Drift of invariant manifolds and transient chaos in memristor Chua's circuit

Mauro Di Marco,[1] Mauro Forti,[1,✉] Luca Pancioni,[1] and Alberto Tesi[2]

[1] *Department of Information Engineering and Mathematics, University of Siena, Siena, Italy*

[2] *Department of Information Engineering, University of Florence, Firenze, Italy*

✉E-mail: forti@dii.unisi.it

The article shows that transient chaos phenomena can be observed in a generalized memristor Chua's circuit where a nonlinear resistor is introduced to better model the real memristor behaviour. The flux-charge analysis method is used to explain the origin of transient chaos, that is attributed to the drift of the index of the memristor circuit invariant manifolds caused by the charge flowing into the nonlinear resistor.

Introduction: Certain classes of nonlinear dynamical systems are found to display transient chaos, i.e. they show complex dynamics only on a finite time interval and eventually and often suddenly they tip to a steady state with periodic or convergent dynamics [1–3]. This phenomenon can be related for instance to the presence of slowly drifting parameters caused by internal or external factors acting on the systems. Transient chaos is relevant to model a wide variety of phenomena in different fields as chemistry, neuroscience, medicine, ecology, and climate research and fluid dynamics. We refer the reader to the special issue [4] for a review of recent advances in the theory and application of transient chaotic systems. A nice application of transient chaos in the neural network field is described in the classic paper [5], where this phenomenon is used for implementing a simulated annealing scheme aimed at improving the search ability and optimization capabilities of discrete-time Hopfield neural networks.

Since the first electronic realization and theoretic investigation in the eighties [6], Chua's circuit became a prototype to display quite complex dynamics and many bifurcation phenomena [7]. Its simple electronic structure has made it possible to use Chua's circuit in several ways, e.g. to generate analogic pseudo-random signals, to build up experiments for investigating secure communication techniques, to develop image processing applications via arrays of Chua's circuits. Recently, a number of contributions investigated the implementation of Chua's circuit using memristors [8, 9]. The memristor, theoretically introduced by Chua [10] as the fourth basic circuit element in addition to the resistor, capacitor and inductor, is nowadays considered as a device with a key role for emerging computing paradigms [11–13].

Recent contributions in the literature [11, 14] have developed a technique, named flux-charge analysis method (FCAM), to analyze a class of nonlinear dynamical circuits with linear resistors, capacitors, inductors and ideal memristors. FCAM is based on analyzing the circuit in the flux–charge domain instead of the standard voltage–current domain. FCAM permits to show that for structural reasons the state space of such circuits can be foliated in invariant manifolds and, as a consequence, there is coexistence of infinitely many different attractors, a phenomenon termed extreme multistability. As a relevant example, the state space of Chua's circuit with an ideal memristor can be foliated in infinitely many invariant manifolds and each manifold can be identified by an index depending upon the initial conditions [8]. The same index appears as a constant forcing term in the equations in the flux-charge domain obtained via FCAM. Therefore, by changing the index we obtain different dynamics and there is coexistence of convergent, periodic and chaotic attractors for a fixed set of circuit parameters.

Goal of this paper is to show the existence of transient chaos phenomena in a class of generalized memristor Chua's circuits where a nonlinear resistor is added in parallel to the ideal memristor. Such a nonlinear resistor can be used to better model the behaviour of real memristor devices implemented in nanotechnology [15-17]. Otherwise, the resistor can be deliberately introduced in a circuit with ideal memristors to achieve the desired transient chaos phenomena.

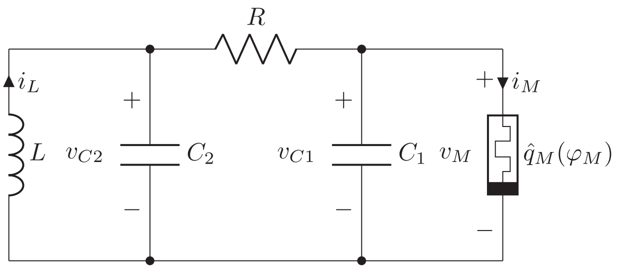

Fig. 1 *Memristor Chua's circuit (MCC) with a locally active flux-controlled memristor*

The considered class of generalized memristor Chua's circuits have fixed parameters and obey an autonomous system of differential equations in the standard voltage–current domain. Therefore, it is unclear when remaining in this domain why there is transient chaos. On the other hand, by using the analysis in the flux-charge domain developed via FCAM, it is possible to unveil the reason why there is the birth of transient chaos. This is attributed to the charge passing through the nonlinear resistor that causes a slow drift of the index of the invariant manifolds where the dynamics of the memristor circuits evolves.

Memristor Chua's circuit: Consider the circuit in Figure 1, obtained from Chua's circuit by replacing the nonlinear resistor (Chua's diode) with an ideal locally-active flux-controlled memristor. This is named memristor Chua's circuit or MCC. Let v_M (resp., i_M) be the voltage (resp., current) in the memristor and also consider the memristor flux $\varphi_M(t) = \int_{-\infty}^{t} v_M(\sigma)d\sigma$ and charge $q_M(t) = \int_{-\infty}^{t} i_M(\sigma)d\sigma$. The memristor obeys the constitutive relation $q_M = \hat{q}_M(\varphi_M) = -a\varphi_M + b\varphi_M^3$, where $a, b > 0$.

In the standard (v, i)-domain, MCC satisfies the fourth-order system of differential equations

$$\begin{cases} C_1 \frac{dv_{C1}}{dt} = \frac{v_{C2}-v_{C1}}{R} - \hat{q}'_M(\varphi_M)v_{C1} \\ C_2 \frac{dv_{C2}}{dt} = \frac{v_{C1}-v_{C2}}{R} + i_L \\ L\frac{di_L}{dt} = -v_{C2} \\ \frac{d\varphi_M}{dt} = v_{C1} \end{cases} \quad (1)$$

for $t \geq t_0$. The state variables in the (v, i)-domain are the two capacitor voltages v_{C1}, v_{C2}, the inductor current i_L and the memristor flux φ_M. The corresponding initial conditions are $v_{C10} = v_{C1}(t_0)$, $v_{C20} = v_{C2}(t_0)$, $i_{L0} = i_L(t_0)$ and $\varphi_{M0} = \varphi_M(t_0)$.

Let us consider the function $\Phi : \mathbb{R}^4 \to \mathbb{R}$ of the state variables in the (v, i)-domain given by

$$\Phi(v_{C1}, v_{C2}, i_L, \varphi_M) = \varphi_M + R\hat{q}_M(\varphi_M) + Li_L + RC_1v_{C1}.$$

It can be checked that the time derivative of Φ along the solutions of (1) vanishes for all time. Hence, we have

$$\Phi\left(v_{C1}(t), v_{C2}(t), i_L(t), \varphi_M(t)\right) = \Phi\left(v_{C10}, v_{C20}, i_{L0}, \varphi_{M0}\right) \doteq \Phi_0$$

for $t \geq t_0$. This means that Φ is an invariant of motion (or first-order integral) for the dynamics in the (v, i)-domain.

Define the sets

$$\mathcal{M}(\Phi_0) = \{(v_{C1}, v_{C2}, i_L, \Phi_M) \in \mathbb{R}^4 : \Phi(v_{C1}, v_{C2}, i_L, \varphi_M) = \Phi_0\} \quad (2)$$

where Φ_0 is any real number. Each set is a three-dimensional invariant manifold for the dynamics of (1), i.e. if a solution of (1) starts on a given manifold $\mathcal{M}(\Phi_0)$ at t_0, then it is constrained to stay on $\mathcal{M}(\Phi_0)$ for all subsequent times. Hence, the state space $\mathbb{R}^4$ in the (v, i)-domain can be decomposed (foliated) in ∞^1 three-dimensional invariant manifolds and each manifold is identified by Φ_0, that is referred to henceforth as the manifold index.

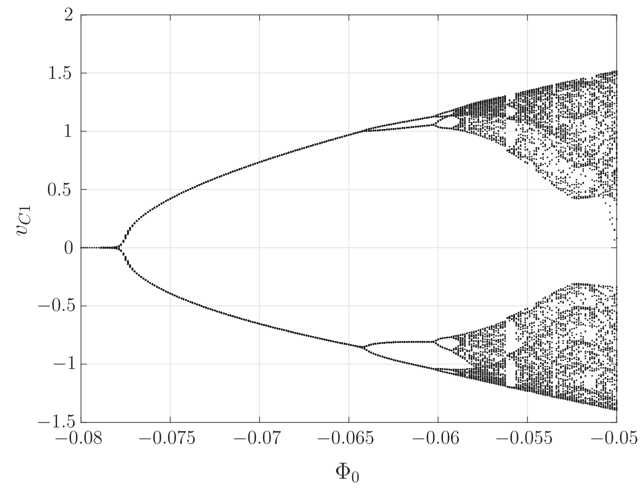

Fig. 2 *Bifurcation diagram of MCC depicting local maxima and minima of the voltage v_{C1} as a function of the index Φ_0 of the invariant manifold*

To obtain the dynamics of MCC on each invariant manifold, it is convenient to write the equations in the (φ, q)-domain. To this end, consider for each element the incremental flux $\varphi(t; t_0) = \int_{t_0}^{t} v(\sigma)d\sigma$ and charge $q(t; t_0) = \int_{t_0}^{t} i(\sigma)d\sigma$. By integrating (1) between t_0 and $t \geq t_0$, and using the change of variables $\bar{\varphi}_{C2}(t) = \varphi_{C2}(t; t_0) - Li_{L0}$ and $\bar{q}_L(t) = q_L(t; t_0) + C_2 v_{C20} - \varphi_{M0}/R + Li_{L0}/R$, we obtain in the (φ, q)-domain the third-order system in the state variables $\varphi_M, \bar{\varphi}_{C2}$ and $\bar{q}_L$ given by

$$\begin{cases} RC_1 \frac{d\varphi_M}{dt} = & \bar{\varphi}_{C2} - \varphi_M - R\hat{q}_M(\varphi_M) + \Phi_0 \\ RC_2 \frac{d\bar{\varphi}_{C2}}{dt} = & \varphi_M - \bar{\varphi}_{C2} + R\bar{q}_L \\ L\frac{d\bar{q}_L}{dt} = & -\bar{\varphi}_{C2}. \end{cases} \tag{3}$$

Note from the first equation that the index Φ_0 of the manifold, which is a function of the initial conditions for the state variables in the (v, i)-domain, appears as a constant forcing term in the equations describing the dynamics in the (φ, q)-domain.

As usual, introduce the adimensional parameters of Chua's circuit given by $\alpha = C_2/C_1$ and $\beta = R^2C_2/L$. Choose normalized values $R = 1$ Ohm and $C_2 = 1$ F, and let $\alpha = 10$ and $\beta = 15$, so that we have $C_1 = 0.1$ F and $L = 1/15$ H. Moreover, for the memristor, let $a = 8/7$ and $b = 4/63$. Figure 2 depicts the bifurcation diagram of MCC obtained with MATLAB, showing local maxima and minima of v_{C1} as a function of the index Φ_0 of the invariant manifold. Note that for low values of Φ_0 there is a stable equilibrium point that undergoes a Hopf bifurcation with the birth of a stable cycle at $\Phi_0 = -0.078$. Such a cycle remains stable for increasing values of Φ_0, until it becomes unstable at $\Phi_0 = -0.0648$ where a first period-doubling occurs, which generates a stable cycle of twice the period. As Φ_0 is further increased, MCC displays a period-doubling sequence route to chaos [18]. Indeed, for $\Phi_0 \in (-0.0648, -0.050)$ we observe a cascade of period-doubling bifurcations followed by the birth of a single-scroll chaotic attractor. All these different dynamics coexist for MCC for the considered fixed set of circuit parameters.

Generalized memristor Chua's circuit: In this section we consider a generalization of MCC where a nonlinear resistor satisfying $i_R = \hat{i}_R(v_R)$ is added in parallel to the ideal memristor (Figure 3). Such a circuit is named generalized MCC or G-MCC. The considered combination of an ideal memristor and a nonlinear resistor, which corresponds to an extended memristor model [], is useful to model a relevant class of real memristor devices. In particular, when the resistor has a rectifying characteristic, it can account for the diode-like effects at the interface between the memristor metal and insulating material [15-17]. It is shown in [17] that by massaging the parameters of the ideal memristor and those of the rectifying nonlinear resistor, it is possible to accurately reproduce the asymmetric pinched hysteresis loops presented by a class of real memristors in response to a sinusoidal input signal. This is in contrast to the symmetric hysteresis loops displayed by ideal memristors as in MCC.

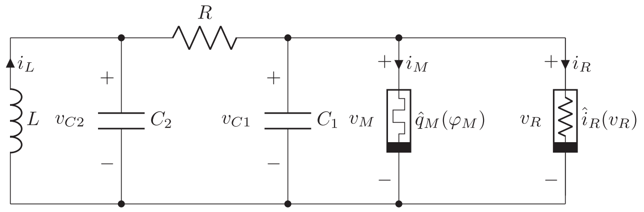

Fig. 3 *Generalized memristor Chua's circuit (G-MCC) where a nonlinear resistor is added in parallel to the ideal memristor*

The equations describing G-MCC in the (v, i)-domain are given by the fourth-order system in the state variables v_{C1}, v_{C2}, i_L and φ_M

$$\begin{cases} C_1 \frac{\mathrm{d}v_{C1}}{\mathrm{d}t} = & \frac{v_{C2}-v_{C1}}{R} - \hat{q}'_M(\varphi_M)v_{C1} - \hat{i}_R(v_{C1}) \\ C_2 \frac{\mathrm{d}v_{C2}}{\mathrm{d}t} = & \frac{v_{C1}-v_{C2}}{R} + i_L \\ L\frac{\mathrm{d}i_L}{\mathrm{d}t} = & -v_{C2} \\ \frac{\mathrm{d}\varphi_M}{\mathrm{d}t} = & v_{C1} \end{cases} \tag{4}$$

for $t \geq t_0$. By integrating these equations between t_0 and $t \geq t_0$, and using the same change of variables as that previously introduced, we obtain in the (φ, q)-domain the fourth-order system in the state variables $\varphi_M, \bar{\varphi}_{C2}, \bar{q}_L$ and $q_R(t; t_0) = \int_{t_0}^{t} \hat{i}_R(\sigma)\, \mathrm{d}\sigma$ given by

$$\begin{cases} RC_1 \frac{\mathrm{d}\varphi_M}{\mathrm{d}t} = & \bar{\varphi}_{C2} - \varphi_M - R\hat{q}_M(\varphi_M) + \Psi(t) \\ RC_2 \frac{\mathrm{d}\bar{\varphi}_{C2}}{\mathrm{d}t} = & \varphi_M - \bar{\varphi}_{C2} + R\bar{q}_L \\ L\frac{\mathrm{d}\bar{q}_L}{\mathrm{d}t} = & -\bar{\varphi}_{C2} \\ \frac{\mathrm{d}q_R}{\mathrm{d}t} = & \hat{i}_R\Big(\frac{1}{RC_1}(\bar{\varphi}_{C2} - \varphi_M - R\hat{q}_M(\varphi_M) \\ & +\Phi_0 - Rq_R(t; t_0))\Big) \end{cases} \tag{5}$$

where we have let

$$\Psi(t) = \Phi_0 - Rq_R(t; t_0). \tag{6}$$

Note that the constant index Φ_0 of MCC in the first equation of Equation (3) is replaced by the time-varying quantity Ψ in the first equation of Equation (5), i.e. Ψ plays the role of a dynamic manifold index of G-MCC.

Manifold drift and transient chaos: We have seen in Section 2 that the dynamics of MCC evolves on a given manifold whose index Φ_0 is determined by the initial conditions for the state variables in the (v, i)-domain. MCC can be considered as a 'frozen' dynamical system whose bifurcation diagram obtained by varying the index Φ_0 is in Figure 2. The diagram shows different dynamics, i.e. convergent, periodic and chaotic dynamics, for different values of Φ_0. Now, consider G-MCC and suppose the nonlinear resistor is such that its charge $q_R(t; t_0)$ is slowly drifting with time. Then, according to Equation (6) and the first equation in Equation (5), this causes a slow drift of the index Ψ of the manifold and so we expect to observe a sequence of bifurcations along the diagram in Figure 2 for the G-MCC dynamics.

Diode-like nonlinear resistor: Suppose that the nonlinear resistor has a rectifying diode-like characteristic. For simplicity we approximate the characteristic with the nonlinear function

$$i_R = \hat{i}_R(v_R) = \begin{cases} 0, & v_R < v_{\mathrm{th}} \\ i_{\max}, & v_R \geq v_{\mathrm{th}} \end{cases} \tag{7}$$

where $v_{\mathrm{th}} > 0$ is a threshold and $i_{\max} > 0$ is the saturation value of the current.

Note that, by construction, we have $\hat{i}_R(v_R) \geq 0$ for any voltage v_R, hence the charge $q_R(t; t_0)$ into the nonlinear resistor is monotonically increasing and, according to Equation (6), this charge causes a decrease of the index Ψ of the invariant manifold. If we further assume that the

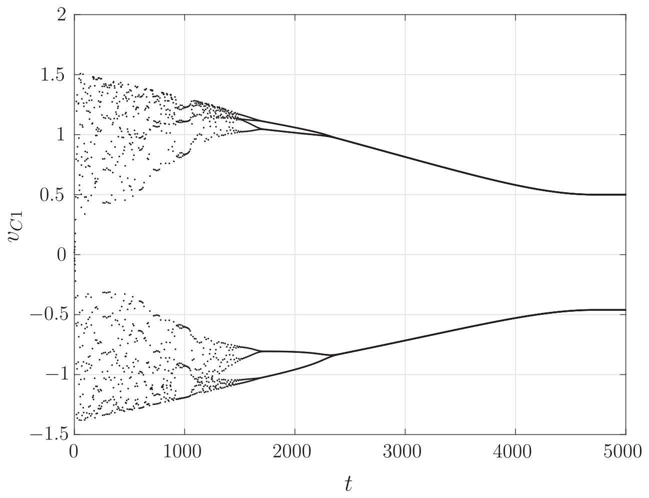

Fig. 4 *Vertical axis: local minima and maxima of v_{C1} for G-MCC with a diode-like nonlinear resistor. Horizontal axis: time in normalized units*

current in the diode has small values for the voltages in G-MCC, we conclude that the diode causes a slow negative drift of the invariant manifold index Ψ as time passes.

In order to illustrate this phenomenon, choose as an example $v_{\text{th}} = 0.5$ V, and $i_{\max} = 20\ \mu$A. Suppose that the initial value of the index is $\Phi_0 = -0.05$ at $t_0 = 0$. Figure 4 depicts the time-domain behaviour of G-MCC as obtained by means of MATLAB simulations. In particular, it shows local minima and maxima of v_{C1} as a function of t in normalized units. It can be seen that at the start G-MCC displays a single-scroll chaotic dynamics corresponding to values of the index $\Phi_0 = -0.05$ or slightly lower. Then, G-MCC undergoes to a sequence of inverse period-doubling bifurcations causing the death of chaos. For larger values of time G-MCC exhibits a periodic behaviour. By comparing Figure 4 with Figure 2, it can be checked that, as expected, G-MCC actually runs dynamically through the bifurcation diagram of MCC starting from $\Phi_0 = -0.05$ and going towards lower values of the manifold index. Correspondingly, G-MCC shows a transient chaos behaviour where the initial single scroll attractor eventually transforms into a periodic attractor.

In addition to better modelling the behaviour of real memristor devices, nonlinear resistor as in Equation (7) may also be deliberately introduced to obtain the desired transient chaos phenomena. Moreover, we note that if we use in G-MCC a reverse diode satisfying

$$i_R = \hat{i}_{\text{rev}}(v_R) = \begin{cases} 0, & v_R > -v_{\text{th}} \\ -i_{\max}, & v_R \leq -v_{\text{th}} \end{cases} \tag{8}$$

then $q_R(t; t_0)$ would be a monotonically decreasing function of time thus causing a drift towards higher values of the index Ψ of the manifold. Finally, it is observed that analogous transient chaos phenomena have been observed with simulations also when the nonlinear resistor (7) is replaced by a Shockley diode with characteristic $i_R(v_R) = I_S(\exp(v_R/\eta V^T) - 1)$, where $I_S > 0$ is the reverse bias saturation current, V^T is the thermal voltage and η is the ideality factor.

Conclusion: The paper has studied transient chaos phenomena in a generalized Chua's circuit with memristor (G-MCC) where a nonlinear resistor in parallel to an ideal memristor is used for better modelling a real-memristor behaviour. The origin of transient chaos is ascribed to the charge flowing into this resistor. Namely, by using the flux-charge analysis method, it is shown that when the resistor has a diode-like characteristic, its charge causes a drift of the index of invariant manifolds. As a consequence, G-MCC can undergo a sequence of bifurcations from chaos to a periodic behaviour, a phenomenon that can be referred to as transient chaos. Future work will be devoted to analyze theoretically the bifurcation phenomena of G-MCC using tools as the harmonic balance. Moreover, we will investigate the use of G-MCC as a source of transient chaos for improving the optimization capabilities of neural networks in solving optimization problems.

Author contributions: Mauro Di Marco: Conceptualization, methodology, writing - original draft. MAURO FORTI: Conceptualization, methodology, writing - original draft. Luca Pancioni: Conceptualization, methodology, writing - original draft. Alberto Tesi: Conceptualization, methodology, writing - original draft.

Acknowledgment: Ministero dell'Università e della Ricerca, (Grant/Award Number: 2017LSCR4K-003)

Conflict of interest statement: The authors declare no conflict of interest.

Data availability statement: Data available on request from the authors.

Received: *28 July 2023* Accepted: *5 September 2023*
doi: 10.1049/ell2.12948

References

1 Grebogi, C., Ott, E., Yorke, J.A.: Crises, sudden changes in chaotic attractors, and transient chaos. *Phys.* **D 7**(1–3), 181–200 (1983)
2 Lai, Y.C., Tél, T.: *Transient Chaos: Complex Dynamics on Finite Time Scales*. Vol. **173**, Springer, Berlin, Heidelberg (2011)
3 Tél, T.: The joy of transient chaos. *Chaos* **25**(9), 097619 (2015)
4 Omel'chenko, E., Tél, T.: Focusing on transient chaos. *J. Phys.: Complexity* **3**(1), 010201 (2022)
5 Chen, L., Aihara, K.: Chaotic simulated annealing by a neural network model with transient chaos. *Neural Networks* **8**(6), 915–930 (1995)
6 Chua, L.O.: The genesis of Chua's circuit. *Archiv Elektronik Ubertragungstechnik.* **46**(4), 250–257 (1992)
7 Madan, R.: Special issue on Chua's circuit: a paradigm for chaos, part I introduction and applications. *J. Circuit Syst. Comput.* **3** (1993)
8 Corinto, F., Forti, M.: Memristor circuits: bifurcations without parameters. *IEEE Trans. Circuits Syst. I, Reg. Pap.* **64**(6), 1540–1551 (2017)
9 Ascoli, A., et al.: Edge of chaos explains Prigogine's instability of the homogeneous. *IEEE J. Emerging Sel. Top. Circuits Syst.* **12**(4), 804–820 (2022). doi:https://doi.org/10.1109/JETCAS.2022.3221156
10 Chua, L.O.: Memristor-the missing circuit element. *IEEE Trans. Circuit Theory* **18**(5), 507–519 (1971)
11 Corinto, F., Forti, M., Chua, L.O.: *Nonlinear Circuits and Systems with Memristors*. Springer, Cham (2021)
12 Huang, T., et al.: Editorial special issue for 50th birthday of memristor theory and application of neuromorphic computing based on memristor-part I. *IEEE Trans. Circuits Syst. I: Reg. Pap*. **68**(11), 4417–4418 (2021)
13 Ascoli, A., et al.: Edge of chaos theory resolves Smale paradox. *IEEE Trans. Circuits Syst. I: Reg. Pap*. **69**(3), 1252–1265 (2022)
14 Corinto, F., Forti, M.: Memristor circuits: flux–charge analysis method. *IEEE Trans. Circuits Syst. I: Reg. Pap.* **63**(11), 1997–2009 (2016)
15 Yang, J.J., et al.: Memristive switching mechanism for metal/oxide/metal nanodevices. *Nat. Nanotechnol*. **3**(7), 429 (2008)
16 Hu, S.G., et al.: Review of nanostructured resistive switching memristor and its applications. *Nanosci. Nanotechnol. Lett*. **6**(9), 729–757 (2014)
17 Corinto, F., Gilli, M., Forti, M.: Flux-charge description of circuits with non-volatile switching memristor devices. *IEEE Trans. Circuits Syst. II: Expr. Briefs* **65**(5), 642–646 (2018)
18 Guckenheimer, J., Holmes, P.: *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer-Verlag, New York (1983)