilon,i}\right)\Big|_{\epsilon=0} - R^\mu_E \\ &= \tfrac{1}{2}E(x^+)\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}(\partial_0\phi^i + \partial_1\phi^i) - (\delta^\mu_0 + \delta^\mu_1)\,\mathcal{L}\right] \\ &= \tfrac{1}{2}E(x^+)\left[T^\mu{}_0 + T^\mu{}_1\right] \ , \end{aligned}$$

where $T^\mu{}_\nu$ is the energy momentum tensor :

(4.29a) $$\begin{aligned} T^\mu{}_\nu &\equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\partial_\nu\phi^i - \delta^\mu_\nu\,\mathcal{L} \\ &= (\eta^{\mu\kappa}\rho_{ij}(\phi) + \epsilon^{\mu\kappa}\tilde{\omega}_{ij}(\phi))\,\partial_\nu\phi^i\partial_\kappa\phi^j \\ &\quad - \delta^\mu_\nu\left[\tfrac{1}{2}\rho_{ij}(\phi)\left(\partial_0\phi^i\partial_0\phi^j - \partial_1\phi^i\partial_1\phi^j\right) + \tilde{\omega}_{ij}(\phi)\,\partial_0\phi^i\partial_1\phi^j\right]. \end{aligned}$$

The ω_{ij}–terms in $T^\mu{}_\nu$ disappear :

(4.29b) $$T^0{}_0 = -T^1{}_1 = \tfrac{1}{2}\rho_{ij}(\phi)\left[\partial_0\phi^i\partial_0\phi^j + \partial_1\phi^i\partial_1\phi^j\right] ,$$

(4.29c) $$T^0{}_1 = -T^1{}_0 = \rho_{ij}(\phi)\,\partial_0\phi^i\partial_1\phi^j \ .$$

These also show that (cf. §2.1.3 of [2])

(4.30) $$\text{trace}\ (T^\mu{}_\nu) \equiv T^0{}_0 + T^1{}_1 = 0 \ ,$$

(4.31) $$T_{\mu\nu} : \text{symmetric, i.e., } T_{01} = T_{10} \quad (T_{\mu\nu} \equiv \eta_{\mu\kappa}T^\kappa{}_\nu) \ .$$

The corresponding Noether charge is

$$(4.32)\qquad I_E^{(\tau)} = \oint I_E^0 = \oint \tfrac{1}{2} E(\tau+\sigma)\left[T^0{}_0 + T^0{}_1\right] = \oint E(\tau+\sigma)\, T_{++}$$

where, putting $T = T_{\mu\nu}\, dx^\mu \otimes dx^\nu$,

(4.33)

$$\begin{aligned} T_{++} &\equiv T(\partial_+, \partial_+) = \tfrac{1}{4}(T_{00} + T_{01} + T_{10} + T_{11}) = \tfrac{1}{2}(T_{00} + T_{01}) \\ &= \rho_{ij}(\phi)\partial_+\phi^i\partial_+\phi^j = \langle g^{-1}\partial_+ g\,,\ g^{-1}\partial_+ g\rangle\,, \end{aligned}$$

giving (1.30).

The cocycle term, which is 0, in (1.32) is obtained using Theorem 3.1 as follows. At $g_0 \in Q$ of (4.21) and $h(\square) = 0$, from (4.25), we have

$$(4.34)\qquad A_E^{(\tau)} \equiv 0\,,\quad \dot{A}_E^{(\tau)} \equiv 0\,.$$

Thus by (3.26) and (4.22), we see that the cocycle term vanishes.

(The calculations based on Theorem 3.1 are the ones up to constant. It may be desirable to prove (1.25) and (1.32) directly. The argument in Appendix D gives (1.25). For (1.32), by (i) of Lemma B.1, we know that $\left\{I_{E_1}^{(\tau)},\ I_{E_2}^{(\tau)}\right\} - I_{[E_1,E_2]}^{(\tau)}$ is a constant on TQ. At $g(\square) = g_0$ and $h(\square) = 0$, it is easy from (1.31) to see that $I_E^{(\tau)} = 0$ and

$$\frac{\delta I_E^{(\tau)}}{\delta g(\sigma)} = \frac{\delta I_E^{(\tau)}}{\delta h(\sigma)} = 0\,,$$

hence the constant is 0.)

Appendix A. Supplement to §3

In this appendix, we fix $\xi \in \underline{\mathbf{G}}$ and drop it from the notations such as $A^{(t)}_{\xi}$, R^{U}_{ξ}, ..., etc. in §3. We argue on the symplectic manifold (TQ, ω_{TQ}) and write ω_{TQ} as ω. A similar calculation is done in Appendix of [5]. See also [4],

Let U be an open subset of Q in the open covering (2.19) and consider the 1–form θ_{TU} on TU given by (2.24) :

$$\theta_{TU} = \tfrac{\partial L^U}{\partial v^i}\, dq^i = \left[\rho_{ki}(q)v^k + \alpha^U_i(q)\right] dq^i \ . \tag{A.1}$$

The key formula of our argument is

$$\mathcal{L}_{V^{(t)}}\, \theta_{TU} = dR^{(t)U} \ , \tag{A.2}$$

where $V^{(t)} = A^{(t)i}\frac{\partial}{\partial q^i} + B^{(t)i}\frac{\partial}{\partial v^i}$ is the vector field on TQ given by (3.9) or (3.10), $R^{(t)U}$ is the function (3.5) on TU and $\mathcal{L}_V$ means the Lie derivative. We use the formulas about $\mathcal{L}_V$ freely. See, for example, p. 121 of [1].

Since $\omega = d\theta_{TU}$ on TU, the formula (A.2) gives

$$V^{(t)} \text{ is symplectic, that is, } \mathcal{L}_{V^{(t)}}\omega = 0 \ , \tag{A.3}$$

in fact $\mathcal{L}_{V^{(t)}}\omega = \mathcal{L}_{V^{(t)}}(d\theta_{TU}) = d(\mathcal{L}_{V^{(t)}}\theta_{TU}) \overset{(A.2)}{=} ddR^{(t)U} = 0$.

The formula (A.2) also yields : the function

$$I^U(t,q,v) = \left[\rho_{ij}(q)v^j + \alpha^U_i(q)\right] A^i(t,q,v) - R^U(t,q,v) \tag{A.4}$$

of (3.16), which is also written as

$$I^{(t)U}(q,v) = \left\langle \theta_{TU}\,,\ A^{(t)}(q,v)\right\rangle - R^{(t)U}(q,v) \ , \tag{A.5}$$

satisfies

$$-dI^{(t)U} = \omega(V^{(t)}, \cdot) \quad \text{on } TU \ , \tag{A.6}$$

because

$$\begin{aligned} \omega(V^{(t)}, \cdot) &= i(V^{(t)})\cdot\omega = i(V^{(t)})\cdot d\theta_{TU} \\ &= \left[\mathcal{L}_{V^{(t)}} - d\cdot i(V^{(t)})\right]\theta_{TU} \overset{(A.2)}{=} dR^{(t)U} - d\left\langle \theta_{TU}\,,\ V^{(t)}\right\rangle \\ &= -d\left[\left\langle \theta_{TU}\,,\ A^{(t)}\right\rangle - R^{(t)U}\right] = -dI^{(t)U} \ . \end{aligned} \tag{A.7}$$

We now prove the formula (A.2). We have assumed $A^{(t)}$ is of the form (3.14) :

$$A^{(t)i}(q,v) = D^{(t)i}(q) + E^{(t)i}{}_j(q)\, v^j \ . \tag{A.8}$$

For $\forall q(\cdot) \in \mathrm{Path}\, Q$, putting $q^\epsilon(\cdot) \equiv (\exp - \epsilon\xi)\cdot(q(\cdot))$, we have (3.2) :

$$\tfrac{d}{d\epsilon}\, q^{\epsilon,i}(t)\Big|_{\epsilon=0} = A^{(t)i}(q(t), \dot q(t)) \ . \tag{A.9}$$

Thus, for the Lagrangian $L^U(q,v)$ of (2.21), we have, omitting ^{U} and $^{(t)}$,

(A.10)

$$\begin{aligned}
\tfrac{d}{d\epsilon}\, L\left(q^\epsilon(t), \dot q^\epsilon(t)\right)\Big|_{\epsilon=0} &= \tfrac{\partial L}{\partial q^i}(q,\dot q)\, A^i(q,\dot q) + \tfrac{\partial L}{\partial v^i}(q,\dot q)\, \tfrac{d}{dt} A^i(q,\dot q) \\
&= \left[-u_{;i}\, D^i + \alpha_i\, \dot D^i\right] \\
&+ \left[-u_{;i}\, E^i{}_\ell + \alpha_{\ell;i}\, D^i + \rho_{i\ell}\, \dot D^i + \alpha_i\, D^i{}_{;\ell} + \alpha_i\, \dot E^i{}_\ell\right] \dot q^\ell \\
&+ \alpha_i\, E^i{}_\ell\, \ddot q^\ell \\
&+ \left[\tfrac{1}{2}\, \rho_{\ell m;i}\, D^i + \alpha_{\ell;i}\, E^i{}_m + \rho_{i\ell}\, \dot E^i{}_m + \rho_{i\ell}\, D^i{}_{;m} + \alpha_i\, E^i{}_{m;\ell}\right] \dot q^\ell \dot q^m \\
&+ \rho_{i\ell}\, E^i{}_m\, \dot q^\ell \ddot q^m + \left[\tfrac{1}{2}\, \rho_{\ell m;i}\, E^i{}_n + \rho_{i\ell}\, E^i{}_{m;n}\right] \dot q^\ell \dot q^m \dot q^n \ ,
\end{aligned}$$

where $u_{;i}$ means $\frac{\partial u}{\partial q^i}$, etc., and $\dot D^i$ means $\frac{\partial}{\partial t} D^i(t,q,v)$, etc.

The function $R^{(t)}(q,v)$ of (3.5) must have the form

$$R^{(t)}(q,v) = W^{(t)}(q) + X^{(t)}_\ell(q)\, v^\ell + \tfrac{1}{2}\, Y^{(t)}_{\ell m}(q)\, v^\ell\, v^m \ , \quad Y^{(t)}_{m\ell} = Y^{(t)}_{\ell m} \ , \tag{A.11}$$

because terms of higher order in v produce, in $\frac{d}{dt} R^{(t)}(q,\dot q)$, terms which are absent in (A.10). For the function (A.11), we have

(A.12)

$$\begin{aligned}
\tfrac{d}{dt}\, R^{(t)}(q,\dot q) &= \dot W + \left[W_{;\ell} + \dot X_\ell\right] \dot q^\ell + X_\ell\, \ddot q^\ell + \left[X_{\ell;m} + \tfrac{1}{2}\, \dot Y_{\ell m}\right] \dot q^\ell \dot q^m \\
&\quad + Y_{\ell m}\, \dot q^\ell \ddot q^m + \tfrac{1}{2}\, Y_{\ell m;n}\, \dot q^\ell \dot q^m \dot q^n \ .
\end{aligned}$$

Since (3.6) is valid for any $q(\cdot) \in \mathrm{Path}\, Q$, (A.10) and (A.12) must be equal term by term, so we have

(A.13a) $$-u_{;i}\, D^i + \alpha_i\, \dot{D}^i = \dot{W}\ ,$$

(A.13b) $$-u_{;i}\, E^i{}_\ell + \alpha_{\ell;i}\, D^i + \rho_{i\ell}\dot{D}^i + \alpha_i\, D^i{}_{;\ell} = W_{;\ell}\ ,$$

(A.13c) $$\alpha_i\, E^i{}_\ell = X_\ell\ ,$$

(A.13d) $$\rho_{\ell m;i}\, D^i + \rho_{i\ell}\, D^i{}_{;m} + \rho_{im} D^i{}_{;\ell} + \dot{E}_{\ell m} + \beta_{i\ell}\, E^i{}_m + \beta_{im} E^i{}_\ell = 0\ ,$$

(A.13e) $$\rho_{i\ell}\, E^i{}_m = Y_{\ell m}\ ,$$

(A.13f)

$$\begin{aligned}
&\rho_{\ell m;i} E^i{}_n + \rho_{mn;i} E^i{}_\ell + \rho_{n\ell;i} E^i{}_m \\
&= -Y_{\ell n;m} - Y_{m\ell;n} - Y_{nm;\ell} \\
&\quad + (\rho_{i\ell;n} + \rho_{in;\ell}) E^i{}_m + (\rho_{i\ell;m} + \rho_{im;\ell}) E^i{}_n + (\rho_{im;n} + \rho_{in;m}) E^i{}_\ell\ .
\end{aligned}$$

We used (A.13c) in (A.13b) and (A.13d). In (A.13d), the coefficient of $\dot{q}^\ell \dot{q}^m$ was compared after symmetrizing in $\ell \leftrightarrow m$, and (A.13f) was also obtained after symmetrizing in $\ell \leftrightarrow m \leftrightarrow n$. By (A.13e), $E_{\ell m} \equiv \rho_{\ell i}\, E^i{}_m$ is symmetric in $\ell \leftrightarrow m$. The derivation of (A.13f) used (A.13e).

Now we have

(A.14a)

$$\begin{aligned}
\mathcal{L}_{V^{(t)}}\, \theta_{TU} &= \left[V^{(t)} \cdot (\rho_{\ell m}\, v^\ell + \alpha_m) \right] dq^m + (\rho_{\ell m}\, v^\ell + \alpha_m)\, d(V^{(t)} \cdot q^m) \\
&= (\mathrm{I})_M\, dq^M + (\mathrm{II})_M\, dv^M
\end{aligned}$$

with

(A.14b)

$$
\begin{aligned}
(\mathrm{I})_M \quad &= \alpha_{M\,;i}\, D^i + \rho_{iM}\, \dot{D}^i - E^k{}_M\, u_{;k} + \alpha_i\, D^i{}_{;M} \\
&+ \Big\{ \rho_{\ell M\,;i}\, D^i + \alpha_{M\,;i}\, E^i{}_\ell + \rho_{iM} \Big(\dot{E}^i{}_\ell + D^i{}_{;\ell} + E^i{}_j\, \rho^{jk}\, \beta_{k\ell} \Big) \\
&\qquad + \rho_{\ell i}\, D^i{}_{;M} + \alpha_i\, E^i{}_{\ell\,;M} \Big\}\, v^\ell \\
&+ \Big\{ \rho_{\ell M\,;i}\, E^i{}_m + \rho_{iM} (E^i{}_{\ell\,;m} - E^i{}_j\, \Gamma^j{}_{\ell m}) + \rho_{\ell i}\, E^i{}_{m\,;M} \Big\}\, v^\ell v^m \quad ,
\end{aligned}
$$

(A.14c) $\quad (\mathrm{II})_M \;=\; \alpha_i\, E^i{}_M + \rho_{\ell i}\, E^i{}_M\, v^\ell \;.$

We also have

(A.15) $\; dR = \Big[W_{;M} + X_{\ell\,;M}\, v^\ell + \frac{1}{2} Y_{\ell m\,;M}\, v^\ell v^m \Big]\, dq^M + \Big[X_M + Y_{\ell M}\, v^\ell \Big]\, dv^M \;.$

We compare the coefficients in (A.14) and (A.15). The coefficients of dv^M are equal by (A.13c) and (A.13e). In the coefficients of dq^M, the terms without v are equal by (A.13b), the terms linear in v by (A.13c) and (A.13d). For the terms containing $v^\ell v^m$, we symmetrize in $\ell \leftrightarrow m$ for (A.14b) and see that they are equal by (4.13f), proving (A.2).

Appendix B. Symplectic Actions on Manifold

Let $\mathbf{G}$ be a (connected) Lie group acting on a C^∞–manifold M. For $\xi \in \underline{\mathbf{G}}$, the Lie algebra of $\mathbf{G}$, we define $V_\xi \in \mathrm{V}(M)$, where $\mathrm{V}(M)$ means the space of all vector fields on M, by

(B.1) $$V_\xi(z) \equiv \tfrac{d}{d\epsilon} (\exp - \epsilon\xi) \cdot z \Big|_{\epsilon=0} \;, \quad z \in M \;.$$

Then we have (Prop. 4.1.26 of [1], remark the sign of Def. 4.1.24)

(B.2) $$[\, V_\xi \,,\, V_\eta \,] = V_{[\xi\,,\eta]} \quad \text{for} \quad \xi, \eta \in \underline{\mathbf{G}} \;.$$

We further assume that there is a symplectic structure ω on M, that is, ω is a non–degenerate closed 2–form on M, and the action of $\mathbf{G}$ on M is symplectic in the sense that $\gamma^* \omega = \omega$ for $\forall \gamma \in \mathbf{G}$, where $\gamma \in \mathbf{G}$ is

regarded as $\gamma : M \longrightarrow M; z \mapsto \gamma \cdot z$. In the Lie algebra level, this means

$$\mathcal{L}_{V_\xi}\,\omega = 0 \quad \text{for} \quad \forall \xi \in \underline{\mathbf{G}}\,. \tag{B.3}$$

See §24, 25, 26 of [3] or Chap. 3 of [11].

From now on, we assume that M is connected and simply connected. Since

$$d\,[\,\omega(V_\xi,\cdot)\,] = d\,[\,i(V_\xi)\cdot\omega\,] = \left[\mathcal{L}_{V_\xi} - i(V_\xi)\cdot d\right]\omega = \mathcal{L}_{V_\xi}\omega - i(V_\xi)\cdot(d\omega) \overset{(B.3)}{=\!=} 0\,,$$

we have a function $f_\xi \in C^\infty(M)$ satisfying

$$-d\,f_\xi = \omega(V_\xi,\cdot)\,. \tag{B.4}$$

The function f_ξ is determined up to a constant on M. The situation is shown by the following exact diagram (p. 184 of [3]) :

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{R} \arrow[r] & C^\infty(M) \arrow[r, "\Phi"] & \mathrm{V}^{\mathrm{H}}(M) \arrow[r] & 0 \\
 & & & \underline{\mathbf{G}} \arrow[u, "\Psi"'] \arrow[ul, "\lambda"] &
\end{tikzcd} \tag{B.5}$$

Here Φ is the mapping $f \in C^\infty(M) \mapsto V_f \in \mathrm{V}(M)$ with

$$-d\,f = \omega(V_f,\cdot) \tag{B.6}$$

which is a Lie algebra homomorphism :

$$[\,\Phi(f)\,,\ \Phi(g)\,] = \Phi(\{f,g\})\ ,\ \text{i.e.,}\quad [\,V_f\,,\ V_g\,] = V_{\{f,g\}}\ , \tag{B.7}$$

where $\{f,g\}$ for $f,\ g \in C^\infty(M)$ is the Poisson bracket defined by

$$\{f,g\} \equiv \omega(V_f,V_g) = V_f\cdot g\,. \tag{B.8}$$

We denote by $\mathrm{V}^{\mathrm{H}}(M)$ the space of Hamiltonian vector fields on M (the image of Φ) which is closed under [,]. In our case of simply connected M, it is equal to the space of symplectic vector fields ($V \in \mathrm{V}(M)$ is called *symplectic* if $\mathcal{L}_V\,\omega = 0$). The mapping Ψ is the correspondence $\xi \mapsto V_\xi$, which is a Lie algebra homomorphism by (B.2), and $\lambda : \underline{\mathbf{G}} \longrightarrow C^\infty(M)$ is the correspondence $\xi \mapsto f_\xi$. Then (B.4) means the commutativity of the diagram (B.5) :

$$\Phi\circ\lambda = \Psi\,. \tag{B.9}$$

We always can take λ as a linear mapping, adding a constant to $f_\xi = \lambda(\xi)$ if necessary (choose a linear basis of $\underline{\mathbf{G}}$, fix f_ξ arbitrarily for them and extend it to the whole $\underline{\mathbf{G}}$ linearly), so hereafter we impose λ to be linear.

Although Φ and Ψ are Lie algebra homomorphisms, λ is not necessarily so in general. This means that in stead of $\{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi, \eta])$, we may have

$$\{\lambda(\xi), \lambda(\eta)\} = \lambda([\xi, \eta]) + c_\lambda(\xi, \eta) . \tag{B.10}$$

To treat such a situation, the concept of the Lie algebra cohomology is particularly suitable (see, for example, §4.1 of [6] or §24 of [3]). A zero cochain c^0 is an element of $\mathbf{R}$, a 1–cochain c^1 is a linear map $c^1 : \underline{\mathbf{G}} \to \mathbf{R}$, i.e., $c^1 \in \underline{\mathbf{G}}^*$, and a 2–cochain c^2 is an antisymmetric bilinear form $\underline{\mathbf{G}} \times \underline{\mathbf{G}} \to \mathbf{R}$, i.e., $c^2 \in \bigwedge^2(\underline{\mathbf{G}}^*)$, and so on. The coboundary operator δ is given by

$$\delta \ : \ \mathbf{R} \to \underline{\mathbf{G}}^* \ ; \ c^0 \mapsto 0 , \tag{B.11a}$$

$$\delta \ : \ \underline{\mathbf{G}}^* \to \bigwedge^2(\underline{\mathbf{G}}^*) \ ; \ c^1 \mapsto \delta c^1 \ : \ \delta c^1(\xi, \eta) = c^1([\xi, \eta]) , \tag{B.11b}$$

$$\delta \ : \ \bigwedge^2(\underline{\mathbf{G}}^*) \to \bigwedge^3(\underline{\mathbf{G}}^*) \ ; \ c^2 \mapsto \delta c^2 \ : \tag{B.11c}$$

$$\delta c^2(\xi, \eta, \zeta) = -c^2([\xi, \eta], \zeta) - c^2([\eta, \zeta], \xi) - c^2([\zeta, \xi], \eta) ,$$

and so on, satisfying $\delta \circ \delta = 0$. A cochain c is called a *cocycle* if $\delta c = 0$. Then we have

Lemma B.1.

(i) *The term* $c_\lambda(\xi, \eta)$ *in* (B.10) :

$$c_\lambda(\xi, \eta) \equiv \{\lambda(\xi), \lambda(\eta)\} - \lambda([\xi, \eta]) \tag{B.12}$$

is constant on M, *hence* c_λ *is regarded as a 2–cochain and shown to be a cocycle.*

(ii) *For two choices of* $\lambda, \lambda' : \underline{\mathbf{G}} \to C^\infty(M)$, *the 2–cocycles are cohomologous :*

$$c_\lambda - c_{\lambda'} = \delta c^1 \quad \textit{for some } c^1 \in \underline{\mathbf{G}}^* . \tag{B.13}$$

(Proof) See Prop. 24.1 of [3]. Q.E.D.

We also have

Lemma B.2.

(i) *For any $z \in M$, we can define the cocycle $c_z \in \bigwedge^2(\underline{\mathbf{G}}^*)$ by*

(B.14) $$c_z(\xi, \eta) \equiv \omega_z(V_\xi, V_\eta) \qquad \text{for } \xi,\ \eta \in \underline{\mathbf{G}}\ .$$

(ii) *For two choices of z, $z' \in M$, the cocycles are cohomologous :*

(B.15) $$c_z - c_{z'} = \delta c^1 \quad \textit{for some } c^1 \in \underline{\mathbf{G}}^*\ .$$

(Proof) See Prop. 3.3.2 of [11]. The cochain c^1 of (B.15) is given by $c^1(\xi) = \int_z^{z'} \omega(V_\xi, \cdot)$, where the path connecting z and z' on M, along which the integral of the 1–form $\omega(V_\xi, \cdot)$ is taken, is arbitrarily chosen. Q.E.D.

Lemma B.3. *For any choice of linear $\lambda : \underline{\mathbf{G}} \to C^\infty(M)$ satisfying* (B.9) *and $z \in M$, the 2–cocycles c_λ given by* (B.12) *and c_z by* (B.14) *are cohomologous :*

(B.16) $$c_\lambda - c_z = \delta c^1 \quad \textit{for some } c^1 \in \underline{\mathbf{G}}^*\ .$$

(Proof) Writing $\lambda(\xi)$ as f_ξ, we put $c^1(\xi) = -f_\xi(z)$. Since $\{ f_\xi\ ,\ f_\eta \} - f_{[\xi,\eta]}$ is constant by (i) of Lemma B.1, we have $c_\lambda(\xi, \eta) = \{ f_\xi\ ,\ f_\eta \}(z) - f_{[\xi,\eta]}(z)$. On the other hand, we get

$$c_z(\xi, \eta) \overset{(B.14)}{=\!=} \omega_z(V_\xi, V_\eta) \overset{(B.9)}{=\!=} \omega_z(V_{f_\xi}, V_{f_\eta}) \overset{(B.8)}{=\!=} \{ f_\xi\ ,\ f_\eta \}(z)$$

yielding $(c_\lambda - c_z)(\xi, \eta) = -f_{[\xi,\eta]}(z) = c^1([\xi\ ,\ \eta]) \overset{(B.11b)}{=\!=} \delta c^1(\xi, \eta)$. Q.E.D.

The argument above is a fancy way of

Proposition B.4. *Let M be a connected and simply connected manifold, having a symplectic 2–form ω on it. We consider a symplectic action of a Lie group $\mathbf{G}$ on M.*

Then, for $\forall \xi \in \underline{\mathbf{G}}$, a function f_ξ on M is determined by (B.4) *up to constant. Furthermore, choosing $z \in M$ arbitrarily and adding constants to f_ξ's if necessary, the correspondence $\xi \mapsto f_\xi$ is linear and we have*

(B.17) $$\{ f_\xi\ ,\ f_\eta \} = f_{[\xi,\eta]} + c_z(\xi, \eta) \qquad \textit{for } \xi,\ \eta \in \underline{\mathbf{G}}\ ,$$

where c_z is the cocycle given by (B.14).

Appendix C. Noether Currents for Gauge and Conformal Symmetry

In this appendix, we give the proof of (4.17) and (4.26). We drop " " of "$\mathcal{L}$" or "ω_{ij}" in (1.9) for simplicity, and put $\widetilde{\omega}_{ij} = 2\lambda^2 K \omega_{ij}$. We also drop G of ρ_G.

To give (4.17), we assume (1.17) : $2\lambda^2 K = 4\pi$. For ϕ^ϵ in (4.13), we have

$$\text{(C.1)} \quad \mathcal{L}(\phi^\epsilon)(\phi)$$

$$= \tfrac{1}{2}\{\, \eta^{\mu\nu}\rho_{ij}(\phi + \epsilon\,\Phi) + \epsilon^{\mu\nu}\widetilde{\omega}_{ij}(\phi + \epsilon\,\Phi)\,\}\,(\partial_\mu\phi^i + \epsilon\,\partial_\mu\Phi^i)(\partial_\nu\phi^j + \epsilon\,\partial_\nu\Phi^j)\ ,$$

hence

$$\text{(C.2)} \quad \tfrac{d}{d\epsilon}\,\mathcal{L}(\phi^\epsilon)\Big|_{\epsilon=0}$$

$$= \tfrac{1}{2}\,\{\,\eta^{\mu\nu}\,\partial_k\rho_{ij}(\phi) + \epsilon^{\mu\nu}\,\partial_k\widetilde{\omega}_{ij}(\phi)\,\}\,L^k{}_a(\phi)\,\xi^a(x^+)\,\partial_\mu\phi^i\,\partial_\nu\phi^j$$

$$+\,\{\eta^{\mu\nu}\,\rho_{ij}(\phi) + \epsilon^{\mu\nu}\,\widetilde{\omega}_{ij}(\phi)\}\,\partial_\mu\phi^i\,\Big[\partial_k L^j{}_a(\phi)\,\partial_\nu\phi^k\,\xi^a(x^+) + L^j{}_a(\phi)\,\partial_\nu\xi^a(x^+)\Big],$$

where we used (4.15). On the other hand, for R^μ_ξ of (4.18), we have

$$\text{(C.3)} \quad \partial_\mu\,R^\mu_\xi \;=\; -\epsilon^{\mu\nu}\,\Big[\partial_k\rho_{ij}(\phi)\,\partial_\mu\phi^k - \partial_k\widetilde{\omega}_{ij}(\phi)\,\partial_\mu\phi^k\Big]\,L^i{}_a(\phi)\,\partial_\nu\phi^j\,\xi^a$$

$$-\;\epsilon^{\mu\nu}(\rho_{ij}(\phi) - \widetilde{\omega}_{ij}(\phi))\,\Big[\partial_k L^i{}_a(\phi)\,\partial_\mu\phi^k\,\partial_\nu\phi^j\,\xi^a + L^i{}_a(\phi)\,\partial_\nu\phi^j\,\partial_\mu\xi^a\Big]\,.$$

Remarking that $\partial_\mu\xi^a = \partial_\nu\xi^a = \xi^{a\prime}$ because $\xi^a = \xi^a(x^+) = \xi^a(x^0 + x^1)$, we obtain

$$\text{(C.4)} \qquad -\epsilon^{\mu\nu}\rho_{ij}(\phi)\,L^i{}_a(\phi)\,\partial_\nu\phi^j\,\partial_\mu\xi^a = \eta^{\mu\nu}\,\rho_{ij}(\phi)\,L^i{}_a(\phi)\,\partial_\mu\phi^j\,\partial_\nu\xi^a\ .$$

Hence

$$\text{(C.5a)} \qquad \tfrac{d}{d\epsilon}\,\mathcal{L}(\phi^\epsilon)\Big|_{\epsilon=0} - \partial_\mu\,R^\mu_\xi \;=\; \text{(I)} + \text{(II)} + \text{(III)}$$

with

$$\text{(C.5b)} \qquad \text{(I)} \;=\; \eta^{\mu\nu}\left[\tfrac{1}{2}\,\partial_k\rho_{ij}\,L^k{}_a\,\partial_\mu\phi^i\,\partial_\nu\phi^j + \rho_{ij}\,\partial_k L^j{}_a\partial_\mu\phi^i\,\partial_\nu\phi^k\right]\xi^a\ ,$$

$$\text{(C.5c)} \qquad \text{(II)} \;=\; \epsilon^{\mu\nu}\,\partial_k\left[\rho_{ij}\,L^i{}_a\right]\partial_\mu\phi^k\,\partial_\nu\phi^j\xi^a\ ,$$

$$\text{(C.5d)} \qquad \text{(III)} \;=\; \tfrac{1}{2}\,\epsilon^{\mu\nu}\,\partial_k\tilde{\omega}_{ij}\,\partial_\mu\phi^i\,\partial_\nu\phi^j\,L^k{}_a\,\xi^a$$

$$-\,\tfrac{1}{2}\,\epsilon^{\mu\nu}\left[\partial_k\tilde{\omega}_{ij}\,\partial_\mu\phi^k\,\partial_\nu\phi^j\,L^i{}_a + \partial_k\tilde{\omega}_{ji}\,\partial_\mu\phi^k\,\partial_\nu\phi^i\,L^j{}_a\right]\xi^a.$$

Remarking that

$$\partial_i L_{ja} - \partial_j L_{ia} \overset{(4.12)}{=}$$

$$\partial_i\left\langle\chi^{-1}(\phi)\,\partial_j\chi(\phi)\,,\,t_a\right\rangle - \partial_j\left\langle\chi^{-1}(\phi)\,\partial_i\chi(\phi)\,,\,t_a\right\rangle = -\tilde{\Omega}_{ijk}(\phi)\,L^k{}_a(\phi)\ ,$$

where $\tilde{\Omega}_{ijk} = 2\lambda^2 K\Omega_{ijk} = 4\pi\Omega_{ijk}$ (see (1.4)), we have

$$\text{(C.6)} \qquad \text{(II)} \;=\; \partial_i L_{ja}(\partial_0\phi^i\,\partial_1\phi^j - \partial_1\phi^i\,\partial_0\phi^j)\xi^a \;=\; -\tilde{\Omega}(\partial_0\phi,\partial_1\phi,\Phi)\ .$$

We also have

(C.7)

$$\text{(III)} \;=\; \tfrac{1}{2}\,\epsilon^{\mu\nu}\left[\partial_k\tilde{\omega}_{ij} + \partial_i\tilde{\omega}_{jk} + \partial_j\tilde{\omega}_{ki}\right]\partial_\mu\phi^i\,\partial_\nu\phi^j\,L^k{}_a\,\xi^a \overset{(1.13)}{=} \tilde{\Omega}(\partial_0\phi,\partial_1\phi,\Phi)\ ,$$

giving (II) + (III) = 0. For (I), we get

$$\text{(C.8)} \qquad \partial_k\rho_{ij}\,L^k{}_a + \rho_{ik}\,\partial_j L^k{}_a + \rho_{jk}\,\partial_i L^k{}_a = 0\ .$$

In fact, the left hand side of (C.8) is written as $(\partial_k\rho_{ij} - \partial_j\rho_{ik} - \partial_i\rho_{jk})L^k{}_a + \partial_j L_{ia} + \partial_i L_{ja}$. We have

$$\partial_k\rho_{ij} - \partial_j\rho_{ik} - \partial_i\rho_{jk}$$

$$\overset{(4.16)}{=} \left\langle\chi^{-1}\partial_k\chi\,,\;\chi^{-1}\partial_i\chi\,\chi^{-1}\partial_j\chi + \chi^{-1}\partial_j\chi\,\chi^{-1}\partial_i\chi - 2\chi^{-1}\,\partial_i\partial_j\chi\right\rangle\ ,$$

and $\partial_i L_{ja} + \partial_j L_{ia} = \langle -\chi^{-1}\partial_i\chi\,\chi^{-1}\partial_j\chi - \chi^{-1}\partial_j\chi\,\chi^{-1}\partial_i\chi + 2\chi^{-1}\,\partial_i\partial_j\chi\,,\;t_a\rangle$. Thus, remarking that $\chi^{-1}\partial_k\chi\cdot L^k{}_a = t_a$, we have (C.8). This implies (I) = 0 , giving (C.5a) = 0 , hence (4.17). Q.E.D.

Next we give the proof of (4.26). In this case we may not impose $2\lambda^2 K = 4\pi$. For an element E of (1.28), we write as $g(x^+,x^-) \to g^\epsilon(x^+,x^-)$ for the

transformation (1.29), and let $\phi(x^+, x^-)$, $\phi^\epsilon(x^+, x^-) = \phi(x^+, x^-) + \epsilon\,\Phi(x^+, x^-)$ be the corresponding ones to $g(x^+, x^-)$, $g^\epsilon(x^+, x^-)$ respectively in the coordinate system. Then

$$\text{(C.9)} \qquad \Phi(x^+, x^-) = E(x^+)\,\partial_+\phi(x^+, x^-) = \tfrac{1}{2}\,E(x^+)\,(\partial_0\phi + \partial_1\phi)\ .$$

Thus we have

$$\begin{aligned}
\text{(C.10)} \quad & \tfrac{d}{d\epsilon}\,\mathcal{L}(\phi^\epsilon)\Big|_{\epsilon=0} \\
&= \tfrac{1}{2}\,[\,\eta^{\mu\nu}\,\partial_k\rho_{ij}(\phi) + \epsilon^{\mu\nu}\,\partial_k\tilde{\omega}_{ij}(\phi)\,]\,\partial_\mu\phi^i\,\partial_\nu\phi^j\,\Phi^k \\
&\quad + [\,\eta^{\mu\nu}\,\rho_{ij}(\phi) + \epsilon^{\mu\nu}\,\tilde{\omega}_{ij}(\phi)\,]\,\partial_\mu\phi^i\,\partial_\nu\Phi^j \\
&= \tfrac{1}{2}\,E(x^+)\,\Big[\,\tfrac{1}{2}\,\{\,\eta^{\mu\nu}\,\partial_k\rho_{ij} + \epsilon^{\mu\nu}\,\partial_k\tilde{\omega}_{ij}\,\}\,\partial_\mu\phi^i\,\partial_\nu\phi^j\,(\partial_0\phi^k + \partial_1\phi^k) \\
&\qquad\qquad + \{\,\eta^{\mu\nu}\,\rho_{ij} + \epsilon^{\mu\nu}\,\tilde{\omega}_{ij}\,\}\,\partial_\mu\phi^i\,(\partial_\nu\partial_0\phi^j + \partial_\nu\partial_1\phi^j)\,\Big] \\
&\quad + E'(x^+)\,\mathcal{L} \\
&= \tfrac{1}{2}\,E(x^+)\,(\partial_0 + \partial_1)\mathcal{L} + E'(x^+)\,\mathcal{L}\ ,
\end{aligned}$$

where we have used

$$\eta^{\mu\nu}\,\rho_{ij}\,\partial_\mu\phi^i(\partial_0\phi^j + \partial_1\phi^j)\,\partial_\nu E(x^+) = \eta^{\mu\nu}\,\rho_{ij}\,\partial_\mu\phi^i\,\partial_\nu\phi^j\,E'(x^+)\ ,$$

thanks to $E = E(x^+)$ being a function of $x^+ = x^0 + x^1$. On the other hand, for R^μ_E of (4.27), we have

$$\text{(C.11)} \quad \partial_\mu\,R^\mu_E \;=\; \tfrac{1}{2}\,(\delta^\mu_0 + \delta^\mu_1)\,\partial_\mu[\,E(x^+)\,\mathcal{L}\,] \;=\; \tfrac{1}{2}\,E(x^+)\,(\partial_0 + \partial_1)\mathcal{L} + E'(x^+)\,\mathcal{L}\ ,$$

proving (4.26). Q.E.D.

Appendix D. Poisson Brackets off $2\lambda^2 K = 4\pi$

In this appendix, we calculate the Poisson bracket of the functions $I^{(\tau)}_\xi$ of (1.24) for arbitrary λ and K, which gives (1.25) with (1.26) when $2\lambda^2 K = 4\pi$.

First, we derive a general formula for Poisson brackets of functions on T^*Q (or TQ) of at most linear order in p (or v). For a vector field $W = W^i(q)\,\frac{\partial}{\partial q^i}$

on Q, we define $f_W \in C^\infty(T^*Q)$ by

(D.1) $$f_W(q,p) = \langle p\,,\ W_q \rangle = W^i(q)\,p_i\ .$$

Functions on Q are considered as functions on T^*Q (or TQ) through the projection $T^*Q \to Q$ (or $TQ \to Q$). We do not distinguish them but write as $\check{f}$ to indicate that it depends only on $q \in Q$. Then, for $\check{f}_1,\ \check{f}_2 \in C^\infty(Q)$, and $W_1,\ W_2 \in \mathrm{V}(Q)$, we have

(D.2) $$\{\, f_{W_1} + \check{f}_1\,,\ f_{W_2} + \check{f}_2 \,\} = f_{[W_1,W_2]} - \beta(W_1,W_2) + W_1 \cdot \check{f}_2 - W_2 \cdot \check{f}_1\ .$$

In fact, by (2.14), we obtain $\{\, f_{W_1}\,,\ f_{W_2} \,\} = W_1^i \frac{\partial}{\partial q^i} W_2^j\, p_j - \frac{\partial}{\partial q^i} W_1^j\, p_j\, W_2^i - \beta_{ij}\, W_1^i\, W_2^j = f_{[W_1,W_2]} - \beta(W_1,W_2)$, $\{\, \check{f}_1\,,\ \check{f}_2 \,\} = 0$, and $\{\, f_W\,,\ \check{f} \,\} = W \cdot \check{f}$. Remark that the formula (D.2) implies : functions on T^*Q of the form $f_W + \check{f}$, that is, of at most linear order in p, are closed under the Poisson bracket $\{\ ,\ \}$.

When we consider on TQ, the functions $\tilde{f}_W$ on TQ corresponding to f_W on T^*Q through (2.16) is given as $\tilde{f}_W(q,v) = \langle W\,,\ v \rangle_q = \rho_{ij}(q)\, W^i(q)\, v^j$, satisfying

(D.3) $$\{\, \tilde{f}_{W_1} + \check{f}_1\,,\ \tilde{f}_{W_2} + \check{f}_2 \,\} = \tilde{f}_{[W_1,W_2]} - \beta(W_1,W_2) + W_1 \cdot \check{f}_2 - W_2 \cdot \check{f}_1\ .$$

For our configuration space $Q = LG$, defining $\check{f}_\xi \in C^\infty(Q)$ by

$$\check{f}_\xi(g(\square)) = \fint \left\langle g^{-1}(\sigma) g'(\sigma)\,,\ \xi(\tau+\sigma) \right\rangle$$

and $W_\xi \in \mathrm{V}(Q)$ by W_ξ at $g(\square) = g(\square)\,\xi(\tau+\square)$, the function $I_\xi^{(\tau)}$ on TQ of (1.24) is written as

(D.4) $$I_\xi^{(\tau)} = \tilde{f}_{W_\xi} + \check{f}_\xi\ ,$$

because, then, $\langle W_\xi\,,\ h(\square) \rangle_{g(\square)} \overset{(4.2)}{=} \fint \left\langle g^{-1}(\sigma) h(\sigma)\,,\ \xi(\tau+\sigma) \right\rangle$.

We now apply (D.3) to our situation. We first obtain

(D.5) $$[\, W_\xi\,,\ W_\eta \,] = W_{[\xi,\eta]} \quad \text{for } \xi,\eta \in \underline{LG}\ ,$$

by (B.2), because the vector field W_ξ is derived through (B.1) for the action of LG on Q : $\gamma(\square) \in LG$; $g(\square) \in Q \mapsto g(\square)\,\gamma^{-1}(\tau+\square)$. We also have

(D.6) $$W_\xi \cdot \check{f}_\eta - W_\eta \cdot \check{f}_\xi = 2\,\check{f}_{[\xi,\eta]} + 2 \fint \langle\, \xi'(\sigma)\,,\ \eta(\sigma) \,\rangle\ ,$$

in fact

$$(W_\xi \cdot \check{f}_\eta)(g(\square)) \;=\; \check{f}_\eta(g(\square) + W_\xi) - \check{f}_\eta(g(\square))$$

$$= \fint \Big\langle \{\, g(\sigma)(1+\xi(\tau+\sigma)\,\}^{-1} \tfrac{d}{d\sigma}\{\, g(\sigma)(1+\xi(\tau+\sigma)\,\} - g^{-1}(\sigma)g'(\sigma)\,,$$

$$\eta(\tau+\sigma)\Big\rangle$$

$$= \fint \Big\langle [\, g^{-1}(\sigma)g'(\sigma)\,,\; \xi(\tau+\sigma)\,] + \xi'(\tau+\sigma)\,,\; \eta(\tau+\sigma)\Big\rangle$$

$$= \check{f}_{[\xi,\eta]}(g(\square)) + \fint \langle\, \xi'(\sigma)\,,\; \eta(\sigma)\,\rangle\;,$$

and use integration by parts. Finally we have

$$\beta(W_\xi, W_\eta) = \tfrac{2\lambda^2 K}{4\pi}\, \check{f}_{[\xi,\eta]}\;, \tag{D.7}$$

in fact,

$$\beta_{g(\square)}(W_\xi, W_\eta) \overset{(4.1),(1.4)}{=\!=\!=} \tfrac{2\lambda^2 K}{4\pi} \fint \Big\langle g^{-1}(\sigma)g'(\sigma)\,,\; [\xi(\tau+\sigma)\,,\; \eta(\tau+\sigma)]\Big\rangle$$

$$= \tfrac{2\lambda^2 K}{4\pi}\, \check{f}_{[\xi,\eta]}(g(\square))\;.$$

Thus we obtain, from (D.5), (D.6) and (D.7),

(D.8)

$$\begin{aligned}
\Big\{ I_\xi^{(\tau)}\,,\; I_\eta^{(\tau)} \Big\} &\overset{(D.4)}{=\!=} \Big\{ \tilde{f}_{W_\xi} + \check{f}_\xi\,,\; \tilde{f}_{W_\eta} + \check{f}_\eta \Big\} \\
&\overset{(D.3)}{=\!=} \tilde{f}_{[W_\xi, W_\eta]} - \beta(W_\xi, W_\eta) + W_\xi \cdot \check{f}_\eta - W_\eta \cdot \check{f}_\xi \\
&= \tilde{f}_{W_{[\xi,\eta]}} + \Big(2 - \tfrac{2\lambda^2 K}{4\pi}\Big)\, \check{f}_{[\xi,\eta]} + 2 \fint \langle\, \xi'(\sigma)\,,\; \eta(\sigma)\,\rangle\;.
\end{aligned}$$

This formula is valid for all λ and K.

For the case $2\lambda^2 K = 4\pi$, the formula (D.8) becomes

(D.9)

$$\begin{aligned}\left\{ I_{\xi}^{(\tau)},\ I_{\eta}^{(\tau)} \right\} &= \tilde{f}_{W_{[\xi,\eta]}} + \check{f}_{[\xi,\eta]} + 2\cdot\frac{1}{2\lambda^2}\int_0^{2\pi} d\sigma\,\langle\,\xi'(\sigma),\ \eta(\sigma)\,\rangle\ . \\ &= I_{[\xi,\eta]}^{(\tau)} + \frac{K}{2\pi}\int_0^{2\pi} d\sigma\,\langle\,\xi'(\sigma),\ \eta(\sigma)\,\rangle\ ,\end{aligned}$$

which is the Kac–Moody relation (1.25) of level K.

This also shows that the cocycle term of (D.9) is originated from $W_1\cdot\check{f}_2 - W_2\cdot\check{f}_1$ part of (D.3), not from $\beta(W_1, W_2)$ part, and the level K appears through the relation $2\lambda^2 K = 4\pi$. See the remark below (4.24). Anyway the formula (D.2) or (D.3) is more or less unnatural. The consideration in the light–cone coordinate like [10] may gives a more symmetric formula, although we have not gotten it yet.

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Geometric Quantization of Wess–Zumino–Witten Model

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Synopsis : We consider the Wess–Zumino–Witten (WZW) model of level $K \in \mathbf{Z}_{>0}$, based on a compact Lie group G. In this note, we regard the WZW model as a *charged system* and quantize it by the geometric quantization. The configuration space of the model is $LG = \mathrm{Map}(S^1, G)$, the loop group. We assert that the space of quantum states of the model is $C^\infty_K(\tilde{L}G)$, where $\tilde{L}G$ is a central extension of LG by $\mathbf{T} = S^1$ and $\psi \in C^\infty_K(\tilde{L}G)$ means $\psi : \tilde{L}G \to \mathbf{C}$ with $\psi(\tilde{g}t) = t^{-K}\psi(\tilde{g})$ for $t \in \mathbf{T}$. We also clarify the reason why the central extension $\tilde{L}G$ and its level K representations appear.

Key words: geometric quantization, WZW model, loop group

§1 Introduction

It is suitable to treat the WZW model [6] as a *charged system* ([2], Chap. 10 of [5] or p. 187 of [7]) .

In general, a charged system $(Q; \langle\ ,\ \rangle, u, \beta)$ consists of a configuration space Q which is a possibly infinite dimensional C^∞–manifold, a Riemannian metric

$\langle\ ,\ \rangle$ on Q, a potential $u : Q \to \mathbf{R}$ and a *closed* 2–form β on Q. We denote a point of the tangent bundle TQ by (q, v) when $v \in T_qQ$. If the 2–form β is *exact*, i.e., there exists a 1–form α on Q with $d\alpha = \beta$, then the Lagrangian function is given by

$$(1.1) \qquad L\ :\ TQ\ \to\ \mathbf{R}\ ;\ (q,v)\ \mapsto\ \frac{1}{2}\langle v,v\rangle_q - u(q) + \langle \alpha, v\rangle_q\ ,$$

where the second $\langle\ ,\ \rangle_q$ is the pairing of a 1–form and a vector on T_qQ. The motion is governed by the Principle of Least Action:

$$(1.2) \qquad \delta\int dt\, L\ =\ 0\quad\Longleftrightarrow\quad \frac{\delta L}{\delta q}\ :=\ \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial v}\ =\ 0\ .$$

If β is *not* exact, then we consider an open covering $\{U, V, \ldots\}$ of Q, on each U of which we have a 1–form α^U on U with $d\alpha^U = \beta_{|U}$. In this case, we only have a local Lagrangian on each TU given by

$$(1.3) \qquad L^U\ :\ TU\ \to\ \mathbf{R}\ ;\ (q,v)\ \mapsto\ \frac{1}{2}\langle v,v\rangle_q - u(q) + \langle \alpha^U, v\rangle_q\ .$$

Though the solution curves of (1.2) with L replaced by L^U or L^V coinside on $U \cap V$.

For the WZW model based on a compact Lie group G, we take the configuration space as $Q = LG := \mathrm{Map}(S^1, G)$, the loop group. We write an element of Q as $g(\square)$, where $\square$ means the dummy of $\sigma \in S^1$, and $(g(\square), \delta g(\square)) \in TQ$ means $\delta g(\square) \in T_{g(\square)}Q$, i.e., $\delta g(\sigma) \in T_{g(\sigma)}G$ for each $\sigma \in S^1$.

We consider a fixed $K \in \mathbf{Z}_{>0}$. The metric is given as

$$(1.4) \qquad \langle \delta_1 g(\square), \delta_2 g(\square)\rangle_{g(\square)}\ =\ \frac{K}{4\pi}\int_0^{2\pi} d\sigma\, \langle \delta_1 g(\sigma), \delta_2 g(\sigma)\rangle_{g(\sigma)}$$

where $\langle\ ,\ \rangle_g$ is the Riemannian metric on G defined by

$$(1.5) \qquad \langle \delta_1 g, \delta_2 g\rangle_g\ :=\ \langle g^{-1}\delta_1 g, g^{-1}\delta_2 g\rangle \quad \text{for}\ \ \delta_1 g,\ \delta_2 g \in T_gG$$

and $\langle \xi, \eta\rangle$ for $\xi,\ \eta \in \mathbf{g}$, the Lie algebra of G, is the positive Killing form : $-\mathrm{tr}\,(\xi\eta)$. The potential $u : Q \to \mathbf{R}$ is given by

$$(1.6) \qquad u(g(\square))\ =\ \frac{K}{8\pi}\int_0^{2\pi} d\sigma\, \langle g'(\sigma), g'(\sigma)\rangle_{g(\sigma)}\ ,\quad g'(\sigma)\ =\ \frac{d}{d\sigma}g(\sigma)\ .$$

Finally, at $g(\square) \in Q$, we give the 2–form β by

$$(1.7) \qquad \beta_{g(\square)}(\delta_1 g(\square), \delta_2 g(\square)) \;=\; K \int_0^{2\pi} d\sigma\, \Omega_{g(\sigma)}(g'(\sigma), \delta_1 g(\sigma), \delta_2 g(\sigma)) \;,$$

where Ω is the 3–form on G defined by

(1.8)

$$\Omega_g(\delta_1 g, \delta_2 g, \delta_3 g) \;=\; \frac{1}{4\pi} \langle g^{-1}\delta_1 g, [g^{-1}\delta_2 g, g^{-1}\delta_3 g] \rangle \quad \text{for } \delta_1 g,\ \delta_2 g,\ \delta_3 g \;\in\; T_g G \,.$$

This form Ω is closed but not exact. The 2–form β of (1.7) is also shown to be closed but not exact. The usual action functional of WZW model can be regarded as the Euclidification of the time integration of the local Lagrangian (1.3) given by the above terms.

In §2, we describe the general formalism of geometric quantization for charged systems. In §3, we consider the symmetry of the WZW model and see that the Noether charges for the gauge symmetry have the suitable shape for the quantization program of §2. In §4, we give the Hilbert space (the space of quantum states) and clarify the reason of the appearance of the central extension $\widetilde{L}G$.

The quantization of the model by Path Integral is considered in [1]. By the mathematical point of view, Path Integral is not fully acceptable. So there is a meaning in the approach of this paper. Also the symplectic structure (9.5.2) of [3] is defined on ΩG, the based loop group. Our symplectic structure is the canonical one on the cotangent bundle of LG. Our line bundle over LG appears naturally by the Schrödinger polarization. Hence our method gives a clear reason of the appearance of the loop group and its central extension.

§2 Geometric quantization of charged systems

We first describe the formalism of the geometric quantization [5] [7] of a general symplectic manifold (M, ω), that is, ω is a closed and non–degenerate 2–form (symplectic 2–form) on M. For $f \in C^\infty(M)$, we define $V_f \in \mathrm{V}(M)$, the space of all vector fields on M, by $-df = \omega(V_f, \cdot)$. Then the Poisson bracket of $f_1,\ f_2 \in C^\infty(M)$ is given by $\{f_1, f_2\} := \omega(V_{f_1}, V_{f_2}) = V_{f_1} \cdot f_2$ and the correspondence $f \mapsto V_f$ is a Lie algebra homomorphism : $[V_{f_1}, V_{f_2}] = V_{\{f_1, f_2\}}$.

The first step is to assign a linear operator $\hat{f}$ for $f \in C^\infty(M)$ so that the following condition by Dirac is satisfied :

$$(2.1) \qquad [\hat{f}_1, \hat{f}_2] \;=\; -i\hat{f}_3 \;, \qquad f_1,\; f_2,\; f_3 = \{f_1, f_2\} \;\in\; C^\infty(M) \;.$$

We impose that

$$(2.2) \qquad [\omega/2\pi] \;\in\; H^2(M;\mathbf{Z}) \;,$$

which is the condition of the existence of a (complex) line bundle $\check{B}$ over M whose first Chern class is $[\omega/2\pi]$. We denote by $\Gamma(\check{B})$ the space of all C^∞ sections of $\check{B}$. The bundle is equipped with the connection $\check{\nabla}$ whose curvature equals to ω in the sense that

$$\check{\nabla}_{V_1}\check{\nabla}_{V_2}\check{s} - \check{\nabla}_{V_2}\check{\nabla}_{V_1}\check{s} - \check{\nabla}_{[V_1,V_2]}\check{s} \;=\; -i\omega(V_1, V_2)\,\check{s}$$

for $V_1,\; V_2 \in \mathrm{V}(M)$ and $\check{s} \in \Gamma(\check{B})$. For $f \in C^\infty(M)$, we define the linear operator $\check{f} : \Gamma(\check{B}) \to \Gamma(\check{B})$ by

$$(2.3) \qquad \check{f}\cdot\check{s} \;:=\; -i\check{\nabla}_{V_f}\check{s} + f\check{s} \;,$$

for $\check{s} \in \Gamma(\check{B})$. Then the condition (2.1), replacing $\hat{}$ with $\check{}$, is satisfied.

The second step is to restrict the sections to those depending on only half the variables of M. In order that we take a *polarization* $\Pi = \{\Pi_m \;;\; m \in M\}$. This means Π_m is a subspace of T_mM and the correspondence $m \mapsto \Pi_m$ is smooth (*distribution*), *integrable* in the following sense : $V_1, V_2 \in \mathrm{V}_\Pi(M) \;\Rightarrow\; [V_1, V_2] \in \mathrm{V}_\Pi(M)$, where $\mathrm{V}_\Pi(M) := \{\; V \in \mathrm{V}(M) \;;\; V_m \in \Pi_m$ for all $m \in M \;\}$, and *Lagrangian* in the sense that Π_m is a Lagrangian subspace of T_mM. Then we consider the space of sections *covariantly constant along* Π : $\Gamma_\Pi(\check{B}) := \{\check{s} \in \Gamma(\check{B}) \;;\; \check{\nabla}_V\check{s} = 0$ for $V \in \mathrm{V}_\Pi(M)\}$. The problem is now whether $\Gamma_\Pi(\check{B})$ is invariant under $\check{f}$ of (2.3). As shown below, this problem is resolved for functions corresponding to gauge currents if we take the Schrödinger polarization for charged system.

We now consider a charged system $(Q; \langle\;,\;\rangle, u, \beta)$. Although Q for the WZW model is infinite dimensional, we explain the procedure for finite dimentional cases and write as $q = (q^j) = (q^1, q^2, \ldots, q^N) \in Q$ and so on.

On the cotangent bundle $M = T^*Q$, whose point is written as $(q,p) = (q^j, p_k)$ when $p \in (T_qQ)^*$, we have the canonical 2–form $\omega_0 := dp_j \wedge dq^j = d\theta_0$, $\theta_0 := p_j dq^j$. This is symplectic. The phase space of the charged system is T^*Q, but as the symplectic 2–form, we take

$$(2.4) \quad \omega_\beta := \omega_0 + \beta = dp_j \wedge dq^j + \frac{1}{2}\beta_{jk} dq^j \wedge dq^k , \quad \beta_{jk} := \beta(\tfrac{\partial}{\partial q^j}, \tfrac{\partial}{\partial q^k}) .$$

Then, taking $\omega = \omega_\beta$, we have $V_f = \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j} - \beta_{jk}\frac{\partial f}{\partial p_j}\frac{\partial}{\partial p_k}$ and

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial p_j}\frac{\partial f_2}{\partial q^j} - \frac{\partial f_1}{\partial q^j}\frac{\partial f_2}{\partial p_j} - \beta_{jk}(q)\frac{\partial f_1}{\partial p_j}\frac{\partial f_2}{\partial p_k} ,$$

for $f, f_1, f_2 \in C^\infty(M)$. The Hamiltonian $H \in C^\infty(M)$ is given by $H(q,p) = \frac{1}{2}\langle p, p\rangle_q + u(q)$, where $\langle\ ,\ \rangle_q$ in this formula is the one given through the isomorphism

$$(2.5) \qquad (q,v) \in TQ \quad \leftrightarrow \quad (q,p) \in T^*Q ; \quad p = \langle v, \cdot\rangle_q .$$

The solution curve of $V_H \in \mathrm{V}(M)$ is equivalent to the solution on Q by the Principle of Least Action for the Lagrangian (1.3). See §2 of [2].

Next quantization. Since ω_0 is exact, the condition (2.2) for $\omega = \omega_\beta$ is equivalent to

$$(2.6) \qquad [\beta/2\pi] \in H^2(Q;\mathbf{Z}) .$$

We assume that the condition is satisfied. As the polarization, we take $\Pi_m := \mathrm{span}\left\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \ldots, \frac{\partial}{\partial p_N}\right\}$ at $m = (q,p) \in M$, called the *Schrödinger polarization.* We say $f \in C^\infty(M)$ is Π–*preserving* if the flow generated by V_f preserves $\Pi : V \in \mathrm{V}_\Pi(M) \Rightarrow \mathcal{L}_{V_f} V \in \mathrm{V}_\Pi(M)$, where $\mathcal{L}_V$ is the Lie derivative. For $W = W^j(q)\frac{\partial}{\partial q^j} \in \mathrm{V}(Q)$, we put

$$(2.7) \qquad f_W = f_W(q,p) := \langle p, W\rangle_q = W^j(q)p_j \in C^\infty(M)$$

and $\mathcal{F}^1(M) := \{f_W + h \ ; \ W \in \mathrm{V}(Q) \text{ and } h \in C^\infty(Q)\}$, the set of all functions on M at most linear in p. Then we have

$$(2.8) \qquad f \in C^\infty(M) : \Pi - \text{preserving} \iff f \in \mathcal{F}^1(M) .$$

We also have

$$(2.9) \quad \{f_{W_1} + h_1, f_{W_2} + h_2\} = f_{[W_1,W_2]} - \beta(W_1, W_2) + W_1 \cdot h_2 - W_2 \cdot h_1 .$$

This implies that $\mathcal{F}^1(M)$ is closed under $\{\ ,\ \}$. For $f \in \mathcal{F}^1(M)$, it is shown that $\Gamma_\Pi(\check{B})$ is invariant under the operator $\check{f}$ given by (2.3) :

$$\check{f} \ : \ \Gamma_\Pi(\check{B}) \ \rightarrow \ \Gamma_\Pi(\check{B}) \ . \tag{2.10}$$

The operators (2.10) satisfy (2.1), replacing $\hat{}$ with $\check{}$.

Regarding $Q \subset M$ as zero sections, we consider $B := \check{B}_{|Q}$, which is a line bundle over Q whose first Chern class is $[\beta/2\pi]$. The connection $\check{\nabla}$ on $\check{B}$ naturally derives a connection ∇ on B, whose curvature is β. Then we have the natural isomorphism :

$$\Gamma_\Pi(\check{B}) \ \rightarrow \ \Gamma(B) \ ; \ \check{s} \ \mapsto \ s := \check{s}_{|Q} \ . \tag{2.11}$$

For $f = f_W + h \in \mathcal{F}^1(M)$, the operator $\check{f}$ of (2.3) can be regarded through (2.11) as $\hat{f} : \Gamma(B) \rightarrow \Gamma(B)$, whose concrete shape is given by

$$\hat{f} \cdot s \ = \ -i\nabla_W s + hs \ . \tag{2.12}$$

These operators $\hat{f}$ for $f \in \mathcal{F}^1(M)$ satisfy (2.1). Thus we have the satisfactory quantization procedure for functions in $\mathcal{F}^1(M)$ representing on the space $\Gamma(B)$. Cf. Chap. 10 of [5].

§3 Symmetry of WZW model

In general, the symmetry of a charged system $(Q; \langle\ ,\ \rangle, u, \beta)$ is described as follows. We omit not so important conditions. See, for details, §3 of [2]. We assume the Q is connected and simply connected for simplicity. We consider the space of all paths $\mathrm{Path}\, Q := \mathrm{Map}(\mathbf{R}, Q)$ whose element is written as $q(\cdot)$. Let $\mathbf{G}$ be a (possibly infinite dimensional) Lie group, which is assumed to be connected, acting on $\mathrm{Path}\, Q$. The action is written as $\gamma \cdot q(\cdot) \in \mathrm{Path}\, Q$ for $\gamma \in \mathbf{G}$ and $q(\cdot) \in \mathrm{Path}\, Q$.

The invariance of the system under the $\mathbf{G}$–action means : for $\xi \in \underline{\mathbf{G}}$, the Lie algebra of $\mathbf{G}$, putting $q^\epsilon(\cdot) = \exp(-\epsilon\xi) \cdot q(\cdot)$, the Lagrangian L^U for each U of the open covering of q is invariant (up to time derivative) in the following sense

$$\frac{d}{d\epsilon} L^U(q^\epsilon(t), \dot{q}^\epsilon(t)) \ |_{\epsilon=0} \ = \ \frac{d}{dt} R^U_\xi(t, q(t), \dot{q}(t)) \tag{3.1}$$

for some $R^U_\xi = R^U_\xi(t,q,v)$. Then the space of all solutions $\mathrm{Sol}\,Q$ is invariant under the $\mathbf{G}$ action. For fixed $t_0 \in \mathbf{R}$, there is the bijection

$$(3.2) \qquad (q_0, v_0) \in TQ \quad \leftrightarrow \quad q(\cdot) \in \mathrm{Sol}\,Q \ \text{ with } q(t_0) = q_0,\ \dot{q}(t_0) = v_0 \ .$$

Hence we have a $\mathbf{G}$ action on TQ derived by the original action on $\mathrm{Sol}\,Q$ through (3.2). We write it as

$$(3.3) \qquad \gamma \cdot (q,v) \in TQ \quad \text{for } \gamma \in \mathbf{G} \text{ and } (q,v) \in TQ \ .$$

The local form of the action (3.3) is easily obtained once the concrete form of the action on $\mathrm{Path}\,Q$ is given.

In general, a $\mathbf{G}$ action on a manifold $X : x \in X \mapsto \gamma \cdot x \in X$ derives the vector field V_ξ for $\xi \in \underline{\mathbf{G}}$ by (remark the sign of $-\epsilon\xi$)

$$(3.4) \qquad V_\xi(x) \ := \ \frac{d}{d\epsilon}[\exp(-\epsilon\xi)\cdot x]_{|\epsilon=0} \ .$$

The correspondence $\xi \mapsto V_\xi$ is a Lie algebra homomorphism :

$$(3.5) \qquad V_{[\xi,\eta]} \ = \ [V_\xi, V_\eta] \ .$$

Then the vector field $V_\xi \in \mathrm{V}(TQ)$ for $\xi \in \underline{\mathbf{G}}$ in the sense of (3.4) for the action (3.3) is written as

$$(3.6) \qquad V_\xi \ = \ A^j_\xi(q,v)\frac{\partial}{\partial q^j} + B^j_\xi(q,v)\frac{\partial}{\partial v^j} \ .$$

This $A^j_\xi(q,v)$ can be read off from the $\mathbf{G}$ action on $\mathrm{Path}\,Q$ as follows : for $\xi \in \underline{\mathbf{G}}$, we have a function $A^j_\xi(t,q,v)$ for which

$$(3.7) \qquad \frac{d}{d\epsilon}\, q^{\epsilon,j}(t)\,|_{\epsilon=0} \ = \ A^j_\xi(t,q(t),\dot{q}(t)) \ , \qquad q^\epsilon(\cdot) = \exp(-\epsilon\xi)\cdot q(\cdot) \ ,$$

and $A^j_\xi(q,v) = A^j_\xi(t_0,q,v)$ where t_0 is the one in (3.2).

We pull back the symplectic structure on T^*Q to TQ through the isomorphism (2.5) and use the same notations ω_β, $\{\ ,\ \}$ or $\mathcal{F}^1(TQ)$. It is seen that the vector field $V_\xi \in \mathrm{V}(TQ)$ is locally Hamiltonian : $\mathcal{L}_{V_\xi}\omega_\beta = 0$. Since we have assumed Q is simply connected, it is globally Hamiltonian, i.e.,

$$(3.8) \qquad V_\xi \ = \ V_{I_\xi} \ \text{ for some } I_\xi \ \in \ C^\infty(TM) \ .$$

The function $I_\xi \in C^\infty(TQ)$ is locally written as

$$(3.9) \qquad I_\xi(q,v) \;=\; \frac{\partial L^U}{\partial v^j}(q,v)A^j_\xi(q,v) - R^U_\xi(t_0,q,v)\ ,$$

where A^j_ξ is in (3.6) and R^U_ξ is in (3.1) with t_0 being in (3.2). The shape of this function is the one by the usual Noether's procedure, hence is invariant along solutions. The functions $I_\xi \in C^\infty(TQ)$ is determined modulo constant. We eliminate the ambiguity for example by requiring them to vanish at a point of TQ which is fixed beforehand. Then we have

$$(3.10) \qquad \{I_\xi, I_\eta\} \;=\; I_{[\xi,\eta]} + c^2(\xi,\eta)\ ,$$

where $c^2 : \underline{\mathbf{G}} \times \underline{\mathbf{G}} \to \mathbf{R}$ is a Lie algebra cocycle which cannot be eliminated in general. There is a method to obtain the cocycle from the shape of the $\mathbf{G}$ action on Path Q (Th. 3.1 of [2]). We call the function I_ξ the *Noether charge* for $\xi \in \underline{\mathbf{G}}$.

For $W \in \mathrm{V}(Q)$, we also write as $f_W \in C^\infty(TQ)$ corresponding to (2.7) through (2.5) :

$$(3.11) \qquad f_W(q,v) \;=\; \langle v, W\rangle_q\ .$$

We now back to the WZW case : $Q = LG$. In this case Path $Q = \mathrm{Map}(\mathbf{R}\times S^1, G)$, that is, the space of G–valued fields $g = g(x) = g(x^0,x^1)$, where $x^0 = \tau\ (= t)$ and $x^1 = \sigma$. On $\mathbf{R}\times S^1$, we introduce the light–cone coodinate : $x^\pm := x^0 \pm x^1$. We also write the field as $g = g(x^+,x^-)$.

The gauge symmetry of the model is described as follows. The symmetry group is $\mathbf{G} = L^+G \times L^-G$, where $L^\pm G$ means the space of G–valued 2π–periodic function of $x^\pm$. The action of $(\gamma^+,\gamma^-) \in L^+G \times L^-G$ on field $g = g(x^+,x^-)$ is given by

$$(3.12) \qquad g(x^+,x^-) \;\to\; \gamma^-(x^-)g(x^+,x^-)\gamma^+(x^+)^{-1}\ .$$

We here collect some notations. For fixed τ_0, whose role is t_0 in (3.2), we consider the correspondences

$$(3.13a) \quad \gamma(\square) \in LG,\ \xi(\square) \in L\mathbf{g} \quad \text{with} \quad \gamma^\pm(\tau_0 \pm \square) \in L^\pm G,\ \xi^\pm(\tau_0\pm\square) \in L^\pm\mathbf{g}\ .$$

For $\xi(\square) \in L\mathfrak{g}$, let $\xi^{(\ell)}$ (respectively $\xi^{(r)}$) be the left invariant (respectively right invariant) vector field on $Q = LG$ whose value at the Lie algebra $L\mathfrak{g}$ is $\xi(\square)$:

$$(3.13b) \qquad \begin{cases} \left[\, \xi^{(\ell)} \text{ at } g(\square) \,\right] = g(\square)\xi(\square) = g(\square)\xi^{+}(\tau_0 + \square) \,, \\ \left[\, \xi^{(r)} \text{ at } g(\square) \,\right] = \xi(\square)g(\square) = \xi^{-}(\tau_0 - \square)g(\square) \,. \end{cases}$$

For the second equalities, see (3.13a). Then, for $\xi,\ \eta \in L\mathfrak{g}$, we have

$$(3.13c) \qquad \left[\xi^{(\ell)}, \eta^{(\ell)}\right] = [\xi, \eta]^{(\ell)} \,, \quad \left[\xi^{(r)}, \eta^{(r)}\right] = -[\xi, \eta]^{(r)} \,.$$

In general, for a 1–form ν and a vector field W on Q, we define the function F^{ν}_{W} on TQ as $f_W + h$ for $h = \langle \nu, W \rangle$:

$$(3.14a) \qquad F^{\nu}_{W}(q, v) := f_W(q, v) + \langle \nu, W \rangle = \langle W, v \rangle_q + \langle \nu, W \rangle \,.$$

For such functions, the relation (2.9) becomes

$$(3.14b) \qquad \{F^{\nu}_{W_1}, F^{\nu}_{W_2}\} = F^{\nu}_{[W_1, W_2]} - (\beta - d\nu)(W_1, W_2) \,,$$

for $W_1,\ W_2 \in \mathrm{V}(Q)$. In the geometric quantization, (2.12) becomes

$$(3.14c) \qquad \hat{F^{\nu}_{W}} \cdot s = -i\nabla_W s + \langle \nu, W \rangle s \,.$$

We let ν_K be the 1–form on $Q = LG$ defined by (cf. proof of (4.4.4) of [4])

$$(3.15) \quad \nu_{K,g(\square)}(\delta g(\square)) := \langle g'(\square), \delta g(\square) \rangle_{g(\square)} = \frac{K}{4\pi} \int_0^{2\pi} d\sigma \, \langle\, g'(\sigma),\ \delta g(\sigma)\, \rangle_{g(\sigma)} \,.$$

We also let c_K be the 2–cocycle on $L\mathfrak{g}$ given by (see (4.2.2) of [4])

$$(3.16a) \qquad c_K(\xi, \eta) = \frac{K}{2\pi} \int_0^{2\pi} d\sigma \langle \xi(\sigma),\ \eta'(\sigma) \rangle \,.$$

For 2–cocycle c on $L\mathfrak{g}$, let $c^{(\ell)}$ (respectively $c^{(r)}$) be the left invariant (respectively right invariant) 2–form on LG, that is,

$$(3.16b) \qquad \begin{cases} c^{(\ell)}_{g(\square)}(\delta_1 g(\square), \delta_2 g(\square)) = c(g^{-1}(\square)\delta_1 g(\square), g^{-1}(\square)\delta_2 g(\square)) \,, \\ c^{(r)}_{g(\square)}(\delta_1 g(\square), \delta_2 g(\square)) = c(\delta_1 g(\square) g^{-1}(\square), \delta_2 g(\square) g^{-1}(\square)) \,. \end{cases}$$

Writing the 2–form (1.7) on $Q = LG$ as β_K, we have

$$\beta_K - d\nu_K = c_K^{(\ell)}, \quad \beta_K + d\nu_K = -c_K^{(r)}. \tag{3.16c}$$

Now consider the x^+ part of the action (3.12) :

$$g(x^+, x^-) \to g(x^+, x^-)\gamma^+(x^+)^{-1}.$$

The Noether charge $I_\xi^+ \in C^\infty(TQ)$ for $\xi(\square) \in L\mathbf{g}$ (or $\xi^+(\tau_0 + \square) \in L^+\mathbf{g}$) is given by ((4.20) in [2])

$$I_\xi^+(g(\square), \delta g(\square)) = \frac{K}{4\pi}\int_0^{2\pi} d\sigma \left\langle \delta g(\sigma) + g'(\sigma),\ g(\sigma)\xi^+(\tau_0 + \sigma) \right\rangle_{g(\sigma)}, \tag{3.17}$$

i.e., $I_\xi^+ = F_{\xi^{(\ell)}}^{\nu_K}$. This is an element of $\mathcal{F}^1(TQ)$. From (3.17), (3.14b), (3.16c) and (3.16b), we have

$$\{I_\xi^+, I_\eta^+\} = I_{[\xi,\eta]}^+ - c_K(\xi, \eta). \tag{3.18}$$

For the x^- part of the action (3.12) : $g(x^+, x^-) \to \gamma^-(x^-)g(x^+, x^-)$, the Noether charge $I_\xi^- \in C^\infty(TQ)$ for $\xi(\square) \in L\mathbf{g}$ or $\xi^-(\tau_0 - \square) \in L^-\mathbf{g}$ is given by

(3.19)

$$I_\xi^-(g(\square), \delta g(\square)) = -\frac{K}{4\pi}\int_0^{2\pi} d\sigma \left\langle \delta g(\sigma) - g'(\sigma),\ \xi^-(\tau_0 - \sigma)g(\sigma) \right\rangle_{g(\sigma)},$$

i.e., $I_\xi^- = F_{-\xi^{(r)}}^{-\nu_K}$, and we have as above

$$\{I_\xi^-, I_\eta^-\} = I_{[\xi,\eta]}^- + c_K(\xi, \eta). \tag{3.20}$$

§4 Hilbert space of WZW model

Let us summarize our situation. We regard the WZW model as the charged system $(Q; \langle\ ,\ \rangle, u, \beta)$ with $Q = LG$, (1.4, 6) and $\beta = \beta_K$ of (1.7). The Hilbert space (the space of quantum states) by the geometric quantization of the system is considered as $\Gamma(B_K)$, where B_K is the line bundle over Q with the connection whose curvature is β_K. We remark that the word *Hilbert space*

used here is not the mathematical one, that is the complete space with respect to a given inner product. It is our future problem to give proper inner product to our space.

The condition (2.6) for $\beta = \beta_K$ corresponds to $K \in \mathbf{Z}$. Recall that we have assumed $K \in \mathbf{Z}_{>0}$. For $\xi \in L\mathbf{g}$, we have the function I_ξ^+ on $T(LG)$ with the relation (3.18). The function is promoted to the operator $\hat{I}_\xi^+ : \Gamma(B_K) \to \Gamma(B_K)$ by (2.12). The relation (3.18) is promoted through (2.1) to

$$\left[\, i\hat{I}_\xi^+ \,,\, i\hat{I}_\eta^+ \,\right] = i\hat{I}_{[\xi,\eta]}^+ - iKc_1(\xi,\eta) \,. \tag{4.1}$$

In order to make the correspondence $\xi \mapsto i\hat{I}_\xi^+ \in \mathrm{End}(\Gamma(B_K))$ a Lie algebra homomorphism (hence a representation of the Lie algebra), we extend $L\mathbf{g}$ to $\tilde{L}\mathbf{g} := L\mathbf{g} \oplus \mathbf{R}$ with the Lie bracket defined by (see §4.2 of [4])

$$[\,(\xi,r)\,,\,(\eta,s)\,] = ([\xi,\eta], c_1(\xi,\eta)) \,. \tag{4.2}$$

Then we have the Lie algebra homomorphism

$$\mathcal{O}^+ : \tilde{L}\mathbf{g} \to \mathrm{End}(\Gamma(B_K)) \;;\; \mathcal{O}^+(\xi,r) := i\left(\hat{I}_\xi^+ - Kr\right) , \tag{4.3}$$

with $\mathcal{O}^+(\xi,0) = i\hat{I}_\xi^+$.

For the x^- part of the action (3.12), the operator $\hat{I}_\xi^- : \Gamma(B_K) \to \Gamma(B_K)$ now satisfies

$$\left[\, i\hat{I}_\xi^- \,,\, i\hat{I}_\eta^- \,\right] = i\hat{I}_{[\xi,\eta]}^- + iKc_1(\xi,\eta) \,. \tag{4.4}$$

Corresponding to (4.3), we have a Lie algebra representation

$$\mathcal{O}^- : \tilde{L}\mathbf{g} \to \mathrm{End}(\Gamma(B_K)) \;;\; \mathcal{O}^-(\xi,r) := i\left(\hat{I}_\xi^- + Kr\right) , \tag{4.5}$$

with $\mathcal{O}^-(\xi,0) = i\hat{I}_\xi^-$.

These Lie algebra representations are derived from Lie group representations as follows. We put $\mathbf{T} := S^1 = \{z \in \mathbf{C}; |z| = 1\}$ and consider the central extension $\tilde{L}G$ of LG by $\mathbf{T}$ (§4.1 of [4]) :

$$\mathbf{T} \to \tilde{L}G \to LG \,, \tag{4.6}$$

whose Lie algebra is $\tilde{L}\mathbf{g}$ with (4.2). This means that $\mathbf{T}$ is considered as a subgroup of $\tilde{L}G$ contained in the center with $\tilde{L}G/\mathbf{T} = LG$. Considering the space

(4.7) $C_K^\infty(\widetilde{L}G)$

$$:= \{ \psi : \widetilde{L}G \to \mathbf{C} \, ; \, \psi(\tilde{g}t) = \psi(t\tilde{g}) = t^{-K}\psi(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{L}G, \, t \in \mathbf{T} \} ,$$

we have

Proposition
There is a natural isomorphism between $\Gamma(B_K)$ *and* $C_K^\infty(\widetilde{L}G)$:

$$\Gamma(B_K) \ni s \leftrightarrow \psi \in C_K^\infty(\widetilde{L}G) , \tag{4.8}$$

through which, the Lie algebra representation (4.3) is derived from the Lie group representation

$$\rho^+ : \widetilde{L}G \to \mathrm{Aut}\left(C_K^\infty(\widetilde{L}G)\right) \, ; \, \tilde{\gamma} \mapsto \left[\rho^+(\tilde{\gamma}) \cdot \psi\right](\tilde{g}) = \psi(\tilde{g}\tilde{\gamma}) , \tag{4.9}$$

and the representation (4.5) is derived from

$$\rho^- : \widetilde{L}G \to \mathrm{Aut}\left(C_K^\infty(\widetilde{L}G)\right) \, ; \, \tilde{\gamma} \mapsto \left[\rho^-(\tilde{\gamma}) \cdot \psi\right](\tilde{g}) = \psi(\tilde{\gamma}^{-1}\tilde{g}) . \tag{4.10}$$

We assert that the Hilbert space of the WZW model is $C_K^\infty(\widetilde{L}G)$ of (4.7). It is seen that the representation (4.9) on the space is of level $-K$ and (4.10) is of level K.

(Proof) The central extension $\widetilde{L}G$ is a principal $\mathbf{T}$-bundle with a connection $\widetilde{\nabla}$ whose curvature is $c_1^{(\ell)}$. The connection $\widetilde{\nabla}$ is given by the first way of §4.5 of [4] as

$$\xi^{(\ell)} \mapsto \tilde{\xi} := (\xi, 0)^{(\ell)} \tag{4.11}$$

for $\xi \in L\mathbf{g}$, where $(\xi, 0)^{(\ell)}$ means the left invariant vector field on $\widetilde{L}G$ whose value at the Lie algebra $\widetilde{L}\mathbf{g}$ is $(\xi, 0)$. See the proof of Prop.4.5.6 of [4].

The line bundle B_K over LG is considered as the bundle associated to $\widetilde{L}G$ by the representation $t \in \mathbf{T} \mapsto t^K \in \mathrm{Aut}\mathbf{C} = \mathbf{C}^\times$. The bundle B_K has the connection ∇ whose curvature is β_K. We also consider the connection ∇^+ on B_K given by adding $-\nu_K$ to ∇. Locally, $\nabla = d - i\alpha$ and $\nabla^+ = d - i(\alpha - \nu_K)$ for $\alpha = d^{-1}\beta_K$. We see that, from (3.14c) and (3.17), the operator $i\hat{I}_\xi^+$:

$\Gamma(B_K) \to \Gamma(B_K)$ is nothing but $\nabla^+_{\xi^{(\ell)}}$. The curvature of ∇^+ is $c_K^{(\ell)}$ by (3.16c). Hence this corresponds to the connection given by (4.11).

The identification (4.8) is the usual one (Prop. 3.5.2 of [3]), through which we have the correspondence

$$(4.12) \qquad i\hat{I}^+_\xi \cdot s \quad \leftrightarrow \quad (\xi, 0)^{(\ell)} \cdot \psi \ .$$

This means (4.3) is derived from (4.9).

For the x^- part, we add ν_K to ∇ to obtain the connection ∇^- on B_K with the curvature $-c_K^{(r)}$ by (3.16c). Then the operator $i\hat{I}^-_\xi : \Gamma(B_K) \to \Gamma(B_K)$ is considered as $\nabla^-_{-\xi^{(r)}}$. On $\widetilde{L}G$, instead of (4.11), we consider the connection $\widetilde{\nabla}^-$ given by $\xi^{(r)} \mapsto (\xi, 0)^{(r)}$, whose curvature is $-c_1^{(r)}$ by (3.13c). Thus, through the identification (4.8), we have in this case

$$(4.13) \qquad i\hat{I}^-_\xi \cdot s \quad \leftrightarrow \quad -(\xi, 0)^{(r)} \cdot \psi \ .$$

From this we see that the representation (4.5) is derived from (4.10). Q.E.D.

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TIME-SPACE DEPENDENT FORMULATION IN THERMOFIELD DYNAMICS

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The recently developed formulation of time-dependent non-equilibrium thermofield dynamics (TFD) is extended to spatially inhomogeneous field systems in this paper. This inhomogeneous formalism is constructed from introducing momentum mixing thermal Bogoliubov transformation. The momentum mixing number density in the Bogoliubov matrix is related to the number density distribution defined at each space-time point which is to be observed.

Extension of thermofield dynamics (TFD) to a non-equilibrium theory is a challenging subject. It was made clear in Ref. 1 that eigenvalues of the total Hamiltonian in TFD, denoted by $\hat{H}$, is unbounded from below and the thermal degree of freedom originates from this property of $\hat{H}$ in TFD formulation. We then attempted to construct a calculational formulation for time-dependent (spatially homogeneous only though) non-equilibrium isolated systems of quantum fields, making thermal Bogoliubov transformation time-dependent (but keeping thermal vacuum time-independent, i.e., using a single Fock space).[2,3] In the previous papers,[2,3] the renormalization condition of this formulation in time representation led naturally to the Boltzmann equation.

The purpose of this paper is to construct the quasiparticle representation in spatially inhomogeneous TFD. This is the first step to extend the whole time-dependent non-equilibrium TFD in Refs. 1–3 to spatially inhomogeneous (i.e., time-space dependent) cases. Such a formulation is crucial to compare non-equilibrium TFD with experiments, as most of the non-equilibrium phenomena, taking place in nature, are accompanied by some sorts of inhomogeneity in space. Heat conduction is an example of immediate application of spatially inhomogeneous TFD.

Recall the basic relations between a- and ξ-operators (the latter is called the vacuum operators, since they specify thermal vacua: $\xi|0\rangle = \tilde{\xi}|0\rangle = 0$ and $\langle 0|\xi^\dagger = \langle 0|\tilde{\xi}^\dagger = 0$) in time-dependent TFD,[2,3]

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$$a_k(t)^\mu = B^{-1}[n_k(t)]^{\mu\nu}\xi^\nu e^{-i\int^t ds\,\omega_k(s)}$$
$$\bar{a}_k(t)^\mu = \bar{\xi}^\nu B[n_k(t)]^{\nu\mu} e^{i\int^t ds\,\omega_k(s)} \ . \tag{1}$$

Here the thermal doublet notation is used and the thermal Bogoliubov matrix takes the following forms for the choice[2,3] $\alpha = 1$:

$$B[n] = \begin{bmatrix} 1+\sigma n & -n \\ -\sigma & 1 \end{bmatrix} , \tag{2}$$

where σ is $+1(-1)$ for boson(fermion). The number parameter $n_k(t)$ is related to the thermal expectation as

$$n_k(t)\delta(\mathbf{k}-\mathbf{l}) = \langle 0|\bar{a}_k(t)^1 a_l(t)^1|0\rangle \ . \tag{3}$$

It is noted in the above choices of $\alpha = 1$ that we have a simple matrix relation which becomes important later:

$$B^{-1}[n]B[n'] = 1 - \sigma(n-n')T_0 \ , \tag{4}$$

where the matrix with constant elements is given by

$$T_0 = \begin{bmatrix} 1 & -\sigma \\ \sigma & -1 \end{bmatrix} \tag{5}$$

and

$$T_0^2 = 0 \ . \tag{6}$$

The first attempt toward formulating spatial dependence in thermal situation might be to make the Bogoliubov matrix $B[n_k(t)]$ in (1) dependent on space coordinate $\mathbf{x}$, suggested by the fact that t-dependent TFD is obtained by simply making B dependent on t. However, this approach, giving rise to $\mathbf{x}$-dependent a_k, does not work, because the commutation relation

$$[a_k(t)^\mu, \bar{a}_l(t)^\nu] = \delta^{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) \tag{7}$$

cannot be preserved once a_k starts depending on $\mathbf{x}$. Thus our challenge in this paper is to introduce somehow $\mathbf{x}$-dependence of thermal situations into a_k, keeping (7).

Another attempt is to introduce Bogoliubov matrix, mixing not only thermal indices but also momentum indices, as we have done in the previous paper.[4] However, the formulation in Ref. 4, without using explicit spatial representation, was not useful, e.g., when a spatial dependence of initial temperature is given. In this paper we will formulate an explicit space dependent formalism below by starting with the method in Ref. 4 and reformulating it. This new formalism matches the recent development in time-dependent non-equilibrium TFD,[1-3] giving us a time-space dependent formalism.

Let us introduce the momentum mixing through thermal Bogoliubov transformation according to Ref. 4: The parameter $n_k(t)$ in B, (2), and its inverse B^{-1} is replaced by $\mathbf{N}(t)_{kl}$. The Bogoliubov matrices thus obtained are denoted by $\mathbf{B}(t)_{kl}^{\mu\nu}$ and $\mathbf{B}^{-1}(t)_{kl}^{\mu\nu}$ and have the momentum suffix $\mathbf{k}$ in addition to the thermal one μ:

$$\mathbf{B}(t)_{kl}^{\mu\nu} = \begin{bmatrix} \delta(\mathbf{k}-\mathbf{l}) + \sigma\mathbf{N}(t)_{kl} & -\mathbf{N}(t)_{kl} \\ -\sigma\delta(\mathbf{k}-\mathbf{l}) & \delta(\mathbf{k}-\mathbf{l}) \end{bmatrix}^{\mu\nu} . \tag{8}$$

As is easily shown from these definitions, the two matrices $\mathbf{B}(t)_{kl}^{\mu\nu}$ and $\mathbf{B}^{-1}(t)_{kl}^{\mu\nu}$ are inverse to each other in the suffix space of $(\mathbf{k},\mu)$:

$$\mathbf{B}^{-1}(t)_{kq}^{\mu\lambda}\mathbf{B}(t)_{ql}^{\lambda\nu} = \delta_{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) \tag{9}$$

$$\mathbf{B}(t)_{kq}^{\mu\lambda}\mathbf{B}^{-1}(t)_{ql}^{\lambda\nu} = \delta_{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) . \tag{10}$$

In spatially inhomogeneous case the a-operators are related to the ξ-operators through

$$\begin{aligned} a_k(t)^\mu &= \mathbf{B}^{-1}(t)_{kl}^{\mu\nu}\xi_l^\nu e^{-i\int^t ds\,\omega_l(s)} \\ \bar{a}_k(t)^\mu &= \bar{\xi}_l^\nu \mathbf{B}(t)_{lk}^{\nu\mu} e^{i\int^t ds\,\omega_l(s)} , \end{aligned} \tag{11}$$

from which it follows that

$$\mathbf{N}(t)_{kl} = \langle 0|\bar{a}_l(t)^1 a_k(t)^1|0\rangle \tag{12}$$

instead of (3) in homogeneous case. Because of (10), (11) can be inverted,

$$\begin{aligned} \xi_k^\mu &= \mathbf{B}(t)_{kl}^{\mu\nu} a_l(t)^\nu e^{i\int^t ds\,\omega_k(s)} , \\ \bar{\xi}_k^\mu &= \bar{a}_l(t)^\nu \mathbf{B}^{-1}(t)_{lk}^{\nu\mu} e^{-i\int^t ds\,\omega_k(s)} . \end{aligned} \tag{13}$$

Equations (9) and (10) guarantee that the canonical commutation relations are preserved when one moves from the a-operator set to the ξ-operator set and vice versa.

The Bogoliubov transformations (11) and (13) lead to the equations of motion,

$$\begin{aligned} i\dot{a}_k(t)^\mu &= \omega_k(t)a_k(t)^\mu - \mathbf{R}(t)_{kl}T_0^{\mu\nu}a_l(t)^\nu \\ i\dot{\bar{a}}_k(t)^\mu &= -\omega_k(t)\bar{a}_k(t)^\mu + \bar{a}_l(t)^\nu \mathbf{R}(t)_{lk}T_0^{\nu\mu} , \end{aligned} \tag{14}$$

where

$$\mathbf{R}(t)_{kl} = \mathbf{R}_0(t)_{kl} + \mathbf{R}_1(t)_{kl} \tag{15}$$

with

$$\mathbf{R}_0(t)_{kl} = i\sigma\dot{\mathbf{N}}(t)_{kl} \tag{16}$$

$$\mathbf{R}_1(t)_{kl} = (\omega_l(t) - \omega_k(t))\sigma\mathbf{N}(t)_{kl} . \tag{17}$$

Equations (14) can be put in the form

$$i\dot{a}_k(t)^\mu = [a_k(t)^\mu,\ \hat{H}_Q(t)]\ , \tag{18}$$

$$i\dot{\bar{a}}_k(t)^\mu = [\bar{a}_k(t)^\mu,\ \hat{H}_Q(t)]\ , \tag{19}$$

where the Hamiltonian $\hat{H}_Q$ is given by

$$\hat{H}_Q(t) = \hat{H}_0(t) - \hat{Q}_0(t) - \hat{Q}_1(t) \tag{20}$$

with

$$\hat{H}_0(t) = \int d^3k\omega_k(t)\bar{a}_k(t)^\mu a_k(t)^\mu \tag{21}$$

and

$$\hat{Q}_i(t) = \bar{a}_k(t)^\mu \mathbf{R}_i(t)_{kl} T_0^{\mu\nu} a_l(t)^\nu \quad (i = 0, 1)\ . \tag{22}$$

Direct calculations prove the following formula,

$$\mathbf{B}(t)^{\mu\lambda}_{kq}[\mathbf{X} + [\mathbf{X}, \sigma\mathbf{N}(t)]T_0]^{\lambda\lambda'}_{qq'}\mathbf{B}^{-1}(t)^{\lambda'\nu}_{q'l} = \delta_{\mu\nu}\mathbf{X}_{kl}\ , \tag{23}$$

for any matrix $\mathbf{X}$ with momentum suffix. Choosing

$$\mathbf{X}_{kl} = \mathbf{W}_{kl} \equiv \omega_k(t)\delta(\mathbf{k} - \mathbf{l}) \tag{24}$$

in (23), we have

$$\hat{H}_0(t) - \hat{Q}_1(t) = \int d^3k\omega_k(t)\bar{\xi}_k\xi_k \equiv \hat{H}_{0\xi}(t)\ , \tag{25}$$

which is diagonal in terms of vacuum operators. It is obvious that $\hat{H}_Q$ is not diagonal in terms of vacuum operators, because it should induce temporal changes. Indeed, the difference between $\hat{H}_Q$ and $\hat{H}_{0\xi}$ is the $\mathbf{R}_0$-term proportional to $\dot{\mathbf{N}}$.

The formulation of momentum-mixing thermal Bogoliubov transformations above works without showing any internal inconsistency to describe spatially inhomogeneous situations. One missing point in the formulation is a relationship between the parameter of the theory, i.e., each element of $\mathbf{N}(t)_{kl}$, and each experimental situation given. Usually, or from the physical intuition, spatially inhomogeneous thermal system is best specified by a number density parameter depending on $\mathbf{x}$, say $n(t, \mathbf{x} : \mathbf{k})$. Thus our next task is to reformulate the above momentum mixing formulation into a form described in terms of $n(t, \mathbf{x} : \mathbf{k})$.

We try to introduce the $\mathbf{x}$-dependent number density through the following Fourier transformation from $\mathbf{N}(t)_{kl}$,

$$\begin{aligned} n(t, \mathbf{x} : \mathbf{k}) &= \int d^3l e^{-i\mathbf{l}\cdot\mathbf{x}}\mathbf{N}(t)_{k+\gamma l, k-(1-\gamma)l} \\ &= \int d^3l e^{-i\mathbf{l}\cdot\mathbf{x}}\langle 0|\bar{a}_{k+\gamma l}(t)^1 a_{k-(1-\gamma)l}(t)^1|0\rangle \end{aligned} \tag{26}$$

or equivalently

$$\mathbf{N}(t)_{kl} \equiv \frac{1}{(2\pi)^3}\int d^3x e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}} n(t,\mathbf{x}:\gamma\mathbf{k}+(1-\gamma)\mathbf{l}) \ . \tag{27}$$

Here the short-handed notation $\gamma\mathbf{k}$ means the transformation of the vector $\mathbf{k}$ by a matrix γ,

$$(\gamma\mathbf{k})_i \equiv \gamma_{ij}k_j \ , \tag{28}$$

and γ is at this stage an arbitrary matrix and will be determined by a self-consistency condition of the theory, as will be shown later. This definition of $n(t,\mathbf{x}:\mathbf{k})$ will be supported by arguments on local density and current density operators later, see (45) and (46). Making use of (26) or (27) combined with (2) and (8), we rewrite (11) and (13),

$$\begin{aligned} a_k(t)^\mu &= \frac{1}{(2\pi)^3}\int d^3l \int d^3x e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}} B^{-1}[n(t,\mathbf{x}:\gamma\mathbf{k}+(1-\gamma)\mathbf{l})]^{\mu\nu}\xi_l(t)^\nu \\ \bar{a}_k(t)^\mu &= \frac{1}{(2\pi)^3}\int d^3l \int d^3x e^{i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}} \bar{\xi}_l(t)^\nu B[n(t,\mathbf{x}:(1-\gamma)\mathbf{k}+\gamma\mathbf{l})]^{\nu\mu} \end{aligned} \tag{29}$$

and

$$\begin{aligned} \xi_k(t)^\mu &= \frac{1}{(2\pi)^3}\int d^3l \int d^3x e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}} B[n(t,\mathbf{x}:\gamma\mathbf{k}+(1-\gamma)\mathbf{l})]^{\mu\nu}a_l(t)^\nu \\ \bar{\xi}_k(t)^\mu &= \frac{1}{(2\pi)^3}\int d^3l \int d^3x e^{i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}} \bar{a}_l(t)^\nu B^{-1}[n(t,\mathbf{x}:(1-\gamma)\mathbf{k}+\gamma\mathbf{l})]^{\nu\mu} \ , \end{aligned} \tag{30}$$

respectively, where we introduced the notation,

$$\begin{aligned} \xi_k(t)^\mu &\equiv \xi_k^\mu e^{-i\int^t ds\,\omega_k(s)} \ , \\ \bar{\xi}_k(t)^\mu &\equiv \bar{\xi}_k^\mu e^{i\int^t ds\,\omega_k(s)} \ . \end{aligned} \tag{31}$$

Equations (29) and (30) constitute basic relations involving $n(t,\mathbf{x}:\mathbf{k})$ for oscillator variables. In spite of the presence of $n(t,\mathbf{x}:\mathbf{k})$, the canonical commutation relations are preserved because of (9) and (10).

Here are a few comments on the time-space dependent TFD for oscillator variables so far: Since the vacuum operators ξ_k and $\bar{\xi}_k$ are independent of time and space, so is their vacuum $|0\rangle$. Note that the quantum energy $\omega_k(t)$ does not depend on $\mathbf{x}$. This is because the quasiparticle unperturbed energy is given by the quantum energy in the asymptotic domain which is infinitely far away from the domain with spatial thermal variation. We find a similar situation in case of states with macroscopic objects such as solitons.[5] However, interaction creates a self-energy which can depend on $\mathbf{x}$. This may create an $\mathbf{x}$-dependent quantum energy. This is not the unperturbed quantum energy. Being the asymptotic energy, the unperturbed

energy includes only that part of the self-energy which is independent of $\mathbf{x}$. This is the energy renormalization for spatially inhomogeneous systems.

We now reformulate the time-space dependent TFD above in terms of the quasi-particle fields,

$$\begin{aligned}\varphi(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k a_k(t)^\mu e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{(2\pi)^{3/2}}\int d^3k \mathbf{B}^{-1}(t)^{\mu\nu}_{kq}\xi_q(t)^\nu e^{i\mathbf{k}\cdot\mathbf{x}} , \\ \bar{\varphi}(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k \bar{a}_k(t)^\mu e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{(2\pi)^{3/2}}\int d^3k \bar{\xi}_q(t)^\nu \mathbf{B}(t)^{\nu\mu}_{qk} e^{-i\mathbf{k}\cdot\mathbf{x}} ,\end{aligned} \tag{32}$$

where (11) are used. Let us also introduce the ξ-fields by

$$\begin{aligned}\xi(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k \xi_k(t)^\mu e^{i\mathbf{k}\cdot\mathbf{x}} , \\ \bar{\xi}(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k \bar{\xi}_k(t)^\mu e^{-i\mathbf{k}\cdot\mathbf{x}} .\end{aligned} \tag{33}$$

The equal-time canonical commutation relations for the fields introduced above are

$$[\varphi(t,\mathbf{x})^\mu, \bar{\varphi}(t,\mathbf{x}')^\nu] = [\xi(t,\mathbf{x})^\mu, \bar{\xi}(t,\mathbf{x}')^\nu] = \delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{x}') . \tag{34}$$

We note a formula deriving from (26) or (27), given by

$$\int d^3k d^3l e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{N}(t)_{kl}e^{-i\mathbf{l}\cdot\mathbf{y}} = \int d^3k n(t,(1-\gamma)\mathbf{x}+\gamma\mathbf{y}:\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} , \tag{35}$$

from which follow

$$\begin{aligned}\int d^3k d^3l e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{B}^{-1}(t)^{\mu\nu}_{kl}e^{-i\mathbf{l}\cdot\mathbf{y}} &= \int d^3k B^{-1}[n(t,(1-\gamma)\mathbf{x}+\gamma\mathbf{y}:\mathbf{k})]^{\mu\nu}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ \int d^3k d^3l e^{i\mathbf{l}\cdot\mathbf{y}}\mathbf{B}(t)^{\nu\mu}_{lk}e^{-i\mathbf{k}\cdot\mathbf{x}} &= \int d^3k B[n(t,\gamma\mathbf{x}+(1-\gamma)\mathbf{y}:\mathbf{k})]^{\nu\mu}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} .\end{aligned} \tag{36}$$

Then $\varphi(t,\mathbf{x})^\mu$ can be expressed in terms of $\xi(t,\mathbf{x})^\mu$,

$$\begin{aligned}\varphi(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^3}\int d^3y d^3q B^{-1}[n(t,(1-\gamma)\mathbf{x}+\gamma\mathbf{y}:\mathbf{q})]^{\mu\nu}\xi(t,\mathbf{y})^\nu e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} , \\ \bar{\varphi}(t,\mathbf{x})^\mu &= \frac{1}{(2\pi)^3}\int d^3y d^3q \bar{\xi}(t,\mathbf{y})^\nu B[n(t,\gamma\mathbf{x}+(1-\gamma)\mathbf{y}:\mathbf{q})]^{\nu\mu}e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} .\end{aligned} \tag{37}$$

From the definitions, (32), and the equations of motion for $a_k(t)^\mu$ and $\bar{a}_k(t)^\mu$, (14), we obtain the field equations,

$$i\dot{\varphi}(t,\mathbf{x})^\mu = \omega(-i\nabla)\varphi(t,\mathbf{x})^\mu + \int d^3y R(t,\mathbf{x},\mathbf{y}) T_0^{\mu\nu}\varphi(t,\mathbf{y})^\nu \ , \tag{38}$$

$$i\dot{\bar{\varphi}}(t,\mathbf{x})^\mu = -\bar{\varphi}(t,\mathbf{x})^\mu \omega(i\overleftarrow{\nabla}) - \int d^3y \bar{\varphi}(t,\mathbf{y})^\nu T_0^{\nu\mu} R(t,\mathbf{y},\mathbf{x}) \ , \tag{39}$$

where

$$R(t,\mathbf{x},\mathbf{y}) = R_0(t,\mathbf{x},\mathbf{y}) + R_1(t,\mathbf{x},\mathbf{y}) \tag{40}$$

with

$$R_i(t,\mathbf{x},\mathbf{y}) \equiv \frac{1}{(2\pi)^3}\int d^3k d^3l e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{R}_i(t)_{kl} e^{-i\mathbf{l}\cdot\mathbf{y}} \ , \tag{41}$$

$\mathbf{R}_i(t)_{kq}(i = 0,1)$ being given in (16) and (17). When Eq. (35) is considered, $R_i(t,\mathbf{x},\mathbf{y})$ can be rewritten as

$$R_0(t,\mathbf{x},\mathbf{y}) = i\frac{1}{(2\pi)^3}\int d^3q \sigma \dot{n}(t,(1-\gamma)\mathbf{x}+\gamma\mathbf{y}:\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \ , \tag{42}$$

$$R_1(t,\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^3}\int d^3q \{\omega(i\nabla_y) - \omega(-i\nabla_x)\} n(t,(1-\gamma)\mathbf{x}+\gamma\mathbf{y}:\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \ . \tag{43}$$

The field equations (39) follow from the canonical formalism with the Hamiltonian $\hat{H}_Q$ in (20), which in terms of the field variables becomes,

$$\hat{H}_Q(t) = \int d^3x \bar{\varphi}(t,\mathbf{x})^\mu \omega(-i\nabla)\varphi(t,\mathbf{x})^\mu - \int d^3x d^3y \bar{\varphi}(t,\mathbf{x})^\mu R(t,\mathbf{x},\mathbf{y}) T_0^{\mu\nu}\varphi(t,\mathbf{y})^\nu \ . \tag{44}$$

Let us study the vacuum expectation values of local density and current density operators in terms of φ. The expectation value of density, most easily calculated from (32), (12) and (35), becomes

$$\langle 0|\bar{\varphi}(t,\mathbf{x})^1\varphi(t,\mathbf{x})^1|0\rangle = \int \frac{d^3k}{(2\pi)^3} n(t,\mathbf{x}:\mathbf{k}) \ . \tag{45}$$

This shows that $n(t,\mathbf{x}:\mathbf{k})$ is real because expectation values of operators consisting solely of φ^1 and $\bar{\varphi}^1$ at a certain time are independent of α and $\varphi^\dagger(=\bar{\varphi}^1)$ is hermite conjugate of φ in $\alpha = 1/2$. The expectation value of current density is found to be

$$\begin{aligned}\langle 0| - i\bar{\varphi}(t,\mathbf{x})^1 \overleftrightarrow{\nabla}_\gamma \varphi(t,\mathbf{x})^1|0\rangle &= \int d^3k d^3l e^{i\mathbf{k}\cdot\mathbf{x}}\{\gamma\mathbf{k}+(1-\gamma)\mathbf{l}\}\mathbf{N}(t)_{kl} e^{-i\mathbf{l}\cdot\mathbf{x}} \\ &= \int \frac{d^3k}{(2\pi)^3}\mathbf{k} n(t,\mathbf{x}:\mathbf{k}) \ ,\end{aligned} \tag{46}$$

where

$$\overleftrightarrow{\nabla}_\gamma = \gamma\nabla - (1-\gamma)\overleftarrow{\nabla} \ . \tag{47}$$

Equations (45) and (46) are consistent with the intuitive interpretation intended in its introduction that $n(t, \mathbf{x} : \mathbf{k})$ is a number density distribution at space point $\mathbf{x}$ at time t: The $n(t, \mathbf{x} : \mathbf{k})$ specifies time-space dependent thermal situation. For example, consider a situation in which the local temperature $T(\mathbf{x})$ is given at a time, say $t = t_0$. This condition is formulated by $n(t_0, \mathbf{x} : \mathbf{k}) = 1/[\exp(\beta(\mathbf{x})\omega_k(t_0)) - 1]$ with $\beta(\mathbf{x}) = 1/T(\mathbf{x})$. This consideration settles the question asking how one may introduce space dependent temperature in a thermal theory. In the following, we proceed without specifying $n(t, \mathbf{x} : \mathbf{k})$. Here is a comment on the constant matrix γ which comes into the $\mathbf{x}$-dependent formulation and has been arbitrary so far. The γ is determined from the tilde conjugation rules: The thermal doublets, a_k^μ and $\bar{a}_k^\mu$, are required to transform under the tilde conjugation as

$$\begin{aligned} (a_k(t)^\mu)^\sim &= \sigma \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\mu\nu} \bar{a}_k(t)^\nu \\ (\bar{a}_k(t)^\mu)^\sim &= a_k(t)^\nu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\nu\mu} , \end{aligned} \tag{48}$$

which implies, combined with (11), that

$$\mathbf{N}^*(t)_{kl} = \mathbf{N}(t)_{lk} \ . \tag{49}$$

Recall (27) and the fact that $n(t, \mathbf{x} : \mathbf{k})$ should be real, we then find

$$\gamma_{ij} = \frac{1}{2}\delta_{ij} \ . \tag{50}$$

The unperturbed one-body propagator follows from the thermal Bogoliubov transformation as follows:

$$\Delta_c(t, \mathbf{x} : t', \mathbf{x}')^{\mu\nu} \equiv -i\langle 0|T[\varphi(t, \mathbf{x})^\mu \bar{\varphi}(t', \mathbf{x}')^\nu]|0\rangle \tag{51}$$

$$= \frac{1}{(2\pi)^3} \int d^3k d^3k' d^3q e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{x}'}$$

$$\times \mathbf{B}^{-1}(t)^{\mu\mu'}_{kq} \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu'\nu'} e^{-i\int_{t'}^t ds\, \omega_q(s)} \mathbf{B}(t)^{\nu'\nu}_{qk'} \tag{52}$$

$$= \frac{1}{(2\pi)^9} \int d^3k d^3k' d^3y d^3y' d^3q e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{y}')} e^{i\mathbf{q}\cdot(\mathbf{y}-\mathbf{y}')}$$

$$\times e^{-i\int_{t'}^t ds\, \omega_l(s)} B^{-1}\left[n\left(t, \mathbf{y} : \frac{\mathbf{k}+\mathbf{q}}{2}\right)\right]^{\mu\mu'}$$

$$\times \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu'\nu'} B\left[n\left(t', \mathbf{x}' : \frac{\mathbf{k}'+\mathbf{q}}{2}\right)\right]^{\nu'\nu} . \tag{53}$$

These are multiple integrations of terms which consist of diagonal matrices sandwiched between Bogoliubov matrices. Concerning the information of spatial dependences of systems, (53) is useful. This propagator is the internal lines of Feynman diagrams of the perturbative calculations. In performing practical calculation of Feynman diagrams, (52) is preferred, because it usually simplifies calculations. Study of corrected one-body propagator and derivation of the equation for time-space behavior of the unperturbed number $n[t, \mathbf{x} : \mathbf{k}]$ from the self-energy renormalization condition is expected to follow the steps taken in the study of time-dependent TFD in Refs. 2 and 3 and to lead us to the entropy law for spatially inhomogeneous systems. These are important immediate problems.

We close this paper by giving some comments on the Hamiltonian and momentum operator. As was shown in Refs. 2 and 3, even when we consider a spatially homogeneous time-dependent situation, the unperturbed Hamiltonian for a-operators, $\hat{H}_Q(t)$, differs from the one for ξ-operators, $\hat{H}_0(t)$. The difference arises from the time dependence through which the dynamical energy does not conserve but some dynamical energy changes into heat energy. In this spatially homogeneous case, too, the unperturbed Hamiltonians for $a(t)_k$- or $\varphi(t,\mathbf{x})$-operators and for $\xi(t)_k$- or $\xi(t,\mathbf{x})$-operators are still different from each other: $\hat{H}_Q(t)$ in (20) and $\hat{H}_{0\xi}(t)$ in (25), respectively. The difference between $\hat{H}_Q(t)$ and $\hat{H}_{0\xi}(t)$ is due to the $\mathbf{R}_0$-term in (16). Since $\mathbf{R}_0$ is proportional time-derivative of $\mathbf{N}_{kl}$, the difference is apparently caused by temporal changes in thermal situation.

Recall that $\hat{H}_0(t)$ in (21) is diagonal in terms of $a_k(t)$-operators but not in terms of $\xi_k(t)$ ones due to the presence of $\mathbf{R}_1$-term, as shown in (25). (Indeed, according to (17), $\mathbf{R}_1$ vanishes in spatially homogeneous cases because then $\mathbf{N}_{kl}$ is diagonal with respect to the momentum suffix.) This difference is expected from the fact that the ξ-particle is the free particle in the domain asymptotically far from domains of spatial variations of thermal situations. It should be emphasized that the free quasiparticle Hamiltonian in time-space dependent TFD is not $\hat{H}_0(t)$ but $\hat{H}_{0\xi}(t)$, which will be crucial when one considers a self-consistent renormalization condition.

The momentum operators have similar structures: There are two generators for space-translation, one is for $\varphi(t,\mathbf{x})$ denoted by $\hat{\mathbf{P}}(t)$ and the other for $\xi(t,\mathbf{x})$ denoted by $\hat{\mathbf{P}}_\xi(t)$, and they are different from each other as

$$\hat{\mathbf{P}}(t) - \int d^3k d^3q \bar{a}_k(t)^\mu \{\mathbf{k} - \mathbf{q}\} \sigma \mathbf{N}(t)_{kq} T_0^{\mu\nu} a_q(t)^\nu = \hat{\mathbf{P}}_\xi(t) \ , \tag{54}$$

where

$$\hat{\mathbf{P}}(t) = \int d^3k \mathbf{k} \bar{a}_k(t)^\mu a_k(t)^\mu \ , \tag{55}$$

$$\hat{\mathbf{P}}_\xi(t) = \int d^3k \mathbf{k} \bar{\xi}_k(t)^\mu \xi_k(t)^\mu \ . \tag{56}$$

Note that not $\hat{H}_0$ and $\hat{\mathbf{P}}$, but $\hat{H}_{0\xi}$ and $\hat{\mathbf{P}}_\xi$ annihilate thermal vacua:

$$\begin{aligned} \hat{H}_{0\xi}(t)|0\rangle &= \hat{\mathbf{P}}_\xi(t)|0\rangle = 0 \\ \langle 0|\hat{H}_{0\xi}(t) &= \langle 0|\hat{\mathbf{P}}_\xi(t) = 0 \ . \end{aligned} \tag{57}$$

In order to look at the relation between $\{\hat{H}_0, \hat{\mathbf{P}}\}$ and $\{\hat{H}_{0\xi}, \hat{\mathbf{P}}_\xi\}$ more closely, we rewrite the expression on the right-hand side of (23) as follows:

$$[\mathbf{X} + [\mathbf{X}, \sigma\mathbf{N}(t)]T_0]^{\mu\nu}_{kl} = [\exp[-\sigma\mathbf{N}(t)T_0]\,\mathbf{X}\,\exp[\sigma\mathbf{N}(t)T_0]]^{\mu\nu}_{kl} \ , \tag{58}$$

where (6) is used. This, when $\mathbf{X}_{kl} = \omega_k\delta(\mathbf{k}-\mathbf{l})$ or $\mathbf{k}\delta(\mathbf{k}-\mathbf{l})$, implies that

$$\hat{H}_{0\xi}(t) = \int d^3k\,\omega_k(t)[\bar{a}(t)\,\exp[-\sigma\mathbf{N}(t)T_0]]^\mu_k\,[\exp[\sigma\mathbf{N}(t)T_0]a(t)]^\mu_k \tag{59}$$

$$\hat{\mathbf{P}}_\xi(t) = \int d^3k\,\mathbf{k}[\bar{a}(t)\,\exp[-\sigma\mathbf{N}(t)T_0]]^\mu_k\,[\exp[\sigma\mathbf{N}(t)T_0]a(t)]^\mu_k \ . \tag{60}$$

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GAUGE THEORY OF THE THERMAL BOGOLIUBOV TRANSFORMATION

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Abstract

Gauge field associated with the space-time dependent thermal Bogoliubov transformation is introduced and the minimal interaction of it is derived.

The subject of this note is a by-product of the work [1] which I did in collaboration with Professor Umezawa and Dr. Yamanaka in Edmonton last summer. There we proposed a way to extend thermo field dynamics (TFD) to spatially inhomogeneous system.

In TFD we have two kinds of fields, the field by which the dynamics of the system is described and one which has the thermal vacuum as its vacuum. These two fields are related with each other by the thermal Bogoliubov transformation which depends on the parameters characterizing the thermal state of the system. For the spatially inhomogeneous system, these parameters and hence the thermal Bogoliubov transformation must depend on the space-time coordinates. The space-time dependence of the thermal Bogoliubov transformation leads the noncommutativity of the derivation with respect to space-time coordinates and the transformation. This motivates us to introduce a gauge field associated with the thermal Bogoliubov transformation to construct the covariant derivative. The purpose of this note is to show how it can be done.

Let us begin by recapitulating the results of Ref.1 to show how to construct the space-time dependent thermal Bogliubov transformation.

As is known [2] , the thermal Bogoliubov transformation matrix B contains the three parameters, n_k, α_k, s_k.

For simplicity, we choose the paticular values of $\alpha_k = 1$ and $s_k = (1/2)\ln(1+n_k)$. For this choice, B has a simple form as

$$B[n_k] = \begin{pmatrix} 1+\sigma n_k & -n_k \\ -\sigma & 1 \end{pmatrix}, \tag{1}$$

with σ being 1 (-1) for bosonic (fermionic) operators.

The transformation by B is generated formally by the following operator transformations:

$$\begin{aligned} a_k^\mu &= B^{-1}[n_k]^{\mu\nu}\xi_k^\nu \\ &= e^{-G[n_k]}e^{-G_0}\xi_k^\mu e^{G_0}e^{G[n_k]}, \\ \bar{a}_k^\mu &= \bar{\xi}_k^\nu B[n_k]^{\nu\mu} \\ &= e^{-G[n_k]}e^{-G_0}\bar{\xi}_k^\mu e^{G_0}e^{G[n_k]}, \end{aligned} \tag{2}$$

where

$$G_0 = \int d\vec{k}\bar{\xi}_k^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{\mu\nu} \xi_k^\nu, \tag{3a}$$

$$G[n_k] = \int d\vec{k} n_k \bar{\xi}_k^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mu\nu} \xi_k^\nu. \tag{3b}$$

Here we use the usual thermal doublet notation [3] .

To generalize the above transformation to spatially inhomogeneous situations, we note that G in eq.(3b) can be rewritten [4] as

$$G[n_k] = \frac{1}{(2\pi)^3} \int d\vec{x} \int d\vec{k} n_k J_k(t, \vec{x}; \vec{k}) \tag{4}$$

with

$$J(t, \vec{x}; \vec{k}) = \frac{1}{(2\pi)^3} \int d\vec{\eta} e^{i\vec{k}\cdot\vec{\eta}} \bar{\xi}(t, \vec{x} + \frac{1}{2}\vec{\eta})^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mu\nu} \xi(t, \vec{x} - \frac{1}{2}\vec{\eta})^\nu. \tag{5}$$

Here the fields $\bar{\xi}(t, \vec{x})^\mu$ and $\xi(t, \vec{x})^\mu$ are defined by

$$\xi(t, \vec{x})^\mu = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \xi_k^\mu e^{i\vec{k}\cdot\vec{x} - i\omega_k t}, \tag{6a}$$

$$\bar{\xi}(t, \vec{x})^\mu = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \bar{\xi}_k^\mu e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}, \tag{6b}$$

which obey the equal time commutation relation,

$$[\xi(t, \vec{x})^\mu,\ \bar{\xi}(t, \vec{y})^\nu]_\sigma = \delta^{\mu\nu} \delta(\vec{x} - \vec{y}), \tag{7}$$

where $[A, B]_\sigma$ means $AB - \sigma BA$.

In the time dependent but spatially homogeneous case [2], we introduce the time dependent Bogoliubov transformation by considering n_k in $G[n_k]$, hence in $B[n_k]$, to be time dependent. For spatially inhomogeneous situation, it seems quite natural to modify the generator $G[n_k]$ by replacing n_k in (4) with the space-time dependent $n_k(t, \vec{x}) \equiv n(t, \vec{x}; \vec{k})$.

The transformations (2) with thus generalized $G[n(t,\vec{x};\vec{k}]$ lead the momentum mixing Bogoliubov transformations [1, 5]

$$a_k(t)^\mu = \frac{1}{(2\pi)^3}\int d\vec{k}' \int d\vec{x} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} B^{-1}[n(t,\vec{x};\frac{1}{2}(\vec{k}+\vec{k}'))]^{\mu\nu} e^{-i\omega_{k'}t}\xi^\nu_{k'}, \quad (8a)$$

$$\bar{a}_k(t)^\mu = \frac{1}{(2\pi)^3}\int d\vec{k}' \int d\vec{x} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \bar{\xi}^\nu_{k'} e^{i\omega_{k'}t} B[n(t,\vec{x};\frac{1}{2}(\vec{k}+\vec{k}'))]^{\nu\mu}, \quad (8b)$$

which assure the commutation relation

$$[a_k(t)^\mu,\ \bar{a}_l(t)^\nu]_\sigma = \delta^{\mu\nu}\delta(\vec{k}-\vec{l}), \quad (9)$$

We note that $n(t,\vec{x};\vec{k})$ is expressed as

$$n(t,\vec{x};\vec{k}) = \int d\vec{K} e^{i\vec{K}\cdot\vec{x}} \langle 0 \,|\, \bar{a}_{k+\frac{1}{2}K}(t)^1\, a_{k-\frac{1}{2}K}(t)^1 \,|\, 0 \rangle.$$

Now let us construct the field operators $\phi(t,\vec{x})$ and $\bar{\phi}(t,\vec{x})$ for the oscillator variables $a_k(t)$ and $\bar{a}_k(t)$ as

$$\phi(t,\vec{x})^\mu = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{k} a_k(t)^\mu e^{i\vec{k}\cdot\vec{x}},$$

$$\bar{\phi}(t,\vec{x})^\mu = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{k} \bar{a}_k(t)^\mu e^{-i\vec{k}\cdot\vec{x}}.$$

Because of eq.(9),the field operators introduced above satisfy the canonical commutation relation for a type I field, i.e. field which contains only the

positive frequency part:

$$[\phi(t,\vec{x})^{\mu},\ \bar{\phi}(t,\vec{y})^{\nu}] = \delta^{\mu\nu}\delta(\vec{x}-\vec{y}). \tag{10}$$

By taking Fourier transform of eq.(8), we get the Bogoliubov transformations which relate the fields $\phi(t,\vec{x})$ and $\bar{\phi}(t,\vec{x})$ to $\xi(t,\vec{x})$ and $\bar{\xi}(t,\vec{x})$:

$$\phi(t,\vec{x})^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu}\xi(t,\vec{y})^{\nu}, \tag{11a}$$

$$\bar{\phi}(t,\vec{x})^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\bar{\xi}(t,\vec{y})^{\nu}B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}. \tag{11b}$$

The inverse transformations to eq.(11) are given by

$$\xi(t,\vec{x})^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu}\phi(t,\vec{y})^{\nu}, \tag{12a}$$

$$\bar{\xi}(t,\vec{x})^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\bar{\phi}(t,\vec{y})^{\nu}B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}. \tag{12b}$$

The space-time depedence of $n(t,\vec{x};\vec{k})$ gives rise to the noncommutativity of the Bogoliubov transfomation (11) and the derivations with respect to space-time coordinates:

$$\begin{aligned}\partial_{\alpha}\phi(t,\vec{x})^{\mu} &+ \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_{\alpha}n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})M_0^{\mu\nu}\phi(t,\vec{y})^{\nu}\\ &= \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu}\partial_{\alpha}\xi(t,\vec{y})^{\nu},\end{aligned}$$

$$\partial_\alpha \bar{\phi}(t,\vec{x})^\mu - \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_\alpha n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})\bar{\phi}(t,\vec{y})^\nu M_0^{\nu\mu} \tag{13a}$$

$$= \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_\alpha \bar{\xi}(t,\vec{y})^\nu B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}, \tag{13b}$$

where

$$M_0 = \begin{pmatrix} \sigma & -1 \\ 1 & -\sigma \end{pmatrix}.$$

Here the derivative operator ∂_α is defined by

$$\partial_\alpha = \begin{cases} \partial/\partial x_i & \alpha = i = 1,2,3 \\ \partial/\partial t & \alpha = 0 \end{cases} \tag{14}$$

and the following notation is also used:

$$\partial_i n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k}) = \frac{\partial}{\partial X_i} n(t,\vec{X};\vec{k})\Big|_{\vec{X}=\frac{1}{2}(\vec{x}+\vec{y})}. \tag{15}$$

The noncommutativity of the derivations and the local (space-time dependent) symmetry transformations is common story in gauge theory formalism. This entice us to introduce gauge fields associated with the space-time dependent thermal Bogoliubov transformation (11).

Let us consider new fields $A_\alpha(t,\vec{x};\vec{k})$ and assume that these A_α are transformed like

$$A_\alpha(t,\vec{x};\vec{k}) \to A_\alpha(t,\vec{x};\vec{k}) - i\partial_\alpha n(t,\vec{x};\vec{k}) \tag{16}$$

under the thermal Bogoliubov transformatoion which transforms

$$\xi(t,\vec{x})^\mu \to \phi(t,\vec{x})^\mu. \tag{17}$$

By using A_α thus introduced, we define the operator D_α by

$$\begin{aligned} D_\alpha f(t,\vec{x})^\mu &\equiv \partial_\alpha f(t,\vec{x})^\mu \\ &-\frac{i}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}A_\alpha(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})M_0^{\mu\nu}f(t,\vec{y})^\nu. \end{aligned} \tag{18}$$

Then, from eq.(13) and eq.(16), the operator D_α thus defined has the simple transformation property

$$D_\alpha\xi(t,\vec{x})^\mu \to D_\alpha\phi(t,\vec{x})^\mu \tag{19}$$

under the transformation (11), so that the opeator D_α in (18) gives the covariant derivative with respect to the termal Bogoliubov transformation (11).

The minimal coupling of the gauge field A_α to the field ϕ can be derived in the usual way, that is, by replacing ∂_α with D_α in the Lagrangian which is invariant under the global transformation.

For the simplest example, we take a free Schrödinger field which has the TFD Larangian density,

$$\hat{\mathcal{L}}_0 = \int d\vec{x}\,\{i\bar{\phi}(t,\vec{x})^\mu \partial_t \phi(t,\vec{x})^\mu - \frac{1}{2m}(\partial_i\bar{\phi}(t,\vec{x})^\mu)(\partial_i\phi(t,\vec{x})^\mu)\}. \tag{20}$$

Clearly $\hat{\mathcal{L}}_0$ is invariant under a global (with n_k not depending on the space-time coordiates) thermal Bogoliubov transformation. We can turn this invariance into one under local (with n_k depending on the space-time coordinate) transformation (11) by replacing ∂_α with D_α to get

$$\begin{aligned}\hat{\mathcal{L}} &= \int d\vec{x}\,\{i\bar{\phi}(t,\vec{x})^\mu D_t\phi(t,\vec{x})^\mu - \frac{1}{2m}(D_i\bar{\phi}(t,\vec{x})^\mu)(D_i\phi(t,\vec{x})^\mu)\} \\ &= \hat{\mathcal{L}}_0 - i\int d\vec{x}\int d\vec{k}\rho_Q(t,\vec{x};\vec{k})A_t(t,\vec{x};\vec{k}) \\ &\quad -i\int d\vec{x}\int d\vec{k} j_{Qi}(t,\vec{x};\vec{k})A_i(t,\vec{x};\vec{k}),.\end{aligned} \tag{21}$$

with

$$\rho_Q = \frac{1}{(2\pi)^3}\int d\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}}\bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^\mu M_0^{\mu\nu}\phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^\nu, \tag{22a}$$

$$\begin{aligned} j_{Qi} &= \frac{1}{(2\pi)^3}\int d\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}}\frac{i}{2m}\{\partial_i\bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^\mu M_0^{\mu\nu}\phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^\nu, \\ &\qquad -\bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^\mu M_0^{\mu\nu}\partial_i\phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^\nu\}.\end{aligned} \tag{22b}$$

We note that, in the r.h.s. of (21), the term quadratic in A_α vanishes because of the identity

$$M_0^2 = 0. \tag{23}$$

The second and the third terms of the r.h.s. of eq.(21) gives the minimal coupling of the gauge field to the field ϕ.

Thus we have been able to introduce the gauge field associated with the thermal Bogoliubov transformation and to construct its minimal interaction along a conventional way.

We remain with the question of the physical interpretation of this gauge field.

Here we note only that the Umezawa's thermal energy [2, 6] $Q(t)$ is written by using ρ_Q in eq.(22a) as

$$Q(t) = i \int d\vec{x} \int d\vec{k} \partial_t n(t, \vec{x}; \vec{k}) \rho_Q(t, \vec{x}; \vec{k}).$$

This suggests the possibility that we may be able to interpret our gauge field as some thermodynamical force. This must be a very interesting problem left in the future.

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PHYSICS LETTERS B

Finite temperature quantum field theory and gauge field

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We study the invariance of thermo field dynamics (TFD) for equilibrium systems under local gauge transformations. We show that when restricted to a conveniently constrained subspace of thermal states, TFD has the structure of a gauge theory, the gauge vector field playing the role of the reservoir response to the system dynamics.

Since many years the development of quantum field theory (QFT) at finite temperature has attracted much interest [1] in view of the increasing number of applications. Originally stimulated by condensed matter physics, the studies of thermal QFT have been extended so to cover subjects of interest to high energy physics, cosmology, phase transitions in the early Universe, etc. On the other hand, non-equilibrium systems and dissipative phenomena, both in energy physics as well as in condensed matter physics, also require a deeper understanding of thermal quantum theories [2]. In recent years much work has been devoted to thermo field dynamics (TFD) [3] which is a real time formulation of thermal QFT for equilibrium [4] as well as for non-equilibrium and dissipative systems [#1] [1,7,8]. In the framework of TFD, new concepts have also been introduced which express a time-dependent quasi-particle picture as the coupling to an external (classical) gauge field [9] and thus focus the attention to the geometrical structure of thermal field theories. Moreover, spatially inhomogeneous thermal phenomena have been the subject of recent work in TFD [7] and a further analysis of the symmetry of the theory under space-dependent Bogoliubov transformations is required.

In this paper we want to touch one more point of contact of thermal quantum theory with gauge theory. Our main observation is the following. In TFD the matrix elements of the lagrangian are always invariant under local gauge (i.e. local phase) transformations provided the space of the physical thermal states is restricted to the subspace H_{th} of states constrained by the condition $(a^\dagger a - \tilde{a}^\dagger \tilde{a})|\text{phys}\rangle = 0$. Here, as usual, the tilde-operators denote the degrees of freedom which are introduced in TFD to "double" the system. In the following the tilde-system will be denoted as the thermal reservoir, according to a possible interpretation of the tilde-degrees, of freedom [3,4], although it could be more appropriate simply to speak of the tilde-symmetric system leaving apart the thermal bath interpretation [5]. As a matter of fact, it is the presence of the tilde-kinematical term in the TFD lagrangian which is responsible for the local gauge invariance of the theory in H_{th}. We will then show that in H_{th} the tilde-kinematical term in the lagrangian may be replaced by the minimal coupling between the system field and a vector field $A_\mu(x)$. Our analysis shows that such a vector field may be identified with the conventional vector field associated with a compact Lie gauge group (U(1) in the abelian case or SU(n) in the non-abelian one).

As a result, TFD appears to be in H_{th} a gauge theory, the gauge field A_μ being related to the thermal

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[#1] Non-equilibrium thermal field theory using TFD formalism has been studied in many papers. Among the oldest ones we quote ref. [5]. A recent one is ref. [6]. See also ref. [7].

bath represented by the tilde-degrees of freedom. For simplicity we only consider here equilibrium systems and therefore Bogoliubov transformations which are independent of space and time. In a subsequent paper we will show that our statement holds also for non-equilibrium systems.

For simplicity and to be specific we start by considering the lagrangian of the massless free Dirac field in standard non-thermal QFT:

$$L=-\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi\,. \tag{1}$$

The U(1) local gauge transformation is

$$\psi(x)\to\exp[\mathrm{i}g\alpha(x)]\psi(x)\,. \tag{2}$$

Under (2) L transforms as

$$L\to L'=L-\mathrm{i}g\,\partial^{\mu}\alpha(x)\bar{\psi}(x)\gamma_{\mu}\psi(x)\,. \tag{3}$$

The common story is that to make L invariant under (2), we have to introduce in L the coupling of the current $j_{\mu}=\mathrm{i}\bar{\psi}\gamma_{\mu}\psi$ with the gauge vector A_{μ} such that, when $\psi(x)$ transforms as in eq. (2), $j^{\mu}(x)A_{\mu}(x)$ transforms as

$$gj^{\mu}(x)A_{\mu}(x)\to gj^{\mu}(x)A_{\mu}(x)+gj^{\mu}(x)\,\partial_{\mu}\alpha(x)\,, \tag{4}$$

i.e.

$$A_{\mu}(x)\to A_{\mu}(x)+\partial_{\mu}\alpha(x)\,. \tag{5}$$

The lagrangian L_g so modified is invariant under the U(1) local gauge transformations (2) and (5):

$$L_g=-\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi+\mathrm{i}g\bar{\psi}\gamma^{\mu}\psi A_{\mu}\,,\quad L_g\to L_g'=L_g\,. \tag{6}$$

If one wants A_{μ} to be a dynamical field then the kinematical term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ has also to be added to the lagrangian (6) in the well known fashion.

We now consider TFD for equilibrium systems. It is well known that one has to duplicate the physical degrees of freedom, which is done by introducing the tilde-field $\tilde{\psi}(x)$ and the tilde lagrangian. The lagrangian for the massless free Dirac theory at finite temperature is then given by

$$\hat{L}=L-\tilde{L}=-\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi+\bar{\tilde{\psi}}\gamma^{\mu}\,\partial_{\mu}\tilde{\psi}\,. \tag{7}$$

The original motivation to introduce the tilde-field $\tilde{\psi}(x)$ is to obtain statistical averages of physical observables A as expectation values of the "thermal vacuum" $|0(\theta)\rangle$:

$$\langle 0(\theta)|A|0(\theta)\rangle=\langle A\rangle=\frac{\mathrm{Tr}(A\exp[-\beta H])}{\mathrm{Tr}(\exp[-\beta H])}\,. \tag{8}$$

In general, the tilde-field is thus an auxiliary field and the tilde-system is a copy (with the same spectrum and couplings) of the physical system. Coupling of the physical field $\psi(x)$ with the tilde field $\tilde{\psi}(x)$ is not allowed in $\hat{L}$. The thermal vacuum, however, appears as a condensate of couples of physical and tilde-field quanta; in a simplified notation $|0(\theta)\rangle$ is written as

$$|0(\theta)\rangle=\prod_{k}(\cos\theta_k+\sin\theta_k\,a_k^{\dagger}\tilde{a}_k^{\dagger})|0\rangle\,. \tag{9}$$

Here $a^{\dagger}$ and $\tilde{a}^{\dagger}$ denote the creation operators associated to ψ and $\tilde{\psi}$, respectively; all quantum number indices are suppressed except momentum. The state $|0(\theta)\rangle$ is the vacuum with respect to the fields $\psi(\theta;x)$ and $\tilde{\psi}(\theta;x)$ obtained by a Bogoliubov transformation from $\psi(x)$ and $\tilde{\psi}(x)$, respectively:

$$\psi(\theta;x)=B^{-1}(\theta)\psi(x)B(\theta)\,,$$
$$\tilde{\psi}(\theta;x)=B^{-1}(\theta)\tilde{\psi}(x)B(\theta)\,. \tag{10}$$

In this paper we only consider Bogoliubov transformations whose parameter θ is independent of space–time. The space of states constructed out of $|0(\theta)\rangle$ by repeated applications of creation operators of $\psi(\theta;x)$ and $\tilde{\psi}(\theta;x)$ will be called the finite temperature representation $\{|0(\theta)\rangle\}$.

The following relation holds for the thermal vacuum:

$$(a_k^{\dagger}a_k-\tilde{a}_k^{\dagger}\tilde{a}_k)|0(\theta)\rangle=0\,. \tag{11}$$

In the following we will consider the subspace $\mathrm{H_{th}}\subset\{|0(\theta)\rangle\}$ made of all the states $|a\rangle_{\mathrm{th}}$, including $|0(\theta)\rangle$, such that eq. (11) holds for any $|a\rangle_{\mathrm{th}}$. In $\mathrm{H_{th}}$ we have for example

$$\langle j_{\mu}(x)\rangle_{\mathrm{th}}=\langle\tilde{j}_{\mu}(x)\rangle_{\mathrm{th}}\,, \tag{12}$$

where $\langle\ \rangle_{\mathrm{th}}$ denotes matrix elements in $\mathrm{H_{th}}$. We observe that $\mathrm{H_{th}}$ is invariant under the dynamics described by $\hat{L}$ (even in the more general case in which some interaction term is present in $\hat{L}$ provided the charge is conserved). We now notice that due to eq. (12) the matrix elements in $\mathrm{H_{th}}$ of the TFD lagrangian equation (7) (as well as of a more general lagrangian than the simple one presently considered) are

invariant under the simultaneous local gauge transformations of the ψ and $\tilde{\psi}$ fields given by eq. (2) and by

$$\tilde{\psi}(x) \to \exp[ig\alpha(x)]\tilde{\psi}(x) \tag{13}$$

respectively. Thus, under (2) and (13)

$$\langle \hat{L} \rangle_{\rm th} \to \langle \hat{L}' \rangle_{\rm th} = \langle \hat{L} \rangle_{\rm th} \quad \text{in } \mathrm{H}_{\rm th} . \tag{14}$$

In the following, equalities between matrix elements in $\mathrm{H}_{\rm th}$, say $\langle A \rangle_{\rm th} = \langle B \rangle_{\rm th}$, will be denoted as $A \cong B$ and called thermal weak (th-w-) equalities. We thus see that the term $\bar{\tilde{\psi}}\gamma^\mu \partial_\mu \tilde{\psi}$ plays a crucial role in the th-w-gauge invariance of $\hat{L}$ under (2) and (13) since it transforms in such a way to compensate the local gauge transformation of the physical kinematical term, i.e.

$$\bar{\tilde{\psi}}(x)\gamma^\mu \partial_\mu \tilde{\psi}(x) \to \bar{\tilde{\psi}}(x)\gamma^\mu \partial_\mu \tilde{\psi}(x) + g\,\partial^\mu \alpha(x)\,\tilde{j}_\mu(x) . \tag{15}$$

This motivates us to introduce the vector field A'_μ by

$$g j^{\bar{\mu}}(x) A'_{\bar{\mu}}(x) \cong \bar{\tilde{\psi}}(x)\gamma^{\bar{\mu}} \partial_{\bar{\mu}} \tilde{\psi}(x) , \quad \bar{\mu} = 1, 2, 3, 4 . \tag{16}$$

The bar over μ means, here and in the following, no summation. Due to eq. (15) A'_μ thus introduced transforms as

$$A'_\mu(x) \to A'_\mu(x) + \partial_\mu \alpha(x) , \tag{17}$$

when (2) and (13) are implemented. This suggests to us to assume that A'_μ can be identified with the conventional U(1) gauge vector field. We shall now confirm that this identification does not contradict (is consistent with) TFD when we restrict ourselves to the th-w-equalities, i.e. to matrix elements in $\mathrm{H}_{\rm th}$. To this aim let us first observe that matrix elements of physical observables, which in TFD are not functions of the tilde-field, but solely of the $\psi(x)$ field, are not changed by the introduction of the assumption (16), and therefore also their statistical averages, whose computation is the main motivation of TFD, are not changed; the reason for this is that the position (16) does not change the thermal vacuum structure. Next, we have to show that the conservation laws implied by the TFD scheme are also preserved as th-w-equalities when the position (16) is adopted. In the simple case of eq. (7) here considered, we have the current conservation laws:

$$\partial^\mu j_\mu(x) = 0 , \quad \partial^\mu \tilde{j}_\mu(x) = 0 . \tag{18a,b}$$

From the Dirac equation for the ψ field coupled to A'_μ,

$$\gamma^\mu \partial_\mu \psi(x) = ig\gamma^\mu \psi(x) A'_\mu(x) , \tag{19}$$

we have,

$$\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) = ig\bar{\psi}(x)\gamma^\mu \psi(x) A'_\mu(x) \cong \bar{\tilde{\psi}}(x)\gamma^\mu \partial_\mu \tilde{\psi}(x) , \tag{20}$$

where the position (16) has also been used. Since

$$\langle \bar{\psi}\gamma^\mu \partial_\mu \psi \rangle_{\rm th} = \langle \bar{\tilde{\psi}}\gamma^\mu \partial_\mu \tilde{\psi} \rangle_{\rm th} , \tag{21}$$

from (20) we obtain

$$\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) \cong ig\bar{\psi}(x)\gamma^\mu \psi(x) A'_\mu(x) . \tag{22}$$

The adjoint of (22) is

$$\bar{\psi}(x)\gamma^\mu \overleftarrow{\partial}_\mu \psi(x) \cong -ig\bar{\psi}(x)\gamma^\mu \psi(x) A'_\mu(x) , \quad \gamma^\dagger_\mu = \gamma_\mu , \tag{23}$$

i.e.

$$\partial^\mu j_\mu(x) \cong 0 . \tag{24a}$$

Similarly, from the th-w-equality in (20) and its adjoint we get

$$\partial^\mu \tilde{j}_\mu(x) \cong 0 . \tag{24b}$$

Of course, we could derive eq. (24a) from $\partial^\mu j_\mu = 0$, which directly follow, from (19). Eq. (24b) is then obtained by use of eq. (12). Such a derivation, however does not involve (16) and thus does not give us informations on its consistency. We observe that subtracting term by term eqs. (22) and (23) we obtain the th-w-equality $ig j^\mu A'_\mu \cong g_{\mu\nu} T^{\mu\nu}$ where $T^{\mu\nu}$ is the energy–momentum tensor [10].

In the case A'_μ does not represent just an external field but is indeed a dynamical field, from the assumption that A'_μ may be identified with the U(1) gauge vector field, we have

$$\partial^\nu F'_{\mu\nu}(x) = -g j_\mu(x) . \tag{25}$$

Then, one more route may be followed to show that the conservation laws implied by TFD are preserved as th-w-equality when the position (16) is adopted. By multiplying both sides of eq. (25) by A'_μ and by using (16) we have

$$[\partial^\nu F'_{\bar\mu\nu}(x)]A'^{\bar\mu}(x) = -gj_{\bar\mu}(x)A'^{\bar\mu}(x)$$
$$\cong -\bar\psi(x)\gamma_{\bar\mu}\,\partial^{\bar\mu}\tilde\psi(x)\,. \tag{26}$$

Under gauge transformation (with $\partial_\mu\alpha(x)\neq 0$) this gives

$$[\partial^\nu F'_{\bar\mu\nu}(x)]\partial^{\bar\mu}\alpha(x) \cong -g\tilde j_{\bar\mu}(x)\,\partial^{\bar\mu}\alpha(x) \tag{27}$$

i.e.

$$\partial^\nu F'_{\mu\nu}(x) \cong -g\tilde j_\mu(x)\,. \tag{28}$$

Use of eq. (12) also gives

$$\partial^\nu F'_{\mu\nu}(x) \cong -gj_\mu(x)\,. \tag{29}$$

From eqs. (28) and (29) we thus obtain the TFD conservation laws, eqs. (18), restricted to H_{th}, namely eqs. (24). Note that eq. (24a) also holds "strongly", as it can be directly derived from eq. (25). Then eqs. (24) may be again derived by using also eq. (12). Eq. (16), however, is not used in this case.

By choosing the Lorentz gauge, from eq. (29) we also obtain

$$\Box A'_\mu(x) \cong -gj_\mu(x)\,, \quad \partial^\mu A'_\mu(x) \cong 0\,. \tag{30}$$

We further observe that under our assumptions observables are invariant under gauge transformations. The fact that eqs. (19) and (29) hold in H_{th} suggests to us to adopt in H_{th} the following lagrangian:

$$\hat L_g = -\tfrac{1}{4}F'^{\mu\nu}F'_{\mu\nu} - \bar\psi\gamma^\mu\,\partial_\mu\psi + ig\bar\psi\gamma^\mu\psi A'_\mu$$
$$\text{in } H_{th}\,. \tag{31}$$

In conclusion the TFD lagrangian (7) has been substituted in H_{th} by the lagrangian (31) which clearly shows the intrinsic gauge properties of TFD. It is interesting that the kinematical term of the tilde-degree of freedom disappeared from the gauge lagrangian (31), although its role is crucial in determining the structure of the states belonging to H_{th} (cf. eqs. (9) and (11)). We also observe that due to the minimal coupling an interaction term has been introduced in (31), although it was absent in the original lagrangian (7) (in our case a free field theory), a feature which is common with the standard introduction of the minimal coupling in the usual gauge theories. In the present case, the fact that the tilde-kinematical term is replaced by the gauge field-current coupling, although in a th-w-sense, suggests that the gauge field A'_μ may play the role of "thermal" reservoir analogous to the one of the tilde-system in the standard TFD. In this respect it is interesting to observe that eq. (28) relates variations of the gauge field tensor $F'_{\mu\nu}$ to the "reservoir" current $\tilde j_\mu$.

Our conclusions also hold true in the case of a massive Dirac field. In such a case in eq. (31) the terms $-m\bar\psi\tilde\psi$ and $-m\bar\psi\psi$ will be added to $\hat L_g$. Finally, we observe that in the case of an interaction term in the starting lagrangian (7) $\tilde L_{tot} = \hat L + \hat L_I$, $\hat L_I = L_I - \tilde L_I$, the above discussion and assumptions still hold provided H_{th} is an invariant subspace under the dynamics described by $\hat L_{tot}$.

In view of the th-w-equivalence (in the sense above specified) either one of the lagrangians (7) or (31) may be used in physical applications, according to computational convenience. The case of the U(1) local gauge can be generalized to a non-abelian compact Lie group G, say for simplicity SU(2). Let ψ and $\tilde\psi$ denote two spinor doublet fields transforming under SU(2) as

$$\psi(x) \to \exp[ig\alpha(x)\cdot t]\psi(x)\,, \tag{32a}$$

$$\tilde\psi \to \exp[ig\alpha(x)\cdot t]\tilde\psi(x)\,, \quad t = \tfrac{1}{2}\tau\,. \tag{32b}$$

Then the above discussion for U(1) is generalized to SU(2) provided the position (16) is replaced by

$$gj^{\bar\mu}\cdot A'_{\bar\mu} \cong \bar\psi\gamma^{\bar\mu}\,\partial_{\bar\mu}\tilde\psi\,, \tag{33}$$

where $j_\mu = i\bar\psi\gamma_\mu t\psi$. Due to (32) the gauge field $A'_\mu(x)$ transforms as

$$A'_\mu(x) \to A'_\mu(x) + \partial_\mu\alpha(x) + gA_\mu(x)\wedge\alpha(x)\,, \tag{34}$$

for infinitesimal transformations, which is indeed the usual infinitesimal transformation property for SU(2) Yang–Mills gauge field.

We have considered spinor fields ψ and $\tilde\psi$ until now. Our discussion can be however extended to include the case of complex scalar field ϕ or else the non-relativistic Schrödinger case. For shortness, we simply report the position which replaces the assumption (16) in the U(1) case; for the scalar field our position is

$$g[(\partial_{\bar\mu}\phi)^\dagger\phi - \phi^\dagger(\partial_{\bar\mu}\phi)]A'^{\bar\mu} - g^2\phi^\dagger\phi A'_{\bar\mu}A'^{\bar\mu}$$
$$\cong (\partial_{\bar\mu}\tilde\phi)^\dagger(\partial^{\bar\mu}\tilde\phi)\,, \quad \bar\mu = 0, 1, 2, 3\,. \tag{35}$$

In the Schrödinger case a position similar to eq. (35) is assumed, provided ϕ denotes the wave-function and $\partial_{\bar\mu}$ represents only the spatial derivatives ($\bar\mu = 1, 2, 3$).

In conclusion we have shown that the tilde-kinematical term in the TFD lagrangian can be replaced by the minimal coupling of the system field with a gauge vector field in the subspace H_{th} of states with even number of tilde and non-tilde particles. This has been possible due to the local gauge invariance of TFD in H_{th}. Our main result is therefore that in H_{th} finite temperature QFT has the structure of a gauge theory, the gauge vector field playing the role of the "reservoir" response to the system dynamics. It is an interesting question to ask if, on the converse, any gauge theory also describes thermal effects in some subspace of states.

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Dissipative and inhomogeneous systems and the gauge field

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We discuss the gauge structure of real time thermal quantum field theory (thermo field dynamics) for dissipative and inhomogeneous systems. We show that a covariant derivative may be conveniently introduced in order to recover the invariance of the lagrangian under space–time-dependent Bogoliubov transformations. In the case of inhomogeneous systems, although we recover canonical commutation relations, the problem of non-locality of the gauge field remains open.

The study of non-equilibrium systems and dissipative phenomena is of interest to high energy physics, cosmology, phase transitions in the early Universe, as well as to solid state physics. Much work has been devoted to the study of quantum field theory (QFT) at finite temperature to analyse the mathematical structure of the theory and to develop a useful computational technique [1,2], both for equilibrium and non-equilibrium systems. In a recent paper [3] it has been shown that the real time formulation of thermal quantum field theory, i.e. thermo field dynamics (TFD) [4,5], has the structure of a gauge theory provided a convenient restriction of the state space is adopted (the thermal state subspace $H_{\rm th}$). There, only equilibrium systems and space–time-independent Bogoliubov transformations were considered. On the other hand, non-equilibrium systems, dissipative regimes and non-homogeneous heat conduction may require the introduction of time-, space- and space–time-dependent Bogoliubov transformations in TFD [#1] [9–12]. The purpose of this paper is to discuss the gauge symmetry properties of TFD with space–time-dependent Bogoliubov transformations. We will use in the following the usual thermal doublet notation

$$\xi_k^\mu(t) = (\xi_k(t), \tilde{\xi}_k^\dagger(t)) , \quad \tilde{\xi}_k^\mu(t) = (\xi_k^\dagger(t), -\tilde{\xi}_k(t)) ,$$

with the operators ξ_k and $\tilde{\xi}_k$ acting on the Fock space with vacuum $|0\rangle$: $\xi_k|0\rangle = 0$, $\tilde{\xi}_k|0\rangle = 0$.

The tilde-operator $\tilde{\xi}_k$ is introduced to "double" the ξ-system and is needed to treat finite temperature problems according to general rules [4–8] on which we do not insist for sake of shortness. The key point in TFD is that thermal statistical averages are computed as expectation values in the thermal vacuum $|0(\theta)\rangle$ and the tilde-operators are crucial in computing traces.

The tilde-degree of freedom has been shown to play an essential role in the gauge invariance properties of the theory in the case studied in ref. [3]. In particular it has been shown that the tilde-kinematical term in the TFD lagrangian may be substituted by the coupling term between the gauge field and the matter field; e.g. in the case of a spinor field we have the equality between the matrix elements in $H_{\rm th}$ [3]:

$$\langle g j^\mu A_\mu(x) \rangle \cong \langle \tilde{\bar{\psi}}(x) \gamma^\mu \partial_\mu \tilde{\psi}(x) \rangle , \tag{1}$$

where the bar over μ means no summation.

[#1] Non-equilibrium thermal field theory using the TFD formalism has been studied in many papers. Among the oldest ones we quote ref. [6]. A recent one is ref. [7]. See also refs. [8,9].

Let $B_k(\theta(x,t))^{\mu\nu}$ be the matrix associated with the space–time-dependent Bogoliubov transformation. The parameter θ also depends on temperature, although this is omitted in our notation. Moreover, in the following we will use the shortened notation $B_k(x,t)^{\mu\nu}$. We write the Bogoliubov transformation as

$$a_k(x,t)^\mu = [B_k^{-1}(x,t)]^{\mu\nu}\xi_k(t)^\nu\,, \tag{2}$$

where

$$\xi_k(t)^\mu = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}s\,\omega_k(s)\right)\xi_k^\mu\,, \tag{3}$$

$$[\xi_k(t)^\mu, \bar{\xi}_q(t)^\nu] = \delta_{kq}\delta^{\mu\nu}\,. \tag{4}$$

The form of $B_k(x,t)^{\mu\nu}$ will be specified in the following.

The fields corresponding to $\xi_k^\mu(t)$ and to $a_k^\mu(t)$ are

$$\psi(x,t)^\mu = (2\pi)^{-3/2}\int \mathrm{d}k\,\xi_k(t)^\mu \exp(-\mathrm{i}kx)\,,\quad \phi(x,t)^\mu = (2\pi)^{-3/2}\int \mathrm{d}k\,a_k(t)^\mu\exp(-\mathrm{i}kx)\,. \tag{5a,b}$$

It immediately appears that, due to space dependence of $B_k(x,t)^{\mu\nu}$, $a_k(x,t)^\mu$ and $\bar{a}_k(x,t)^\mu$ do not satisfy the equal time canonical commutations relations; in fact we have

$$[a_k(x,t)^\mu, \bar{a}_l(x',t)^\nu] = \delta_{kl}[B_k^{-1}(x,t)B_k(x',t)]^{\mu\nu} \neq \delta_{kq}\delta^{\mu\nu}\,. \tag{6}$$

We will see in the following how this can be cured [9].

We will divide our discussion in two parts leaving the inhomogeneous case at the end. We will first consider the case of dependence of $B_k^{\mu\nu}$ only on time (and temperature), $B_k^{\mu\nu} = B_k(t)^{\mu\nu}$, which is the case of dissipation in spatially homogeneous systems. In this case $a_k(t)^\mu$ and $\bar{a}_k(t)^\mu$ satisfy canonical commutation relations.

The time derivative of $a_k(t)^\mu$ is given by

$$\mathrm{i}\,\partial_t a_k(t)^\mu = \omega_k a_k(t)^\mu + \mathrm{i}[\dot{B}_k^{-1}(t)]^{\mu\nu}[B_k(t)]^{\nu\sigma}a_k(t)^\sigma = [a_k(t)^\mu, \hat{H}_0(t)] + \mathrm{i}[\dot{B}_k^{-1}(t)]^{\mu\nu}[B_k(t)]^{\nu\sigma}a_k(t)^\sigma\,, \tag{7}$$

with the hamiltonian $\hat{H}_0(t)$ given by

$$\hat{H}_0(t) = \int \mathrm{d}k\,\omega_k(t)\bar{\xi}_k^\mu\xi_k^\mu = \int \mathrm{d}k\,\omega_k(t)\bar{\xi}_k(t)^\mu\xi_k(t)^\mu = \int \mathrm{d}k\,\omega_k(t)\bar{a}_k(t)^\mu a_k(t)^\mu\,. \tag{8}$$

In eq. (7) we used also

$$\mathrm{i}\dot{\xi}_k(t)^\mu = [\xi_k(t)^\mu, \hat{H}_0(t)]\,. \tag{9}$$

Eq. (7) shows that the hamiltonian $\hat{H}_0(t)$ does not completely control the time evolution of $a_k(t)^\mu$. The Heisenberg equation for $a_k(t)^\mu$ is modified by the introduction of the last term in the RHS of eq. (7). We may introduce the operator

$$Q = \mathrm{i}\int \mathrm{d}k\,\bar{a}_k(t)^\mu[\dot{B}_k^{-1}(t)B_k(t)]^{\mu\nu}a_k(t)^\nu\,, \tag{10}$$

and thus include this term in the commutator:

$$\mathrm{i}\dot{a}_k(t)^\mu = [a_k(t)^\mu, \hat{H}_0(t) + Q]\,. \tag{11}$$

The operator Q is called the heat operator [6–9]. Eq. (11) looks like the standard Heisenberg equation for time evolution of operators in QFT. However, the operator $F = \hat{H}_0(t) + Q$ is to be understood not as the hamiltonian but as the free energy operator since it includes the heat contribution [10]. It can be shown that $B_k(t)^{\mu\nu}$ may be given by

$$B_k(t)=\begin{pmatrix}1+n_k(t) & -n_k(t)\\ -1 & 1\end{pmatrix}, \tag{12}$$

with $n_k(t)$ the particle number. The heat operator is then

$$Q=\mathrm{i}\int \mathrm{d}k\, \dot{n}_k(t)\bar{a}_k(t)^{\mu}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu} a_k(t)^{\nu}\,. \tag{13}$$

Perturbation techniques in terms of the Feynman diagrams may be developed when Q is included in the unperturbed hamiltonian and an additional perturbation term H_1 is considered. The renormalization condition leads then to the kinetic equation for $n_k(t)$ [9].

The above discussion shows that the theory is not invariant under time-dependent Bogoliubov transformations. In particular, time evolution is non-hamiltonian as shown by eq. (7) and as expected, since we are dealing with dissipative phenomena. When dealing with physical states dissipation of course produces damping; for example, the vacuum is unstable under time evolution [10]. Therefore we may ask the question whether a covariant time derivative may be consistently introduced to recover the invariance of the lagrangian under time-dependent Bogoliubov transformations. We will show that this is possible. We will follow two different ways which lead us to the same result.

First we introduce the field $A_0(t,k)$ which transforms as

$$A_0(t,k)\to A_0(t,k)-\mathrm{i}\,\partial_t n_k(t)\,, \tag{14a}$$

when

$$\xi_k(t)^{\mu}\to a_k(t)^{\mu}=[B^{-1}(t)]^{\mu\nu}\xi_k(t)^{\nu}\,. \tag{14b}$$

The covariant derivative $\mathrm{D}_t(x,t)^{\mu\nu}$ is then defined as follows:

$$\mathrm{D}_t(x,t)^{\mu\nu}\psi(x,t)^{\nu}$$
$$\equiv\int \mathrm{d}y\left[\delta^{\mu\nu}\delta(x-y)\partial_t-\mathrm{i}(2\pi)^{-3}\int \mathrm{d}k\exp[\mathrm{i}k(x-y)]A_0(t,k)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\right]\psi(x,y)^{\nu}\,. \tag{15}$$

By using the relation

$$[\dot{B}_k^{-1}(t)B_k(t)]^{\mu\nu}=\dot{n}_k(t)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}, \tag{16}$$

we see indeed that $\mathrm{D}_k(x,t)^{\mu\nu}$ has the proper transformation property when (14) are implemented:

$$\mathrm{D}_t(x,t)^{\mu\nu}\psi(x,t)^{\nu}=(2\pi)^{-3/2}\int \mathrm{d}k\,\mathrm{D}_t(t,k)^{\mu\nu}\xi_k(t)^{\nu}\exp(\mathrm{i}kx)$$
$$\to(2\pi)^{-3/2}\int \mathrm{d}k\,[B^{-1}(t)]^{\nu\sigma}\mathrm{D}_t(t,k)^{\mu\nu}\xi_k(t)^{\sigma}\exp(\mathrm{i}kx)=\mathrm{D}_t(x,t)^{\mu\nu}\phi(x,t)^{\nu}\,, \tag{17a}$$

$$\mathrm{D}_t(t,k)^{\mu\nu}=\delta^{\mu\nu}\partial_t-\mathrm{i}gA_0(t,k)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}. \tag{17b}$$

This fact together with eq. (14a) suggests to identify $A_0(t,k)$ as a gauge field [13]. If we consider for example the kinetic term $\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{i}\bar{\psi}(x,t)^{\mu}\partial_t\psi(x,t)^{\mu}$ in the Schrödinger lagrangian, the "minimal coupling" introduced through the covariant derivative $\mathrm{D}_t(x,t)^{\mu\nu}$ is

$$(2\pi)^{-3}\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}k\,gA_0(t,k)\rho(x,y,t)\exp[\mathrm{i}k(x-y)]\,, \tag{18}$$

with the density $\rho(x, y, t)$ given by

$$\rho(x, y, t)=\bar{\psi}(x, t)^{\mu}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\psi(y, t)^{\nu}. \tag{19}$$

In conclusion we have shown that if we want to recover the invariance under time-dependent Bogoliubov transformation we may proceed as in the conventional case where the invariance under time-dependent phase transformation may be recovered through the introduction of a gauge field.

We now observe that by using the result of ref. [3] we may also follow an alternative way to construct the covariant derivative reaching the same result as above.

First we "double" the ψ degree of freedom by introducing the $\psi''(x, t)^{\mu}$ field. We use the doublet notation $\psi''^{\mu}=(\psi'' \quad \tilde{\psi}''^{\dagger})$, $\bar{\psi}''^{\mu}=(\psi''^{\dagger} \quad -\tilde{\psi}'')$ and put $\psi^{\mu i}=(\psi^{\mu} \quad \tilde{\psi}''^{\mu})=(\psi \quad \tilde{\psi}^{\dagger} \quad \psi''^{\dagger} \quad -\tilde{\psi}'')$, $\bar{\psi}^{\mu i}=(\bar{\psi}^{\mu} \quad -\psi''^{\mu})=(\psi^{\dagger} \quad -\tilde{\psi} \quad -\psi''-\tilde{\psi}''^{\dagger})$, with $i=1, 2$ labelling the ψ and the ψ'' degrees of freedom, respectively. With this choice the Bogoliubov transformation $B(t)$ acts on ψ'' and $\tilde{\psi}''$ in the same way as on ψ and $\tilde{\psi}$. We then introduce the Bogoliubov transformation $C(\tau)$ with the parameter τ independent of space–time. For fixed μ we write

$$\psi(\tau; x, t)^{\mu i}=C^{-1}(\tau)^{ij}\psi(x, t)^{\mu j}, \quad \text{fixed}\,\mu, \quad \bar{\psi}(\tau; x, t)^{\mu i}=\bar{\psi}^{\mu j}C^{-1}(\tau)^{ij}, \quad \text{fixed}\,\mu. \tag{20a,b}$$

Let us denote by $H_{\text{th}}=\{|0(t, \tau)\rangle\}$ the subspace of the states $|\ \rangle_{\text{th}}$ such that

$$\bar{\psi}^{\mu i}\psi^{\mu i}|\ \rangle_{\text{th}}=[(\psi^{\dagger}\psi-\tilde{\psi}^{\dagger}\tilde{\psi})-(\psi''^{\dagger}\psi''-\tilde{\psi}''^{\dagger}\tilde{\psi}'')]|\ \rangle_{\text{th}}=0. \tag{20c}$$

We note that the form $\bar{\psi}^{\mu i}\psi^{\mu i}$ is invariant under both the Bogoliubov transformations, B and C.

We then observe that the lagrangian $\hat{L}=\hat{L}-\hat{L}''$ is invariant in H_{th} under time-dependent Bogoliubov transformation $B_k(t)^{\mu\nu}$ due to the ψ''-kinematical term. We introduce next the field $A_0(t, k)^{\mu\nu}$ by the following position:

$$\langle g\bar{\xi}_k(t)^{\mu}A_0(t, k)^{\mu\nu}\xi_k(t)^{\nu}\rangle=\langle \mathrm{i}\bar{\xi}''_k(t)^{\mu}\delta^{\mu\nu}\,\partial_t\xi''_k(t)^{\nu}\rangle \quad \text{in } H_{\text{th}}. \tag{21}$$

From the position (21) it follows that, when $\xi(t)$ transforms under $B_k(t)^{\mu\nu}$, $A_0(t, k)^{\mu\nu}$ transforms as

$$A_0(t, k)^{\mu\nu}\to A_0(t, k)^{\mu\nu}-\mathrm{i}\dot{n}_k(t)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}, \tag{22}$$

i.e. by putting

$$A_0(t, k)^{\mu\nu}=A_0(t, k)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu} \tag{23}$$

we have

$$A_0(t, k)\to A_0(t, k)-\mathrm{i}\dot{n}_k(t), \tag{24}$$

which is the same as eq. (14a). The covariant derivative $\mathrm{D}_t(t, k)^{\mu\nu}$ is therefore constructed as in eq. (17b) and when working with the field $\psi(x, t)^{\mu}$ we come again to eq. (15).

We now turn to the second part of our discussion. Our task is to generalize the above analysis to the case of Bogoliubov transformation depending also on space. We will follow ref. [9] (see also ref. [13]). We have $n_k=n(x, t, k)$ and $B_k^{\mu\nu}=B_k(x, t)^{\mu\nu}$. We write eq. (2) as

$$a_k(t)^{\mu}=\exp(-G_n)\exp(G_0)\xi_k(t)^{\mu}\exp(G_0)\exp(G_n), \tag{25}$$

with

$$G_0=\int \mathrm{d}k\,\bar{\xi}_k^{\mu}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}^{\mu\nu}\xi_k^{\nu}, \quad G_n=\int \mathrm{d}x\,\mathrm{d}k\,n(x, t, k)J(x, t, k), \tag{26a,b}$$

$$j(x,t,k)=\frac{1}{(2\pi)^3}\int \mathrm{d}\eta \exp(-\mathrm{i}k\eta)\bar{\psi}(t,x+\tfrac{1}{2}\eta)^{\mu}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^{\mu\nu}\psi(t,x-\tfrac{1}{2}\eta)^{\nu}. \tag{26c}$$

The canonical commutation relations are now satisfied:

$$[a_k(t)^{\mu},\bar{a}_q(t)^{\nu}]=\delta_{kq}\delta^{\mu\nu}. \tag{27}$$

The heat operator Q is in this case

$$Q(t)=\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}k \exp[\mathrm{i}k(x-y)]\{\rho(x,y,t)\,\partial_t n[\tfrac{1}{2}(x+y),t,k]+j_i(x,y,t)\,\partial_{x_i} n[\tfrac{1}{2}(x+y),t,k]\}, \tag{28}$$

where the density $\rho(x,y,t)$ and the current $j_i(x,y,t)$ are

$$\rho(x,y,t)=\bar{\psi}(x,t)^{\mu}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\psi(y,t)^{\nu}, \tag{29a}$$

$$j_i(x,y,t)=\tfrac{1}{2}\mathrm{i}\left[\partial_{x_i}\bar{\psi}(x,t)^{\mu}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\psi(y,t)^{\nu}-\bar{\psi}(x,t)^{\mu}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\partial_{y_i}\psi(y,t)^{\nu}\right]. \tag{29b}$$

We introduce the field $A_\alpha(x,t,k)$, $\alpha=t,i$; $i=1,2,3$, which we assume transforming as

$$A_\alpha(x,t,k)\to A_\alpha(x,t,k)-\mathrm{i}\,\partial_\alpha n(x,t,k), \tag{30}$$

when (25) is implemented.

The covariant derivative $\mathrm{D}_\alpha(x,t)^{\mu\nu}$ is now defined by

$$\mathrm{D}_\alpha(x,t)^{\mu\nu}\psi(x,t)^{\nu}$$
$$\equiv\left\{\int \mathrm{d}y\left[\delta^{\mu\nu}\delta(x-y)\,\partial_\alpha-\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}K \exp[\mathrm{i}K(x-y)]\,gA_\alpha[\tfrac{1}{2}(x+y),t,K]\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}^{\mu\nu}\right]\right\}\psi(y,t)^{\nu}, \tag{31}$$

and we have

$$\mathrm{D}_\alpha(x,t)^{\mu\nu}\psi(x,t)^{\nu}\to\mathrm{D}_\alpha(x,t)^{\mu\nu}\phi(x,t)^{\nu}. \tag{32}$$

By considering as an example the Schrödinger-like lagrangian we have

$$L_g=\int \mathrm{d}t\,\mathrm{d}x\,\{\mathrm{i}\bar{\psi}(x,t)^{\mu}\,\mathrm{D}_t^{\mu\nu}\psi(x,t)^{\nu}-\tfrac{1}{2}[\mathrm{D}_i^{\mu\nu}\bar{\psi}(x,t)^{\nu}][\mathrm{D}_i^{\mu\nu}\psi(x,t)^{\nu}]\}$$
$$=\int \mathrm{d}t\,\mathrm{d}x\,\{\mathrm{i}\bar{\psi}(x,t)^{\mu}\,\partial_t\psi(x,t)^{\mu}-\tfrac{1}{2}[\partial_{x_i}\bar{\psi}(x,t)^{\mu}][\partial_{x_i}\psi(x,t)^{\mu}]\}$$
$$+\frac{g}{(2\pi)^3}\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}K \exp[\mathrm{i}K(x-y)]\{A_t[\tfrac{1}{2}(x-y),t,K]\,\rho(x,y,t)+A_i[\tfrac{1}{2}(x+y),t,K]\,j_i(x,y,t)\}. \tag{33}$$

The lagrangian (33) is invariant under (25).

Again we obtain the above result by following ref. [3]: we introduce the ψ''-system and the Bogoliubov transformation eq. (20). In the TFD lagrangian $\hat{\tilde{L}}=\hat{L}-\hat{L}''$ we adopt the substitution

$$\langle gA_t(x,t,k)\rho(x,t)\rangle=\langle \mathrm{i}\bar{\psi}''(x,t)^{\mu}\delta^{\mu\nu}\,\partial_t\psi''(x,t)^{\nu}\rangle, \tag{34a}$$

$$\langle gA_{\bar{i}}(x,t,k)\,j_{\bar{i}}(x,t)\rangle=\langle \tfrac{1}{2}[\partial_{x_{\bar{i}}}\bar{\psi}''(x,t)^{\mu}][\partial_{x_{\bar{i}}}\psi''(x,t)^{\mu}]\rangle, \tag{34b}$$

where the bar over i means no summation and matrix elements are computed in H_{th}. From eqs. (34) it follows that, when (25) is implemented, $A_\alpha(x,t,k)$ transforms as in eq. (30) and the covariant derivative is then constructed as in eq. (31).

Finally we note that the kinematical term for the gauge field may be introduced in the lagrangian as $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with the definition

$$F_{\alpha\beta}=\partial_\alpha \int \mathrm{d}y\,\mathrm{d}K A_\beta[\tfrac{1}{2}(x+y), x, K]\exp[\mathrm{i}K(x-y)]-\partial_\beta \int \mathrm{d}y\,\mathrm{d}K A_\alpha[\tfrac{1}{2}(x+y), t, K]\exp[\mathrm{i}K(x-y)]\,. \qquad (35)$$

We would like to close this paper with two comments. Although we have shown that it is formally possible to recover the gauge invariance under space–time Bogoliubov transformations in the theory for dissipative inhomogeneous systems, and it is also possible to rewrite the gauge coupling terms in terms of doubling the degrees of freedom in the sense above specified, the problem of non-locality of the gauge field in the case of inhomogeneous systems remains open. The recovering of canonical commutation relations (cf. eq. (6) and (27)), and thus of causality, is only a first step forward in dealing with non-local operators.

Our second comment deals with the possibility of expressing the gauge coupling terms in terms of doubling the degrees of freedom. In refs. [4,5] the doubling of the degrees of freedom is crucial in order to introduce finite temperature in canonical, real time quantum field theory and the doubled (tilde) degree of freedom has been interpreted sometime as the reservoir degree of freedom, or as the "hole" corresponding to a physical particle, or as a fundamental symmetry of the theory associated with the thermal degree of freedom [6–8]. In any case, it has been possible to associate some physical interpretation to the mathematical procedure of "doubling" the physical system. In the case of a damped harmonic oscillator this procedure is known to be of crucial relevance if one pretends to write a lagrangian for the system [10]. In ref. [3] and in the present paper a new feature of this procedure emerges, namely it appears that it is related with the local gauge structure of the theory. However, while in ref. [3] a more or less conventional [4–8] interpretation could be attached to the doubled (tilde) degree of freedom, in the present paper it seems more difficult to give a more direct physical interpretation, if any, to the ψ', except its relation with the theory gauge structure. This point deserves further investigation.

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THERMO FIELD DYNAMICS IN TIME REPRESENTATION

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Making use of the thermo field dynamics (TFD) we formulate a calculable method for time-dependent nonequilibrium systems in a time representation (t-representation) rather than in the k_0-Fourier representation. The corrected one-body propagator in the t-representation has the form of B^{-1} (diagonal matrix) B (B being a thermal Bogoliubov matrix). The number parameter in B here is the observed number (the Heisenberg number) with a fluctuation. With the usual definition of the on-shell self-energy a self-consistent renormalization condition leads to a kinetic equation for the number parameter. This equation turns out to be the Boltzmann equation, from which the entropy law follows.

1. Introduction

Thermal quantum field theories are now attracting the attention of physicists in many areas.[1] Thermal quantum field theories for equilibrium states are now well established.[2,3] However, there are many problems which are waiting for a practically manageable formalism for time-dependent thermal phenomena. If there should appear, in a short time, high temperature intermediate states in high energy particle reaction as many physicists have speculated, this is a time-dependent thermal phenomenon in an isolated system. Theoretical description of evolution of the Universe also requires thermal physics in an isolated system. There are many other problems which are concerned with our common experiences. Many phase transitions accompany change in thermal situation. These phenomena are of particular significance for thermal quantum field theories, because many phase transitions are changes in ordered states which are results of spontaneous breakdown of symmetries. The spontaneous breakdown of symmetries is a phenomenon intrinsically associated with quantum field theory, because it is caused by an infinite number of degrees

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of freedom in quantum field systems. Therefore, it suggests the significance of combining quantum field theories and thermal degrees of freedom, leading us to thermal quantum field theories for time-dependent phenomena. It is observed that when a thermal situation is not homogeneous in space, a heat conduction starts.[4] If we were able to extend the thermal quantum field theories to space- and time-dependent phenomena, such theories would provide a basis for heat conduction. Ultimately, we may challenge such a program. However, we wish to cultivate thermal quantum field theories step by step. We discuss in this paper an extension of thermal quantum field theories to time-dependent phenomena exclusively, postponing space-dependent problems to the future.

There are many formulations of thermal quantum field theories. We have been playing our game with thermo field dynamics (TFD)[2] which includes both operator formalism and Green's function formalism. According to our experiences, this approach provides a good physical intuition of what we are doing. Equilibrium TFD is formulated and has been applied to practical problems successfully. On the other hand, nonequilibrium TFD formulation is not settled yet despite our preceding attempts of Refs. 5–8. However, it turns out that our recent considerations given in Ref. 9 and, in preliminary forms, in Ref. 7, lead us to a simple in principle and systematic calculable formalism of nonequilibrium TFD. In this paper we for the first time give this new formalism extensively, following the preliminary considerations given in Ref. 9 and including the derivation of the kinetic equation. The paper is made self-contained, so that the reader can follow it without detailed knowledge of our previous papers.

The structure of the paper is as follows. We explain our basic ideas and notations in the next section, where we introduce the operator $\hat{Q}(t)$, which initiates a time-dependent process. In Sec. 3 the unperturbed propagator in the time-dependent TFD is presented. In Sec. 4 we pick up the best choice for the parameters involved in the thermal Bogoliubov matrix. With this matrix we identify the thermal term represented by $\hat{Q}(t)$. In Sec. 5 we illustrate the time representation formalism through its application to equilibrium TFD. In Sec. 6, applying the time representation formalism to time-dependent TFD we analyze the one-body propagator including interaction effects (corrected propagator). There it will be shown that the corrected propagator has the structure in which a diagonal propagator is sandwiched between the thermal Bogoliubov matrices B^{-1} and B, in which the number parameter is modified by the interaction. This corrected number parameter turns out to be the observed number (the Heisenberg number) with a fluctuation. This result shows how the observed number is related to the unperturbed number. In Sec. 7 a general form of the proper self-energy with two vertices is presented. The proper self-energy turns out to have an extremely simple structure: $B^{-1}(t)V(t-t')B(t')$. Here $V(t-t')$ is a diagonal propagator depending only on $(t-t')$. Applying the usual definition of the on-shell self-energy to $V(t-t')$, we extract the on-shell part $V^0\delta(t-t')$ from $V(t-t')$, thus obtaining the on-shell self-energy $B^{-1}(t)V^0B(t)$.

In Sec. 8 we introduce a self-consistent renormalization condition. Applying this condition to the above on-shell self-energy we obtain the Boltzmann equation from which the entropy law follows. Comments in Sec. 9 conclude the paper.

We again emphasize that the nonequilibrium TFD formalism in this paper is new and different from any previous ones in many respects: (i) the operator $\hat{Q}$ is fully utilized as responsible for time-dependent processes; (ii) the corrected propagator is analyzed systematically in the time representation; (iii) a self-consistent renormalization condition is formulated without ambiguity, and the kinetic equation is derived from it.

2. Basic Ideas

Briefly speaking, the basic idea is the following. When TFD replaced the thermal average with vacuum expectation value, it was suggested how the quantum field theory should be modified in order for the vacuum to carry thermal degrees of freedom. It was dictated that the so-called tilde freedom should be associated with every freedom in accordance with the tilde conjugation rule. Thus, to any operator A is associated its tilde conjugate $\tilde{A}$. The tilde conjugation rules have been presented in many papers. However, to make this paper self-contained, we here list the tilde conjugation rules:

$$(AB)^{\sim} = \tilde{A}\tilde{B}\,, \tag{2.1}$$

$$(c_1 A + c_2 B)^{\sim} = c_1^* \tilde{A} + c_2^* \tilde{B}\,, \tag{2.2}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger\,, \tag{2.3}$$

$$(\tilde{A})^{\sim} = \sigma A\,, \tag{2.4}$$

$$|0\rangle^{\sim} = |0\rangle\,, \tag{2.5}$$

$$\langle 0|^{\sim} = \langle 0|\,, \tag{2.6}$$

for any operators, A and B, any c-numbers, c_1 and c_2, and the thermal vacua, $|0\rangle$ and $\langle 0|$. Here σ is $+1$ (-1) for a bosonic (fermionic) operator A. Any c-number changes into its complex conjugate under the tilde conjugation.

When a Lagrangian is given, the dynamical energy operator H is constructed according to the Hamilton–Jacobi formalism. This H generates time translation of nontilde operators. However, the Hamiltonian in TFD, denoted by $\hat{H}$, should generate the time translation, not only of nontilde fields, but also of tilde fields. According to the above tilde conjugation rules, the tilde conjugation of the Heisenberg equation $i\partial\psi_H/\partial t = [\psi_H, H]$ gives $i\partial\tilde{\psi}_H/\partial t = [\tilde{\psi}_H, -\tilde{H}]$, which indicates that $\hat{H}$ is

$$\hat{H}_H = H_H - \tilde{H}_H\,. \tag{2.7}$$

Here the suffix H means Heisenberg operators. This explicitly shows

$$(\hat{H}_H)^{\sim} = -\hat{H}_H\,, \tag{2.8}$$

which implies that $\exp[i\hat{H}t]$ is invariant under the tilde conjugation. This is consistent with the tilde conjugation rules, (2.5) and (2.6), i.e. that the thermal vacua are invariant under the tilde conjugation. This $\hat{H}_H$ generates the Heisenberg equations of motion for both nontilde and tilde operators:

$$i\frac{\partial}{\partial t}A_H(t) = [A_H(t), \hat{H}_H]\,. \tag{2.9}$$

It is essential to make a distinction between $\hat{H}_H$ and H_H, whose eigenvalues or expectation values are denoted by $\hat{E}$ (*the hat-energy*) and E (*the dynamical energy*), respectively. A crucial feature of thermal field theory is the continuous degeneracy of thermal vacua in hat-energies.

The thermal vacua $|0\rangle$ for stationary systems are zero hat-energy eigenstates with the properties (2.5) and (2.6):

$$\hat{H}_H|0(\theta)\rangle = 0\,. \tag{2.10}$$

The continuous parameter θ in the thermal vacuum of (2.10), which may have multi-components, is intended to label each member in the set of continuously degenerate thermal vacua, $\{|0(\theta)\rangle\}$. This continuous degeneracy originates from the negative sign of $\tilde{H}$ in (2.7).

Thermal degrees of freedom can be seen when and only when we compare the thermal states with different θ. To understand this, let us compare two vacua with different θ. Although they have the same hat-energy, they carry different dynamical energy densities which are given by the vacuum expectation value of the energy density operator of the nontilde fields. This change of energy should be the heat energy (the first law of thermodynamics).[7] In other words a change in $\tilde{H}$ induces a heat energy.

Since we compare thermal vacua with different θ, we might try to introduce an operator which generates the movement through the thermal vacuum set. We are aware of the fact that, since the Fock spaces with different θ are inequivalent to each other, such a generator, $\hat{G}_H$, has no mathematical sense. Indeed, we carefully avoid use of such an operator when we construct a concrete formulation of time-dependent TFD. However, use of $\hat{G}_H$ provides a good intuitive understanding, as it always does in the study of spontaneous breakdown of symmetries.

To connect a thermal vacuum with another, we introduce the operator $\hat{G}_H$ through the following operation:

$$|0(\theta + d\theta)\rangle = e^{id\theta\hat{G}_H}|0(\theta)\rangle\,, \tag{2.11}$$

$$\langle 0(\theta + d\theta)| = \langle 0(\theta)|e^{-id\theta\hat{G}_H}\,. \tag{2.12}$$

The $\hat{G}_H$ can be chosen to satisfy

$$[\hat{H}_H, \hat{G}_H] = 0\,. \tag{2.13}$$

This dictates two statements. The first is that $\hat{G}_H$ is an operator of zero hat-energy mode. The second is that there is the invariance under the $\hat{G}$-transformation, (2.13). Furthermore, this $\hat{G}$-symmetry is ***spontaneously broken***, according to (2.11) and (2.12), which show that

$$\hat{G}_H|0(\theta)\rangle \neq 0. \tag{2.14}$$

The degeneracy of the thermal vacua in hat-energies is associated with breakdown of this symmetry and $\hat{G}_H$ acts as the Goldstone mode operator.

In this paper we confine our interest to those realizations associated with a Fock space of certain quasiparticles (quasiparticle picture). The operators for free quasiparticles, which are the unperturbed ones in the interaction representation when interactions come in, do not carry the suffix H.

Consider a Fock space which is built on a thermal vacuum $|0\rangle$ and write its annihilation operators by ξ_k and $\tilde{\xi}_k$,

$$\xi_k|0\rangle = \tilde{\xi}_k|0\rangle = 0\,. \tag{2.15}$$

These operators and their dagger conjugates are called the vacuum operators, because their choice is made by choice of the thermal vacuum. In equilibrium TFD we formulate the perturbation calculation scheme by using the following unperturbed free Hamiltonian:

$$\hat{H}_0(\theta) = H_0(\theta) - \tilde{H}_0(\theta) \tag{2.16}$$

with

$$H_0(\theta) = \int d^3k\,\omega_k(\theta)a_k^\dagger(\theta)a_k(\theta)\,, \tag{2.17}$$

$$\tilde{H}_0(\theta) = \int d^3k\,\omega_k(\theta)\tilde{a}_k^\dagger(\theta)\tilde{a}_k(\theta)\,. \tag{2.18}$$

We adapt the self-consistency concept formulated in terms of the renormalization theory. Thus, the particle energy $\omega_k(\theta)$ should include the self-energy. In this formulation the relation (2.13) reads

$$[\hat{H}_0(\theta), \hat{G}] = 0\,. \tag{2.19}$$

The G does not carry the suffix H because it is an operator, not in the Heisenberg picture, but in the interaction picture. Because of (2.19) we also have

$$\hat{H}_0(\theta) = \int d^3k\,\omega_k(\theta)[\xi_k^\dagger\xi_k - \tilde{\xi}_k^\dagger\tilde{\xi}_k]\,, \tag{2.20}$$

which depends on θ only through $\omega_k(\theta)$. The θ-dependence in ω_k comes from the renormalization. Furthermore, the commutation relation (2.19) enables us to find[9] that the $\hat{G}$-transformation should be the thermal Bogoliubov transformation, i.e. the linear relation between $(a_k, \tilde{a}_k)$ and $(\xi_k, \tilde{\xi}_k)$ given by

$$\begin{aligned} \xi_k^\mu &= B_k(\theta)^{\mu\nu}a_k^\nu\,, \\ \tilde{\xi}_k^\mu &= \tilde{a}_k^\nu B_k^{-1}(\theta)^{\nu\mu}\,, \end{aligned} \tag{2.21}$$

where the thermal doublet notation, $a_k^1 = a_k$, $a_k^2 = \tilde{a}_k^\dagger$, $\bar{a}_k^1 = a_k^\dagger$, $\bar{a}_k^2 = -\tilde{a}_k$, and similarly for ξ-operators, is used and $B_k(\theta)$ is the 2×2 matrix. Note that once we acquire these linear relations instead of the operator relations using $\hat{G}$ we do not need $\hat{G}$ any more, as will be seen in this paper below. The Bogoliubov matrix $B_k(\theta)$ can generally be written by three real parameters:

$$B_k(\theta)^{\mu\nu} = (1+\sigma n_k)^{1/2} e^{s_k \tau_3} \begin{bmatrix} 1 & -\sigma\left(\dfrac{\sigma n_k}{1+\sigma n_k}\right)^{\alpha_k} \\ -\sigma\left(\dfrac{\sigma n_k}{1+\sigma n_k}\right)^{1-\alpha_k} & 1 \end{bmatrix}, \quad (2.22)$$

using the three parameters, n_k, α_k and s_k. Here τ_i $(i = 1-3)$ are the Pauli matrices. The parameter n_k is the number density defined by

$$n_k(\theta)\delta(\mathbf{k}-\mathbf{l}) = \langle 0|a_k^\dagger a_l|0\rangle\,. \quad (2.23)$$

The θ-dependence of $n_k(\theta)$ originates from the Bogoliubov matrix which depends on θ. The parameter α_k corresponds to the one appearing in the density matrix formalism due to the cyclic property of ρ under the trace formula. The parameter s_k is a parameter related to the freedom in choosing the norm of ket-vacuum; this norm is arbitrary, because its change can be compensated by change of norm of bra-vacuum. In the following, we choose a particular s_k for practical convenience in computation [see (4.10)]. The most relevant parameter is the number parameter $n_k(\theta)$.

It is then natural to assume that a time-dependent situation is treated by a time-dependent Bogoliubov matrix. In other words the time-independent Bogoliubov matrix in equilibrium TFD is replaced by a time-dependent one when we deal with a time-dependent thermal process. This assumption is tried in this paper.

We now apply this assumption to a free quasiparticle field. For simplicity we consider the case

$$\psi(x)^\mu = \int \frac{d^3k}{(2\pi)^{(3/2)}} e^{i\mathbf{k}\cdot\mathbf{x}} a_k(t)^\mu \quad (2.24)$$

with the equal time (anti)commutation relation

$$[a_k(t)^\mu, \bar{a}_l(t)^\nu]_\sigma = \delta^{\mu\nu}\delta(\mathbf{k}-\mathbf{l})\,, \quad (2.25)$$

where $[A,B]_\sigma = AB - \sigma BA$. In the usual quantum field theory we have $a_k(t) = a_k \exp[-i\omega_k t]$, and in equilibrium TFD we have $a_k(t)^\mu = B_k^{-1}(\theta)^{\mu\nu}\xi_k^\nu \exp[-i\omega_k t]$ with the Bogoliubov matrix $B_k(\theta)$. To describe time-dependent thermal phenomena, we make θ a function of time. Since the quantum energy is also a function of time, we replace $\exp[-i\omega_k t]$ by $\exp[-i\int_{t_0}^t ds\omega_k(s)]$, in which $\omega_k(s)$ is the time-dependent quantum energy. Thus in time-dependent nonequilibrium TFD we write

$$\xi_k^\mu = B_k(t)^{\mu\nu} \exp\left[i\int_{t_0}^{t} ds\omega_k(s)\right] a_k(t)^\nu\,, \tag{2.26}$$

$$\bar{\xi}_k^\mu = \bar{a}_k(t)^\nu \exp\left[-i\int_{t_0}^{t} ds\omega_k(s)\right] B_k^{-1}(t)^{\nu\mu}\,, \tag{2.27}$$

where the definition of the thermal doublet is found in the last section.

Since ξ_k^μ and $\bar{\xi}_k^\mu$ are independent of time, we have

$$\frac{d}{dt}\xi_k^\mu = \frac{d}{dt}\bar{\xi}_k^\mu = 0\,, \tag{2.28}$$

which together with the above Bogoliubov transformation gives[10,9]

$$\left[\frac{d}{dt} + i\omega_k(t) - iP_k(t)\right]^{\mu\nu} a_k(t)^\mu = 0\,, \tag{2.29}$$

$$\bar{a}_k^\nu \left[\frac{\overleftarrow{d}}{dt} - i\omega_k(t) + iP_k(t)\right]^{\nu\mu} = 0\,, \tag{2.30}$$

where

$$P_k(t) \equiv iB_k^{-1}(t)\frac{d}{dt}B_k(t)\,. \tag{2.31}$$

The combination $[d/dt - iP_k(t)]$ was called the thermal covariant derivative[10] by considering an analogy to the gauge theory.

The above equations for $a_k(t)^\mu$ and $\bar{a}_k(t)^\mu$ show that the unperturbed Hamiltonian is[9]

$$\hat{H}_Q(t) = \hat{H}_0(t) - \hat{Q}(t)\,, \tag{2.32}$$

where

$$\hat{H}_0(t) = \int d^3k\bar{a}_k(t)\omega_k(t)a_k(t) \tag{2.33}$$

and

$$\hat{Q}(t) = \int d^3k\,\bar{a}_k(t)P_k(t)a_k(t)\,, \tag{2.34}$$

because this gives

$$i\frac{d}{dt}a_k(t) = [a_k(t), \hat{H}_Q]\,, \tag{2.35}$$

$$i\frac{d}{dt}\bar{a}_k(t) = [\bar{a}_k(t), \hat{H}_Q]\,. \tag{2.36}$$

The structure of the unperturbed Hamiltonian $\hat{H}_Q(t)$ is the same as the one (if the dissipative term is dropped) derived in Ref. 6. Although the unperturbed Hamiltonian is $\hat{H}_Q$, the Hamiltonian for the quasiparticle operators $\xi_k(t)^\mu = \xi_k^\mu \exp[-i\omega_k t]$ is the free one that is $\hat{H}_0$.

It is important to note that the operator $\hat{Q}$ [which is proportional to $\dot{n}_k(t)$, as will be shown later] initiates a time-dependent process. The situation is the same

as the cases of spontaneous breakdown of symmetries: we need an unperturbed Hamiltonian with broken symmetries to initiate a perturbation calculation. In time-dependent TFD we need the unperturbed Hamiltonian $\hat{H}_Q$ in which $\hat{Q}$ breaks the time-translation symmetry.

The time-dependent Bogoliubov matrix $B_k(t)$ is obtained from $B_k(\theta)$ by making the three parameters (α_k, s_k, n_k) dependent on time, because the parameter θ stands for these three parameters. We will determine the parameters (α_k, s_k) when we formulate the Feynman diagram method in Sec. 4.[11] Until then we keep unspecified (α_k, s_k, n_k).

Formulation of perturbation calculation is straightforward when we extend the consideration in Ref. 11 to time-dependent situation. The unperturbed Hamiltonian is not $\hat{H}_0$ but $\hat{H}_Q$ in time-dependent TFD. When we denote the interaction Hamiltonian in the usual quantum field theory for nontilde field by H_{int}, the interaction Hamiltonian in equilibrium TFD is given by

$$\hat{H}_{\text{int}} = H_{\text{int}} - \tilde{H}_{\text{int}} \,. \tag{2.37}$$

Since the total Hamiltonian is $\hat{H} = H - \tilde{H}$ even in time-dependent TFD, the change in the unperturbed Hamiltonian should be compensated by the corresponding change in the interaction Hamiltonian. Thus the interaction Hamiltonian $\hat{H}_I$ in time-dependent TFD is

$$\hat{H}_I = \hat{H}_{\text{int}} + \hat{Q} \,. \tag{2.38}$$

In this paper we study the structure of the one body propagator including interaction effects and the proper self-energy by means of the Feynman diagram method formulated in the interaction representation with the unperturbed Hamiltonian $\hat{H}_Q$ and interaction Hamiltonian $\hat{H}_I$. This study together with a self-consistent renormalization condition leads to the Boltzmann equation from which the entropy law follows.

In equilibrium TFD most of the calculations have been formulated in the temporal Fourier representation, because the time-translation symmetry is preserved. In time-dependent cases the temporal Fourier representation is not useful. We therefore formulate our calculation scheme in the time representation. With a particular choice of the parameters (α_k, s_k) we frequently find that calculations in the time representation are very simple.

In the time-dependent TFD formalism developed above it was assumed that a time-dependent process can be described in one Fock space built on the time-independent vacuum $|0\rangle$. This vacuum is to be chosen from an uncountable number of candidates for thermal vacua. This choice is made by a self-consistent renormalization condition which requires that the quasiparticle Hamiltonian $\hat{H}_0 + \delta\hat{H}$ is diagonal in the quasiparticle operators ξ_k^μ. Here $\delta\hat{H}$ is the shift of the Hamiltonian caused by the on-shell self-energy.

3. Unperturbed Propagator

In this section we study the unperturbed propagator controlled by the Hamiltonian $\hat{H}_Q$ in (2.32). From now on in this paper we assume that operators are bosonic. The extension to the fermionic case is straightforward.

Let us begin with the unperturbed propagator for the oscillator operator $a_k(t)^\mu$, introduced in (2.26) and (2.27) through the time-dependent Bogoliubov transformation with the Bogoliubov matrix given by (2.22). For the unperturbed propagator, defined by

$$\Delta_k(t,t')^{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) = -i\langle 0|\mathbf{T}[a_k(t)^\mu \bar{a}_l(t')^\nu]|0\rangle\,, \tag{3.1}$$

where $|0\rangle$ is the thermal vacuum annihilated by ξ_k and $\tilde{\xi}_k$, we have[5,6,10]

$$\Delta_k(t,t')^{\mu\nu} = \left[B_k^{-1}(t)\mathcal{G}_k(t,t')B_k(t')\right]^{\mu\nu}\,, \tag{3.2}$$

where $\mathcal{G}_k(t,t')$ is the diagonal matrix defined by

$$\mathcal{G}_k(t,t') = \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix} e^{-i\int_{t'}^{t} ds\omega_k(s)}\,. \tag{3.3}$$

Note a remarkable structure of this propagator: *it has the form in which a diagonal propagator is sandwiched between the Bogoliubov transformation.* Thus temporal changes in thermal effects the [B-matrix part in (3.2)] appear at each vertex in the Feynman diagrams, while the wave propagation part [the $\mathcal{G}$-matrix part in (3.2)] does not have a thermal mixing. In later sections we prove that this structure is true for any propagator corrected by interactions.

So far we described the unperturbed propagator for time-dependent thermal phenomena in terms of oscillator operator $a_k(t)$. We now translate this result in terms of the quasiparticle field ψ. Here we consider the simplest form of field, that is

$$\begin{aligned} \psi(x)^\mu &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}} a_k(t)^\mu\,, \\ \bar{\psi}(x)^\mu &= \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \bar{a}_k(t)^\mu\,. \end{aligned} \tag{3.4}$$

The propagator is defined by

$$\Delta_c(x,x')^{\mu\nu} = -i\langle 0|\mathbf{T}[\psi(x)^\mu \bar{\psi}(x')^\nu]|0\rangle \tag{3.5}$$

and can be computed by means of the above result for $\Delta_k(t,t')$ as

$$\Delta_c(x,x')^{\mu\nu} = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \Delta_k(t,t')^{\mu\nu}\,. \tag{3.6}$$

4. The Bogoliubov Matrix in t-Representation

In the last section we showed that the unperturbed propagator has the form in which a diagonal propagator is sandwiched between the Bogoliubov transformation. In the next two sections we show that any one-body propagator with interaction effect has a similar structure. Before doing this, we prepare the perturbative calculation scheme in the interaction representation, extending the consideration in Ref. 11 to time-dependent situation. Through the course of this preparation we determine our choice for the parameters (α_k, s_k).

Let us consider the corrected two-point Green's function

$$G_k(t,t')^{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) = -i\langle 0|\mathbf{T}[a_{H,k}(t)^{\mu}\bar{a}_{H,l}(t')^{\nu}]|0\rangle\,. \tag{4.1}$$

Rewriting this quantity in terms of operators in the interaction representation, we have

$$G_k(t,t')^{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) = -i\langle 0|\hat{U}(t_0,\infty)\mathbf{T}[a_k(t)^{\mu}\bar{a}_l(t')^{\nu}\hat{S}]\hat{U}(-\infty,t_0)|0\rangle\,. \tag{4.2}$$

Here we use the interaction representation with $\hat{H}_Q(t)$ as unperturbed Hamiltonian and $\hat{H}_I$ as interaction. The time evolution operator $\hat{U}$ and the operator $\hat{S}$ are introduced as

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0)\,, \tag{4.3}$$

$$\hat{U}(t_0,t_0) = 1\,, \tag{4.4}$$

and

$$\hat{S} \equiv \hat{U}(\infty,-\infty) \tag{4.5}$$

$$= \mathbf{T}\left[\exp\left[-i\int_{-\infty}^{\infty} ds\,\hat{H}_I(s)\right]\right], \tag{4.6}$$

where t_0 is the coincidence time of the Heisenberg and interaction representations. To eliminate the two operators $\hat{U}$ outside the **T**-product in (4.2), we have to choose particular values for α and t_0.[11] There are only two solutions; one is ($\alpha = 1$, $t_0 = -\infty$), in which

$$\langle 0|\hat{U}(t,t') = \langle 0| \tag{4.7}$$

from $\langle 0|\hat{H}_I(t) = 0$, and the other is ($\alpha = 0, t_0 = \infty$), in which

$$\hat{U}(t,t')|0\rangle = |0\rangle \tag{4.8}$$

from $\hat{H}_I(t)|0\rangle = 0$. In both cases we have

$$G_k(t,t')^{\mu\nu}\delta(\mathbf{k}-\mathbf{l}) = -i\langle 0|\mathbf{T}[a_k(t)^{\mu}\bar{a}_l(t')^{\nu}\hat{S}]|0\rangle\,. \tag{4.9}$$

With any other choice for (α, t_0), the **T**-product Feynman diagram method is not applicable.

We write down the explicit form of the B-matrix for $\alpha = 1$ here, as the choice ($\alpha = 1, t_0 = -\infty$) has been employed conventionally by most people's calculations. In a later section we will find that the entropy law excludes the choice ($\alpha = 0, t_0 = \infty$). With $\alpha = 1$ we choose the parameter $s_k(t)$ as

$$s_k(t) = \ln\left[1 + \sigma n_k(t)\right]^{1/2}, \tag{4.10}$$

yielding

$$e^{s_k(t)\tau_3} = \begin{bmatrix} [1+\sigma n_k(t)]^{1/2} & 0 \\ 0 & [1+\sigma n_k(t)]^{-1/2} \end{bmatrix}. \tag{4.11}$$

(Note that in Ref. 6 we used $s_k = 0$.) With this the relation (2.22) for the Bogoliubov matrix takes the form

$$B_k(t) = \begin{bmatrix} 1+\sigma n_k(t) & -n_k(t) \\ -\sigma & 1 \end{bmatrix}, \tag{4.12}$$

the inverse of which is

$$B_k^{-1}(t) = \begin{bmatrix} 1 & n_k(t) \\ \sigma & 1+\sigma n_k(t) \end{bmatrix}. \tag{4.13}$$

The moral in choosing s_k here is that the B-matrix is linear in n. Then, the matrix P_k takes the following simple form:

$$P_k(t) = i\sigma \dot{n}_k(t) T_0, \tag{4.14}$$

where

$$T_0 \equiv \begin{bmatrix} 1 & -\sigma \\ \sigma & -1 \end{bmatrix}. \tag{4.15}$$

5. Equilibrium Propagator in t-Representation

In this section we study the corrected Green's function in the t-space, (4.9), in equilibrium TFD, in which $G_k^{\mu\nu}$ is a function of $t - t'$. Those in time-dependent nonequilibrium TFD will be discussed in the next section. Several matrix formulae involving B are listed in App. A and will be utilized below in the text.

In equilibrium TFD, we can derive the spectral representations for the two-point Green's function and self-energy function,[2] which are very similar to the spectral representation found in the usual quantum field theory except that one has a 2×2 matrix structure in TFD. The spectral representation for the corrected two-point Green's function, introduced in (4.1), reads

$$G_k(t-t')^{\mu\nu} = \int \frac{dk_0}{2\pi} e^{-ik_0(t-t')} G_k(k_0)^{\mu\nu} \tag{5.1}$$

with

$$G_k(k_0)^{\mu\nu} = \int_{-\infty}^{\infty} dw \left[B^{-1}[n(w)] \frac{\rho(w,\mathbf{k})}{k_0 - w + i\epsilon\tau_3} B[n(w)] \right]^{\mu\nu}, \tag{5.2}$$

where $n(w)$ is given by the Boltzmann distribution with the temperature β^{-1},

$$n(w) = \frac{1}{e^{\beta w} - 1} . \tag{5.3}$$

We remark that this $n(w)$ on the on-shell, $w = \omega_k$, is the number density parameterizing the B-matrix in the unperturbed propagator. The function ρ called the spectral function satisfies

$$\int dw\, \rho(w, \mathbf{k}) = 1 . \tag{5.4}$$

Equation (5.1) with (5.2) is rewritten after the integration over k_0 as

$$G_k(t-t')^{\mu\nu} = \int dw \left[B^{-1}[n(w)] \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix} e^{-iw(t-t')} \rho(w, \mathbf{k}) B[n(w)] \right]^{\mu\nu} . \tag{5.5}$$

We now rewrite (5.5) according to the so-called diagonal propagation requirement, which states that the one-body propagator should have a form in which a diagonal propagator is sandwiched between two Bogoliubov matrices. The unperturbed propagator (3.2) in nonequilibrium as well as in equilibrium situations satisfies this condition, while (5.5) does not because its Bogoliubov matrices are not outside of the w-integration. Using the formula (A.15) in App. A twice, we get uniquely the following expression meeting the condition:

$$\begin{aligned} G_k(t-t')^{\mu\nu} &= B^{-1}[N_k(t-t')]^{\mu\mu'} \\ &\quad \times \int dw\, e^{-iw(t-t')} \rho(w, \mathbf{k}) \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu'\nu'} \\ &\quad \times B[N_k(t-t')]^{\nu'\nu} , \end{aligned} \tag{5.6}$$

where

$$N_k(t-t') = \frac{\int dw\, e^{-iw(t-t')} \rho(w, \mathbf{k}) n(w)}{\int dw\, e^{-iw(t-t')} \rho(w, \mathbf{k})} . \tag{5.7}$$

We divide $N_k(t-t')$ into two parts,

$$N_k(t-t') = N_{R,k} + \nu_k(t-t') , \tag{5.8}$$

in such a way that

$$\nu_k(t-t'=0) = 0 . \tag{5.9}$$

This specifies $N_{R,k}$ and $\nu_k(t-t')$ as

$$N_{R,k} = N_k(t-t'=0) = \int dw\, \rho(w, \mathbf{k}) n(w) , \tag{5.10}$$

$$\nu_k(t-t') = N_k(t-t') - N_{R,k} , \tag{5.11}$$

where (5.4) has been used.

The significance of this division becomes clear when we recall the true particle number density $n_{H,k}$ defined by the Heisenberg operator as

$$n_{H,k}\delta(\mathbf{k}-\mathbf{l}) \equiv \langle 0|\bar{a}_{H,k}(t)^1 a_{H,l}(t)^1|0\rangle\,. \tag{5.12}$$

Taking the limit $t \longrightarrow t'$ with $t \geq t'$ or $t' \geq t$ in (5.6) and considering (4.1), (5.4) and (5.12), we find that

$$N_{R,k} = n_{H,k}\,. \tag{5.13}$$

The deviation of $N_k(t-t')$ from the true number density $n_{H,k}$, namely $\nu_k(t-t')$, is called the number fluctuation.

Here is a physical implication, coming from the expression of the Heisenberg number density in terms of the unperturbed one which is seen from (5.10) and (5.13). To investigate this matter, we introduce the spectral representation for the self-energy in the Fourier space:

$$\Sigma(k_0,\mathbf{k})^{\mu\nu} = \int dw\,\sigma(w,\mathbf{k})\left[B^{-1}[n(w)]\frac{1}{k_0-w+i\epsilon\tau_3}B[n(w)]\right]^{\mu\nu}\,, \tag{5.14}$$

which is subject to

$$G = \Delta + \Delta\Sigma G\,. \tag{5.15}$$

The real and imaginary parts of Σ on the on-shell, $k_0 = \omega_k$, represents the energy shift and the dissipation, respectively,

$$\begin{aligned}\delta\omega_k &= \mathrm{Re}\left[\Sigma\left(k_0=\omega_k,\mathbf{k}\right)\right]\\ &= r(\nu=\omega_k,\mathbf{k})\,,\end{aligned} \tag{5.16}$$

$$\mathrm{Im}\left[\Sigma\left(k_0=\omega_k,\mathbf{k}\right)\right] = -i\kappa_k A[n(\omega_k)]\,, \tag{5.17}$$

where the matrix A is defined in (A.22),

$$r(w,\mathbf{k}) = \int dw'\,\mathcal{P}\frac{\sigma(w',\mathbf{k})}{w-w'}\,, \tag{5.18}$$

and the κ_k is called the dissipative coefficient given by

$$\kappa_k = \pi\sigma(\omega_k,\mathbf{k})\,. \tag{5.19}$$

The spectral function ρ is related to this spectral function σ through

$$\rho(w,\mathbf{k}) = \frac{\sigma(w,\mathbf{k})}{\{w-\omega_{0k}-r(w,\mathbf{k})\}^2+\pi^2\sigma^2(w,\mathbf{k})} \tag{5.20}$$

$$\simeq \frac{1}{\pi}\frac{\kappa_k}{\{w-\omega_k\}^2+\kappa_k^2}\,. \tag{5.21}$$

The last approximate expression is obtained by replacing $\sigma(w,\mathbf{k})$ and $r(w,\mathbf{k})$ with their on-shell values. Substituting (5.21) into (5.10), we see that when κ_k is very small (that is, ρ has a sharp peak around $w=\omega_k$), $n_{H,k}$ is well approximated by the unperturbed number $n(\omega_k)$ while $\nu_k(t-t')$ becomes negligibly small. As thermal effects become more dominant, meaning larger value of κ_k, $n_{H,k}$ deviates more from $n(\omega_k)$. This deviation is attributed to the increase in the energy uncertainty caused by thermal instability.

The above result in this section shows that even in equilibrium situations a single quasiparticle in a short time propagation feels thermal fluctuations given by $\nu_k(t-t')$.

6. Nonequilibrium Propagator in t-Representation

In this section we turn our attention to the corrected propagator defined by (4.1) in time-dependent nonequilibrium situations with the choice of $(\alpha=1, t_0=-\infty)$.

From (4.9) we have the following relation:

$$\begin{aligned} F_k(t,t')^{\mu\nu} &\equiv B[n_k(t)]^{\mu\mu'} G_k(t,t')^{\mu'\nu'} B^{-1}[n_k(t')]^{\nu'\nu} \\ &= -i\theta(t-t')\langle 0|\xi_k(t)^\mu \hat{U}(t,t')\bar{\xi}_l(t')^\nu \hat{U}(t',-\infty)|0\rangle \\ &\quad + i\theta(t'-t)\langle 0|\bar{\xi}_l(t')^\nu \hat{U}(t',t)\xi_k(t)^\mu \hat{U}(t,-\infty)|0\rangle\,, \end{aligned} \tag{6.1}$$

where $\xi_k(t)^\mu = \xi_k^\mu \exp[-i\int ds\,\omega_k(s)]$. Here we have used the relation in our choice of $\alpha=1$, (4.7). Since ξ_k^2 and $\bar{\xi}_k^1$ annihilate the bra-vacuum, we have

$$\begin{aligned} F_k(t,t')^{21} &= 0\,, \\ F_k(t,t')^{11} &= -i\theta(t-t')g(t,t':\mathbf{k})^{11}\,, \\ F_k(t,t')^{22} &= i\theta(t'-t)g(t,t':\mathbf{k})^{22}\,, \end{aligned} \tag{6.2}$$

while F_k^{12} is not vanishing in general,

$$F_k(t,t')^{12} \equiv g(t,t':\mathbf{k})^{12} \neq 0\,. \tag{6.3}$$

Here $g(t,t':\mathbf{k})^{11}$, $g(t,t':\mathbf{k})^{22}$ and $g(t,t':\mathbf{k})^{12}$ are functions which are expressed as the integrals of internal times $\{s_i\}$ with integration ranges from $-\infty$ to t, t' and $\max\{t,t'\}$, respectively. In this way we obtain a general structure of $G_k(t,t')^{\mu\nu}$ in (4.9) as

$$\begin{aligned} &G_k(t,t')^{\mu\nu} \\ &= B^{-1}[n_k(t)]^{\mu\mu'} \begin{bmatrix} -i\theta(t-t')g(t,t':\mathbf{k})^{11} & g(t,t':\mathbf{k})^{12} \\ 0 & i\theta(t'-t)g(t,t':\mathbf{k})^{22} \end{bmatrix}^{\mu'\nu'} B[n_k(t')]^{\nu'\nu}\,. \end{aligned} \tag{6.4}$$

We may express $g(t,t':\mathbf{k})^{12}$ by

$$g(t,t':\mathbf{k})^{12} \equiv -i\theta(t-t')g_+(t,t':\mathbf{k}) + i\theta(t'-t)g_-(t,t':\mathbf{k})\,. \tag{6.5}$$

When the tilde conjugation rules (2.1)–(2.6) are applied to the both sides of (4.1) with (6.4) and (6.5), we find the relations

$$g^*(t,t':\mathbf{k})^{11} = g(t',t:\mathbf{k})^{22}\,, \tag{6.6}$$

$$g^*_+(t,t':\mathbf{k}) = -g_-(t',t:\mathbf{k})\,. \tag{6.7}$$

An interesting feature of (6.4) is that one can rewrite it, introducing arbitrary new "number" density parameters $N_{j,k}$ $(j=1,2)$ depending on t and t', as follows:

$$\begin{aligned} G_k(t,t')^{\mu\nu} &= B^{-1}[N_{2,k}(t,t')]^{\mu\mu'} \\ &\times \begin{bmatrix} -i\theta(t-t')g(t,t':\mathbf{k})^{11} & g'(t,t':\mathbf{k})^{12} \\ 0 & i\theta(t'-t)g(t,t':\mathbf{k})^{22} \end{bmatrix}^{\mu'\nu'} \\ &\times B[N_{1,k}(t,t')]^{\nu'\nu} \end{aligned} \tag{6.8}$$

with

$$\begin{aligned} g'(t,t':\mathbf{k})^{12} \equiv\ & g(t,t':\mathbf{k})^{12} \\ & - i\theta(t-t')\{N_{1,k}(t,t') - n_k(t')\}g(t,t':\mathbf{k})^{11} \\ & - i\theta(t'-t)\{N_{2,k}(t,t') - n_k(t)\}g(t,t':\mathbf{k})^{22}\,, \end{aligned} \tag{6.9}$$

which can be easily derived from the formulae

$$B[n]B^{-1}[N] = \begin{bmatrix} 1 & N-n \\ 0 & 1 \end{bmatrix} \tag{6.10}$$

and

$$D_2 \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} D_1^{-1} = \begin{bmatrix} a & b - ad_1 + cd_2 \\ 0 & c \end{bmatrix}\,, \tag{6.11}$$

D_i being the matrix

$$\begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix}\,. \tag{6.12}$$

The argument above shows that the parameters $N_{j,k}(t,t')$ introduced in (6.8) are arbitrary functions, changing in conjunction with the change in choice of $g'(t,t':\mathbf{k})^{12}$. We now show that this arbitrariness in the parameters $N_{j,k}(t,t')$ can be completely removed by requiring the diagonal propagation property, already discussed in the last section. To impose the diagonal propagation requirement on (6.8), meaning

$$\begin{aligned} G_k(t,t')^{\mu\nu} &= B^{-1}[N_{2,k}(t,t')]^{\mu\mu'} \\ &\times \begin{bmatrix} -i\theta(t-t')g(t,t':\mathbf{k})^{11} & 0 \\ 0 & i\theta(t'-t)g(t,t':\mathbf{k})^{22} \end{bmatrix}^{\mu'\nu'} \\ &\times B[N_{1,k}(t,t')]^{\nu'\nu} \end{aligned} \tag{6.13}$$

amounts to the condition

$$g'(t,t':\mathbf{k})^{12} = 0\,. \tag{6.14}$$

This forms a set of equations determining the parameters $N_{j,k}$ uniquely. We find their solutions, using (6.5), as

$$N_{1,k}(t,t') = n_k(t') - \frac{g_+(t,t':\mathbf{k})}{g(t,t':\mathbf{k})^{11}}\,, \tag{6.15}$$

$$N_{2,k}(t,t') = n_k(t) + \frac{g_-(t,t':\mathbf{k})}{g(t,t':\mathbf{k})^{22}}\,. \tag{6.16}$$

The condition requires (6.15) and (6.16) only for $t \geq t'$ and $t' \geq t$, respectively. We extend the definitions of $N_{j,k}$ to the other ranges of t and t' ($t < t'$ and $t' < t$, respectively) by the relations (6.15) and (6.16), so that $N_{j,k}$ are defined as continuous functions on the whole domains of t and t'. The parameters $N_{j,k}$, when they are conditioned by (6.14), are called the Bogoliubov parameters, since the diagonal propagation condition separates the thermal effects (the B-matrices) from the propagation of quasiparticles [the diagonal g-matrix in $G_k(t,t')^{\mu\nu}$], as is in (6.13). Note that we need the two Bogoliubov parameters depending on t and t' in time-dependent nonequilibrium situations while in equilibrium we had a single $N_k(t-t')$.

Copying the steps from (5.8) to (5.11) in the equilibrium case, we rewrite the Bogoliubov parameters as

$$N_{1,k}(t,t') = N_{R1,k}(t') + \nu_{1,k}(t,t')\,, \tag{6.17}$$

$$N_{2,k}(t,t') = N_{R2,k}(t) + \nu_{2,k}(t,t')\,, \tag{6.18}$$

in such a way that

$$\nu_{j,k}(t,t) = 0\,, \quad j = 1,2\,. \tag{6.19}$$

In other words, $N_{Rj,k}$ and $\nu_{j,k}$ are expressed in terms of g-functions as

$$N_{R1,k}(t) = n_k(t) - \frac{g_+(t,t:\mathbf{k})}{g(t,t:\mathbf{k})^{11}}\,, \tag{6.20}$$

$$N_{R2,k}(t) = n_k(t) + \frac{g_-(t,t:\mathbf{k})}{g(t,t:\mathbf{k})^{22}}\,, \tag{6.21}$$

$$\nu_{1,k}(t,t') = -\left(\frac{g_+(t,t':\mathbf{k})}{g(t,t':\mathbf{k})^{11}} - \frac{g_+(t',t':\mathbf{k})}{g(t',t':\mathbf{k})^{11}}\right)\,, \tag{6.22}$$

$$\nu_{2,k}(t,t') = \frac{g_-(t,t':\mathbf{k})}{g(t,t':\mathbf{k})^{22}} - \frac{g_-(t,t:\mathbf{k})}{g(t,t:\mathbf{k})^{22}}\,. \tag{6.23}$$

It is easy to see the following relations from the properties of g-functions under the tilde conjugation, (6.6) and (6.7),

$$N^*_{1,k}(t,t') = N_{2,k}(t',t)\,, \tag{6.24}$$

$$N^*_{R1,k}(t) = N_{R2,k}(t)\,, \tag{6.25}$$

$$\nu^*_{1,k}(t,t') = \nu_{2,k}(t',t)\,. \tag{6.26}$$

Furthermore, the equality between $N_{Rj,k}(t,t')$ and the Heisenberg number density $n_{H,k}(t)$ defined by (5.12) (t-dependent in this case) can be derived similarly as in the last section. We combine (4.1) with (6.13) and take the limits of $t \longrightarrow t'$ in two ways, i.e. $t > t'$ and $t < t'$; then we obtain

$$1 + n_{H,k}(t) = g(t,t:\mathbf{k})^{11}(1 + N_{R1,k}(t))\,, \tag{6.27}$$

$$n_{H,k}(t) = g(t,t:\mathbf{k})^{22} N_{R2,k}(t)\,, \tag{6.28}$$

where (6.17), (6.18) and (6.19) have been considered and the Heisenberg number $n_{H,k}(t)$ was defined in (5.12). We take the complex conjugate of (6.27), and use (6.6), (6.25) and (6.28) to get

$$g(t,t:\mathbf{k})^{22} = 1\,. \tag{6.29}$$

In deriving this, we take account of the fact that $n_{H,k}(t)$ should be real. Similarly we have

$$g(t,t:\mathbf{k})^{11} = 1\,. \tag{6.30}$$

Note that (6.29) and (6.30) in the equilibrium limit are nothing but (5.4). Then the substitutions of (6.29) and (6.30) into (6.27) and (6.28) lead to

$$N_{R1,k}(t) = N_{R2,k}(t) = n_{H,k}(t)\,. \tag{6.31}$$

This shows that the Bogoliubov number parameters with $t = t'$ naturally become the Heisenberg particle number again when the diagonal propagation condition operates. This justifies the diagonal propagation condition in time-dependent nonequilibrium situation.

We can carry over physical implications of the results in the last section to those of time-dependent nonequilibrium calculation in this section, which are summarized below. It is found that the corrected propagator $G_k(t,t')^{\mu\nu}$, the diagonalization propagation condition being imposed, has the matrix structure of $B^{-1}\times$(a diagonal matrix)$\times B$, which is similar to the one of the unperturbed propagator $\Delta_k(t,t')^{\mu\nu}$ in (3.2). This leads us to a significant revision in perturbation calculation in problems of time-dependent thermal phenomena. In nonequilibrium TFD formulated up to now, the B-matrices have been parameterized by the unperturbed number $n_k(t)$. Now we know that the B-matrices are parameterized by more complicated ones, i.e. the Bogoliubov parameters given by $N_{j,k}(t,t')$ ($j = 1,2$), reflecting the interaction effects. The $N_{j,k}(t,t')$ depend on the two times, because the time translational invariance is broken. However, as (6.17)–(6.19) show, they are separated into two

parts. One, depending on a single time, turned out be the Heisenberg number density $n_H(t)$. The other, denoted by $\nu_j(t,t')$, represents thermal fluctuation effects. Thus we have

$$N_{j,k}(t,t') = n_{H,k}(t) + \text{thermal fluctuation effects}\,. \tag{6.32}$$

Putting aside the ν-term, we still see the change in the number density parameter from $n_k(t)$ to $n_{H,k}(t)$. This change is due to the thermal and quantum fluctuation effects induced by the interaction. The functions $n_{H,k}(t)$ and $\nu_{j,k}(t,t')$ are calculable perturbatively in terms of $n_k(t)$ and the other renormalized parameters up to the order we desire.

7. The Product Rule

The self-energy diagrams with two vertices in Fig. 1 have particularly simple forms. This is called the product rule in nonequilibrium TFD. We derive this rule here.

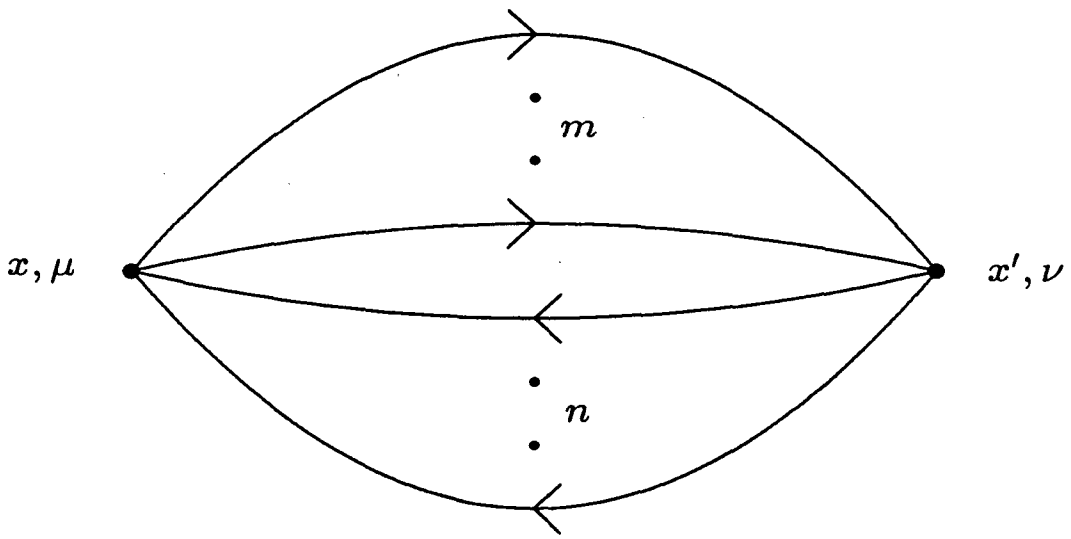

Fig. 1. The self-energy diagram for which the product rule holds.

Let us consider the Feynman diagram of self-energy, shown in Fig. 1. The general expression for the self-energy Feynman diagram in Fig. 1 is given by

$$\Sigma_{m,n}(x,x')^{\mu\nu} = i^{m+n-1}C$$
$$\begin{bmatrix} (\Delta_c(x,x')^{11})^m(\Delta_c(x',x)^{11})^n & (-1)^{m+1}(\Delta_c(x,x')^{12})^m(\Delta_c(x',x)^{21})^n \\ (-1)^n(\Delta_c(x,x')^{21})^m(\Delta_c(x',x)^{12})^n & (-1)^{m+n+1}(\Delta_c(x,x')^{22})^m(\Delta_c(x',x)^{22})^n \end{bmatrix}, \tag{7.1}$$

where C is a real positive constant, and m and n are positive integers. Remember that the unperturbed propagator $\Delta_c(x,x')^{\mu\nu}$ is defined in (3.6).

We study the simplest cases, $(m=2, n=0)$ and $(m=1, n=1)$, and then generalize the results for any m and n. Equation (7.1) becomes in the Fourier space with respect to $\mathbf{x}-\mathbf{x}'$

$$\Sigma_{2,0}(t,t',\mathbf{k})^{\mu\nu} = iC\int\frac{d^3q}{(2\pi)^3}\begin{bmatrix}\Delta_{k-q}(t,t')^{11}\Delta_q(t,t')^{11} & -\Delta_{k-q}(t,t')^{12}\Delta_q(t,t')^{12}\\ \Delta_{k-q}(t,t')^{21}\Delta_q(t,t')^{21} & -\Delta_{k-q}(t,t')^{22}\Delta_q(t,t')^{22}\end{bmatrix} \tag{7.2}$$

for $m=2$ and $n=0$, and

$$\Sigma_{1,1}(t,t',\mathbf{k})^{\mu\nu} = iC\int\frac{d^3q}{(2\pi)^3}\begin{bmatrix}\Delta_{k-q}(t,t')^{11}\Delta_{-q}(t',t,)^{11} & \Delta_{k-q}(t,t')^{12}\Delta_{-q}(t,t')^{21}\\ -\Delta_{k-q}(t,t')^{21}\Delta_{-q}(t,t')^{12} & -\Delta_{k-q}(t,t')^{22}\Delta_{-q}(t,t')^{22}\end{bmatrix} \tag{7.3}$$

for $m=1$ and $n=1$, respectively, where $\Delta_k(t,t')$ is given by (3.2). We refer to (A.19), (A.20) and (A.21); then each matrix element is rewritten according to

$$\begin{aligned}&\Delta_{k-q}(t,t')^{\mu\nu}\Delta_q(t,t')^{\mu\nu}\\ &\quad= \mathcal{G}_{k-q}(t,t')^{11}\mathcal{G}_q(t,t')^{11}A_{+,k-q}(t')^{\mu\nu}A_{+,q}(t')^{\mu\nu}\\ &\qquad+\mathcal{G}_{k-q}(t,t')^{22}\mathcal{G}_q(t,t')^{22}A_{-,k-q}(t)^{\mu\nu}A_{-,q}(t)^{\mu\nu}\,,\end{aligned} \tag{7.4}$$

$$\begin{aligned}&\Delta_{k-q}(t,t')^{\mu\nu}\Delta_{-q}(t',t)^{\nu\mu}\\ &\quad= \mathcal{G}_{k-q}(t,t')^{11}\mathcal{G}_{-q}(t',t)^{22}A_{+,k-q}(t')^{\mu\nu}A_{-,-q}(t')^{\nu\mu}\\ &\qquad+\mathcal{G}_{k-q}(t,t')^{22}\mathcal{G}_{-q}(t',t)^{11}A_{-,k-q}(t)^{\mu\nu}A_{+,-q}(t)^{\nu\mu}\,,\end{aligned} \tag{7.5}$$

where the dummy indices μ and ν are not summed over. Recalling (A.20) and (A.21) with (4.15), (A.5) and (A.6), we calculate each (μ,ν)-matrix element of (7.4) and (7.5) to obtain

$$\begin{aligned}&\Delta_{k-q}(t,t')^{\mu\nu}\Delta_q(t,t')^{\mu\nu}\\ &\quad= \mathcal{G}_{k-q}(t,t')^{11}\mathcal{G}_q(t,t')^{11}\frac{n_{k-q}(t')n_q(t')}{\mathcal{N}_{2,0}(t')}[\{T_+ + \mathcal{N}_{2,0}(t')T_0\}\tau_3]^{\mu\nu}\\ &\qquad-\mathcal{G}_{k-q}(t,t')^{22}\mathcal{G}_q(t,t')^{22}\frac{n_{k-q}(t)n_q(t)}{\mathcal{N}_{2,0}(t)}[\{T_- - \mathcal{N}_{2,0}(t)T_0\}\tau_3]^{\mu\nu}\,,\end{aligned} \tag{7.6}$$

$$\begin{aligned}&\Delta_{k-q}(t,t')^{\mu\nu}\Delta_{-q}(t',t)^{\nu\mu}\\ &\quad= -\mathcal{G}_{k-q}(t,t')^{11}\mathcal{G}_{-q}(t',t)^{22}\frac{n_{k-q}(t')(1+n_{-q}(t'))}{\mathcal{N}_{1,1}(t')}[\tau_3\{T_+ + \mathcal{N}_{1,1}(t')T_0\}]^{\mu\nu}\\ &\qquad+\mathcal{G}_{k-q}(t,t')^{22}\mathcal{G}_{-q}(t',t)^{11}\frac{n_{k-q}(t)(1+n_{-q}(t))}{\mathcal{N}_{1,1}(t)}[\tau_3\{T_- - \mathcal{N}_{1,1}(t)T_0\}]^{\mu\nu}\,,\end{aligned} \tag{7.7}$$

where

$$\mathcal{N}_{2,0}(t) \equiv \mathcal{N}_{2,0}(t : \mathbf{k}-\mathbf{q}, \mathbf{q}) = \frac{n_{k-q}(t) n_q(t)}{1 + n_{k-q}(t) + n_q(t)}\,,$$
$$\mathcal{N}_{1,1}(t) \equiv \mathcal{N}_{1,1}(t : \mathbf{k}-\mathbf{q}, -\mathbf{q}) = \frac{n_{k-q}(t)(1 + n_{-q}(t))}{n_{-q}(t) - n_{k-q}(t)}\,, \tag{7.8}$$

or

$$\mathcal{N}_{m,n}(t) \equiv \frac{\mathcal{F}_{m,n}(t)}{1 - \mathcal{F}_{m,n}(t)} \tag{7.9}$$

with

$$\mathcal{F}_{2,0}(t) \equiv \mathcal{F}_{2,0}(t : \mathbf{k}-\mathbf{q}, \mathbf{q}) = f_{k-q}(t) f_q(t)\,, \tag{7.10}$$

$$\mathcal{F}_{1,1}(t) \equiv \mathcal{F}_{1,1}(t : \mathbf{k}-\mathbf{q}, -\mathbf{q}) = f_{k-q}(t) f_{-q}^{-1}(t)\,, \tag{7.11}$$

while the function f is related to n through

$$n_k(t) = \frac{f_k(t)}{1 - f_k(t)}\,. \tag{7.12}$$

For notational simplicity, the dependence of $\mathcal{N}_{m,n}$ on momenta ($\mathbf{k}$ and $\mathbf{q}$) is suppressed above and below. We substitute (3.3), (7.6) and (7.7) into (7.2) and (7.3), and use (A.15) again to obtain

$$\begin{aligned}
\Sigma_{2,0}(t,t',\mathbf{k})^{\mu\nu} &= C \int \frac{d^3q}{(2\pi)^3} \\
&\times \left[B^{-1}[\mathcal{N}_{2,0}(t)] \begin{bmatrix} -i\theta(t-t') \dfrac{n_{k-q}(t') n_q(t')}{\mathcal{N}_{2,0}(t')} & 0 \\ 0 & i\theta(t'-t) \dfrac{n_{k-q}(t) n_q(t))}{\mathcal{N}_{2,0}(t)} \end{bmatrix} \right. \\
&\times B[\mathcal{N}_{2,0}(t')]]^{\mu\nu}\, e^{-i\int_{t'}^{t} ds(\omega_{k-q}(s)+\omega_q(s))}\,,
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
\Sigma_{1,1}(t,t',\mathbf{k})^{\mu\nu} &= C \int \frac{d^3q}{(2\pi)^3} \\
&\times \left[B^{-1}[\mathcal{N}_{1,1}(t)] \begin{bmatrix} -i\theta(t-t') \dfrac{n_{k-q}(t')(1+n_{-q}(t'))}{\mathcal{N}_{1,1}(t')} & 0 \\ 0 & i\theta(t'-t) \dfrac{n_{k-q}(t)(1+n_{-q}(t))}{\mathcal{N}_{1,1}(t)} \end{bmatrix} \right. \\
&\times B[\mathcal{N}_{1,1}(t')]]^{\mu\nu}\, e^{-i\int_{t'}^{t} ds(\omega_{k-q}(s)-\omega_{-q}(s))}\,.
\end{aligned} \tag{7.14}$$

It is now easy to induce the form of $\Sigma_{m,n}$ with arbitrary m and n from the results of the particular cases, (7.13) and (7.14): For example, when one wishes to calculate the $\Sigma_{m=3,n=0}$, one can follow the algebraic manipulations in obtaining $\Sigma_{2,0}$ from the products of two Δ above, just replacing one Δ with $\Sigma_{2,0}$ which is

written in the same matrix structure with the parameter $\mathcal{N}_{2,0}$ instead of n as Δ. Thus we have for (7.1)

$$\Sigma_{m,n}(t,t',\mathbf{k})^{\mu\nu} = C\int\prod_{i=1}^{m+n}\left(\frac{d^3q_i}{(2\pi)^3}\right)(2\pi)^3\delta(\mathbf{k}-\Sigma_{i=1}^m\mathbf{q}_i+\Sigma_{j=1}^n\mathbf{q}_{m+j})$$

$$\times\left[B^{-1}[\mathcal{N}_{m,n}(t)]\begin{bmatrix}-i\theta(t-t')g_{m,n}(t') & 0\\ 0 & i\theta(t'-t)g_{m,n}(t)\end{bmatrix}B[\mathcal{N}_{m,n}(t')]\right]^{\mu\nu}$$

$$\times\exp\left[-i\int_{t'}^{t}ds(\Sigma_{i=1}^m\omega_{q_i}(s)-\Sigma_{j=1}^n\omega_{-q_{m+j}}(s))\right], \tag{7.15}$$

where $\mathcal{N}_{m,n}$ is given by (7.9) with

$$\mathcal{F}_{m,n}(t) = \frac{f_{q_1}(t)\cdots f_{q_m}(t)}{f_{-q_{m+1}}(t)\cdots f_{-q_{m+n}}(t)}, \tag{7.16}$$

and the function g is defined by

$$g_{m,n}(t) = \frac{n_{q_1}(t)\cdots n_{q_m}(t)(1+n_{-q_{m+1}}(t))\cdots(1+n_{-q_{m+n}}(t))}{\mathcal{N}_{m,n}(t)}. \tag{7.17}$$

This compact expression is an extension of the spectral representation in equilibrium TFD to nonequilibrium TFD in the sense that its matrix structure is decomposed into $B^{-1}(t)$ and $B(t)$ and a diagonal matrix in between.

8. The Renormalization Condition

We have developed the calculable formulation of the time-dependent nonequilibrium TFD in the interaction representation. Now any expectation value or Green's function can be expressed in this formulation in terms of the time-dependent parameters, i.e. the renormalized energy $\omega_k(t)$ and the number density $n_k(t)$ which characterize the unperturbed representation, and the other parameters of models such as coupling constants. So far $\omega_k(t)$ and $n_k(t)$ remain unknown. Thus to determine them is vital to making theoretical predictions in the present formalism. The $\omega_k(t)$ is obtained from a calculation of the real part of the self-energy, as will be shown bellow. A self-consistent renormalization applied to the imaginary part of the self-energy leads to a kinetic equation which controls the temporal behavior of $n_k(t)$.

We illustrate the self-consistent renormalization through its application to $\Sigma^{\mu\nu}$ in (7.15). We start our calculation with the parameter ω_k independent of time. In this approach the time dependence of $\omega_k(t)$ is determined by the time-dependent energy shift $\delta\omega_k(t)$ as

$$\omega_k(t) = \omega_k + \delta\omega_k(t), \tag{8.1}$$

which we will see soon. We rewrite the self-energy in (7.15), suppressing the suffix $\{m, n\}$ and using simplified notations:

$$\Sigma(t,t',\mathbf{k})^{\mu\nu} = \int [dq] \left[B^{-1}[\mathcal{N}(t)] \begin{bmatrix} 1 & 0 \\ 0 & R(t) \end{bmatrix} V(t-t') \begin{bmatrix} R(t') & 0 \\ 0 & 1 \end{bmatrix} B[\mathcal{N}(t')] \right]^{\mu\nu} . \tag{8.2}$$

Here

$$[dq] = \prod_{i=1}^{m+n} \left(\frac{d^3 q_i}{(2\pi)^3} \right) (2\pi)^3 \delta(\mathbf{k} - \Sigma_{i=1}^m \mathbf{q}_i + \Sigma_{j=1}^n \mathbf{q}_{m+j}) , \tag{8.3}$$

$$R(t) = C g_{m,n}(t) , \tag{8.4}$$

$$V(t-t')^{\mu\nu} = \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu\nu} \exp[-iW(t-t')] , \tag{8.5}$$

$$W = \Sigma_{i=1}^m \omega_{q_i} - \Sigma_{j=1}^n \omega_{-q_{m+j}} . \tag{8.6}$$

The diagonal matrix $V(t-t')$ describes the wave propagation of the quasiparticle and has the Fourier form of

$$\begin{aligned} V(t-t') &\equiv \int \frac{dk_0}{2\pi} e^{-ik_0(t-t')} V(k_0) , \\ V(k_0) &= \frac{1}{k_0 - W + i\epsilon\tau_3} . \end{aligned} \tag{8.7}$$

The $V(t-t')^{\mu\nu}$ is of the same structure as the self-energy itself in ordinary field theory. Therefore, we apply the same procedure to $V^{\mu\nu}$ to extract its on-shell part: Put k_0 on the shell, $k_0 = \omega_k$, in the $V^{\mu\nu}(k_0)$ to get

$$V^{(0)}(t-t') = V(k_0 = \omega_k)\delta(t-t') . \tag{8.8}$$

Substituting this for $V(t-t')$ in (8.2), we have the on-shell part

$$\Sigma(t,t',\mathbf{k})^{(0)\mu\nu} \equiv \int [dq]\, R(t) \left[B^{-1}[\mathcal{N}(t)] V(k_0 = \omega_k) B[\mathcal{N}(t)] \right]^{\mu\nu} . \tag{8.9}$$

We remark that t and t' are equal in $\mathcal{N}$ and R because of the presence of $\delta(t-t')$ coming from $V^{(0)}(t-t')$.

Use of the formula $1/(x + i\epsilon\tau_3) = \mathcal{P}1/x - i\pi\tau_3\delta(x)$ in (8.9) leads to the real and imaginary parts:

$$\mathrm{Re}\left[\Sigma(t,t',\mathbf{k})^{(0)\mu\nu}\right] = \delta(t-t') \int [dq]\, R(t)\, \mathcal{P} \frac{1}{\omega_k - W} \delta^{\mu\nu} , \tag{8.10}$$

$$\mathrm{Im}\left[\Sigma(t,t',\mathbf{k})^{(0)\mu\nu}\right] = -i\delta(t-t') \left[\int [dq]\, R(t) \pi \delta(\omega_k - W) A[\mathcal{N}(t)] \right]^{\mu\nu} , \tag{8.11}$$

where A is the matrix in (A.22).

In addition to the above on-shell self-energy the $\hat{Q}$ in the interaction Hamiltonian $\hat{H}_I = \hat{H}^{\rm int} + \hat{Q}$ gives rise to a self-energy counterterm. The total on-shell self-energy will be denoted by $\Sigma^0_{\rm tot}$. The real part is not modified by $\hat{Q}$:

$$\delta\omega_k(t) \equiv \int [dq]\, R(t)\, \mathcal{P}\frac{1}{\omega_k - W}. \tag{8.12}$$

However, $\hat{Q}$ modifies the imaginary part. The imaginary part of the total on-shell self-energy becomes

$$\mathrm{Im}\left[\Sigma_{\rm tot}(t,t',\mathbf{k})^{(0)\mu\nu}\right] = \delta\Sigma(t,\mathbf{k})^{\mu\nu}\delta(t-t') \tag{8.13}$$

with

$$\delta\Sigma(t,\mathbf{k})^{\mu\nu} = -i\left[-\dot{n}_k(t)T_0 + \int [dq]\, R(t)\pi\delta(\omega_k - W)A[\mathcal{N}(t)]\right]^{\mu\nu}. \tag{8.14}$$

This introduces the imaginary self-energy contribution $\delta\hat{H} = \int d^3k\bar{a}_k(t)\delta\Sigma(t,\mathbf{k})$ $a_k(t)$ in the quasiparticle Hamiltonian. Then the quasiparticle Hamiltonian becomes $\hat{H}_0 + \delta\hat{H}$, in which the quantum energy in $\hat{H}_0$ is $\omega_k(t)$. The self-consistent renormalization condition requires that this quasiparticle Hamiltonian (and therefore the imaginary self-energy part $\delta\hat{H}$) be diagonal in the quasiparticle operators ξ^μ_k. It follows from the tilde conjugation rules that only the possible diagonal term to be included in the imaginary part of the Hamiltonian is $i\bar{\xi}_k\tau_3\xi_k$ (with a real coefficient depending on t and $\mathbf{k}$), which according to (2.26), (2.27) and (A.22) becomes $i\bar{a}_k(t)A_k(t)a_k(t)$. Thus the self-consistent renormalization condition applied to $\delta\hat{H}$ is expressed by the relation

$$\left[-\dot{n}_k(t)T_0 + \int [dq]\, R(t)\pi\delta(\omega_k - W)\; A[\mathcal{N}(t)]\right]^{\mu\nu} = \eta_k(t)A_k(t)^{\mu\nu} \tag{8.15}$$

with a real quantity $\eta_k(t)$. Use of (A.22) rewrites this as

$$\begin{aligned}&\left[-\dot{n}_k(t)T_0 + \int [dq]\, R(t)\pi\delta(\omega_k - W)\{T_+ - T_- + 2\mathcal{N}(t)T_0\}\right]^{\mu\nu}\\ &\quad = \eta_k(t)\left[T_+ - T_- + 2n_k(t)T_0\right].\end{aligned} \tag{8.16}$$

Comparing the coefficients in front of $(T_+ - T_-)$ and T_0, we get

$$\eta_k(t) = \kappa_k(t) \equiv \int [dq]\, R(t)\pi\delta(\omega_k - W) \tag{8.17}$$

and

$$\dot{n}_k(t) - 2\int [dq]\, R(t)\pi\delta(\omega_k - W)\mathcal{N}(t) + 2\kappa_k(t)n_k(t) = 0. \tag{8.18}$$

The $\kappa_k(t)$ is the time-dependent dissipative coefficient. Equation (8.18) is the kinetic equation for the unperturbed Bogoliubov parameter $n_k(t)$.

To make our point clearer, let us consider the model with $\psi^\dagger\psi^\dagger\psi\psi$-type interaction. The self-energy $\Sigma^{\mu\nu}_{\{m,n\}}$, (7.15), for the model, related essentially to the present issue, is the two-loop self-energy diagram corresponding to Fig. 1 with $m = 2$ and $n = 1$. With this self-energy part, as we have

$$R(t) = C\frac{n_{q_1}(t)n_{q_2}(t)[1 + n_{q_3}(t)]}{\mathcal{N}(t)}\,, \tag{8.19}$$

the kinetic equation (8.18) reduces to

$$\dot{n}_k(t) = -2\kappa_k(t)n_k(t) + 2\pi C\int[dq]\,\delta(\omega_k - W)n_{q_1}(t)n_{q_2}(t)[1 + n_{q_3}(t)]\,, \tag{8.20}$$

with

$$\kappa_k(t) = \pi C\int[dq]\,\delta(\omega_k - W)\,\{[1 + n_{q_1}(t)][1 + n_{q_2}(t)]n_{q_3}(t) - n_{q_1}(t)n_{q_2}(t)[1 + n_{q_3}(t)]\}\,. \tag{8.21}$$

Equation (8.20) is the kinetic equation for the Bogoliubov parameter $n_k(t)$ in the two-loop self-energy approximation without vertex correction. Within this approximation *the kinetic equation has the structure of the Boltzmann equation.*

Let us investigate the characteristics of the Boltzmann equation (8.20) a bit further. Multiplying an arbitrary function of $\mathbf{k}$, ϕ_k, on both sides of (8.20), and integrating it with respect to $\mathbf{k}$, we have

$$\int d^3k\,\phi_k\dot{n}_k(t) = \frac{(2\pi)^7C}{2}\int\prod_{i=0}^{3}\left(\frac{d^3q_i}{(2\pi)^3}\right)\delta(0-1-2+3) \times(\phi_3+\phi_2-\phi_1-\phi_0)\,n_0n_1(1+n_2)(1+n_3)\,, \tag{8.22}$$

where we introduced the abbreviations $\omega_i = \omega_{q_i}$, $\phi_i = \phi_{q_i}$, $n_i = n_{q_i}(t)$, and

$$\delta(0-1-2+3) = \delta(\mathbf{q}_0-\mathbf{q}_1-\mathbf{q}_2+\mathbf{q}_3)\,\delta(\omega_0-\omega_1-\omega_2+\omega_3)\,. \tag{8.23}$$

To obtain the expression of the right-hand side of (8.22), we changed the integration variables. We notice immediately that, for $\phi_k = 1$, the right-hand side of (8.22) disappears. Furthermore, using the property of delta function, $x\delta(x) = 0$, we see that, for $\phi_k = \mathbf{k}$ or ω_k, the right-hand side of (8.22) is equal to zero. These properties indicate that the number density $\mathcal{N}$, the momentum density $\mathcal{P}$ and the energy density $\mathcal{E}$ are conserved quantities. Here, they are defined, respectively, by

$$\mathcal{N} = \int d^3k\,n_k(t)\,, \tag{8.24}$$

$$\mathcal{P} = \int d^3k\,\mathbf{k}\,n_k(t), \tag{8.25}$$

$$\mathcal{E} = \int d^3k\,\omega_k\,n_k(t). \tag{8.26}$$

The conservation of these quantities reflects the fact that the system is an isolated system.

Let us further check the behavior of the entropy density $\mathcal{S}(t)$, defined by

$$\mathcal{S}(t) = \int d^3k \left\{ [1 + n_k(t)] \ln[1 + n_k(t)] - n_k(t) \ln n_k(t) \right\}. \tag{8.27}$$

Changing the integration parameters, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{S}(t) = & \frac{(2\pi)^7 C}{4} \int \prod_{i=0}^{3} \left(\frac{d^3 q_i}{(2\pi)^3} \right) \delta(0 - 1 - 2 + 3) \\ & \times [n_0 n_1 (1 + n_2)(1 + n_3) - (1 + n_0)(1 + n_1) n_2 n_3] \\ & \times \ln \frac{n_0 n_1 (1 + n_2)(1 + n_3)}{(1 + n_0)(1 + n_1) n_2 n_3}. \end{aligned} \tag{8.28}$$

As $(x - y)\ln(x/y) \geq 0$ for positive x and y, Eq. (8.28) shows that the entropy always increases in time. This is the TFD derivation of the second law of thermodynamics. Thus we conclude that with our choice $(\alpha_k = 1, t_0 = -\infty)$ the time-independent Fock space built on the vacuum $|0\rangle$ takes care of time-dependent processes.

As an important aside: The choice $(\alpha_k = 0, t_0 = \infty)$ leads to the incorrect result that the entropy decreases. Thus this choice should be avoided.

9. Comments

In this paper we presented a t-representation formalism for nonequilibrium TFD by means of time-dependent Bogoliubov matrix. To deal with interactions we reviewed the interaction representation in Sec. 4 where only two choices $(\alpha = 1, t_0 = -\infty)$ and $(\alpha = 0, t_0 = \infty)$ are permitted as long as one insists on the use of the **T**-product Feynman method. It was shown in Sec. 8 that the choice of $(\alpha = 1, t_0 = -\infty)$ gives rise to the Boltzmann equation which leads to the right entropy law while the choice of $(\alpha = 0, t_0 = \infty)$ does not. We chose α by demanding that the **T**-product Feynman diagram method should be available. If we would abandon this demand, we will still be able to establish some calculational method for any α which will become very much complicated. However, t_0 should be $t_0 = -\infty$ whatever the choice of α would be.

Derivation of the Boltzmann equation in the density operator formalism has attracted the attention of many physicists. There are a vast number of papers on this subject. The same subject in the closed path formalism too has been studied intensively. Reference 14 presents a recent work on this subject. Since the usual formulation of the closed path formalism does not have the parameter s_k, which plays a crucial role in the TFD formalism in this paper, it is not obvious how the

consideration in this paper is related to the study of the kinetic equation in the closed path formalism.

The whole consideration in this paper is based on the general structure of the two-point Green's function in the t-representation. The corrected Green's functions can be put in the form $B^{-1} \times$ (a diagonal matrix) $\times B$. The parameter appearing in the Bogoliubov matrix are divided into two terms; one is the Heisenberg number density and the other, denoted by ν, represents thermal fluctuation effects on the number density. It is important to note that these properties are common in equilibrium and nonequilibrium cases. In other words, there appear fluctuations in the number density even in equilibrium situations in the t-representation.

A similar fluctuation is expected to happen also in particle collisions. An interesting application of the present formalism is found in the physics of high energy particle reactions such as quark–gluon plasma both in equilibrium and in nonequilibrium. As an example, we can calculate the transition matrix element of reactions with two incoming and two outgoing particles, using the unperturbed propagator in Sec. 3 and the interaction representation in Sec. 4 in the t-representation. Then the propagation of the center of mass point may also feel a short time thermal fluctuation. If this fluctuation becomes very strong, we might find a signal of a phase transition.

Most of time-dependent nonequilibrium phenomena which we observe in nature are of spatial inhomogeneity. Heat conduction is a typical example.[4] It is expected that spatial inhomogeneity even in the example of particle reactions above plays a significant role in giving rise to the nonequilibrium process. The formalism in this paper can be generalized to cover spatially inhomogeneous systems, which we report in a separate paper.[12]

It may be worth pointing out that extension of thermal quantum field theories to time–space-dependent phenomena seems to suggest a gauge theory formalism, though not much has been done in this line of approach. Making use of the thermal covariant derivative with all three parameters (n_k, α_k, s_k) depending on time, Henning and his collaborators have recently started a study of a gauge theory in the case of time-dependent Bogoliubov matrices.[13] This is another open subject in nonequilibrium TFD.

Appendix A. Matrix Formulae

We list here some matrix formulae involving the Bogoliubov matrices to be used in the text.

A straightforward matrix calculation for the B-matrix in (4.12) gives

$$\frac{1+\tau_3}{2} B_k(t) = \frac{1+\tau_3}{2} + \sigma n_k(t)(T_0 + T_-)\,, \tag{A.1}$$

$$B_k^{-1}(t)\,\frac{1+\tau_3}{2} = T_+\,, \tag{A.2}$$

$$\frac{1-\tau_3}{2} B_k(t) = T_- , \tag{A.3}$$

$$B_k^{-1}(t) \frac{1-\tau_3}{2} = \frac{1-\tau_3}{2} - \sigma n_k(t)(T_0 - T_+) . \tag{A.4}$$

Here the notations are

$$T_+ = \begin{bmatrix} 1 & 0 \\ \sigma & 0 \end{bmatrix} , \tag{A.5}$$

$$T_- = \begin{bmatrix} 0 & 0 \\ -\sigma & 1 \end{bmatrix} . \tag{A.6}$$

The three matrices, $T_\pm$ above and T_0 in (4.15), play the basic roles in practical calculations. A remarkable fact is that these matrices satisfy some simple algebraic relations. They are

$$T_0^2 = 0 , \tag{A.7}$$

$$T_+^2 = T_+ , \tag{A.8}$$

$$T_-^2 = T_- , \tag{A.9}$$

$$T_+ T_0 = T_0 T_- = T_0 , \tag{A.10}$$

$$T_0 T_+ = T_- T_0 = T_+ T_- = T_- T_+ = 0 , \tag{A.11}$$

$$T_+ + T_- = 1 . \tag{A.12}$$

Using these relations we can easily show that

$$B_k^{-1}(t) \frac{1+\tau_3}{2} B_k(t') = T_+ + \sigma n_k(t') T_0 , \tag{A.13}$$

$$B_k^{-1}(t) \frac{1-\tau_3}{2} B_k(t') = T_- - \sigma n_k(t) T_0 . \tag{A.14}$$

These two relations, when combined, lead to

$$B^{-1}[N_2] \begin{bmatrix} \mathcal{G}_1 & 0 \\ 0 & \mathcal{G}_2 \end{bmatrix} B[N_1] = \mathcal{G}_1(T_+ + \sigma N_1 T_0) + \mathcal{G}_2(T_- - \sigma N_2 T_0) . \tag{A.15}$$

Apply this formula to the unperturbed propagator in (3.2), i.e.

$$N_1 = n_k(t') , \qquad N_2 = n_k(t) , \tag{A.16}$$

$$\mathcal{G}_1 = \mathcal{G}_k(t,t')^{11} , \quad \mathcal{G}_2 = \mathcal{G}_k(t,t')^{22} , \tag{A.17}$$

$$\sigma = 1 , \tag{A.18}$$

we have

$$\Delta_k(t,t') = \mathcal{G}_k(t,t')^{11} A_{+,k}(t') + \mathcal{G}_k(t,t')^{22} A_{-,k}(t) , \tag{A.19}$$

where the matrices $A_\pm$ are defined by

$$A_{+,k}(t) = T_+ + n_k(t)T_0\,, \tag{A.20}$$

$$A_{-,k}(t) = T_- - n_k(t)T_0\,. \tag{A.21}$$

We frequently make use of the matrix A defined by

$$\begin{aligned} A_k(t) &\equiv B_k^{-1}(t)\tau_3 B_k(t) \\ &= A_{+,k}(t) - A_{-,k}(t) = T_+ - T_- + 2n_k(t)T_0\,. \end{aligned} \tag{A.22}$$

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ENTROPY LAW FROM INHOMOGENEOUS THERMOFIELD DYNAMICS

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In the framework of thermofield dynamics for spatially inhomogeneous time-dependent nonequilibrium situations, we derive the kinetic equation, which is different from the Boltzmann equation due to quantum effects. It is shown that this kinetic equation leads to the entropy law. An expression for the entropy flow is found.

The development of thermofield dynamics (TFD) in the past few years[1–4] has made it a powerful and systematic tool with a solid foundation to tackle time-dependent nonequilibrium systems of quantum fields. We see the reasons for this from the following aspects of TFD: (i) In TFD time-dependent thermal situation is described by time-dependent unperturbed (quasiparticle) number density, whereas thermal vacuum is kept *time-independent.*[3,4] Thus the whole time-dependent process can be represented in a single Hilbert space using a single thermal vacuum, because of which TFD calculation of Green's functions is a mathematically well-defined procedure. (ii) There are redundant parameters in thermal Bogoliubov transformation. It was shown that under a particular choice of the redundant parameters the Feynman diagram method is available in calculating Green's functions of Heisenberg operators without relying on the Gell–Mann–Low relation.[3–5] (iii) Therefore, it is possible to study general structures of Green's functions in time representation, as was done in Ref. 4. Moreover, a simple method of perturbative calculation of diagrams, called TFD recipe, was invented[6,7] and is very helpful for studying Green's functions. (iv) As TFD is formulated properly as an operator formalism of quantum fields because of (i) and as the Green's functions in time representation are put into a compact form (allowing us a physical interpretation of each part) owing to (ii) and (iii), the renormalization becomes a transparent procedure in time-dependent TFD. As witnessed in Refs. 3 and 4, the renormalization condition in the spatially homogeneous time-dependent thermal situation derived an equation for the unperturbed number density, which coincides with the kinetic equation of

Boltzmann type and consequently leads to the entropy increase law, i.e. the second law of thermodynamics.

We then ask whether the spatially homogeneous time-dependent TFD can be extended to the inhomogeneous one including the entropy law. Such an extension is vital for comparison with experiments, since most nonequilibrium phenomena are associated with inhomogeneity. The formulation of inhomogeneous TFD for the unperturbed system was already given in Refs. 8 and 9. An attempt to derive a kinetic equation from the renormalization was made in Ref. 10. In this letter we derive a kinetic equation using a renormalization condition different from the previous one[10] and show the entropy law from it.

The spatial inhomogeneity in TFD is introduced through the momentum-mixing thermal Bogoliubov transformation[8,9]: It reads for bosonic oscillator variables a_k (the discussions in this letter is restricted to bosonic operators for simplicity, as extension to fermionic ones is straightforward)

$$\begin{aligned} a_k(t)^\mu &= \mathbf{B}^{-1}(t)^{\mu\nu}_{kq}\xi^\nu_q e^{-i\int^t ds\,\omega_q(s)}\,, \\ \bar{a}_k(t)^\mu &= e^{i\int^t ds\,\omega_q(s)}\bar{\xi}^\nu_q\,\mathbf{B}(t)^{\nu\mu}_{qk}\,, \end{aligned} \tag{1}$$

where the suffices k, q stand for momentum variables, μ, ν are thermal indices, i.e. $a^1 = a$, $a^2 = \tilde{a}^\dagger$, $\bar{a}^1 = a^\dagger$, $\bar{a}^2 = -\tilde{a}$. The ξ_k-operators define the time-space-independent thermal vacua $|0\rangle$ and $\langle 0|$ as $\xi_k|0\rangle = \tilde{\xi}_k|0\rangle = 0$ and $\langle 0|\xi^\dagger_k = \langle 0|\tilde{\xi}^\dagger_k = 0$. We take the following form for the thermal Bogoliubov matrix $\mathbf{B}(t)$,

$$\mathbf{B}(t)^{\mu\nu}_{kq} = \begin{bmatrix} \delta(\mathbf{k}-\mathbf{q}) + \mathbf{N}(t)_{kq} & -\mathbf{N}(t)_{kq} \\ -\delta(\mathbf{k}-\mathbf{q}) & \delta(\mathbf{k}-\mathbf{q}) \end{bmatrix}^{\mu\nu}, \tag{2}$$

with

$$\mathbf{N}(t)_{kq} = \langle 0|\bar{a}(t)^1_q a(t)^1_k|0\rangle\,. \tag{3}$$

This parametrization of the thermal matrix is the same as that chosen in homogeneous case,[4]

$$B[n_k(t)]^{\mu\nu} = \begin{bmatrix} 1+n_k(t) & -n_k(t) \\ -1 & 1 \end{bmatrix}^{\mu\nu}, \tag{4}$$

except for momentum mixing, and is crucial for the availability of Feynman method later.

We recall from Ref. 8 that the expression for unperturbed bosonic field of Schrödinger type using $a_k(t)^\mu$ in (1),

$$\varphi(t,\mathbf{x})^\mu = \frac{1}{(2\pi)^{3/2}}\int d^3k\, a_k(t)^\mu e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{5}$$

which yields the unperturbed 2×2 matrix propagator,

$$\Delta(x,x')^{\mu\nu} \equiv -i\langle 0|T[\varphi(x)^{\mu}\bar{\varphi}(x')^{\nu}]|0\rangle$$
$$= \int \frac{d^3k d^3k' d^3q}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{B}^{-1}(t)^{\mu\mu'}_{kq} \mathcal{G}(t,t':\mathbf{q})^{\mu'\nu'} \mathbf{B}(t')^{\nu'\nu}_{qk'} e^{-i\mathbf{k}'\cdot\mathbf{x}}, \tag{6}$$

where

$$\mathcal{G}(t,t':\mathbf{q})^{\mu\nu} = e^{-i\int_{t'}^{t} ds\, \omega_q(s)} \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu\nu}. \tag{7}$$

The unperturbed Hamiltonian governing the time evolution of the above φ is given by

$$\hat{H}_Q(t) = \int d^3x\, \bar{\varphi}(t,\mathbf{x})^{\mu}\omega(-i\nabla)\varphi(t,\mathbf{x})^{\mu} - \hat{Q}_R(t),$$
$$\hat{Q}_R(t) \equiv \int d^3x d^3y\, \bar{\varphi}(t,\mathbf{x})^{\mu} R(t,\mathbf{x},\mathbf{y}) T_0^{\mu\nu} \varphi(t,\mathbf{y})^{\nu}, \tag{8}$$

where

$$T_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{9}$$

and

$$R(t,\mathbf{x},\mathbf{y}) = \int \frac{d^3q}{(2\pi)^3} \left[i\frac{\partial}{\partial t} + \{\omega(i\nabla_y) - \omega(-i\nabla_x)\} \right] n\left(t, \frac{\mathbf{x}+\mathbf{y}}{2} : \mathbf{q}\right) e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}. \tag{10}$$

Here the number density parameter $n(x:\mathbf{k})$ is related to $\mathbf{N}(t)_{kq}$ through the Fourier transformation

$$n(t,\mathbf{x}:\mathbf{k}) \equiv \int d^3q e^{i\mathbf{q}\cdot\mathbf{x}} \mathbf{N}(t)_{k+\frac{q}{2},\, k-\frac{q}{2}} \tag{11}$$

or

$$\mathbf{N}(t)_{kk'} \equiv \int \frac{d^3x}{(2\pi)^3} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} n\left(t, \mathbf{x} : \frac{\mathbf{k}+\mathbf{k}'}{2}\right). \tag{12}$$

The total Hamiltonian $\hat{H}$ in TFD, given by

$$\hat{H}(t) = H(t) - \tilde{H}(t), \tag{13}$$

is divided into the unperturbed part $\hat{H}_Q(t)$ in (8) and interaction one $\hat{H}_I(t)$,

$$\hat{H}(t) = \hat{H}_Q(t) + \hat{H}_I(t), \tag{14}$$

$$\hat{H}_I(t) = \hat{H}_{\text{int}} + \hat{Q}_R(t). \tag{15}$$

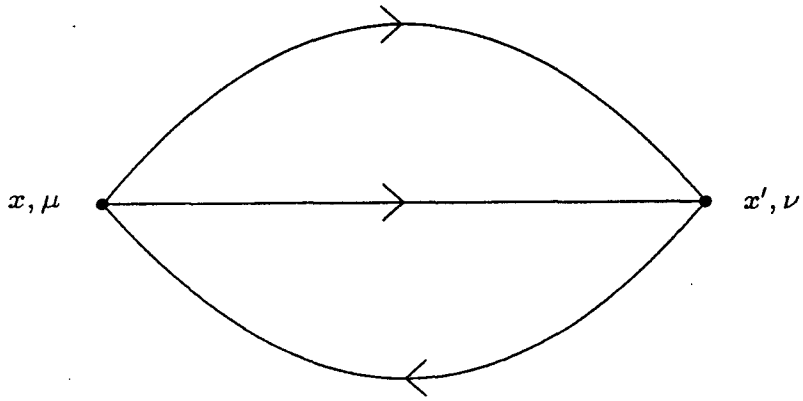

Fig. 1. The loop self-energy diagram without vertex correction.

The second term in (15) is the counterterm, while $\hat{H}_{\rm int}(t)$ consists of nonlinear terms and the usual counterterms such as the energy counterterm. For definiteness, the following model of the interaction density is taken in this letter:

$$\begin{aligned} \hat{H}_{\rm int}(t) &= H_{\rm int}(t) - \tilde{H}_{\rm int}(t)\,, \\ H_{\rm int}(t) &= g\int d^3x\,(\varphi^\dagger(x)\varphi(x))^2\,. \end{aligned} \tag{16}$$

Here and below the usual counterterms in $\hat{H}_{\rm int}(t)$, which are not essential in our following discussions, are suppressed.

On the basis of the division (14) of $\hat{H}(t)$, one can develop the interaction representation. As was shown in Ref. 4, the Feynman diagram method is available in the calculation of the full propagator,

$$G(x,x')^{\mu\nu} \equiv -i\langle 0|T[\varphi_{\rm H}(x)^\mu \bar{\varphi}_{\rm H}(x')^\nu]|0\rangle\,, \tag{17}$$

$\varphi_{\rm H}(x)^\mu$ being the Heisenberg operator. For this the choice of the form for $\mathbf{B}(t)_{kq}$ in (2) is crucial.[4]

The 2×2-matrix self-energy $\Sigma(x,x')^{\mu\nu}$ is defined through the Dyson–Schwinger equation,

$$G(x,x')^{\mu\nu} = \Delta(x,x')^{\mu\nu} + \int d^4y d^4y' \Delta(x,y)^{\mu\mu'}\Sigma(y,y')^{\mu'\nu'}G(y',x')^{\nu'\nu}\,. \tag{18}$$

The loop self-energy without vertex corrections in our model, (16), corresponding to the Feynman diagram in Fig. 1, is obtained as

$$\Sigma_{\rm loop}(x,x')^{\mu\nu} = -8g^2 \begin{bmatrix} (\Delta(x,x')^{11})^2\Delta(x',x)^{11} & -(\Delta(x,x')^{12})^2\Delta(x',x)^{21} \\ -(\Delta(x,x')^{21})^2\Delta(x',x)^{12} & (\Delta(x,x')^{22})^2\Delta(x',x)^{22} \end{bmatrix}. \tag{19}$$

Substitute (6) in this, we then find in the Fourier space with respect to $\mathbf{x}$ and $\mathbf{x}'$,

$$\mathbf{\Sigma}_{\rm loop}(t,t')^{\mu\nu}_{kk'} \equiv \int \frac{d^3x d^3x'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \Sigma_{\rm loop}(x,x')^{\mu\nu} e^{i\mathbf{k}'\cdot\mathbf{x}'} \tag{20}$$

$$\begin{aligned} &= C_g \int \prod_{i=1}^{3} (d^3k_i d^3k_i') \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3') \\ &\quad \times \Big[-i\theta(t-t') e^{-i\int_{t'}^{t} ds[\omega_{k_1}(s)+\omega_{k_2}(s)-\omega_{k_3}(s)]} \\ &\quad \times (\mathbf{M}_1(t')_{k_1k_1'k_2k_2'k_3'k_3} T_+ + \mathbf{M}_0(t')_{k_1k_1'k_2k_2'k_3'k_3} T_0) \\ &\quad + i\theta(t'-t) e^{-i\int_{t'}^{t} ds[\omega_{k_1'}(s)+\omega_{k_2'}(s)-\omega_{k_3'}(s)]} \\ &\quad \times (\mathbf{M}_1(t)_{k_1k_1'k_2k_2'k_3'k_3} T_- - \mathbf{M}_0(t)_{k_1k_1'k_2k_2'k_3'k_3} T_0) \Big] , \end{aligned} \tag{21}$$

where

$$C_g \equiv \frac{8g^2}{(2\pi)^6} , \tag{22}$$

$$\begin{aligned} \mathbf{M}_0(t)_{k_1k_1'k_2k_2'k_3'k_3} &= \mathbf{N}(t)_{k_1k_1'} \mathbf{N}(t)_{k_2k_2'} (\mathbf{I}_{k_3'k_3} + \mathbf{N}(t)_{k_3'k_3}) , \\ \mathbf{M}_1(t)_{k_1k_1'k_2k_2'k_3'k_3} &= (\mathbf{I}_{k_1k_1'}\mathbf{I}_{k_2k_2'} + \mathbf{N}(t)_{k_1k_1'}\mathbf{I}_{k_2k_2'} + \mathbf{I}_{k_1k_1'}\mathbf{N}(t)_{k_2k_2'})\mathbf{N}(t)_{k_3'k_3} \\ &\quad - \mathbf{N}(t)_{k_1k_1'}\mathbf{N}(t)_{k_2k_2'}\mathbf{I}_{k_3'k_3} , \end{aligned} \tag{23}$$

with the notation of $\mathbf{I}_{kk'} = \delta(\mathbf{k}-\mathbf{k}')$. The expression in (21) can be rewritten in the form of $B^{-1} \times$ (a diagonal matrix) $\times B$, using the B-matrix in (4), when the formulas,

$$\begin{aligned} B^{-1}[n(t)]^{\mu\mu'} \left[\frac{1+\tau_3}{2}\right]^{\mu'\nu'} B[n(t')]^{\nu'\nu} &= T_+^{\mu\nu} + n(t')T_0^{\mu\nu} , \\ B^{-1}[n(t)]^{\mu\mu'} \left[\frac{1-\tau_3}{2}\right]^{\mu'\nu'} B[n(t')]^{\nu'\nu} &= T_-^{\mu\nu} - n(t)T_0^{\mu\nu} , \end{aligned} \tag{24}$$

with (9) and

$$T_+ = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} , \qquad T_- = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} , \tag{25}$$

are used, namely,

$$\begin{aligned} &\mathbf{\Sigma}_{\rm loop}(t,t')^{\mu\nu}_{kk'} \\ &= C_g \int \prod_{i=1}^{3} (d^3k_i\, d^3k_i') \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3') \\ &\quad \times B^{-1}[\mathcal{N}(t)]^{\mu\mu'} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{M}_1(t) \end{bmatrix}^{\mu'\lambda} V(t,t')^{\lambda\lambda'} \begin{bmatrix} \mathbf{M}_1(t') & 0 \\ 0 & 1 \end{bmatrix}^{\lambda'\nu'} B[\mathcal{N}(t')]^{\nu'\nu} , \end{aligned} \tag{26}$$

where

$$V(t,t')^{\mu\nu} = \begin{bmatrix} -i\theta(t-t')e^{-i\int_{t'}^{t} ds[\omega_{k_1}(s)+\omega_{k_2}(s)-\omega_{k_3}(s)]} & 0 \\ 0 & i\theta(t'-t)e^{-i\int_{t'}^{t} ds[\omega_{k'_1}(s)+\omega_{k'_2}(s)-\omega_{k'_3}(s)]} \end{bmatrix}^{\mu\nu}, \tag{27}$$

$$\mathcal{N}(t) = \frac{\mathbf{M}_0(t)_{k_1k'_1k_2k'_2k'_3k_3}}{\mathbf{M}_1(t)_{k_1k'_1k_2k'_2k'_3k_3}}. \tag{28}$$

Note that for simplicity we drop the dependencies of $\mathbf{M}_1$, $\mathcal{N}$ and V on k_i's and k'_i's.

To the loop self-energy is added the counterterm self-energy, denoted by $\Sigma_Q(x,x')^{\mu\nu}$, to make the total self-energy,

$$\mathbf{\Sigma}(t,t')^{\mu\nu}_{kk'} = \mathbf{\Sigma}_{\text{loop}}(t,t')^{\mu\nu}_{kk'} + \mathbf{\Sigma}_Q(t,t')^{\mu\nu}_{kk'}. \tag{29}$$

The term $\hat{Q}_R(t)$ in (15) contributes to $\Sigma_Q(x,x')^{\mu\nu}$ as

$$\mathbf{\Sigma}_Q(t,t')^{\mu\nu}_{kk'} = \mathbf{R}(t)_{kk'}\delta(t-t')T_0^{\mu\nu}, \tag{30}$$

where

$$\mathbf{R}(t)_{kk'} = i\dot{\mathbf{N}}(t)_{kk'} + (\omega_{k'}(t) - \omega_k(t))\mathbf{N}(t)_{kk'}. \tag{31}$$

The next step is to extract the *on-shell* part from the total self-energy obtained above, on which a renormalization condition will be imposed. For this, consider the diagonal matrix function $V(t,t')^{\mu\nu}$ included in $\mathbf{\Sigma}_{\text{loop}}(t,t')^{\mu\nu}_{kk'}$, representing the propagation of the quasiparticles ξ. We assume that ω_k is independent of time, then V depends only on $t-t'$. Then one may define its Fourier transformation as

$$V(t)^{\mu\nu} = \int \frac{dk_0}{2\pi} e^{-ik_0t} V(k_0)^{\mu\nu} \tag{32}$$

to find that

$$V(k_0)^{\mu\nu} = \begin{bmatrix} \dfrac{1}{k_0 - \omega_{k_1} - \omega_{k_2} + \omega_{k_3} + i\varepsilon} & 0 \\ 0 & \dfrac{1}{k_0 - \omega_{k'_1} - \omega_{k'_2} + \omega_{k'_3} - i\varepsilon} \end{bmatrix}^{\mu\nu}. \tag{33}$$

Let us define its on-shell by putting

$$\begin{aligned} k_0\delta^{\mu\nu} &= \left(\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2}\right)\delta^{\mu\nu} \\ &\quad + \begin{bmatrix} \omega_{k_1} + \omega_{k_2} - \omega_{k_3} & 0 \\ 0 & \omega_{k'_1} + \omega_{k'_2} - \omega_{k'_3} \end{bmatrix}^{\mu\nu} \end{aligned} \tag{34}$$

there, which means that the on-shell part of $V(t-t')^{\mu\nu}$ is given by

$$V^{(\mathrm{on})}(t-t')^{\mu\nu} = \delta(t-t')\left[\frac{1}{\omega_{(k+k')/2}-\omega_{(k_1+k_1')/2}-\omega_{(k_2+k_2')/2}+\omega_{(k_3+k_3')/2}+i\varepsilon\tau_3}\right]^{\mu\nu}, \quad (35)$$

where τ_3 is the Pauli matrix. Thus the on-shell part of the loop self-energy, denoted by $\mathbf{\Sigma}^{(\mathrm{on})}_{\mathrm{loop}}(t,t')^{\mu\nu}_{kk'}$, follows from substituting this $V^{(\mathrm{on})}(t-t')^{\mu\nu}$ into $V(t,t')^{\mu\nu}$ inside $\mathbf{\Sigma}_{\mathrm{loop}}(t,t')^{\mu\nu}_{kk'}$,

$$\begin{aligned}\mathbf{\Sigma}^{(\mathrm{on})}_{\mathrm{loop}}(t,t')^{\mu\nu}_{kk'} &= C_g\int\prod_{i=1}^{3}(d^3k_i\,d^3k_i')\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3')\delta(t-t')\mathbf{M}_1(t)\\ &\quad\times B^{-1}[\mathcal{N}(t)]^{\mu\mu'}\left[\frac{1}{\omega_{(k+k')/2}-\omega_{(k_1+k_1')/2}-\omega_{(k_2+k_2')/2}+\omega_{(k_3+k_3')/2}+i\varepsilon\tau_3}\right]^{\mu'\nu'}\\ &\quad\times B[\mathcal{N}(t)]^{\nu'\nu} \qquad (36)\end{aligned}$$

with (23) and (28). The on-shell counterterm is $\mathbf{\Sigma}_Q(t,t')^{\mu\nu}_{kk'}$ in (30) itself, therefore the on-shell part of the total self-energy is given by

$$\mathbf{\Sigma}^{(\mathrm{on})}(t,t')^{\mu\nu}_{kk'} = \mathbf{\Sigma}^{(\mathrm{on})}_{\mathrm{loop}}(t,t')^{\mu\nu}_{kk'} + \mathbf{\Sigma}_Q(t,t')^{\mu\nu}_{kk'}\,. \quad (37)$$

Extending the arguments in our previous papers of the spatially homogeneous case,[3,4] we require the renormalization condition that the total on-shell self-energy in (37) should be diagonal in terms of the quasiparticle operators ξ^μ_p and $\bar{\xi}^\mu_p$, i.e.

$$\begin{aligned}&\mathbf{B}(t)^{\mu\mu'}_{pk}\mathbf{\Sigma}^{(\mathrm{on})}(t,t')^{\mu'\nu'}_{kk'}\mathbf{B}^{-1}(t)^{\nu'\nu}_{k'p'}\\ &\qquad = (\text{diagonal with respect to the thermal indices})\,. \qquad (38)\end{aligned}$$

Although this is a 2×2-matrix relation, it turns out that the following single equation suffices to satisfy the condition (38):

$$\begin{aligned}&\dot{\mathbf{N}}(t)_{kk'} - i(\omega_{k'}-\omega_k)\mathbf{N}(t)_{kk'}\\ &\quad = -i\int d^3q\{\delta\mathbf{W}(t)_{kq}\mathbf{N}(t)_{qk'} - \mathbf{N}(t)_{kq}\delta\mathbf{W}^*(t)_{qk'}\}\\ &\qquad + 2\pi C_g\int\prod_{i=1}^{3}(d^3k_i\,d^3k_i')\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3')\\ &\qquad\times\delta(\omega_{(k+k')/2}-\omega_{(k_1+k_1')/2}-\omega_{(k_2+k_2')/2}+\omega_{(k_3+k_3')/2})\mathbf{M}_0(t)\,, \qquad (39)\end{aligned}$$

where

$$\delta\mathbf{W}(t)_{kk'} \equiv C_g \int \prod_{i=1}^{3}(d^3k_i\, d^3k_i')\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3')$$
$$\times \frac{\mathbf{M}_1(t)}{\omega_{(k+k')/2}-\omega_{(k_1+k_1')/2}-\omega_{(k_2+k_2')/2}+\omega_{(k_3+k_3')/2}+i\varepsilon}\,. \tag{40}$$

Equation (39) is the kinetic equation.

Let us translate (39) into that in $(\mathbf{x},\mathbf{k})$-representation. To do this, we prepare the following formula: When the $*$-product in double $\mathbf{k}$-representation is defined by

$$(\mathbf{F}_1 * \mathbf{F}_2)_{k_1k_2} = \int d^3q \mathbf{F}_{1k_1q}\mathbf{F}_{2qk_2}\,, \tag{41}$$

one has

$$F_1 * F_2(\mathbf{x}:\mathbf{k}) = \int \frac{d^3q_1 d^3q_2 d^3y_1 d^3y_2}{(2\pi)^6} e^{i(\mathbf{q}_1\cdot\mathbf{y}_1+\mathbf{q}_2\cdot\mathbf{y}_2)}$$
$$\times F_1\left(\mathbf{x}+\frac{\mathbf{y}_2}{2}:\mathbf{k}+\mathbf{q}_1\right)F_2\left(\mathbf{x}-\frac{\mathbf{y}_1}{2}:\mathbf{k}+\mathbf{q}_2\right)$$
$$= \exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_1}\cdot\frac{\partial}{\partial\mathbf{k}_2}-\frac{\partial}{\partial\mathbf{x}_2}\cdot\frac{\partial}{\partial\mathbf{k}_1}\right)\right]F_1(\mathbf{x}:\mathbf{k})F_2(\mathbf{x}:\mathbf{k}) \tag{42}$$

in terms of

$$F(\mathbf{x}:\mathbf{k}) \equiv \int d^3q e^{i\mathbf{q}\cdot\mathbf{x}}\mathbf{F}_{k+\frac{q}{2},\,k-\frac{q}{2}}\,. \tag{43}$$

In the last expression of (42), $\partial/\partial\mathbf{x}_i$ and $\partial/\partial\mathbf{k}_i$ $(i=1,2)$ operate on arguments in $F_i(\mathbf{x}:\mathbf{k})$. With the help of (42), (39) is rewritten in $(\mathbf{x},\mathbf{k})$-representation as

$$\dot{n}(t,\mathbf{x}:\mathbf{k}) - i\left\{\omega\left(\mathbf{k}-\frac{1}{2i}\nabla_x\right)-\omega\left(\mathbf{k}+\frac{1}{2i}\nabla_x\right)\right\}n(t,\mathbf{x}:\mathbf{k})$$
$$= -i\exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\right]\delta W(t,\mathbf{x}:\mathbf{k})n(t,\mathbf{x}:\mathbf{k})$$
$$+ i\exp\left[\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\right]n(t,\mathbf{x}:\mathbf{k})\delta W^*(t,\mathbf{x}:\mathbf{k})$$
$$+ i\delta W_0(t,\mathbf{x}:\mathbf{k})\,, \tag{44}$$

where

$$\delta W(t,\mathbf{x}:\mathbf{k}) = C_g \int d^3q \prod_{i=1}^{3}(d^3k_i\, d^3k_i')e^{i\mathbf{q}\cdot\mathbf{x}}$$
$$\times \delta\left(\mathbf{k}+\frac{\mathbf{q}}{2}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3\right)\delta\left(\mathbf{k}-\frac{\mathbf{q}}{2}-\mathbf{k}_1'-\mathbf{k}_2'+\mathbf{k}_3'\right)$$
$$\times \frac{\mathbf{M}_1(t)}{\omega_{(k+k')/2}-\omega_{(k_1+k_1')/2}-\omega_{(k_2+k_2')/2}+\omega_{(k_3+k_3')/2}+i\varepsilon} \tag{45}$$

$$= C_g \int \prod_{i=1}^{3} d^3L_i \delta(\mathbf{k} - \mathbf{L}_1 - \mathbf{L}_2 + \mathbf{L}_3) M_1(t, \mathbf{x} : \mathbf{L}_i)$$

$$\times \frac{1}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3} + i\varepsilon}, \tag{46}$$

$$\delta W_0(t, \mathbf{x} : \mathbf{k}) = -2\pi i C_g \int d^3q \prod_{i=1}^{3} (d^3k_i\, d^3k_i') e^{i\mathbf{q}\cdot\mathbf{x}}$$

$$\times \delta\left(\mathbf{k} + \frac{\mathbf{q}}{2} - \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3\right) \delta\left(\mathbf{k} - \frac{\mathbf{q}}{2} - \mathbf{k}_1' - \mathbf{k}_2' + \mathbf{k}_3'\right)$$

$$\times \delta(\omega_{(k+k')/2} - \omega_{(k_1+k_1')/2} - \omega_{(k_2+k_2')/2} + \omega_{(k_3+k_3')/2}) \mathbf{M}_0(t) \tag{47}$$

$$= -2\pi i C_g \int \prod_{i=1}^{3} d^3L_i \delta(\mathbf{k} - \mathbf{L}_1 - \mathbf{L}_2 + \mathbf{L}_3) M_0(t, \mathbf{x} : \mathbf{L}_i)$$

$$\times \delta(\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}), \tag{48}$$

with

$$M_0(t, \mathbf{x} : \mathbf{L}_i) = n(t, \mathbf{x} : \mathbf{L}_1) n(t, \mathbf{x} : \mathbf{L}_2)(1 + n(t, \mathbf{x} : \mathbf{L}_3)),$$

$$M_1(t, \mathbf{x} : \mathbf{L}_i) = (1 + n(t, \mathbf{x} : \mathbf{L}_1) + n(t, \mathbf{x} : \mathbf{L}_2)) n(t, \mathbf{x} : \mathbf{L}_3) \tag{49}$$

$$- n(t, \mathbf{x} : \mathbf{L}_1) n(t, \mathbf{x} : \mathbf{L}_2).$$

For comparison, we write down the famous Boltzmann equation,

$$\dot{n}(t, \mathbf{x} : \mathbf{k}) + \mathbf{v} \cdot \nabla_x n(t, \mathbf{x} : \mathbf{k}) = \mathrm{St}[n](t, \mathbf{x} : \mathbf{k}), \tag{50}$$

where the collision integral is given by

$$\mathrm{St}[n](t, \mathbf{x} : \mathbf{k}) \equiv \int \prod_{i=1}^{3} d^3L_i w(\mathbf{k}, \mathbf{L}_3 : \mathbf{L}_1, \mathbf{L}_2)$$

$$\times \{ n(t, \mathbf{x} : \mathbf{L}_1) n(t, \mathbf{x} : \mathbf{L}_2)(1 + n(t, \mathbf{x} : \mathbf{k}))(1 + n(t, \mathbf{x} : \mathbf{L}_3)$$

$$- n(t, \mathbf{x} : \mathbf{k}) n(t, \mathbf{x} : \mathbf{L}_3)(1 + n(t, \mathbf{x} : \mathbf{L}_1))(1 + n(t, \mathbf{x} : \mathbf{L}_2)\}. \tag{51}$$

For our present model, the transition rate w in the Born approximation takes the form

$$w(\mathbf{k}, \mathbf{L}_3 : \mathbf{L}_1, \mathbf{L}_2) = 2\pi C_g \delta(\mathbf{k} - \mathbf{L}_1 - \mathbf{L}_2 + \mathbf{L}_3) \delta(\omega_k - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}). \tag{52}$$

As is well known, the entropy law follows from the Boltzmann equation (50).

In spatially homogeneous case, the kinetic equation derived from time-dependent TFD completely coincides with the above Boltzmann equation without $\mathbf{x}$-dependence.[3,4] Therefore, it was easy to show that the entropy law holds for spatially homogeneous time-dependent TFD.

Our kinetic equation in inhomogeneous case (44) differs from the Boltzmann equation. We see the following three sources of such differences.

(i) The term $-i\{\omega(\mathbf{k}-\frac{1}{2i}\nabla_x)-\omega(\mathbf{k}+\frac{1}{2i}\nabla_x)\}n$ replaces $\mathbf{v}\cdot\nabla_x n$ in the Boltzmann equation which represents the classical kinematical effect of particle flow. We expand it around $\mathbf{k}$ as

$$-i\left\{\omega\left(\mathbf{k}-\frac{1}{2i}\nabla_x\right)-\omega\left(\mathbf{k}+\frac{1}{2i}\nabla_x\right)\right\}=\frac{\partial\omega}{\partial\mathbf{k}}\cdot\nabla_x+\cdots$$
$$=\left[\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\left(\frac{1}{2}\frac{\partial}{\partial\mathbf{k}_\omega}\cdot\nabla_x\right)^{2j}\right]\frac{\partial\omega}{\partial\mathbf{k}}\cdot\nabla_x\,, \tag{53}$$

where the suffix ω in $\frac{\partial}{\partial\mathbf{k}_\omega}$ is put to remind that this differentiation operates on ω but not on n. If we identify the group velocity of the wave,

$$\mathbf{v}_g=\frac{\partial\omega}{\partial\mathbf{k}}\,, \tag{54}$$

with the classical velocity $\mathbf{v}$, neglecting the higher derivative terms, we see a correspondence between our kinetic equation and the Boltzmann equation regarding the kinematical effect of particle flow. Our kinetic equation includes the nature of quantum wave.

(ii) The presence of the factor $\exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\right]$ on the right-hand side of (44) is due to the effect of spontaneous change in momentum of propagating particle. As a matter of fact, in (39) we get the terms like $\int d^3q\delta\mathbf{W}(t)_{pq}\mathbf{N}(t)_{qp'}$, from which the factor on the top of this paragraph originates.

(iii) The presence of $1/(\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}+i\varepsilon)$ in (46) reflects the effect of the energy uncertainty.

First consider a very crude approximation of our kinetic equation (44), namely, neglect (ii) (then automatically no (iii)), and keep only the $\mathbf{v}_g$ in (i). Then (44) simply reduces to the Boltzmann equation (50) built on a classical picture. It is natural since all the effects neglected above are of quantum origin.

Let us treat (44). We just rewrite it as

$$\dot{n}(t,\mathbf{x}:\mathbf{k})-i\left\{\omega\left(\mathbf{k}-\frac{1}{2i}\nabla_x\right)-\omega\left(\mathbf{k}+\frac{1}{2i}\nabla_x\right)\right\}n(t,\mathbf{x}:\mathbf{k})$$
$$=\mathrm{St}\,[n](t,\mathbf{x}:\mathbf{k})+\Delta\,\mathrm{St}\,[n](t,\mathbf{x}:\mathbf{k})\,, \tag{55}$$

where $\mathrm{St}[n](t,\mathbf{x}:\mathbf{k})$ is found in (51) and

$$\begin{aligned}
&\Delta\,\mathrm{St}[n](t,\mathbf{x}:\mathbf{k})\\
&\equiv -i\left(\exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\right]-1\right)\delta W(t,\mathbf{x}:\mathbf{k})n(t,\mathbf{x}:\mathbf{k})\\
&\quad+i\left(\exp\left[\frac{i}{2}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\right]-1\right)n(t,\mathbf{x}:\mathbf{k})\delta W^*(t,\mathbf{x}:\mathbf{k})\,. \qquad (56)
\end{aligned}$$

Expanding the exponential factors and using (46), we have

$$\begin{aligned}
&\Delta\,\mathrm{St}[n](t,\mathbf{x}:\mathbf{k})\\
&=C_g\int\prod_{i=1}^{3}d^3L_i\left[\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(2j+1)!2^{2j}}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)^{2j+1}\right.\\
&\quad\times\delta(\mathbf{k}-\mathbf{L}_1-\mathbf{L}_2+\mathbf{L}_3)\mathcal{P}\frac{1}{\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}}\\
&\quad+\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(2j)!2^{2j}}\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)^{2j}\delta(\mathbf{k}-\mathbf{L}_1-\mathbf{L}_2+\mathbf{L}_3)\\
&\quad\left.\times\,2\pi\delta(\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3})\right]M_1(t,\mathbf{x}:\mathbf{L}_i)n(t,\mathbf{x}:\mathbf{k}) \qquad (57)\\
&\simeq -C_g\int\prod_{i=1}^{3}d^3L_i\left(\frac{\partial}{\partial\mathbf{x}_W}\cdot\frac{\partial}{\partial\mathbf{k}_n}-\frac{\partial}{\partial\mathbf{x}_n}\cdot\frac{\partial}{\partial\mathbf{k}_W}\right)\delta(\mathbf{k}-\mathbf{L}_1-\mathbf{L}_2+\mathbf{L}_3)\\
&\quad\times\mathcal{P}\frac{M_1(t,\mathbf{x}:\mathbf{L}_i)}{\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}}n(t,\mathbf{x}:\mathbf{k})\,, \qquad (58)
\end{aligned}$$

where only the lowest gradient term is picked up in the last expression.

Now introduce the entropy density $S(t,\mathbf{x})$ by

$$S(t,\mathbf{x})\equiv\int d^3k s(t,\mathbf{x}:\mathbf{k})\,, \qquad (59)$$

$$s(t,\mathbf{x}:\mathbf{k})\equiv(1+n(t,\mathbf{x}:\mathbf{k}))\ln(1+n(t,\mathbf{x}:\mathbf{k}))-n(t,\mathbf{x}:\mathbf{k})\ln n(t,\mathbf{x}:\mathbf{k})\,. \qquad (60)$$

Its time derivative becomes

$$\dot{S}(t,\mathbf{x})=\int d^3k\dot{n}(t,\mathbf{x}:\mathbf{k})\ln\frac{1+n(t,\mathbf{x}:\mathbf{k})}{n(t,\mathbf{x}:\mathbf{k})}\,. \qquad (61)$$

We substitute (55) with (53) and (57) or (58). Here we explicitly show the result up to the first non-trivial gradient terms, although *inclusion of higher spatial derivative*

terms does not change our final conclusion of the entropy law below. Noting that $\partial n/\partial \mathbf{k} \ln(1+n)/n = \partial s/\partial \mathbf{k}$ and $\partial n/\partial \mathbf{x} \ln(1+n)/n = \partial s/\partial \mathbf{x}$ and performing integral by parts with respect to $\mathbf{k}$, we manipulate

$$\int d^3k \Delta \, \mathrm{St}[n](t,\mathbf{x}:\mathbf{k}) \ln \frac{1+n(t,\mathbf{x}:\mathbf{k})}{n(t,\mathbf{x}:\mathbf{k})}$$
$$= \nabla_x \cdot \left[-C_g \int d^3k \prod_{i=1}^{3} d^3L_i \left\{ \delta(\mathbf{k}-\mathbf{L}_1-\mathbf{L}_2+\mathbf{L}_3) \right.\right.$$
$$\left.\left. \times \mathcal{P} \frac{M_1(t,\mathbf{x}:\mathbf{L}_i)}{\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}} \frac{\partial s(t,\mathbf{x}:\mathbf{k})}{\partial \mathbf{k}} \right\} + O(\nabla_x) \right] . \quad (62)$$

Thus we have the following equation for the entropy density,

$$\dot{S}(t,\mathbf{x}) + \nabla_x \cdot \mathbf{J}_s(t,\mathbf{x}) = \mathrm{Pr}[n](t,\mathbf{x}) , \quad (63)$$

where

$$\mathbf{J}_s(t,\mathbf{x}) \equiv \int d^3k \left[\frac{\partial s(t,\mathbf{x}:\mathbf{k})}{\partial \mathbf{k}} \left\{ \mathbf{v}_g + C_g \int \prod_{i=1}^{3} d^3L_i \delta(\mathbf{k}-\mathbf{L}_1-\mathbf{L}_2+\mathbf{L}_3) \right.\right.$$
$$\left.\left. \times \mathcal{P} \frac{M_1(t,\mathbf{x}:\mathbf{L}_i)}{\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}} \right\} + O(\nabla_x) \right] , \quad (64)$$

$$\mathrm{Pr}[n](t,\mathbf{x}) \equiv \int d^3k \, \mathrm{St}[n](t,\mathbf{x}:\mathbf{k}) \ln \frac{1+n(t,\mathbf{x}:\mathbf{k})}{n(t,\mathbf{x}:\mathbf{k})} . \quad (65)$$

As $\mathrm{St}[n](t,\mathbf{x}:\mathbf{k})$ inside $\mathrm{Pr}[n](t,\mathbf{x})$, given by (51), is the collision integral of the classical Boltzmann equation, it can be proved that $\mathrm{Pr}[n](t,\mathbf{x})$ is the production of the entropy per unit time and unit volume:

$$\mathrm{Pr}[n](t,\mathbf{x}) \geq 0 . \quad (66)$$

In summary, we have the equation of continuity (63) with the production source for entropy $\mathrm{Pr}[n](t,\mathbf{x})$ in (65), which leads to the conclusion that the total entropy,

$$\mathcal{S}(t) \equiv \int d^3x S(t,\mathbf{x}) \quad (67)$$

increases in time,

$$\frac{d}{dt}\mathcal{S}(t) \geq 0 . \quad (68)$$

The source term $\mathrm{Pr}[n](t,\mathbf{x})$ remains the same as that in the classical Boltzmann equation. During this derivation of the entropy law we found a natural definition of the entropy flow $\mathbf{J}_s(t,\mathbf{x})$ given by (64).

The definition of the on-shell part of self-energy in (36) and (37) and the renormalization condition on it in (38) were crucial. The basis of an argument at more profound level to justify the choice of the on-shell part is open.

In spatially inhomogeneous case the momentum is not a good quantum number, which is the very reason for momentum-mixing in the present formalism. It is then a natural attempt to look for a good quantum number in inhomogeneous situation, by which quasiparticle states are labeled. This possibility should be pursued in future study.

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ENTROPY IN INHOMOGENEOUS TFD

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ABSTRACT

Using thermo field dynamics (TFD), we derive the kinetic equation for spatially inhomogeneous time-dependent nonequilibrium system of quantum field. The entropy law, i.e., the increase of entropy in time follows from this kinetic equation which is different from the classical Boltzmann equation due to quantum effects. Our analysis also gives an expression for the entropy current.

1. Inhomogeneous TFD

Recently thermo field dynamics (TFD)[1,2] has been formulated in a way that it accomodates time-dependent (though spatially homogeneous) nonequilibrium phenomena of quantum fields[3,4,2]. There we derived the kinetic equation of Boltzmann type through the renormalization procedure, which leads to the entropy law[3,4,2]. The purpose of this report is to show that also in spatially inhomogeneous case one can derive the kinetic equation and the entropy law from it, using the formulation of inhomogeneous TFD[5,6].

Let us summarize the formulation of inhomogeneous TFD very briefly from Ref. 5. The unperturbed bosonic field of Schrödinger type is given by

$$\varphi(t,\vec{x})^{\mu} = \frac{1}{(2\pi)^{3/2}}\int d^3k\, a_k(t)^{\mu} e^{i\vec{k}\cdot\vec{x}}, \tag{1}$$

where the mechanism of the *momentum-mixing in thermal Bogoliubov transformation* gives rise to spatial inhomogeneity:

$$a_k(t)^{\mu} = B^{-1}(t)^{\mu\nu}_{kq}\xi^{\nu}_{q} e^{-i\int^t ds\,\omega_q(s)}, \qquad \bar{a}_k(t)^{\mu} = e^{i\int^t ds\,\omega_q(s)}\bar{\xi}^{\nu}_{q} B(t)^{\nu\mu}_{qk}, \tag{2}$$

the suffices k, q stand for momentum variables, $\mu, \nu(=1,2)$ are thermal indices, i.e., $a^1 = a, a^2 = \tilde{a}^{\dagger}, \bar{a}^1 = a^{\dagger}, \bar{a}^2 = -\tilde{a}$. The ξ_k-operators define the time-space-independent thermal vacuum $|0\rangle$ as $\xi_k|0\rangle = \tilde{\xi}_k|0\rangle = 0$. We take the following form for the thermal Bogoliubov matrix,

$$B(t)^{\mu\nu}_{kq} = \begin{bmatrix} \delta(\vec{k}-\vec{q}) + N(t)_{kq} & -N(t)_{kq} \\ -\delta(\vec{k}-\vec{q}) & \delta(\vec{k}-\vec{q}) \end{bmatrix}^{\mu\nu}, \tag{3}$$

with $N(t)_{kq} = \langle 0|\tilde{a}(t)^1_q a(t)^1_k|0\rangle$, as this parameterization enables us to use the Feynman diagram method[7,4]. The matrix B is a generalization of the thermal Bogoliubov matrix B in homogeneous case[2,3,4],

$$B[n]^{\mu\nu} = \begin{bmatrix} 1+n & -n \\ -1 & 1 \end{bmatrix}^{\mu\nu}. \tag{4}$$

The unperturbed 2×2 matrix propagator is calculated as

$$\Delta(x,x')^{\mu\nu} \equiv -i\langle 0|T[\varphi(x)^\mu \bar{\varphi}(x')^\nu|0\rangle = \int \frac{d^3k d^3k' d^3q}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}$$
$$\times B^{-1}(t)^{\mu\mu'}_{kq} e^{-i\int_{t'}^{t} ds\, \omega_q(s)} \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu'\nu'} B(t')^{\nu'\nu}_{qk'} e^{-i\vec{k}'\cdot\vec{x}'} \tag{5}$$

with $x = (t, \vec{x})$. The unperturbed Hamiltonian governing the time evolution of φ is

$$\begin{aligned} \hat{H}_Q(t) &= \int d^3x\, \bar{\varphi}(t,\vec{x})^\mu \omega(-i\vec{\nabla})\varphi(t,\vec{x})^\mu - \hat{Q}_R(t) \\ \hat{Q}_R(t) &\equiv \int d^3x d^3y \bar{\varphi}(t,\vec{x})^\mu R(t,\vec{x},\vec{y}) T_0^{\mu\nu} \varphi(t,\vec{y})^\nu, \qquad T_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \end{aligned} \tag{6}$$

where

$$R(t,\vec{x},\vec{y}) = \int \frac{d^3q}{(2\pi)^3} \left[i\frac{\partial}{\partial t} + \{\omega(i\vec{\nabla}_y) - \omega(-i\vec{\nabla}_x)\} \right] n(t, \frac{\vec{x}+\vec{y}}{2} : \vec{q}) e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \tag{7}$$

$$n(t,\vec{x}:\vec{k}) \equiv \int d^3q\, e^{i\vec{q}\cdot\vec{x}} N(t)_{k+\frac{q}{2},k-\frac{q}{2}}. \tag{8}$$

The total Hamiltonian $\hat{H}$ in TFD, $\hat{H}(t) = H(t) - \tilde{H}(t)$, is divided as $\hat{H}(t) = \hat{H}_Q(t) + \hat{H}_I(t)$. Here in the interaction part, $\hat{H}_I(t) = \hat{H}_{\rm int}(t) + \hat{Q}_R(t)$, the second term is the counter term, while $\hat{H}_{\rm int}(t)$ consists of non-linear terms. For definiteness, the following model of the interaction density is taken below,

$$\mathcal{H}_{\rm int}(x) = g\left(\varphi^\dagger(x)\varphi(x)\right)^2. \tag{9}$$

Now the calculation in the interaction representation can be formulated, where the Feynman diagram method is available[4].

2. Self-Energy Calculations

We have the g^2-order loop self-energy without vertex corrections in our model (9) as

$$\Sigma_{loop}(x,x')^{\mu\nu} = -8g^2 \begin{bmatrix} (\Delta(x,x')^{11})^2 \Delta(x',x)^{11} & -(\Delta(x,x')^{12})^2 \Delta(x',x)^{21} \\ -(\Delta(x,x')^{21})^2 \Delta(x',x)^{12} & (\Delta(x,x')^{22})^2 \Delta(x',x)^{22} \end{bmatrix}, \tag{10}$$

whose Fourier transform w. r. t. $\vec{x}$ and $\vec{x}'$ reduces to

$$\Sigma_{loop}(t,t')^{\mu\nu}_{kk'} = C_g \int \prod_{i=1}^{3} (d^3k_i d^3k'_i)\delta(\vec{k}-\vec{k}_1-\vec{k}_2+\vec{k}_3)\delta(\vec{k}'-\vec{k}'_1-\vec{k}'_2+\vec{k}'_3)$$
$$\times B^{-1}[\mathcal{N}(t)]^{\mu\mu'}\begin{bmatrix}1 & 0\\ 0 & M_1(t)\end{bmatrix}^{\mu'\lambda} V(t,t')^{\lambda\lambda'}\begin{bmatrix}M_1(t') & 0\\ 0 & 1\end{bmatrix}^{\lambda'\nu'} B[\mathcal{N}(t')]^{\nu'\nu} \quad (11)$$

where $C_g \equiv (8g^2)/(2\pi)^6$, B in (4) and

$$V(t,t')^{\mu\nu} =$$
$$\begin{bmatrix} -i\theta(t-t')e^{-i\int_{t'}^{t} ds[\omega_{k_1}(s)+\omega_{k_2}(s)-\omega_{k_3}(s)]} & 0 \\ 0 & i\theta(t'-t)e^{-i\int_{t'}^{t} ds[\omega_{k'_1}(s)+\omega_{k'_2}(s)-\omega_{k'_3}(s)]}\end{bmatrix}^{\mu\nu} \quad (12)$$

$$M_1(t)_{k_1k'_1k_2k'_2k'_3k_3} = (I_{k_1k'_1}I_{k_2k'_2} + N(t)_{k_1k'_1}I_{k_2k'_2} + I_{k_1k'_1}N(t)_{k_2k'_2})N(t)_{k'_3k_3}$$
$$-N(t)_{k_1k'_1}N(t)_{k_2k'_2}I_{k'_3k_3}, \quad (13)$$

$$\mathcal{N}(t) = \frac{N(t)_{k_1k'_1}N(t)_{k_2k'_2}(I_{k'_3k_3}+N(t)_{k'_3k_3})}{M_1(t)_{k_1k'_1k_2k'_2k'_3k_3}}, \quad (14)$$

with the notation of $I_{kk'} = \delta(\vec{k}-\vec{k}')$. Note that the above expression is put in the form of $B^{-1}\times$(a diagonal matrix)$\times B$.

To the loop self-energy is added the counter-term, denoted by $\Sigma_Q(x,x')^{\mu\nu}$, to make the total self-energy,

$$\Sigma(t,t')^{\mu\nu}_{kk'} = \Sigma_{loop}(t,t')^{\mu\nu}_{kk'} + \Sigma_Q(t,t')^{\mu\nu}_{kk'}, \quad (15)$$

where

$$\Sigma_Q(t,t')^{\mu\nu}_{kk'} = i\dot{N}(t)_{kk'} + (\omega_{k'}(t)-\omega_k(t))N(t)_{kk'}\delta(t-t')T_0^{\mu\nu}. \quad (16)$$

3. Renormalization and Kinetic Equation

The *on-shell* part of the total self-energy, on which a renormalization condition will be imposed, is found in the following procedure: The diagonal matrix function $V(t,t')^{\mu\nu}$ included in $\Sigma_{loop}(t,t')^{\mu\nu}_{kk'}$ represents the propagation of the ξ-particle. We assume that ω_k is independent of time, then V depends only on $t-t'$. Then consider its Fourier transform $V(k_0)^{\mu\nu} = [1/\{k_0-\omega_{k_1}-\omega_{k_2}+\omega_{k_3}+i\epsilon\tau_3\}]^{\mu\nu}$, where τ_3 is the Pauli matrix. Let us define its on-shell by putting

$$k_0\delta^{\mu\nu} = \left(\omega_{(k+k')/2}-\omega_{(k_1+k'_1)/2}-\omega_{(k_2+k'_2)/2}+\omega_{(k_3+k'_3)/2}\right)\delta^{\mu\nu}$$
$$+\begin{bmatrix}\omega_{k_1}+\omega_{k_2}-\omega_{k_3} & 0\\ 0 & \omega_{k'_1}+\omega_{k'_2}-\omega_{k'_3}\end{bmatrix}^{\mu\nu} \quad (17)$$

(cf. Ref. 8), meaning that the on-shell part of $V(t-t')^{\mu\nu}$ is given by

$$V^{(on)}(t-t')^{\mu\nu} = \delta(t-t')\left[\frac{1}{\omega_{(k+k')/2}-\omega_{(k_1+k'_1)/2}-\omega_{(k_2+k'_2)/2}+\omega_{(k_3+k'_3)/2}+i\epsilon\tau_3}\right]^{\mu\nu}. \quad (18)$$

Thus the on-shell part of the loop self-energy, denoted by $\Sigma^{(on)}_{loop}(t,t')^{\mu\nu}_{kk'}$, follows from substituting this $V^{(on)}(t-t')^{\mu\nu}$ into $V(t,t')^{\mu\nu}$ inside $\Sigma_{loop}(t,t')^{\mu\nu}_{kk'}$,

$$\begin{aligned}\Sigma^{(on)}_{loop}(t,t')^{\mu\nu}_{kk'} &= C_g \int \prod_{i=1}^{3}(d^3k_i d^3k'_i)\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2+\vec{k}_3)\delta(\vec{k}'-\vec{k}'_1-\vec{k}'_2+\vec{k}'_3) \\ &\quad \times M_1(t)\left[B^{-1}[\mathcal{N}(t)]V^{(on)}(t-t')B[\mathcal{N}(t)]\right]^{\mu\nu}. \end{aligned} \tag{19}$$

The on-shell counter term is $\Sigma_Q(t,t')^{\mu\nu}_{kk'}$ itself, therefore the on-shell part of the total self-energy is given by

$$\Sigma^{(on)}(t,t')^{\mu\nu}_{kk'} = \Sigma^{(on)}_{loop}(t,t')^{\mu\nu}_{kk'} + \Sigma_Q(t,t')^{\mu\nu}_{kk'}. \tag{20}$$

Extending the arguments in our previous papers of the spatially homogeneous case[3,4], we require the renormalization condition that the total on-shell self-energy (20) should be diagonal in terms of ξ^μ_p and $\bar{\xi}^\mu_p$, i.e., $[B(t)_{pk}\Sigma^{(on)}(t,t')_{kk'}B^{-1}(t)_{k'p'}]^{\mu\nu} =$ (diagonal with respect to the thermal indices). Although this is a 2×2-matrix relation, it turns out that the condition is satisfied by the single equation, which after some manipulations becomes in $(\vec{x},\vec{k})$-representation

$$\begin{aligned}\dot{n}(t,\vec{x}:\vec{k}) - i\{\omega(\vec{k}-\frac{1}{2i}\vec{\nabla}_x) - \omega(\vec{k}+\frac{1}{2i}\vec{\nabla}_x)\}n(t,\vec{x}:\vec{k}) &= i\delta W_0(t,\vec{x}:\vec{k}) \\ + \left[-i\exp\{-\frac{i}{2}(\frac{\partial}{\partial\vec{x}_W}\cdot\frac{\partial}{\partial\vec{k}_n} - \frac{\partial}{\partial\vec{x}_n}\cdot\frac{\partial}{\partial\vec{k}_W})\}\,\delta W(t,\vec{x}:\vec{k})n(t,\vec{x}:\vec{k}) + \text{h.c.}\right], \end{aligned} \tag{21}$$

where $\partial/\partial\vec{x}_W$ and $\partial/\partial\vec{x}_n$, e.g., operate only on δW and n, respectively and

$$\delta W(t,\vec{x}:\vec{k}) = C_g \int \prod_{i=1}^{3} d^3L_i \frac{\delta(\vec{k}-\vec{L}_1-\vec{L}_2+\vec{L}_3)M_1(t,\vec{x}:\vec{L}_i)}{\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}+i\epsilon} \tag{22}$$

$$\begin{aligned}\delta W_0(t,\vec{x}:\vec{k}) &= -2\pi i C_g \int \prod_{i=1}^{3} d^3L_i\,\delta(\vec{k}-\vec{L}_1-\vec{L}_2+\vec{L}_3)\,M_0(t,\vec{x}:\vec{L}_i) \\ &\quad \times\delta(\omega_{L_1+L_2-L_3}-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}) \end{aligned} \tag{23}$$

with $M_0(t,\vec{x}:\vec{L}_i) = n_1n_2(1+n_3)$ and $M_1(t,\vec{x}:\vec{L}_i) = (1+n_1+n_2)n_3 - n_1n_2$ when the short-handed notation $n_i = n(t,\vec{x}:\vec{L}_i)$ is used. Equation (21) is our kinetic equation.

4. Entropy Law from Kinetic Equation

Our kinetic equation has a different form from the classical Boltzmann equation, modified by quantum effects. Equation (21) in spatially homogeneous limit becomes the Boltzmann equation without $\vec{x}$-dependence[3,4] and the entropy law is evident then. Our new result to be shown below is that *the entropy law is derived* from (21) with spatial inhomogeneity.

To see the differences from the Boltzmann equation, we rewrite (21),

$$\dot{n}(t,\vec{x}:\vec{k}) + \{\vec{v}_g + \Delta\vec{v}_g\} \cdot \vec{\nabla}_x n(t,\vec{x}:\vec{k}) = \mathrm{St}[n](t,\vec{x}:\vec{k}) + \Delta\mathrm{St}[n](t,\vec{x}:\vec{k}). \tag{24}$$

Here $\vec{v}_g = \partial\omega/\partial\vec{k}$ is the group velocity, the collision integral is given by

$$\mathrm{St}[n] \equiv \int \prod_{i=1}^{3} d^3L_i\, w \left\{ n_1 n_2 (1+n_3)(1+n(t,\vec{x}:\vec{k})) - n(t,\vec{x}:\vec{k}) n_3 (1+n_1)(1+n_2) \right\}, \tag{25}$$

with $w = 2\pi C_g \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3)\delta(\omega_k - \omega_{L_1} - \omega_{L_2} + \omega_{L_3})$, and the terms $\Delta\vec{v}_g$ and $\Delta\mathrm{St}[n]$ are their respective corrections of higher derivatives w. r. t. $\vec{x}$ and represent deviations from the Boltzmann equation. We only write down the first leading correction term in $\Delta\mathrm{St}[n]$, because the expression of $\Delta\vec{v}_g$ is irrelevant to our derivation of the entropy law below and inclusion of general higher-derivative terms in $\Delta\mathrm{St}[n]$ do not alter our conclusion on the entropy law:

$$\begin{aligned}\Delta\mathrm{St}[n](t,\vec{x}:\vec{k}) = -C_g \int \prod_{i=1}^{3} d^3L_i \Bigg[\frac{\partial}{\partial\vec{x}} \Bigg\{ \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \\ \times \mathcal{P} \frac{M_1(t,\vec{x}:\vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \Bigg\} \cdot \frac{\partial n(t,\vec{x}:\vec{k})}{\partial\vec{k}} - \left\{ \frac{\partial}{\partial\vec{x}} \leftrightarrow \frac{\partial}{\partial\vec{k}} \right\} + O(\vec{\nabla}_x) \Bigg]\end{aligned} \tag{26}$$

Now introduce the entropy density $S(t,\vec{x})$ by

$$S(t,\vec{x}) \equiv \int d^3k\, s(t,\vec{x}:\vec{k}) \tag{27}$$

$$s(t,\vec{x}:\vec{k}) \equiv (1+n(t,\vec{x}:\vec{k}))\ln(1+n(t,\vec{x}:\vec{k})) - n(t,\vec{x}:\vec{k})\ln n(t,\vec{x}:\vec{k}). \tag{28}$$

Its time derivative becomes

$$\dot{S}(t,\vec{x}) = \int d^3k\, \dot{n}(t,\vec{x}:\vec{k}) \ln \frac{1+n(t,\vec{x}:\vec{k})}{n(t,\vec{x}:\vec{k})}. \tag{29}$$

Noting that $\partial n/\partial\vec{k} \ln\frac{1+n}{n} = \partial s/\partial\vec{k}$ and $\partial n/\partial\vec{x} \ln\frac{1+n}{n} = \partial s/\partial\vec{x}$ and performing integral in part with respect to $\vec{k}$, we obtain the formula

$$\begin{aligned}\int d^3k\, \Delta\mathrm{St}[n](t,\vec{x}:\vec{k}) \ln \frac{1+n(t,\vec{x}:\vec{k})}{n(t,\vec{x}:\vec{k})} = \vec{\nabla}_x \cdot \Bigg[-C_g \int d^3k \prod_{i=1}^{3} d^3L_i \\ \times \mathcal{P} \frac{\delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) M_1(t,\vec{x}:\vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \frac{\partial s(t,\vec{x}:\vec{k})}{\partial\vec{k}} \Bigg\} + O(\vec{\nabla}_x) \Bigg].\end{aligned} \tag{30}$$

Substitute (24) into (29) and apply (30), then we derive the equation for the entropy density,

$$\dot{S}(t,\vec{x}) + \vec{\nabla}_x \cdot \vec{J}_s(t,\vec{x}) = \mathrm{Pr}[n](t,\vec{x}) \equiv \int d^3k\, \mathrm{St}[n](t,\vec{x}:\vec{k}) \ln \frac{1+n(t,\vec{x}:\vec{k})}{n(t,\vec{x}:\vec{k})} \tag{31}$$

where the entropy current $\vec{J}_s(t,\vec{x})$ is given by

$$\vec{J}_s(t,\vec{x}) \equiv \int d^3k \left[\frac{\partial s(t,\vec{x}:\vec{k})}{\partial \vec{k}} \left\{ \vec{v}_g + C_g \int \prod_{i=1}^{3} d^3L_i\, \delta(\vec{k}-\vec{L}_1-\vec{L}_2+\vec{L}_3) \right.\right.$$
$$\left.\left. \times \mathcal{P} \frac{M_1(t,\vec{x}:\vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \right\} + O(\vec{\nabla}_x) \right]. \tag{32}$$

The term $\mathrm{Pr}[n](t,\vec{x})$ is interpreted as the production of the entropy par unit time and unit volume. As $\mathrm{St}[n](t,\vec{x}:\vec{k})$ inside $\mathrm{Pr}[n](t,\vec{x})$ is the collision integral in the classical Boltzmann equation, it can be proved that

$$\mathrm{Pr}[n](t,\vec{x}) \geq 0. \tag{33}$$

In summary, we have the equation of continuity with production source for entropy, which leads to the conclusion that the total entropy $\mathcal{S}(t)$ increases in time:

$$\mathcal{S}(t) \equiv \int d^3x\, S(t,\vec{x}), \quad \frac{d}{dt}\mathcal{S}(t) \geq 0. \tag{34}$$

This conclusion remains true even when we include all the higher derivative terms in $\Delta\mathrm{St}[n]$.

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Quasi Particle Modes in Spatially Inhomogeneous Thermo Field Dynamics

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ABSTRACT

A new formulation to deal with spatially inhomogeneous thermal system of fields is proposed within the framework of time-dependent thermo field dynamics. A key for this is to introduce quasi particle modes which diagonalize thermal Bogoliubov matrix except for thermal indices. We give a systematic method of perturbative calculations of Green's functions, and a prescription to formulate a self-consistent renormalization condition leading to a kinetic equation.

1. Introduction

In our previous papers[1,2], we proposed a way to extend thermo field dynamics (TFD)[3,4] to describe spatially inhomogeneous thermal systems of quantum fields. A key trick in such a generalization is to introduce a momentum mixing thermal Bogoliubov transformation which is given by

$$B(t)^{\mu\nu}_{kl} = \begin{bmatrix} \delta_{kl} + \sigma N(t)_{kl} & -N(t)_{kl} \\ -\sigma\delta_{kl} & \delta_{kl} \end{bmatrix}^{\mu\nu} \tag{1}$$

where $\mu, \nu = 1, 2$ are thermal indices, k, l stand for momentum vectors $\boldsymbol{k}, \boldsymbol{l}$ and $\sigma = 1(-1)$ for boson (fermion) operators. The parameter $N(t)_{kl}$ above is related to the number density of a particle with momentum $\boldsymbol{k}$ at $\boldsymbol{x}$ and t, denoted by $n(t, \boldsymbol{x} : \boldsymbol{k})$, as

$$N(t)_{kl} = \frac{1}{V}\int d^3x\, e^{-i(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{x}} n(t, \boldsymbol{x} : \frac{\boldsymbol{k}+\boldsymbol{l}}{2}). \tag{2}$$

This momentum mixing, however, causes much complexity to the formulation. A more important factor to demand a reformulation comes from the fact that the mixing of momentum indices stems from the breakdown of the space translational invariance of the system which we are dealing with: Evidently the momentum is not

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a good quantum number in spatially inhomogeneous system from the beginning. It is natural to look for an adequate quantum number which characterizes quasi particle mode in each spatially inhomogeneous system, and to rewrite the formulation in terms of the new quantum number. In this paper we give a TFD formulation towards this direction.

The formation of the quasi particle modes is controlled by both the dynamics and the thermal (space- and time-dependent) situation under consideration. Their characterization requires a self-consistent treatment, which will be a renormalization condition in quantum field theory in our case.

Another important aspect in our problem is that the characterization of the quasi particle modes inevitably depends on time. This new type of temporal dependence appears in our formulation, as will be seen.

This paper is organized as follows: In Sec. 2 the concept of quasi particle modes in spatially inhomogeneous thermal systems is introduced through the diagonalization procedure of $N_{kl}(t)$. The wave functions associated with the quasi particle modes are given. The creation and annihilation operators which define the representation space of operators are related to those of the quasi particle modes by means of the thermal Bogoliubov transformation in Sec. 3. We then derive the equations of motion for the operators of the quasi particle modes. In Sec. 4 the field operators of Schrödinger type are constructed from the wave functions in Sec. 2 and the operators introduced in Sec. 3. Section 5 is devoted to the formulation of interaction picture. We sketch in Sec. 6 a prescription of a self-consistent renormalization condition, i.e., a diagonalization condition on full Green's function, which should lead to a kinetic equation for quasi particle number density with determinations of quasi particle modes at each time. The results are summarized in Sec. 7.

2. Quasi Particle Mode

We start with the diagonalization of the matrix $N_{kl}(t)$ in (1) which is hermitian. There must exist a unitary matrix $u_{ki}(t)$ which diagonalizes $N_{kl}(t)$ at each time t, i.e.,

$$\sum_{k,l} u_{ik}^{-1}(t) N_{kl}(t) u_{lj}(t) = n_i(t)\delta_{ij}. \tag{3}$$

Here and in the following, we treat momenta as discrete variables in order to simplify our presentations, it is trivial to rewrite them as continuous ones.

Let us introduce the concept of a *quasi particle mode* through this diagonalization, since only after diagonalization one can identify modes whose number can be counted. Thus we call the mode characterized by i ($N_{kl}(t)$ being diagonalized as in (3)) a *quasi particle mode*. The diagonal element $n_i(t)$ can be regarded as the number density of the quasi particles in mode i at t.

Next we define a wave function $f_i(\boldsymbol{x}, t)$ associated with the quasi particle mode i by

$$f_i(\boldsymbol{x}, t) \equiv \frac{1}{\sqrt{V}} \sum_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} u_{ki}(t). \tag{4}$$

Because of the unitarity of the matrix $u_{ki}(t)$, the wave functions $\{f_i(\boldsymbol{x},t)\}$, thus defined, makes a complete and orthonormal set at each time t:

$$\sum_i f_i(\boldsymbol{x},t)f_i^*(\boldsymbol{y},t) = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{5}$$

$$\int d^3x\, f_i^*(\boldsymbol{x},t)f_j(\boldsymbol{x},t) = \delta_{ij}. \tag{6}$$

For later convenience we write the "equation of motion" for $f_i(\boldsymbol{x},t)$

$$i\dot{f}_i(\boldsymbol{x},t) = \sum_j f_j(\boldsymbol{x},t)\Omega_{ji}(t) \tag{7}$$

with

$$\Omega_{ji}(t) \equiv i\sum_k u_{jk}^{-1}(t)\dot{u}_{ki}(t) \tag{8}$$

where the dot means a derivative with respect to time t.

3. Representation of Particle and Quasi Particle Operators

How to choose the representation space for field operators is a crucial question in quantum field theory. Here, as in the previous papers[4], we realize all the operators in a *single* Fock space: This Fock space is associated with a set of stable free particles with a momentum index, described by the space-time-independent annihilation operators $(\xi_k, \tilde{\xi}_k)$ and the creation operators $(\xi_k^\dagger, \tilde{\xi}_k^\dagger)$. They satisfy the commutation relation,

$$\left[\xi_k^\mu, \tilde{\xi}_l^\nu\right]_\sigma = \delta^{\mu\nu}\delta_{kl} \tag{9}$$

where we use the thermal doublet notation as

$$\begin{aligned} \xi_k^1 &= \xi_k & \xi_k^2 &= \tilde{\xi}_k^\dagger \\ \tilde{\xi}_k^1 &= \xi_k^\dagger & \tilde{\xi}_k^2 &= -\sigma\xi_k^\dagger. \end{aligned} \tag{10}$$

The σ-commutator in (9) is defined by

$$[A,B] = AB - \sigma BA. \tag{11}$$

Denoting the vacuum states of this Fock space, called as thermal vacua and denoted by $|0\rangle$ and $\langle 0|$, we have

$$\begin{aligned} \xi_k|0\rangle &= \tilde{\xi}_k|0\rangle = 0 \\ \langle 0|\xi_k^\dagger &= \langle 0|\tilde{\xi}_k^\dagger = 0. \end{aligned} \tag{12}$$

We shall call the particles described by ξ_k^μ's *representation particles* as they define the representaion space.

Now consider the operators $\alpha_i(t)^\mu$ and $\bar{\alpha}_i(t)^\mu$ which describe the quasi particle in the mode i at time t. From the definition of the quasi particle mode i given

in the previous section, one postulates that these operators $\alpha_i(t)^\mu$'s (i) satisfy the commutation relation

$$[\alpha_k(t)^\mu, \bar{\alpha}_l(t)^\nu]_\sigma = \delta^{\mu\nu}\delta_{kl} \tag{13}$$

and (ii) give the relation

$$n_i(t) = \langle 0|\bar{\alpha}_i(t)^1\alpha_i(t)^1|0\rangle. \tag{14}$$

Recall here that $n_i(t)$ is the number density of the quasi particle with i, which is given by the eigenvalues of the matrix $N_{kl}(t)$ (see eq.(3)).

We can realize the operators $\alpha_i(t)^\mu$ and $\bar{\alpha}_i(t)^\mu$ satisfying the above postulates by a linear combination of the representation particle operators ξ_k^μ and $\bar{\xi}_k^\mu$ in the following way:

$$\begin{aligned}
\alpha_i(t)^\mu &= \mathcal{B}^{-1}[n_i(t)]^{\mu\nu}\, e^{-i\int^t \omega_i(s)ds} \sum_k u_{ik}^{-1}(t)\xi_k^\nu \\
\bar{\alpha}_i(t)^\mu &= \sum_k \bar{\xi}_k^\nu u_{ki}(t)\, e^{i\int^t \omega_i(s)ds}\, \mathcal{B}[n_i(t)]^{\nu\mu}
\end{aligned} \tag{15}$$

where $\omega_i(t)$ is the time-dependent energy of the quasi particle in the mode i which is determined later by a renormalization procedure, and $\mathcal{B}[n_i(t)]$ is a 2×2 matrix as

$$\mathcal{B}[n_i(t)] = \begin{bmatrix} 1+\sigma n_i(t) & -\sigma n_i(t) \\ -\sigma & 1 \end{bmatrix}. \tag{16}$$

This matrix has the same form as the one for the thermal Bogoliubov transformation we used in spatially homogeneous time-dependent situations[5,6] except that $n_k(t)$ is replaced by $n_i(t)$. The difference from spatially homogeneous cases is that the time-dependence of $\alpha_i(t)$ comes not only from the thermal Bogoliubov matrix $\mathcal{B}[n_i(t)]$ but also from the diagonalizing matrix $u_{ki}(t)$. Thus the equations of motion for $\alpha_i(t)^\mu$'s read

$$\begin{aligned}
i\dot{\alpha}_i(t)^\mu &= \omega_i(t)\alpha_i(t)^\mu - i\sigma\dot{n}_i(t)T_0^{\mu\nu}\alpha_i(t)^\nu \\
&\quad + \sum_j \Omega_{ij}(t)\, e^{-i\int^t\{\omega_i(s)-\omega_j(s)\}ds}\,[\delta^{\mu\nu} - \sigma(n_i(t)-n_j(t))T_0^{\mu\nu})]\,\alpha_j(t)^\nu
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
i\dot{\bar{\alpha}}_i(t)^\mu &= -\omega_i(t)\bar{\alpha}_i(t)^\mu + i\sigma\dot{n}_i(t)\bar{\alpha}_i(t)^\nu T_0^{\nu\mu} \\
&\quad - \sum_j \bar{\alpha}_j(t)^\nu\Omega_{ji}(t)\, e^{-i\int^t\{\omega_j(s)-\omega_i(s)\}ds}\,[\delta^{\mu\nu} - \sigma(n_j(t)-n_i(t))T_0^{\nu\mu})]
\end{aligned} \tag{18}$$

with

$$T_0 = \begin{bmatrix} 1 & -\sigma \\ \sigma & -1 \end{bmatrix}. \tag{19}$$

Note that the terms including $\Omega_{ij}(t)$ in (17) and (18) are due to the time-dependence of the diagonalizing matrix $u_{ki}(t)$.

4. Construction of Field Operators

We now turn to constructing field operators. For simplicity, only Schrödinger type fields (both bosonic and ferminionic) are considered. Combining the operators $\alpha_i(t)$ introduced in the last section with the complete set of wave functions $f_i(\boldsymbol{x},t)$ in Sec. 3, we can construct the following field operators $\phi(\boldsymbol{x},t)^\mu$ and $\bar{\phi}(\boldsymbol{x},t)^\mu$ as

$$\begin{aligned}\phi(\boldsymbol{x},t)^\mu &= \sum_i \alpha_i(t)^\mu f_i(\boldsymbol{x},t)\\ \bar{\phi}(\boldsymbol{x},t)^\mu &= \sum_i \bar{\alpha}_i(t)^\mu f_i^*(\boldsymbol{x},t).\end{aligned} \tag{20}$$

It is easy to check that these field operators satisfy the equal-time commutation relations,

$$\left[\phi(\boldsymbol{x},t)^\mu, \bar{\phi}(\boldsymbol{y},t)^\nu\right]_\sigma = \delta^{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{21}$$

owing to the commutation relation (13) of $\alpha_i(t)^\mu$'s and the completeness (5) of the wave functions $f_i(\boldsymbol{x},t)$.

The field equation for $\phi(\boldsymbol{x},t)$ is also derived from the equation of motion for $\alpha_i(t)^\mu$ and $f_i(\boldsymbol{x},t)$, i.e., eqs.(17) and (18), and eq.(7):

$$\begin{aligned}i\dot{\phi}(\boldsymbol{x},t)^\mu &= \int d^3y\, K(\boldsymbol{x},\boldsymbol{y},t)^{\mu\nu}\phi(\boldsymbol{y},t)^\nu\\ i\dot{\bar{\phi}}(\boldsymbol{x},t)^\mu &= -\int d^3y\, \bar{\phi}(\boldsymbol{y},t)^\nu K(\boldsymbol{y},\boldsymbol{x},t)^{\nu\mu}\end{aligned} \tag{22}$$

where

$$K(\boldsymbol{x},\boldsymbol{y},t)^{\mu\nu} = \sum_{i,j} f_i(\boldsymbol{x},t)K_{ij}(t)^{\mu\nu}f_j^*(\boldsymbol{y},t) \tag{23}$$

with

$$\begin{aligned}K_{ij}(t)^{\mu\nu} &= \delta^{\mu\nu}\left[\omega_i(t)\delta_{ij} + \left(e^{-i\int^t\{\omega_i(s)-\omega_j(s)\}ds} - 1\right)\Omega_{ij}(t)\right]\\ &\quad -\sigma T_0^{\mu\nu}\left[i\dot{n}_i\delta_{ij} + e^{-i\int^t\{\omega_i(s)-\omega_j(s)\}ds}\,\Omega_{ij}(t)\left(n_i(t)-n_j(t)\right)\right].\end{aligned} \tag{24}$$

The field equations (22) follow from the Lagrangian,

$$\begin{aligned}\hat{L}_Q(t) &= \int d^3x\, i\bar{\phi}(\boldsymbol{x},t)^\mu\dot{\phi}(\boldsymbol{x},t)^\mu\\ &\quad -\int d^3x d^3y\, \bar{\phi}(\boldsymbol{x},t)^\mu K(t,\boldsymbol{x},\boldsymbol{y})^{\mu\nu}\phi(\boldsymbol{y},t)^\nu.\end{aligned} \tag{25}$$

We note that the canonical field conjugate to $\phi(\boldsymbol{x},t)^\mu$ derived from this Lagrangian is $i\bar{\phi}(\boldsymbol{x},t)^\mu$ and hence eq.(21) guarantees the canonical commutation relations.

According to the canonical formalism, we have the Hamiltonian $\hat{H}_Q$ as

$$\hat{H}_Q(t) = \int d^3x d^3y\, \bar{\phi}(\boldsymbol{x},t)^\mu K(t,\boldsymbol{x},\boldsymbol{y})^{\mu\nu}\phi(\boldsymbol{y},t)^\nu. \tag{26}$$

In terms of $\hat{H}_Q$, the equations of motion (22) are rewritten as

$$i\dot{\phi}(\boldsymbol{x},t)^{\mu} = \left[\phi(\boldsymbol{x},t)^{\mu}, \hat{H}_Q(t)\right]$$
$$i\dot{\bar{\phi}}(\boldsymbol{x},t)^{\mu} = \left[\bar{\phi}(\boldsymbol{x},t)^{\mu}, \hat{H}_Q(t)\right]. \tag{27}$$

5. Interaction Picture

By the construction of $\phi(\boldsymbol{x},t)$'s, their time development controlled by eqs. (27) can be understood as "free" motion of the quasi particles.

For perturbative calculations, we have to devide the Hamiltonian of the given system into an unperturbed part and a perturbation. But the way of the division is not trivial. In usual quantum field theory (at zero temperature), we take the part which takes care of the free motion of asymptotic particles as unperturbed one and determine the parameters specifying the asymptotic particles like mass and charge self-consistently by a renormalization procedure. However, in thermal situations, one can not define the asymptotic particles because particles do not become independent of each other asymptotically. In the present case, it should be the quasi particles introduced above that play a role of the asymptotic particles in zero temperature quantum field theory. Therefore it is reasonable to consider $\hat{H}_Q$ to be the unperturbed part of the total Hamiltonian.

Thus our scheme of perturbative calculation is developed in the following way. For a given system, we have its Lagrangian L from which we obtain the Hamiltonian of the system H according to the canonical formalism. Then the TFD Hamiltonian is given by[4]

$$\hat{H} = H - \tilde{H}. \tag{28}$$

On the other hand, we regard $\hat{H}_Q$ given by (26) as the unperturbed Hamiltonian, so that the total Hamiltonian $\hat{H}$ is divided into two parts, $\hat{H}_Q$ and $\hat{H}$-$\hat{H}_Q$:

$$\begin{aligned} \hat{H} &= \hat{H}_Q + \hat{H}_I \\ \hat{H}_I &= \hat{H} - \hat{H}_Q. \end{aligned} \tag{29}$$

We take the latter $\hat{H}_I$ to be the perturbation. This division of $\hat{H}$ leads us to look upon $\phi(\boldsymbol{x},t)$ and $\bar{\phi}(\boldsymbol{x},t)$ whose time evolution is generated by the unperturbed Hamiltonian $\hat{H}_Q$ as the field operators in the interaction picture. Hence the unperturbed Green's function of the field $\phi(\boldsymbol{x},t)$ is calculated by using eqs. (15) and (20) with (12) as

$$\begin{aligned} \Delta(\boldsymbol{x},t,\boldsymbol{x}',t')^{\mu\nu} &\equiv -i\langle 0|\mathrm{T}\left(\phi(\boldsymbol{x},t)^{\mu}\bar{\phi}(\boldsymbol{x}',t')^{\nu}\right)|0\rangle \\ &= \sum_i\sum_j f_i(\boldsymbol{x},t)\Delta_{ij}(t,t')^{\mu\nu}f_j^*(\boldsymbol{x}',t') \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Delta_{ij}(t,t')^{\mu\nu} &= \mathcal{B}^{-1}[n_i(t)]^{\mu\mu'} \\ &\quad\times\begin{bmatrix} -i\theta(t-t')g_{ij}(t,t') & 0 \\ 0 & i\theta(t'-t)g_{ij}(t,t') \end{bmatrix}^{\mu'\nu'} \mathcal{B}[n_j(t')]^{\nu'\nu} \end{aligned} \tag{31}$$

with

$$g_{ij}(t,t') = e^{-i\int^t \omega_i(s)ds} \sum_k u_{ik}^{-1}(t) u_{kj}(t')\, e^{i\int^{t'} \omega_j(s')ds'}. \tag{32}$$

It is remarkable that, as in the spatially homogeneous case[4,5,6], the above Green's function has the form of a diagonal matrix with respect to thermal indices, sandwiched between the thermal Bogoliubov matrices. In the formalism with the momentum mixing Bogoliubov transformation[2], the dynamical changes and the thermal ones twine with each other even in the unperturbed propagation. However, the above structure of the Green's function means that the thermal changes are taken care of only at each vertex and the field propagates with no thermal mixing, when it is used for the lines in Feynman diagrams. We should also note that at equal time

$$g_{ij}(t,t) = \delta_{ij}. \tag{33}$$

With the unperturbed one-body propagator in (30) and the perturbation Hamiltonian $\hat{H}_I$ in(29), we can formulate the perturbation calculation in terms of the Feynman diagrams (for the validity of the Feynman diagram method, see Section 7.5.1 of ref. 4). A crucial step in this formulation is the renormalization procedure which leads to the self-consistent equation for the parameters specifying the quasi particle modes.

6. Self-consistent Renormalization Scheme

To sketch out the renormalization procedure, we consider the following Lagrangian

$$L = \int d^3x\, \phi^\dagger(\boldsymbol{x},t) \left(i\frac{\partial}{\partial t} - \omega(-i\nabla) \right) \phi(\boldsymbol{x},t) - W[\phi^\dagger(\boldsymbol{x},t), \phi(\boldsymbol{x},t)], \tag{34}$$

where W is a polynomial function, representing a nonlinear interaction. The TFD Hamiltonian follows from this Lagrangian as

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_{int}(t) \tag{35}$$

with

$$\hat{H}_0(t) = \int d^3x\, \bar{\phi}(\boldsymbol{x},t)^\mu\, \omega(-i\nabla)\, \phi(\boldsymbol{x},t)^\mu \tag{36}$$

and

$$\hat{H}_{int}(t) = W[\bar{\phi}(\boldsymbol{x},t)^1, \phi(\boldsymbol{x},t)^1] - W[\phi(\boldsymbol{x},t)^2, -\sigma\bar{\phi}(\boldsymbol{x},t)^2] \tag{37}$$

where $\omega(\boldsymbol{k})$ denotes a bare energy. Hence the perturbation $\hat{H}_I$ consists of two terms;

$$\hat{H}_I(t) = \hat{H}_c(t) + \hat{H}_{int}(t) \tag{38}$$

where the renormalization counter term $\hat{H}_c(t)$ is

$$\hat{H}_c(t) = \hat{H}_0(t) - \hat{H}_Q(t). \tag{39}$$

Putting (20) into (26) and (36), we have an expression of $\hat{H}_c(t)$ in the interaction representation as

$$\hat{H}_c(t) = \sum_{i,j} \bar{\alpha}_i(t)^\mu \left[\omega_{ij}(t)\delta^{\mu\nu} - K_{ij}(t)^{\mu\nu}\right] \alpha_j(t)^\nu \tag{40}$$

where

$$\omega_{ij}(t) = \sum_k u_{ik}^{-1}(t)\omega(k)u_{kj}(t) \tag{41}$$

and $K_{ij}(t)^{\mu\nu}$ is given by (24).

It should be noted here that the factor $\omega_{ij}(t)\delta^{\mu\nu} - K_{ij}(t)^{\mu\nu}$ in eq.(40) is a linear combination of $\delta^{\mu\nu}$ and $T_0^{\mu\nu}$.

Now our formalism goes on in a similar way to the one we have developed in the spatially homogeneous cases[4-10]. One calculates the full Green's function, employing the interaction picture,

$$G_{ij}(t,t')^{\mu\nu} = -i\langle 0|\mathbf{T}\left(\alpha_i(t)^\mu \bar{\alpha}_j(t')^\nu \hat{S}\right)|0\rangle \tag{42}$$

where

$$\hat{S} = \mathbf{T}\left[\exp(-i\int_{-\infty}^{\infty} \hat{H}_I(s)ds)\right]. \tag{43}$$

We can argue a general form for this full Green's function, as in the previous papers[4,9,11], with additional indices i, j in this case: Simple algebraic manipulations provide the result of

$$G_{ij}(t,t')^{\mu\nu} = \left[\mathcal{B}^{-1}[n_i(t)]^{\mu\mu'} \begin{bmatrix} \mathcal{G}_{ij}^{(r)}(t,t') & \mathcal{G}_{ij}(t,t') \\ 0 & \mathcal{G}_{ij}^{(a)}(t,t') \end{bmatrix} \mathcal{B}[n_j(t')]\right]^{\mu\nu}. \tag{44}$$

With the form of the full Green's function in (44), we may impose a diagonalization (with respect to thermal indices) condition on it. Extending a self-consistent renormalization condition to thermal Green's function (2×2-matrix Green's function) naturally leads us to such a diagonalization condition[7,8,9,10]. Let us apply the method proposed in a recent paper[10], for example, to the present case, namely we require

$$\mathcal{G}_{ij}(t,t) = 0. \tag{45}$$

This condition should give us simultaneous equations to determine the time-evolution of the quasi particle number density $n_i(t)$ and the unitary matrix $u_{ki}(t)$ once the initial condition at $t = t_I$, in terms of $N_{kl}(t_I)$, is given. Detailed account of the diagonalization condition in (45) and the resultant equations will be reported in a future publication.

7. Summary

In this paper we have proposed a new formulation to deal with spatially inhomogeneous thermal system of quantum fields. The key concept is the introduction of quasi particle modes by diagonalizing momentum-mixing thermal Bogoliubov matrix except for thermal indices.

The operators representing these quasi particle modes are written explicitly, and their dynamical properties are investigated. Filed operators are also constructed from these quasi particle operators.

Using the quasi particle operators or the field operator associated with them as unperturbative representation, we have formulated the interaction picture as a basis for perturbative calculations.

Finally the self-consistent renormalization condition which is the diagonalization condition of the full Green's function is illustrated. This condition is expected to provide a kinetic equation for quasi particle number density together with equations determining quasi particle modes at each time. We elaborate on this point in a future study.

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Diffusion in Inhomogeneous TFD

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ABSTRACT

Solutions of the kinetic equation, derived from the renormalization condition on the on-shell self-energy in inhomogeneous TFD previously, are applied to the number density parameter in the expressions of particle and energy flows. Under certain approximations or coarse-graining diffusion type equations for particle and energy densities are derived. A systematic way of calculating higher order corrections is developed.

1. Introduction

Thermo field dynamics (TFD) has been extended from quantum field theory of thermal equilibrium[1] solely to one of nonequilibrium in the last decade.[2]

First of all it is pointed out that the basic structures of equilibrium and nonequilibrium (inhomogeneous as well as homogeneous) TFD are common[2]: (A) Every degree of freedom is doubled according to the tilde conjugation rules, $A \Longrightarrow (A, \tilde{A})$. (B) The (dynamical) observable A is given by

$$\langle A \rangle = \langle 0|A|0\rangle \tag{1}$$

where $|0\rangle$ is the thermal vacuum. (C) The total Hamiltonian $\hat{H}$ is

$$\hat{H} = H - \tilde{H}. \tag{2}$$

The important fact is that the spectrum of $\hat{H}$ is *unbounded from below*, which is contrary to the usual quantum field theory. This implies the degeneracy in thermal vacua, from which arises the degree of the thermal Bogoliubov transformation in TFD. This thermal degeneracy is connected with the first law of thermodynamics and is the origin of dissipation.[3,4]

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The choice of the representation space is crucial in quantum field theory. Our strategy to deal with time-space-dependent thermal situations is to simply use a thermal Bogoliubov matrix, *staying in a single representation space.* As is shown previously,[5-7] we introduce a momentum mixing thermal Bogoliubov matrix with time-dependent parameter.

In the formulation of time-dependent TFD (though spatially homogeneous) we obtained the kinetic equation of Boltzmann type through the renormalization procedure[8,9,2] (or the diagonalization procedure[10,11]), leading to the entropy law. This work has been extended to inhomogeneous case as well.[12-14]

In this report we discuss particle and energy flows, using the kinetic equation derived in inhomogeneous TFD. Our main conclusion is that diffusion type equations for particle and energy densities come out under certain approximations (or coarse-graining) with expressions for diffusion constants. In addition, it will be seen that instead of Noether currents we naturally define new currents, which simplify the equations for currents. We develop a systematic way to include higher order terms using the concept of the dynamical map.[1,2]

2. Formulation of the Inhomogeneous Time-Dependent TFD

Let us briefly summarize the formulation of the inhomogeneous time-dependent TFD.[6] For definiteness we take throughout this paper the following self-interacting model of Schrödinger type bosonic field $\psi(x)$,

$$\mathcal{L} = \psi^\dagger(x)\left[i\frac{\partial}{\partial t} - \omega_0(-i\nabla)\right]\psi(x) - g\left(\psi^\dagger(x)\psi(x)\right)^2 \tag{3}$$

where $x = (t, \boldsymbol{x})$.

The unperturbed field operators, denoted by $\varphi(x)^\mu$ and $\bar{\varphi}(x)^\mu$ $(\mu, \nu = 1, 2)$ in TFD notation, i.e, $\varphi^1 = \varphi, \varphi^2 = \tilde{\varphi}^\dagger, \bar{\varphi}^1 = \varphi^\dagger, \bar{\varphi}^2 = -\tilde{\varphi}$, are given by

$$\begin{aligned}
\varphi(x)^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k\, a_k(t)^\mu e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3k\, B^{-1}(t)^{\mu\nu}_{kq}\xi^\nu_q e^{-i\int^t ds\,\omega_q(s)} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \\
\bar{\varphi}(x)^\mu &= \frac{1}{(2\pi)^{3/2}}\int d^3k\, \bar{a}_k(t)^\mu e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{i\int^t ds\,\omega_q(s)}\bar{\xi}^\nu_q B(t)^{\nu\mu}_{qk} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}.
\end{aligned} \tag{4}$$

Here the time-independent operators ξ's annihilate thermal vacua, $\xi|0\rangle = \tilde{\xi}|0\rangle = 0$ and $\langle 0|\xi^\dagger = \langle 0|\tilde{\xi}^\dagger = 0$, and therefore are called to describe *particle representation.* The momentum-mixing Bogoliubov matrix $\boldsymbol{B}$ is

$$B(t)^{\mu\nu}_{kq} = \begin{bmatrix} \delta(\boldsymbol{k}-\boldsymbol{q}) + N(t)_{kq} & -N(t)_{kq} \\ -\delta(\boldsymbol{k}-\boldsymbol{q}) & \delta(\boldsymbol{k}-\boldsymbol{q}) \end{bmatrix}^{\mu\nu}$$

$$B^{-1}(t)^{\mu\nu}_{kq} = \begin{bmatrix} \delta(\boldsymbol{k}-\boldsymbol{q}) & N(t)_{kq} \\ \delta(\boldsymbol{k}-\boldsymbol{q}) & \delta(\boldsymbol{k}-\boldsymbol{q})+N(t)_{kq} \end{bmatrix}^{\mu\nu}. \tag{5}$$

The number density parameter $n(x : \boldsymbol{k})$ is related to $N(t)_{kq}$ through the Fourier transformation

$$n(t,\boldsymbol{x}:\boldsymbol{k}) \equiv \int d^3q\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} N(t)_{k+\frac{q}{2},k-\frac{q}{2}} \tag{6}$$

or

$$N(t)_{kk'} \equiv \int \frac{d^3x}{(2\pi)^3} e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} n(t,\boldsymbol{x}:\frac{\boldsymbol{k}+\boldsymbol{k}'}{2}). \tag{7}$$

The equations of motion for φ is

$$i\partial_0\varphi(x)^\mu = \omega(-i\nabla)\varphi(x)^\mu - \int d^3y\, R(t,\boldsymbol{x},\boldsymbol{y}) T_0^{\mu\nu}\varphi(t,\boldsymbol{y})^\nu, \tag{8}$$

where

$$R(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left\{i\frac{\partial}{\partial t} + \omega(i\nabla_y) - \omega(-i\nabla_x) n(t,\frac{\boldsymbol{x}+\boldsymbol{y}}{2}:\boldsymbol{k})\right\} \tag{9}$$

with

$$T_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{10}$$

The field equations (8) follow from the Lagrangian, given by

$$\hat{L}_Q(t) = \int d^3x\, \bar{\varphi}(x)^\mu \left\{i\frac{\partial}{\partial t} - \omega(-i\nabla)\right\}\varphi(x)^\mu + \int d^3x d^3y \bar{\varphi}(t,\boldsymbol{x})^\mu R(t,\boldsymbol{x},\boldsymbol{y}) T_0^{\mu\nu}\varphi(t,\boldsymbol{y})^\nu, \tag{11}$$

from which we obtain the Hamiltonian according to the canonical formalism,

$$\begin{aligned} \hat{H}_Q(t) &= \int d^3x\, \bar{\varphi}(x)^\mu \omega(-i\nabla)\varphi(x)^\mu - \hat{Q}_R(t) \\ \hat{Q}_R(t) &= \int d^3x d^3y \bar{\varphi}(t,\boldsymbol{x})^\mu R(t,\boldsymbol{x},\boldsymbol{y}) T_0^{\mu\nu}\varphi(t,\boldsymbol{y})^\nu. \end{aligned} \tag{12}$$

The scheme of perturbative calculations (the interaction representation) is developed as usual, since we have the unperturbed formulation above.[15] The unperturbed propagator is derived as

$$\begin{aligned} \Delta(x,x')^{\mu\nu} &\equiv -i\langle 0|\dot{\mathrm{T}}[\varphi(x)^\mu\bar{\varphi}(x')^\nu]|0\rangle \\ &= \int \frac{d^3k d^3k' d^3q}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} B^{-1}(t)^{\mu\mu'}_{kq} \mathcal{G}_q(t,t')^{\mu'\nu'} B(t')^{\nu'\nu}_{qk'} e^{-i\vec{k}'\cdot\vec{x}'} \end{aligned} \tag{13}$$

where

$$\mathcal{G}_q(t,t')^{\mu\nu} = e^{-i\int_{t'}^{t} ds\, \omega_q(s)} \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu\nu}. \tag{14}$$

The total Hamiltonian $\hat{H}(t)$ is divided into $\hat{H}_Q(t)$ and

$$\hat{H}_I(t) = \hat{H}_{int}(t) + \hat{Q}_R(t) \tag{15}$$

$$\hat{H}_{int} = g\int d^3x \left\{ \left(\varphi^\dagger(x)\varphi(x)\right)^2 - \left(\tilde{\varphi}^\dagger(x)\tilde{\varphi}(x)\right)^2 \right\}. \tag{16}$$

Then the time-evolution operator $\hat{U}$ is a solution of

$$i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0) \tag{17}$$

with $\hat{U}(t_0,t_0) = 1$. The full propagators is given by

$$\begin{aligned} G(x,x')^{\mu\nu} &\equiv -i\langle 0|\mathrm{T}[\psi_H(x)^\mu\bar{\psi}_H(x')^\nu]|0\rangle \\ &= -i\langle 0|\mathrm{T}[\hat{U}(\infty,-\infty)\varphi(x)^\mu\bar{\varphi}(x')^\nu]|0\rangle. \end{aligned} \tag{18}$$

3. Derivation of the Kinetic Equation

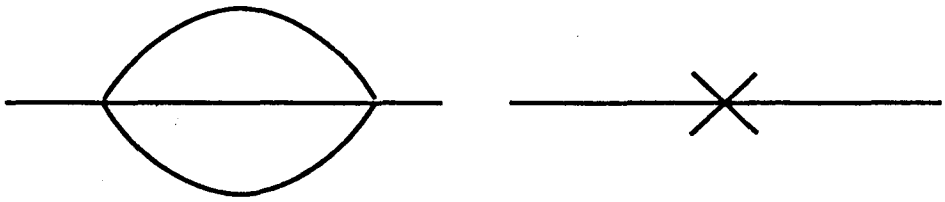

Fig. 1. Self-energy contributions of loop and counter term (the $\hat{Q}_R(t)$ term)

In our previous papers[13,14] we calculated the total self-energy (loop contributions + counter term, see Figure) and imposed the renormalization condition on its on-shell part of the total self-energy (equivalently the diagonalization of the full propagator[10,11]). This procedure provided us with a single equation for $n(x:\boldsymbol{k})$ (in short $n_k(x)$), i.e., the kinetic equation of Boltzmann type:

$$\left[\frac{\partial}{\partial t} + \frac{\boldsymbol{k}}{m}\cdot\nabla\right] n_k(x) = \mathrm{St}_k(x) + \delta\mathrm{St}_k(x). \tag{19}$$

We divide the collision integral into two terms, the first one is the classical collision integral $\mathrm{St}_k(x)$ and the remaining one, representing quantum corrections to it, is denoted by $\delta\mathrm{St}_k(x)$. They read

$$\mathrm{St}_k(x) = \int \prod_{i=1}^{3} d^3L_i\, w(\boldsymbol{k},\boldsymbol{L}) N(x:\boldsymbol{k},\boldsymbol{L}) \tag{20}$$

$$\begin{aligned}\delta\mathrm{St}_k(x) &= -\int \prod_{i=1}^{3} d^3L_i \left[\sin\left\{\frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_W}\cdot\frac{\partial}{\partial \boldsymbol{k}_n} - \frac{\partial}{\partial \boldsymbol{x}_n}\cdot\frac{\partial}{\partial \boldsymbol{k}_W}\right)\right\}\right. \\ &\qquad \times \{u(\boldsymbol{k},\boldsymbol{L})M_1(x:\boldsymbol{L})\}_W \\ &\quad \left. + \left(\cos\left\{\frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_W}\cdot\frac{\partial}{\partial \boldsymbol{k}_n} - \frac{\partial}{\partial \boldsymbol{x}_n}\cdot\frac{\partial}{\partial \boldsymbol{k}_W}\right)\right\} - 1\right)\{w(\boldsymbol{k},\boldsymbol{L})M_1(x:\boldsymbol{L})\}_W\right] n_k(x)\end{aligned} \tag{21}$$

with the notations

$$\begin{aligned} C_g &= \frac{8g^2}{(2\pi)^6} > 0 \\ w(\boldsymbol{k},\boldsymbol{L}) &= 2\pi C_g \delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3)\delta(\omega_k-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}) \\ u(\boldsymbol{k},\boldsymbol{L}) &= 2C_g\delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3)\mathcal{P}\frac{1}{\omega_k-\omega_{L_1}-\omega_{L_2}+\omega_{L_3}} \\ N(x:\boldsymbol{k},\boldsymbol{L}) &= \{n_{L_1}(x)n_{L_2}(x)(1+n_k(x))(1+n_{L_3}(x)) \\ &\quad -n_k(x)n_{L_3}(x)(1+n_{L_1}(x))(1+n_{L_2}(x))\} \\ M_1(x:\boldsymbol{L}) &= (1+n_{L_1}(x)+n_{L_2}(x))\,n_{L_3}(x) - n_{L_1}(x)n_{L_2}(x) \\ M_0(x:\boldsymbol{L}) &= n_{L_1}(x)n_{L_2}(x)\,(1+n_{L_3}(x))\,. \end{aligned} \tag{22}$$

The suffices W and n in the derivatives of (21) indicate that the differentiations operate only on quantities inside $\{\cdots\}_W$ and the last factor $n_k(x)$, respectively.

In deriving (19) we assumed (i) that ω_k are independent of time, (ii) that the calculation of the loop self-energy is done at the g^2-order without vertex correction, as in the Figure, and (iii) the simple dispersion relation

$$\omega_k = \frac{k^2}{2m} - \mu \quad (\mu : \text{chemical potential}). \tag{23}$$

Note that because of (i) and (ii) our kinetic equation is *local in time.*

For later convenience we list some formulae involving $\mathrm{St}_k(x)$ and $\delta\mathrm{St}_k(x)$. Let us integrate $\mathrm{St}_k(x)$ over $\boldsymbol{k}$ with some weight functions below, then we have

$$\int \frac{d^3k}{(2\pi)^3}\left\{\begin{array}{c}1\\ k_a\\ \omega_k\end{array}\right\}\mathrm{St}_k(x) = 0. \tag{24}$$

Next the integration of $\delta\mathrm{St}_k(x)$ over $\boldsymbol{k}$ becomes

$$\int \frac{d^3k}{(2\pi)^3}\,\delta\mathrm{St}_k(x) = -\nabla\cdot\delta\boldsymbol{J}(x) \tag{25}$$

with

$$\delta J(x) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\prod_{i=1}^{3} d^3L_i \left[u(\boldsymbol{k},\boldsymbol{L})\, F_s\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right) + w(\boldsymbol{k},\boldsymbol{L})\, F_c\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right)\right]\frac{\partial}{\partial \boldsymbol{k}} N(x:\boldsymbol{k},\boldsymbol{L}), \tag{26}$$

$F_s(x) \equiv \sin x/x$ and $F_c(x) \equiv (\cos x - 1)/x$. It is remarked that the integration in (25) becomes a divergence of a vector $\delta J(x)$, which will be important later. In the case of the presence of ω_k in the integrand, we have

$$\int \frac{d^3k}{(2\pi)^3}\,\omega_k\,\delta \mathrm{St}_k(x) = -\nabla\cdot\delta J_\omega(x) + P_\omega(x) \tag{27}$$

where

$$\begin{aligned}\delta J_\omega(x) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\prod_{i=1}^{3} d^3L_i \Bigg[&u(\boldsymbol{k},\boldsymbol{L})\left\{F_s\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}_n}\right)(\omega_k - \frac{1}{8m}\nabla_n^2)\frac{\partial n_k}{\partial \boldsymbol{k}}(x)\right.\\ &\left.+F_c\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}_n}\right)\left(\frac{\boldsymbol{k}}{2m}\cdot\nabla_n\frac{\partial n_k}{\partial \boldsymbol{k}}(x)\right)\right\} M_1(x:\boldsymbol{L})\\ &+w(\boldsymbol{k},\boldsymbol{L})\left\{F_c\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}_n}\right)(\omega_k - \frac{1}{8m}\nabla_n^2)\frac{\partial n_k}{\partial \boldsymbol{k}}(x)\right.\\ &\left.-F_s\left(\frac{\nabla}{2}\cdot\frac{\partial}{\partial \boldsymbol{k}_n}\right)\left(\frac{\boldsymbol{k}}{2m}\cdot\nabla_n\frac{\partial n_k}{\partial \boldsymbol{k}}(x)\right)\right\} M_1(x:\boldsymbol{L}) + u(\boldsymbol{k},\boldsymbol{L})\frac{\boldsymbol{k}}{m}N(x:\boldsymbol{k},\boldsymbol{L})\Bigg]\end{aligned} \tag{28}$$

and

$$P_\omega(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{\kappa_k(x)}{8m}\nabla^2 n_k(x) \tag{29}$$

with the *dissipative coefficient*

$$\kappa_k(x) = \frac{1}{2}\int \prod_{i=1}^{3} d^3L_i\, w(\boldsymbol{k},\boldsymbol{L}) M_1(x:\boldsymbol{L}). \tag{30}$$

On the r.h.s. of (27) there is a source term P_ω together with a divergence of a current δJ_ω.

4. Current and Energy-Momentum Tensor

The dynamics described by (3) is invariant under the global transformation $\psi \to e^{i\theta}\psi$, and the corresponding Noether current is given by

$$\begin{aligned} j_0(x) &= \psi^\dagger\psi \\ \boldsymbol{j}(x) &= \frac{1}{2mi}\left[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi\right]. \end{aligned} \tag{31}$$

For (3) there is also a conserved energy-momentum tensor. We write down those related to energy flow:

$$\begin{aligned} t_{00}(x) &= \psi^\dagger \omega_0 \psi + g(\psi^\dagger \psi)^2 \\ t_{a0}(x) &= -\frac{1}{2m}\left\{(\partial_0 \psi^\dagger)(\partial_a \psi) + (\partial_a \psi^\dagger)(\partial_0 \psi)\right\} \end{aligned} \tag{32}$$

$(a = 1, 2, 3)$ and $\partial_0 = \partial/\partial t$.

In TFD ψ is doubled as $\psi^1 = \psi$ and $-\bar{\psi}^2 = \tilde{\psi}$, and observable quantities depend on ψ^1 only, not on $\bar{\psi}^2$. Generally the local conservation laws for thermal averages of the non-tilde quantities above are *violated* in nonequilibrium situations, due to the mixing between non-tilde and tilde degrees of freedom and due to the breakdown of space-time-translational invariance through the parameter $n_k(x)$. This can be understood from the fact that our representation space is the Fock space spanned by ξ's and the observable (non-tilde) quantities are expressed by powers of both of ξ and $\tilde{\xi}$ with space-time dependent coefficients. We will explicitly see such violations at the unperturbed level in the following sections.

Let us take thermal averages of the current and energy-momentum tensor at the unperturbed level, namely, set $g = 0$ and replace ψ and $\psi^\dagger$ with the unperturbed fields φ^1 and $\bar{\varphi}^1$ in the expressions of (31) and (32). The results after some manipulations are

$$\begin{aligned} \langle j_0^{unp}(x)\rangle &= \int \frac{d^3k}{(2\pi)^3} n_k(x) \\ \langle j^{unp}(x)\rangle &= \int \frac{d^3k}{(2\pi)^3} \frac{\boldsymbol{k}}{m} n_k(x) \end{aligned} \tag{33}$$

and

$$\begin{aligned} \langle t_{00}^{unp}(x)\rangle &= \int \frac{d^3k}{(2\pi)^3} \left[\left\{\omega_k - \frac{1}{8m}\nabla^2\right\} n_k(x)\right] \\ \langle t_{a0}^{unp}(x)\rangle &= \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{m}\left\{-\frac{1}{4}\nabla_a \partial_0 + \left(\omega_k - \frac{1}{8m}\nabla^2\right) k_a\right\} n_k(x)\right]. \end{aligned} \tag{34}$$

In deriving these expressions, we perform such symmetrizations as $\bar{\varphi}\varphi \Longrightarrow (\bar{\varphi}\varphi + \varphi\bar{\varphi})/2$ and $\bar{\varphi}\partial_0\varphi \Longrightarrow (\bar{\varphi}\partial_0\varphi - (\partial_0\bar{\varphi})\varphi)/2$.

We emphasize that these calculations at the unperturbed level should not be confused with the usual quasi-particle or free approximation, because $n_k(x)$ is determined by the renormalization condition on the self-energy which includes interaction effects. We now investigate physical implications of (33) and (34) combined with the kinetic equation (19) in the following two sections.

5. Particle Flow — Fick's Law

As remarked in the last section on general grounds, the unperturbed current (33) is not conserved. As a matter of fact, we derive, using (19), (24) and (25),

$$\begin{aligned}\partial_0\langle j_0^{unp}(x)\rangle + \nabla\cdot\langle \boldsymbol{j}^{unp}(x)\rangle &= \int\frac{d^3k}{(2\pi)^3}\left\{\mathrm{St}_k(x)+\delta\mathrm{St}_k(x)\right\}\\ &= -\nabla\cdot\delta\boldsymbol{J}(x)\neq 0.\end{aligned}\tag{35}$$

See the definition for $\delta\boldsymbol{J}(x)$ in (26). Thus the naive conservation law is violated due to $\delta\mathrm{St}_k$.

But when a new effective current,

$$\boldsymbol{J}(x)\equiv\langle \boldsymbol{j}^{unp}(x)\rangle+\delta\boldsymbol{J}(x),\tag{36}$$

is introduced, then a new conservation law is recovered,

$$\partial_0\langle j_0^{unp}(x)\rangle+\nabla\cdot\boldsymbol{J}(x)=0.\tag{37}$$

This in turn suggests that the average local velocity, denoted by $\boldsymbol{v}_P$, is not given by $\langle \boldsymbol{j}^{unp}\rangle/\langle j_0^{unp}\rangle$ but by

$$\boldsymbol{v}_P(x)=\frac{\boldsymbol{J}(x)}{\langle j_0^{unp}(x)\rangle}\neq\frac{\langle \boldsymbol{j}^{unp}(x)\rangle}{\langle j_0^{unp}(x)\rangle}.\tag{38}$$

Let us assume that the $\boldsymbol{k}$-dependence of $n_k(x)$ comes only through ω_k,

$$n_k(x)=n(x,\omega_k),\tag{39}$$

including the special case of local equilibrium with space-time dependent temperature

$$n_k(x:\boldsymbol{k})=1/\left\{e^{\beta(x)\omega_k}-1\right\}.\tag{40}$$

This assumption is expected to be good when temporal and spatial inhomogeneity of thermal system is small, and corresponds to performing a coarse-graining. Under this assumption, while $\langle \boldsymbol{j}^{unp}\rangle=\boldsymbol{0}$ is easily seen, the lowest two terms of $\delta\boldsymbol{J}$ in gradient expansion read

$$\begin{aligned}\delta\boldsymbol{J}^{(0)}(x) &= \boldsymbol{0}\\ \delta J_a^{(1)}(x) &= -\frac{1}{4}\frac{\partial}{\partial x_b}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{\partial^2}{\partial k_a\partial k_b}\kappa_k(x)\right\}n_k(x)=-\frac{\partial}{\partial x_b}\left[D_{ab}(x)\langle j_0^{unp}(x)\rangle\right]\end{aligned}\tag{41}$$

where

$$\begin{aligned}D_{ab}(x) &= \frac{1}{4}<<\frac{\partial^2\kappa_k}{\partial k_a\partial k_b}>>(x)\\ <<\cdots>> &\equiv \int\frac{d^3k}{(2\pi)^3}\cdots n_k(x)/\int\frac{d^3k}{(2\pi)^3}n_k(x).\end{aligned}\tag{42}$$

Thus we reach the Fick's law,

$$J_a(x) = -\frac{\partial}{\partial x_b}[D_{ab}(x)\langle j_0^{unp}(x)\rangle] + O(\nabla^2) \tag{43}$$

leading to the diffusion type equation for the particle density,

$$\left\{\partial_0 - \frac{\partial^2}{\partial x_a \partial x_b} D_{ab}(x)\right\}\langle j_0^{unp}(x)\rangle \simeq 0, \tag{44}$$

the diffusion constant D_{ab} being related to the dissipative constant κ (see (30)) as in (42).

6. Energy Flow

Arguments, similar to those in the last section for the particle flow, can be applied to the energy flow. Firstly the following equation, violating conservation, comes out:

$$\partial_0\langle t_{00}^{unp}(x)\rangle + \partial_a\langle t_{a0}^{unp}(x)\rangle = -\nabla\cdot\left\{\delta J_\omega(x) - \frac{1}{8m}\nabla^2\delta J(x)\right\} - \frac{1}{4m}\nabla^2\partial_0\langle j_0^{unp}(x)\rangle + P_\omega(x). \tag{45}$$

As for δJ_ω and P_ω, see (28) and (29). Naturally we define

$$\begin{aligned} T_{00}(x) &\equiv \langle t_{00}^{unp}(x)\rangle + \frac{1}{4m}\nabla^2\langle j_0^{unp}(x)\rangle \\ T_{a0}(x) &\equiv \langle t_{a0}^{unp}(x)\rangle + \delta J_{\omega,a}(x) - \frac{1}{8m}\nabla^2\delta J_a(x) \end{aligned} \tag{46}$$

and obtain an equation of continuity with a source term,

$$\partial_0 T_{00}(x) + \partial_a T_{a0}(x) = P_\omega(x). \tag{47}$$

Under the assumption (39) and keeping the first non-trivial order in gradient expansion, we calculate as

$$\begin{aligned} T_{00}(x) &\simeq << \omega_k >> (x)\langle j_0^{unp}(x)\rangle \\ T_{a0}(x) &\simeq -\nabla_b\left\{<< \frac{1}{4}\frac{\partial^2}{\partial k_a \partial k_b}(\omega_k\kappa_k) >> (x)\langle j_0^{unp}(x)\rangle\right\} \\ &\quad + \int \frac{d^3k}{(2\pi)^3}\frac{\partial}{\partial k_a}\left\{\frac{k_b\kappa_k(x)}{2m}\right\}\frac{\partial n_k}{\partial x_b}(x). \end{aligned} \tag{48}$$

If the spatial change of $\kappa_k(x)$ is rather slow, a further simplification is possible:

$$\begin{aligned} T_{a0}(x) &\simeq -\nabla_b\left[<< \frac{1}{4}\frac{\partial^2}{\partial k_a \partial k_b}(\omega_k\kappa_k) - \frac{\partial}{\partial k_a}\left\{\frac{k_b\kappa_k}{2m}\right\} >> (x)\langle j_0^{unp}(x)\rangle\right] \\ P_\omega(x) &\simeq \nabla^2\left[<< \frac{\kappa_k}{8m} >> (x)\langle j_0^{unp}(x)\rangle\right] \equiv -\nabla_a\delta T_{a0}(x). \end{aligned} \tag{49}$$

A new coefficient C_{ab} is introduced as

$$C_{ab}(x) = \frac{<< \frac{1}{4}\frac{\partial^2}{\partial k_a \partial k_b}(\omega_k \kappa_k) - \frac{\partial}{\partial k_a}\left\{\frac{k_b \kappa_k}{2m}\right\} + \delta_{ab}\frac{\kappa_k}{8m} >> (x)}{<< \omega_k >> (x)}, \tag{50}$$

and we have

$$T_{a0}(x) + \delta T_{a0}(x) \simeq -\nabla_b \left\{C_{ab}(x) T_{00}(x)\right\}. \tag{51}$$

Combining (47) with (51), one has a diffusion type equation for the energy density

$$\left\{\partial_0 - \frac{\partial^2}{\partial x_a \partial x_b} C_{ab}(x)\right\} T_{00}(x) \simeq 0. \tag{52}$$

As $T_{00}(x)$ and $T_{a0}(x) + \delta T_{a0}(x)$ satisfy the equation of continuity approximately, the local velocity associated with the energy flow, denoted by $\boldsymbol{v}_E$, should be

$$v_{aE}(x) = \frac{T_{a0}(x) + \delta T_{a0}(x)}{T_{00}(x)}. \tag{53}$$

7. Heisenberg Current

So far our discussion for the current is confined to the unperturbed level. However, true observable currents should be expressed in terms of the Heisenberg non-tilde operators, ψ^1 and $\bar{\psi}^1$.

The general scheme of computing the Heisenberg operators is provided by the concept of the *dynamical map*,[1,2] which states that a Heisenberg operator is expanded in terms of unperturbed operators. In our context this implies

$$\begin{aligned} \psi(x)^1 &= \psi^1[\varphi's] = \left[Z^{1/2}\varphi\right]^1(x) + \cdots && (54) \\ &= \hat{U}^{-1}(t, t_0)\varphi(x)\hat{U}(t, t_0). && (55) \end{aligned}$$

In perturbation the last expression is useful to evaluate corrections to the unperturbed current.

For the Heisenberg particle flow, the full propagator defined in (18) provides sufficient information:

$$\begin{aligned} \langle j_{H,0}(x)\rangle &= iG(x,x) \\ \langle \boldsymbol{j}_H(x)\rangle &= \left[\frac{\nabla_x - \nabla_{x'}}{2mi} iG(x,x')^{11}\right]_{x=x'}. \end{aligned} \tag{56}$$

The crucial point is to simplify the full propagator, using the kinetic equation for $n_k(x)$ in (19), or in other words to use the full propagator *after the renormalization.*

For illustration here is a perturbative result to g^2-order (taken account of (19)).

$$\begin{aligned}\left[iG(x,x')^{11}\right]_{t=t'} \simeq & \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left\{\frac{1}{2} + n_k(t, \frac{\boldsymbol{x}+\boldsymbol{y}}{2})\right\} + \int d^4y d^4y' \\ & [d(x,y)\{\sigma_>(y,y')d(y',x') - \sigma_+(y,y')d_<(y',x')\}\theta(t-s)\theta(s-s')\theta(t-s') \\ & + \{d(x,y)\sigma_<(y,y') - d_>(x,y)\sigma_-(y,y')\} d(y',x')\theta(t-s)\theta(s'-s)\theta(t-s')] \\ & - \int ds\, \theta(t-s) \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} e^{\left(-2i\left\{\omega_k - \frac{1}{2m}\nabla_x^2\right\}(t-s)\right)} [\mathrm{St}_k + \delta\mathrm{St}_k](s, \frac{\boldsymbol{x}+\boldsymbol{y}}{2}), \quad (57)\end{aligned}$$

where

$$\begin{aligned}
d(x,y) &= e^{\left(-i\omega(\frac{1}{i}\nabla_x)(t-s)\right)}\delta(\boldsymbol{x}-\boldsymbol{y}) \\
d_>(x,y) &= \int \frac{d^3k}{(2\pi)^3} e^{\left(-i\omega(\frac{1}{i}\nabla_x)(t-s)\right)} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} n_k(s, \frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \\
d_<(x,y) &= \int \frac{d^3k}{(2\pi)^3} e^{\left(-i\omega(\frac{1}{i}\nabla_y)(t-s)\right)} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} n_k(t, \frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \\
\sigma_+(x,y) &= C_g \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \int \prod_{i=1}^{3} (d^3L_i)\, \delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3) \\
&\quad \times e^{\left(-i\left\{\omega(\boldsymbol{L}_1+\frac{1}{i}\nabla_x)+\omega(\boldsymbol{L}_2+\frac{1}{i}\nabla_x)-\omega(\boldsymbol{L}_3-\frac{1}{i}\nabla_x)\right\}\right)} M_1(s, \frac{\boldsymbol{x}+\boldsymbol{y}}{2} : \boldsymbol{L}) \\
\sigma_-(x,y) &= C_g \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \int \prod_{i=1}^{3} (d^3L_i)\, \delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3) \\
&\quad \times e^{\left(-i\left\{\omega(\boldsymbol{L}_1-\frac{1}{i}\nabla_x)+\omega(\boldsymbol{L}_2-\frac{1}{i}\nabla_x)-\omega(\boldsymbol{L}_3+\frac{1}{i}\nabla_x)\right\}\right)} M_1(t, \frac{\boldsymbol{x}+\boldsymbol{y}}{2} : \boldsymbol{L}) \\
\sigma_>(x,y) &= C_g \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \int \prod_{i=1}^{3} (d^3L_i)\, \delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3) \\
&\quad \times e^{\left(-i\left\{\omega(\boldsymbol{L}_1+\frac{1}{i}\nabla_x)+\omega(\boldsymbol{L}_2+\frac{1}{i}\nabla_x)-\omega(\boldsymbol{L}_3-\frac{1}{i}\nabla_x)\right\}\right)} \times M_0(s, \frac{\boldsymbol{x}+\boldsymbol{y}}{2} : \boldsymbol{L}) \\
\sigma_<(x,y) &= C_g \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \int \prod_{i=1}^{3} (d^3L_i)\, \delta(\boldsymbol{k}-\boldsymbol{L}_1-\boldsymbol{L}_2+\boldsymbol{L}_3) \\
&\quad \times e^{\left(-i\left\{\omega(\boldsymbol{L}_1-\frac{1}{i}\nabla_x)+\omega(\boldsymbol{L}_2-\frac{1}{i}\nabla_x)-\omega(\boldsymbol{L}_3+\frac{1}{i}\nabla_x)\right\}\right)} M_0(t, \frac{\boldsymbol{x}+\boldsymbol{y}}{2} : \boldsymbol{L}). \quad (58)
\end{aligned}$$

From this result we see that memory effects come in, as is expected from general considerations.[16]

8. Summary

In this report we employed spatially inhomogeneous time-dependent TFD, which is a thermal quantum field theory with doubled degrees of freedom and is realized in the representation characterized by the *representation particle* ξ. The physical

implications of the kinetic equation, previously derived, are investigated in application to the particle and energy flows.

Interaction effects are twofold in our treatment: (A) Interaction determines the total self-energy, the self-consistent renormalization condition is imposed on the on-shell part (or the full propagator, on which the diagonalization condition is imposed[10,11]). This leads to the kinetic equation for $n_k(x)$ in (19). (B) Interaction determines the way the Heisenberg operator should be expanded in terms of the unperturbed operators (dynamical map). Physical interpretation for these double roles of interaction is as follows: An excitation particle here is not a bare particle but a dreessed (renormalized) particle, affected by interactions with all the other particles. This "all the other particles" as a whole may be called "heat bath", although presence of real heat bath is not required in TFD approach. This is the process described by (A). In addition we have particle-particle interaction in the thermal background, which is the process of (B). In our approximation of the self-energy calculation, no memory effect appears in (A), still one has memory effect from (B), as was shown. In general both (A) and (B) give rise to memory effects.[16]

Our main result is that the diffusion type equations for the particle and energy densities are derived under appropriate approximation (coarse graining) in the unperturbed approach. A derivation of a diffusion equation for the particle density in TFD approach was demonstrated previously.[17] While only initial nonequilibrium state conditions with local temperature cause approach of the systems to equilibrium,[17] our method faithfully follows temporal and spatial changes of thermal systems, governed by the kinetic equation. In addition higher order corrections can be calculated systematically in our approach.

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THERMAL FIELD THEORY IN NON-EQUILIBRIUM STATES

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Conventional transport theory is not really applicable to nonequilibrium systems which exhibit strong quantum effects. We present two different approaches to overcome this problem. Firstly we point out how transport equations may be derived that incorporate a nontrivial spectral function as a typical quantum effect, and test this approach in a toy model of a strongly interacting degenerate plasma. Secondly we explore a path to include nonequilibrium effects into quantum field theory through momentum mixing transformations in Fock space. Although the two approaches are completely orthogonal, they lead to the same coherent conclusion.

"Dedicated to the memory of Hiroomi Umezawa,
Friend and mentor,
Who had the vision to unify
Quantum field theory and statistical mechanics."

1. Introduction

In the past two decades the interest in nonequilibrium quantum systems has grown tremendously. It is therefore natural, that more and more examples have been found for which the applicability of traditional *transport theory* is doubtful, i.e. macroscopic Boltzmann equations or Vlasov–Uehling–Uhlenbeck equations fail to describe the system adequately. Indeed some of these examples, like dynamical features of the early universe, collective effects in fusion plasmas, ultrashort phenomena in semiconductors, nuclear matter in ultrarelativistic heavy-ion collisions and plasma drops composed of quarks and gluons have two things in common that require a new paradigm for their theoretical understanding.

The first of these common factors is the (space and time) density of interactions between the components of the system. Some of them are interacting weakly in the

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sense of having a small coupling constant, e.g. as between electrons and photons. However, even for weak coupling, a system of *many* particles experiences synergetic effects like *collective* participation in movement. As a consequence, the systems listed above have excitation spectra which differ significantly from those of rarefied gases of free particles. The latter however is a basic requirement for the applicability of traditional transport theory.

The second common factor is the importance of the nonequilibrium aspect on the level of the system components: Inhomogeneities occur on spatial and temporal scales that are comparable to the intrinsic scales of the physical problem. Particularly interesting examples are ultrarelativistic heavy-ion collisions, e.g. between lead nuclei ^{208}Pb at 160×208 GeV laboratory energy. Here, time scales are fm/c and characteristic sizes are a few fm — not significantly different from the diameter of the nucleons which compose the initial nuclei. Such a significant difference however is another requirement for the description of the system in terms of *macroscopic* equations: Traditional transport theory is correct only to first order of gradients in density, temperature etc.

We may summarize the two common aspects of the systems listed above as (1) the breakdown of the quasiparticle picture, and (2) the non-separation of space-time scales. The necessary theoretical description therefore must account for these two aspects.

With the present paper we wish to contribute to this new paradigm from two sides: In Sec. 2 we shall point out a way to incorporate nontrivial excitation spectra into transport theory (which already contains some nonequilibrium features), and in Sec. 3 we will clean a path for the inclusion of nonequilibrium effects (gradients in external parameters) into quantum field theory. These two completely different approaches lead to the same consistent conclusion, summarized in Sec. 4.

2. From Standard to Quantum Transport Theory

We perform this step by studying the reverse direction, i.e. starting from a Schwinger–Dyson equation for the full two-point function of a fermionic quantum field we derive transport equations. In contrast to the standard treatment however, we do *not* perform the usual quasiparticle approximation.

As has been pointed out by various authors, (see Ref. 1 for an overview) the description of dynamical quantum phenomena in a statistical ensemble necessitates a formalism with a doubled Hilbert space. For our purpose the relevant content of this formalism is, that its two-point Green functions are 2×2 matrix-valued. We prefer the technically simpler method of thermo field dynamics (TFD),[2] but will keep our derivation sufficiently general to perform similar calculations in the Schwinger–Keldysh, or Closed-Time Path (CTP) formalism.[3]

Within this matrix formulation, we consider the Schwinger–Dyson equation for the full quark propagator $S = S_0 + S_0 \odot \Sigma \odot S$, where S_0 is the free and S the full two-

point Green function of the quark field, Σ is the full self-energy and the generalized product $\odot$ is a matrix product (thermal and spinor indices) and an integration (each of the matrices is a function of two space coordinates). Throughout this paper we use the convention to write space-time and momentum variables also as lower indices, e.g. $\Sigma_{xy} \equiv \Sigma(x,y)$.

In the CTP formulation as well as in the $\alpha = 1$ parameterization of TFD,[4] the matrix elements of S, S_0 and Σ obey

$$S^{11}_{(0)} + S^{22}_{(0)} = S^{12}_{(0)} + S^{21}_{(0)}\,, \quad \Sigma^{11} + \Sigma^{22} = -\Sigma^{12} - \Sigma^{21}\,. \tag{1}$$

Therefore the four components of the Schwinger–Dyson equation are not independent, the matrix equation can be simplified by a linear transformation which one may conveniently express as a matrix product.[5] It achieves a physical interpretation only in the TFD formalism, see Ref. 4. The transformation matrices $\mathcal{B}$ are

$$\mathcal{B}(n) = \begin{pmatrix} (1-n) & -n \\ 1 & 1 \end{pmatrix}, \tag{2}$$

depending on one parameter only. For example, the third term in the Schwinger–Dyson equation becomes

$$\mathcal{B}(n)\tau_3 S_0 \odot \Sigma \odot S\,(\mathcal{B}(n))^{-1} = \begin{pmatrix} S_0^R \odot \Sigma^R \odot S^R & \text{something} \\ & S_0^A \odot \Sigma^A \odot S^A \end{pmatrix}. \tag{3}$$

Here, $\tau_3 = \mathrm{diag}(1,-1)$, $\Sigma^{R,A}$ are the retarded and advanced full self-energy function, and $S^{R,A}$ are the retarded and advanced full propagator (similarly for S_0)

$$\begin{aligned} \Sigma^R &= \Sigma^{11} + \Sigma^{12}\,, \quad \Sigma^A = \Sigma^{11} + \Sigma^{21}\,, \\ S^R &= S^{11} - S^{12}\,, \quad S^A = S^{11} - S^{21}\,. \end{aligned} \tag{4}$$

The diagonal elements of the transformed equation therefore are *retarded* and *advanced* Schwinger–Dyson equation. The off-diagonal element is a *transport equation*.

We now switch to the mixed (or Wigner) representation of functions depending on two space-time coordinates: $\tilde{\Sigma}_{XP} = \int d^4(x-y)\exp(iP_\mu(x-y)^\mu)\Sigma_{xy}$ with $X = (x+y)/2$, the $\tilde{}$-sign will be dropped henceforth. The Wigner transform of the convolution $\Sigma \odot G$ is a nontrivial step: Formally it may be expressed as a gradient expansion

$$\int d^4(x-y)\exp(iP_\mu(x-y)^\mu)\Sigma_{xz} \odot G_{zy} = \exp(-i\diamond)\tilde{\Sigma}_{XP}\tilde{G}_{XP}\,. \tag{5}$$

$\diamond$ is a 2nd order differential operator acting on both functions appearing behind it in the form of a Poisson bracket $\diamond A_{XP}B_{XP} = \frac{1}{2}(\partial_X A_{XP}\partial_P B_{XP} - \partial_P A_{XP}\partial_X B_{XP})$. We will henceforth use the infinite-order differential operator $\exp(-i\diamond) = \cos\diamond - i\sin\diamond$. Propagator and self-energy are split into real Dirac matrix valued functions

$$S^{R,A}_{XP} = G_{XP} \mp i\pi\mathcal{A}_{XP} \quad \Sigma^{R,A}_{XP} = \mathrm{Re}\,\Sigma_{XP} \mp i\pi\Gamma_{XP}\,. \tag{6}$$

$\mathcal{A}_{XP}$ is the generalized spectral function of the quantum field.

Now consider the equations obtained by action of Dirac differential operators (= *inverse free propagators*) on the matrix-transformed Schwinger–Dyson equation.[6] The diagonal components are

$$\begin{aligned} \mathrm{Tr}[(P^\mu\gamma_\mu - m)\mathcal{A}_{XP}] &= \cos\diamond\,\mathrm{Tr}[\mathrm{Re}\,\Sigma_{XP}\mathcal{A}_{XP} + \Gamma_{XP}G_{XP}]\,, \\ \mathrm{Tr}[(P^\mu\gamma_\mu - m)G_{XP}] &= \mathrm{Tr}[1] + \cos\diamond\,\mathrm{Tr}[\mathrm{Re}\,\Sigma_{XP}G_{XP} - \pi^2\Gamma_{XP}\mathcal{A}_{XP}]\,. \end{aligned} \tag{7}$$

Two important facts about these equations have to be emphasized. First notice that these equations do not in general admit a δ-function solution for $\mathcal{A}_{XP}$ even in zero order of $\diamond$. In fact, in contrast to other papers[7] we find that there is no such thing as a mass shell constraint in quantum transport theory.

Secondly, the equations do not contain odd powers of the differential operator $\diamond$. This implies, that when truncating the Schwinger–Dyson equation to first order in $\diamond$ (the usual order for the approximations leading to *kinetic* equations), the spectral function $\mathcal{A}_{XP}$ may still be obtained as the solution of an algebraic equation.

2.1. *Transport equation*

The off-diagonal component of the transformed Schwinger–Dyson equation reads, after acting on it with the inverse free propagator[4,6]

$$\widehat{S}_0^{-1}S^K_{xy} = \Sigma^R_{xz}\odot S^K_{zy} - \Sigma^K_{xz}\odot S^A_{zy}\,, \tag{8}$$

with kinetic components $S^K = (1-n)S^{12} + nS^{21}$ and $\Sigma^K = (1-n)\Sigma^{12} + n\Sigma^{21}$. Inserting the real functions defined before, this leads to a differential equation, which contains all the features of our desired quantum transport theory[4,6]:

$$\begin{aligned} &\mathrm{Tr}[(\partial^\mu_X\gamma_\mu + 2\sin\diamond\,\mathrm{Re}\,\Sigma_{XP} + \cos\diamond\,2\pi\Gamma_{XP})S^K_{XP}] \\ &\quad = 2i\,\mathrm{Tr}[i\sin\diamond\,\Sigma^K_{XP}G_{XP} - \cos\diamond\,\Sigma^K_{XP}\,i\pi\mathcal{A}_{XP}]\,. \end{aligned} \tag{9}$$

Note, that here even as well as odd powers of the operator $\diamond$ occur, and the solution in zero order $\diamond$ is not trivial. To see this more clearly, we *define* the generalized covariant distribution function N_{XP} through the equation

$$(1 - N_{XP})\,S^{12}_{XP} + N_{XP}\,S^{21}_{XP} = 0\,. \tag{10}$$

It may be proven easily from the Kubo–Martin–Schwinger boundary condition,[8] that with this definition of N_{XP} one reaches the proper equilibrium limit $N_{XP} \to n_{\mathrm{F}}(E)$, where $n_{\mathrm{F}}(E)$ is the Fermi–Dirac equilibrium distribution function at temperature T,

$$n_{\mathrm{F}}(E) = (e^{\beta(E-\mu)} + 1)^{-1}\,. \tag{11}$$

With this definition, $S^K_{XP} = 2\pi i(N_{XP} - n)\mathcal{A}_{XP}$, and therefore N_{XP} is the parameter which diagonalizes the the full nonequilibrium matrix-valued propagator through the Bogoliubov matrix $\mathcal{B}$ from (2)[4]:

$$\mathcal{B}(N_{XP})\tau_3 S_{XP}(\mathcal{B}(N_{XP}))^{-1} = \begin{pmatrix} G_{XP} - i\pi\mathcal{A}_{XP} & \\ & G_{XP} + i\pi\mathcal{A}_{XP} \end{pmatrix}\,. \tag{12}$$

Although one may use the generalized distribution function directly in the above Eq. (9), we wish to compare this equation to *traditional transport theory*. Hence, one more step has to be performed, which is the consistent expansion to first order in the operator $\diamond$.

As outlined before, we may then use a spectral function which is the solution of an algebraic equation. One may argue, that in nonequilibrium states a spectral representation of the propagator does not exist in general,[2] but one may still exploit the fact that retarded and advanced propagator are by definition analytical functions of the energy parameter in the upper or lower complex energy half-plane:

$$S^{R,A}(E,\mathbf{p},X) = \mathrm{Re}\, G_{XP} \mp \pi i \mathcal{A}_{XP} = \int_{-\infty}^{\infty} dE' \mathcal{A}(E',\mathbf{p},X) \frac{1}{E-E' \pm i\epsilon}, \tag{13}$$

which is nothing but the Wigner transform of $S_{xy}^{R,A} = \mp 2\pi i \Theta(\pm(x_0 - y_0))\mathcal{A}_{xy}$.

Inserting this into the quantum transport equation yields, correct to first order in $\diamond$ (see Refs. 6 and 4 for details):

$$\begin{aligned}
&\mathrm{Tr}[\mathcal{A}_{XP}\{(P_\mu\gamma^\mu - m - \mathrm{Re}\,\Sigma_{XP}), N_{XP}\}] \\
&\quad = i\,\mathrm{Tr}[\mathcal{A}_{XP}(N_{XP}\Sigma^{21}_{XP} - (N_{XP}-1)\Sigma^{12}_{XP})] \\
&\qquad - i\int_{-\infty}^{0} d\tau \int \frac{dE}{2\pi} \sin(\tau E)\mathrm{Tr}\Big[\Big\{\mathcal{A}(X;P_0+E,\mathbf{P}),\, (N_{XP}\Sigma^{21}(t+\tau/2,\,\mathbf{X};P) \\
&\qquad - (N_{XP}-1)\Sigma^{12}(t+\tau/2,\,\mathbf{X};P))\Big\}_N\Big].
\end{aligned} \tag{14}$$

In this equation, $\{\cdot,\cdot\}$ denotes the Poisson bracket, the index N means that the derivatives are not acting on N_{XP}.

The integral over the history of our system constitutes a *memory term* of this generalized transport equation. The memory effect therefore can be attributed to the nonzero spectral width of the physical excitations in our system.

However, even when we send this spectral width parameter to zero in the above equation, a new effect remains. Using a simple quasiparticle spectral function

$$\begin{aligned}
\mathcal{A}_{XP} &= (P_\mu\gamma^\mu - m - \mathrm{Re}\,\Sigma_{XP}) \\
&\quad \delta(P_0 - E(t,\mathbf{x},\mathbf{P}))\left|\frac{\partial((P^\mu - \mathrm{Re}\,\Sigma^\mu_{XP})^2 - (M - \mathrm{Re}\,\Sigma^s_{XP})^2)}{\partial P_0}\right|^{-1},
\end{aligned} \tag{15}$$

where $E(t,\mathbf{x},\mathbf{P})$ is the (space-time and momentum dependent) generalized energy of the corresponding "particle"-like state, one obtains after integration over P_0

$$\begin{aligned}
&\left(\frac{\partial N(\mathbf{x},\mathbf{p},t)}{\partial t} + \frac{\partial E(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{p}}\frac{\partial N(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}} - \frac{\partial E(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}}\frac{\partial N(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{p}}\right) \\
&\qquad = St[N(t,\mathbf{x},\mathbf{p})] + \delta St[N(t,\mathbf{x},\mathbf{p})].
\end{aligned} \tag{16}$$

The L.H.S. of this equation is the Vlasov (or streaming) part of the standard transport equation, it is free of dissipation.

The first piece on the R.H.S. is the standard "collision integral" of the kinetic equation, while δSt is a *time-local* correction which now contains the three-dimensional Poisson bracket of the Σ^{12}, Σ^{21} self-energies with the function $N(t, \mathbf{x}, \mathbf{p})$.[9] We will consider this equation in more detail in Sec. 3 of this paper.

Before *solving* the generalized transport equation for a simple example, we would like to point out that with our derivation we have indeed taken into account the two topics mentioned in the introductory statements. The breakdown of the quasi-particle approximation is accounted for by the nontrivial spectral function, and the non-separation of time scales is accounted for by the Poisson bracket on the R.H.S. of Eq. (14).

2.2. *Solution of the generalized transport equation*

In the following, we will concentrate on the memory effects hidden in Eq. (14). To isolate them from the gradients in the generalized collision integral $\delta St[N(t, \mathbf{x}, \mathbf{p})]$ of Eq. (16), we consider a very simple model system, tailored to mimic the thermalization of a plasma composed of quarks and gluons. We cannot possibly explain all the relevant physics in the present short paper, and rather refer to the literature on such systems.[10] In view of the previous derivations, we are interested in a calculation of relaxation time scales for such a plasma, and make some simple approximations:

1. We assume, that a gas of bosons (gluons) is instantaneously heated to a very high temperature. In this gas then eventually quark–antiquark pairs start to pop up, until at the very end a thermal equilibrium in the sense of a degenerate plasma is reached.

2. We assume, that the self-energy function for the quarks is dominated by gluonic contributions, and that it does not depend on the energy of the quarks nor on the space coordinates.

3. The gluon background is dominated by external conditions, i.e. we neglect the back-reaction of quarks on the gluon distribution.

4. We neglect the influence of antiquarks in the spectral function. This restriction is removed in an extended version of this application, Ref. 11.

We summarize these assumptions in the following ansatz for the imaginary part of the self-energy function and for the spectral function of quarks:

$$\Gamma_{XP} \equiv \Gamma_t = \gamma^0 g T(t) = \gamma^0 g (T_i \Theta(-t) + T_f \Theta(t)) \,, \tag{17}$$

$$\mathcal{A}(t, E, \mathbf{p}) = \frac{\gamma^0}{\pi} \frac{\gamma_t}{(E - \omega_t)^2 + \gamma_t^2} \,. \tag{18}$$

Hence, we approximate the quark spectral function by two time-dependent parameters ω_t and γ_t, which we may interpret as effective energy and effective spectral width. Arguments for the validity of this approach are given in Ref. 11.

With the above spectral function the coupled system (7) reduces to *a single* nonlinear equation for γ_t, plus the condition $\omega_t^2 = \omega_0^2 = \mathbf{p}^2 + m^2$. This latter condition is more complicated, when the antiparticle piece of the spectral function is taken into account.[11] The energy parameter is chosen as $E = \omega_0$, which yields instead of Eq. (7) as the Schwinger–Dyson equation for the retarded (or advanced) two-point function of the quarks:

$$\gamma_t = gT_i + g(T_f - T_i)\Theta(t)(1 - e^{-2\gamma_t t}) . \tag{19}$$

In Fig. 1, the solution of this equation is plotted in comparison to the time dependent imaginary part of the self-energy function from Eq. (17). It is obvious, that the solution of the nonlinear equation (19) approaches the imaginary part of the self-energy function with a characteristic delay time. In Ref. 11 it is discussed how this delay time is calculated from the system parameters.

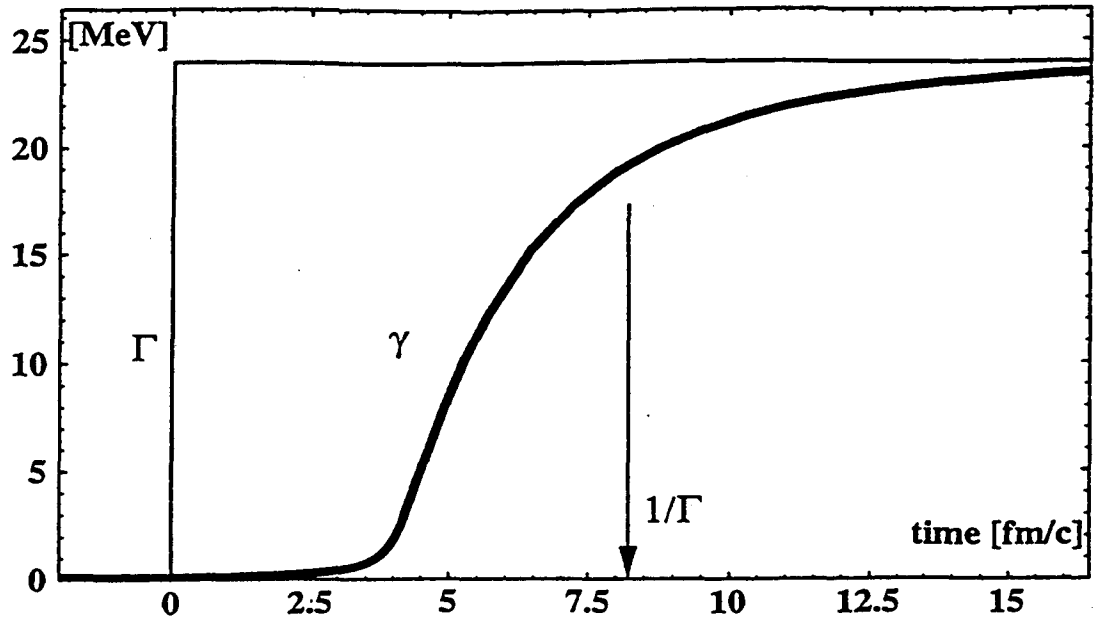

Fig. 1. Time dependent spectral width parameter γ_t. Parameters are $g = 0.12$, $T_i = 1$ MeV, $T_f = 200$ MeV, $m = 10$ MeV. Thin line: Γ_t from Eq. (17), thick line: γ_t from Eq. (19).

We now consider three different levels of transport theory for this model, the corresponding generalized distribution functions relabeled to N_t, N_t^B and N_t^G. First of all, due to the simplicity of our ansatz we obtain as the full quantum transport equation (9):

$$\frac{d}{dt}N_t = -2\gamma_t(N_t - n_F(m, T(t))) \tag{20}$$

with $T(t)$ as defined in Eq. (17). This equation looks surprisingly similar to a kinetic equation in relaxation time approach. However, this similarity is superficial:

The *kinetic* equation, or Boltzmann equation, derived for our simple model system reads

$$\frac{d}{dt}N_t^B = -2\Gamma_t(N_t^B - n_{\rm F}(m, T(t)))\,. \tag{21}$$

Finally, the generalized transport equation (14) is, correct up to first order in the gradients:

$$\frac{d}{dt}N_t^G = -2\Gamma_t(N_t^G - n_{\rm F}(m, T(t))) + 4t\Theta(t)(g(T_f - T_i))^2 \exp(-2\Gamma_t t) \times \left(N_t^G - \frac{n_{\rm F}(m, T_f)T_f - n_{\rm F}(m, T_i)T_i}{T_f - T_i}\right)\,. \tag{22}$$

In Fig. 2 we show the numerical solution for N_t^G, and compare it to the Boltzmann solution N_t^B as well as the full quantum transport solution N_t. This comparison of the three methods shows, that the full quantum transport equation results in a *much* slower equilibration process than the Boltzmann equation. This result is in agreement with other attempts to solve the quantum relaxation problem[12,13]: The quantum system exhibits a memory, it behaves in an essentially non-Markovian way.

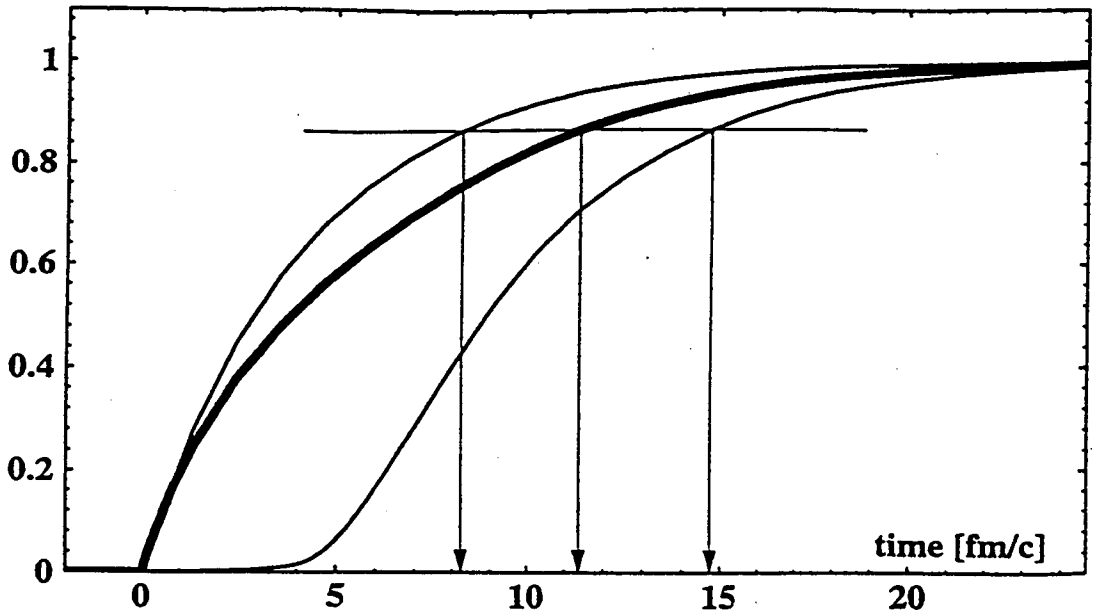

Fig. 2. Normalized time dependent fermionic distribution function for slow quarks. Parameters as in Fig. 1; thin lines: left $N_t^B/n_F(m,T_f)$ from the Boltzmann equation (21), right $N_t/n_F(m,T_f)$ from the quantum transport equation (20); thick lines: $N_t^G/n_F(m,T_f)$ from the generalized kinetic equation (22).

In particular, for the physical scenario studied here, the time to reach $1-1/e^2 \approx$ 86% of the equilibrium quark occupation number is almost doubled (14.7 fm/c as

compared to 8.2 fm/c in the Boltzmann case). We furthermore find, that with the generalized transport equation one does at least partially describe the memory effects in a quantum system (the characteristic time now is 11.4 fm/c).

Thus, although we have only used a toy model, it might turn out that quantum effects (= memory as described in this contribution) substantially hinder the thermalization of a strongly interacting plasma over long time scales. A more thorough discussion of this physical result is carried out in Ref. 11.

3. From Quantum Field Theory to Inhomogeneous TFD

In the following we describe a system of charged bosons in a nonequilibrium state. For simplicity we first consider a space-inhomogeneous situation, but neglect the spectral width of the bosons. This view therefore is orthogonal to the previous section, where we ended in considering a time-dependent system with translational invariance. While this gave us a window to isolate the memory effects of quantum transport theory, we will isolate some nonequilibrium quantum features in the present picture.

As we have pointed out, the description of nonequilibrium systems requires to double the Hilbert space. The reason is, that the occupation number, i.e. the interpretation of a state as "particle" or "hole" (and hence its temporal boundary condition) may change from point to point in order to ensure causality.

The proper formulation is achieved by expressing each physical (causal) excitation operator as the superposition of an operator evolving forward in time and a backward evolving operator. Such a superposition is known in the Liouville space, we refer to the literature for an introduction.[14,15] Obviously, there is also a corresponding anti-causal (orthogonal) linear combination. Denoting the causal single-particle creation and annihilation operators by a_{ql}, $a^\dagger_{ql}$ (the two different charges are distinguished by a lower index $l = \pm$) and the anti-causal operators with $\tilde{}$-signs, one may conveniently express this superposition as a matrix in three-momentum space[16,17]:

$$\begin{aligned} \begin{pmatrix} a_{kl}(t) \\ \tilde{a}^\dagger_{kl}(t) \end{pmatrix} &= \int d^3\mathbf{q} (\mathcal{B}_l^{-1}(t,\mathbf{q},\mathbf{k}))^* \begin{pmatrix} \xi_{ql} \\ \tilde{\xi}^\#_{ql} \end{pmatrix} e^{-iE_\mathbf{q}t}\,, \\ \begin{pmatrix} a^\dagger_{kl}(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix}^T &= \int d^3\mathbf{q} \begin{pmatrix} \xi^\#_{ql} \\ -\tilde{\xi}_{ql} \end{pmatrix}^T \mathcal{B}_l(t,\mathbf{q},\mathbf{k}) e^{iE_\mathbf{q}t}\,. \end{aligned} \tag{23}$$

$\mathbf{k}$ is the three-momentum of the modes, therefore in this notation $a^\dagger_{k-}(t)$ creates a negatively charged physical excitation with momentum $\mathbf{k}$, while $a_{k+}(t)$ annihilates a positive charge.

The operator $\xi^\#_{ql}$ creates a mode with momentum $\mathbf{q}$ and energy $E_\mathbf{q} \equiv E(\mathbf{q})$, which propagates forward in time. $\tilde{\xi}_{\mathbf{q}l}$ annihilates a state with the same momentum and energy, but propagating backwards in time. These two operators therefore may

be combined in a Bogoliubov transformation. with a 2×2 matrix (compare to the fermion case, Eq. (12)):

$$B_l(t,\mathbf{q},\mathbf{k}) = \begin{pmatrix} (\delta^3(\mathbf{q}-\mathbf{k}) + N_l(t,\mathbf{q},\mathbf{k})) & -N_l(t,\mathbf{q},\mathbf{k}) \\ -\delta^3(\mathbf{q}-\mathbf{k}) & \delta^3(\mathbf{q}-\mathbf{k}) \end{pmatrix} . \tag{24}$$

For simplicity, let us choose a local equilibrium state, i.e. $N(t,\mathbf{q},\mathbf{k})$ is the Fourier transform of a space-local Bose–Einstein distribution function

$$\begin{aligned} N_l(t,\mathbf{q},\mathbf{k}) &= \frac{1}{(2\pi)^3}\int d^3\mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} n_l(t,\mathbf{z},(\mathbf{q}+\mathbf{k})/2) \\ n_l(t,\mathbf{z},\mathbf{p}) &= [e^{\beta(t,\mathbf{z})(E_p-\mu_l(t,\mathbf{z}))} - 1]^{-1} . \end{aligned} \tag{25}$$

With these creation and annihilation operators we construct two mutually commuting complex scalar fields ϕ_x, $\widetilde{\phi}_x$ see Refs. 1, 4 and 13 for details:

$$\begin{aligned} \phi_x &= \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}(a^\dagger_{k-}(t)e^{-i\mathbf{kx}} + a_{k+}(t)e^{i\mathbf{kx}}) \\ \widetilde{\phi}_x &= \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}(\widetilde{a}^\dagger_{k-}(t)e^{i\mathbf{kx}} + \widetilde{a}_{k+}(t)e^{-i\mathbf{kx}}) . \end{aligned} \tag{26}$$

Each of these is a representation of the canonical commutation relations, and they may be combined in a statistical doublet $\Phi_x = \left(\phi_x, \widetilde{\phi}_x\right)^T$.

Before proceeding we would like to emphasize that the above heuristic formulation has led us to the creation and annihilation operators known from thermo field dynamics (TFD,[2]). However, the construction of two commuting representations in nonequilibrium field theory is also achieved in other formulations,[1] and their existence has been noticed independently by several authors.[14,15]

3.1. *Effective interaction and diffusion*

In this subsection we will rely more on results obtained in the TFD formalism than in the other parts of this paper. However, we believe that the approach described in the following is also valid and useful for other methods used in thermal field theory.

First of all we note, that due to the introduction of the momentum mixing terms the equation of motion for the physical creation/annihilation operators is

$$i\frac{\partial}{\partial t}\begin{pmatrix} a_{kl}(t) \\ \widetilde{a}^\dagger_{kl}(t) \end{pmatrix} = \left[\begin{pmatrix} a_{kl}(t) \\ \widetilde{a}^\dagger_{kl}(t) \end{pmatrix}, \widehat{H} + \widehat{Q}\right] , \tag{27}$$

with a "bare Hamiltonian"

$$\begin{aligned} \widehat{H} &= \sum_{l=\pm}\int d^3\mathbf{k} E_k \begin{pmatrix} a^\dagger_{kl}(t) \\ -\widetilde{a}_{kl}(t) \end{pmatrix}^T \begin{pmatrix} a_{kl}(t) \\ \widetilde{a}^\dagger_{kl}(t) \end{pmatrix} \\ &= \sum_{l=\pm}\int d^3\mathbf{k} E_k (a^\dagger_{kl}(t)a_{kl}(t) - \widetilde{a}^\dagger_{kl}(t)\widetilde{a}_{kl}(t)) \end{aligned} \tag{28}$$

and a second part that vanishes for homogeneous systems:

$$\widehat{\mathcal{Q}} = \sum_{l=\pm} \int d^3\mathbf{k} d^3\mathbf{q} \begin{pmatrix} a^\dagger_{kl}(t) \\ -\widetilde{a}_{kl}(t) \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (-i\partial_t + E_k - E_q) N_l(t, \mathbf{k}, \mathbf{q}) \begin{pmatrix} a_{ql}(t) \\ \widetilde{a}^\dagger_{ql}(t) \end{pmatrix} . \tag{29}$$

Consequently, by introducing the above linear combinations for the *physical* creation and annihilation operators, the physical fields are no longer free. The time evolution acquires a term mixing causal and anti-causal field.

Let us emphasize at this point, that one should think of $\widehat{H} + \widehat{Q}$ as the *Liouville* operator of our quantum system, which is the generator of the time evolution for the density matrix. Its spectral properties then are immediately obvious,[14,15,18,19] e.g. it is not bounded from below.

The mixing part $\widehat{\mathcal{Q}}$ of the full Liouvillean vanishes when $N_l(t, \mathbf{k}, \mathbf{q})$ is time independent and proportional to $\delta(\mathbf{k} - \mathbf{q})$, which according to (25) is the case when $n_l(t, \mathbf{z}, \mathbf{p})$ does not depend on the space-time coordinates $(t, \mathbf{z})$. Consequently $\widehat{\mathcal{Q}}$ couples the systems to *gradients* in the function $n_l(t, \mathbf{z}, \mathbf{p})$.

To see this more explicitly, we henceforth introduce a "trivial" dispersion relation $E_q = \sqrt{\mathbf{q}^2 + m^2}$ and obtain

$$(E_k - E_q) N_l(t, \mathbf{k}, \mathbf{q}) = -i \frac{\mathbf{Q}}{2E_Q (2\pi)^3} \int d^3\mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} \nabla_z n_l(t, \mathbf{z}, \mathbf{Q}) + \mathcal{O}(\nabla_z^2) , \tag{30}$$

where $\mathbf{Q} = (\mathbf{q} + \mathbf{k})/2$. One may argue, that this is an *ad hoc* introduction of a coupling into the system dynamics. However, this formulation has a deeper foundation: Consider for a moment an equilibrium system, because then the Liouville operator is identical to Eq. (28) and invariant under the thermal Bogoliubov transformation as $(a^\dagger_k a_k - \widetilde{a}^\dagger_k \widetilde{a}_k) = (\xi^\#_k \xi_k - \widetilde{\xi}^\#_k \widetilde{\xi}_k)$. This invariance constitutes of a continuous symplectic symmetry of the Liouville space — and such a global symmetry can be made *local*: We search for those operators ξ, which diagonalize the Liouville operator *locally* in space and time. Such a requirement connects the parameter $n_l(t, \mathbf{z}, \mathbf{p})$ of the Bogoliubov transformation with the system dynamics.

Therefore, this process of gauging the symplectic symmetry is entirely equivalent to the space-time local Bogoliubov transformation *diagonalizing* the single-particle propagator that was carried out in Sec. 2.

On the other hand, it is well known that gauging of a symmetry automatically introduces the coupling of the field ϕ_x to gradients in an external scalar field. Here, this external scalar field is the distribution function "field" $n_l(t, \mathbf{z}, \mathbf{p})$, it is classical and does not possess any dynamical feature. To summarize these arguments: The new gradient terms in the time evolution occur, because we are implementing *causal* boundary conditions locally in space and time.

Let us now turn to the generalized transport equation used in the previous section. Rewritten for bosons, with an effective energy parameter E that is independent of $\mathbf{x}$ it reads

$$\left(\frac{\partial n_l(t,\mathbf{x},\mathbf{k})}{\partial t}+\frac{\partial E(\mathbf{k})}{\partial \mathbf{k}}\frac{\partial n_l(t,\mathbf{x},\mathbf{k})}{\partial \mathbf{x}}\right)=St[n_l(t,\mathbf{x},\mathbf{k})]+\delta St[n_l(t,\mathbf{x},\mathbf{k})]\,. \tag{31}$$

According to our derivation in Sec. 2 we know that δSt contains a three-dimensional Poisson bracket of n and some interaction probability. Obviously this implies that

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int d^3\mathbf{x}\,\delta St[n_l(t,\mathbf{x},\mathbf{k})]=0 \tag{32}$$

and therefore we may write the momentum integral as the three-dimensional divergence of a current:

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\delta St[n_l(t,\mathbf{x},\mathbf{k})]=-\nabla_x\,\mathbf{J}_l^\delta(t,\mathbf{x}) \tag{33}$$

with $\nabla_x\equiv\partial/\partial\mathbf{x}$. We use the fact, that the momentum integral over the "standard" collision term $St[n(t,\mathbf{x},\mathbf{k})]$ is zero,[20] and therefore obtain for the integrated transport equation

$$\partial_t n_l(t,\mathbf{x})+\nabla_x(\mathbf{j}_l(t,\mathbf{x})+\mathbf{J}_l^\delta(t,\mathbf{x}))=0\,. \tag{34}$$

This is the expression for *current conservation*, with

$$\mathbf{j}_l(t,\mathbf{x})=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}}{E_k}n_l(t,\mathbf{x},\mathbf{k}) \tag{35}$$

as the convective part of this current. Equation (34) implies, that in our boson gas a current $\mathbf{J}_l^\delta(t,\mathbf{x})$ arises even for zero "convection" $\mathbf{j}_l(t,\mathbf{x})$: An inhomogeneous temperature distribution gives rise to diffusion. For small gradients of the temperature distribution the diffusion current $\mathbf{J}_l^\delta(t,\mathbf{x})$ will be proportional to $\nabla_x n_l(t,\mathbf{x})$. The momentum integrated transport equation then is nothing but Fick's law.

More detailed derivations within a non-relativistic field model, as well as exemplaric calculations of particle flow and energy flow may be found in Ref. 21.

3.2. *Calculation of the diffusion coefficient*

Having outlined the diffusion problem in the previous subsection, we need to generalize the momentum mixing to excitations with continuous mass spectrum. This amounts to an approximation of the fully interacting quantum fields by *generalized free fields*[19] as already discussed in Sec. 2, and will certainly complicate matters very much.

On the other hand we have outlined above, that only the spatial gradient terms enter into the diffusion problem. Hence, for the following we may neglect the time dependence of the mixing parameter n. Instead of Eq. (23) we therefore write

$$\begin{aligned}\begin{pmatrix}a_{kl}(t)\\ \widetilde{a}_{kl}^\dagger(t)\end{pmatrix}&=\int_0^\infty dE\int d^3\mathbf{q}\mathcal{A}_l^{1/2}(E,\mathbf{k})(\mathcal{B}_l^{-1}(E,\mathbf{q},\mathbf{k}))^*\begin{pmatrix}\xi_{Eql}\\ \widetilde{\xi}_{Eql}^\#\end{pmatrix}e^{-iEt}\\ \begin{pmatrix}a_{kl}^\dagger(t)\\ -\widetilde{a}_{kl}(t)\end{pmatrix}^T&=\int_0^\infty dE\int d^3\mathbf{q}\mathcal{A}_l^{1/2}(E,\mathbf{k})\begin{pmatrix}\xi_{Eql}^\#\\ -\widetilde{\xi}_{Eql}\end{pmatrix}^T\mathcal{B}_l(E,\mathbf{q},\mathbf{k})e^{iEt}\,,\end{aligned} \tag{36}$$

where the 2×2 Bogoliubov matrices $\mathcal{B}$ have the same form as already given in Eq. (24), but N_l is replaced by

$$N_l(E, \mathbf{q}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{z} e^{-i(\mathbf{q}-\mathbf{k})\mathbf{z}} n_l(E, \mathbf{z}) ,$$
$$n_l(E, \mathbf{z}) = [e^{\beta(\mathbf{z})(E-\mu_l(\mathbf{z}))} - 1]^{-1} . \tag{37}$$

The ξ-operators have commutation relations

$$[\xi_{Ekl}, \xi^{\#}_{E'k'l'}] = \delta_{ll'}\delta(E - E')\delta^3(\mathbf{k} - \mathbf{k}') . \tag{38}$$

Similar relations hold for the $\widetilde{\xi}$ operators, all other commutators vanish, see Ref. 19.

The weight functions $\mathcal{A}_l(E, \mathbf{k})$ are positive and have support only for positive energies, their normalization is

$$\int_0^\infty dE E \mathcal{A}_l(E, \mathbf{k}) = \frac{1}{2} , \quad \int_0^\infty dE \mathcal{A}_l(E, \mathbf{k}) = Z_{kl} . \tag{39}$$

The principles of this expansion have been discussed in Ref. 19, its generalization to nonequilibrium states was introduced in Ref. 4. For equilibrium states the combination $\mathcal{A}_B(E, \mathbf{k}) = \mathcal{A}_+(E, \mathbf{k})\Theta(E) - \mathcal{A}_-(-E, -\mathbf{k})\Theta(-E)$ is the spectral function of the boson field ϕ_x and the limit of free particles with mass m is recovered when $\mathcal{A}_B(E, \mathbf{k}) \longrightarrow \mathrm{sign}(E)\,\delta(E^2 - \mathbf{k}^2 - m^2) = \mathrm{sign}(E)\,\delta(E^2 - \omega_k^2)$. As we have seen in the previous section, the concept of a local spectral function may be applied to nonequilibrium states up to first order in the gradient operator $\diamond$.

Using these relations we are then able to express the current *response* of the quantum field to the gradients in the distribution function by using the Heisenberg equation of motion for time-dependent operators, Eq. (27), to obtain

$$\mathbf{J}^\delta(t, \mathbf{x}) = i \int_{t_0}^t d\tau \langle [\widehat{\mathbf{j}}(t - \tau, \mathbf{x}), \widehat{\mathcal{Q}}] \rangle \tag{40}$$

where $\widehat{\mathbf{j}}(x) = i(\phi_x^\dagger \nabla_x \phi_x - \phi_x \nabla_x \phi_x^\dagger)$ is the current operator.

As has been pointed out in Refs. 4 and 13, the ith vector component of the l-charged currents generated by the inhomogeneity of the system is

$$\mathbf{J}_l^{\delta(i)}(t, \mathbf{x}) = 2\pi \int \frac{d^3\mathbf{Q}}{(2\pi)^3} \frac{\mathbf{Q}^{(i)}}{2Z_{Ql}^2} \int dE (\mathcal{A}_l(E, \mathbf{Q}))^2$$
$$\times \int dE' E' \left\{ \frac{\partial n_B^l(E', \mathbf{Q}, \mathbf{x})}{\partial \mathbf{x}^{(j)}} \frac{\partial \mathcal{A}_l(E', \mathbf{Q})}{\partial \mathbf{Q}^{(j)}} \right\} . \tag{41}$$

In principle, this equation may be used to calculate the diffusion constant for interacting boson fields. The result of such a calculation made for pions coupled to nuclear matter has led to results which are bigger than the semi-classical transport

coefficients[22,20] by factors of 10–100.[13,4] Since the transport coefficients are proportional to the relaxation time of the system, this corresponds to the result obtained in Sec. 2: Quantum effects lead to a slowdown of the relaxation process.

In order to establish our approach more safely, we now insert a boson spectral function with a constant spectral width. However, in contrast to the fermionic case discussed in Sec. 2, we may then no longer neglect the anti-particles. Hence we use

$$\mathcal{A}_B(E,\mathbf{k}) = \frac{1}{\pi}\frac{2E\gamma_B}{(E^2 - E_k^2 - \gamma_B^2)^2 + 4E^2\gamma_B^2}\,. \tag{42}$$

Similar to the fermionic spectral function (18), one obtains this form by summing over four simple poles in the complex energy plane, each with the same distance from the real E-axis and with the same residue. A short calculation then gives the approximate results

$$\begin{aligned} \frac{1}{Z_{Ql}^2}\int dE(\mathcal{A}_l(E,\mathbf{Q}))^2 &= \frac{1}{2\pi\gamma_B} + \frac{2}{\pi^2 E_Q} + \mathcal{O}(\gamma_B) \\ \int dE' E' \frac{\partial n_l(E',\mathbf{Q},\mathbf{x})}{\partial \mathbf{x}^{(j)}}\frac{\partial \mathcal{A}_l(E',\mathbf{Q})}{\partial \mathbf{Q}^{(j)}} &= \frac{\mathbf{Q}^{(j)}}{|\mathbf{Q}|}\frac{\partial}{\partial|\mathbf{Q}|}\frac{\partial}{\partial \mathbf{x}^{(j)}} n_l(E_Q,\mathbf{x})\,, \end{aligned} \tag{43}$$

such that an integration by parts gives, up to first order in the gradients of the distribution function,

$$\mathbf{J}_l^\delta(\mathbf{x}) = -\left(\frac{1}{2\gamma_B} + \mathcal{O}(1)\right)\nabla_x\, n_l(t,\mathbf{x}) = -D\,\nabla_x n_l(t,\mathbf{x})\,. \tag{44}$$

The diffusion coefficient D therefore diverges when $\gamma_B \to 0$. In traditional calculations of transport coefficients this diffusion coefficient is obtained as $D = \tau\left\langle \mathbf{v}^2\right\rangle/3$, i.e. as the product of relaxation time and square average of the particle velocity.[20] As we have shown before, the relaxation time of a nonequilibrium quantum system is infinite if the particles do not have a spectral width — and therefore our result has the correct free-particle limit.

For more detailed considerations however, we would have to define a model for an interacting system, which e.g. yields the spectral width parameter as function of the system characteristics (temperature, chemical potential etc.). This would overstress the goal set for the present paper.

4. Conclusion

With the present paper we have demonstrated, how one may apply modern techniques of "thermal" field theory to nonequilibrium states. This has become important because of the quantum character of nonequilibrium systems studied today: A definite breakdown of the quasiparticle approximation and the emergence of space-time inhomogeneities on scales comparable to those of system components are the reasons for the non-applicability of "standard" transport theory.

We have started with the derivation of a generalized transport equation, which accounts for the breakdown of the quasiparticle approximation. In Fig. 2 we have compared the solution of this equation with the full Schwinger–Dyson equation as well as with standard transport theory. We find, that the nontrivial spectral function one encounters when going beyond the quasiparticle picture leads to strong *memory effects* in transport phenomena. In particular, we studied a system which exhibits a characteristic delay time. The memory effects are partially taken into account by the generalized transport equation — but it turns out, that gradients in the external parameters are equally important to describe the physical systems listed in the introduction to this work.

In the second major part of this work we therefore pointed out, how one may incorporate gradient effects into "thermal" field theory. This is achieved by introducing a mixing of operators in momentum space. Firstly we have shown that such an approach indeed leads to relaxation currents in inhomogeneous systems. Secondly we have used the mixing transformation to achieve a linear response result for transport coefficients.

This transport coefficient calculation was not pursued numerically. However, we may rely on previous numerical estimates of transport coefficients achieved with this method.[4] Those numerical estimates give a result very similar to the one obtained in the quantum transport model of Sec. 2: Quantum systems are relaxing *slower* than anticipated from standard transport theory.

Hence, the two approaches we have demonstrated in the present paper lead to a coherent result, which is also in agreement with more traditional calculations from statistical mechanics. We therefore believe that a clear path exists towards an improved treatment of quantum systems in nonequilibrium states, and we express our gratitude to our teacher Hiroomi Umezawa for leading us on this path.

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Chaos Causes Perspective Reversals for Ambiguious Patterns

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Abstract. A new chaotic mechanism is found to cause perspective reversal for ambiguious patterns. And the new senario to cause perspective reversal is proposed with use of chaos, by reexamining the psychological parameters of schema. The psychological parameters are incorporated into a PDP schema model proposed by Rumelhart et al., and a simple one-dimensional map is derived in a mean-field approximation. The numerical experiments for a one-dimensional logistic map are performed, and the results are compared with new psychological experiments of perspective reversals for a Necker cube by 96 students. It is shown that these are qualitatively good agreement with each other. This gives a concrete example of the role of chaos in cognitive science.

1 Introduction

The study of ambiguous patterns such as a Necker cube[1] (Fig. 1) has intrigued psychologists for a long time. Peculiar is oscillatory behaviors between possible two percepts (Fig. 2) in the Necker cube. The best known hypothesis thus far to explain these findings is that of saturation or fatigue proposed by Köhler[2]. Upon this basis, a variety of theories has been published[3, 4, 5]. But there are two cricial discrepancies between experimental facts and the fatigue assumption discussed below.
It has been known by experiments[7] that there are two types of observers, *i.e.*, fast and slow observers. Fast observer has larger frequency of perspective reversals than slow observer. We here point out that the slow observers become faster as experiments are repeated. In fact, the observers with no experiences are almost all slow observers at the first time. This is supported in our experiments, since only 5 persons are fast in all the 96 observers who was made experiment at the first time. Listed below are the experimental facts which conflict with fatigue effect:

Fact 1. The persistent times (durations) staying one of the two percepts are of the order of one second or at most ten seconds.

Fact 2. The persistent times become shorter and shorter as experiments are repeated.

The *Fact 1* indicates that the fatigue effect appeares within several seconds or more. If we intend to interprete this fact by the fatigue effect, the following question becomes arose:

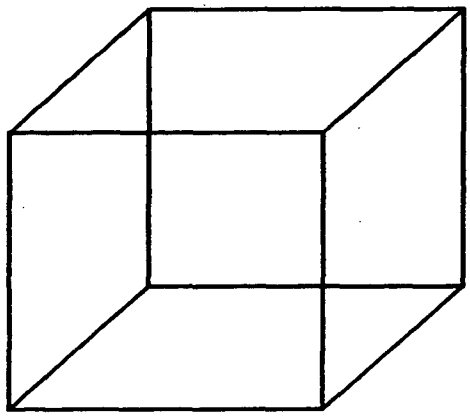

Fig. 1. Necker cube.

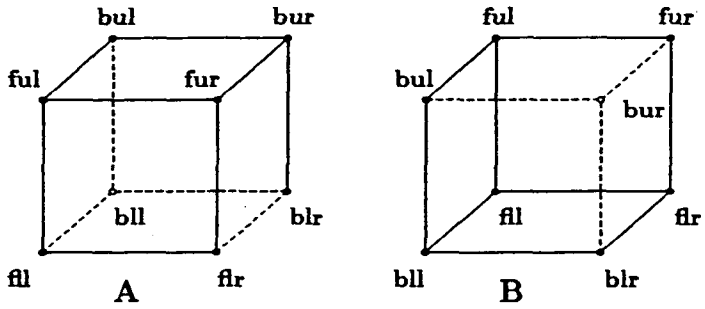

Fig. 2. Two perspects of a Necker cube, and 16 hypotheses of the vertices which we assign to network units.Here 'ful' is meant by 'forward-upper-left', 'blr' by 'backward-lower-right', and so on.

Question 1. *Does the fatigue effect works so quick?*

On the other hand, *Fact 2* means that the observer familiar with the Necker cube become easy to be fatigued. And the next question is taken place:

Question 2. *Does the observer used to seeing become easy to be fatigued?*

If say yes, this conflicts with the fact that the observer used to seeing recover from fatigue quickly.
In Sec. 2, we reexamine the psychological schema parameters so as to give consistent interpretations with the above, and show a new experiment to clarify psychological state. In Sec. 3, a network model is introduced, and a one-dimensional map is derived in a mean-field approximation, finally the associative schema model is proposed. Sec. 4 is devoted to discussions and conclusions. The preliminary reports are presented in Refs. [10], [12] – [14].

2 Perspective Reversal

We here discuss psychological parameters of a schema model, and clarify the mechanism responsible for perspective reversals by estimating characteristic time of change in these parameters. And we propose a new senario of perspecitve reversals.

2.1 Psychological Schema Parameters

We propose two parameters of schama as follows.

(a)*Strength of schema formation* which denotes tendency to have a simpler interpretation of line figure of Necker cube as cubic figure.

This is weak for an observer at a first experience, and strong for one used to seeing the Necker cube. Thus the characteristic time of change is at most of the order of number of experimental trials. In a experimental sequence of one trial (10 min.), there are many cases in which the frequency of oscillation becomes large. These give a minimum estimation of the characteristic time of formation strength to be at least several minites.
On the other hand, the characteristic time of perspective reversals is of the order of second, and the reversal mechanism is not governed by this strength of schema formation. The strength of schema formation is reflected to the frequency of perspective reversals as discuss in Sec. 4.
As the another parameter of schema, we propose

(b)*Strength of schema conviction* which denotes tendency to be convinced of one of schemata.

The conviction strength is thought to be weak immediately after an occurrence of the perspective reversal, since new schema just occurred is unexpected one. But soon later, the new shcema seems to be plausible for an observer, and the conviction strength becomes larger. The characteristic time of conviction strength is thought to be of the order of second, since changes in conviction strength are accompanied with perspective reversals. We may now conclude that the perspective reversals is closely related to a dynamical process of conviction strength.
Although the characteristic time of conviction strength coincides with that of perspective reversals, the larger conviction strength never bring reversal process by itself. In order to show oscillatory behaviors in perspective reversals, some kinds of switching mechanism are required which has been thus far due to a fatigue effect. We propose in Sec. 3.3 such a switching mechanism as a new chaotic switching making use of nonlinear effect.

2.2 Uncertain States of Percepts

Our senario of perspective reversals is as follows. Immediately after an occurrence of the perspective reversal, the conviction strength is weak. But soon after, it becomes large. Further increments of the conviction strength make its state unstable, and chaotic responses occur due to a nonlinear effect. Through this chaotic wandering, a new state corresponding to another percept is found and stabilized. The above processes are repeated, and perspective reversals are realized.
In order to justify this senario, we have to find the chaotic wandering states in a sequence of psychological experiment. These states are considered to be uncertain, and neither one of percepts nor another is certain. Since the published experiments

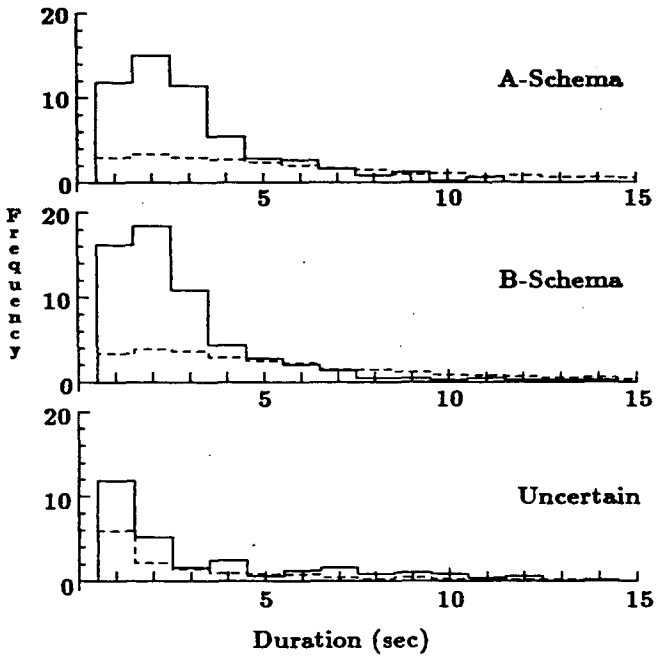

Fig. 3. Frequencies in experiments by observers.The solid line denotes averaged frequencies of five fast observers with averaged durations less than 3 sec. The broken line denotes those of ninety-one slow observers greater than 3 sec.

of perspective reversal[7, 8] are not concerned to the existence of uncertain state, we have made experiments with a new method to extract the uncertain state of percepts.

The basic concept of our experiment is to take three kinds of times, *i.e.*, (1)times persisting A-schema, (2)B-shcema and (3)uncertain. The frequencies of these times are plotted in Fig 3. The observers are 96 students aged 18 to 22 years, who did not know the aims of the experiments. They are required not to move their eyes from the center of the Necker cube figure. The experimental run lasted at 10 min.

As shown in Fig. 3, the distribution of the persistent time in uncertain state is quite different from those of A- and B-schema. The uncertain distribution is a monotonically decreasing, although the others have a hump. This type of monotonically decreasing features is known to appear when chaotic burst[9] takes place. Therefore this could be seen as a candidate for an experimental evidence of chaotic behaviors.

3 Associative Schema Model

We now construct a theoretical model describing the psychological phenomenon of perspective reversals. Starting from a connectionist model proposed by Rumelhart *et al.*[6] of PDP (Parallel Distributed Processing) group, we relate the psychological parameters of formation and conviction strength to their schema model. Their original model does not show perspective reversals, but does moderate dynamics finding each one of perspects. Therefore we will add new dynamics corresponding to that of the conviction strength, and realize oscillatory behaviors of the perspective reversals.

3.1 PDP Schema Model

The PDP schema model[6] is a simple constraint satisfaction model of a Necker cube (Fig. 1). We assume that 16 units represent hypotheses about the correct interpretations of the vertices as shown in Fig. 2. In the PDP schema model, these hypotheses are taken to be units whose activation denotes certainty for the hypothesis.
The activation rule of a single unit is given by[6]

$$a_i(t+1) = \{1 - |net_i(t)|\}\, a_i(t) + net_i(t)\, \theta(net_i(t)) \ , \tag{1}$$

where $net_i(t)$ indicates a net-input from other units, defined by

$$net_i(t) = \sum_j w_{ij}\, a_j(t) + bias_i \ . \tag{2}$$

This model is called an *interactive activation model*[11]. Here w_{ij}'s denote inter-unit weights between i-th and j-th units, $bias_i$ a bias of i-th unit. These weights have been chosen so as to guarantee the balance condition (sum rule [10]) for each i,

$$\sum_{j=\{A\}} w_{ij} + \sum_{k=\{B\}} w_{ik} = 0 \ , \tag{3}$$

between A- and B-schemata. Furthermore, the symmetricity is also assumed at each unit i and k in A- and B-schemata, respectively,

$$\sum_{j=\{A\}}^{(i\in\{A\})} w_{ij} = \sum_{l=\{B\}}^{(k\in\{B\})} w_{kl} \ . \tag{4}$$

There are two kinds of parameters in PDP schema model, *i.e.*, the interconnection weight w_{ij} and bias $bias_i$. The weight parameter denotes firmness of the schema formation, and we may regard it as the strength of schema formation. On the other hand, the unit becomes easy to activate as the bias grows, and it is natural to assign the bias to the strength of schema conviction.

3.2 Derivation of One-Dimensional Map

We here indicate that the interactive activation model implies chaos, because a one-dimensional logistic map $X(t+1) = AX(t)(1-X(t))$ can be extracted from the above activation rule (1) with (2) in a mean-field approximation as followings. For simplicity, we first assume the simplest case of $bias_i = 0$ and $w_{ij} = w > 0$. This is single schema case in which all units are equivalent and cooperative with others. The net-input (2) thus becomes positive definite, *i.e.*, $net_i(t) = w\sum_j a_j(t) = Nw\langle a(t)\rangle > 0$. Here $N = \sum_j 1$ indicates total number of units, and $\langle a(t)\rangle$ is meant by taking an average of activations $\sum_j a_j(t)/\sum_j 1$. By taking an average of the activation rule (1), we obtain $\langle a(t+1)\rangle = (1+Nw)\langle a(t)\rangle - Nw\langle a(t)\rangle^2$. This becomes a well-known one-dimensional logistic map $X(t+1) = AX(t)(1-X(t))$, where $X(t) = (Nw/(1+Nw))\langle a(t)\rangle$ and $A = 1+Nw$.
From this discussion, we can conclude in a mean-field approximation that ***PDP schema model implies chaos***, since the logistic map is a representative one to cause chaos. It should be noted that the unit itself has no origin to cause chaos .

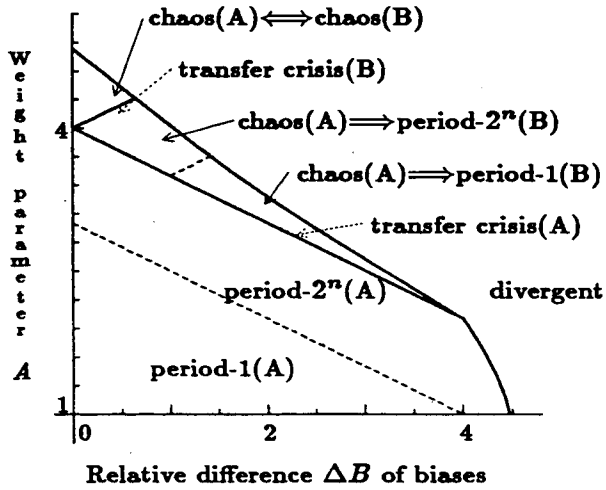

Fig. 4. Phase diagram calculated by a one-dimensional logistic map (7). The initial value of $X(0)$ is set to be 0.001.'period-2^n' denotes a period-doubling route to chaos.

We extend this discussion to two schemata case such as in the Necker cube[10, 12, 13]. Assuming a bias in A-schema to be $+\Delta B/2$ and in B-schema $-\Delta B/2$, the net-inputs (2) can be read as

$$\langle net_i(t)\rangle_{\mathrm{A}} = -\langle net_i(t)\rangle_{\mathrm{B}} = 3c\,\Delta a(t) + \frac{\Delta B}{2}\,, \tag{5}$$

where

$$\Delta a(t) = \langle a(t)\rangle_{\mathrm{A}} - \langle a(t)\rangle_{\mathrm{B}} = \frac{\sum_{i=\{\mathrm{A}\}} a_i(t)}{\sum_{i=\{\mathrm{A}\}} 1} - \frac{\sum_{i=\{\mathrm{B}\}} a_i(t)}{\sum_{i=\{\mathrm{B}\}} 1}\,. \tag{6}$$

We have assumed no self-connections of units ($w_{ii} = 0$), and the sum-rules[10] for inter-unit weights have been used. The parameter c characterizes a weight parameter between units, and is related to an average of weights within same schema.
We now introduce scaled variables $X(t) = \{(A-1)/A\}\,\Delta a(t)$and $Z = \Delta B/(2A)$. The map functions $X(t+1) = F(X(t))$ is obtained with

$$F(X) = AX\{1 - |X + Z|\} + (A-1)\,Z\,. \tag{7}$$

Here the definition $A \equiv 1+3c$ has been used. It is apparent that Eq. (7) is reduced to the well-known one-dimensional logistic map with vanishing Z for positive X, and it is confirmed that the PDP schema model surely shows chaotic responses in a mean-field manner.
Period-1 fixed points of Eq. (7) is determined by $X^* = F(X^*)$ which gives three fixed points, $X^* = X_1^*, -X_1^*$ and $-Z$ where we have introduced an period-1 fixed point $X_1^* = (A-1)/A$ in a one-dimensional logistic map $f(X) = AX(1-X)$. The theoretical phase diagram calculated by Eq. (7) is plotted in Fig. 4, from which the relative difference ($\Delta B = 2AZ$) of biases between schemata is also appeared to be a bifurcation parameter besides the weight parameter $A = 1+3c$. It is noted that Eq. (7) is invariant under interchanges $X \to -X$ and $Z \to -Z$. This means that

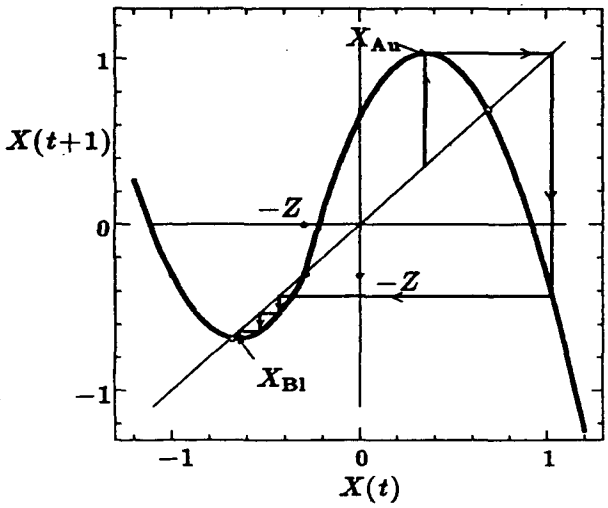

Fig. 5. A typical situation to occur a transfer crisis. The chaotic orbit has a chance transferring to B-schema, through the narrow channel $[1, F(X_{Au})]$. It is noted that, at this parameter set, the orbit falling in B-schema becomes stable at a period-1 fixed point near the lowermost point X_{Bl}.

an interchange of A- and B-schemata holds good if Z is also interchanged into $-Z$. Therefore, we hereafter concentrate ourselves only on a case of $Z = \Delta B/(2A) > 0$. Characteristics of the phase diagram Fig. 4 are the occurrence of *transfer crises*[12, 15]. The typical situation to take place the transfer crisis is shown in Fig. 5. This is a kind of 'local' boundary crisis against the chaotic attractor of A-schema. The *transfer crisis* is found and named by Yamaguchi and Sakai[15] in 1983 for a forced one-demensional logistic map by an alternating disturbance. This phenomenon has been found in physical system of a Josephson junction[16, 17] by numerical calculations. Eq. (7) is now found to be the simplest map showing the *transfer crisis* between coexisting two attractors.

Peculiar is a hysteresis[15] associated with the transfer crisis, which is seen in Fig. 4 along the line at constant weight parameter A. Starting from vanishing bias $\Delta B = 0$, the orbit is confined in A-schema. When the ΔB is increasing along the constant A line, the orbit becomes unstable and chaotic. Further increase takes an occurrence of the *transfer crisis*, and the orbit is falled into B-schema, when ΔB exceeds the critical line of 'transfer crisis(A)'. But, the reverse process never hold because of the hysteresis effect, even if ΔB is switched back to decrease.

The hysteresis is arose from breaking of the symmetry in Eq. (7). This symmetry breaking is taken from the existence of the relative difference ΔB of biases between A- and B-schemata. At the next subsection, we make use this peculiar feature of a hysteresis, and the new interpretation is given for the oscillatory behaviors in psychological phenomena observed such as in a Necker cube.

3.3 Dynamics of Perspective Reversal

We are now at the stage to relate PDP schema model and psychological schema parameters of formation and conviction strength.

In order to realize the oscillatory behaviors of perspective reversals, we should incor-

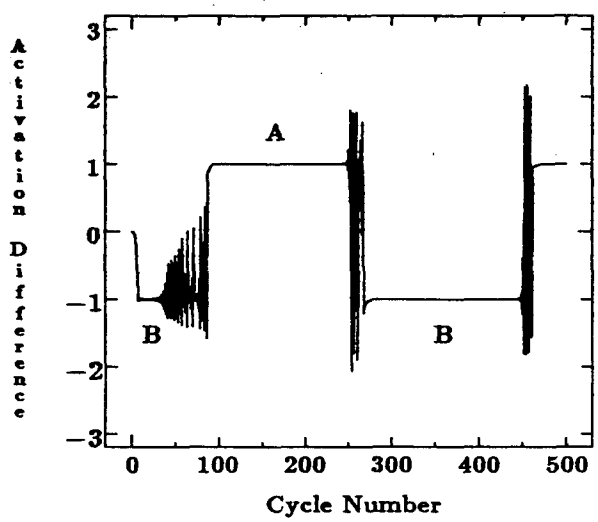

Fig. 6. Time-series profile of an activation difference $\Delta a(t)$ by maps (7) and (9). $\Delta a = 1$ and -1 denotes A- and B-schema, respectively. The adopted initial parameters are $c = 0.6(A = 2.8), \epsilon = 0.01, \Delta B(t=0) = Z = 0, X(t=0) = -0.001$. Note that plotted are not $X(t)$ but $\Delta a(t) = X(t)/X_1^*$ which is chosen because the period-1 fixed points are not affected by the bifurcation parameter A.

porate some dynamics into the conviction strength (bias). We adopt the following update rule for biases[13],

$$bias_i(t+1) = (1-\epsilon)\, bias_i(t) + \epsilon\, net_i(t)\ , \tag{8}$$

where ϵ is a positive smallness parameter to govern the velocity of changes.
The cognitive meanings of Eq. (8) is as followings. When the particular hypothesis (say, i-th unit) is feasible ($net_i > 0$), enhanced is the tendency ($bias_i$) to believe that the hypothesis holds good.
Eq. (8) can be rewritten in a mean-field approximation as

$$Z(t+1) = Z(t) + \epsilon X(t)\ , \tag{9}$$

where $Z(t) = \Delta B(t)/(2A)$. An introduction of the enhancement mechanism of Eq. (9) leads to movement along a line of constant weight parameter A on the phase diagram Fig. 4. If the initial state is located at period-1(A), it moves to the right and finally falls into the window of B-schema. In Fig. 6, we plot the time-series profile calculated by maps Eqs. (7) and (9). The oscillatory behaviors are obtained as required in realizing perspective reversals. An alternation or switching phenomena seen in Fig. 6 is caused by a tranfer crisis[12, 15]. It is seen as an abrupt change of orbits from chaos confined in one attractor to a stable fixed point apart from the chaotic attractor.
The dynamical behaviors in Fig. 6 might be interpreted in a cognitive sense as followings. Once a certain schema is confirmed, the confirmation becomes greater (corresponding to a growth of $|\Delta B|$ according to (8) or (9)). Further confirmation gives rise to a loss of confidence corresponding to chaotic state, and leads to a sudden discovery of an another perspect via chaotic transition due to transfer crisis.

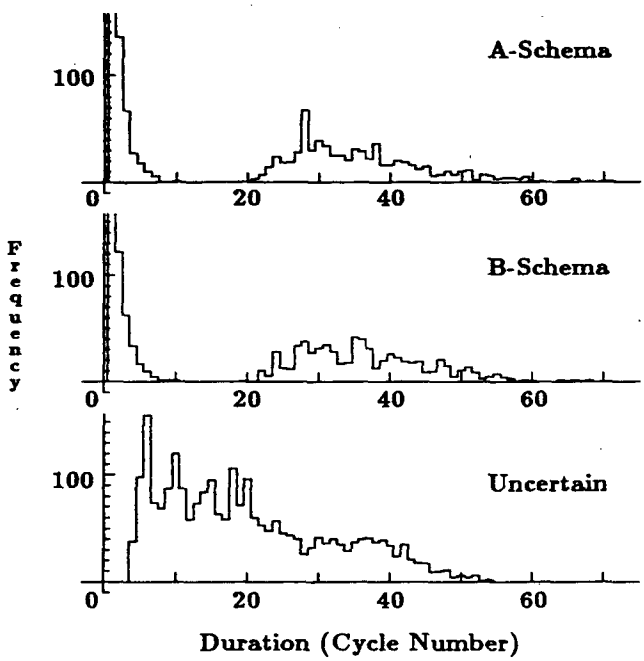

Fig. 7. Frequency distributions by maps (7) and (9). Performing 100,000 iterations, the resulting time-series are classified into three categorizes: A-schema if $\Delta a = 1 \pm 0.1$, B-schema if $\Delta a = -1 \pm 0.1$, otherwise, 'Uncertain'.

The characteristic features in Fig. 6 are the existence of uncertain region in the time-series, *i.e.*, the chaotic wandering region. The numerical calculations are performed up to 100,000 iterations with use of Eqs. (7) and (9), and are plotted in Fig. 7. Comparing with Fig. 3, the qualitative agreement is obtained, if we neglect the lower duration below 20 cycles in Fig. 7. These neglections are rather meaningful, because such a shorter duration should not really recognized by an observer.

4 Discussions

We here discuss the effect of formation strength. As indicated in *Fact 2* in Sec. 1, the frequency of perspective reversals are larger for an observer used to seeing the Necker cube. This can be explained by our model as follows. The observer used to seeing has a larger strength of schema formation which is corresponding to the bifurcation parameter $A = 1+3c$ in the phase diagram of Fig. 4. Since the horizontal width of ΔB at constant A is propotional to the average period of chaotic switching, the larger A causes lager frequency of oscillatory behaviors in perspective reversals. In summary, we propose a new idea to interpret perspective reversals of ambiguous patters such as a Necker cube. Based on a PDP schema model, we derive a one-dimensional logistic map in a mean-field approximation, the obtained map shows a varieties of dynamics including chaos and *transfer crises*[12, 15] with hysteresis. These are based on the following findings: (1) PDP schema model implies chaos. (2) The chaos can be specified by a one-dimensional map in a mean-field approximation. (3) A transfer of an orbit can be taken place from the chaotic attractor of one schema into another attractor corresponding to different schema.

It should be noted that the origin of chaos does not come from single unit itself but from averaged behaviors of units. In this sense, this type of chaos should be called as a *mean-field chaos*.
We have introduced a new experiment to take three categories of times for perspective reversal, *i.e.*, (1)one percept with confidence, (2)alternative percept, and (3)uncertain percept without confidence. A qualitatively good agreements are obtained between the experiments by 96 observers and numerical results by the derived logistic maps. A new interpretation is given such that the uncertain times could be a chaotic wandering or searching state in a competitive-cooperative network system.

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PAPER Special Section on Nonlinear Theory and Its Applications

Chaotic Responses in a Self-Recurrent Fuzzy Inference with Nonlinear Rules

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SUMMARY It is shown that a self-recurrent fuzzy inference can cause chaotic responses at least three membership functions, if the inference rules are set to represent nonlinear relations such as *pie-kneading transformation*. This system has single input and single output both with crisp values, in which membership functions is taken to be triangular. Extensions to infinite memberships are proposed, so as to reproduce the continuum case of one-dimensional logistic map $f(x)=Ax(1-x)$. And bifurcation diagrams are calculated for number N of memberships of 3, 5, 9 and 17. It is found from bifurcation diagrams that different periodic states coexist at the same bifurcation parameter for $N \geqq 9$. This indicates multistability necessarily accompanied with hysteresis effects. Therefore, it is concluded that the final states are not uniquely determined by fuzzy inferences with sufficiently large number of memberships.

key words: *fuzzy inference, chaos, logistic map, bifurcation diagram, hysteresis*

1. Introduction

Recently nonlinear aspects of fuzziness have been got much attensions [1]-[5]. It is apparent that fuzzy theory is closely related to nonlinearities as shown in the following some cases. (a) Fuzzy set theory contains nonlinear operations in compositions of inferred membership functions, if max-min composition such as Mamdani's inference method [6] is adopted. (b) Fuzzy measure theory could be regarded to represent nonlinear composition and/or decomposition in a qualitative manner. Besides these immanent nonlinearities in fuzzy theories, (c) fuzzy inference rules could represent nonlinearities by a suitable way.

On the other hand, characteristic features of nonlinearities are accompanied with dynamical aspects. Representative phenomena of well known nonlinear effects are soliton (solitary wave) and chaos [8].

Therefore, in order to investigate the dynamical aspects of fuzziness, we thus prepare some dynamical mechanism with nonlinearity and investigate I/O (input-output) characteristics. In this situation, there are possible four categories as follows. (i) Crisp I/O with crisp mechanism, (ii) Fuzzy I/O with crisp mechanism, (iii) Crisp I/O with fuzzy mechanism, (iv) Fuzzy I/O with fuzzy mechanism.

For case (i), the simplest one of nonlinear mechanisms to cause chaos is a logistic map $f(x)=Ax(1-x)$ which was discussed by May [7] in 1976, and got much attensions to scientists thus far. The extensions from crisp I/O to fuzzy I/O have been discussed [1], [3] for case (ii) with piecewise linear map [3], and extend it to case (iv). On the other hand, (iii) has been discussed by Teodorescu [1] with cyclic rules.

In this paper, we investigate the case (iii) in which the fuzzy mechanism is taken to be fuzzy inference, since the process of fuzzy inferences could be regarded as a dynamical process. The way incorporating nonlinearity into fuzzy inference is that fuzzy inference rules are taken to be nonlinear as discussed below [9].

The organizations of this paper is as follows. Section 2 is devoted how to incorporate nonlinearity into a fuzzy inference. The simplest case to cause chaos is first shown, and this is extended to include the continuum case corresponding to one-dimensional logistic map. In Sect. 3, the time-series profiles are shown, and the results are summerized in a bifurcation diagram. Section 4 is devoted to discussions. The preliminary report have been presented in Ref. [9].

2. Incorporation of Nonlinearity

Since we hereafter discuss the case (iii) in Sect. 1, i.e., I/O is crisp but the mechanism is fuzzy, we now introduce nonlinearity into fuzzy inference rules so as to cause chaos.

We at first indicate what kind of nonlinearity could cause chaos. The simplest mechanism necessary to cause chaos is so called *pie-kneading transformation* with both processes of *expanding* and *folding*. Furthermore, the iterative processes of *expanding* and *folding* lead to chaotic responses, if the expanding ratio is sufficiently large. Here, the expanding ratio is called *bifurcation parameter* which is a control parameter to govern the dynamics.

2. 1 Simplest Three Rules Case

At this step, we can express the above mechanism by fuzzy inference rules. In order to represent the folding

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process, we must have at least three fuzzy labels for which we here label 'SMALL', 'MEDIUM' and 'BIG' as usually. And the above mechanism can be expressed by the following inference rules with MAX operator standing for OR.

(A) if *input* is SMALL, then *output* is SMALL.
OR (B) if *input* is MEDIATE, then *output* is BIG.
OR (C) if *input* is BIG, then *output* is SMALL.

When we consider that the above labels indicate crisp values, all inputs are finally falling into 'SMALL' state and no complex dynamics can occur. This stands also for fuzzy memberships if the overlapping are vanishing between neighbouring memberships. On the other hand, when the memberships are allowed to overlap, chaos can be taken place as shown below due to nonlinear rules reflecting *pie-kneading transformation*.

We hereafter take membership functions as isosceles triangles whose position of the tail coinsides with a peak position of neighbouring memberships. In this case, the inputs are categorized into following two cases whether the input is between two peaks of 'SMALL' and 'MEDIUM' or between 'MEDIUM' and 'BIG'.

In case of an input between 'SMALL' and 'MEDIUM', the input is amplified according to the inference rules (A) and (B), and goes to some value between 'SMALL' and 'BIG'. On the other hand, for larger input, the folding effect is brought by the inference rules (B) and (C), and the input is folding into a certain value between 'BIG' and 'SMALL'. Thus, the chaotic *pie-kneading* mapping can be obtained.

The resulting chaotic time-series profile is plotted in Fig. 1 where the I/O is restricted within [0, 1], and the peak positions of triangular memberships labelled 'SMALL', 'MEDIUM' and 'BIG' are set to be 0, 0.5 and 1, respectively. Here, composition of inferred memberships is made by Mamdani's inference method [6], and defuzzification is by a center of gravity method of min-max.

Whether these rules cause chaos or not is easily checked in this case by plotting a return map as shown in Fig. 2. From this figure, we can conclude that this is surely chaos, because the orbit is not spread over the phase space $(x(t), x(t+1))$ but restricted within the narrow confined region (chaotic attractor) showing one-dimensional order. In fact, we can easily obtain an analytic form of the one-dimensional map function $x(t+1)=f(x(t))$ corresponding to Fig. 2 by

$$f(x)=\frac{4x(1-x)}{1+8\left|x-\frac{1}{2}\right|\left(\frac{1}{2}-\left|x-\frac{1}{2}\right|\right)}. \tag{1}$$

Thus the map function derived from the fuzzy inference could be called as a *piecewise nonlinear map* whose piece represents nonlinear interporation due to fuzzy inference. As shown below, these nonlinear pieces are responsible for coexistences of attractors when we extend the fuzzy inference to many rules case.

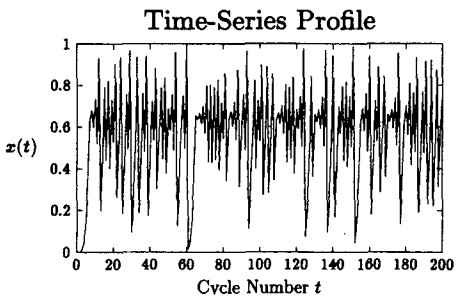

Fig. 1 Time-series profile of chaotic response. The simplest case with three labels, i.e., 'SMALL', 'MEDIUM' and 'BIG', of triangular membership. Adopted inference rules are (A)-(C) in the text.

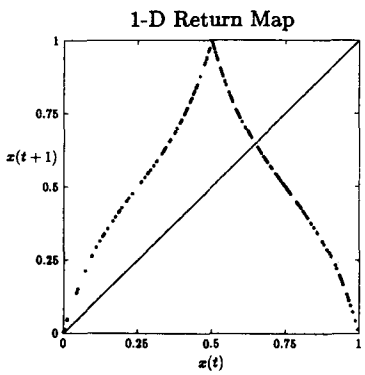

Fig. 2 Return map of the chaotic time-series (Fig. 1) at scale index $j=1$.

2.2 Extensions to Many Rules Cases

We now systematically extend the above case of three labels to infinite labels. The leading principle is described as followings. If the number of labels is increased to infinity, the width of single membership becomes infinitely small. And the inference rules representing the relations between labels are set so as to coincide with a logistic map in this limit. To do this, we have to systematically define (a) how to divide memberships, and (b) how to make inference rules between many labels.

In order to describing (a), we at first assume a *conservation principle* for memberships at any division levels: when all the memberships are algebraically summing up at some division level, the resulting value becomes unity for the defined interval. This stands for

the above three rules case in Sect. 2. 1 with isosceles membership functions.

Under the assumption of the conservation principle, we can show a systematic determination of membership functions at any division levels making use of a *scale function* [10]. The scale function satisfys

$$\sum_k \phi(2^j x-k)=1, \tag{2}$$

which means that at some division level indexed by j the scaled $\phi(x)$ is summing up and gives unity with all the translations indexed by k.

Although the fundamental scale function $\phi(x)$ cannot be uniquely determined, one of the simplest is a isosceles triangle with range [0, 2] and unit height:

$$\phi(x)=\frac{1}{2}|x|-|x-1|+\frac{1}{2}|x-2|, \tag{3}$$

At this stage we regard the scale function $\phi(2^j x-k)$ to be a membership function with division level j and membership label k. For three labels case in the previous section, since the triangular memberships have width unity, 'SMALL' is corresponding to $\phi(2x+1)$, 'MEDIUM' to $\phi(2x)$, and 'BIG' to $\phi(2x-1)$, respectively. Thus the division level is indicated by $j=1$, and the membership label 'SMALL', 'MEDIUM', and 'BIG' are indicated by translation indices $k=-1$, 0, 1, respectively.

It is thus easy to extend to any division levels. If the I/O of the fuzzy inference mechanism is confined within an interval [0, 1], the scale index j must be greater than unity, and the translation index k is also confined within $[-1, 2^j-1]$, by which memberships are labelled. Then the total number N of membership labels within [0, 1] are given by $N=2^j+1$. In case of arbitrary scale index j, it is more convenient to label the membership function by integer k than the linguistic label of memberships. In Table 1, we show an example of $N=3$, in which the scale index j is set to be unity.

We now describe how to determine the inference rules systematically. Although there are infinitely large choices to determine the rules for single-input and single-output, we adopt our leading principle such that the I/O correspondence is equivalent to the one-dimensional logistic map function $f(x)=Ax(1-x)$ in the limit of infinite scale index j, i.e., infinite number of labels.

According to this leading principle, we relate label k of *if*-part to that of *then*-part making use of the one-dimensional logistic map as the following procedures.

(1) At first the division level is chosen by specifying the scale index j. Then the input label $k_{\rm in}$'s are spanned on integer space from -1 to 2^j-1.

(2) A given input label $k_{\rm in}$ of *if*-part is converted into a real number $\hat{x}_{\rm in}$ within [0, 1], which is meant by the peak position of the membership function.

$$\hat{x}_{\rm in}=\frac{1+k_{\rm in}}{2^j}. \tag{4}$$

(3) In order to make a nonlinear rules, $\hat{x}_{\rm in}$ is mapped into a real number $\hat{y}_{\rm out}$ through

$$\hat{y}_{\rm out}=f(\hat{x}_{\rm in}). \tag{5}$$

Here the map function f represents nonlinear mechanism, and is taken to be one-dimensional logistic map defined by

$$f(x)=Ax(1-x), \tag{6}$$

where A denotes a bifurcation parameter.

(4) The next step is to discritize the real number $\hat{y}_{\rm out}$ into the corresponding output label $\hat{k}_{\rm out}$ of an integer. At first,

$$\hat{k}_{\rm out}=2^j\hat{y}_{\rm out}-1, \tag{7}$$

and the resulting real number $\hat{k}_{\rm out}$ is discretized into an output label by taking a nearest integer $k_{\rm out}$ by

$$k_{\rm out}=\mathrm{sign}(\hat{k}_{\rm out})\,\mathrm{int}\left(|\hat{k}_{\rm out}|+\frac{1}{2}\right). \tag{8}$$

where $\mathrm{sign}(x)$ is a signature function giving $+1$ for $x>0$, and -1 for $x\leqq 0$. The function $\mathrm{int}(x)$ is meant by taking largest integer smaller than x.

Thus we obtain desired inference rules: if *input* is

Table 1 Relations between membership labels and translation indices in $N=3$. Scale index j is set to be unity in Eq. (2), and the number of membership labels $N=3$, the maximum translation index $k=1$.

Label	k	peak position
SMALL	−1	0
MEDIUM	0	0.5
BIG	1	1

Table 2 Fuzzy inference rule for k-label in case $j=2$, $N=5$. Labels $k_{\rm in}=(-1, 0, 1, 2, 3)$ of *if*-part are mapped into $k_{\rm out}$ of *then*-part when the bifurcation parameter A in Eq. (6) is specified.

	$k_{\rm in}$					A
	−1	0	1	2	3	
	−1	−1	−1	−1	−1	[0.0 – 0.5]
	−1	−1	0	−1	−1	[0.5 – 0.66667]
	−1	0	0	0	−1	[0.6667 – 1.5]
$k_{\rm out}$	−1	0	1	0	−1	[1.5 – 2.0]
	−1	1	1	1	−1	[2.0 – 2.5]
	−1	1	2	1	−1	[2.5 – 3.3333]
	−1	2	2	2	−1	[3.3333 – 3.5]
	−1	2	3	2	−1	[3.5 – 4.0]

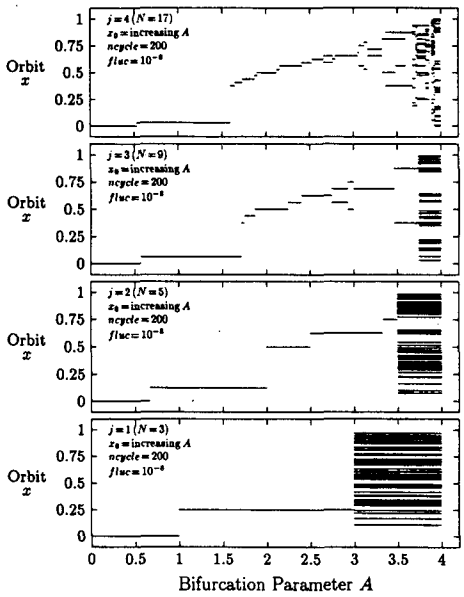

Fig. 3 Bifurcation diagram with increasing bifurcation parameter A. Plotted are orbits without initial transients. The number of iterations are performed up to 200. The initial orbits are taken to be the final orbit with neighboring smaller bifurcation parameter. j denotes scale index of membership function. $N=2^j+1$ indicates number of membership functions. Random fluctuations of $\pm 10^{-8}$ are added to avoid unstable periodic orbits.

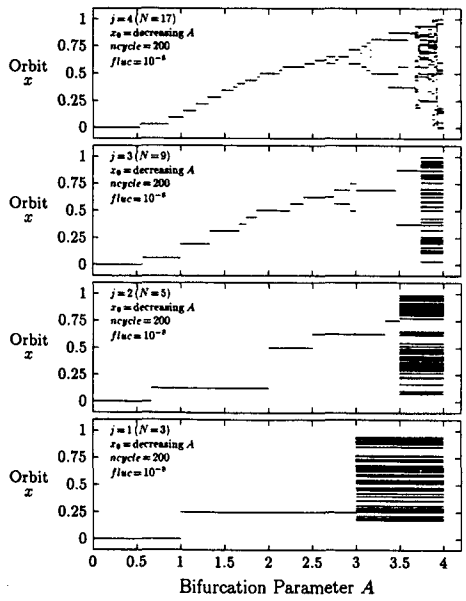

Fig. 4 Bifurcation diagram with decreasing bifurcation parameter A. The initial orbits are taken to be the final orbit with neighboring larger bifurcation parameter. The other situations are same as in Fig. 3.

label-k_{in}, then *output* is label-k_{out}. The one example is listed in Table 2. It should be noted that for a given input label k_{in}, there are finite interval in A falling into the same output label k_{out}, because k_{out} is an integer but the nonlinear mechanism (6) gives a real number. This will lead a discontinuous change although the bifurcation parameter A is continuously changed.

3. Bifurcation Diagram

In order to clarify the dynamical behaviors of self-recurrent fuzzy inference, we show bifurcation diagrams in Figs. 3 and 4. Comparing these with well-known bifurcation diagram of a logistic map, it is found that (a) periodic states tend to prevail against chaos as the number of memberships is increasing, i.e., the chaotic parameter range in A becomes narrower for larger j, (b) period-doubling route to chaos is not assured as the bifurcation parameter is increasing, and a variety of kind of periods are observed for large bifurcation parameter.

The following peculiarities are also found. (c) Existence of unstable periodic states and (d) co-

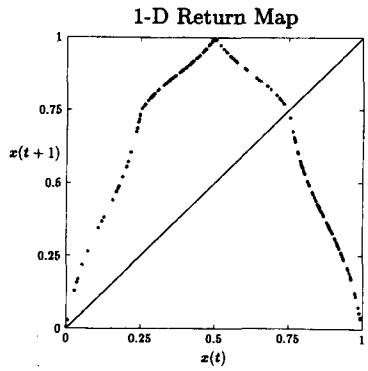

Fig. 5 Return map of the chaotic time-series at scale index $j=2$.

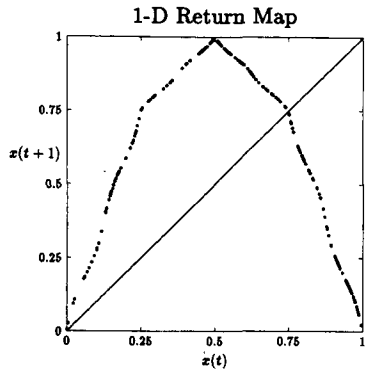

Fig. 6 Return map of the chaotic time-series at scale index $j=3$.

existence of periodic attractors. These are obviously seen as hystereses in Figs. 3 and 4 for the scale index $j \geqq 3$. The effects of hysteresis are remarkable near the bifurcation points.

It is concluded from the above observations that the stability tendencies are meant by (a) and (b). On the other hand, (c) and (d) imply unstable and multistable tendencies, respectively.

It is also found that chaos is observed at the largest bifurcation parameter for every scale index j. The case for $j=1$ has been shown in Fig. 2. And we now show return maps of chaotic time-serieses for $j=2$ and $j=3$ in Figs. 5 and 6, respectively. It is noticed that the shapes of these one-dimensional maps are not smooth but patchy. These interval of the patch is corresponding to the interval between two peaks of neighboring memberships. Since these patches are not linear shaped but nonlinear, we name these maps as *piecewise nonlinear maps*.

4. Discussions

The effects of *fuzzy division* appears in (a) existence of unstable fixed points, and (b) coexistence of periodic attractors. Although the unstable fixed points are also present in the logistic map, characteristic are those positions at which the memberships have high symmetry. For example, in case of scale index $j=1$, there are five high symmetric points in fuzzy memberships, i.e., three of them are corresponding to peak positions of memberships, and two are cross points of neighboring memberships. It is easily shown that these five are connected with others, because two cross points are mapped onto the peak of 'MEDIUM' which is mapped onto the peak of 'BIG'. And the peak of 'BIG' is further mapped onto the peak of 'SMALL'. It should be noted that, although these mappings hold good in chaotic region with bifurcation parameter $A \geqq 3$, the nearby orbits are all chaotic. Then, the above mappings are necessarily unstable against infinitely small fluctuations.

The another peculiarity is strong dependence on the initial values. This is observed, for example, in case of the scale index $j=4$ with bifurcation parameter $A=3.55$. When we change its initial value, three final states of period-1, -2 and -4 are found, for example, with initial values 0.71875, 0.25 and 0.5, respectively, as shown in Fig. 7. We find in Fig. 7 that period-1 and -4 are stable, but unstable is period-2 falling finally into the stable period-4. The coexistence of stable period-1 and -4 implies multistability which is accompanied with hysteresis as changing bifurcation parameter A as shown in Figs. 3 and 4. It is also noted that tunning or learning of memberships may cause this type of hysteresis since changing bifurcation parameter A is corresponding to tuning.

These results of initial value dependences associated with coexistence of periodic attractors are mainly due to nonlinearity of fuzzy interporation. Then these are general consequences so long as the fuzzy inference are concerned and repeatedly applied.

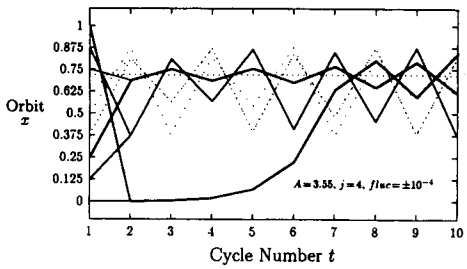

Fig. 7 Initial transients of orbits under random fluctuation $\pm 10^{-3}$. Ten orbits are plotted, which have initial values $x_0=$ 0.125, 0.25, 0.375, 0.5, 0.625, 0.71875, 0.75, 0.875, 1.0. After 3 cycles, three period-4 orbits, two period-2 and single period-1 remain.

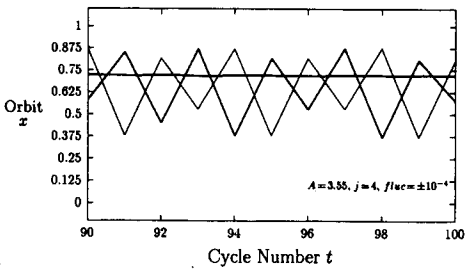

Fig. 8 Final orbits under random fluctuation $\pm 10^{-3}$. Four orbits in Fig. 7 are unstable, and finally only three orbit (two period-4 and single period-1) are stabilized.

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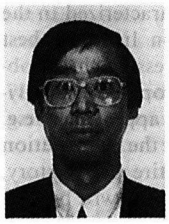

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研究論文(15)　　9639247

自動車運転時のドライバーの脈波のゆらぎ*

片山　硬[1)]　　橋本　博[2)]　　阪井　和男[3)]

Fluctuation of Capillary Blood Pulse in Driving on Public Road

Tsuyoshi Katayama　　Hiroshi Hashimoto　　Kazuo Sakai

Capillary blood pulses of the finger were measured while subjects were driving on public roads. These data were analyzed by means of return-map in order to discuss fluctuation of blood flow. It was found that capillary blood pulses show various fluctuations in different driving conditions. The nature of the pulse fluctuation is available for estimating driver's internal states. Indices suitable for expressing the fluctuation are moment and density of return-map.

Key words : Human Engineering, Driver Behavior / Accident Prevention, Capillary Blood Pulse, Internal State, Fluctuation

1．まえがき

最近の交通事故の増加に伴い，事故予防の重要性が認識されている．事故予防では，車両側の対策とともに，運転者に関する対策も重要な項目の一つである．なかでも運転者の心身状態を推定し，事故防止に役立てる手法が従来より提案されている[(1)～(2)]．これらは，心身状態のうちで，事故に直結する覚醒度を推定し，いねむり運転を防止するものである．心身状態の変化は循環器系によく現われるとされ，心電位を基にした心拍数の変動と覚醒度に関する研究が進められてきた[(3)～(4)]．また，毛細血管に流れる血液も心身状態に依存した多彩なゆらぎを示すことが知られている．特に指先の血流を表わす指先脈波は覚醒度等により顕著なゆらぎを示す[(5)]．

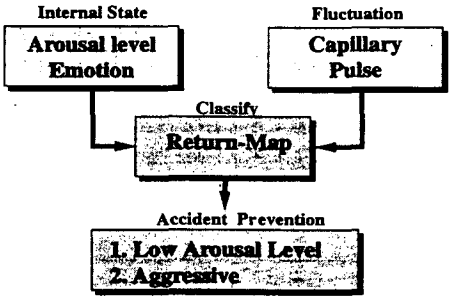

図１　心身状態の推定と予防安全

本研究では，図１に示すように，ゆらぎを視覚的に把握可能なリターンマップの手法[(6)]を用いて運転者の心身状態と指先脈波のゆらぎとの関係を明らかにし，心身状態を脈波のゆらぎにより推定することにより，居眠り運転，闘争的な運転等の事故に結びつきやすい運転を防止する方法を見い出すことを最終目的としている．

2．室内実験

実走行時の計測に先立ち，情動変化と脈波のゆらぎに関する基礎特性を得るために室内実験を実施した．40歳代の男性２名および女性１名を被験者とし，爽快感，不快感，闘争的な気分および恐怖感を生じるビデオ映像を提示し，その時の指先脈波を計測した．

実験手順としては，図２に示すように，まず比較データとなる安静閉眼時，クイズ解答時，安静開眼時の計測を行い，次にビデオ映像を提示する．その後ビデオ映像等に関するアンケート調査を実施し，最後に爽快感を与えるビデオ映像を提示する．この様な手法を用いて，各被験者に対して１日に1条件の測定を実施し

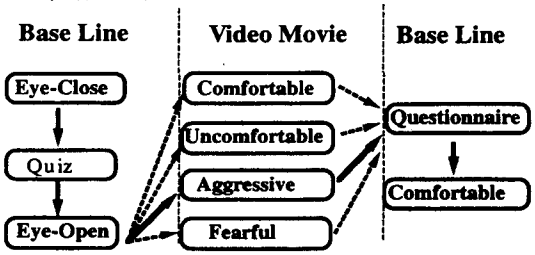

図２　室内実験の実験手順

*1996年１月９日受理．　1995年９月28日自動車技術会秋季学術講演会において発表．

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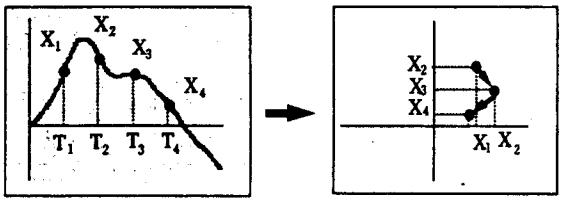
3) 明治大学法学部（168　杉並区永福１-９-１）．

た．

計測した時系列データは，図３に示す手法を用いて，リターンマップの形式に表現した．すなわち，時系列データを，ある時間間隔でサンプリングし，隣り合う数値をＸ座標－Ｙ座標に繰り返しプロットし，リターンマップを作成した．

なお本研究で使用するリターンマップのＸ軸およびＹ軸の目盛りは任意スケールであり，同一の被験者に対しては同一のスケールを使用することにする．

典型的なリターンマップの例を図４に示す．これは恐怖感を生じるビデオ映像を提示した際の指先脈波リターンマップである．最上段の２種類および中段の左図は比較用データであり，それぞれ安静閉眼時，クイズ解答時および安静開眼時のマップである．中段の右のマップが恐怖ビデオ鑑賞時のものを表している．下段の二種類の図は，それぞれビデオ鑑賞後のアンケート回答時および実験終了間際に提示する爽快ビデオ鑑賞時のマップである．



Time Series　　Return-Map

図３　時系列データからリターンマップの作成

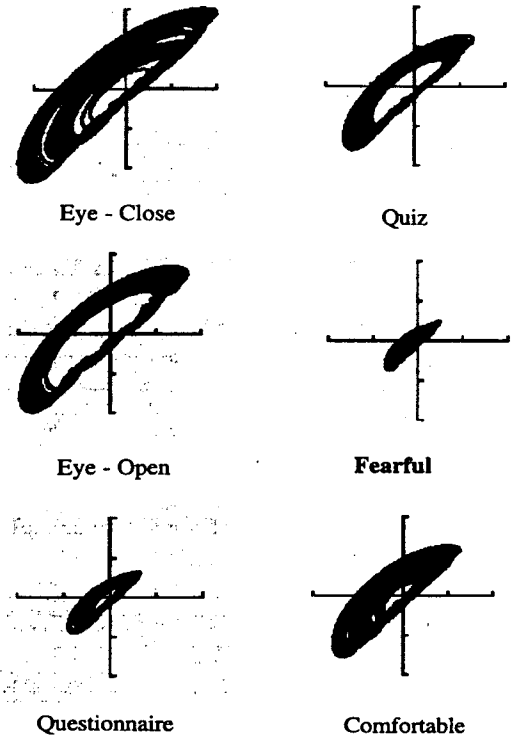
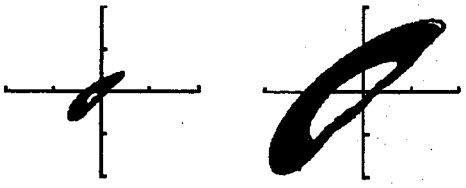


図４　室内実験でのリターンマップ例

同図が示すように，それぞれの心身状態によってリターンマップの大きさおよび形状等が異なることが分かる．そこで，これらの形状を分類する指標として，ここではモーメントおよび稠密度と呼ぶ２種類の量を導入する．以下にその定義と内容を述べる．

（１）モーメント

二次元のリターンマップの重心点まわりの極モーメントを用いる．図５に，その概念図を示す．同図の左のマップはモーメントが小さく，右側の図は大きな値をとる．この指標は，指先の血流量を表現している．すなわち血流量が増大するとモーメントが大きくなる．

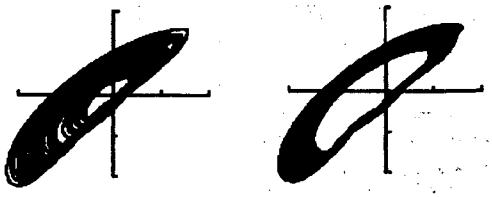


Small Moment　　Big Moment

図５　リターンマップのモーメントの大小

（２）稠密度

モーメントをマップのＸ座標の最大値の二乗に比例した数値で正規化した量で定義する．すなわち $D= M/ (AX_m^2)$ と定義する．ここでＤは稠密度，Ｍはモーメント，X_m はマップのＸ座標の最大値を表し，Ａは比例係数である．したがってマップが外縁部に集中するにつれ，稠密度が増大する．稠密度の大小を表わす概念図を図６に示す．この量は血流量の規則性に関連したもので，規則性が高まると稠密度が大きくなり，不規則になるにつれて小さな値をとる．



Low Density　　High Density

図６　リターンマップの稠密度の大小

各状態におけるモーメントおよび稠密度を図７および図８に示す．図７にみるように，不快ビデオ，闘争的ビデオ，恐怖ビデオ鑑賞時およびアンケート回答時のモーメントは他に比較して小さくなることが分かる．一方，図８が示すように，不快ビデオ，闘争的ビデオおよび恐怖ビデオ鑑賞時には稠密度が低下する．

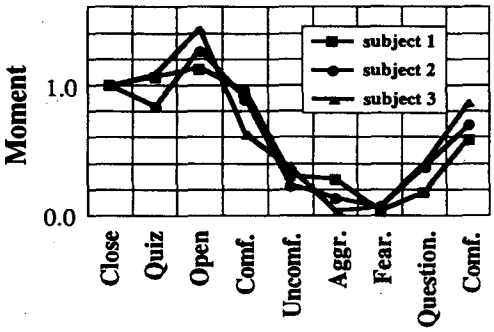


図７　各状態におけるモーメントの値

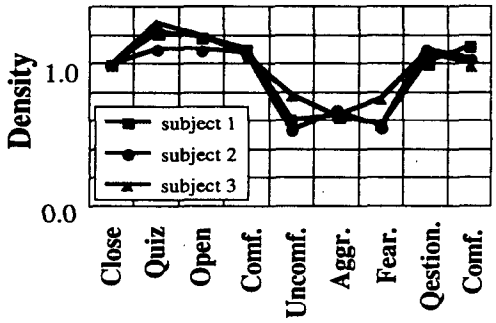


図８　各状態における稠密度の値

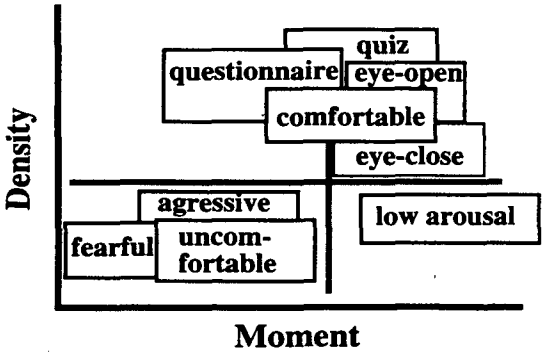


図９　心身状態とモーメント - 稠密度マップ

この結果より，図９に示すように，モーメントおよび稠密度を用いて，心身状態のクラス分けが可能であることが分かる．

３．運転時の脈波

自動車運転時の脈波を解析するために，被験者９名を用い実交通路での脈波計測を実施した．被験者は２０歳代～４０歳代の男性７名女性２名とした．また実交通路として，つくば市内の一般道，常磐自動車道および首都高速道路を選択した．

実験中にはさまざまな交通環境に出合った．その際の指先脈波のリターンマップは個人差は現れるものの，９名の被験者とも定性的には同様の傾向が見られた．その代表的なリターンマップの例を図１０に示す．同図の最上段は安静閉眼時および一般道走行時のリターンマップである．第二段左の図は一般道において右折行動を起こす直前のものであり，右図は常磐自動車道でのトンネル走行時のものである．第三段左図は小菅において他ルートとの合流時のものを表しており，右図は長期の渋滞時のマップである．首都高速道路銀座線走行中および目的地であるレインボーブリッジ走行中のリターンマップはそれぞれ最下段に示され

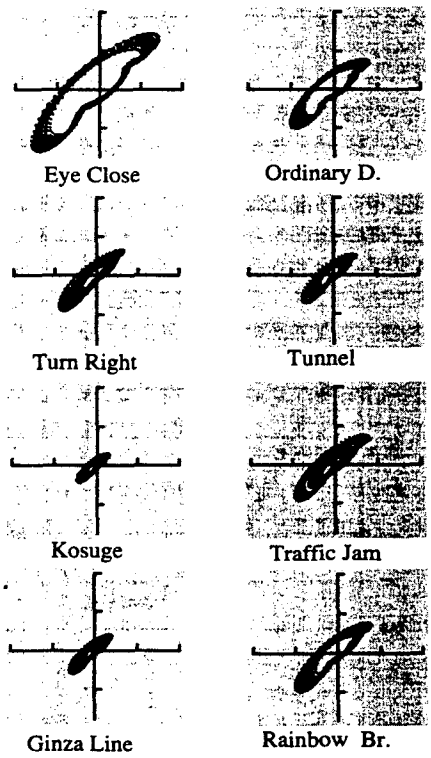


図１０　実走行時のリターンマップの例

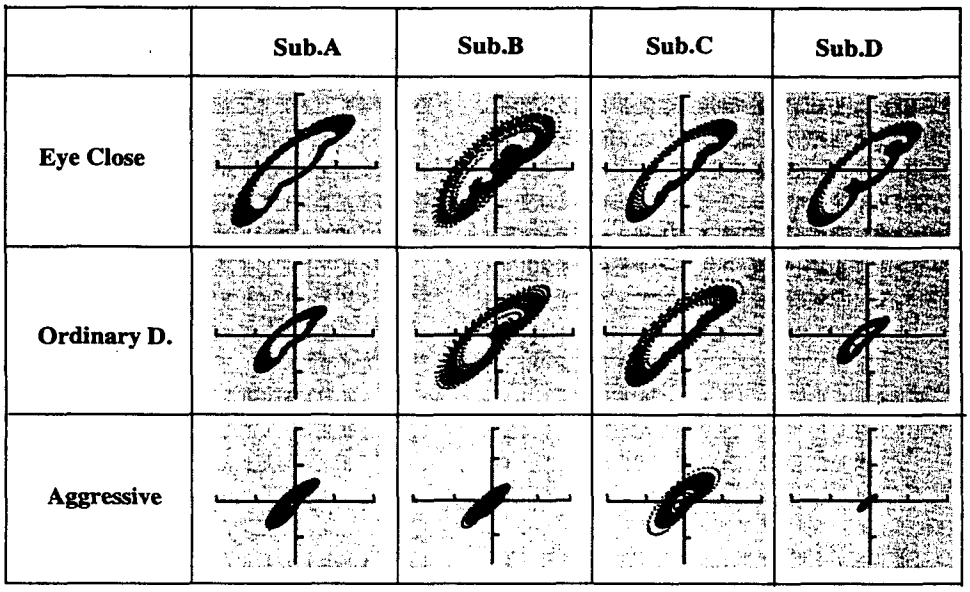


図１１　闘争的な車両に遭遇時のリターンマップ

ている．このように実交通路走行中には，指先脈波は多彩なゆらぎを示すことが分かる．

また，本計測中に４名の被験者は，料金所等の合流時において非常に闘争的な車に遭遇している．その際の脈波のリターンマップを図１１に示す．この図中の各列に，それぞれの被験者のマップが示されている．最上段の図は安静閉眼時，中段の図は一般道における通常走行時，下段の図は闘争的な車両に遭遇した際のマップである．

同図が示すように，各ドライバとも安静閉眼時に比べ，通常走行時にはマップが小さくなる．また闘争的な車両に遭遇すると，マップが更に小さくなり，かつ不規則性が強くなることが分かる．

この４名のドライバの代表的な交通場面におけるリターンマップのモーメントおよび稠密度の値の変化を図１２および図１３に示す．

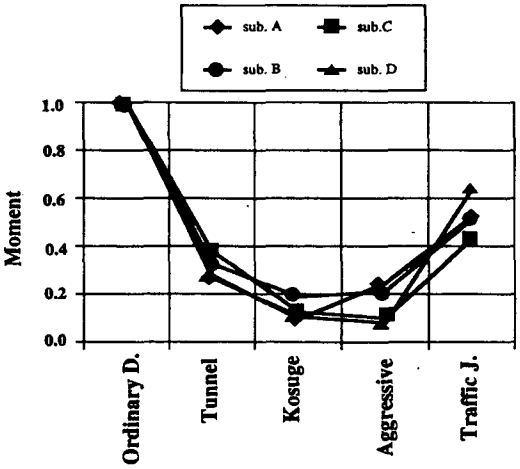


図１２　実走行時のモーメントの値

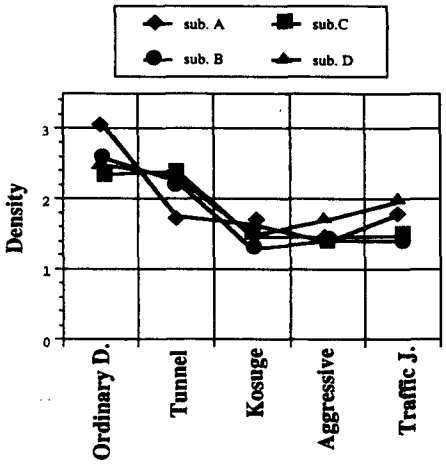


図１３　実走行時の稠密度の値

図１２は各場面におけるモーメントの値を，通常走行時の値を基に換算したグラフである．この図が示すように，トンネル走行時，小菅での合流時および闘争的な車両に遭遇時には，モーメントの値が著しく低下することが分かる．また渋滞時には各被験者ともモーメントが増大している．一方図１３より，小菅の合流，闘争的な車両に遭遇時および渋滞時にはリターンマップの空密度が低下しているといえる．

実走行時のここで得られた結果を基に，上で導入したモーメントおよび稠密度を用いて図１４に示すように，各状態をクラス分けすることができる．すなわち，通常走行時を基準にして，闘争的な車両に遭遇等の異常に緊張した場面では，モーメントは小さく（血流量が少く）稠密度が低く（ゆらぎが大きく）なる．またトンネル走行等で緊張する場合にはモーメントは小さくなるが，稠密度の大幅な低下は起こらない．一方渋滞時等で覚醒度が低下した場合には，稠密度が大幅に低下するがモーメントの大きな低下は生じない．

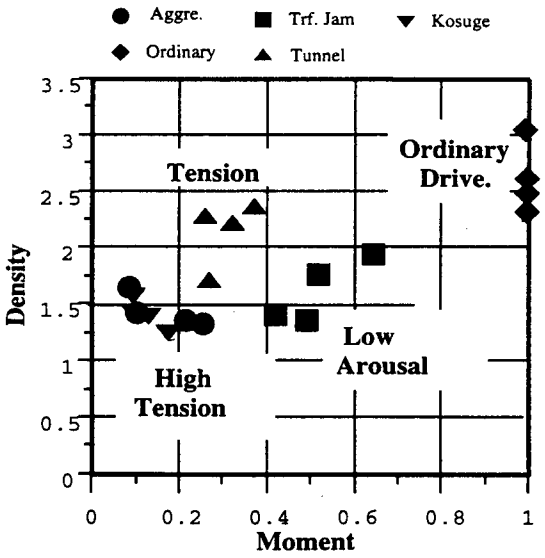


図１４　実走行時のモーメント - 稠密度マップ

４．まとめ

以上の結果は，被験者が10名程度と少数ではあるが，指先脈波のリターンマップの形状により，心身状態が推定可能であることを示している．車の予防安全という観点からこの結果をみると，次の二つの場合が注意を要する状態であると考えられる．

（１）モーメントが大きく（血流量が多く）稠密度が低い（ゆらぎが大）．この場合には，いねむり運転の可能性がある．

（２）モーメントが小さく（血流量が少なく）稠密度が低い（ゆらぎが大）．この場合には，闘争的，恐怖感あるいは不快感等を感じて運転している可能性がある．

したがって，運転中のドライバーの指先脈波を計測し，そのリターンマップのモーメントおよび稠密度が，上記（１）あるいは（２）の状態になると，ドラバーは通常時に比べて，事故を起こし易い状況に近づいていると推定される．

このように，指先脈波のゆらぎを基にして，ドライバーの心身状態がある程度推定可能であるといえる．しかし，現実の事故予防に役立てるには，更に多数の実験研究を重ねる必要がある．

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(No. 94035)
《研究速報》

心身状態と脈波のゆらぎ[*1]

Fluctuation of Capillary Pulse as an Index for Internal States

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Abstract

Fluctuations of finger capillary pulse are discussed in different internal states, especially in different emotional states. Capillary pulses of finger were measured while subjects were watching video movies. Fluctuations of capillary pulse were analyzed by means of return-map. It is found that fluctuations of capillary pulse are available for estimating driver's internal states. Suitable indices for fluctuation are moment and density of return-map.

1．まえがき

最近の交通事故の増加に伴い，事故予防の重要性が再認識されている．事故予防では，車両側の対策とともに，運転者に関する対策も重要な項目の一つである．なかでも運転者の心身状態を推定し，事故防止に役立てる手法が従来より提案されている[1)~2)]．これらは，心身状態のうち，事故に直結する覚醒度を推定し，いねむり運転を防止するものである．心身状態の変化は循環器系によく現れ，心電位を基にした心拍数の変動と覚醒度に関する研究が進められてきた[3)~4)]．また，毛細血管に流れる血液も多彩なゆらぎを示すことが知られている．特に，指先の血流を示す指先脈波は，覚醒度などの心身状態に依存したゆらぎを示す[5)]．

本研究では，図1に示すように，運転者の心身状態により変化する指先脈波のゆらぎをリターンマップ[6)]の形状により分類し，運転者の心身状態を指先脈波のゆらぎにより推定し，いねむり運転，闘争的な運転などの事故に結びつきやすい運転を防止することを最終目標としている．

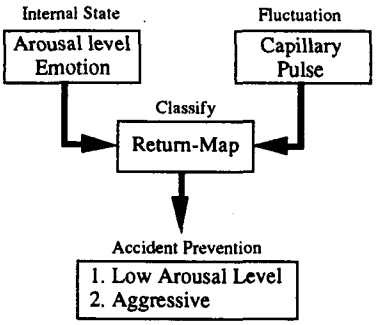

図1　心身状態と事故予防

その第一段階として，既報[5)]での覚醒度と脈波のゆらぎに関する研究に引き続き，情動変化と脈波のゆらぎとの関連を調べる．ここで用いる手法は爽快感，不快感，闘争的な気分および恐怖感を生じるビデオ映像を被験者に提示し，その時の指先脈波を計測し，そのゆらぎと情動状態との関連を調査する．

2．実験方法

この研究で実施した実験手順の概要を図2に示す．主要な実験は，異なる4種類の情動状態を生じるビデオ画像鑑賞時の指先脈波計測である．最

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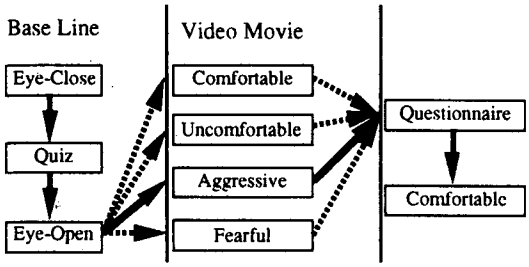

図２　実験手順

初に比較データとなる安静閉眼時（７分間），クイズ解答時（７分間）および安静開眼時（７分間）の脈波計測を行い，次にそれぞれの情動状態を生じるビデオ画像を30分～60分間提示し，計測を実施した．その後に，提示したビデオ画像についての主観評価に関するアンケート回答時の測定を行う．最後に，再び，爽快感を伴うビデオ画像を30分間提示する．各被験者に対して，１日に１情動状態の計測，すなわち，４日間に分けて測定している．実験は，ビデオを用いて爽快感，不快感，闘争的な気分，恐怖感を生じる実験の順序で実施した．また計測は毎回午後１時30分に開始するように設定している．

実験環境として，外部の音声を遮断した空調設備の整った音響実験室を選択した．

被験者は40歳代の男性２名および女性１名である．

３．実験結果

計測結果の代表例を，図３および図４に示す．これらは，指先脈波40秒間のデータを基に作成したリターンマップである．リターンマップは，ゆらぎを視覚的に理解することに有効なもので，時系列データを，ある時間間隔でサンプリングし，隣り合う数値をX座標－Y座標に繰り返しプロットしたものである[6]．図３の上段には，ベースラインデータとなる安静閉眼時，クイズ回答時および安静開眼時のリターンマップが示されている．下段左側に主要な結果，すなわち闘争的なビデオ鑑賞時のマップが示されている．同様のリターンマップが，図４に示されている．この場合は下段左側に，恐怖感を生じるビデオ鑑賞時のものが表示されている．

図３および図４が示すように，それぞれの状態によってリターンマップの大きさおよび形状などが異なることがわかる．そこで，これらの違いを分類する指標として，ここではモーメントおよび稠密度と呼ぶ２種類の指標を導入する．以下に，

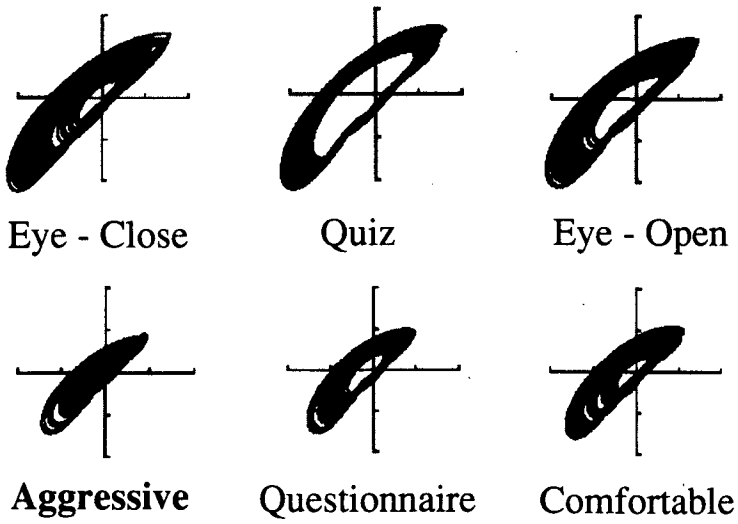

図３　指先脈波のリターンマップ例

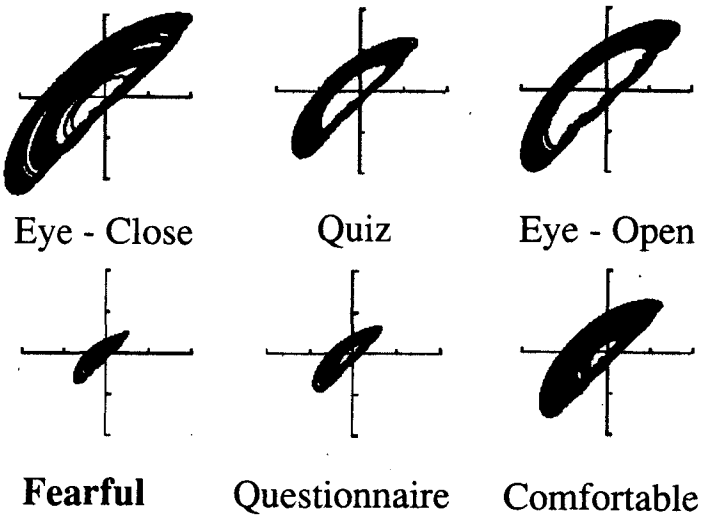

図4　指先脈波のリターンマップ例

その定義と内容を簡単に述べる．

(1) モーメント

二次元のリターンマップの重心点まわりの極モーメントを用いる．これは，指先の血流量を表現している．すなわち，血流量が増大するとモーメントが大きくなる．

(2) 稠密度

モーメントをマップの最大値の二乗で正規化した量で定義する．マップが外縁部に集中するにつれ，稠密度が増大する．この量は血流量のゆらぎの大きさに関連している．すなわち，ゆらぎが減少すると稠密度が増加する．

各状態におけるモーメントおよび稠密度を図5および図6に示す．これらは安静閉眼時の数値で正規化したものである．

図5にみるように，安静閉眼時，クイズ解答時，安静開眼時および爽快ビデオ鑑賞時のモーメントは大きく，不快ビデオ鑑賞時，闘争ビデオ鑑賞時，恐怖ビデオ鑑賞時およびアンケート回答時のモーメントは，安静閉眼時に比較して約40%以下となる．

一方稠密度は，図6が示すように，安静閉眼時，クイズ解答時，安静開眼時，爽快ビデオ鑑賞時およびアンケート回答時には高い数値をとり，

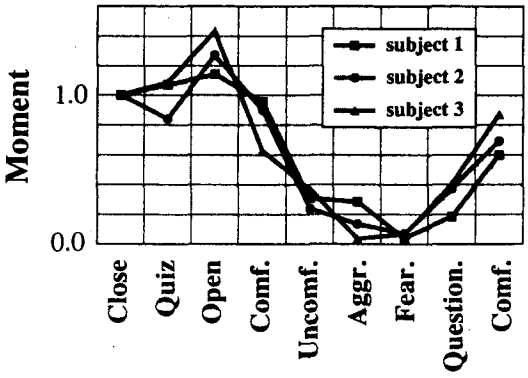

図5　心身状態とリターンマップのモーメント

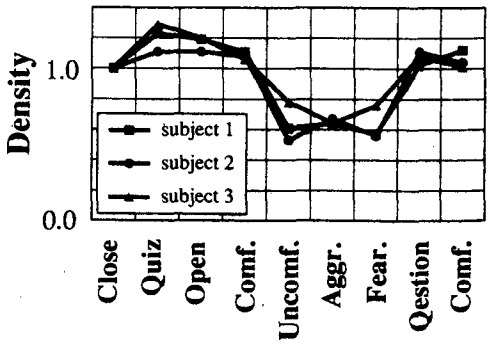

図6　心身状態とリターンマップの稠密度

不快ビデオ鑑賞時，闘争ビデオ鑑賞時および恐怖ビデオ鑑賞時には低い数値を示す．

このように情動を伴う身心状態は，指先脈波のモーメントおよびその稠密度を用いてクラス分けが可能であることがわかる．既報[5]で報告した覚醒度に関する結果を含めて，各心身状態と指先脈波のモーメントと稠密度による概念図を図7のようにまとめることができる．これは，横軸にモーメントをとり，縦軸に稠密度をとっている．安静開眼時およびクイズ解答時にはモーメントが大きく(血流量が多く)，かつ，稠密度が高い(ゆらぎが小さい)．覚醒度が低下すると，モーメントが大きく，稠密度が低くなる．また，アンケート回答時にはモーメントが小さく，稠密度が高くなる．一方，闘争・不快・恐怖を伴うビデオ鑑賞時には，モーメントが小さく(血流量が少なく)，かつ，稠密度が低下する(ゆらぎが大きくなる)と言える．

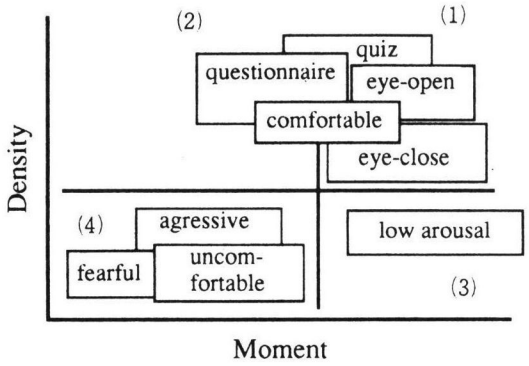

図7　心身状態とモーメント－稠密度図

4．まとめ

以上の結果は，被験者が3名と少数ではあるが，指先脈波のリターンマップの形状により，心身状態推定の可能性を示している．車の予防安全という観点からこの結果をみると，次の二つの場合が注意を要する状態であると言える．

(1) モーメントが大きく(血流量が多く)稠密度が低下(ゆらぎが大)．この場合には，いねむり運転の可能性があり，

(2) モーメントが小さく(血流量が少なく)稠密度が低下(ゆらぎが大)．この時には闘争的，恐怖感あるいは不快感を感じていると推定できる．

したがって，運転中のドライバの指先脈波をモニターし，そのリターンマップのモーメントおよび稠密度が，上記(1)および(2)の状態になると，このドライバは通常時に比べて，事故を起こしやすい状況に近づいていると言える．

このように，指先脈波のゆらぎを基にして，ドライバの心身状態が推測可能である．しかし，現実の事故予防に役立てるには，更に多数の実験研究を重ねる必要がある．

今後の課題は，実交通状況における脈波の計測と，その解析および車載可能な脈波ゆらぎ計測システムの開発である．

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片山　硬　　阪井和男　　中島和子

指先の血流は，顔面などのものに比べ著しくゆらいでおり，そのゆらぎにより人の心身状態が推定できそうである．この結果を基に，交通安全に役立つドライバの心身状態モニターシステムを完成させたいと考えています．

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Synchronization of nonlinear systems with distinct parameters: Phase synchronization and metamorphosis

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Rather than the synchronization between identical chaotic systems, the "phase synchronization" among two or many nonlinear systems with distinct nonlinear parameters is investigated. It is observed that the dynamics of globally coupled N_1 periodic and N_2 chaotic systems can be reduced to that between matrix-coupled chaotic and periodic systems as the result of a two-step realization of the synchronization among the former systems. The same situation holds even for the array of distinct systems with a nearest-neighbor interaction.

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The brain is a network of a vast number of chaotic neurons and for the pattern recognition the requisite condition is supposed to be the synchronization among neurons and the appearance of the macroscopic coherent mode in the network. For instance, Ott, Grebogi, and Yorke [1] have presented recently an intriguing view that the intelligence is supported by a chaotic switch. That is, chaos lives with infinitely many unstable periodic orbits and a periodic orbit transits to another most efficiently via the chaos. A nice demonstration of this possibility is given for a single unit chaos. Thus if the cells in the network synchronize we can regard them as one unit and the regularity of a few units may also be applied to the network. This possibility is the case which we study in this article.

There exist interesting investigations on the synchronization of chaotic cells [2–5]. In particular, Pecora and Carroll [2] found that two chaotic attractors with exactly the same (slightly different) nonlinear parameters synchronize perfectly (keeping tiny distance) when they are coupled together in a way that one of the dynamical variables of the master flow is substituted for the corresponding one of the slave flow. One may regard this shared variable as a common driving term and others (two subsystems) as under the influence of it via feedback. Their finding that even the chaotic flows synchronize triggered further studies including the synchronization under common noise terms [6]. As for the coupled map system Kaneko found that the globally coupled map lattice is endowed with remarkably rich clustering structure and that the positive-negative switch can be realized among the clusters by input [3].

We should note that all of the previous investigations are directed to the finding of the precise synchronization between identical units. In this article we address ourselves to the question of what the outcome would be if we couple together the same systems but with distinct nonlinear parameters. By constructing a simple coupling model based on the globally coupled map lattice we find an amazing phenomena that the systems flow synchronizing in the phase but with different sizes and/or positions. This phenomena may be termed as the *phase synchronization*. Actually in the brain what is important is presumably the phase locking between the neurons and is not the precise status of each. Thus our observation of the phase synchronization may have an important consequence to the activity of the brain.

For two systems we find that two nonlinear systems evolve in interaction in two new phase synchronizing flows in the phase space, the new pattern depending on the coupling, even if the set of parameters in one system is set in the chaotic regime and that in the other in the periodic regime. The rule of the dependence is in general quite simple; if the coupling favors the chaotic (periodic) system the new pattern is generally also chaotic (periodic). For large number of systems we prepare N systems in two categories; N_1 chaotic and N_2 periodic systems. We find that a rule like the above holds that the majority wins the minority. We also find that in a certain way the population ratio (N_1/N) can be related to the matrix parameter θ in the $N=2$ model. The implication of this relation is the main theme of this article.

Let us first describe our model for the simple case of $N=2$. We take two nonlinear systems with variables more than one. As an example we take the Lorenz systems and we treat them as discretized maps with the parameters for one system in the chaotic regime $(r_1=28, b_1=8/3, P_1=10)$ and for the other in the periodic regime $(r_2=270, b_2=8/3, P_2=10)$. At each time step the systems $(i=1,2)$ first evolve via the flow equations

$$\begin{aligned} x_i(t+\Delta t)&=x_i(t)+P_i(y_i-x_i)(t)\Delta t\ ,\\ y_i(t+\Delta t)&=y_i(t)+(-x_iz_i+r_ix_i-y_i)(t)\Delta t\ ,\\ z_i(t+\Delta t)&=z_i(t)+(x_iy_i-b_iz_i)(t)\Delta t\ . \end{aligned} \tag{1}$$

Then they interact each other by a simple matrix with two continuous parameters ϵ and θ in only one of the three variables x,y,z.

For instance, for the "x coupling,"

$$\begin{aligned} (1-\epsilon_2)x_1+\epsilon_2x_2&\mapsto x_1'\ ,\\ (1-\epsilon_1)x_2+\epsilon_1x_1&\mapsto x_2'\ , \end{aligned} \tag{2}$$

with $\epsilon_1=\theta\epsilon$, $\epsilon_2=(1-\theta)\epsilon$ $(0\le\epsilon,\theta\le 1)$. The mechanism is that system 1 receives the effect of system 2 with a coupling constant ϵ_2 and vice versa. The systems evolve repeating this two-step process of map and interaction. Roughly the parameter ϵ gives the coupling strength and the parameter θ acts as a weight factor between the two systems. The sequence of map and interaction is the usual one in the coupled map analysis [3]. The difference is that we couple only one dimension to create a 'mean field' and the other subsystems are evolving under the influence of this mean field. We should add that in our model we define the driving term as a weighted mean between the x_1 and x_2 rather than $\dot{x}_1$ and $\dot{x}_2$. The subsystems evolve as nonautonomous flows (discretized in the time step Δt) and the mean field is calculated as in the usual map model. Thus our model is a multidimensional map model with coupling in one dimension. In this respect we are using the (discretized) Lorenz system as a typical sample of the multidimensional map. However, interestingly, the model reduces to a smooth flow at the limit $\Delta t\to 0$ which can be explained from the following argument. The interaction (2) serves to focus the motion of units to the mean field while the nonlinear evolution (1) acts in defocusing direction. Under a certain balance the systems fall into the attractor. This is just the same with the usual coupled map model but our model differs from them in that the subdimensions [e.g., y_1 and y_2 in (1)] have distinct nonlinear parameters (r_i). Near the onset of the attractor the driving variables (x_1 and x_2) come close to each other and the variations due to the interaction (2) become as small as the variation in (1). As the effect the orbits of our model become continuous flow. In fact we have checked that all the following results are unchanged for any choice of sufficiently small Δt (typically $\Delta t\approx 10^{-4}$).

The above matrix form facilitates a way to interpolate various important limits using the parameters ϵ and θ. For instance in the limit of $\epsilon=1$ and $\theta=0$ or 1 our model reproduces the original model of Pecora and Carroll but with a drastic extension that the nonlinear parameters for each system are set at completely distinct values.

To explain our analysis of N systems, let us briefly look at the analysis of the globally coupled map (GCM) by Kaneko [3]. In his model N identical maps first evolve as

$$x_i(n+1)=f(x_i(n))=1-a[x_i(n)]^2 \tag{3}$$

and then interact via a mean field with a coupling (ϵ),

$$(1-\epsilon)x_i+\frac{\epsilon}{N}\sum_j^N x_j\mapsto x_i' \ . \tag{4}$$

Under a certain balance of the nonlinearity (a) and the coherence (ϵ) these maps divide into two clusters and fall into two attractors moving with periodicity two as $(+-+-+\cdots)$ and $(-+-+-\cdots)$. Here the $+$ $(-)$ denotes that the value of the attractor is larger (smaller) than the unstable fixed point x^* of the map $[x^*=(\sqrt{1+4a}-1)/2a]$ at even n. In this two cluster regime the maps (4) reduce to

$$\begin{aligned}(1-\epsilon_-)x_+ +\epsilon_- x_- &\mapsto x'_+\\ (1-\epsilon_+)x_- +\epsilon_+ x_+ &\mapsto x'_-\end{aligned} \tag{5}$$

and $\epsilon_\pm=[N_\pm/(N_+ +N_-)]\epsilon$, where N_+ (N_-) denotes the number of $+$ $(-)$ cells.

The main interest in our large N analysis is the possibility of phase synchronization among many systems with distinct parameters. As the simplest setup we take N_1 systems in the chaotic regime and N_2 systems in the periodic regime. After the evolution (1) in one time step Δt they interact by the same equation as (4) and the y and y' (z and z') do not interact directly. Suppose that the N_1 systems and the N_2 systems synchronize among each and change into two clusters. Then the complicated dynamics expressed by N coupled equations is expected to reduce to a more simple $N=2$ dynamics. This will allow us to repeat formally the reduction from (4) to (5) and relate the dynamics of (N_1,N_2) systems to that of the $N=2$ systems by a simple rule

$$\frac{N_1}{N_1+N_2}=\theta \tag{6}$$

and the subsequent phase synchronization between the two clusters will be observed. This should be checked by the observation that the synchronizing trajectories in the large N systems with certain population ratio N_1/N agree with those in $N=2$ case at the weight factor θ given by the rule (6). We will see in the following that the two cluster formation does take place and that the conjectured reduction holds perfectly in our model. Of course the dynamical requirements enabling the reduction of N to 2 are completely different between the GCM and our model. In the GCM the requirement is that *identical* systems organize themselves into two attractors. In our model the requirement is that N_1 systems and N_2 systems, which are *distinct* in the nonlinear parameters, fall into two clusters before the phase synchronization between the two. This crucial difference should not be overlooked. Now we are ready to present our results in order.

(1) $N=2$ *and the strongest* ($\epsilon=1$) *and one-way* [$\theta=0$ or 1] *limit.* First we briefly review our results for $\epsilon=1$. The coupling for $\epsilon=1$ is

$$\begin{aligned}\theta x_1+(1-\theta)x_2&\mapsto x_1\\ (1-\theta)x_2+\theta x_1&\mapsto x_2 \ .\end{aligned} \tag{7}$$

The limit $\epsilon=1$ and $\theta=0$ means $x_2\mapsto x_1$ and the limit $\epsilon=1$ and $\theta=1$ means $x_1\mapsto x_2$. Thus at $\epsilon_1=1$, $\theta=1$ system 1 becomes the master of the slave system 2. This is nothing but the case studied in the pioneering work by Pecora and Carroll who found that the two chaotic systems with (almost) the same parameters synchronize. We also analyze this extreme coupling case, but we set the two systems at completely different nonlinear parameters. What would the consequence be for the tight, one-way

coupling between periodic and chaotic systems? Do they synchronize somehow or other? This question may look absurd since in the periodic system the phase is naturally defined while in the chaotic system only the trajectory length or diffeomorphic transform of it may be regarded as such. But we find that an amazing answer comes out. The slave system actually *metamorphose* into the same character with the master system and there motion is in complete *phase synchronization* with the master. For instance the periodic slave turned chaotic under the influence of the chaotic master and vice versa. This is by no means trivial, since the systems are coupled by only one of the dynamical variables and the other degrees of freedom are not in direct interaction. We present the figures of the metamorphoses in the following generic case.

(2) *The case for generic* ϵ $(0<\epsilon<1)$ *and both-way* θ $(0\leq\theta\leq 1)$ *coupling.* We show in Fig. 1 the orbits of the coupled Lorenz systems for $\epsilon=0.3$. Figures 1(a) and 1(b) show the result of the coupling for $\theta=0.2$ and $\theta=0.8$, respectively. System 1 is set in the chaotic regime ($r_1=28$, $b_1=8/3$, $P_1=10$) and system 2 is in the periodic regime ($r_2=270$, $b_2=8/3$, $P_2=10$). To be explicit the coupling matrix(2) for the x variable at $\theta=0.2$ is

$$\begin{aligned} &0.76x_1+0.24x_2\mapsto x_1\ ,\\ &0.06x_1+0.94x_2\mapsto x_2 \end{aligned} \tag{8}$$

and at $\theta=0.8$

$$\begin{aligned} &0.94x_1+0.06x_2\mapsto x_1\ ,\\ &0.24x_1+0.76x_2\rightarrow x_2\ . \end{aligned} \tag{9}$$

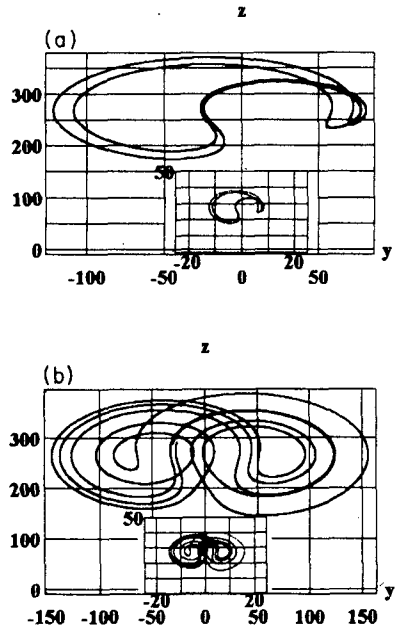

FIG. 1. The phase synchronizing orbits of x-coupled Lorenz systems in the y-z plane. $\epsilon=0.3$ and $\Delta t=10^{-4}$. System 1 in the chaotic regime ($r_1=28, b_1=8/3, P_1=10$) and system 2 in the periodic regime ($r_2=270, b_2=8/3, P_2=10$). Two systems have turned into mutually organized new shapes. (a) The coupling parameter $\theta=0.2$ [See Eq. (8)]. System 1 (chaotic) has metamorphosed under the effect of system 2 (periodic). Note the difference in the scale for each orbit. (b) The coupling parameter $\theta=0.8$. System 2 (periodic) has followed the system 1 (chaotic).

Metamorphosis in this case is in the way that both systems turned into mutually organized new shapes. In the small θ case [Fig. 1(a)] we find that the shape of the periodic system (system 2) is not affected much while the chaotic system (system 1) metamorphoses into periodic flow in due course of interaction. Thus, for small θ, system 2 wins in the competition of the structure decision. In contrast for large θ the winner is system 1 [see Fig. 1(b)]. The synchronization is so perfect that we do not need to show the Lissajous plots. The Lissajous contour is simply a diagonal line of a rectangle for any initial points in the basin. We also have checked that the phase synchronization occurs independently from the choice of the coupling variables as far as the sub-Lyapunov exponent is not positive [2]. However, there seems no unique way to compare the resulting two orbits. For instance, in the case of the x-coupled Lorenz models depicted in Fig. 1, we need two scaling factors in both the y and z directions in order to compare the two orbits. On the other hand in the y-coupled Lorenz system the orbits turn out with almost the same size. Furthermore, in the case of the Rössler model there is no sizable difference but some parallel shift makes the two trajectories almost overlap. Despite these differences we find that in every case the phase synchronization is perfect.

The most important question now is why the parameter θ determines which is the winner among the chaotic and the periodic systems. At the limit $\epsilon=1$, this can be naturally understood because as we saw above $\theta=1$ (0) means system 1 is the master (slave) and the θ smoothly interpolates between these limits. However, at the generic ϵ the case is more subtle as is seen in the nontrivial form of the coupling matrix. For instance at $\theta=0.2$ the first equation of (8) dictates that system 1 wins the game while the second equation dictates the other way round and one cannot tell which is the fate of the flows. We will obtain a plausible answer in the analysis of the N coupled systems.

(3) *The Poincaré map and Lyapunov exponents.* The role of the weight θ is best illustrated by the Poincaré map and the Lyapunov exponents in Figs. 2(a), and 2(b), respectively. The Poincaré map is evaluated by the cut of the contours with the conditions $\dot{z}_i=0$ and $\ddot{z}_i<0$ $(i=1,2)$.

The Lyapunov exponents are evaluated by the basic method [7] by keeping track of the expanding rate of volume of the parallelepiped in the six dimensional phase space. The θ is varied from 0 to 1 in step 0.002 and at any θ the systems remain in phase synchronization. Figure 2 shows that in the small θ region the systems synchronize in periodic flows and in the large θ region in

chaotic flows. The parameter θ indeed acts as a novel control parameter of the nonlinearity of the whole two systems. There are clear periodic windows among the chaos and the agreement in the position of the windows in Figs. 2(a) and 2(b) is perfect.

(4) *The adiabatic change from period down to chaos.* It is interesting to test the ability of the parameter θ to control the nonlinearity of the phase synchronizing systems. By ability we mean that we can rapidly change θ (for instance from $\theta=0.2$ to $\theta=0.8$ within only ten turns of the orbits) keeping perfect phase synchronization between the systems. It is fun to press the up and down keys for the θ parameter and watch the dance of the synchronizing orbits in the display. The phase synchronization is quite tight. In Fig. 3 we show some example of this. The parameters of the models are the same as those for Fig. 2. We adopt the y coupling and show the x-z plot. Just for the purpose of illustration the θ is varied continuously with the time t by $\theta=2/\pi \tan^{-1}(t)$.

(5) *The N-coupled systems.* Our model consists of N-coupled Lorenz systems: the N_1 systems with parameters in the chaotic regime and the N_2 systems in the periodic regime and they evolve from completely random starting points.

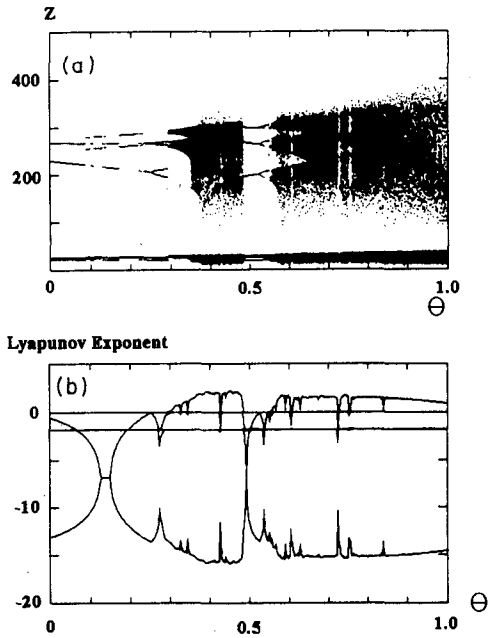

FIG. 2. The same Lorenz systems as Fig. 1 (but $r_2=300$). For small θ the two systems synchronize in periodic flows and for large θ in chaotic flows. The agreement in the position of the windows is perfect. (a) The Poincaré map versus θ with the cut conditions $\dot{z}_i=0$ and $\ddot{z}_i<0$ $(i=1,2)$. (b) The six Lyapunov exponents versus θ. One exponent is at zero and the line around $\lambda=-2$ is a twofold degenerate. The minimum exponent moves far down from the frame.

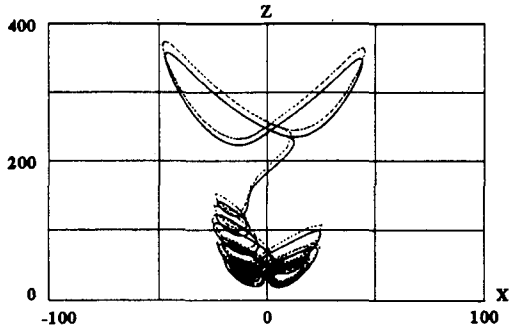

FIG. 3. The adiabatic change of y-coupled flows in the x-z plane in perfect phase synchronization from period down to chaotic regime. The parameters of the models are the same as those for Fig. 2. The solid curve for system 1 and the dashed curve for system 2. The $\theta=2/\pi \tan^{-1}(t-t_0)$ and t_0 is the time that the flows stabilized in the periodic orbits.

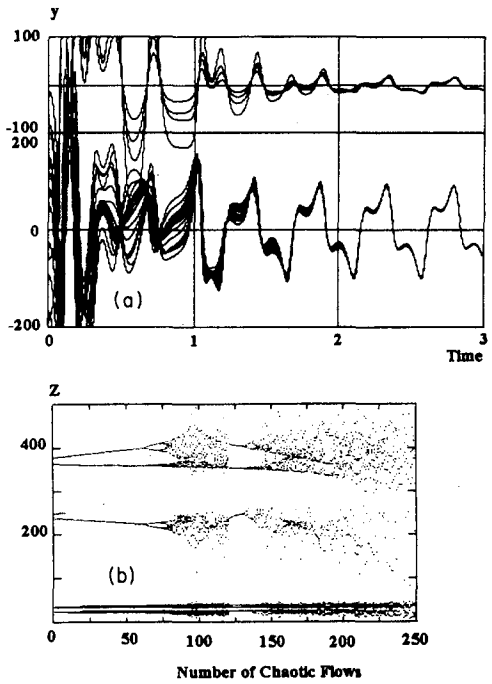

FIG. 4. The N Lorenz system globally coupled in the variable x_i; N_1 chaotic systems $(r=28, b=8/3, P=10)$ and the N_2 periodic systems $(r=270, b=8/3, P=10)$. Random start. (a) The two-step process to the perfect phase synchronization. Parameters are the same as Fig. 1. The $N_1=4$ and $N_2=16$ systems gradually bunch together among each during $t=0$ to $t=2$ and the two bunches mutually derives themselves into phase synchronization after $t=2$. (b) The Poincaré map of totally $N=250$ Lorenz systems versus the number of chaotic members N_1. This agrees with the $N=2$ Poincaré map in Fig. 2(a), especially as for the pattern of the periodic windows structure and gives an additional verification of the two-step dynamics via the relation [see Eq. (6)].

Globally coupling. First let us investigate the mean field type coupling (4) and the y_i and y_i' (z_i and z_i') do not interact directly. We observe that the initial N trajectories soon change into two perfectly phase synchronizing orbits which look just like those in Fig. 1. As is clearly seen in Fig. 4(a) the synchronization proceeds in two steps. First each of the N_1 and N_2 systems bunch together among each. Then the two bunches mutually interact—just like the two units in the $N=2$ analysis—and finally the whole N systems derive themselves into two phase synchronizing orbits in the phase space, one consist of N_1 units and the other of N_2 units. As we have discussed at (6) this two-step process should also be checked by the correspondence between the population ratio in the N systems and the weight factor θ in the $N=2$ matrix coupling model (2). Figure 4(b) exhibits the Poincaré map of the globally coupled N_1 chaotic and N_2 periodic systems with respect to the number of chaotic systems (N_1). The agreement with the $N=2$ Poincaré map in Fig. 2(a)—even as for the pattern of the periodic windows structure— gives further verification of this two-step dynamics via the relation (6).

In the $N=2$ analysis we wished to seek out the real reason of the tendency that the large θ favors system 1. As relation (6) is now established, the tendency that the large (small) θ favors system 1 (2) can be rephrased in terms of the N systems that the majority wins over the minority systems. This gives a plausible answer to the question posed in the $N=2$ analysis.

N systems with nearest-neighbor couplings. In the globally coupled model there is not notion of the distance. By this simplification the essential feature of the synchronization can be best studied in a scale invariant manner but the lack of the distance forbids the analysis of the spatiotemporal structure of the synchronization. Therefore, we set N_1 (chaotic) and N_2 (periodic) systems in random combinations on a circle and let them interact with the nearest-neighbor both-way coupling

$$\left[1-\frac{\epsilon}{2}\right]x_i+\frac{\epsilon}{2}x_{i+1}\mapsto x_i\ ,$$
$$\left[1-\frac{\epsilon}{2}\right]x_{i+1}+\frac{\epsilon}{2}x_i\mapsto x_{i+1}\quad (i=1,\ldots,N)\ . \tag{10}$$

Figure 5 shows the spatiotemporal plot of the evolution of the $N=20$ systems with $N_1=16$ and $N_2=4$ from the random initial values and $\epsilon=0.3$. Despite that the four periodic systems are far separated from each other by the chaotic systems, we see that amazingly they quickly turn into mutual synchronization as is observed as synchronizing peaks. This "barrier penetration" also occurs for the other 16 chaotic systems and the chaotically oscillating sheet exhibits their synchronizing motion. This first step—the cluster formation—completes roughly in 2 sec and then, as the second step, the periodic cluster metamorphose into perfect phase synchronization with the chaotic cluster. For illustration we can only show the small N case but we have numerically confirmed this route to the synchronization in larger systems and also in the random spatial distribution of the N_1 and N_2 systems. Thus the phase synchronization in the nearest-neighbor model also proceeds in two steps.

In conclusion we investigated the coupled distinct nonlinear systems. We found that they derive themselves into *phase synchronization.* For $N=2$ we constructed a simple matrix model and found that we can control the synchronization by a single parameter in the model θ. For large N we considered the coupling of N_1 systems with parameters in the chaotic regime and N_2 systems in the periodic regime. We again found a complete phase synchronization. Furthermore, we found that the synchronization in many systems proceeds in two steps; the like systems first synchronize among themselves forming

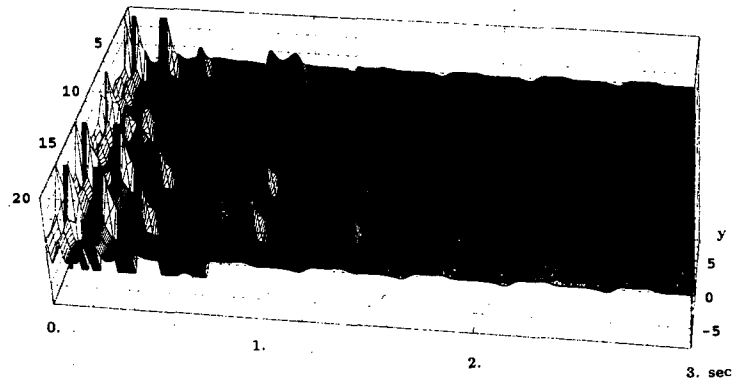

FIG. 5. The $N=20$ systems with nearest-neighbor both-way ($\theta=0.5$) x couplings ($\epsilon=0.3$) and the spatiotemporal structure of the two-step process to the phase synchronization. $N_1=16$ (chaotic) and $N_2=4$ (periodic) systems. The y coordinates of the latter are scaled down by factor 20. The four periodic systems are at $i=2$, 8, 13, 17 and are far separated from each other by the other 16 chaotic systems. The first process completes in 2 sec; the periodic systems quickly turn into four synchronizing peaks due to penetrating interaction across the chaotic systems and the chaotic systems form a coherently oscillating sheet. In the second process, the periodic peaks metamorphose into perfect phase synchronization with the chaotic cluster.

two clusters and then as two units they start the metamorphosis to the final phase synchronization. We verified this two-step process by the direct inspection as well as by the check of the validity of the relation (6) between the dynamics of the N systems and the two systems.

We often come across the case that the synchronization in the phase is the important issue while the magnitudes of components are not much relevant. For instance in the path integral the classical trivial and nontrivial configurations dominate the amplitude as all nearby paths contributes coherently with different magnitudes. We have relaxed the notion of synchronization to the phase synchronization and in this freedom we have seen an interesting possibility of a reduction of the large system dynamics to the small system dynamics.

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流れ素子の大域的ネットワーク・ダイナミックスと位相同期現象

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Phase Synchronization and Dynamics of a Global Network of Distinct Flows

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Synopsis：
It is shown that the dynamics of globally coupled N_1 chaotic and N_2 periodic systems is reduced to that of matrix-coupled chaotic and periodic systems as the result of the synchronization among the former systems. The synchronization proceeds in two steps: the formation of the two clusters and the metamorphosis of the attractors into phase synchronizing orbits. By a key observation that final attractors represent the motion of the 'center of gravity' of the network, the mechanism of the dynamics reduction is clarified. The phase synchronization admits final attractors with variety of non-linearity and some possibility for the technical application is also considered.
Key word：
Phase-synchronization
Reduction of Dynamics
Neural-Network
Globally Coupled Oscillator

目　次

1. はじめに

脳は，ニューロンを素子とする神経回路網－ニューラル・ネットワーク－である．ニューロンは，強い非線形性をもつカオス素子である．従って，パターン認識や記憶などの知的活動においては，カオス素子の結合系で，ある秩序形成が行われていると考えられる．我々は，非線形素子の結合系での集団運動の形成について一連の解析をしているが[1,2]，本論文では，

特に連続的非線形素子を大域的に結合させた場合に生ずる我々が最近見いだした位相同期現象[1]の機構の解明をする．我々の扱う連続的非線形素子は，微分方程式で記述され，流れの特質を失わないような微小の時間間隔で差分化し，各ステップごとに相互作用させる．これまで同期現象については，全く等しい非線形性を持つ素子間の完全な同期にのみ焦点が向けられてきた．一方で，最近の大域的離散写像素子の解析では構造発生を中心として大きな進展が起こっている．我々は，位相同期という新しい視点を導入し，異なる非線形性パラメータを持つ素子の同期現象を調べ，この系が離散写像素子の大域的結合系と同等のバラエティに富むアトラクターを持つことを述べる．本論文では，以後，連続的非線形素子を流れ素子，離散写像素子をマップと呼ぶ．

ニューラル・ネットワークのダイナミックスに対して Hopfield が後に彼の名を冠された統計力学模型を提案したのは1982年にさかのぼる[3]．彼の模型は，Little によるニューラル・ネットワークのダイナミックスの確率的解釈[4]とニューロン結合定数に対する Hebb の学習則に基づいている．そのダイナミックスは物理学者にとって興味の尽きないスピン・グラス系に対応している．この Hopfield 模型の統計力学は，Amit, Gutfreund, Sompolinsky が与えたレプリカ法による平均場理論によって定式化され[5]，簡明な原理に基づき徹底した理論分析が可能である模型として一つの基準点となっている．本論文次節以降では，具体的な結合模型の解析に議論を絞るが，我々の研究は，Hopfield 模型以降のニューラル・ネットワーク研究の中で，同期現象[6]を中心とする新しい発展の流れ[7]の中にある．我々の一連の研究は，一方では，脳の認識機構の解明を目指し，また，他方では，非線形素子を力学変数にする統計力学的模型への場の理論的アプローチを目指している．そこで，Hopfield 模型を基準点として，この論文の内容をはじめに概述しよう．

Hopfield模型のニューラル・ネットワークのダイナミックスは，次のようなものである．N個のニューロンからなる神経網を単純化して考えると，各ニューロン（$i=1,\cdots,N$）は，その内部の電位に非線形に依存する発火の頻度V_iで特徴づけられる．従って，ある時刻での神経網の状態はN次元ベクトル$V(t)$で記述される．1つのニューロン(番号j)の発火信号は、軸索を電気化学的に伝わり軸索終端のシナプシス小胞から化学伝達物質を放出させ，この伝達液が次のニューロン（番号i）の中に拡散し，シナプシスの結合効率J_{ij}の重みでニューロン(i)の電位を$J_{ij}V_j$だけ変化させる．ニューロンiの内部電位は，神経網のすべての他のニューロンjの影響で，内部電位変化が加算的に変化し，しきい値Uを越えた分に対応して非線形性を持ったsigmoid型関数$f(x)$のフィルターを通して，発火をするようになる．すなわち，神経回路網の活性状態の変化は，平均的なプロセッシングの時間をΔtとして，

$$V_i(t+\Delta t) = f\left(\sum_{j=1}^{N} J_{ij}V_j(t) - U\right) \qquad (J_{ii}=0) \tag{1}$$

で記述される．Hopfieldは，このダイナミックスをスピン系の統計力学にマップする．すなわち，ニューロンの状態をもっとも高い発火率にあるか($V_i \to 1$)，発火を全くしていないか($V_i \to 0$)の2値に限ることにし，変数S_iを$S_i = 2V_i - 1$によって導入すれば，変数S_iは，±1の値を持ち，あたかも，スピン変数の様に見なせる事になる。同時に，sigmoid型関数$f(x)$も，区間［−1，1］から，その上へのsigmoid型関数に規格化をし（そのもっとも堅い極限がHeavisideの階段関数$\theta(x)$)，さらに$\frac{1}{2}J_{ij}$を単にJ_{ij}と書くことにすれば，(1)は

$$\begin{aligned} S_i(t+\Delta t) &= f(h_i(t)) \\ h_i(t) &= \sum_{j=1}^{N} J_{ij}S_j(t) + h_i^{\mathrm{ext}} \\ h_i^{\mathrm{ext}} &= \sum_{j=1}^{N} J_{ij} - U \end{aligned} \tag{2}$$

となる．ここで，上の式(2)の$h_i(t)$の第1項が，スピン系の平均場をあらわし，第2項が外場と見なせることに着目し，立場を変えて温度$kT = \frac{1}{\beta}$のスピン系の統計力学を考えると，次の段階でのスピンS_iの期待値は，

$$\langle S_i \rangle = \frac{(+1)e^{\beta h_i(t)} + (-1)e^{-\beta h_i(t)}}{e^{\beta h_i(t)} + e^{-\beta h_i(t)}} = \tanh(\beta h_i(t)) \tag{3}$$

となり，やはりsigmoid型関数$\tilde{f}(x) = \tanh(x)$によって与えられる．従って上のスピン系の発展方程式(2)における関数$f(x)$を，ある入力に対する出力が0になるか，1になるかの確率を与えるものと解釈すれば[4]，式(2)のニューラル・ネットワークのダイナミックスを結合係数J_{ij}を持つ統計力学模型として取り扱えることになる．Hopfield模型の第2の要素は，Hebbの学習則であり，p個のパターン$V_i = \xi_i^{\mu}$ ($\mu = 1, \cdots, p$)に対して，連結効率J_{ij}が，

$$J_{ij} = \lambda \sum_{\mu=1}^{p} \xi_i^{\mu}\xi_j^{\mu} \tag{4}$$

で与えられるとする．ニューラル・ネットワークでは，$J_{ij} > 0$の興奮性シナプシスと$J_{ij} < 0$の抑制性シナプシスの両者が混在するから，対応するスピン系の物理は，強磁性と反磁性の混在したスピン・グラスになる．Hopfield模型の成功は，この系が熱的ノイズのもとでバラエティに富むパターンをアトラクター（記憶銘記状態）として持つことにある．

さて，本論文で考察する非線形素子の結合模型でも，基本的に式（1）を採用しているが，系の結合定数J_{ij}を記銘パターンによって変化させる学習則は採用しない．例えば，2.1節におけるマップの場合，式（6)，(7）を変数$y_n(i) = f(x_n(i))$について書けば，

$$y_{n+1}(i) = f\left((1-\varepsilon) y_n(i) + \frac{\varepsilon}{N} \sum_{j=1}^{N} y_n(j) \right) \qquad (5)$$

となり，式（1）と同等なダイナミックスを扱っていることが分かるが，結合定数は，ただ1つの固定されたパラメータ ε で与えられている．我々の興味は結合定数 J_{ij} を一定に保ち，かつ，各素子の非線形性も一定値に保っても，なお結合系全体が同期現象のために自発的に秩序・構造発生を行い，1つのアトラクターに落ち込んでいく可能性にある．

本論文第2節では，最近の非線形素子の同期現象の研究の進展を述べる．2.1節では，大域的マップ結合系の金子[8]の仕事を見る．多数の非線形ロジスティック写像素子を平均場を通して相互作用させると，各素子の非線形性のためのカオス的傾向と相互作用のための秩序性のバランスのもとで，系は構造発生を起こし，バラエティにとむアトラクターを有することが示される．2.2節では，2個の流れ素子に対する Pecora-Carroll の模型を調べる．この模型では，一方の素子の変数の一つを他方の対応する変数に代入してしまうので，本論文では，成分代入結合系と呼ぶ．この模型は，カオス領域にある流れ非線形素子でも，非線形性の大きさが全く同じに設定された場合に同期現象が起こることを示す重要なものである．2.3節では，結合を成分代入によるものでなく，脳の模型にふさわしいように両方向型とし，N 個の流れ素子の大域的ネットワークの振る舞いを調べる．第3節では，一貫して，流れ素子の非線形性が異なる場合の結合系のダイナミックスを解析する．3.1節では，N 個の素子のうち，N_1 個はカオス領域に，N_2 個は周期的領域に非線形パラメータを設定した (N_1, N_2) 型の流れ結合系を考察し，この系は，2つのクラスターへまず同期し，続いて形態変化をして位相同期したアトラクターへと2段階の発展をすることを見る．3.2節，3.3節では，2つの流れ素子の行列型結合系を解析し，前節の (N_1, N_2) 系と対比して，(N_1, N_2) 系の第1段階の同期は非常に強く，系のダイナミックスは2素子のダイナミックスに簡略化していることを確かめる．最後の3.4節では，さらに結合系の最終的アトラクターのダイナミックスは，“重心運動”に帰着していることを示し，結合系の位相同期現象を理論的に解明する．また，(N_1, N_2) の構成比，もしくは，結合係数を変化させることでこの重心運動の非線形性の様相を自在に制御できることを示す．第4節では，この同期・形態変化したアトラクターが敏感に制御に応答すること，また大域的結合でなくても隣接的な相互作用の場合でも同様な形態変化・位相同期が生ずることを示す．

2. 等しいパラメータを持つ非線形系の結合

2.1 マップの大域的ネットワーク

大域的ネットワークの定式化

ここでは，等しい非線形性を持つマップの結合系として，金子のモデル[8]を例示する．このモデルでは，N 個のマップを平均場を通して大域的に相互作用させる．結合系の発展は次の2段階で行われる．まず，共通の非線形パラメータ a を持つロジスティック写像素子 N 個を同時に

$$\begin{aligned} x_{n+1}(i) &= f(x_n(i)) \\ &= 1 - a x_n^2(i) \end{aligned} \tag{6}$$

で発展させる，次に各ロジスティック素子を，全系のつくる平均場を通して重み ε ($0 \leqq \varepsilon \leqq 1$) で

$$x_{n+1}(i) \leftarrow (1-\varepsilon) x_{n+1}(i) + \frac{\varepsilon}{N} \sum_{j=1}^{N} x_{n+1}(j) \tag{7}$$

により相互作用させる．この $n \to n+1$ の発展を繰り返させる．この2段階発展（6），（7）は，次のように捉えることができる．まず第1段階（6）では，各素子本来のもつ非線形性が結合系の発散を促す．それに対して第2段階（7）では，素子が重み ε だけ平均場の方向に引き寄せられるので，系の収束が促される（Fig. 1 参照）．

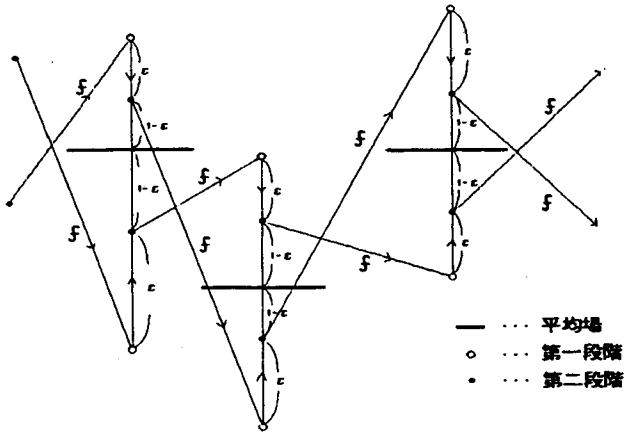

Fig.1 マップの大域的ネットワークの2クラスター構造における二段階発展の概念図
白丸，黒丸は，それぞれ第1段階（$f(x_n) = 1 - a x_n^2$），第2段階（平均場を通した相互作用）の発展を表す．平均場が上下しているのは，各反復回数ごとの平均場がゆらいでいることを表している．

結合による系の安定秩序化と構造発生

Fig. 2 では，ロジスティック写像を $N=100$個用い，非線形パラメータは各素子が単独の場合にカオスの振る舞いをする $a=1.98$にとり，縦軸にそれぞれの素子の漸近値，横軸に ε をとった Poincaré マップを示す．この Fig. 2 及び上述の 2 段階発展から，結合定数 ε に対するこの系の振る舞いは，以下のように捉えることができる．まず ε が小さいとき（$0 \leqq \varepsilon \leqq 0.2$）は，平均場の収束の効果が小さく，各素子の非線形性が高い（$a=1.98$）ため，それぞれの素子は別個にカオスの振る舞いをする（乱れの相）．また逆に ε が大きいとき（$0.5 \leqq \varepsilon \leqq 1.0$）は，平均場の収束の効果がそれぞれの素子を支配するので，全体が 1 つの集団に同期して 1 つのカオスになる（コヒーレントカオス相）．中間の領域においては，2 段階発展の発散と収束の微妙なバランスによって興味深い現象が起こる．例えば $\varepsilon=0.3$の場合では，N 個の素子が 2 つのクラスターに別れて，それらが互いに逆位相 2 周期に振る舞う．つまり，カオス素子系が 2 つの集団の周期構造を発生をする．Fig. 1 は，この構造発生状態における 2 段階発展の概念図である．平均場を通して相互作用させた（6），（7）の 2 段階の発

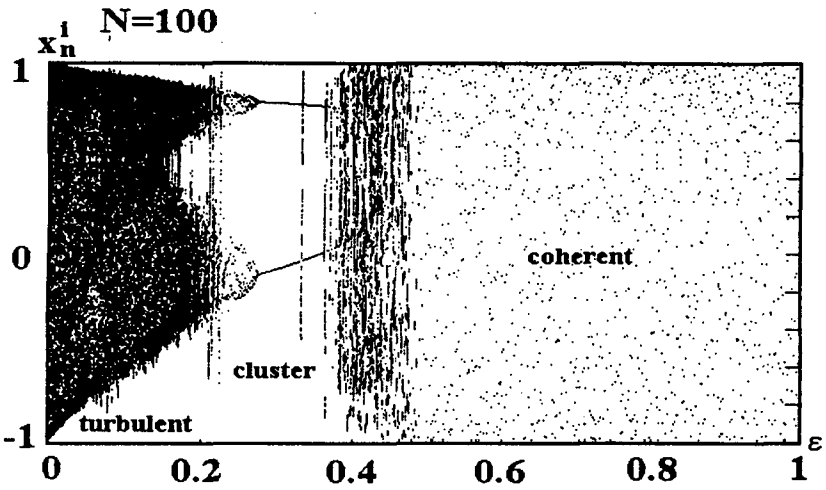

Fig.2　マップ（$N=100$）の大域的ネットワークの ε に対する 1 パラメータ分岐図（Poincaré マップ）

横軸には ε を取り，各 ε に対する素子 $i=1,\cdots,N$ の漸近値（反復回数 n：1000から1256）を縦軸に取ってある．ただし，$0 \leqq \varepsilon \leqq 0.5$までは，不安定固定点 x^*の上下に属する素子の数が等しい場合のみをデータとして採用する．一方，$0.5 \leqq \varepsilon \leqq 1.0$の範囲はコヒーレントカオス層であり，そのような場合分けはしない．およそ$0.28 \leqq \varepsilon \leqq 0.38$の範囲での 2 クラスター構造がはっきりわかる．また，$0.5 \leqq \varepsilon \leqq 1.0$では，点の数が少ないことから，全体がまとまったコヒーレントカオスであることがわかる．

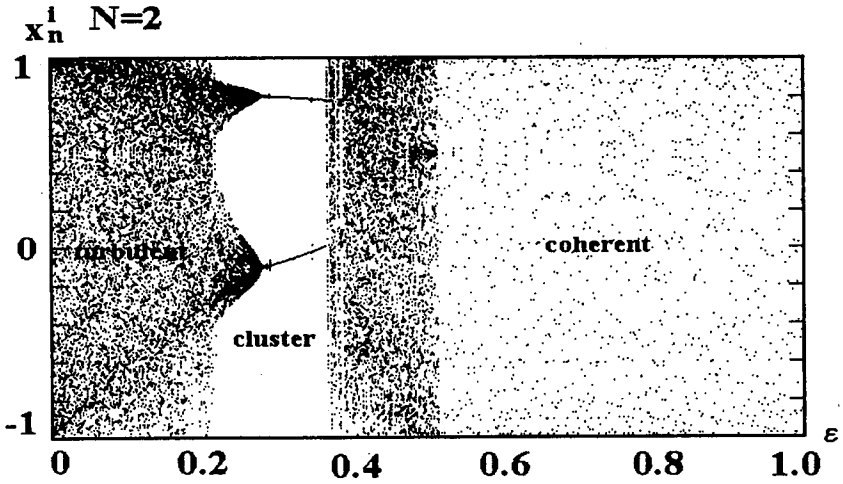

Fig.3 マップ（$N=2$）の大域的ネットワークの ε に対する1パラメータ分岐図（Poincaré マップ）
$0 \leqq \varepsilon \leqq 0.5$におけるデータの場合分けは Fig. 2 と同様である。乱れの層(turbulent) でのカオスの運動範囲が Fig. 2 に比べて大きいことがわかる．これは，素子が多数の場合には平均場の揺らぎが小さいためである．また，Fig. 2 と同様に，およそ$0.28 \leqq \varepsilon \leqq 0.38$の範囲での2つのクラスター構造が確認できることから，多数素子の場合（Fig. 2）の構造は，2つの素子の場合に帰着されることがわかる．

展では，各素子の性質を変えないでも，相互作用の強さ ε を操作することによって系の集団運動としての非線形性を変化させることが可能である．

構造発生によるダイナミックスの簡略化

Fig. 3 には，$N=2$ の場合のPoincaréマップを示す．Fig. 2 とこの図を比較すると，系の2つのクラスター構造の発生が素子数Nに依存しないことを確認できる．つまり，この系の振る舞いは本質的に2つの素子のダイナミックスに帰着する．そこで，この大域的ネットワークが2つのクラスター構造を発生した時に限って考察を進める．各素子がどちらのクラスターに属するかを区別するため，写像の反復回数nが偶数の時に，不安定固定点$x^*=(\sqrt{1+4a}-1)/2a$ より大きいクラスターに属する素子の値を x_+ と呼び，その数を N_+ とする．同時に、x^*よりも小さい素子の値を x_- と呼び，その数を N_- とする（$N_+ + N_- = N$）．2つのクラスター集団のダイナミックスは，次のような2通りの記述ができる．

〔1〕平均場を $\bar{f}=(N_+f(x_+)+N_-f(x_-))/N$ とする．すると，2段階発展（6），（7）は，まとめて

$$x_+ \leftarrow (1-\varepsilon)f(x_+)+\varepsilon\bar{f}$$
$$x_- \leftarrow (1-\varepsilon)f(x_-)+\varepsilon\bar{f} \qquad (8)$$

と書き換えられる．従って，大域的ネットワークは，2つのマップが平均場 $\bar{f}$ と相互作用している系と同等である（Fig. 1 参照）．

〔2〕$\varepsilon_+=\varepsilon\frac{N_+}{N}$，$\varepsilon_-=\varepsilon\frac{N_-}{N}$という2つの量を定義すると，大域的ネットワークの2段階発展（6），（7）は，

$$x_+ \leftarrow (1-\varepsilon_-)f(x_+)+\varepsilon_- f(x_-)$$
$$x_- \leftarrow (1-\varepsilon_+)f(x_-)+\varepsilon_+ f(x_+) \qquad (9)$$

のような2つの素子の行列型結合系に帰着できる．

式（8），（9）は，大域的ネットワークでクラスター発生が起こると，系全体のダイナミックスが少数系（$N=2$）のダイナミックスで表現できる，という簡略化を端的に示している．逆位相2周期の2つのクラスター構造が発生したときには，2つのクラスターの間の素子の遷移を促す外部パルスの入力によりポジ・ネガの反転が起こることが金子によって発見されている[8]．これはパターン認識の機構の理解への一つの手がかりと考えられる．また，これらの簡略化は2つのクラスター構造の時に限らない．もし大域的ネットワークが k 個のクラスター集団を発生するときは，式（8），（9）は k 個の連立した式に書き直すことができる。結合系は，各クラスターに属する素子数 N_i を成分とするベクトル $(N_1, \cdots, N_k)$ によってコード化できる．

2.2 流れ素子（微分方程式系）の成分代入結合系

成分代入結合系の定式化

流れ素子（微分方程式系）2つの結合系としてまず Pecora-Carroll のモデル[7]を調べる．まず，典型的な流れ素子として，Lorenz モデルを用いる．

$$\frac{dx^{(i)}}{dt}=f(x^{(i)}, y^{(i)}, z^{(i)})=P(y^{(i)}-x^{(i)})$$
$$\frac{dy^{(i)}}{dt}=g(x^{(i)}, y^{(i)}, z^{(i)})=-x^{(i)}z^{(i)}+rx^{(i)}-y^{(i)} \qquad (10)$$
$$\frac{dz^{(i)}}{dt}=h(x^{(i)}, y^{(i)}, z^{(i)})=x^{(i)}y^{(i)}-bz^{(i)}$$

上式における上添字は系の番号を表し（$i=1, 2$），またパラメータ P，b，r は2つの系で共通にする（$P=10$，$b=8/3$，$r=60$）．このパラメータ領域で，各 Lorenz モデルは単独ではカオスの振る舞いをする．これらの一つの変数を一方から他方に代入して結合させる．変数 x の場合を例にとると，

$$\frac{dx^{(1)}}{dt}=f(x^{(1)};\ y^{(1)},z^{(1)}),\quad \frac{dy^{(1)}}{dt}=g(x^{(1)};\ y^{(1)},z^{(1)}),\quad \frac{dz^{(1)}}{dt}=h(x^{(1)};\ y^{(1)},z^{(1)})$$
$$\frac{dx^{(2)}}{dt}=f(\underline{x^{(1)}};\ y^{(2)},z^{(2)}),\quad \frac{dy^{(2)}}{dt}=g(\underline{x^{(1)}};\ y^{(2)},z^{(2)}),\quad \frac{dz^{(2)}}{dt}=h(\underline{x^{(1)}};\ y^{(2)},z^{(2)})$$

となる．このモデルは，以下の2つの捉え方を許す．まず，2つの3次元非線形素子が連立しているという捉え方をする（Fig. 4 (1)）．すると，系1の変数 $x^{(1)}$が系2の $x^{(2)}$に代入されることから，系1をMaster，系2をSlaveと呼ぶことができる．MasterはSlaveからのフィードバックを受けずに系本来の振る舞いを示し，一方，SlaveはMasterからの影響が加わった振る舞いをする．また，共通なDrive信号を持つ2つの系という見方も有効である(Fig. 4 (2))．この立場では，xはDrive信号であり，2つのResponse系($y^{(1)}$, $z^{(1)}$)，($y^{(2)}$, $z^{(2)}$)が，Drive変数にフィードバックを与えつつ，相互作用の下で発展することになる．

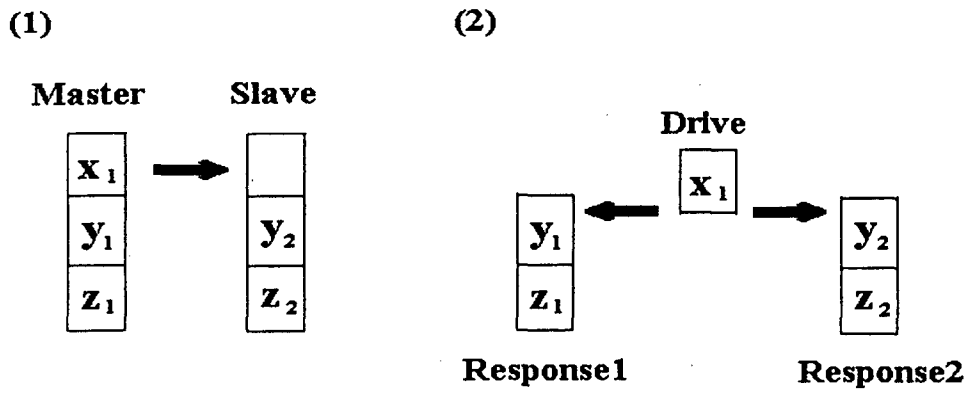

Fig.4 Pecora-Carrollによる成分代入結合系の概念図
(1)は，二つの素子が連立したMaster-Slaveの捉え方，(2)は，共通のDrive信号を持つ二つのResponseという捉え方の概念図である，

同期現象

Fig. 5には，2次元 ($y^{(i)}$, $z^{(i)}$) 平面上での軌道の発展（$i=1,2$）を示す．違った初期値 (20, 20, 20)，(−100, −100, 0）から出発した2つの軌道が同じアトラクターへ収束し，やがて同期をする様子がわかる．Fig. 6では，この各素子の間の変数の差 ($\Delta y\equiv|y^{(1)}-y^{(2)}|$, $\Delta z\equiv|z^{(1)}-z^{(2)}|$) の時系列により，同期現象の過程を示す．結合していない場合のΔy，Δzは，常に10^1の程度にとどまっているのに対して，結合した場合の Δy，Δz は指数関数的に急速に減少していることがわかる．

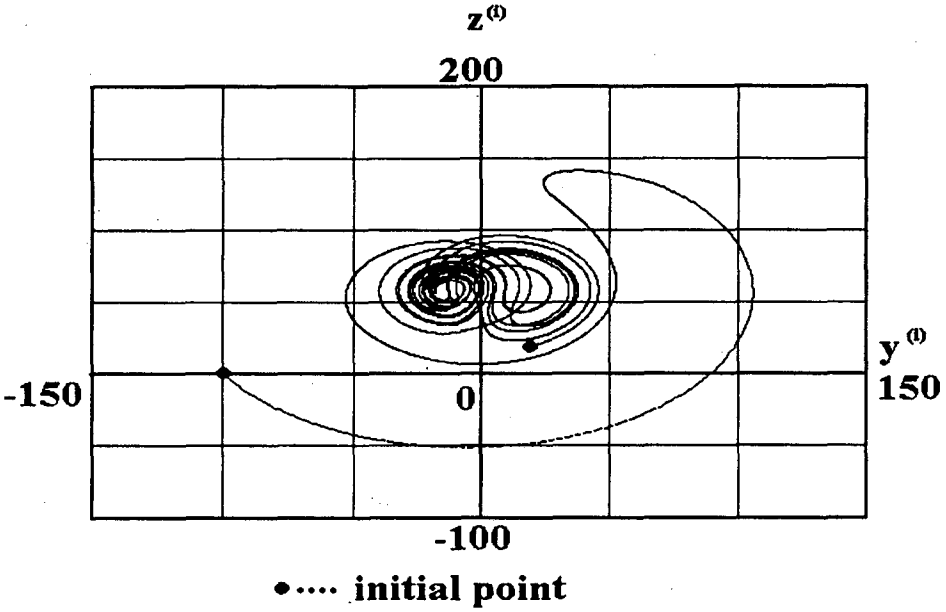

Fig.5　成分代入結合系（Lorenz モデル）の相平面上での振る舞い
全く異なる初期値（図中 2 つの黒丸）から発展した二つの軌道が，同じアトラクターに引き込まれ，次第に同期していく様子がわかる．

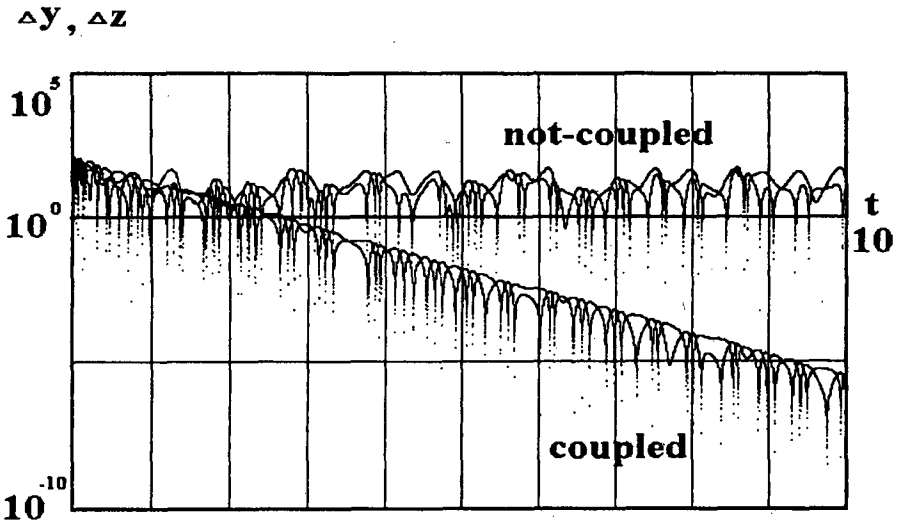

Fig.6　成分代入結合系（Lorenz モデル）の変数間の差（Δy，Δz）の時系列
横軸には時間をとり，縦軸には変数の差を対数で取る．結合させた場合，結合させない場合，ともに同じ初期値で発展させる．結合させた場合は，変数の差が指数関数的に減少して同期しているのに対し，結合させない場合は，10^1のオーダーあたりで滞在しており，同期をしていないことがわかる．

同期現象の必要条件

同期現象が起こるためには，Driveを除いた他の2変数のLyapunov指数（sub-Lyapunov指数）が負である必要がある．Lorenzモデルの結合系と，Rösslerモデルの結合系のsub-Lyapunov指数を表1に示す．この表によると，Lorenzモデルにおいて，zがDrive信号の時には，xの発散性のために同期現象が起きないことがわかる．また，Rösslerモデルの場合には，Drive信号にy変数を用いた場合にしか同期現象が起きないことがわかる．

表1

system	Drive	Response	Sub-Lyapunov exponents
Rössler	x	(y,z)	(+0.2,8.89)
a=0.2 b=0.2	y	(x,z)	(-0.056,-8.81)
c=9.0	z	(x,y)	(+0.1,+0.1)
Lorenz	x	(y,z)	(-1.81,-1.86)
P=10 b=8/3	y	(x,z)	(-2.67,-9.99)
r=60	z	(x,y)	(+0.0108,-11.01)

共通の非線形性を持った2つの流れ素子が完全な同期現象をカオス領域でも示すことは，カオス的なニューロンのネットワークで，脳の知的な活動を保証する可能性である．この点でPecora-Carrollの先駆的仕事の意義は大きい．

非線形性が異なる場合：位相同期

それでは，非線形性が異なる場合には，同期現象はどのようになるのであろうか．これが，本論文の主題であるが，この点についてPecora-Carrollは非線形性が僅かに異なる場合の結果のみを与えている．

上と同じLorenz素子の結合模型においてパラメータrの差は5％にとり，$r_1=60$，$r_2=57$の場合のy変数，及びz変数の差（$\Delta y=|y^{(1)}-y^{(2)}|$，$\Delta z=|z^{(1)}-z^{(2)}|$）の場合の時系列をFig. 7に示す．この場合，2つの系は，10^2程度の距離から，前とほぼ同様に指数関数的な接近をし，その後，10^1程度の一定の距離を保って発展することが見られる．このことから，Pecora-Carrollは，2つの系の間の非線形性の相違が10パーセント程度以内であれば同期が起こると結論している．

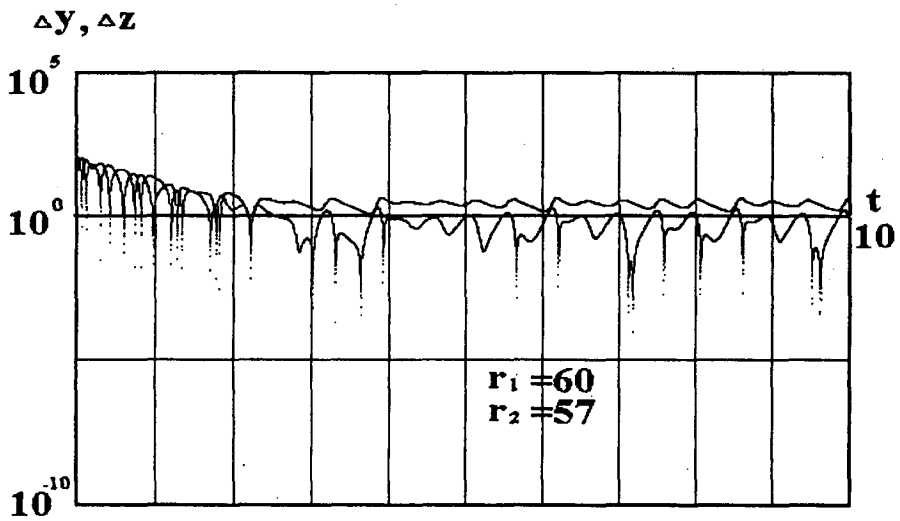

Fig.7　異なる非線形性($r_1=60$, $r_2=57$)を持つ Lorenz モデルの行列型結合系
軸の取り方や，初期値は Fig. 6 に同じである．$t \leq 2$ までに Δz の距離が10^1のオーダーまで減少し，以降はそのあたりの距離を保っていることがわかる．一見すると，この系は同期していないように見えるが，この図による解析は位相同期現象を見失うことになっている（本文参照）．

しかしながら，このような Δy や Δz の様な距離差での解析は，非常に大きな危険性を含んでいる．実際，２つの系の間の非線形性の相違がもっと大きい場合，例えば本論文の次節以降で考察する $r_1=28$, $r_2=300$の様な場合，同期現象は見られないということが結論されてしまう．実は，第３節で報告するように，位相空間で２つの系の軌道の全プロファイルを観察すれば，それらは完全に相似形のアトラクターを描き，位相的に完全に同期していることが見られる．この興味深い現象を本論文では位相同期現象と呼ぶ．Δy や Δz の様な距離差を使った同期の解析は，この重要な現象を我々の目から隠してしまう．実際，我々の知る限り，Pecora-Carroll のカオス系の同期現象のレポート以降の，ネットワークにおける引き込み現象の研究は，非線形性が同一の素子の系のそれに主力を注がれているように思われるが，第１節で強調した様に，ネットワークの協同現象で重要なのは，２つの系の軌道の接近ではなく位相的な同期現象である．本論文では，次節以降で，視点を広げ，様々の非線形性を持つ多数の素子から成るネットワークにおける位相同期現象を考察する．

2.3 等しい非線形性を持つ流れ素子の大域的ネットワーク

カスケード模型から大域的ネットワークへ

Pecora-Carrollは，共通の非線形性を持つ流れ素子のネットワークも考えている．それは，１つのMaster（M）から順に変数代入をしていき，一方方向結合の鎖を作る($M \rightarrow S^{(1)} \rightarrow S^{(2)} \rightarrow S^{(3)} \rightarrow \cdots$) カスケード型のネットワークである．しかし，成分代入結合系はどうして

も１つの素子が全体を支配することになってしまう．この状況を脳に置き換えると脳全体の振る舞いを１つの素子が決定してしまうことになり，成分代入結合系は脳のモデルとしては適切ではない．すでに2.1節で見たように，マップの大域的ネットワークでは，双方向の相互作用のために，全素子が対等に関与する協同現象が起こり，カオス素子の結合系でも，周期的な構造発生を起こす．そこで，流れ素子の大域的ネットワークの振る舞いを調べよう．

流れ素子として，ここでは N 個の Lorenz モデル (10) を用いることにする．まず，前節のマップの第１段階（６）に対応して各素子を

$$
\begin{aligned}
x^{(i)}(t+\Delta t) &= x^{(i)}(t) + P(y^{(i)} - x^{(i)})\Delta t \\
y^{(i)}(t+\Delta t) &= y^{(i)}(t) + (-x^{(i)}z^{(i)} + r^{(i)}x^{(i)} - y^{(i)})\Delta t \qquad (i=1,\ \cdots,\ N) \quad (11) \\
z^{(i)}(t+\Delta t) &= z^{(i)}(t) + (x^{(i)}y^{(i)} - bz^{(i)})\Delta t
\end{aligned}
$$

によって発展させる，第２段階では，（７）に対応して．x，y，z の３変数のうち，例えば，x について

$$
x^{(i)} \leftarrow (1-\varepsilon)x^{(i)} + \frac{\varepsilon}{N}\sum_{i=1}^{N} x^{(i)} \qquad (12)
$$

により，その平均場を通して大域的に相互作用させる．

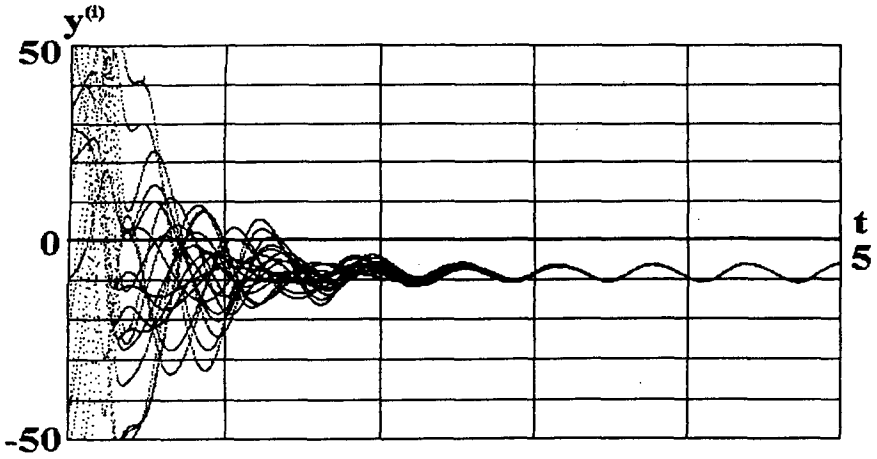

Fig.8 流れ素子(Lorenz モデル)の大域的ネットワークの時系列
20個の流れ素子の非線形性は等しく，それらの初期値は乱数で発生させる．変数 x を平均場を通して相互作用させた20本の Lorenz モデルの軌道が，$t=3$ までには同期をしていることがわかる．

連続的流れの極限の存在

ここで流れを差分化した式（11）について考える．この系は，相互作用させた変数についてはマップとして発展し，直接相互作用させていない残りの2変数についてはΔtで離散化された非自律系として発展する．よってこの系は，1次元について相互作用した多次元マップということができる．しかし，この系は、$\Delta t \to 0$の極限において，以下のことから連続的な振る舞いを示す．Lorenzモデル(11)の各変数は，微小時間Δtごとに，Δtのオーダーの変化で微小な非線形発展をするのに対し，平均場を通した相互作用（12）は平均場の方向に座標変数のオーダーで系を収束させる．これは，速度変数でなく座標変数で相互作用させたことに依っている．この系がアトラクターに収束するまでは，座標変数の大きさはΔtよりもずっと大きいために，系は離散的に振る舞う．しかし，一定の時間が経過すると，Drive変数であるx変数は，その平均場へと指数関数的に急速に収束していき，その収束性は微小時間発展中の非線形効果と競合するようになる．ここまで発展が進んだときに，系の軌道は1つのアトラクターに漸近し，そのアトラクターは，系の発散性と収束性のバランスで決定される．この時に，各微小時間ごとの軌道の変化は，発散性，収束性のそれぞれがΔtのオーダーの大きさになっているので系の軌道は連続した流れになるのである．このときのバランスは，離散化する際のΔtが連続的な流れを保証するだけ微小であれば，その値には全く依存しない．以下の数値計算においては，このΔtの値を$\Delta t \simeq 10^{-4}$にとるが，すべての計算の結果は，これ以下のΔtについてすべて成立する．

同期現象とその条件

Fig. 8には，この系を$N=20$，各素子のパラメータを$r=28$(カオス)，$P=10$，$b=8/3$として発展させた時のy変数の時系列を示す．この図により，すべての20本の軌道が同期をして，最終的には相空間上で1本のカオス軌道になることがわかる．この同期現象は、各素子を周期に設定しても，同様に1本の周期軌道になる。相互作用させる変数をx以外の変数にした場合や，2つ以上の変数を組み合わせた場合，以下のようになる．

x，y，(x, y) (y, z) (x, y, z)での結合　→　同期現象が起きる

z，(x, z)での結合　→　同期しない

このように，非線形性が共通の流れ素子の大域的ネットワークでは，各素子に異なる初期値を設定したのにもかかわらず，すべての素子が同じ変数値を持つようになり，結合系は1つのアトラクター上を発展するようになる．この協同現象は，脳の記憶保持において重要な役割を果たすと思われる．

流れ素子の特性

マップの大域的ネットワークの場合には，各マップ素子が共通の高い非線形性を持っている場合に，平均場を通した相互作用が与えられると，クラスター化が起こり，それぞれのクラスターは，周期的な運動をするという興味深い協同現象が起こることは2.1節で見たとおりである．これに対して，各素子が共通の非線形性を持つ流れ素子の大域的ネットワークでは，連続的な流れが実現される程度に Δt を微小にとる限りでは，Δt ごとの変数の変化量が小さいため，我々の結果は，マップの場合と異なりこのようなクラスター発生が起こらないことを示している．我々は，次節で異なる非線形性を持つ流れ素子のネットワークを調べる．この場合，共通の非線形性をもつ素子の成す部分集合ごとに同期が起こり，その部分集合同士の相互作用によってネットワークが新しい非線形状態に移行することを見る．この点で，大域的結合模型は変数代入によるカスケード型模型に比べて多くの様相を包含している．

3．異なる非線形性をもつ流れ素子の結合系

基本的な確率関数 $\tilde{f}=\tanh(\beta h)$ は，h の値について異なる非線形性の分布を持つ．同様に，簡易化した脳のモデルでも，すべての素子が全く等しい非線形状態にあるとは考えにくい．よって，ここでは異なる非線形性を持った素子のネットワークを考えていこう．

3.1 異なる非線性を持つ Lorenz モデルの大域的ネットワーク

2段階引き込み現象

流れ素子として Lorenz モデルを N 個用いて，簡単のため，N 個のうち N_1個にはカオス的（$r=28$），N_2個には周期的（$r=300$）な非線形パラメータを設定する．またその他のパラメータについてはすべて共通にする（$P=10$，$b=8/3$）．そして（11），（12）により，変数 x について，これら（N_1，N_2）個の素子を平均場を通して相互作用させる．

Fig.9，10にこの系の数値計算結果を示す．Lorenz モデルは $N=20$個用い，また，(12)における平均場との相互作用のパラメータは $\varepsilon=0.3$とする。Fig.9に示すように，ランダムな初期値からの系の時間発展は，次の２段階の発展をしている．

〔１〕まず，互いに等しい非線形パラメータを持つもの同士が同期を始め（t≃0.4），系は２つのそれぞれ同期したクラスターに分解し，結果的に軌道は２本になる（t≃2.0）．この段階

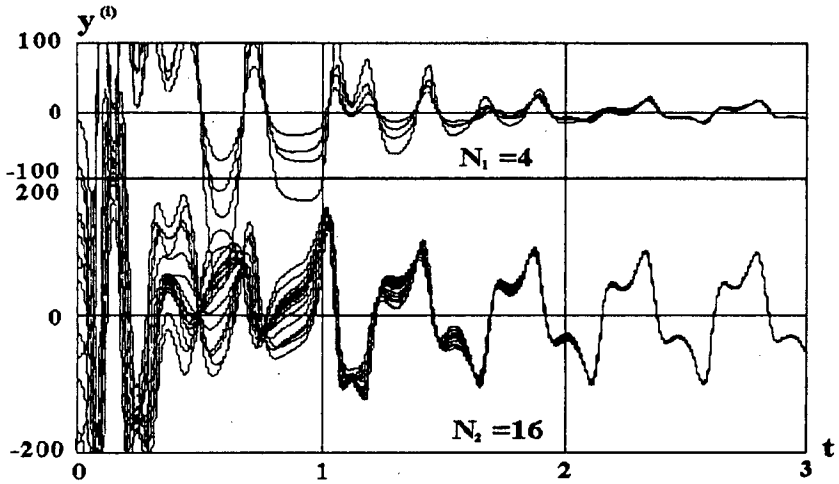

Fig.9　異なる非線形性を持つ流れ素子（Lorenz モデル）の大域的ネットワークの時系列
非線形パラメータを $N_1=4$ 個については $r=28$，$N_2=16$個については $r=300$とする．この図の縦軸の原点は，N_1，N_2それぞれについて取ることにする．$t\simeq1.8$までに N_1，N_2それぞれが２段階引き込み現象の〔1〕の同期現象を起こし，$t>2.2$以降は２段階引き込み現象の〔2〕の形態変化を起こしていることがわかる．

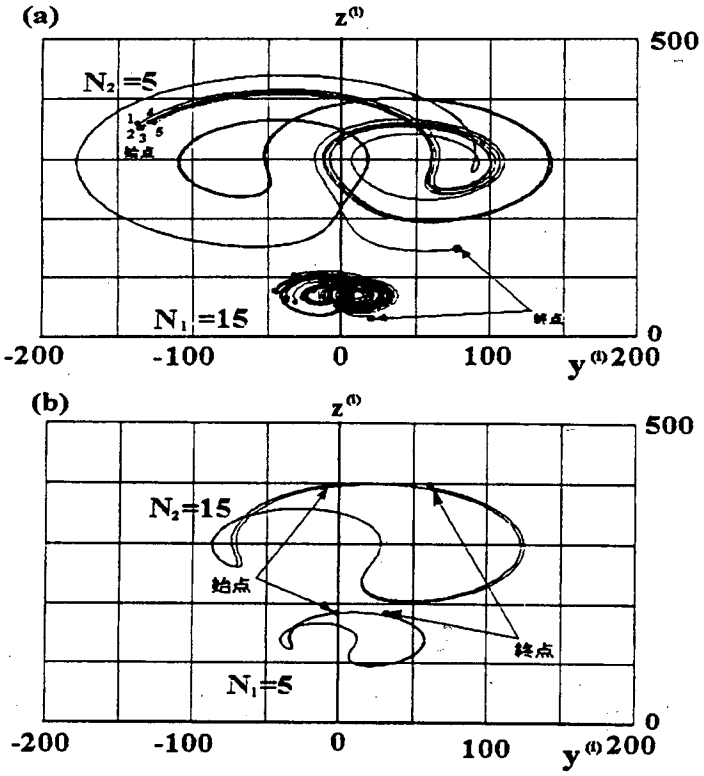

Fig.10　異なる非線形性を持つ流れ素子（Lorenz モデル）大域的ネットワークの相平面上での振る舞い

10a においては，非線形パラメータを N_1=15個については r=28, N_2=5個については r=300とし，初期値は乱数で発生させる．$z^i\simeq100$あたりの15本の軌道は，２倍の大きさで描いてある．図中の始点は t=1.5での軌道の切断点を表す．次第に N_1本，N_2本の軌道がそれぞれ一本になり，位相同期した相似形のカオスアトラクターを描く．終点では位相同期のため，それぞれが一点になっている．

10b においては，非線形パラメータを N_1=5個については r=28, N_2=15個については r=300とし，初期値は乱数で発生させる．10b における軌道も，$z^i\simeq150$あたりの軌道は，３倍の大きさで描いてある．図中の始点は，20本の軌道が十分収束した時（t=5）の切断点を表す．収束した結果，20本の軌道が２本になり，その２本の軌道が周期軌道で位相同期をしている様子がわかる．

では2本の軌道はそれぞれの非線形性に固有な軌道を描く．つまり，多数系のダイナミックスは，2つの系によって記述できるものとなる．

〔2〕2つのクラスター間の相互作用のもとで，系の非線形性が変化し($2.2<t<2.4$)，それらは，互いに違った位置.違った大きさを持つ相似形のアトラクターをつくる($t>2.4$)．

〔2〕で見られる相似形を描く同期現象をここでは"**位相同期**"と名付ける．Fig.10a、10bには，それぞれ $(N_1, N_2)=(15, 5), (5, 15)$ の場合の (y, z) 平面上での最終的なアトラクターを示す．〔1〕で2本にまとまった軌道が〔2〕で収束するアトラクターの種類は，カオスの数の方が周期よりも多い場合（Fig.10a）は，位相同期したカオスに，周期の数の方がカオスよりも多い場合（Fig.10b）はともに周期になる．つまり，クラスターの素子数の大小は，カオス⟷周期の間の非線形性の変化を引き起こしている．我々は，非線形性の変化と位相同期をもたらす〔2〕の過程を，"**形態変化**"と名付ける．上述の2段階を経て実現する引き込み現象は，簡潔にいうと，同期現象に引き続く形態変化ということができる．なお，ここで取り上げた2段階引き込み現象は，$N=20$に限らずもっと多数の素子の場合でも同様に起こる．

構成比に対するPoincaréマップ

2段階引き込み現象によって系が収束するアトラクターの種類には，素子の数 N におけるカオス(N_1)と周期(N_2)の間の割合が関係している．そこで，構成比として

$$\eta=\frac{N_1}{N_1+N_2} \tag{13}$$

を定義し，アトラクターの η 依存性を調べる．Fig.11のPoincaréマップの縦軸は N 個の素子の $z^{(i)}$変数，横軸は η ($0\leqq\eta\leqq1$)である．互いに相似なPoincaréマップは，系が形態変化を起こした N_1個と N_2個のクラスターになっていることを示している．またこのPoincaréマップは，構成比 η が変わるにつれて，この系のアトラクターが，周期，カオス，3周期の窓の構造発生などのバラエティに富む様相を持つことを端的にしめしている．

N 個の異なる非線形性（N_1, N_2）を持つ素子の大域的ネットワークは，2段階引き込み現象によって，わずか1つのパラメータ η で指定される位相同期しているアトラクターに形態変化する．

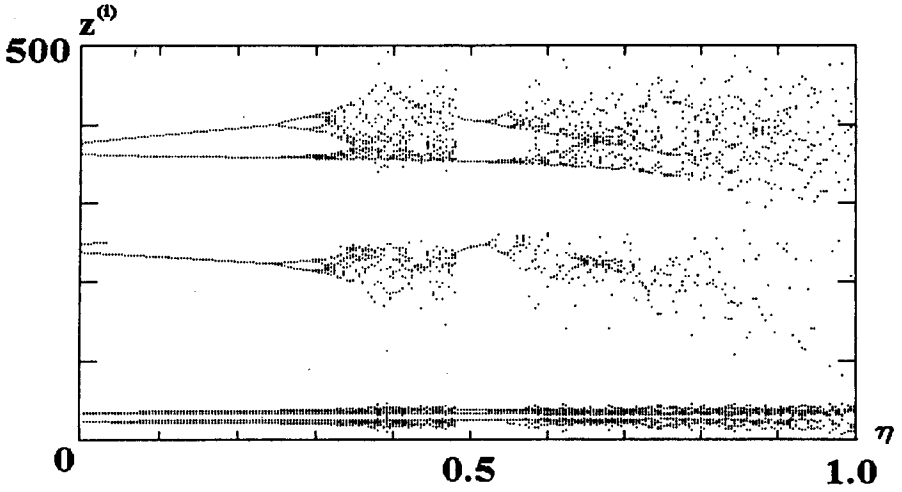

Fig.11　異なる非線形性を持つ流れ素子（Lorenz モデル）の大域的ネットワークの Poincaré マップ

素子数 $N=250$とする．横軸には構成比 $\eta=N_1/(N_1+N_2)$を取り，縦軸には各 η ごとの素子全体の極限軌道を時間範囲$10\leq t\leq 15$について取る．極限軌道をプロットする条件は，z 変数の傾きがゼロの時にする．上下に相似な Poincaré マップがあることから，位相同期が確認できる．また，各構成比ごとのアトラクターは，周期，カオス，三周期の窓など，バラエティーに富んでいる．

3.2 異なる非線形パラメータを持つ２つの流れ素子の行列型結合系

行列型結合系の定式化

異なる非線形性を持った流れ素子の大域的ネットワークは，前節の２段階引き込み現象における第１段階の同期現象によって，我々の最も簡単な設定では２つのクラスターに分離し，その２つのクラスター間のダイナミックスに帰着する．それと比べるために，ここでは，異なる非線形パラメータを持つただ２つの Lorenz モデルの双方向の相互作用モデルを考察する．結合には，（９）の様なマップでよく使われる行列型結合を採用する．

まず２つの Lorenz モデルを用い，それぞれの非線形パラメータ r をカオスと周期にそれぞれ設定する（$r_1=28$, $r_2=300$）．また，その他のパラメータは共通にする（$P=10$, $b=8/3$）．そして，この２つの系を式（９）のように変数 x について結合させる．

$$x^{(1)} \leftarrow (1-\varepsilon_2)x^{(1)}+\varepsilon_2 x^{(2)}$$
$$x^{(2)} \leftarrow (1-\varepsilon_1)x^{(2)}+\varepsilon_1 x^{(1)} \qquad (14)$$

ただし、$\varepsilon_1=\theta\varepsilon$，$\varepsilon_2=(1-\theta)\varepsilon$ とする（$0\leq\varepsilon\leq 1$，$0\leq\theta\leq 1$）．この写像で，

系 i ($i=1,2$) は，他方の素子に ε_i で影響を与え，パラメータ ε はその相互作用の強さを表す．この系を，異なる非線形性を持つ流れ素子の行列型結合系と呼ぶ．

パラメータ極限の解析と系の形態変化

次に，この系をパラメータ ε，θ の取り方によって解析していこう．

(1) $\varepsilon=1$，$\theta=1$ (又は0) の場合

$\varepsilon=1$，$\theta=1$ というパラメータの設定は，系の結合が最も強く，結合による影響は，系1から系2という一方方向になる．実際に $\varepsilon=1$，$\theta=1$ の場合，式 (14) は，

$$x^{(1)} \leftarrow x^{(1)}$$
$$x^{(2)} \leftarrow x^{(1)} \tag{15}$$

となる．すなわちこの極限は，Pecora-Carroll の変数代入結合系に他ならない．

(2)一般の ε，θ の場合

一般の ε，θ では，ε が系の結合の強さを決め，θ が2系の双方向の相互作用の方向性を制御している．以下では，2つのパラメータ設定 ($\varepsilon=0.3$ で $\theta=0.2$ 及び $\theta=0.8$) を例として解析する．Fig.12a，12b には，それぞれ $\theta=0.2$ と $\theta=0.8$ の場合の (y，z) 平面上での極限軌道を，また Fig.12c には $\theta=0.2$ の場合の極限軌道のリサージュ図形を示す．まず Fig.12a では，系のアトラクターは2つの周期軌道になり，Fig.12b ではカオス軌道になっている．この2つの例は，この系において，θ の大小によりカオス⟷周期の非線形性の変化が起きていることを示している．また Fig.12c において，リサージュ図形が原点を通る直線になっていることから，2つの軌道が位相同期をしていることがわかる．つまり，この行列型結合系でも，前節の2段階引き込み現象における形態変化が起きている．

異なる非線形性を持つ2つの Lorenz モデルの行列型結合系は，
形態変化を起こし，最終的に、ただ1つのパラメータ θ で
記述される位相同期したアトラクターへと発展する．

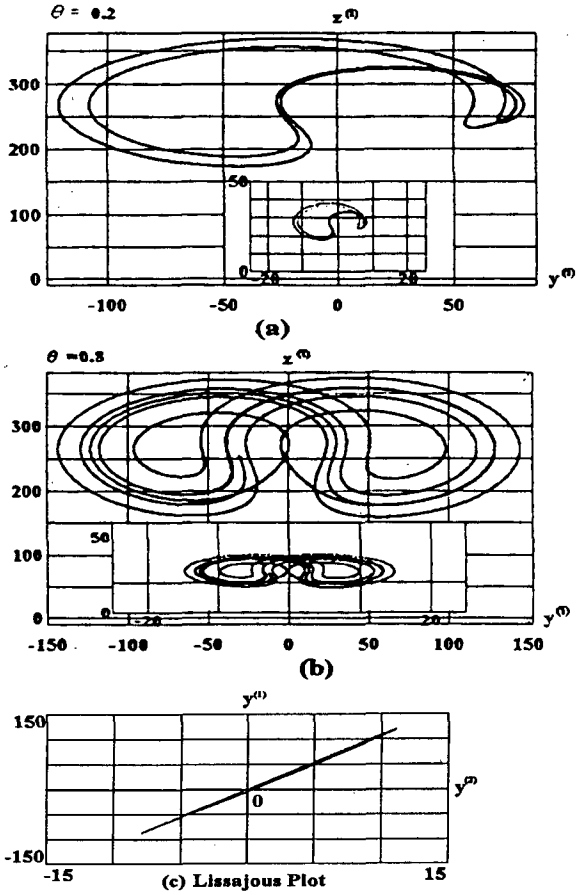

Fig.12　Lorenz モデルの行列型結合系（$\epsilon = 0.3$）の相平面上での極限軌道と，その Lissajous 図形

二つの Lorenz モデルの初期値は乱数で発生させ，また非線形パラメータはそれぞれ，r_1=28，r_2=300 とする．

12a において，$\theta = 0.2$ では，r_1=28 の非線形性が形態変化し，周期軌道で位相同期現象が起きている．

同様に，12b において，$\theta = 0.8$ では，r_2=300 の非線形が形態変化し，カオスで位相同期現象が起きている．

12c には $\theta = 0.2$ の場合の Lissajous 図形が示してある．この図によって，2 つの流れ素子の間の位相同期が確かめられる．

3.3 大域的ネットワーク（η）と行列型結合系（θ）の間の対応

3.1節で見たように，異なる非線形性を持つ流れ素子の大域的ネットワークにおいては，N_1 個と N_2個がそれぞれまとまったクラスターを形成し，構成比 η によってこの系の形態変化がコントロールされる．また3.2節で見たように，異なる非線形性を持つ Lorenz モデルの行列型結合系においては，パラメータ θ によって系の形態変化はコントロールされる．従って，これら 2 つのパラメータ η，θ の間には，何らかの対応関係が存在するはずである．もし，3.1節で述べた大域的模型での 2 段階の引き込み現象で第 1 段階のクラスター化が強く，それ以後の発展において，(N_1，N_2) 型模型の素子のダイナミックスが完全に 2 つの素子のダイナミックスに帰着するとすれば，マップの場合の式（7）から（9）の導出を形式的にこの場合も適用できて，η と θ のアトラクターを決定する上での役割は，全く一致するはずである．

ここでは，パラメータη，θに対するそれぞれのPoincaréマップを比較することで，この仮定の正しさを確認しよう．前者は，すでにFig.11で与えられているので，Fig.13に，

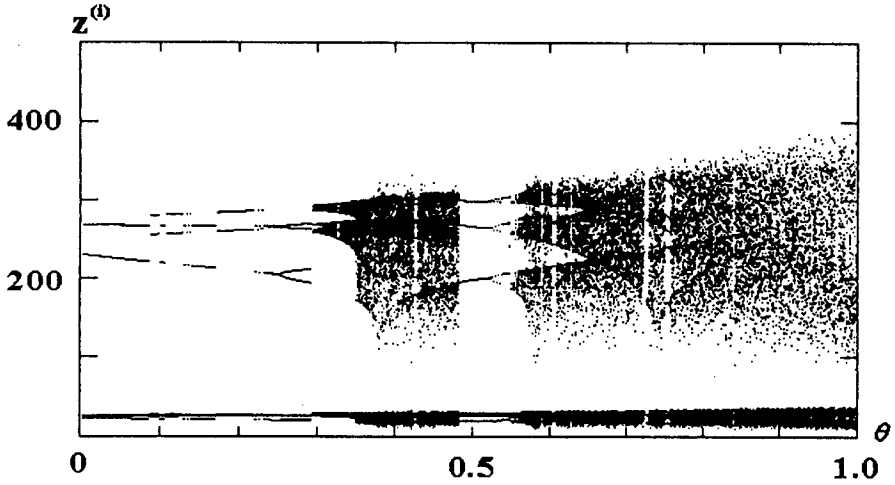

Fig.13 Lorenz モデルの行列型結合系($\varepsilon=0.3$)の Poincaré マップ
横軸には θ を取り，縦軸には各 θ ごとの流れ素子の漸近値を取る．漸近値をプロットする条件は，z 変数の傾きがゼロの時にする．上下に相似な Poincaré マップがあることから，位相同期が確認できる．また，*Fig.*11との比較によって，結合定数 θ と構成比 η の対応がわかる．

Fig.11の場合と全く同じに取った行列型結合系の Poincaré マップを示す．分岐の様子や周期の窓に至るまで，２つの図は確かに完全に対応している．すなわち，行列型結合系におけるパラメータ θ と，大域的ネットワークにおける定数 η の間には，完全な１対１の対応があり，２段階引き込み現象における第１段階のクラスター状態は非常に強く，引き続く形態変化過程において，それぞれのクラスターは独立な単一素子と見なすことが可能である．

3.4 結合パラメータ θ と非線形パラメータ r の対応

少しだけ，ここでの研究の当初の事情を述べる．この論文におけるこの節の意味が明確になるからである．我々が，流れ素子の結合系を解析し始めた頃の第１の目標は，２個のカオス的振る舞いを持つ流れ素子に行列型の結合をさせて，周期的な流れを実現することにあった．写像素子の場合には，確かに，それを実現できる．実際，２個の写像素子がカオス的領域にあっても相互作用を適当に強くすれば平均化のために結合系は周期的アトラクターにはいる．それと同じことが流れ素子の場合でも起こるはずだ，という期待にもとづいて様々な模型，初期値，パラメータや変数の組み合わせを試みたが，当時はすべて，失敗に終わった．

実は，それは，全く簡単なことであった．初期値についての依存性もないし，模型もLorenz素子でもRössler素子でもかまわない．流れ素子の行列型結合系では形態変化をつかさどるのは式（14）のパラメータ θ であり，単一の流れ素子の非線形性を決定するのは，Lorenz

素子では，パラメータ r である．答えの鍵は，この2つのパラメータ θ と r が，完全に1：1で対応しているという事である．また，$(N_1,\ N_2)$ 型の流れ素子の結合模型では，第1段階の同期過程でダイナミックスが2つのクラスターのダイナミックスに簡略化され，そのため，続く形態変化過程は，2つの流れ素子の行列型結合系の位相同期過程に帰着することを見てきたが，この1：1対応は，最後の仕上げをする．位相同期している $(N_1,\ N_2)$ 系の最終的なアトラクターは，これらの模型においては，この1：1対応のために，単一の流れ素子の力学で記述されるものとなる．この節では，まずこの1：1対応の事実を数値計算で確認しよう．

Fig.14aに，$r_1=28$，$r_2=300$ を持つ2つのLorenzモデルの $\varepsilon=0.3$ での行列型結合系のLyapunov指数の θ 依存性を示す[9]．2つのLorenzモデルの結合系であるために，結合系は6次元相空間中の流れであり，6個のLyapunov指数を持つ．一方，Fig.14bは，単一のLorenzモデルのLyapunov指数を $r=28$ から $r=300$ の間で示している．勿論，r を除くほかのパラメータは，すべて共通にとる $(P=10,\ b=8/3)$．両図を比較すれば明白なように，結合系のLyapunov指数6個の中，3個の指数の特定の θ での値は，単一の流れのLyapunov指数3個の

$$r=r_1+\theta(r_2-r_1) \tag{16}$$

における値と完全に一致し，他の固有値は，全く θ 依存性を持たぬtrivialな自由度を記述している．

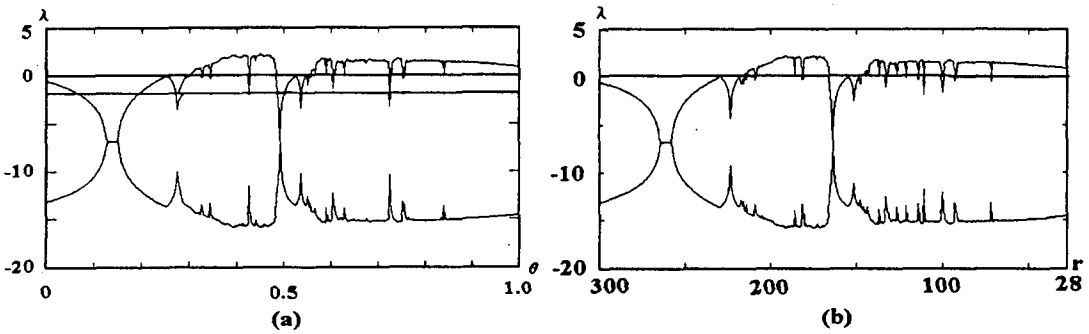

Fig.14　行列型結合系($\varepsilon=0.3$)(a)及び相互作用のないLorenzモデル(b)のLyapunov指数
14a, 14bともに縦軸はLyapunov指数を同じ範囲で取る($-20\leq\lambda\leq5$).
14aについては6次元のLyapunov指数を各 $\theta(0\leq\theta\leq1.0)$ ごとに求める．$\lambda\simeq-2.0$ でのほぼ一定の直線上には，二つのLyapunov指数が属している．また，図の範囲外の $\lambda<-20$ にはもう一つのLyapunov指数がある．
14bについては3次元のLyapunov指数を各 $r(300\geq r\geq28)$ ごとに求める．14aとの対応を見るために，14bでの横軸 r は増加とは逆の方向に取る。
14aにおける自明に一定なLyapunov指数を除けば，14aと14bから，θ と r の間に対応があることがわかる．

すなわち，結合系の力学的自由度の中，ちょうど半分が単一の素子の力学に帰着している事を示している．この1：1対応を直接に確認する事もできる．上の換算式により，例えば，行列型結合系（$r_1=28$，$r_2=300$）での $\theta=0.5$での振る舞いは，単一の Lorenz モデルの $r=(28+300)/2=164$での振る舞いと一致する訳である．すべての θ の値についても，位相同期した結合系の極限軌道のプロファイルが，対応する r での単一の軌道のそれと一致していることが確認されている．

この1：1対応を利用すればこの節の冒頭で述べた我々の当初の目標も容易に実現できる．Fig.15が Rössler 素子の場合を例に取った答えを与える．すなわち，2つのカオス的な流れ素子を結合させて周期的な流れを実現するには，まず単一の流れの最大 Lyapunov 指数が 0（周期的）となる非線形パラメータ値をまず特定し，次にその値を含むパラメータの区間を両端点での最大 Lyapunov 指数が正（カオス的）になるように選び，最後に2つの流れ素子のパラメータを端点それぞれの値に設定すればよい．この事の工学的価値は明らかである．流れ素子の結合系の位相同期現象は，結合定数で完全に制御できる．非線形素子の非線形性

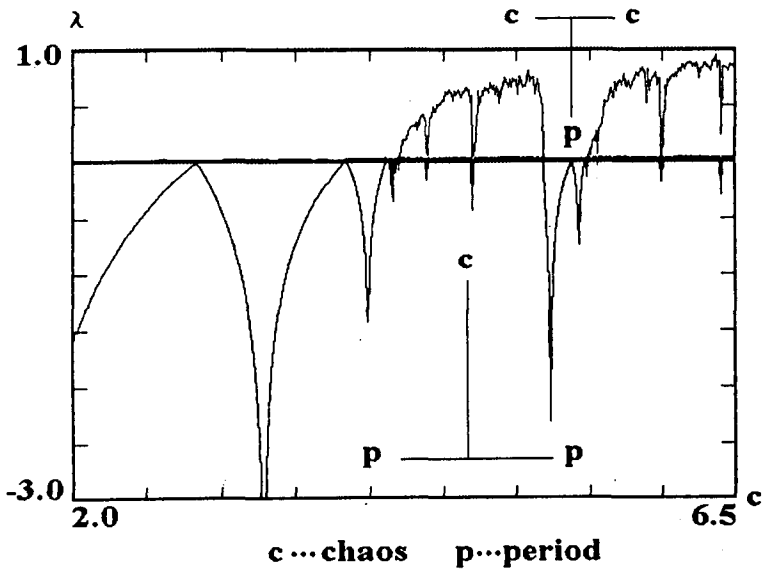

Fig.15 2つの流れ素子から結合定数でアトラクターを制御する方法
例として，標準的な Rössler モデル($a=0.2, b=0.2, c\epsilon[2.0, 6.5]$)をとり，その最大 Lyapunov 指数のグラフ上に本文中の制御法を図示する．

を機械的に操作しないでも，２つの異なる領域での非線形素子を用意して加算回路で出力を合成・フィードバックすればよい．そうすれば，カオスとカオスから位相同期した周期を作ることもできるし，周期と周期からカオスを作ることもできる．この意味で，流れ素子の結合系は，写像の結合系と同等に豊富なバラエティに富むアトラクターを提供している．生物の進化の全過程が，胎児の成長にみることができるように，非線形性パラメータの値が r_1 から r_2 に至るまでの流れのすべての様相が，２つの素子のパラメータ値を r_1 と r_2 に固定しておいても，その間の系のすべての振る舞いを結合定数を変えるだけで実現できる．

この1：1対応の原因は，次のように理解される．再び２つの Lorenz 素子に戻る．我々は，それぞれの流れ素子の x，y，z の変数の中，x のみを式（14）によって結合させている．この式は，（８），（９）の対応により，２つの系の“重心”の x 座標

$$\bar{x}=\theta x^{(1)}+(1-\theta)x^{(2)} \tag{17}$$

を与える式と見なせる。初期値としては３変数とも全く各系で異なる値に設定するので，発展の初期には、(17)の重み付きの平均操作は系に非連続的な発展を与えてしまうが，平均値への収束は指数関数的に急速なので，結合系は直ちに連続的な流れになることは，2.2節で詳述したとおりであり，それが位相同期した２つの流れであることは3.2節で確認されている．この連続な流れの成立のあとで，２つの流れの重心—(x_1, y_1, z_1) と (x_2, y_2, z_2) との式(17)による重み付き平均—の従う発展方程式を式(11)から，右辺で $x^{(1)}\simeq x^{(2)}\simeq\bar{x}$ として求めると，まさに式(16)で決まる r を持つ単一の流れの発展方程式と一致する．すなわち，２つの流れの重心は，単一の Lorenz 流れ素子の r での力学に従う．しかも，位相同期は，２つの流れ素子が相似の軌道を持つことを意味するので，結局２つの流れは，r を持つ単一の流れと相似になる．このことは，Rössler 模型でも同様である．従って，結合系を構成する２つの流れの相対運動の自由度は位相同期とともに死んで，生き残る自由度は，重心運動の自由度である．Lyapunov 指数が示す動的な自由度は，重心運動の自由度である．

なお，1：1対応の解析的な説明で，重心の従う方程式が，単一の流れのそれと一致している部分は，流れの方程式で x の係数が y, z の１次式であることによる．我々は，もっと一般の非線形流れで，どの程度まで重心運動への簡略化が成立するかを現在数値的に調べている．

4．位相同期現象の剛性 (robustness)

4.1　θ による流れ素子結合系の非線形性コントロール

パラメータ θ を与えるごとに，２つの流れ素子の行列型結合系の非線形性が決定される，

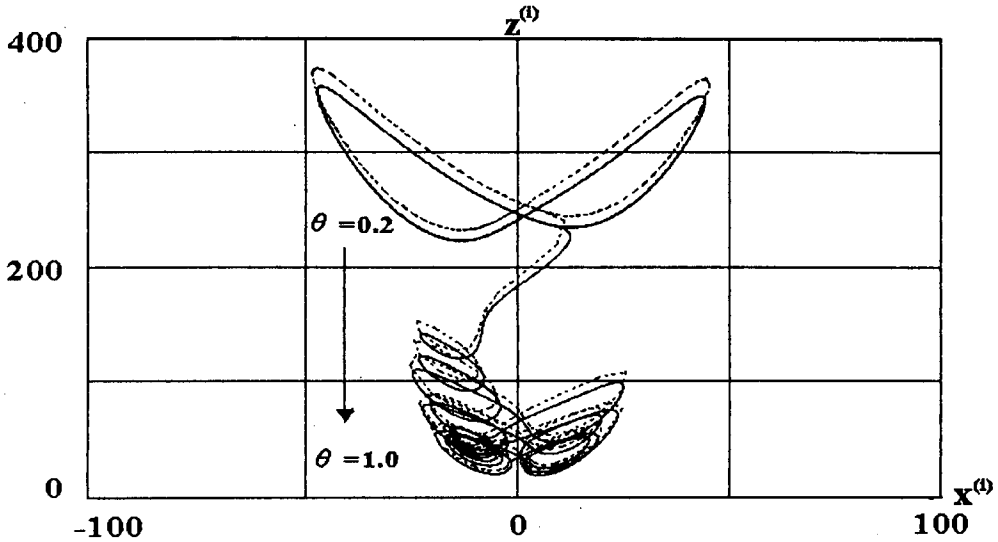

Fig.16 θ の連続的変化に対する Lorenz モデルの行列型結合系の相平面上での振る舞い
二つの Lorenz モデルの非線形パラメータはそれぞれ，$r_1=28$，$r_2=300$とする．この場合，行列型結合によって相互作用させる変数を y とする．$\theta=0.2$で十分に収束した軌道に対して，$\theta=\frac{2}{\pi}\tan^{-1}(t)$ で θ を変化させる．急激な θ の増加にも関わらず，位相同期が保たれていることから，この位相同期が極めて剛的であることがわかる．

この θ の急激な変化に対して，位相同期が十分速くレスポンスすることが工学的な応用には決め手となる．

結合方法やパラメータは，3.2節の (14) と同じであるが，結合変数としてここでは y を用いることにする．Fig.16には，$(x,\ z)$ 平面上での軌道の変化を示してある．この図における θ は次のように変化させる．まず，$\theta=0.2$で行列型結合系の 2 つの軌道を発展させる．急速に，カオス軌道は形態変化し，2 本の軌道は周期的アトラクターに位相同期をする（図の上部）．ここで，パラメータ θ を時間の関数 $\theta=\frac{2}{\pi}\tan^{-1}(t)$ で連続的に増加させる．Fig.16を見ると，2 つの流れ素子は，$\theta=0.2\rightarrow1.0$の急激な変化に完全に応答して，位相同期を保ちつつ周期からカオスに形態変化をすることがわかる．すなわち，位相同期は非線形性の連続的な変化に対して極めて剛的である．ここでは再現性を与えるために θ を決まった時間の関数で変化させたが，θ を計算機のキーを押して人為的に様々に変化させても，位相同期を保ちつつ，軌道が発展する．

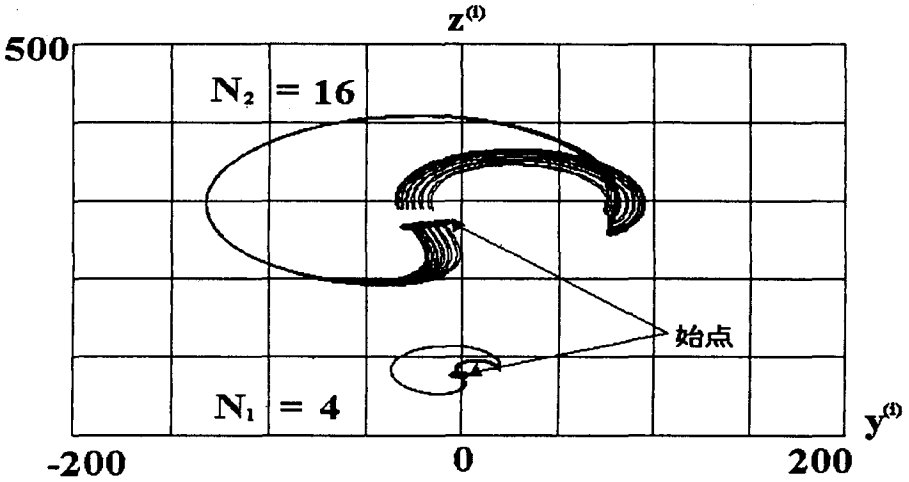

Fig.17　行列型結合からなる円環型の系の相平面上での振る舞い
$N_1=4$ 個についての非線形パラメータを $r=28$ とし，$N_2=16$ 個については $r=300$ とする．また，$N_1=4$ 個は，$i=2, 8, 13, 17$ 番目に配置する．この図における軌道は，十分収束させている．この系は，$r=28$ を持つ 4 本の軌道が，残りの $r=300$ を持つ16本に引きずられて形態変化を起こし，位相同期を起こしていることがわかる．

4.2 流れ素子の隣接相互作用系

第 3 節で見たように，異なる非線形性を持つ Lorenz モデルの大域的ネットワークは，結果的に重心運動に帰着でき，相互作用のない単一の Lorenz モデルに対応している．ここでは，行列型の結合から成る円環型の系の場合を調べる．

まず，素子として Lorenz モデルを $N=20$ 個用いる．r をのぞくパラメータは共通にし（P＝10，b＝8/3），$N_1=4$ 個はカオス（$r=28$），$N_2=16$ 個は周期（$r=300$）にパラメータを設定する。次に，これらの Lorenz モデルを円環型に配置し，(11) によって発展させた後，隣り合うもの同士を，周期境界条件のもとで式 (14) の行列型結合により双方向に相互作用させる．

$$x^{(i)} \leftarrow (1-\varepsilon)x^{(i)} + \varepsilon x^{(i+1)}$$

$$x^{(i+1)} \leftarrow (1-\varepsilon)x^{(i+1)} + \varepsilon x^{(i)}$$

以下では，$\varepsilon=0.15$ とし，4 つのカオス的な素子は，円環上のランダムな位置（図では $i=2$，8，13，17）に配置する．Fig.17に，(y，z) 平面上での20個の流れ素子の極限軌道を示す．大域的結合の場合は，同じ非線形性をもつ素子の集団は厳密に 1 つのアトラクターに入るが，この隣接的相互作用の場合の軌道は，あたかもランデブー・フライトのように極めて接近し

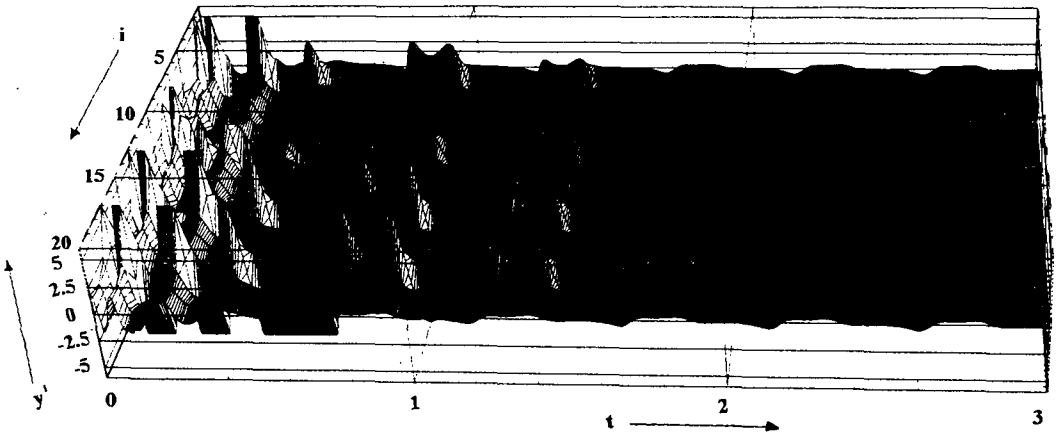

Fig.18 行列型結合からなる円環型の系の時系列
素子のパラメータや配置はFig.17と同様である．ただし，$r=300$を持つ16本の軌道については，プロットするy変数を1/20でスケールダウンさせる．この系の引き込み現象は次の二段階で理解できる．$t<2.0$までに，周期の4本についてはピークが揃い，カオスの16本についてはカオスの帯がなめらかに同期している。次に，周期のピークが形態変化を起こし，カオスの帯と完全に位相同期をしている。

た位置で同期をしていることがわかる．そして，2つのフライト集団の間の相互作用により，本来，カオス的である4本の軌道は周期性を持つ16本の軌道に大きく引きずられて，両集団の形態変化した結果としての最終的なアトラクターは位相同期をした2つのランデブー・フライトとなる．Fig.17には，この航跡のスナップ・ショットを示すが，明白に位相同期を示している．Fig.18には，この系の時系列を示す．大域的模型の場合と同様に，同じパラメータをもつ軌道集団が同期し($t<1.5$)，続いて4個のカオス軌道が周期軌道に形態変化していく現象を見て取ることができる．この現象は周期とカオスの構成比にのみ依存し，円環上の位置関係には依存しない．同じ非線形パラメータを持つものを，円環上に連続的に配置しても離散的に配置しても全く同様の形態変化が観察できる．

5．おわりに

本論文では，まず第2節で，マップの大域的結合模型と，2つの流れ素子の結合模型の最近の発展を対比した後，流れ素子の大域的結合模型，特に，異なる非線形性領域にある流れ素子の結合系の解析の動機を述べた．さらに，我々の考察する流れ素子の結合模型が，差分

化された発展の各ステップごとに座標変数を結合させるにもかかわらず，連続的な流れとしての極限を持っていることを述べた．

第3節では，最も簡単な例として，N_1のカオス的なLorenz流れ素子とN_2個の周期的なLorenz流れ素子との大域的な結合模型を構成して，我々の導入した新しい概念である位相同期の立場から解析を行った．3.1節では，この(N_1，N_2)型の大域的結合系では，同期現象が2段階の過程を経て実現されることを述べた．すなわち，第1段階では，結合定数は素子を区別しないにも関わらず，まず非線形性が等しい素子が同期を行い，第2段階では，これらの集団同士の相互作用のもとで全体系は形態変化を遂げ，新しい位相同期をしている2つのアトラクターを作る．このような2段階の発展機構は，第1段階の後に，(N_1，N_2)型の多数の素子からなる結合系の力学は，わずか2個のクラスター間の力学に帰着され，従って，第2段階の形態変化・位相同期過程は，2つの素子間の結合系の場合と同等であることを暗示する．

3.2節では，このことを確認するために，異なる非線形領域にある2個の流れ素子の行列型結合系を調べた．続く3.3節で，大域的模型での構成比 $\eta=N_1/(N_1+N_2)$ と，行列型結合系でのパラメータ θ が，正しい対応を持つことから，(N_1，N_2) 結合系の力学が，2素子の力学へ簡略化されていることを確認した．最後に3.4節で，このような2つの非線形流れ素子の結合では，位相同期のために，系の力学はその“重心運動”で代表され，本質的に単一の流れ素子の運動と一致することを示した．しかも，その重心運動は，流れにかける重みに相当する結合定数を制御することで，様々なアトラクターを実現することを示した．

第4節では，このような位相同期現象が工学的に応用するに足る剛性を持つこと，また，必ずしも，大域的結合でなくても，両方向の相互作用を許しさえすれば，隣接相互作用のみを持つイジング型の結合模型でも実現することを示した．

我々は，しばしば，系の力学変数の絶対値でなくその位相が本質的な意味を持つ物理学現象に遭遇する．例えば，超伝導現象に重要なのは，BCS対の波動関数の位相ロッキングであり，また量子場の径路積分を古典解やインスタントン解が支配するのは，やはり，それらの回りの径路が同位相で積分に寄与するからである．このような集団モード（秩序変数）が形成されると，系の物理は，集団の重心座標の運動（零モード）で支配される．この物理は，保存則のために自明なものになることもあるが，境界条件によって非自明になり，位相幾何学的に興味深い内容を持つことがある．この論文で明らかにした非線形素子の結合系のバラエティに富む非自明な位相同期したアトラクターの存在は，非可積分系の特質を反映しているように思われる．

謝辞

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Phase synchronization and nonlinearity decision in the network of chaotic flows

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The dynamics of a network of globally coupled chaotic flows is reduced to that of a single chaotic flow as the result of the phase synchronization. The mechanism of the nonlinearity decision among the flows is clarified and a simple decision rule is presented which holds in almost the entire range of the couplings and in a wide class of nonlinear flows. The key observation is that final attractors represent the "motion of the center of mass" of the network. The nonlinearity of the final attractors can be controlled by couplings. [S1063-651X(97)04203-7]

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I. INTRODUCTION

The brain is a network of a huge number of chaotic neurons and it is supposed that the synchronization over cells of the network is requisite for intelligence and pattern recognition (e.g., [1,2]). In a recent paper [3] we reported a phenomenon called *phase synchronization*. We observed it in a simple network of globally coupled N nonlinear flows in which the parameters of N_1 flows are set in the chaotic regime, and others ($N_2=N-N_1$) in the periodic regime. Starting from random initial values, the like N_1 and N_2 flows first synchronize among themselves and form two clusters, one chaotic and the other periodic, each with the original nonlinearity, and then the two clusters metamorphose into two final attractors, with some mutually decided nonlinearity. The final attractors are perfectly synchronizing each other in the phase and their orbits are precisely similar but different in the size and in positions in the phase space.

Thus the phase synchronization we found concerns flows of distinct nonlinearities and consists of two parts; the formation of new metamorphosed attractors and the precise phase locking among final attractors. Rosenblum, Pikovsky and Kurths [4] independently found similar phenomena where two or many flows with the same nonlinearity synchronize in phase even if they are given by different angular velocities with the difference of some few tens of percent. We should note that difference in the angular velocities ($\Delta\omega$) only amounts to the difference in the linear contributions ($\dot{x}_i=-\omega_i y_i+\cdots, \dot{y}_i=\omega x_i+\cdots,\ i=1,2$). Thus their analysis mainly concerns flows of the same nonlinearity. Also there is some difference of interest between the two works. Our main interest is the possibility of precise phase locking between flows with completely different nonlinear parameters, and the possibility of formation of phase-locking states with new nonlinearity from them depending on the coupling weight. On the other hand, the main interest in [4] is the possibility of the phase synchronization in an extended sense rather than the precise phase locking when flows of identical nonlinearities are coupled together with equal weight. They call it "phase synchronization" if the phase difference is varying but does not grow, that is, when there is no difference in the average angular velocities over long time. They found an interesting threshold in the coupling strength depending on the difference of angular velocities; beyond it the phase difference does not grow over 2π and below it the phase difference grows indefinitely. They also showed that two different flows may phase synchronize in this weak sense when the difference of the two flows may be regarded as a large effective noise term as in the case of the coupling of Rössler and Mackey-Glass systems.

Our phase synchronization (the metamorphosis and the phase locking) may be regarded as an interesting case of the dynamics reduction in the complex system. The global coupling is not essential; even with the nearest-neighbor bothway coupling we observe similar phase synchronization over the network. Most amazing is the fact that even the flows with completely distinct nonlinear parameters can phase synchronize. In this article we clarify the mechanism of this reduction of the network dynamics and investigate how the flows mutually decide their final nonlinearity. In Sec. II we recapitulate some of our previous results. We in particular discuss the condition for the smooth flow limit of the evolution of our network model. In Sec. III we present an intriguing observation that the final phase-synchronizing attractors represent essentially the motion of the "center of mass" of the network. The relative motions are swept away by the overwhelming coherence over the network as a result of the synchronization among the flows, which is reminiscent of the appearance of the order parameter in the superconductor or the Higgs condensation in particle physics. Most interestingly the nonlinearity of the final phase-synchronizing attractors is decided mutually among the flows by a simple decision rule so that the final phase-synchronizing attractors may be controlled by the coupling parameters between the flows and/or by the population ratios between unlike flows. In Sec. IV we conclude with some remarks on the flow-map correspondence.

II. THE PHASE SYNCHRONIZATION

First let us present a simple network of globally coupled nonlinear flows. Our model is a natural extension of the globally coupled one-dimensional map lattice by Kaneko [2] to the network of flows and to the higher dimensions. As a canonical example of nonlinear flow with more than one variable, let us take N Lorenz flows and set the parameters for the first set of flows $[(x_i(t),y_i(t),z_i(t)),\ i=1,\ldots,N_1]$

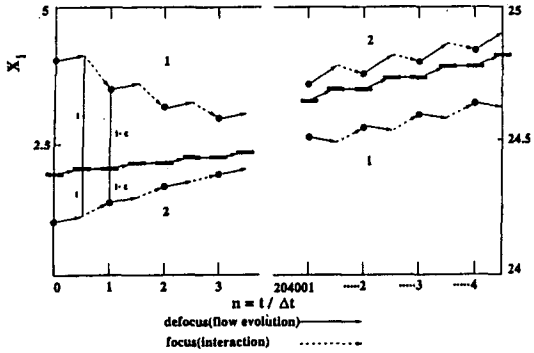

FIG. 1. A sample result of the x-coupled two distinct Lorenz flows, $r_1=28, r_2=300$, and the coupling $\epsilon=0.3, \theta=0.3$, illustrating the iteration of the two steps; the evolution (the arrowed solid line) and the interaction (the arrowed dashed line). $b=\frac{8}{3}, P=10$ for both flows, and only the directly coupled variables x_1 and x_2 are exhibited. The flows (circles) are pulled to the mean field (solid bar) at the fixed rate $1-\epsilon$ by the interaction while the mean field is not affected. For the first few iterations (left) the interaction violates the smoothness of the flows. After sufficient focus (right) the smoothness is realized. Note the change in the scale. The numbers along the horizontal axis represent the iteration steps; each step takes $\Delta t=10^{-4}$. The interaction is instantaneous, but to illustrate the invariance of the mean field, it is represented with half-width of the Δt.

in the chaotic regime and for the second set $[(x_i(t),y_i(t),z_i(t)),\ i=N_1+1,\ldots,N_1+N_2]$ in the periodic regime. Typically we choose $r^{(1)}=28$, $r^{(2)}=300$, $b^{(1)}=b^{(2)}=\frac{8}{3}$, $P^{(1)}=P^{(2)}=10$. At each time step all flows first evolve independently via the flow equations,

$$x_i(t+\Delta t)=x_i(t)+P(y_i-x_i)\Delta t,$$

$$y_i(t+\Delta t)=y_i(t)+(-x_iz_i+r_ix_i-y_i)\Delta t,$$

$$z_i(t+\Delta t)=z_i(t)+(x_iy_i-bz_i)\Delta t,$$

$$r_i=\begin{cases} r^{(1)} & \text{for } i=1,\ldots,N_1 \\ r^{(2)} & \text{for } i=N_1+1,\ldots,N_1+N_2. \end{cases}$$

$$(i=1,\ldots,N) \tag{1}$$

Then, they interact with each other in only one dimension via their mean field. For instance, choosing x for this dimension, the interaction is given by

$$x_i \leftarrow (1-\epsilon)x_i+\epsilon\bar{x}, \tag{2}$$

where $\bar{x}$ is the mean field,

$$\bar{x}=\frac{1}{N}\sum_{i=1}^{N} x_i, \tag{3}$$

and ϵ is the coupling strength. The network evolves repeating this two-step process of nonlinear evolution (1) and interaction (2). We illustrate in Fig. 1 the iteration of the two

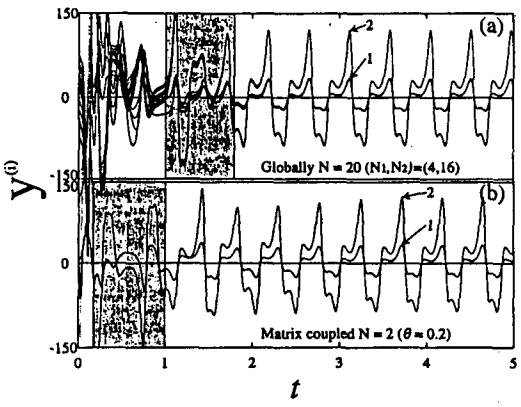

FIG. 2. The evolution of Lorenz flows in time: (a) Globally coupled flows; $N_1=4$ flows with $r=28$ and $N_2=16$ flows with $r=300$, both with $b=\frac{8}{3}, P=10$. Flow 2 is scaled up by factor 3. (b) Matrix-coupled two flows [see Eq. (9)] at $\epsilon=0.3$ and $\theta=0.2=[4/(4+16)]$. Both from random start. The flows in (a) first form two clusters and passing the decision process (shadowed area) they metamorphose into perfectly phase-synchronizing periodic attractors similar to the final attractors of the matrix model in (b).

steps schematically for the x variables in the case of $N=2$. In the first step the flows evolve independently of each other from different values and with distinct nonlinearity. Thus the nonlinear evolution in the first step acts in general as a *defocusing lens*. In the second step the interaction serves as a *focusing lens* with a fixed rate $1-\epsilon$ to the mean field $\bar{x}$. A few remarks are in order here. First, the interaction in the second step preserves the value of the mean field,

$$\frac{1}{N}\sum_i\left((1-\epsilon)x_i+\frac{\epsilon}{N}\sum_j x_j\right)=\frac{1}{N}\sum_i x_i=\bar{x}. \tag{4}$$

This invariance of the mean field under the interaction plays an important role in guaranteeing the continuous motion of the center of mass of the network as is discussed below. Secondly, in our two-step evolution model, the mean field determines the next positions of the flows directly rather than via the velocities of the flows. This is a crucial difference between our model and some similar network model [4,5]. Only in our approach can the dynamics of the network of flows have a direct correspondence with the globally coupled map lattice [2]. Thirdly, although the sequence of nonlinear evolution and interaction is similar with that in the coupled map lattice, we couple the variables in only one dimension to create the mean field and the variables in other dimensions are evolving under the influence of this mean field. Hence it is legitimate to regard the mean field $\bar{x}(t)$ as the master system and the other variables as the slave system [5,6].

Now, let us derive a crucial condition for the smoothness of the evolution of flows under Eqs. (1) and (2). For some period the interaction in the second step pulls the x_i variables to their mean value $\bar{x}$ drastically at every iteration. But, if the focusing is operated on them sufficiently frequently, all of them are soon focused around the mean value $\bar{x}$ and the change of the x_i becomes compatible with or smaller than

Δt. Then the invariance of $\bar{x}$ under the interaction guarantees the smoothness of the evolution of x_i variables to the same extent with the smoothness of other slave variables. Let us denote the time between two Poincaré shots as T. The focusing rate during this period may be estimated as

$$(1-\epsilon)T/\Delta t \approx e-T/\Delta t. \tag{5}$$

This should overcome the defocusing rate by the nonlinearity, which is exp $(\lambda_{max}T)$ where λ_{max} is the largest eigenvalue of the Froquet matrix. Thus the condition that focusing works on the x_i variables sufficiently may be estimated

$$\epsilon\frac{T}{\Delta t}>\lambda_{max}. \tag{6}$$

After this smooth flow limit is reached we can study the response of the slave system y_i and z_i under the influence of the master system $\bar{x}$. For our case of the Lorenz flows λ_{max} is of the order of 1 and study the region $0.001 \leq \epsilon \leq 1$, so we need $\Delta t \leq 10^{-3}$. Of course such Δt is small enough so that the difference equation (1) approximates the differential equation. With this consideration we take $\Delta t=10^{-4}$ throughout this article.

In Fig. 2(a) we show the evolution of the globally coupled $N=20$ flows from random start where $N_1=4$ (chaotic) and $N_2=16$ (periodic). For the first period $(0.2 \leq t \leq 1)$ the like flows synchronize among themselves, forming two clusters and, passing the decision period $(1 \leq t \leq 1.8)$, the two clusters metamorphose into two final attractors. Let us verify that the first synchronization $(t \leq 1)$ among like flows is sufficiently robust so that the N flows really pass the decision stage as tightly bound two clusters. Such a test of the dynamics reduction from N to 2 may be devised by constructing a model of two flows representing the two clusters, respectively. If such reduction really occurs, that is, if

$$(x_i,y_i,z_i)=\begin{cases}(x^{(1)},y^{(1)},z^{(1)} & \text{for } i=1,\ldots,N_1\\ (x^{(2)},y^{(2)},z^{(2)}) & \text{for } i=N_1+1,\ldots,N_1+N_2\end{cases}$$

holds all the time, the evolution equation reduces to

$$x^{(1)}(t+\Delta t)=x^{(i)}(t)+P(y^{(1)}-x^{(i)})\Delta t,$$

$$y^{(1)}(t+\Delta t)=y^{(i)}(t)+(-x^{(i)}z^{(i)}+r^{(i)}x^{(i)}-y^{(i)})\Delta t,$$

$$z^{(i)}(t+\Delta t)=z^{(i)}(t)+(x^{(i)}y^{(i)}-bz^{(i)})\Delta t$$

$$(i=1,2)$$

and the interaction is simply

$$x^{(i)}\leftarrow(1-\epsilon)x^{(i)}+\epsilon\bar{x} \quad (i=1,2), \tag{7}$$

where $\bar{x}$ is the population-ratio-weighted average,

$$\bar{x}=\eta x^{(1)}+(1-\eta)x^{(2)}, \tag{8}$$

with $\eta=N_1/N$ and $1-\eta=N_2/N$. Now let us introduce an $N=2$ flow model with the evolution as given by Eq. (7) and with an interaction in only one dimension,

$$x^{(1)}\leftarrow(1-\epsilon_2)x^{(1)}+\epsilon_2x^{(2)},$$

$$x^{(2)}\leftarrow(1-\epsilon_1)x^{(2)}+\epsilon_1x^{(1)}, \tag{9}$$

with $\epsilon_1=\theta\epsilon, \epsilon_2=(1-\theta)\epsilon$. This is a natural extension of the one-way coupling model by Pecora and Carroll [6] to both-way coupling with an interpolation parameter θ. Now we have the interaction described by Eqs. (7) and (8) with the population ratio η on one hand, and the interaction (9) with the interpolation parameter η on the other hand. By simple arithmetic we can show these agree with each other when $\theta=\eta$ [3]. That is, if the reduction from N to 2 really occurs, the evolution of the clusters thereafter, in particular the metamorphosis (decision), should proceed just in the same way with the evolution of the simple $N=2$ matrix model with the interpolation parameter θ set at the value of the population ratio η of the clusters.

In Fig. 2(b) we show the evolution of the matrix-coupled $N=2$ model. The agreement between Figs. 2(a) and 2(b) is remarkable. The flows of the global network in Fig. 2(a) take some time for the formation of two clusters $(0.2 \leq t \leq 1)$ and their evolution thereafter into the final attractors is just the same with that of the two flows in Fig. 2(b), with θ adjusted at the population ratio $\eta(=\frac{4}{20}=0.2)$ of the $N=20$ flows in Fig. 2(a).

Till now we have for simplicity divided the flows into two groups, one in the chaotic and the other in the periodic regime. Everything goes the same way with the other combinations, chaotic and chaotic but with different chaoticness and so on. Furthermore we have checked the case of three groups, four groups, ..., N groups (N flows each with its own nonlinearity). The flows again first synchronize among like flows forming clusters (except for the last case), and then metamorphose into final phase-synchronizing three, four, ...,N attractors. Now let us proceed to the target in this article, namely, the analysis of the nonlinearity decision in the metamorphosis.

III. THE METAMORPHOSIS AND NONLINEARITY DECISION

Since we have verified the reduction of the dynamics of the flows to the dynamics of clusters, we hereafter mainly consider the nonlinearity decision between two flows (two clusters) with distinct nonlinearity. This is the study of the second reduction from $N=2$ to 1. (Every discussion below, in particular the extraction of the center of mass degree of freedom, can be extended to the case of general N. Recall the classical mechanics where the two body problem has all the essential ingredients of the N body problem apart from the more subtle case of integrability.) Previously we reported that the population ratio $\eta=N_1$ in the globally coupled network of flows, or equivalently the interpolation parameter θ in the matrix-coupled two flows, serves as a control parameter of the nonlinearity of the final attractors. Now we show that there exists a simple rule of the nonlinearity decision which is valid for a wide class of nonlinear flows.

In the limit $\theta=0$ or 1 our matrix model reduces to precisely the master-slave model proposed by Pecora and Carroll in their pioneering work of the chaos synchronization [6]. (In preparing this article we find that these authors also

suggested in another paper [7] the study of "large parameter variation" which is precisely the target of our work.)

At $\theta=1$ the flow (cluster) 1 wins and at $\theta=0$ the flow (cluster) 2 wins the decision game [3,6]. Thus the rule for $\theta\approx 0$ and $\theta\approx 1$ may be schematically expressed

$$F(r^{(1)})\otimes F(r^{(2)})|_\theta \to \begin{cases} \mathrm{A}(r^{(1)}) & \text{if } \theta\approx 1 \\ \mathrm{A}(r^{(2)}) & \text{if } \theta\approx 0, \end{cases} \tag{10}$$

with obvious symbols. For instance, $\mathrm{A}(r^{(1)})$ means two phase-synchronizing periodic attractors with nonlinearity at $r^{(1)}$. Surprisingly, we find that by adjusting the coupling θ appropriately, it is possible to create from two chaotic flows the phase-synchronizing two periodic attractors

$$F(C,r^{(1)})\otimes F(C,r^{(2)})\to \mathrm{A}(P)$$

or even to create from the coupling of the two periodic flows the phase-synchronizing two chaotic attractors

$$F(P,r^{(1)})\otimes F(P,r^{(2)})\to \mathrm{A}(C).$$

Here we added the symbol C or P as a memento of the nonlinearity (chaotic or periodic) of the single flow (cluster) F at the value of $r^{(i)}$ written to the right of it. We assert that the nonlinearity decision rule in general is

$$F(r^{(1)})\otimes F(r^{(2)})|_\theta \to \mathrm{A}(r(r^{(1)},r^{(2)},\theta)),$$

$$r(r^{(1)},r^{(2)},\theta)\approx \bar{r}\equiv \theta(r)^{(1)}+(1-\theta)r^{(2)}. \tag{11}$$

The rule is as follows. First, the final two phase-synchronizing attractors are modulo a scale factor the same with the attractor of the original single flow with new nonlinearity. Second, the decided nonlinearity parameter value $r(r^{(1)},r^{(2)},\theta)$ is essentially given by the weighted average $\bar{r}$. (For the network of the flows with two groups of distinct nonlinearity it suffices to replace θ by the population η. The decision rule holds even for the negative θ or for $\theta\geq 1$ though for the η such an extension is immaterial.) This means that by varying the parameter θ the whole pattern of the attractor of the single flow in the range $r\in[r^{(1)},r^{(2)}]$ can be produced. The decision rule (11) is so simple that things might look trivial. However, we should note that the fact that the two flows (two clusters) with completely distinct nonlinearity synchronize in phase is already a surprise and that averaging or interpolating the nonlinearity parameters is a completely new notion.

Let us now present evidence for the decision rule. In Fig. 3 we show the Poincaré map of the coupled Lorenz flows for the whole range of the coupling θ with both flows set in the chaotic regime ($r^{(1)}=28$ and $r^{(2)}=200$). Figures 3(a) and 3(b) are for the x-coupled case and for the y-coupled case, respectively, and these should be compared with the Poincaré map of a single Lorenz flow in Fig. 3(c) for $r\in[28,200]$. In Figs. 3(a) and 3(b) the clouds of lower (upper) points represent the Poincaré section for the flow 1 (2). At any value of θ the lower and upper clouds agree with each other after a scaling as the result of the phase synchronization. Around $\theta=1(0)$ the master flow is the system 1 (2) and around these regions the Poincaré map of the final attractors certainly agrees with the Poincaré map of the master flow in Fig. 3(c) as the natural consequence of the rule (10). More interestingly we find in Fig. 3(a) an outstanding periodic window around $0.2\leq\theta\leq 0.3$ and also many other smaller windows. This shows that two chaotic flows have metamorphosed into phase-synchronizing periodic attractors at certain values of θ. The spectrum of the periodic windows in θ in Fig. 3(a) precisely agrees with the spectrum of the windows of the single flow in $\bar{r}$ in Fig. 3(c). Furthermore the z distributions of the clouds in Fig. 3(a) at any θ agree with the z distributions in Fig. 3(c) at the corresponding value of $\bar{r}$, which means that the final phase-synchronizing attractors are nothing but the attractor of the single flow at the corresponding $\bar{r}$. Summing up we observe that in the case of the x coupling the extended decision rule (11) holds precisely in the form

$$F(C,r^{(1)}=28)\otimes F(C,r^{(2)}=200)|_\theta$$
$$\to \mathrm{A}(\bar{r}=28\theta+200(1-\theta)). \tag{12}$$

Notably the spectrum of the periodic windows in Fig. 3(b) is almost the same as that in Fig. 3(c) but the positions of the windows in θ are shifted to the smaller θ direction about 0.15. This on one hand reveals that the decision rule holds in general approximately and on the other hand indicates the nontriviality of the rule. The agreement in the Lyapunov exponents between Figs. 3(d), 3(e) (coupled flows), and 3(f) (a single flow) also confirms the extended decision rule.

Now let us clarify why the decision rule holds. We divide the argument into items.

(1) *Decomposition of the variables.* When the final phase-synchronizing attractors are formed after the metamorphosis, the variables $x^{(1)}$ and $x^{(2)}$ have already focused around the mean value $\bar{x}$ while the other variables $y^{(1)},z^{(1)},y^{(2)}$, and $z^{(2)}$ evolve nonautonomously under the influence of $\bar{x}$. In order to analyze their motion let us introduce the "center of mass" variables and the relative variables just as in the classical mechanics. That is,

$$(\bar{x},\bar{y},\bar{z})=\theta(x^{(1)},y^{(1)},z^{(1)}+(1-\theta)(x^{(2)},y^{(2)},z^{(2)}),$$

$$(x_R,y_R,z_R)=(x^{(1)},y^{(1)},z^{(1)})-(x^{(2)},y^{(2)},z^{(2)}). \tag{13}$$

Some explanation of the term "center of mass" is in order here. We have defined $\bar{x}$ in Eq. (3) for the network of flows which leads to $\bar{x}$ in Eq. (8) for the two clusters (and the $\bar{x}$ above via the identification $\theta\equiv\eta$). These $\bar{x}$ might have been called the x coordinate of the center of mass of N flows and that of two clusters, respectively, with an assignment of a unit mass to each flow. However, we carefully called them the mean field and the population-ratio-weighted average, respectively, since they are concerned with only one of the dimensions. Here we are newly defining $\bar{y}$ and $\bar{z}$ in accord with $\bar{x}$ to define a vector $(\bar{x},\bar{y},\bar{z})$. Hence we may now call this the center of mass. We admit that the decomposition above is really a simple algebraic redefinition of the variables but we use the term *center of mass* hereafter in order to emphasize the conceptual jump that we consider not only the active variable $\bar{x}$ (active in the sense that it is used in the model to impose the focusing on the flows) but also fictitious $\bar{y}$ and $\bar{z}$ together. We are aware that it is rather radical to talk

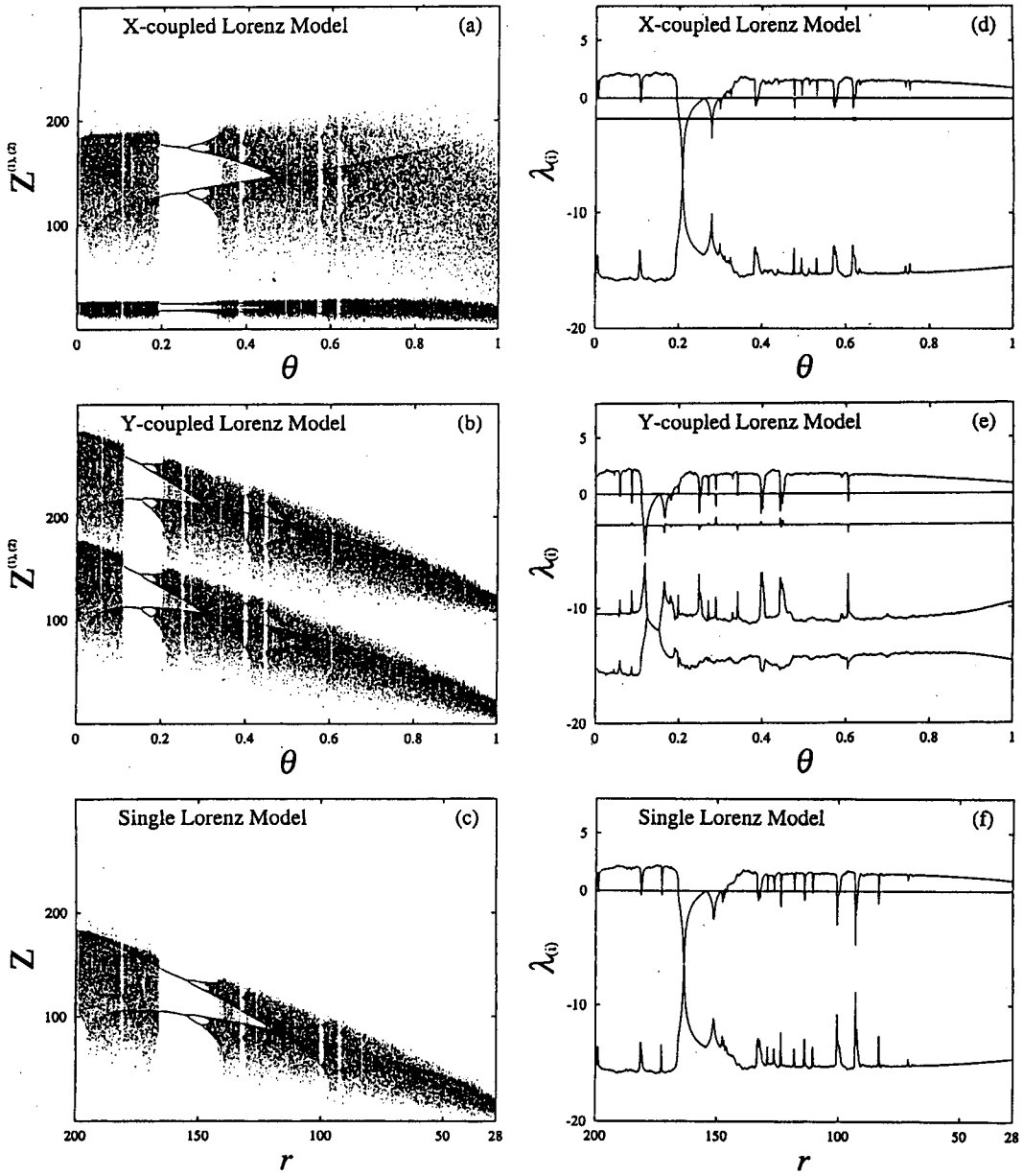

FIG. 3. The Poincaré map of two chaotic Lorenz flows ($r_1=28$, $r_2=200$, $b=\frac{8}{3}$, $P=10$) coupled by a matrix with $\epsilon=0.3$ and $\theta \in [0,1]$ [see Eq. (9)], (a) x coupling, (b) y coupling, to be compared with the Poincaré map of the single flow $r \in [200,,28])$ in (c). (For the y coupling the map of flow 2 is shifted upwards to avoid the overlap.) The maps are sampled by the conditions $dz/dt=0$ and $d^2z/dt^2=0$. The periodic window spectrum of (a) [(b)] agrees perfectly (approximately) with that of (c). The corresponding Lyapunov exponents in (d), (e), and (f) confirm the observation. See the decision rule (12).

about the center of the mass of the attractors but this analogy actually turns out vital in the discussion below. We also decompose $r^{(1)}$ and $r^{(2)}$ as

$$\bar{r}=\theta r^{(1)}+(1-\theta)r^{(2)},$$

$$\Delta r=r^{(1)}-r^{(2)}.$$

(2) *The motion of the center of mass.* With these new variables we can rewrite Eq. (1) as

$$\bar{x} \leftarrow \bar{x} + P(\bar{y} - \bar{x})\Delta t,$$

$$\bar{y} \leftarrow \bar{y} + (-\overline{xz} + \overline{rx})\Delta t + \theta(1-\theta)(-x_R z_R + x_R \Delta r)\Delta t, \tag{14}$$

$$\bar{z} \leftarrow \bar{z} + (\overline{xy} - b\bar{z})\Delta t + \theta(1-\theta)x_R y_R \Delta t.$$

The flows (clusters) evolve under the iteration of the two-step process of the evolution and interaction. For the x coupling the x_R becomes quickly of the order of Δt in the iteration. Taking $x_R \to 0$ in Eq. (14) we find that the terms proportional to $\theta(1-\theta)$ all vanish and the evolution of the center of the mass $(\bar{x},\bar{y},\bar{z})$ in the first step reduces to the evolution of the single Lorenz flow with the nonlinearity parameter $\bar{r}$. As for the second step the invariance condition (4) comes into play. It assures that the center of the mass $(\bar{x},\bar{y},\bar{z})$ is not affected by the interaction. Hence the $(\bar{x},\bar{y},\bar{z})$ evolves only by just the first step in each iteration. Thus the center of the mass of the two clusters and the center of mass of the N flows should form the Lorenz attractor with nonlinearity $\bar{r}$.

(3) *The motion of phase-synchronizing orbits and the decision rule.* The final two attractors $(x^{(1)},y^{(1)},z^{(1)})$ and $(x^{(2)},y^{(2)},z^{(2)})$ are phase synchronizing with each other. Hence their orbits must agree with the orbits of their center of mass modulo a scale factor. Hence the final attractors must be those of the single Lorenz flow with nonlinearity $\bar{r}$. This is the explanation of the precise decision rule for the x-coupling case.

(4) *The reduction from $N=2$ to 1.* The original degree of freedom of two flows (clusters) was 6, and after $x_R \to 0$ it becomes 5, the $\bar{x},\bar{y},\bar{z}$, and the rotation and expansion around the origin in the $x_R - y_R$ plane.

The phase synchronization, namely, the synchronizing motion around similar orbits, implies that the last two freedoms are essentially also lost. This is clearly seen in the Lyapunov exponents in Figs. 3(d) and 3(e). There are six eigenvalues among which three precisely agree with the exponents of the single Lorenz flow in Fig. 3(f) in the whole range of $\bar{r}$. They change sensitively with the variation of the coupling θ and represent the active degree of freedom of $(\bar{x},\bar{y},\bar{z})$. The other exponents do not vary with θ and represent the nonactive degree of freedom. From the nonautonomous evolution equations for (x_R,y_R,z_R) we can easily verify the independence of their Lyapunov exponents from θ.

(5) *Generic cases.* In the above argument we used the fact that x_R is focused to nearly zero in the x coupling. This removes the terms proportional to $\theta(1-\theta)$ and reduces Eq. (14) to the evolution equation of the single Lorenz flow at $\bar{r}$. The sufficient condition for the same argument which works in the generic case is that the evolution equation does not have nonlinear terms in other variables than the one chosen for the coupling. In short the slave dimensions must be linear among themselves. In the case of the above x-coupled Lorenz system, the nonlinear terms are xz and xy, both of which are linear in y and z. Hence the sufficient condition is satisfied and the decision rule holds precisely. Actually the synchronization of the coupled flows can be realized in general when the Lyapunov exponents for the driven system are less than zero [6]. For the Lorenz flows we can construct the coupled flow network not only by the coupling in x but also in y. For the y-coupled Lorenz flows, the xz term does not satisfy the sufficient condition. However, as we find in the numerical calculation, the nonlinearity decision is approximately made following the rule in Eq. (11) with some negotiation (the quoted shift by ~ 0.15). This can be understood as due to the low nonlinearity in the slave dimensions. We have numerically tested various known nonlinear flows. In all cases the decision rule holds precisely when the sufficient condition is satisfied, and approximately otherwise so far as the synchronization occurs. For instance, the Rössler flows can be synchronized by the y coupling. The nonlinear term xz violates the sufficient condition and we observe the negotiated decision (see Fig. 4 below). The Brusselator can be synchronized by both x and y coupling. The nonlinear term x^2y satisfies the condition for the x coupling but violates it for the y coupling. We indeed observe that the rule holds precisely in the x coupling but approximately in the y coupling.

Let us present two more figures in order to demonstrate how the decision rule works and to indicate the feasibility of the technical application of it. In Fig. 4 we choose the y-coupled Rössler model which exhibits the negotiated decision. The bottom box represents the Lyapunov exponents in the range $c \in [2,6.5]$. The other parameters a and b are both fixed at 0.2. We pick two flows, the flow 1 with $c=3.5$ and the flow 2 with $c=5.3$.

Before the coupling both flows are in the periodic regime as shown in the middle two boxes. By the coupling with $\epsilon=0.3$ and $\theta=0.2$ they should metamorphose following the rule

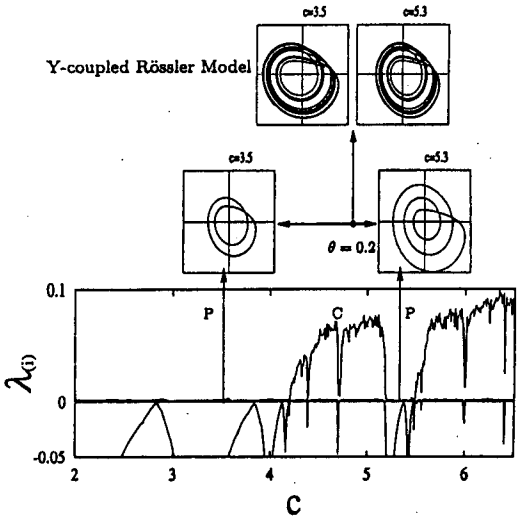

FIG. 4. An illustration of the decision rule in Eq. (15). We pick two Rössler flows $c_1=3.5, c_2=5.3, a=b=0.2$; the attractors are both periodic (middle two boxes). The coupling with $\epsilon=0.3$ and $\theta=0.2$ makes chaotic phase-synchronizing two attractors (top two boxes).

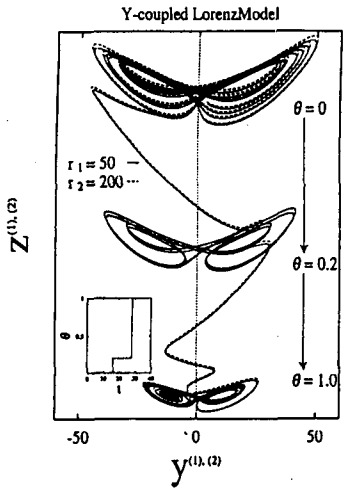

FIG. 5. A sample of control of phase-synchronizing two coupled attractors (solid and dashed lines, $\epsilon=0.3$) by the coupling θ. Flows 1 and 2 are both chaotic. $r_1=50$, $r_2=200$, $b=\frac{8}{3}$, $P=10$. Responding to the quick change of the coupling depicted in the little box they swiftly decide their mutual nonlinearity by the decision rule (16). At the rest period $t\in[16.2,28]$ they freely make periodic attractors (the middle profile).

$$\mathcal{F}_{\text{Rössler}}(P,c^{(1)}=3.5)\otimes\mathcal{F}_{\text{Rössler}}(P,c^{(2)}=5.3)|_{\theta=0.2}\rightarrow \mathrm{A}(c), \tag{15}$$

where $\mathcal{F}_{\text{Rössler}}$ represents the Rössler flow, and $c\approx\bar{c}=3.5\times0.2+0.2+5.3\times0.8=4.94$. The single flow at $c\approx4.94$ is chaotic, as can be seen in the bottom box with the maximum Lyapunov exponent $\lambda_{\max}=0.0693$. The final phase-synchronizing attractors shown in the top two boxes are indeed chaotic with $\lambda_{\max}=0.071$ and the decision rule is respected by two Rössler flows well even in the y coupling.

In Fig. 5 we exhibit the phase-synchronizing attractors of the y-coupled Lorenz flows ($\epsilon=0.3$) controlled by the decision rule

$$\mathcal{F}_{\text{Lorenz}}(C,r^{(1)}=50)\otimes\mathcal{F}_{\text{Lorenz}}(C,r^{(2)}=200)|_{\theta}$$
$$\rightarrow A(r(r^{(1)},r^{(2)},\theta)),$$

$$r(r^{(1)},r^{(2)},\theta)\approx\bar{r}\equiv\theta r^{(1)}+(1-\theta)r^{(2)}, \tag{16}$$

where $\mathcal{F}_{\text{Lorenz}}$ represents the Lorenz flow. [This supplements our previous analysis of the simpler case $\mathcal{F}_{\text{Lorenz}}(C,r=28)\otimes\mathcal{F}_{\text{Lorenz}}(P,r=300)|_{\theta}$. See Fig. 2 of [3].] Actually this figure is a record of controlling the phase-synchronizing attractors by varying θ by pressing the key of a personal computer. In order to avoid the overlap of the trajectories the attractors are constantly scrolled down in the display. Most intriguing is the robustness of the phase synchronization under the rapid variation of the coupling θ. See the change of θ during the period $t\in[16,16.2]$ and $t\in[28,28.2]$ reproduced in the little box. We tried a game to produce the periodic attractors from two chaotic flows by control. Watching the two phase-synchronizing attractors dancing in the display it was quite easy to figure out the value of θ necessary, which was around 0.2. Thus in the run for this figure we let the two flows move around freely with θ fixed at 0.2 for the period $t\in[16.2,28]$. The orbits in the middle of the figure are the resulting periodic attractors.

The single Lorenz flow has a prominent periodic window around $r=160$. The decision rule gives $\bar{r}=50\theta+200(1-\theta)\approx170$ for $\theta=0.2$ and we have certainly caught this window in the control. How much is the decision rule negotiated by the y-coupled Lorenz flows? The Lyapunov exponents of these attractors are $(\underline{0},-\underline{1.46},-2.93,-8.69,-\underline{13.6},-326)$ and the underlined three exponents agree with three exponents of the single Lorenz flow at $r=165.6$. While the precise rule under the sufficient condition dictates the formation of the flows with $r=170$ the actual attractors have $r=165.6$. Thus the negotiation is $170\rightarrow165.6$.

IV. CONCLUSION

The dynamics of a network of globally coupled flows with distinct nonlinearity is reduced first into that of clusters formed by like flows by synchronization among them and then further reduction occurs by the decision of the nonlinearity among the clusters. The clusters metamorphose into the final phase-synchronizing attractors which are essentially the attractor of a single flow with mutually decided nonlinearity among the clusters. We have shown that the decision rule can be written in a simple form in Eq. (11). The sufficient condition for the decision rule is that the nonlinear terms of the flow equation are linear in the variables other than the one chosen for the coupling. Even if the condition is not satisfied the decision rule is respected with some negotiation as long as the synchronization can be realized between the flows (clusters). We have shown ample examples for both cases and have demonstrated the feasibility of technical application of the rule.

We close this article by pointing out that our network model of globally coupled flows may be closely linked to the network of globally coupled maps. Recall the intriguing fact that the flow of a dissipative system may be in the same universality class with the map essentially because the Poincaré section of the flow becomes one dimensional due to dissipation [8]. We have used the condition (6) to determine Δt so that the focusing (2) is applied to the system frequently enough and that the smooth flow limit is guaranteed after Δx becomes very small. Thus the analysis presented in this article is the dynamics in the strong focus regime where the dynamical reduction overwhelms the network. If, on the other hand, ϵ is extremely small, as small as $O(\Delta t)$, the effect of interaction will not affect the smoothness of the orbits, and the nonlinearity of the flows and the coherence due to the interaction via the mean field will make a subtle balance on the Poincaré section. In such a weak focus regime, the network of flows will mimic the network of maps on the Poincaré section. In a preliminary analysis of the coupled Duffing oscillators in the universality class of the May map, we indeed observed the formation of spatial clusters. An extensive study in this regime is underway.

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3.　総合研究を顧みて

一つの概念は，「ことば」できちんと捉えられたときに発展性を持つ．場の理論は，その無限自由度の力学を記述することばとして，柔軟さと強力さを併せ持ち，素粒子論をはじめとしてその周辺の物理学での概念形成に欠かせないものになっている．

当時，和泉校舎の自然科学系で場の理論を共通言語とする研究者４人が，学部の枠を越えて，互いに緩やかな相互作用を繰り返しながら，研究を進めて見ようではないかという発想をしたことから本研究は立ち上がった．この試みは，明治大学科学技術研究所の総合研究として認められ，1992年4月の発足後，３年の研究期間を経て，本報告に掲載した多くの論文が示すように十分な成果を上げて，1995年3月に完了した．場の理論の数学的構造を厳密に追求する事，場の理論を弦理論をふまえて再構築しようとすること，さらに，場の理論を隣接分野で駆使しその新しい機能を探ろうとすること，など，総合研究ならではの成果が上がった．．場の理論の内包する言語としての躍動性のためか，研究成果には，むしろ場の理論にベースを置きつつ，周辺に切り込んでいく種類のものが多く含まれているが，表題は，研究所で認められた当初のものとした．

代表者として成果を取りまとめつつ，この研究が実ったことを大変嬉しく感じている．

参加研究者は，

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である．発足当時，筆者(島田)は，本学商学部に所属していたが，研究場所を生田に移す事になった．本研究に対して，同研究所から支給された研究費は，次の通りである．

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期間中の，科学技術研究所の惜しみないサポートに深く感謝申し上げる．