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Mean field limit for one dimensional opinion dynamics with Coulomb interaction and time dependent weights

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ABSTRACT

The mean field limit with time dependent weights for a 1D singular case, given by the attractive Coulomb interactions, is considered. This extends recent results (Ayi and Duteil, 2021; Duteil, 2022) for the case of regular interactions. The approach taken here is based on transferring the kinetic target equation to a Burgers-type equation through the distribution function of the measures. The analysis leading to the stability estimates of the latter equation makes use of Kruzkov entropy type estimates adapted to deal with nonlocal source terms.

1. Introduction

1.1. General background

In this paper, we are concerned with analysing the mean field limit of the following system of $2N$ ODEs:

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} m_j^N(t)V'(x_j^N(t) - x_i^N(t)), & x_i^N(0) = x_i^{0,N} \\ \dot{m}_i^N(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)), & m_i^N(0) = m_i^{0,N}. \end{cases} \tag{1.1}$$

The notation is as follows: the unknowns $x_i^N \in \mathbb{R}$ and $m_i^N \in \mathbb{R}$ are referred to as the *opinions* and *weights* respectively. The evolution of the opinions is given in terms of the weights and a function $V : \mathbb{R} \to \mathbb{R}$ which is called *the interaction* modulating the value of the opinion x_i^N by the presence of the other opinions x_j^N. The evolution of the weights is given by means of functions $\psi_i^N : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ where we apply the notation

$$\mathbf{x}_N(t) := (x_1^N(t), \dots, x_N^N(t)), \qquad \mathbf{m}_N(t) := (m_1^N(t), \dots, m_N^N(t)).$$

The weights $m_i^N(t)$ can be interpreted as the proportion of the total population with opinion $x_i^N(t)$. How the system (1.1) originates from real-life phenomena is beyond the scope of this work. Just to mention a few works which explain how this system models phenomena in biology and the social sciences, we refer to [3,13,16]. The system (1.1) is a weighted version of the first order N-body problem (simply by taking all the weights to be identically equal to 1), to which we now briefly draw our attention to. As to give

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an extensive review of the relevant literature, we momentarily consider arbitrary $d \geq 1$, although our main result is in 1D. By now, the mean field limit of the N-body problem

$$\dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^N \nabla V(x_j^N(t) - x_i^N(t)), \qquad x_i^N(0) = x_i^{0,N} \tag{1.2}$$

is fairly well understood, even in the case of interactions with strong singularities near the origin — we will comment more about this later on. This mean field limit is understood in terms of the empirical measure which is defined by

$$\mu_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^N(t)}.$$

Thanks to the work of Dobrushin [7], and assuming ∇V is Lipschitz, it is possible to show that $\mu_N(t)$ converges to the solution μ of the Vlasov equation

$$\partial_t \mu(t,x) - \operatorname{div}(\mu \nabla V \star \mu)(t,x) = 0, \qquad \mu(0,\cdot) = \mu^0 \tag{1.3}$$

with respect to the Wasserstein metric (provided this is true initially of course). When time dependent weights are included the problem is rendered difficult, since now the candidate for the empirical measure is

$$\mu_N(t) := \frac{1}{N}\sum_{i=1}^N m_i^N(t)\delta_{x_i^N(t)}$$

and formal considerations (see Proposition 15 in [7]) suggest the following transport equation with self-consistent source term as the target equation

$$\partial_t \mu(t,x) - \operatorname{div}(\mu \nabla V \star \mu)(t,x) = h[\mu](t,x), \qquad \mu(0,\cdot) = \mu^0. \tag{1.4}$$

Here $h[\mu]$ is the self-consistent source term which arises from the inclusion of weights, and for the moment we do not specify it. Already at the level of the well posedness theory of the target Eq. (1.4) and the system (1.1) some care is needed, since apriori it is not entirely that the solution stays a probability density for all times. The well-posedness of Eqs. (1.1) and (1.4) as well as the weighted mean field limit has been successfully established in [3,8] in arbitrary dimension and for interactions with Lipschitz gradient. The approach in [8] is based on stability estimates for the Wasserstein distance, which imply both well-posedness and mean field limit, whereas the approach in [3] recovers to some extent the mean field limit obtained in [8], and is based on the graph limit method. We refer also to [14,15] for more details about the graph limit regime and its link with the mean field limit. Other works which consider weighted opinion dynamics are [10], in which the weights are taken to be time-independent, but may vary from one opinion to another, and [16] which serves as a general survey.

1.2. Main results

All the existing literature reviewed so far concerns arbitrary dimension and relatively well behaved potentials in terms of regularity, typically with at least locally Lipschitz gradient. It is the aim of this work to investigate how to overcome the challenges created due to singular potentials in 1D. In particular, we consider the attractive 1D Coulomb interaction $V(x) = |x|$ (so that $\partial_x V = \operatorname{sgn}(x)$, with the convention $\operatorname{sgn}(0) = 0$). In addition, we add further limitations on the equation governing the weights, namely we take

$$\psi_i^N(\mathbf{x}_N, \mathbf{m}_N) = \frac{1}{N}\sum_{j=1}^N m_i^N m_j^N S(x_j^N - x_i^N) \tag{1.5}$$

where $S \in C_0^\infty(\mathbb{R})$ is assumed to be odd. From the opinion modelling point of view, the value $|S(x_j^N - x_i^N)|$ in (1.5) can be interpreted as the rate of change from opinion x_i^N to x_j^N at a given time. In view of this interpretation the oddness of the function S is natural from the assumption of conservation of the total population of individuals. In fact, it means that the proportion of individuals that change their opinion from value x_i^N to value x_j^N is the opposite of the individuals that change their opinion from value x_j^N to value x_i^N at a given time. Note that at a first glance it seems we allow that at any given time the individuals can change abruptly from opinion x_i^N to opinion x_j^N. However, if the rate function S is of compact support, as in our main result in Theorem 4.1, then the allowed change of opinion is local and smooth. As we already mentioned, this parity condition turns out to be important for the purpose of guaranteeing preservation of the total mass. For these choices, the system (2.1) takes the form

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^N m_j^N(t)\operatorname{sgn}(x_j^N(t) - x_i^N(t)), & x_i^N(0) = x_i^{0,N}, \\ \dot{m}_i^N(t) = \frac{1}{N}\sum_{j=1}^N m_i^N(t) m_j^N(t) S(x_j^N(t) - x_i^N(t)), & m_i^N(0) = m_i^{0,N}. \end{cases} \tag{1.6}$$

In this case, the mean field equation takes the form

$$\partial_t \mu(t,x) - \partial_x\left(\mu(t,x)\int_{-\infty}^x \mu(t,y)dy - \mu(t,x)\int_x^\infty \mu(t,y)dy\right) = \mu(t,x) S \star \mu(t,x), \qquad \mu(0,x) = \mu^0.$$

When the kernel S is odd one expects that μ stays a probability density for all times and therefore the equation formally transforms to

$$\partial_t \mu(t,x) - \partial_x \left(\mu(t,x) \left(2 \int_{-\infty}^{x} \mu(t,y)dy - 1 \right) \right) = \mu(t,x) S \star \mu(t,x), \qquad \mu(0,x) = \mu^0. \tag{1.7}$$

Setting $F(t,x) = -\frac{1}{2} + \int_{-\infty}^{x} \mu(t,y)dy$ and integrating both sides of Eq. (1.7), we arrive (still at the formal level) at the following non local Burgers type equation for F

$$\partial_t F + \partial_x (A(F)) = \mathbf{S}[F](t,x) \tag{1.8}$$

where $A(F) := -F^2$ and

$$\mathbf{S}[F](t,x) := F(t,x)(\phi \star F)(t,x) - \int_{-\infty}^{x} F(t,z)(\partial_z \phi \star F)(t,z)dz, \ \phi := \partial_x S.$$

Of course, the definition of $\mathbf{S}[F]$ is motivated by integrating by parts the expression $\int_{-\infty}^{x} \partial_z F(t,z)(S \star \partial_z F)(t,x)dz$. The idea of analysing the equation for the primitive of μ stems from the work [4] which studies the homogeneous Burgers equation. The advantage of Eq. (1.8) in comparison to (1.7) is that the flux term is local, which raises the hope that a well posedness theory is in reach. We refer to [6] for a well posedness theory in the case of non-local fluxes (yet with no source term). By closely adapting the method introduced in [12] (which has its roots in the classical Kruzkov entropy) we are able to simultaneously prove the BV-well posedness of Eq. (1.8), and derive the unique solution of (1.8) in the limit as $N \to \infty$ of the primitive of the empirical measure (shifted by $-\frac{1}{2}$ in order to ensure proper cancellation), namely

$$F_N(t,x) := -\frac{1}{2} + \frac{1}{N} \sum_{k=1}^{N} m_k^N(t) H(x - x_k^N(t))$$

where H is the Heaviside function. A notable difference with respect to [12] is that we deal with time dependent fluxes for the approximation sequence of conservation laws with sources satisfied by F_N associated to the particle system (1.6), see (3.2) below. In addition, we also face technical difficulties in checking the entropy conditions of the accumulation limits of the sequence F_N due to the time dependency. Other differences which put our work in variance with [12] are reflected in the well-posedness for the system (1.1), the handling of the different source term, and the verification that F_N is an entropy solution of (a discretized version of) Eq. (1.8). We stress that the 1D settings are essential for the analysis to be carried out properly. In the context of higher dimensions, we mention the recent breakthrough [18], in which the author succeeded in deriving Eq. (1.3) from the system (1.2) for interaction which may have even stronger singularities than Coulomb. The modulated energy strategy in [18] is well suited for higher dimensions, which raises the interesting question whether this strategy can be extended to the framework of the present work in which time dependent weights are considered.

In Section 2 we fix some notation, prove the well posedness of the system (1.1) for the case of the 1D attractive Coulomb interaction and record other basic properties of the solutions of the ODE system. Section 3 is aimed at constructing a discretized flux and showing that F_N satisfies the Burgers equation associated with this flux. Section 4 is devoted towards proving Theorem 4.1, the main result of this work. The mean field limit and the BV-well posedness of Eq. (1.8) are the content of Theorem 4.1.

2. Preliminaries

For readability, the upper index N of the opinions/weights appears implicitly in the sequel. Consider the weighted N-body problem

$$\begin{cases} \dot{x}_i(t) = \frac{1}{N} \sum\limits_{1 \le j \le N: j \ne i} m_j(t) \mathrm{sgn}(x_j(t) - x_i(t)), & x_i(0) = x_i^0, \\ \dot{m}_i(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)), & m_i(0) = m_i^0. \end{cases} \tag{2.1}$$

For the purpose of establishing well posedness of the system (2.1) on short times it is not strictly necessary to take ψ_i^N to be of the form (1.5). However, in order to have well-posedness for all times this special form, and specifically the oddness of S seems to be needed at least to some extent. The hypotheses that we impose on ψ_i^N in the sequel are identical to those specified in [3] and are recalled below.

Hypothesis (H1). There is a constant $L > 0$ such that for all $(\mathbf{x}_N, \mathbf{y}_N, \mathbf{m}_N, \mathbf{p}_N) \in \mathbb{R}^{4N}$ the inequalities

$$\left| \psi_i^N(\mathbf{x}_N, \mathbf{m}_N) - \psi_i^N(\mathbf{y}_N, \mathbf{m}_N) \right| \le L \left| \mathbf{x}_N - \mathbf{y}_N \right|,$$

$$\left| \psi_i^N(\mathbf{x}_N, \mathbf{m}_N) - \psi_i^N(\mathbf{x}_N, \mathbf{p}_N) \right| \le L \left| \mathbf{m}_N - \mathbf{p}_N \right|$$

hold. In addition, there is a constant $C > 0$ such that and for each $(\mathbf{x}_N, \mathbf{m}_N) \in \mathbb{R}^{4N}$ it holds that

$$\left| \psi_i^N(\mathbf{x}_N, \mathbf{m}_N) \right| \le C(1 + \max_{1 \le k \le N} |m_k|). \tag{2.2}$$

We shall hereafter supress the dependence on N of the opinions/weights whenever this dependence is irrelevant. Since we wish to prove well posedness on a time interval which does not shrink to zero as $N \to \infty$, a natural assumption to impose is that the opinions are separated initially, i.e.

$$\forall N \in \mathbb{N} : x_1^0 < \cdots < x_N^0. \tag{2.3}$$

2.1. Filippov solutions

Before embarking on the task of establishing the stability estimates, we must first prove the existence of a solution to Eq. (1.6) on some time interval $[0, T]$, where T *is independent of* N (because eventually we wish to take $N \to \infty$). It is the aim of this section to explain how this is achieved — in fact we will prove existence for arbitrarily large times. First, let us be lucid about the notion of solution that we work with. We start by examining the more general ODE (2.1) and we specify later the stage in which the special form (1.5) comes into play.

Definition 2.1. A classical solution of the system (2.1) on $[0, T]$ is an absolutely continuous curve $t \mapsto (\mathbf{x}_N(t), \mathbf{m}_N(t)) \in \mathrm{AC}([0,T]; \mathbb{R}^{2N})$ such that the system (2.1) is satisfied for a.e. $t \in [0, T]$. Equivalently, for all $t \in [0, T]$ it holds that

$$x_i(t) = x_i^0 + \int_0^t \frac{1}{N} \sum_{1\le j\le N: j\neq i} m_j(\tau)\,\mathrm{sgn}\left(x_j(\tau) - x_i(\tau)\right) d\tau,$$

$$m_i(t) = m_i^0 + \int_0^t \psi_i^N(\mathbf{x}_N(\tau), \mathbf{m}_N(\tau)) d\tau.$$

In order to prove existence of solutions in the sense of Definition 2.1 it is convenient to use the machinery of differential inclusions as developed by Filippov [9]. The special form (1.6) is irrelevant for what concerns the abstract theory of Filippov. First we must review some basic definitions and facts from convex analysis. Let us recall how to view the system (2.1) as a differential inclusion. Let $\Omega \subset \mathbb{R}^l$ be some domain. Given a vector field $f : \Omega \to \mathbb{R}^l$ whose set of discontinuities is D we assign to it a set valued map $\mathcal{S}[f] : \Omega \to 2^{\mathbb{R}^l}$ defined as

$$\mathcal{S}[f](X) := \left\{ \lim_{\substack{X^*\to X\\ X^*\notin D}} f(X^*) \right\}.$$

With this notation define the set valued map $\mathcal{F}[f] : \Omega \to 2^{\mathbb{R}^l}$ by

$$\mathcal{F}[f](X) := \overline{\mathrm{co}}\,\mathcal{S}[f](X), \tag{2.4}$$

where co stands for the convex hull.

Remark 2.1. In the context of system (2.1) we take $f(\mathbf{x}_N, \mathbf{m}_N)$ to be the vector

$$\left(\frac{1}{N} \sum_{1\le j\le N} m_1 \mathrm{sgn}(x_j - x_1), \ldots, \frac{1}{N} \sum_{1\le j\le N} m_N \mathrm{sgn}(x_j - x_N), \psi_1^N(\mathbf{x}_N, \mathbf{m}_N), \ldots, \psi_N^N(\mathbf{x}_N, \mathbf{m}_N) \right). \tag{2.5}$$

Next, we recall the notion of upper-semicontinuity.

Definition 2.2. For each closed sets $A, B \subset \Omega$ let $\beta(A, B) := \sup_{a\in A} \inf_{b\in B} |a - b|$. A function $\mathcal{F} : \Omega \to 2^{\mathbb{R}^l}$ is said to be upper-semicontinuous at $p \in \Omega$ if $\lim_{p'\to p} \beta(\mathcal{F}(p), \mathcal{F}(p')) = 0$. If $\mathcal{F}$ is upper-semicontinuous for each $p \in \Omega$ then we say that it is upper-semicontinuous.

On the other hand, recall the notion of a solution to a differential inclusion

Definition 2.3. Let $\mathcal{F} : \Omega \to 2^{\mathbb{R}^l}$ be a set valued map. A solution on $[\underline{T}, \overline{T}]$ to the differential inclusion

$$\dot{X}(t) \in \mathcal{F}(X(t)) \tag{2.6}$$

is an absolutely continuous map $X : [\underline{T}, \overline{T}] \to \mathbb{R}^l$ such that (2.6) holds for a.e. $t \in [\underline{T}, \overline{T}]$.

The existence theory of differential inclusions has been developed by several different groups (Just to mention a few, see [1,2,9]). We follow [9], and by no means aim to give an exhaustive overview of this rich theory. The following theorem is the main tool that we need to get existence for the opinion dynamics.

Theorem 2.1 (*[9], Theorem 1, Page 77*)**.** *Let $\mathcal{F} : \mathbb{R}^l \to 2^{\mathbb{R}^l}$ be such that*

1. *$\mathcal{F}(x) \neq \emptyset$ for all $x \in \Omega$.*
2. *$\mathcal{F}(x)$ is compact and convex for all $x \in \Omega$.*
3. *$\mathcal{F}$ is upper-semicontinuous.*

Then for any $\overline{T} > 0$ and any $X_{\underline{T}} \in \mathbb{R}^l$ there exist a solution (in the sense of Definition 2.3) to the differential inclusion

$$\dot{X}(t) \in \mathcal{F}(X(t)),\ \ X(\underline{T}) = X_{\underline{T}}$$

on the time interval $[\underline{T}, \overline{T}]$.

Here is a convenient criteria for upper semi-continuity

Lemma 2.1 (*[9], Lemma 3, Page 67*). *Let $f : \Omega \to \mathbb{R}^l$ be piecewise continuous and let $\mathcal{F}[f]$ be defined as in (2.4). Then $\mathcal{F}[f]$ is upper-semicontinuous.*

We can now gather all of the above reminders to get the following important conclusion

Corollary 2.1. *Let ψ_i^N satisfy (H1). Let $f : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be given by (2.5). For any $\overline{T} > 0$ and any $(\mathbf{x}_N^{\underline{T}}, \mathbf{m}_N^{\underline{T}}) \in \mathbb{R}^{2N}$ there exist a solution to the differential inclusion*

$$(\dot{\mathbf{x}}_N, \dot{\mathbf{m}}_N) \in \mathcal{F}[f](\mathbf{x}_N(t), \mathbf{m}_N(t)), \qquad (\mathbf{x}_N(\underline{T}), \mathbf{m}_N(\underline{T})) = (\mathbf{x}_N^{\underline{T}}, \mathbf{m}_N^{\underline{T}})$$

on $[\underline{T}, \overline{T}]$.

Proof. That $\mathcal{S}[f](p)$ is compact is an immediate consequence of the easy observation that $\mathcal{S}[f]$ is finite set-valued map. For the same reason $\mathcal{F}[f](p)$ is the set of all convex combinations of the points of $\mathcal{S}[f](p)$, and is therefore compact (see, e.g., page 62 in [9]). By Lemma 2.1, $\mathcal{F}[f]$ is upper-semicontinuous, so that by Theorem 2.1 the claim follows. □

It is important to observe that when f is continuous, the map $\mathcal{F}[f]$ is single valued and as a result the corresponding differential inclusion reduces to an ODE in the sense of Definition 2.1. This observation, as well as the special structure which arises from the attractive interaction and forces opinions to stick together, allows us to construct solutions to the system (2.1) in the sense of Definition 2.1. We now return to the form (1.6). The following proposition is the main result of this section.

Proposition 2.1. *Let assumption (2.3) hold and suppose $m_i^0 > 0$ and $\frac{1}{N}\sum_{i=1}^N m_i^0 = 1$. Let S be odd, continuous and bounded. For each $\overline{T} > 0$ there exist a solution on $[0, \overline{T}]$ to the system (1.6) (in the sense of Definition 2.1).*

Proof. Step 1(a). Preservation of total mass. Let $\overline{T} > 0$ and let $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ be some solution on $[0, \overline{T}]$ given by Corollary 2.1. For each $1 \le i \le 2N$ let $\mathfrak{f}_i : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be the vector field whose ith component identifies with the ith component of f and is 0 in all other components, so that $f = \sum_{i=1}^{2N} \mathfrak{f}_i$. By subadditivity of $\mathcal{F}[f]$ we have

$$\mathcal{F}[f] \subset \bigcup_{i=1}^{2N} \mathcal{F}[\mathfrak{f}_i].$$

Since the functions ψ_i^N are continuous, $\mathcal{F}[\mathfrak{f}_i] = \{\mathfrak{f}_i\}$ for each $N+1 \le i \le 2N$, so that the weights are governed by an ODE, namely for a.e. $t \in [0, \overline{T}]$ one has

$$\dot{m}_i(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)), \qquad m_i(0) = m_i^0.$$

We claim that the total mass is preserved. Indeed, using that S is odd, let us compute

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^N \dot{m}_k(t) &= \frac{1}{N^2}\sum_{k=1}^N\sum_{j=1}^N m_k(t)m_j(t)S(x_j(t)-x_k(t)) = -\frac{1}{N^2}\sum_{k=1}^N\sum_{j=1}^N m_k(t)m_j(t)S(x_k(t)-x_j(t)) \\ &= -\frac{1}{N^2}\sum_{j=1}^N m_j(t)\sum_{k=1}^N m_k(t)S(x_k(t)-x_j(t)) = -\frac{1}{N}\sum_{j=1}^N \dot{m}_j(t).\end{aligned}$$

As a result we get

$$\frac{d}{dt}\frac{1}{N}\sum_{k=1}^N m_k(t) = 0,$$

so that

$$\frac{1}{N}\sum_{k=1}^N m_k(t) = \frac{1}{N}\sum_{k=1}^N m_k^0 = 1.$$

Step 1(b). Weights remain positive. We claim that $m_i(t) > 0$ for all $t \in [0, \overline{T}]$. Indeed using preservation of total mass we obtain

$$\left|\frac{d}{dt}\frac{1}{2}\log\left(m_i^2+\varepsilon^2\right)\right| = \left|\frac{m_i(t)\dot{m}_i(t)}{m_i^2+\varepsilon^2}\right| = \left|\frac{m_i^2(t)}{(m_i^2(t)+\varepsilon^2)N}\sum_{j=1}^N m_j(t)S(x_j(t)-x_i(t))\right| \le \|S\|_\infty\,,$$

hence

$$-\|S\|_\infty \le \frac{1}{2}\frac{d}{dt}\log\left(m_i^2(t)+\varepsilon^2\right) \le \|S\|_\infty\,.$$

Integration in time yields the for all $t \in [0, \overline{T}]$ the estimate

$$-2\,\|S\|_\infty\, t + \log\left((m_i^0)^2 + \varepsilon^2\right) \le \log\left(m_i^2 + \varepsilon^2\right) \le 2\,\|S\|_\infty\, t + \log\left((m_i^0)^2 + \varepsilon^2\right),$$

and consequently

$$\left((m_i^0)^2 + \varepsilon^2\right)\exp(-2\,\|S\|_\infty\, t) \le m_i^2(t) + \varepsilon^2 \le \left((m_i^0)^2 + \varepsilon^2\right)\exp(2\,\|S\|_\infty\, t).$$

Letting $\varepsilon \to 0$ gives

$$(m_i^0)^2 \exp(-2\,\|S\|_\infty\, t) \le m_i^2(t) \le (m_i^0)^2 \exp(2\,\|S\|_\infty\, t).$$

In particular $m_i(t)$ does not vanish on $[0, \overline{T}]$, which by continuity and the assumption $m_i^{\mathrm{in}} > 0$ implies $m_i(t) > 0$ for all $t \in [0, \overline{T}]$ with the estimate

$$m_i^0 \exp(-\,\|S\|_\infty\, t) \le m_i(t) \le m_i^0 \exp(\|S\|_\infty\, t).$$

Step 2. **Construction of a classical solution on** $[0, \overline{T}]$. The idea is to apply an iteration argument which terminates after at most N steps. This iteration proceeds as follows. The assumption that $\forall i \neq j\,:\, x_i^0 \neq x_j^0$ implies that $\forall i \neq j\,:\, x_i(t) \neq x_j(t)$ on some sufficiently (possibly N-dependent) short time interval (by continuity). As a result the solution is classical on some short time — by the same argument at the beginning of Step 1(a). Let

$$T_1^* = \sup\left\{0 < T \le \overline{T}\,\middle|\,\forall t \in [0, T), \forall i \neq j\,:\, x_i(t) \neq x_j(t)\right\}.$$

If $T_1^* = \overline{T}$ we are done. If $T_1^* < \overline{T}$, consider the set of all collisions at time T_1^* with the ith particle, i.e.

$$J_1^i := \left\{j \neq i\,\middle|\,x_i(T_1^*) = x_j(T_1^*)\right\},$$

and set

$$f_i^1(\mathbf{x}_N, \mathbf{m}_N) := \frac{1}{N}\sum_{j \notin J_1^i} m_j \operatorname{sgn}(x_j - x_i),$$

and

$$f^1 := (f_1^1, \ldots, f_N^1).$$

Obviously $J_1^i = J_1^j$ iff $j \in J_1^i$. From the same consideration as before the solution of the differential inclusion

$$\dot{\mathbf{x}}_{N,1} \in \mathcal{F}[f^1](\mathbf{x}_{N,1}(t)),\ \mathbf{x}_{N,1}(T_1^*) = \mathbf{x}_N(T_1^*)$$

satisfies $x_i(t) \neq x_j(t)$ for all $1 \le i \le N$ and $j \in [N] \setminus J_1^i$ with $i \neq j$, and is classical (both of these conclusions hold on some short time of course). Let

$$T_2^* = \sup\left\{T_1^* < T \le \overline{T}\,\middle|\,\forall t \in [T_1^*, \overline{T}), \forall 1 \le i \le N, \forall j \in [N] \setminus J_1^i, i \neq j\,:\, x_i(t) \neq x_j(t)\right\}.$$

If $T_2^* = \overline{T}$, then consider the curve

$$\mathbf{y}_N(t) = \begin{cases} \mathbf{x}_N(t),\ t \in [0, T_1^*] \\ \mathbf{x}_{N,1}(t)\ t \in (T_1^*, \overline{T}] \end{cases}.$$

In this case we make use of the attractive structure, which implies that opinions which collided remain so, in order to show that $\mathbf{y}_N(t)$ is a solution:

Claim 2.1. *The curve* $(\mathbf{y}_N(t), \mathbf{m}_N(t))$ *is a solution to the system* (1.6) *on* $[0, \overline{T}]$.

Proof. The curve $\mathbf{y}_N(t)$ solves the Eq. (1.6) on $[0, T_1^*]$. To see that $\mathbf{y}_N(t)$ solves the equation on $(T_1^*, \overline{T}]$ we need to explain why particles that collided at time T_1^* remain collided for all $t > T_1^*$. Indeed, suppose on the contrary that $x_{i_0}(T_1^*) = x_{j_0}(T_1^*)$ for some $i_0 \neq j_0$, but

$$\tau := \inf\left\{T_1^* < t \le \overline{T}\,\middle|\,x_{i_0}(t) \neq x_{j_0}(t)\right\} > 0,$$

so that $x_{i_0}(\tau) = x_{j_0}(\tau)$. We may assume with no loss of generality that $x_{i_0}(t) - x_{j_0}(t) > 0$ on some sufficiently small time interval $(\underline{\tau}, \overline{\tau}) \subset (\tau, \overline{T}]$. Let

$$\tau_* := \inf\left\{\,\tau < \tau' < \underline{\tau}\,\middle|\,\forall t \in (\tau', \overline{\tau})\,:\, x_{i_0}(t) - x_{j_0}(t) > 0\right\},$$

so that $x_{i_0}(t) - x_{j_0}(t) > 0$ for all $t \in (\tau_*, \overline{\tau})$ and $x_{i_0}(\tau_*) = x_{j_0}(\tau_*)$. On the other hand, since $J_1^{i_0} = J_1^{j_0}$, for a.e. $t \in (\tau_*, \overline{\tau})$ we compute that

$$\begin{aligned}\dot{x}_{i_0}(t) - \dot{x}_{j_0}(t) &= \frac{1}{N} \sum_{j \notin J_1^{i_0}} m_j(t)(\mathrm{sgn}(x_j(t) - x_{i_0}(t)) - \mathrm{sgn}(x_j(t) - x_{j_0}(t))) \\ &= \frac{1}{N} \sum_{j \notin J_1^{i_0} : x_{i_0}(t) \geq x_j \geq x_{j_0}(t)} m_j(t)(\mathrm{sgn}(x_j - x_{i_0}) - \mathrm{sgn}(x_j - x_{j_0})) \leq 0,\end{aligned} \tag{2.7}$$

which implies that for all $t \in (\tau_*, \overline{\tau})$

$$x_{i_0}(t) - x_{j_0}(t) \leq x_{i_0}(\tau_*) - x_{j_0}(\tau_*) = 0,$$

which is absurd. Remark that Inequality (2.7) is thanks to the fact the weights are positive (step 1(b)). Therefore $x_{i_0}(t) = x_{j_0}(t)$ for all $t > T_1^*$ which shows that for all $T_1^* < t \leq \overline{T}$ it holds that

$$\frac{1}{N} \sum_{j \notin J_1^{i_0}} \mathrm{sgn}(x_j(t) - x_i(t)) = \frac{1}{N} \sum_{j=1}^{N} \mathrm{sgn}(x_j(t) - x_i(t)),$$

so that $\mathbf{y}_N(t)$ is a solution to the system (1.1) on $(T_1^*, \overline{T}]$ as well. □

If $T_2^* < \overline{T}$, then we continue according to the algorithm described above, which must terminate after at most N steps, thereby yielding an absolutely continuous curve $\mathbf{y}_N(t)$ on $[0, \overline{T}]$. The same argument demonstrated in Claim 2.1 shows that $\mathbf{y}_N(t)$ is a solution to the system (1.6). □

We finish this section by observing a few elementary properties of solutions to (1.6). First we stress that *any* classical solution on some given time interval to the system (1.6) must satisfy the "sticky opinions property", namely that opinions that collide stay collided. As a result there exist a finite partition of the time interval into sub-intervals on each of which no new collisions occur. Moreover, opinions preserve the initial ordering. This is summarized in the following

Proposition 2.2. *Let the assumptions of Proposition 2.1 hold. Let $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ be a solution to system (1.6) (in the sense of Definition 2.1) on some time interval $[0, \overline{T}]$. There exist $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = \overline{T}$ $(1 \leq k \leq N)$ such that for any given $1 \leq l \leq k$ it holds that for any given $1 \leq i, j \leq N$ either $\forall t \in [T_{l-1}, T_l) : x_i(t) = x_j(t)$ or $\forall t \in [T_{l-1}, T_l) : x_i(t) \neq x_j(t)$. Moreover, $x_1(t) \leq \cdots \leq x_N(t)$, $t \in [0, \overline{T}]$.*

A careful examination of the argument in Proposition 2.1 reveals that the proof of Proposition 2.2 is in fact implicitly included in that of Proposition 2.1 , and is therefore omitted. This is just to clarify that the "sticky opinions property" is not some kind of an extra assumption. In addition, for the mean field limit it is important to observe that

$$F_N(t, x) := -\frac{1}{2} + \frac{1}{N} \sum_{k=1}^{N} m_k(t) H(x - x_k(t)), \tag{2.8}$$

becomes constant for any x outside some interval, which is a consequence of preservation of total mass — this is precisely the place where we use the assumption that S is odd. This would allow to obtain stability estimates globally in L^1 (rather only in L^1_{loc}) and is recorded in the following simple

Lemma 2.2. *Let the assumptions of Proposition 2.1 hold. Suppose also there is some $\overline{X}$ such that for all $N \in \mathbb{N}$ and $1 \leq i \leq N$ it holds that $\left|x_i^0\right| \leq \overline{X}$. Then, there exist some $\overline{R} = \overline{R}(\|S\|_\infty, \overline{X}, \overline{T}) > 0$ such that for any $F_N(t, \pm x) = \pm\frac{1}{2}$ for all $t \in [0, \overline{T}]$ and $|x| > \overline{R}$.*

Proof. Thanks to the equation for the opinions, the bound for the masses in step 1 of Proposition 2.1 and the assumption on the initial opinions, we have

$$\left|x_i(t)\right| \leq \left|x_i^0\right| + \frac{1}{N} \sum_{1 \leq j \leq N : j \neq i} \int_0^t \left|m_j(\tau)\right| d\tau \leq \overline{X} + \overline{MT},$$

for some $\overline{M} = \overline{M}(\|S\|_\infty, \overline{T})$. Set $\overline{R} = \overline{R}(\overline{T}, \overline{X}, \overline{M}) := \overline{X} + \overline{MT}$. Then according, for all $x \geq \overline{R}$ it holds that

$$H(x - x_k(t)) = 1$$

while for all $x < -\overline{R}$ it holds that

$$H(x - x_k(t)) = 0.$$

Keeping in mind step 1 of Proposition 2.1 we conclude that for all $x > R$ and $t \in [0, \overline{T}]$

$$F_N(t, x) = -\frac{1}{2} + \frac{1}{N} \sum_{k=1}^{N} m_k(t) = -\frac{1}{2} + 1 = \frac{1}{2},$$

while for all $x < -\overline{R}$ and $t \in [0, \overline{T}]$

$$F_N(t, x) = -\frac{1}{2} + 0 = -\frac{1}{2}. \quad \square$$

3. The discretized version of the Burgers-type equation

In this section we construct a discretization of the flux by means of the weights, and then show that F_N is an entropy solution for the corresponding Burgers like equation — that is Eq. (1.8), with the only difference being that the flux A is replaced by a discretized approximation thereof. As usual, we assume that the opinion are ordered increasingly, i.e.,

$$x_1^0 < x_2^0 < \cdots < x_N^0.$$

Throughout this section we always work under the assumptions of Proposition 2.1 and take $t \mapsto (\mathbf{x}_N(t), \mathbf{m}_N(t))$ to be a solution of the system (1.6). For each $0 \le i \le N$ we set

$$\theta_i(t) := \begin{cases} -\frac{1}{2} + \frac{1}{N}\sum_{j=1}^{i} m_j(t), & 1 \le i \le N, \\ -\frac{1}{2}, & i = 0. \end{cases}$$

Note that since the weights are positive for all times (step 1 in Proposition 2.1) the θ_i are ordered increasingly

$$-\frac{1}{2} \equiv \theta_0(t) < \cdots < \theta_N(t) \equiv \frac{1}{2}.$$

The discretized flux, denoted $A_N(t,x)$, is defined for each t to be the (unique) continuous, piecewise linear function with break points only at $\left(\theta_i(t)\right)_{i=1}^{N-1}$ such that $A(t,\theta_i(t)) = A(\theta_i(t))$. In other words for each $x \in (\theta_{i-1}(t), \theta_i(t))$

$$A_N(t,x) := \left(\frac{A(\theta_i(t)) - A(\theta_{i-1}(t))}{\theta_i(t) - \theta_{i-1}(t)}\right)(x - \theta_i(t)) + A(\theta_i(t)).$$

Although the A_N are time dependent, the following simple lemma shows that they approximate the time independent flux A.

Lemma 3.1. *For each $t \in [0,T]$ and each $x \in [-\frac{1}{2}, \frac{1}{2}]$ it holds that*

$$\left|A_N(t,x) - A(x)\right| \le \frac{2\overline{M}\,\mathrm{Lip}(A|_{[-\frac{1}{2},\frac{1}{2}]})}{N}.$$

where $\overline{M}$ is as in Lemma 2.2 .

Proof. Keep $t \in [0,T]$ fixed. Let $x \in [-\frac{1}{2}, \frac{1}{2}]$, and pick $1 \le i \le N$ such that $x \in [\theta_{i-1}(t), \theta_i(t)]$. By the above formula for A_N we get

$$\left|A_N(t,x) - A(x)\right| \le \left|\frac{A(\theta_i(t)) - A(\theta_{i-1}(t))}{\theta_{i-1}(t) - \theta_i(t)}\left(x - \theta_i(t)\right)\right| + \left|A(\theta_i(t)) - A(x)\right|.$$

Clearly

$$\left|\frac{x - \theta_i(t)}{\theta_{i-1}(t) - \theta_i(t)}\right| \le 1,$$

so that

$$\begin{aligned}\left|A_N(t,x) - A(x)\right| &\le \left|A(\theta_i(t)) - A(\theta_{i-1}(t))\right| + \left|A(\theta_i(t)) - A(x)\right| \\ &\le 2\mathrm{Lip}(A|_{[-\frac{1}{2},\frac{1}{2}]})\left|\theta_i(t) - \theta_{i-1}(t)\right| \le \frac{2\overline{M}\,\mathrm{Lip}(A|_{[-\frac{1}{2},\frac{1}{2}]})}{N}. \quad \square\end{aligned}$$

The definition of entropy solutions originated in the celebrated work [11]. We recall the notion of entropy solution for conservation laws with non-local source terms, which will be the notion of solution that we will use for what concerns the mean field limit and the well posedness for the Cauchy problem of the Burgers equation.

Definition 3.1. Let $A \in C([0,T]; \mathrm{Lip}[-\frac{1}{2}, \frac{1}{2}])$. A function $F \in BV([0,T]\times\mathbb{R})$ is called an entropy solution of the equation

$$\partial_t F + \partial_x(A(t,F)) = \mathbf{S}[F](t,x)$$

with initial data $F^0 \in BV(\mathbb{R})$ if

(1) $\forall t \in [0,T], \forall x \ge R : F(t,\pm x) = \pm\frac{1}{2}$.
(2) The map $x \mapsto F(t,x)$ is non-decreasing for any $t \in [0,T]$.
(3) $F(t,\cdot)$ converges to F^0 as $t \to 0^+$ in the sense of distributions.
(4) For any $\chi \in C_0^\infty((0,T)\times\mathbb{R})$ and any $\alpha \in \mathbb{R}$ it holds that

$$\int_0^T \int_{\mathbb{R}} \left(|F - \alpha|\,\chi_t + \mathrm{sgn}(F-\alpha)\,(A(F) - A(\alpha))\,\partial_x\chi + \mathrm{sgn}(F-\alpha)\chi\mathbf{S}[F](t,x)\right) dx\,dt \ge 0. \tag{3.1}$$

Remark 3.1. A particular byproduct of the requirements (1) and (2) is that for any $t \in [0,T]$ the total variation $\left|\partial_x F(t,\cdot)\right|$ is a probability measure.

Remark 3.2. We emphasize that we can keep the classical definition of the convolution for the term $\mathbf{S}[F]$ unlike in [12]. That is, if $\mathrm{supp}(S) \subset [-r, r]$ we take $R = \max\left\{r, \overline{R}\right\}$($\overline{R}$ as in Lemma 2.2) and realizing that

$$\phi \star F(t, z) := \int_{-2R}^{2R} \phi(z - \zeta)F(t, \zeta)d\zeta,$$

due to the definition of $S \star F$, $\phi = \partial_z S$, and the fact that F is constant outside the interval $[-2R, 2R]$. It is now evident that $\phi \star F, \partial_z\phi \star F \in C_0([0, T] \times \mathbb{R})$, which enables to make sense of the last term in the left hand side of inequality (3.1).

Next we claim that F_N is an entropy solution to our Burgers equation but with a discretized flux,. i.e. the equation

$$\partial_t F_N + \partial_x(A_N(t, F_N)) = \mathbf{S}[F_N](t, x). \tag{3.2}$$

To verify this we need to check that

i. F_N is a classical solution on finitely many regions which form a disjoint partition of the whole domain $(0, T) \times \mathbb{R}$
ii. The Rankine–Hugoniot condition
iii. The Oleinik conditions.

It is classical that the verification of i-iii imply the integral inequality (3.1)- see Appendix for more details about this implication. We start by verifying point 1. Let us recall that by Proposition 2.2 we know there exist finitely many times $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = T$ such that on each (T_{j-1}, T_j) collision does not occur. More specifically, we know that for each $1 \le j \le k$ there is a disjoint partition of $[N]$ into subsets $I^j_1, \ldots, I^j_{m_j} \subset [N]$ (in brief $\sqcup_{i=1}^{m_j} I^j_i = [N]$) such that:

(a) Given $1 \le i \le m_j$ it holds that $x_\alpha(t) = x_\beta(t)$ for each $\alpha, \beta \in I^j_i$ and each $t \in [T_{j-1}, T_j)$.
(b) For each $1 \le i < i' \le m_j$ it holds that $x_\alpha(t) < x_\beta(t)$ for each $\alpha \in I^j_i, \beta \in I^j_{i'}$ and each $t \in [T_{j-1}, T_j)$.

For each $1 \le j \le k$ and $1 \le i \le m_j$ we let $i^*(j)$ be the maximal index inside I^j_i, i.e.

$$i^*(j) := \max\left\{r \,\middle|\, r \in I^j_i\right\}.$$

With this notation we have the following simple

Lemma 3.2. *Let F_N be given by (2.8). For each $1 \le j \le k$ and $1 \le i \le m_j$ set*

$$V^{i,j}_L := \left\{(t, x) \,\middle|\, t \in [T_{j-1}, T_j), x_{(i-1)^*(j)}(t) \le x < x_{i^*(j)}(t)\right\}$$

and

$$V^{i,j}_R := \left\{(t, x) \,\middle|\, t \in [T_{j-1}, T_j), x_{i^*(j)}(t) < x \le x_{(i+1)^*(j)}(t)\right\}.$$

(with the convention $x_{0^*} = -\infty$ and $x_{(m_j+1)^*} = +\infty$. Then, for each $1 \le j \le k$ and $1 \le i \le m_j$, F_N is a classical solution on both $V^{i,j}_L$ and $V^{i,j}_R$.

Proof. For any $(t, x) \in V^{i,j}_L$ we have

$$F_N(t, x) = -\frac{1}{2} + \frac{1}{N}\sum_{r=1}^{(i-1)^*(j)} m_r(t) = \theta_{(i-1)^*(j)}(t)$$

and for any $(t, x) \in V^{i,j}_R$ we have

$$F_N(t, x) = -\frac{1}{2} + \frac{1}{N}\sum_{r=1}^{i^*(j)} m_r(t) = \theta_{i^*(j)}(t).$$

In particular, note that $A_N(t, F_N(t, x))$ is constant in x on both regions, so that its x-derivative vanishes. In addition for each $x_{(i-1)^*(j)}(t) \le x < x_{i^*(j)}(t)$ and $t \in [T_{j-1}, T_j)$ a routine calculation shows that

$$\begin{aligned}\mathbf{S}[F_N](t, x) &= \frac{1}{N^2}\sum_{l,r} H(x - x_l(t))m_l(t)m_r(t)S(x_l(t) - x_r(t)) \\ &= \frac{1}{N^2}\sum_{r=1}^{N}\sum_{l=1}^{(i-1)^*(j)} m_l(t)m_r(t)S(x_l(t) - x_r(t)),\end{aligned}$$

while

$$\begin{aligned}\frac{d}{dt}F_N(t, x) &= \frac{1}{N^2}\sum_{l=1}^{(i-1)^*(j)}\sum_{r=1}^{N} m_l(t)m_r(t)S(x_l(t) - x_r(t)) \\ &= \frac{1}{N^2}\sum_{r=1}^{N}\sum_{l=1}^{(i-1)^*(j)} m_l(t)m_r(t)S(x_l(t) - x_r(t)) = \mathbf{S}[F_N].\end{aligned}$$

This shows that F_N is a classical solution on $V^{i,j}_L$. The same calculation shows that F_N is a classical solution on $V^{i,j}_R$. $\square$

We move to verify conditions (ii) and (iii).

Lemma 3.3. *The function F_N is an entropy solution to Eq. (3.2) in the sense of Definition 3.1*

Proof. Condition (1) is from Lemma 2.2 and that $x \mapsto F_N(t,x)$ is non-decreasing is immediate from the fact that the weights are positive, which gives condition (2). Validating the requested weak inequality is slightly longer and rests upon the Rankine–Hugoniot and Oleinik conditions (which as already remarked, are recapped in Appendix). With the same notation of Lemma 3.2 , keep $1 \le j \le k$ and $1 \le i \le m_j$ fixed and consider the curve

$$\Gamma_{i,j} := \left\{ (t, x_{i^*(j)}(t)) \middle| t \in [T_{j-1}, T_j) \right\}.$$

The Rankine–Hugoniot Condition. We wish to show that

$$\frac{A(t, F^{i,j}_{N,L}(t)) - A(t, F^{i,j}_{N,R}(t))}{F^{i,j}_{N,L}(t) - F^{i,j}_{N,R}(t)} = \dot{x}_{i^*(j)}(t).$$

To make the equations a bit lighter let us abbreviate $i^*(j) = \bar{i}, (i-1)^*(j) = \underline{i}$. It is clear that

$$F^{i,j}_{N,L}(t) = -\frac{1}{2} + \frac{1}{N}\sum_{r=1}^{\underline{i}} m_r(t) = \theta_{\underline{i}}(t)$$

and

$$F^{i,j}_{N,R}(t) = -\frac{1}{2} + \frac{1}{N}\sum_{r=1}^{\bar{i}} m_r(t) = \theta_{\bar{i}}(t).$$

Since $\frac{1}{N}\sum_{r=1}^{N} m_r(t) = 1$ we may rewrite

$$F^{i,j}_{N,L}(t) = \frac{1}{2} - \frac{1}{N}\sum_{r>\underline{i}} m_r(t).$$

Therefore, using that $\frac{1}{N}\sum_{j=1}^{N} m_j(t) = 1$ and the fact that the initial order is preserved, the following identities hold

$$\begin{aligned}
\frac{A_N(t, F^{i,j}_{N,L}(t)) - A_N(t, F^{i,j}_{N,R}(t))}{F^{i,j}_{N,L}(t) - F^{i,j}_{N,R}(t)} &= \frac{\theta^2_{\bar{i}}(t) - \theta^2_{\underline{i}}(t)}{\theta_{\underline{i}}(t) - \theta_{\bar{i}}(t)} = -(\theta_{\bar{i}}(t) + \theta_{\underline{i}}(t)) \\
&= 1 - \frac{1}{N}\sum_{r=1}^{\bar{i}} m_r(t) - \frac{1}{N}\sum_{r=1}^{\underline{i}} m_r(t) \\
&= 1 - \left(1 - \frac{1}{N}\sum_{r>\bar{i}} m_r(t)\right) - \frac{1}{N}\sum_{r=1}^{\underline{i}} m_r(t) = \frac{1}{N}\sum_{r>\bar{i}} m_r(t) - \frac{1}{N}\sum_{r=1}^{\underline{i}} m_r(t) \\
&= \frac{1}{N}\sum_{r>\bar{i}} \operatorname{sgn}(x_r(t) - x_{\bar{i}}(t)) m_r(t) + \frac{1}{N}\sum_{r=1}^{\underline{i}} \operatorname{sgn}(x_r(t) - x_{\bar{i}}(t)) m_r(t) \\
&\quad + \frac{1}{N}\sum_{r=\underline{i}+1}^{\bar{i}} \operatorname{sgn}(x_r(t) - x_{\bar{i}}(t)) m_r(t) \\
&= \frac{1}{N}\sum_{r=1}^{N} m_r(t) \operatorname{sgn}(x_r(t) - x_{\bar{i}}(t)) = \dot{x}_{\bar{i}}(t).
\end{aligned}$$

The Oleinik Condition. Let $\theta \in (\theta_{\underline{i}}(t), \theta_{\bar{i}}(t))$. We wish to show

$$\frac{A_N(t,\theta) - A_N(t, \theta_{\underline{i}}(t))}{\theta - \theta_{\underline{i}}(t)(t)} \ge \dot{x}_{\bar{i}}(t).$$

Note that if we pick $\underline{i} + 1 \le m \le \bar{i}$ such that $\theta \in [\theta_{m-1}(t), \theta_m(t)]$ then since A_N is piecewise linear we clearly have

$$\frac{A_N(t,\theta) - A_N(t, \theta_{\underline{i}}(t))}{\theta - \theta_{\underline{i}}(t)} \ge \min\left\{ \frac{A_N(t,\theta_m(t)) - A_N(t, \theta_{\underline{i}}(t))}{\theta_m(t) - \theta_{\underline{i}}(t)}, \frac{A_N(t,\theta_{m-1}(t)) - A_N(t, \theta_{\underline{i}}(t))}{\theta_{m-1}(t) - \theta_{\underline{i}}(t)} \right\}.$$

Therefore it suffices to check the inequality for $\theta = \theta_k(t)$ where $\underline{i} \le k \le \bar{i}$. We have

$$\frac{A_N(t,\theta_k(t)) - A_N(t, \theta_{\underline{i}}(t))}{\theta_k(t) - \theta_{\underline{i}}(t)} = \frac{A(\theta_k(t)) - A(\theta_{\underline{i}}(t))}{\theta_k(t) - \theta_{\underline{i}}(t)} = -(\theta_k(t) + \theta_{\underline{i}}(t)) \ge -(\theta_{\bar{i}}(t) + \theta_{\underline{i}}(t)) = \dot{x}_{\bar{i}}(t)$$

where the inequality is because $\theta_{\bar{i}}(t) \ge \theta_k(t)$, and the last identity is a byproduct of the previous step. □

4. The mean field limit

In this section we prove well-posedness, stability estimates and mean field limit for the equation

$$\partial_t F + \partial_x(A(t,F)) = \mathbf{S}[F](t,x) \tag{4.1}$$

where

$$\mathbf{S}[F](t,x) := F(t,x)(\phi \star F)(t,x) - \int_{-\infty}^{x} F(t,z)(\partial_z \phi \star F)(t,z)dz, \ \phi := \partial_x S.$$

This equation has a slightly more general form than Eq. (1.8), since here the flux is time dependent. As already explained, the strategy of proof is a modification of the argument in [12], and can be divided into the following steps

1. Extraction of a converging subsequence from F_N with a limit F.
2. Showing that the limit F obtained in 1. is an entropy solution.
3. Provided steps 1 + 2 are successfully established, it remains to prove stability estimates (from which the remaining parts – that is uniqueness and mean field limit – would follow). This is one step where our argument differs from the one in [12], since the source term in question carries a different form. Some technical modifications appear in step 2 as well.

Before detailing the proof of the plan proposed above, we recall the following chain rule for BV functions, which will be used in the course of the proof

Lemma 4.1 (*[5], Lemma A.21*)**.** *Suppose $W \in BV_{\mathrm{loc}}(\mathbb{R})$ and f is Lipschitz. Then $f \circ W \in BV_{\mathrm{loc}}(\mathbb{R})$ and*

$$\left|\frac{d}{dx} f \circ W\right| \leq |f|_{\mathrm{Lip}} \left|\frac{d}{dx} W\right|$$

in the sense of measures.

In order to treat the source term we will be forced to verify the entropy inequality only on dense subset of the real line, which is sufficient, as observed in the following simple

Lemma 4.2. *The entropy inequality holds (3.1) iff it holds for some dense set $D \subset \mathbb{R}$.*

Proof. Let $D \subset \mathbb{R}$ be dense and suppose (3.1) holds all $\beta \in D$, and let $\alpha \in \mathbb{R}$. Then, taking a sequence $(\beta_k) \subset D$ such that $\beta_k \to \alpha$, we have that for all k

$$\int_0^T \int_{\mathbb{R}} \mathbf{1}_{F>\beta_k} \left((F-\beta_k)\chi_t + \left(A(F) - A(\beta_k)\right)\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt$$
$$- \int_0^T \int_{\mathbb{R}} \mathbf{1}_{F<\beta_k} \left((F-\beta_k)\chi_t + \left(A(F) - A(\beta_k)\right)\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt.$$

As $k \to \infty$ the first integral tends to

$$\int_0^T \int_{\mathbb{R}} \mathbf{1}_{F\geq\alpha} \left((F-\alpha)\chi_t + (A(F) - A(\alpha))\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt,$$

whereas the second integral tends to

$$- \int_0^T \int_{\mathbb{R}} \mathbf{1}_{F\leq\alpha} \left((F-\alpha)\chi_t + (A(F) - A(\alpha))\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt,$$

so that

$$\int_0^T \int_{\mathbb{R}} \left(|F-\alpha|\, \chi_t + \mathrm{sgn}(F-\alpha)(A(F) - A(\alpha))\partial_x \chi + \mathrm{sgn}(F-\alpha)\chi \mathbf{S}(F)(t,x)\right) dxdt$$
$$= \int_0^T \int_{\mathbb{R}} \mathbf{1}_{F\geq\alpha} \left((F-\alpha)\chi_t + (A(F) - A(\alpha))\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt$$
$$- \int_0^T \int_{\mathbb{R}} \mathbf{1}_{F\leq\alpha} \left((F-\beta_k)\chi_t + \left(A(F) - A(\beta_k)\right)\partial_x \chi + \chi \mathbf{S}(F)(t,x)\right) dxdt \geq 0. \quad \square$$

We are now in a good position to prove the main result, which we now state. Remark that we insist on including time dependency in the flux in order to enable applying the belowstated stability estimate for the fluxes $A_N(t,x)$ and $A(x)$ as defined in Section 3.

Theorem 4.1. *Let $S \in C_0^\infty(\mathbb{R})$ be such $\mathrm{supp}(S) \subset [-r,r]$ for some $r > 0$. Let $A(t,x), \widetilde{A}(t,x) \in C([0,T]; \mathrm{Lip}([-\frac{1}{2},\frac{1}{2}]))$. Suppose $F^0 \in BV(\mathbb{R})$ is non-decreasing and there is some $R > 0$ such that*

$$\forall x \geq R : F^0(\pm x) = \pm\frac{1}{2}. \tag{4.2}$$

1. (Well-posedness) There exist a unique entropy solution (in the sense of Definition 3.1) to the problem (4.1).

2. (Stability) If $F, \widetilde{F}$ are two entropy solutions with initial datas $F^0, \widetilde{F^0}$ respectively satisfying (4.2) then there is some $C = C(r, R, \|\partial_z \phi\|_\infty, \|\phi\|_\infty) > 0$ such that

$$\left\|F(t,\cdot) - \widetilde{F}(t,\cdot)\right\|_1 \leq e^{Ct}\left(\left\|F^0 - \widetilde{F^0}\right\|_1 + t \sup_{t\in[0,T]}\left|A(t,\cdot) - \widetilde{A}(t,\cdot)\right|_{\text{Lip}}\right).$$

3. (Mean field limit) It holds that $F_N - F \underset{N\to\infty}{\to} 0$ in $C([0,T]; L^1(\mathbb{R}))$ provided $F_N^0 - F^0 \to 0$ in $L^1(\mathbb{R})$.

Proof. Step 1. Extracting a converging subsequence from F_N. Keep $t \in [0,T]$ fixed. We know that $F_N(t,\cdot)$ is a sequence of non-decreasing functions with uniformly bounded (with respect to N) total variation on each compact subset. Therefore, by Helly's selection theorem there is a subsequence, still labeled $F_N(t,\cdot)$, and some $F(t,\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ such that $F_N(t,\cdot) \underset{N\to\infty}{\to} F(t,\cdot)$ pointwise a.e. By diagonalization we may find a subsequence, still labeled $F_N(t)$, such that $F_N(t,\cdot)$ converge to $F(t,\cdot)$ for all $t \in \mathbb{Q}$. Recall that by Lemma 2.2 , as long as $x > R$, $F_N(t, \pm x) = \pm\frac{1}{2}$ for all t. As a result, the same conclusion is true for the limit $F(t,\cdot)$, so that we have $F_N(t,\cdot) - F(t,\cdot) \underset{N\to\infty}{\to} 0$ in $L^1(\mathbb{R})$ for all $t \in \mathbb{Q}$. Next, we upgrade the convergence to irrational times as well.

$$\begin{aligned}
\int_{\mathbb{R}} \left|F_N(t,x) - F_N(s,x)\right| dx &= \int_{|x|\leq R}\left|\frac{1}{N}\sum_{i=1}^N m_i(t)\mathbf{1}_{\{x|x\geq x_i(t)\}} - \frac{1}{N}\sum_{i=1}^N m_i(s)\mathbf{1}_{\{x|x\geq x_i(s)\}}\right| dx \\
&\leq \int_{|x|\leq R}\left|\frac{1}{N}\sum_{i=1}^N m_i(t)\mathbf{1}_{\{x|x_i(t)\leq x\leq x_i(s)\}}\right| dx \\
&\quad + \int_{|x|\leq R}\left|\frac{1}{N}\sum_{i=1}^N (m_i(t) - m_i(s))\mathbf{1}_{\{x|x\geq x_i(s)\}}\right| dx.
\end{aligned}$$

The first integral is

$$\leq \overline{M} \max_{1\leq i\leq N}\left|x_i(t) - x_i(s)\right| \leq \overline{M}^2 |t-s|,$$

whereas the second integral is

$$\leq 2R \max_{1\leq i\leq N}\left|m_i(t) - m_i(s)\right| \leq 2RC(1+\overline{M})|t-s|,$$

so that

$$\left\|F_N(t,\cdot) - F_N(s,\cdot)\right\|_1 \leq c|t-s|, \tag{4.3}$$

for some suitable constant $c = c(C, \overline{M}, R)$. The estimate (4.3) implies that $F_N(t,\cdot)$ is a Cauchy sequence in $L^1((-R,R))$ for any $t \in [0,T]$. Indeed, keep t fixed and let $\{t_k\}_{k=1}^\infty \subset \mathbb{Q}$ such that $t_k \underset{k\to\infty}{\to} t$. Let $\varepsilon > 0$. We estimate

$$\begin{aligned}
\left\|F_N(t,\cdot) - F_{N+p}(t,\cdot)\right\|_1 &\leq \left\|F_N(t,\cdot) - F_N(t_k,\cdot)\right\|_1 + \left\|F_N(t_k,\cdot) - F(t_k,\cdot)\right\|_1 \\
&\quad + \left\|F_{N+p}(t,\cdot) - F_{N+p}(t_k,\cdot)\right\|_1 + \left\|F_{N+p}(t_k,\cdot) - F(t_k,\cdot)\right\|_1 \\
&\leq 2c\left|t - t_k\right| + \left\|F_N(t_k,\cdot) - F(t_k,\cdot)\right\|_1 + \left\|F_{N+p}(t_k,\cdot) - F(t_k,\cdot)\right\|_1.
\end{aligned}$$

Pick k large enough so that $|t - t_k| < \varepsilon$ and pick N_0 large enough so that for any $N \geq N_0$ one has $\|F_N(t_k,\cdot) - F(t_k,\cdot)\|_1 < \varepsilon$. Then for these choices we get

$$\left\|F_N(t,\cdot) - F_{N+p}(t,\cdot)\right\| \lesssim \varepsilon$$

and therefore there is some $\widetilde{F}(t,\cdot) \in L^1((-R,R))$ such that $F_N(t,\cdot) \underset{N\to\infty}{\to} \widetilde{F}(t,\cdot)$ in $L^1((-R,R))$. In particular, for each t, the sequence $F_N(t,\cdot)$ is confined in a compact set of $L^1((-R,R))$. Therefore the estimate (4.3) makes the theorem of Arzelá-Ascoli available (Theorem 1.1 in [17] for example), thereby ensuring the existence of a subsequence, still labeled F_N, and an element $\widetilde{\widetilde{F}} \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ such that $F_N - \widetilde{\widetilde{F}} \underset{N\to\infty}{\to} 0$ in $C([0,T]; L^1(\mathbb{R}))$. It is clear that $\widetilde{\widetilde{F}} \in BV([0,T]\times\mathbb{R})$ and $\forall x \geq R : \widetilde{\widetilde{F}}(t, \pm x) = \pm\frac{1}{2}$. To minimize cumbersome notation we shall hereafter denote by F the limit function $\widetilde{\widetilde{F}}$ that we constructed.

Step 2. The Limit function F is an entropy solution. We wish to show that the limit function F obtained in step 1. is an entropy solution to the original equation, which will establish the existence claim. By Lemma 3.3 we know that

$$\int_0^T \int_{\mathbb{R}} \left|F_N - \alpha\right| \chi_t + \text{sgn}(F_N - \alpha)\left(A_N(t, F_N) - A_N(t,\alpha)\right)\partial_x \chi + \text{sgn}(F_N - \alpha)\chi \mathbf{S}[F_N](t,x)\, dx\, dt \geq 0,$$

and we wish now to pass to the limit as $N \to \infty$. That

$$\int_0^T \int_{\mathbb{R}} \left|F_N - \alpha\right| \chi_t \to \int_0^T \int_{\mathbb{R}} |F - \alpha|\, \chi_t$$

is due to dominated convergence theorem. In addition, Lemma 3.1 and the identity $\text{sgn}(a-b)(a^2 - b^2) = |a-b|(a+b)$ entails

$$\left|\int_0^T \int_{\mathbb{R}} \text{sgn}(F_N - \alpha)\left(A_N(t, F_N) - A_N(t,\alpha)\right)\partial_x \chi - \int_0^T \int_{\mathbb{R}} \text{sgn}(F - \alpha)(A(F) - A(\alpha))\partial_x \chi\right|$$

$$\begin{aligned}
&\leq \left|\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F_N-\alpha)\left(A_N(t,F_N)-A_N(t,\alpha)\right)\partial_x\chi - \int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F_N-\alpha)\left(A(F_N)-A(\alpha)\right)\partial_x\chi\right| \\
&\quad + \left|\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F_N-\alpha)\left(A(F_N)-A(\alpha)\right)\partial_x\chi - \int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F-\alpha)\left(A(F)-A(\alpha)\right)\partial_x\chi\right| \\
&\leq \frac{4\overline{M}\operatorname{Lip}(A|_{[-\frac{1}{2},\frac{1}{2}]})}{N}\int_0^T\int_{\mathbb{R}} |\partial_x\chi|\,dxdt \\
&\quad + \left|\int_0^T\int_{\mathbb{R}} |F_N-\alpha|(F_N+\alpha)\partial_x\chi - \int_0^T\int_{\mathbb{R}} |F-\alpha|(F+\alpha)\partial_x\chi\right| \underset{N\to\infty}{\to} 0.
\end{aligned}$$

The last term is slightly more subtle. Observe that

$$\int_{-2R}^{2R}\phi(z-\zeta)F_N(t,\zeta)d\zeta \underset{N\to\infty}{\to} \int_{-2R}^{2R}\phi(z-\zeta)F(t,\zeta)d\zeta,$$

so that

$$F_N(t,x)\phi\star F_N(t,x) \underset{N\to\infty}{\to} F(t,x)\phi\star F(t,x)$$

pointwise a.e. Recall that $\phi'\star F_N$ is compactly supported and therefore dominated convergence is applicable for the second integral as well which implies

$$\int_{-\infty}^{x} F_N(t,z)\phi'\star F_N(t,z)dz \underset{N\to\infty}{\to} \int_{-\infty}^{x} F(t,z)\phi'\star F(t,z)dz.$$

By Lemma 4.2 it suffices to verify the entropy condition on dense set. Note that since F is locally summable it holds that $\lambda(\{(t,x)\,|\,F(t,x)=\alpha\})=0$ for a.e. α. In particular there is dense set $D\subset\mathbb{R}$ such that $\lambda(\{F(t,x)=\alpha\})=0$ for all $\alpha\in D$. Therefore, since $F_N\to F$ and $\mathbf{S}[F_N](t,x)\to\mathbf{S}[F](t,x)$ pointwise a.e., we conclude that for all $\alpha\in D$

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F_N-\alpha)\chi\mathbf{S}[F_N](t,x)dxdt &= \int_0^T\int_{\mathbb{R}} \mathbf{1}_{\{F(t,x)\neq\alpha\}}\operatorname{sgn}(F_N-\alpha)\chi\mathbf{S}[F_N](t,x)dxdt \\
&\underset{N\to\infty}{\to} \int_0^T\int_{\mathbb{R}} \operatorname{sgn}(F-\alpha)\chi\mathbf{S}[F](t,x)dxdt
\end{aligned}$$

as desired.

Step 3. Uniqueness and Stability. Let $F,\widetilde{F}$ be two entropy solutions. Keep $(s,y)\in[0,T]\times\mathbb{R}$ fixed. Then by definition we have

$$\begin{aligned}
0 &\leq \iint \left|F(t,x)-\widetilde{F}(s,y)\right|\partial_t\chi(t,x,s,y)dxdt \\
&\quad + \iint \operatorname{sgn}(F(t,x)-\widetilde{F}(s,y))\left(A(t,F(t,x))-A(t,\widetilde{F}(s,y))\right)\partial_x\chi(t,x,s,y)dxdt \\
&\quad + \iint \operatorname{sgn}(F(t,x)-\widetilde{F}(s,y))\chi(t,x,s,y)\mathbf{S}[F](t,x)dxdt.
\end{aligned}$$

Exchanging the roles of F and $\widetilde{F}$ we also have the inequality

$$\begin{aligned}
0 &\leq \iint \left|\widetilde{F}(s,y)-F(t,x)\right|\partial_s\chi(t,x,s,y)dyds \\
&\quad + \iint \operatorname{sgn}(\widetilde{F}(s,y)-F(t,x))\left(\widetilde{A}(s,\widetilde{F}(s,y))-\widetilde{A}(s,F(t,x))\right)\partial_y\chi(t,x,s,y)dyds \\
&\quad + \iint \operatorname{sgn}(\widetilde{F}(s,y)-F(t,x))\chi(t,x,s,y)\mathbf{S}[\widetilde{F}](s,y)dyds.
\end{aligned}$$

Integrating both of the above inequalities over the free variables and summing up gives

$$\begin{aligned}
0 &\leq \iiiint \left|F(t,x)-\widetilde{F}(s,y)\right|(\partial_t\chi+\partial_s\chi) \\
&\quad + \iiiint \operatorname{sgn}(F(t,x)-\widetilde{F}(s,y))\left(A(t,F(t,x))-A(t,\widetilde{F}(s,y))\right)\partial_x\chi \\
&\quad + \iiiint \operatorname{sgn}(\widetilde{F}(s,y)-F(t,x))\left(\widetilde{A}(s,\widetilde{F}(s,y))-\widetilde{A}(s,F(t,x))\right)\partial_y\chi \\
&\quad + \iiiint \left(\chi\operatorname{sgn}(F(t,x)-\widetilde{F}(s,y))\mathbf{S}[F](t,x)+\chi\operatorname{sgn}(\widetilde{F}(s,y)-F(t,x))\mathbf{S}[\widetilde{F}](s,y)\right)dxdtdyds.
\end{aligned} \tag{4.4}$$

Now consider the variable change

$$\bar{y}=\frac{x-y}{2},\quad \bar{s}=\frac{t-s}{2},\quad \bar{x}=\frac{x+y}{2},\quad \bar{t}=\frac{t+s}{2},$$

and take χ to be of the form

$$\chi(t,x,s,y)=b_\varepsilon\left(\frac{x-y}{2}\right)b_\varepsilon\left(\frac{t-s}{2}\right)g\left(\frac{x+y}{2}\right)h_\delta\left(\frac{t+s}{2}\right)$$

$$= b_\varepsilon\left(\bar{y}\right) b_\varepsilon\left(\bar{s}\right) g\left(\bar{x}\right) h_\delta\left(\bar{t}\right).$$

where $b_\varepsilon, g, h_\delta$ are chosen as follows. Keep $\sigma < \tau \in (0,T)$ fixed. For each $\varepsilon > 0, \delta > 0$ such that $0 < \varepsilon + \delta < \min(\sigma, T-\tau)$ we define the functions as follows.

1. b_ε is an approximation of the identity, i.e. $b_\varepsilon = \frac{1}{\varepsilon}\zeta(\frac{x}{\varepsilon})$ for some radial $0 \le \zeta \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\zeta) \Subset B_1(0)$.
2. $g(x) = 1$ for all $x \in [-R, R]$ and $g \in C_0^\infty(\mathbb{R})$.
3. Let

$$h_\delta(t) := \begin{cases} 1, & t \in [\sigma,\tau] \\ -\frac{1}{\delta}t + \frac{\tau+\delta}{\delta}, & t \in [\tau, \tau+\delta] \\ \frac{1}{\delta}t - \frac{\sigma-\delta}{\delta}, & t \in [\sigma-\delta, \sigma] \\ 0, & t \notin [\sigma-\delta, \tau+\delta] \end{cases}.$$

With this choice we easily observe that $\chi(t,x,s,y)$ is compactly supported in the variables (t,x) for any fixed (s,y), and viceversa.

Claim 4.1. *For each fixed $(s,y) \in (0,T)\times\mathbb{R}$ the function $(t,x) \mapsto \chi(t,x,s,y)$ is compactly supported in $(0,T)\times\mathbb{R}$. Vice versa, for each fixed $(t,x) \in (0,T)\times\mathbb{R}$ the function $(s,y) \mapsto \chi(t,x,s,y)$ is compactly supported in $(0,T)\times\mathbb{R}$.*

Proof. That $x \mapsto \chi(t,x,s,y)$ vanishes outside some finite interval is clear. In addition if $s > 2\varepsilon$ then $b_\varepsilon(\frac{t-s}{2}) = 0$ for all $t \in (0, s-2\varepsilon)$, while if $s \le 2\varepsilon$ then $h_\delta(\frac{t+s}{2}) = 0$ then since $\sigma > \varepsilon+\delta$ it follows that $2\sigma - 2\delta - s > 0$, so that $h_\delta(\frac{t+s}{2}) = 0$ for all $t \in (0, 2\sigma-2\delta-s)$. Hence $\chi(t,x,s,y) = 0$ for all $t \in (0, \mathbf{1}_{s<2\varepsilon}(s-2\varepsilon) + \mathbf{1}_{s\ge 2\varepsilon}2\sigma - 2\delta - s)$ (note that the right end of the interval is strictly positive). If $s < T - 2\varepsilon$ then $b_\varepsilon(\frac{t-s}{2}) = 0$ for all $t \in (s+2\varepsilon, T)$ while if $s \ge T-2\varepsilon$ then $2\tau + 2\delta - s \le 2\tau + 2\delta + 2\varepsilon - T < 2T - T = T$ because $\tau < T - (\delta+\varepsilon)$. Therefore $\chi(t,x,s,y) = 0$ for all $t \in (\mathbf{1}_{s<T-2\varepsilon}(s+2\varepsilon) + \mathbf{1}_{s\ge T-2\varepsilon}(2\tau+2\delta-s), T)$. To conclude $t \mapsto \chi(t,x,s,y)$ vanishes outside some interval compactly supported in $(0,T)$. Since b_ε is radial, by symmetry the same argument shows that for any fixed (t,x) the function $(s,y) \mapsto \chi(t,x,s,y)$ is compactly supported. □

In addition, it is readily checked that

$$\partial_t + \partial_s = \partial_{\bar{t}}, \quad \partial_x + \partial_y = \partial_{\bar{x}}, \quad \partial_x - \partial_y = \partial_{\bar{y}}.$$

Under this variable change, the second and third term in (4.4) can be grouped together as follows (to make the equations lighter the arguments are of F and $\widetilde{F}$ are always $(\bar{t}+\bar{s}, \bar{x}+\bar{y})$ and $(\bar{t}-\bar{s}, \bar{x}-\bar{y})$ respectively, and are implicit)

$$\begin{aligned}
&\iiiint \mathrm{sgn}(F(t,x) - \widetilde{F}(s,y))\left(A(t,F(t,x)) - A(t,\widetilde{F}(s,y))\right)\partial_x\chi \\
&\quad + \iiiint \mathrm{sgn}(\widetilde{F}(s,y) - F(t,x))\left(\widetilde{A}(s,\widetilde{F}(s,y)) - \widetilde{A}(s,F(t,x))\right)\partial_y\chi \\
&= \frac{1}{2}\iiiint \mathrm{sgn}(F-\widetilde{F})\left(A(\bar{t}+\bar{s},F) - A(\bar{t}+\bar{s},\widetilde{F})\right)(\partial_{\bar{x}} + \partial_{\bar{y}})\chi \\
&\quad + \frac{1}{2}\iiiint \mathrm{sgn}(\widetilde{F}-F)\left(\widetilde{A}(\bar{t}-\bar{s},\widetilde{F}) - \widetilde{A}(\bar{t}-\bar{s},F)\right)(\partial_{\bar{x}} - \partial_{\bar{y}})\chi \\
&= \frac{1}{2}\iiiint \mathrm{sgn}(F-\widetilde{F})\left(A(\bar{t}+\bar{s},F) + \widetilde{A}(\bar{t}-\bar{s},F) - A(\bar{t}+\bar{s},\widetilde{F}) - \widetilde{A}(\bar{t}-\bar{s},\widetilde{F})\right)\partial_{\bar{x}}\chi \\
&\quad + \frac{1}{2}\iiiint \mathrm{sgn}(F-\widetilde{F})\left(A(\bar{t}+\bar{s},F) - \widetilde{A}(\bar{t}-\bar{s},F) - A(\bar{t}+\bar{s},\widetilde{F}(s,y)) + \widetilde{A}(\bar{t}-\bar{s},\widetilde{F}(s,y))\right)\partial_{\bar{y}}\chi \\
&= \iiiint \mathrm{sgn}(F-\widetilde{F})\left(A_+(\bar{t}+\bar{s},\bar{t}-\bar{s},F) - A_+(\bar{t}+\bar{s},\bar{t}-\bar{s},\widetilde{F})\right)\partial_{\bar{x}}\chi \\
&\quad + \iiiint \mathrm{sgn}(F-\widetilde{F})\left(A_-(\bar{t}+\bar{s},\bar{t}-\bar{s},F) - A_-(\bar{t}+\bar{s},\bar{t}-\bar{s},\widetilde{F})\right)\partial_{\bar{y}}\chi,
\end{aligned}$$

where we have set

$$A_\pm(t,s,x) := \frac{A(t,x) \pm \widetilde{A}(s,x)}{2}.$$

The Inequality (4.4) is then transformed to

$$\begin{aligned}
0 &\le I_{\varepsilon,\delta} + II_{\varepsilon,\delta} + III_{\varepsilon,\delta} + IV_{\varepsilon,\delta} \\
&:= \iiiint \left|F(\bar{t}+\bar{s},\bar{x}+\bar{y}) - \widetilde{F}(\bar{t}-\bar{s},\bar{x}-\bar{y})\right|\partial_{\bar{t}}\chi(\bar{t},\bar{x},\bar{s},\bar{y})d\bar{t}d\bar{x}d\bar{s}d\bar{y} \\
&\quad + \iiiint \mathrm{sgn}(F(\bar{t}+\bar{s},\bar{x}+\bar{y}) - \widetilde{F}(\bar{t}-\bar{s},\bar{x}-\bar{y}))\left(A_+(\bar{t}+\bar{s},\bar{t}-\bar{s},F) - A_+(\bar{t}+\bar{s},\bar{t}-\bar{s},\widetilde{F})\right)\partial_{\bar{x}}\chi(\bar{t},\bar{x},\bar{s},\bar{y})d\bar{t}d\bar{x}d\bar{s}d\bar{y} \\
&\quad + \iiiint \mathrm{sgn}(F(\bar{t}+\bar{s},\bar{x}+\bar{y}) - \widetilde{F}(\bar{t}-\bar{s},\bar{x}-\bar{y}))\left(A_-(\bar{t}+\bar{s},\bar{t}-\bar{s},F) - A_-(\bar{t}+\bar{s},\bar{t}-\bar{s},\widetilde{F})\right)\partial_{\bar{y}}\chi(\bar{t},\bar{x},\bar{s},\bar{y})d\bar{t}d\bar{x}d\bar{s}d\bar{y} \\
&\quad + \iiiint \chi\,\mathrm{sgn}(F(\bar{t}+\bar{s},\bar{x}+\bar{y}) - \widetilde{F}(\bar{t}-\bar{s},\bar{x}-\bar{y}))\left(\mathbf{S}[F](\bar{t}+\bar{s},\bar{x}+\bar{y}) - \mathbf{S}[\widetilde{F}](\bar{t}-\bar{s},\bar{x}-\bar{y})\right)d\bar{t}d\bar{x}d\bar{s}d\bar{y}.
\end{aligned} \tag{4.5}$$

We handle separately each one of the above integrals.

The integral $I_{\varepsilon,\delta}$. Letting $\varepsilon \to 0$ we get the 2D integral

$$I_\delta := \lim_{\varepsilon\to 0} I_{\varepsilon,\delta} = \int\int \left|F(\bar t,\bar x) - \widetilde F(\bar t,\bar x)\right| g\left(\bar x\right) h'_\delta\left(\bar t\right) d\bar x d\bar t.$$

We can pass to the limit as $\delta \to 0$ to find that

$$\begin{aligned} I_\delta &= \int\int \left|F(\bar t,\bar x) - \widetilde F(\bar t,\bar x)\right| g\left(\bar x\right) h'_\delta\left(\bar t\right) \\ &= \frac{1}{\delta}\int_{\sigma-\delta}^{\sigma}\int \left|F(\bar t,\bar x) - \widetilde F(\bar t,\bar x)\right| g\left(\bar x\right) d\bar x d\bar t - \frac{1}{\delta}\int_{\tau}^{\tau+\delta}\int \left|F(\bar t,\bar x) - \widetilde F(\bar t,\bar x)\right| g\left(\bar x\right) d\bar x d\bar t \\ &\underset{\delta\to 0}{\to} \int \left|F(\sigma,\bar x) - \widetilde F(\sigma,\bar x)\right| g\left(\bar x\right) d\bar x - \int \left|F(\tau,\bar x) - \widetilde F(\tau,\bar x)\right| g\left(\bar x\right) d\bar x \\ &= \int \left|F(\sigma,\bar x) - \widetilde F(\sigma,\bar x)\right| d\bar x - \int \left|F(\tau,\bar x) - \widetilde F(\tau,\bar x)\right| d\bar x, \end{aligned} \tag{4.6}$$

where in the last equation we relied on the choice of g as well as fact that $F - \widetilde F$ is 0 outside $[-R, R]$.

The integral $II_{\varepsilon,\delta}$. Letting $\varepsilon \to 0$ the integral becomes

$$II_\delta = \lim_{\varepsilon\to 0} II_{\varepsilon,\delta} = \iint \operatorname{sgn}(F(\bar t,\bar x) - \widetilde F(\bar t,\bar x))\left(A_+(\bar t,\bar t, F(\bar t,\bar x)) - A_+(\bar t,\bar t,\widetilde F(\bar t,\bar x))\right) g'\left(\bar x\right) h_\delta\left(\bar t\right) d\bar t d\bar x.$$

Observe that $g'\left(\bar x\right) = 0$ for all $\bar x \in \operatorname{supp}(A_+(\bar t,\bar t, F(\bar t,\cdot)) - A_+(\bar t,\bar t,\widetilde F(\bar t,\cdot))) \subset [-R, R]$, where the latter inclusion is because of the uniform in time Lipschitz continuity assumed for $A, \widetilde A$. We therefore infer that the above integral vanishes identically, i.e.

$$II_\delta = 0. \tag{4.7}$$

The integral $III_{\varepsilon,\delta}$. The following claim is a straightforward adaptation of Lemma 5.4 in [12]

Claim 4.2. *The function* $\gamma(\bar t, F, \widetilde F) := \operatorname{sgn}(F - \widetilde F)(A_-(\bar t,\bar t, F) - A_-(\bar t,\bar t,\widetilde F))$ *is uniformly in time Lipschitz with respect to the first and second variable with the bounds*

$$\left|\gamma(\bar t,\cdot,\widetilde F)\right|_{\text{Lip}} \le \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}}, \ \left|\gamma(\bar t, F,\cdot)\right|_{\text{Lip}} \le \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}}.$$

Proof. Keep $\bar t \in [0, T]$, $\widetilde F \in [-\frac12, \frac12]$ fixed. Let $F_1, F_2 \in [-\frac12, \frac12]$. If $\operatorname{sgn}(F_1 - \widetilde F) = \operatorname{sgn}(F_2 - \widetilde F)$ then we have

$$\left|\gamma(\bar t, F_1,\widetilde F) - \gamma(\bar t, F_2,\widetilde F)\right| = \left|A_-(\bar t,\bar t, F_1) - A_-(\bar t,\bar t, F_2)\right| \le \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \left|F_1 - F_2\right|.$$

If $\operatorname{sgn}(F_1 - \widetilde F) \neq \operatorname{sgn}(F_2 - \widetilde F)$ then

$$\begin{aligned} \left|\gamma(\bar t, F_1,\widetilde F) - \gamma(\bar t, F_2,\widetilde F)\right| &= \left|A_-(\bar t,\bar t, F_1) - A_-(\bar t,\bar t,\widetilde F) + A_-(\bar t,\bar t, F_2) - A_-(\bar t,\bar t,\widetilde F)\right| \\ &\le \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} (\left|F_1 - \widetilde F\right| + \left|F_2 - \widetilde F\right|) \\ &= \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \left|F_1 - \widetilde F + \widetilde F - F_2\right| = \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \left|F_1 - F_2\right|. \quad \square \end{aligned}$$

We can apply this observation together with Lemma 4.1 in order to integrate by parts and obtain

$$III_{\varepsilon,\delta} = -\iiiint \partial_{\bar y}\gamma(t, F,\widetilde F)\chi.$$

Moreover we get the bound

$$\begin{aligned} &\left|\iiiint \partial_{\bar y}\gamma(t, F,\widetilde F)\chi\right| \\ &\le \sup_{\bar t\in[0,T]} \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \iiiint \left(\left|\partial_1 F(\bar t + \bar s,\bar x + \bar y)\right| + \left|\partial_1 \widetilde F(\bar t - \bar s,\bar x - \bar y)\right|\right) b_\varepsilon\left(\bar y\right) b_\varepsilon\left(\bar s\right) g\left(\bar x\right) h_\delta\left(\bar t\right). \end{aligned}$$

Letting $\varepsilon \to 0$ and setting $III_\delta := \lim_{\varepsilon\to 0} III_{\varepsilon,\delta}$ we arrive at

$$\left|III_\delta\right| \le \sup_{\bar t\in[0,T]} \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \iint \left(\left|\partial_1 F(\bar t,\bar x)\right| + \left|\partial_1 \widetilde F(\bar t,\bar x)\right|\right) g\left(\bar x\right) h_\delta\left(\bar t\right) d\bar x d\bar t.$$

Letting $\delta \to 0$ this becomes

$$\lim_{\delta\to 0}\left|III_\delta\right| \le \sup_{\bar t\in[0,T]} \left|A_-(\bar t,\bar t,\cdot)\right|_{\text{Lip}} \int_\sigma^\tau\int \left(\left|\partial_1 F(\bar t,\bar x)\right| + \left|\partial_1 \widetilde F(\bar t,\bar x)\right|\right) d\bar x d\bar t.$$

Since for any fixed $\bar t$

$$\int \left(\left|\partial_1 F(\bar t,\bar x)\right| + \left|\partial_1 \widetilde F(\bar t,\bar x)\right|\right) = 2,$$

we conclude

$$\lim_{\delta\to 0}\left|III_\delta\right| \le 2\sup_{\bar{t}\in[0,T]}\left|A\_(\bar{t},\bar{t},\cdot)\right|_{\text{Lip}}(\tau-\sigma) = \sup_{\bar{t}\in[0,T]}\left|A(\bar{t},\cdot)-\widetilde{A}(\bar{t},\cdot)\right|_{\text{Lip}}(\tau-\sigma). \tag{4.8}$$

The integral $IV_{\varepsilon,\delta}$. The treatment of this integral reflects the novelty for what concerns the stability estimate, since the non-local source term in question stands in variance to the one in [12]. Letting $\varepsilon\to 0$ we get the 2D integral

$$IV_\delta := \iint \operatorname{sgn}(F(\bar{t},\bar{x})-\widetilde{F}(\bar{t},\bar{x}))\left(\mathbf{S}[F](\bar{t},\bar{x})-\mathbf{S}[\widetilde{F}](\bar{t},\bar{x})\right) g\left(\bar{x}\right) h_\delta\left(\bar{t}\right) d\bar{t}d\bar{x}.$$

The integral IV_δ is now to be bounded by 4 terms, each of which is mastered separately.

$$\begin{aligned}
\left|IV_\delta\right| &\le \iint \left|\mathbf{S}[F](t,x)-\mathbf{S}[\widetilde{F}](t,x)\right| g(x)\,h_\delta(t)\,dxdt\\
&\le \iint \left|F(t,x)\phi\star F(t,x)-\widetilde{F}(t,x)\phi\star\widetilde{F}(t,x)\right| g(x)\,h_\delta(t)\,dxdt\\
&\quad+\iint\left|\int_{-\infty}^{x}\widetilde{F}(t,z)\partial_z\phi\star\widetilde{F}(t,z)-F(t,z)\partial_z\phi\star F(t,z)dz\right| g(x)\,h_\delta(t)\,dxdt\\
&\le \iint\left|\left(F(t,x)-\widetilde{F}(t,x)\right)\phi\star F(t,x)\right| g(x)\,h_\delta(t)\,dxdt\\
&\quad+\iint\left|\widetilde{F}(t,x)\phi\star(F-\widetilde{F})(t,x)\right| g(x)\,h_\delta(t)\,dxdt\\
&\quad+\iint\left|\int_{-\infty}^{x}\widetilde{F}(t,z)(\partial_z\phi\star(\widetilde{F}-F)(t,z))dz\right| g(x)\,h_\delta(t)\,dxdt\\
&\quad+\iint\left|\int_{-\infty}^{x}\partial_z\phi\star F(t,z)(\widetilde{F}-F)(t,z)dz\right| g(x)\,h_\delta(t)\,dxdt := \sum_{k=1}^{4}J_k.
\end{aligned}$$

Estimate on J_1.

$$\left|\left(F(t,x)-\widetilde{F}(t,x)\right)\phi\star F(t,x)\right| \le 2R\left\|\phi\right\|_\infty\left|F(t,x)-\widetilde{F}(t,x)\right|,$$

so that

$$\begin{aligned}
\left|J_1\right| &\le 2R\left\|\phi\right\|_\infty\iint\left|F(t,x)-\widetilde{F}(t,x)\right| g(x)\,h_\delta(t)\,dxdt\\
&\le 2R\left\|\phi\right\|_\infty\int\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 h_\delta(t)\,dt \underset{\delta\to 0}{\to} 2R\left\|\phi\right\|_\infty\int_\sigma^\tau\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 dt.
\end{aligned}$$

Estimate on J_2.

$$\left|\phi\star(F-\widetilde{F})(t,x)\right| \le \left\|\phi\right\|_\infty\left\|(F-\widetilde{F})(t,\cdot)\right\|_1,$$

so that

$$\begin{aligned}
\left|J_2\right| &\le \frac{1}{2}\left\|\phi\right\|_\infty\iint\left\|(F-\widetilde{F})(t,\cdot)\right\|_1 g(x)\,h_\delta(t)\,dxdt\\
&\le 2R\left\|\phi\right\|_\infty\int\left\|(F-\widetilde{F})(t,\cdot)\right\|_1 h_\delta(t)\,dt \underset{\delta\to 0}{\to} 2R\left\|\phi\right\|_\infty\int_\sigma^\tau\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 dt.
\end{aligned}$$

Estimate on J_3. Since $\operatorname{supp}(\partial_z\phi\star(\widetilde{F}-F))\subset\operatorname{supp}(\phi)+\operatorname{supp}(\widetilde{F}-F)\subset[-(r+R),r+R]$ it follows that

$$\begin{aligned}
\left|\int_{-\infty}^{x}\widetilde{F}(t,z)(\partial_z\phi\star(\widetilde{F}-F)(t,z))dz\right| &\le \left|\int_{-(r+R)}^{r+R}\widetilde{F}(t,z)(\partial_z\phi\star(\widetilde{F}-F)(t,z))dz\right|\\
&\le (r+R)\left\|\partial_z\phi\right\|_\infty\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left|J_3\right| &\le (r+R)\left\|\partial_z\phi\right\|_\infty\iint\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 g(x)\,h_\delta(t)\,dxdt\\
&\le (r+R)\left\|\partial_z\phi\right\|_\infty\iint\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 g(x)\,h_\delta(t)\,dxdt\\
&\le 4R(r+R)\left\|\partial_z\phi\right\|_\infty\int\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 h_\delta(t)\,dt\\
&\underset{\delta\to 0}{\to} 4R(r+R)\left\|\partial_z\phi\right\|_\infty\int_\sigma^\tau\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 dt.
\end{aligned}$$

Estimate on J_4.

$$\left|\int_{-\infty}^{x}\partial_z\phi\star F(t,z)(\widetilde{F}-F)(t,z)dz\right| \le 2R\left\|\partial_z\phi\right\|_\infty\left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1.$$

As a result

$$\begin{aligned}\left|J_4\right| &\le 2R\left\|\partial_z\phi\right\|_\infty \iint \left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 g(x)\,h_\delta(t)\,dx\,dt\\ &\le 8R^2\left\|\partial_z\phi\right\|_\infty \int \left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 h_\delta(t)\,dt\\ &\underset{\delta\to 0}{\to} 8R^2\left\|\partial_z\phi\right\|_\infty \int_\sigma^\tau \left\|\widetilde{F}(t,\cdot)-F(t,\cdot)\right\|_1 dt.\end{aligned}$$

Summarizing

$$\left|\lim_{\delta\to 0}\lim_{\varepsilon\to 0} IV_{\varepsilon,\delta}\right| \le C(r,R,\|\phi\|_\infty,\|\partial_z\phi\|_\infty)\int_\sigma^\tau \left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 dt. \tag{4.9}$$

Step 4. Conclusion. The combination of (4.6), (4.7), (4.8), (4.9) and (4.5) yields the inequality

$$\sup_{t\in[0,T]}\left|A(t,\cdot)-\widetilde{A}(t,\cdot)\right|_{\mathrm{Lip}}(\tau-\sigma)+C\int_\sigma^\tau\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 dt + \int\left|F(\sigma,x)-\widetilde{F}(\sigma,x)\right|dx-\int\left|F(\tau,x)-\widetilde{F}(\tau,x)\right|dx\ge 0,$$

for some constant $C=C(r,R,\left\|\partial_z\phi\right\|_\infty,\|\phi\|_\infty)$. In particular we get

$$\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 \le t\sup_{t\in[0,T]}\left|A(t,\cdot)-\widetilde{A}(t,\cdot)\right|_{\mathrm{Lip}}+\left\|F(0,\cdot)-\widetilde{F}(0,\cdot)\right\|_1+C\int_0^t\left\|F(s,\cdot)-\widetilde{F}(s,\cdot)\right\|_1 ds,$$

which by Gronwall implies

$$\left\|F(t,\cdot)-\widetilde{F}(t,\cdot)\right\|_1 \le e^{Ct}\left(\left\|F(0,\cdot)-\widetilde{F}(0,\cdot)\right\|_1+t\sup_{t\in[0,T]}\left|A(t,\cdot)-\widetilde{A}(t,\cdot)\right|_{\mathrm{Lip}}\right),$$

as desired. □

Remark 4.1. Given initial data F^0, it is not difficult to construct explicitly initial weights m_i^0 and initial opinions x_i^0 such that $\left\|-\frac{1}{2}+\frac{1}{N}\sum_{i=1}^N m_i^0 H(x-x_i^0)-F^0\right\|_1 \underset{N\to\infty}{\to} 0$, thereby witnessing the fact that the assumption $\left\|F_N^0-F^0\right\|_1 \underset{N\to\infty}{\to} 0$ is reasonably typical. See Lemma 5.3 in [12] for a guidance how to do this.

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Appendix. The Rankine–Hugoniot and Oleinik conditions revisited

The Rankine–Hugoniot and Oleinik conditions are a very standard tool in the theory of conservation laws. However it seems that the literature typically formulates these conditions for conservation laws with time independent fluxes, no source terms and when there is only a single curve of discontinuity. All of these additions are completely harmless, but for the sake of completeness we revisit the derivation of the entropy inequality subject in this slightly more general settings. The equation is as usual

$$\partial_t F+\partial_x(A(t,F))=\mathbf{S}[F](t,x). \tag{A.1}$$

Proposition A.1 (*Oleinik Condition*)**.** *Suppose there are times $0=T_0<T_1<\cdots<T_{k-1}<T_k=T$ such that for each $1\le j\le k$ there are curves $\left\{(t,s_i^j(t))\right\}_{i=1}^{m_j}$ ($m_j\in\mathbb{N}$) such that F is a classical solution to Eq. (A.1) on*

$$V_L^{i,j}:=\left\{(t,x)\,\middle|\,t\in[T_{j-1},T_j),\,s_{i-1}^j(t)\le x<s_i^j(t)\right\}$$

and

$$V_R^{i,j}:=\left\{(t,x)\,\middle|\,t\in[T_{j-1},T_j),\,s_i^j(t)<x\le s_{i+1}^j(t)\right\},$$

with the convention $s_0^j(t)=-\infty$ and $s_{m_j+1}^j(t)=+\infty$. For each $t\in[T_{j-1},T_j)$ let

$$F_L^{i,j}(t):=\lim_{x\nearrow s_i^j(t)}F(t,x),\quad F_R^{i,j}(t):=\lim_{x\searrow s_i^j(t)}F(t,x).$$

Suppose that for each $t \in [T_{j-1}, T_j)$ and each $\theta \in (F_L^{i,j}(t), F_R^{i,j}(t))$ it holds that

$$\frac{A(t, F_L^{i,j}(t)) - A(t, F_R^{i,j}(t))}{F_L^{i,j}(t) - F_R^{i,j}(t)} = \dot{s}_i^j(t) \tag{A.2}$$

and

$$\frac{A(t, \theta) - A(t, F_L^{i,j}(t))}{\theta - F_L^{i,j}(t)} \geq \dot{s}_i^j(t). \tag{A.3}$$

Then for each $\alpha \in \mathbb{R}$ F satisfies the entropy inequality

$$\iint_V \partial_t \chi (F - \alpha)\mathrm{sgn}(F - \alpha) + \partial_x \chi \left(A(F) - A(\alpha)\right) \mathrm{sgn}(F - \alpha) + \chi \mathrm{sgn}(F - \alpha)\mathbf{S}[F](t, x) dx dt \geq 0. \tag{A.4}$$

Proof. Let us first consider $\eta : \mathbb{R} \to \mathbb{R}, \psi : \mathbb{R} \to \mathbb{R}$ where $\eta \in C^{1,1}(\mathbb{R})$ is convex and $\psi' = \eta' A'$. Then we have that

$$\begin{aligned}\iint \partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \mathbf{S}[F](t, x) dx dt = &\sum_{j=1}^{k} \sum_{i=1}^{m_j} \iint_{V_L^{i,j}} \left(\partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt \\ &+ \sum_{j=1}^{k} \sum_{i=1}^{m_j} \iint_{V_R^{i,j}} \left(\partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt.\end{aligned}$$

Set $\Gamma_{i,j} := \left\{(t, s_i^j(t)) \middle| t \in [T_{j-1}, T_j) \right\}$. Keep $1 \leq j \leq k$ and $1 \leq i \leq m_j$ fixed and take a test function $\chi \in C_0^\infty((0, T) \times \mathbb{R})$. For readability, we omit the indices i, j. Using that F is a classical solution on V_L, V_R we get

$$\begin{aligned}&\iint_{V_L} \left(\partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt \\ &\quad + \iint_{V_R} \left(\partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt \\ &= \iint_{V_L} \left(-\chi \partial_t \eta(F) - \chi \partial_x \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt \\ &\quad + \iint_{V_R} \left(-\chi \partial_t \eta(F) - \chi \partial_x \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)\right) dx dt \\ &\quad + \int_\Gamma \left(\chi \eta(F_L)\nu^1 + \chi \psi(F_L)\nu^2\right) d\sigma - \int_\Gamma (\chi \eta(F_R)\nu^1 + \chi \psi(F_R)\nu^2) d\sigma \\ &= \int_\Gamma \left(\chi(\psi(F_L) - \psi(F_R))\nu^1 + \chi(\eta(F_L) - \eta(F_R))\nu^2\right) d\sigma.\end{aligned}$$

Here

$$(\nu^1, \nu^2) = \frac{1}{\sqrt{1 + \dot{s}^2}}(-\dot{s}, 1),$$

and the last equality is because of the identity

$$\partial_t \eta(F) + \partial_x \psi(F) + \eta'(F)\mathbf{S}[F] = 0,$$

which is easily derived using that F is a classical solution on each region. We claim that

$$(\eta(F_L) - \eta(F_R)) - (\psi(F_L) - \psi(F_R))\dot{s}(t) \geq 0.$$

First, we integrate by parts to find that

$$\begin{aligned}\psi(F_R) - \psi(F_L) &= \int_{F_L}^{F_R} \eta'(y)A'(t, y) dy \\ &= -\int_{F_L}^{F_R} \eta''(y)(A(t, y) - A(t, F_L(t))) dy + \eta'(\cdot)(A(t, \cdot) - A(t, F_L))\Big|_{F_L}^{F_R} \\ &= -\int_{F_L}^{F_R} \eta''(y)(A(t, y) - A(t, F_L)) dy + \eta'(F_R(t))(A(t, F_R(t)) - A(t, F_L(t))),\end{aligned}$$

and so thanks to the assumption (A.3) and the convexity of η we have the inequality

$$\begin{aligned}\psi(F_L) - \psi(F_R) &= \int_{F_L}^{F_R} \eta''(y)(A(t, y) - A(t, F_L)) dy - \eta'(F_R)(A(t, F_R) - A(t, F_L)) \\ &\geq \dot{s}(t) \int_{F_L}^{F_R} \eta''(y)(y - F_L) dy - \eta'(F_R)(A(t, F_R)) - A(t, F_L).\end{aligned} \tag{A.5}$$

On the other hand

$$\eta(F_R) - \eta(F_L) = \int_{F_L}^{F_R} \eta'(y)dy = -\int_{F_L}^{F_R} \eta''(y)(y - F_L)dy + (y - F_L)\eta'|_{F_L}^{F_R}$$
$$= -\int_{F_L}^{F_R} \eta''(y)(y - F_L(t))dy + (F_R - F_L)\eta'(F_R).$$

Together with inequality (A.5) the last equation entails

$$\psi(F_L) - \psi(F_R) - \dot{s}(t)(\eta(F_L) - \eta(F_R)) \geq \eta'(F_R)\left(\dot{s}(t)(F_R - F_L) - A(t, F_R) - A(t, F_L)\right) = 0,$$

where the last equality is because of the assumption (A.2). It follows that

$$\int \partial_t \chi \eta(F) + \partial_x \chi \psi(F) + \chi \eta'(F)\mathbf{S}[F](t, x)dxdt \geq 0. \tag{A.6}$$

To finish, we use a standard approximation argument. For each $\varepsilon > 0$ consider the convex function $\mathfrak{s}_\varepsilon \in C^{1,1}(\mathbb{R})$ defined by

$$\mathfrak{s}_\varepsilon(z) := \begin{cases} \frac{1}{2\varepsilon}z^2, & |z| \leq \varepsilon, \\ |z| - \frac{\varepsilon}{2}, & |z| > \varepsilon. \end{cases}$$

and for each $\alpha \in \mathbb{R}$ let

$$\eta_\varepsilon(z) := \mathfrak{s}_\varepsilon(z - \alpha).$$

It is clear that we have pointwise convergence

$$\eta_\varepsilon \underset{\varepsilon\to 0}{\to} |z - \alpha|, \ \eta'_\varepsilon \underset{\varepsilon\to 0}{\to} \operatorname{sgn}(z - \alpha).$$

Furthermore we can take

$$\psi_\varepsilon(F) = \int_\alpha^F \eta'_\varepsilon A'(t, y)dy = \int_\alpha^F \eta'_\varepsilon (A(t, y) - A(t, \alpha))'dy,$$

and as $\varepsilon \to 0$ we get

$$\psi_\varepsilon(F) \underset{\varepsilon\to 0}{\to} \int_\alpha^F \operatorname{sgn}(y - \alpha)(A(t, y) - A(t, \alpha))'dy = \operatorname{sgn}(F - \alpha)(A(t, F) - A(t, \alpha))$$

pointwise. Therefore testing inequality (A.6) with $\eta_\varepsilon, \psi_\varepsilon$ and taking the limit as $\varepsilon \to 0$ yields the asserted inequality (A.4). □

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