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Autoría ditelliana: Marenco, Javier

Otra autoría: Blaum, Manuela

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Valid inequalities and complete characterizations of the 2-domination and $\mathcal{P}_3$-hull number polytopes

Manuela Blaum

Instituto de Ciencias, Universidad Nacional de General Sarmiento
Buenos Aires, Argentina

Javier Marenco

Instituto de Ciencias, Universidad Nacional de General Sarmiento

Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires
Buenos Aires, Argentina

Abstract

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is 2-*dominating* if every vertex in $\overline{S}$ has at least two neighbors in S. The minimum cardinality of such a set is called the 2-*domination number* of G. Consider a process in discrete time that, starting with an initial set of marked vertices S, at each step marks all unmarked vertices having two marked neighbors. In such a process, the minimum number of initial vertices in S such that eventually all vertices are marked is called the $\mathcal{P}_3$-*hull number* of G. In this work, we explore a polyhedral relation between these two parameters and, in addition, we provide new families of valid inequalities for the associated polytopes. Finally, we give explicit descriptions of the polytopes associated to these problems when G is a path, a cycle, a complete graph, or a tree.

Keywords: polyhedral combinatorics, $\mathcal{P}_3$-convexity, hull-number, 2-domination number.

Let $G = (V, E)$ be a simple connected graph, with vertex set V and edge set E. We say that a subset $S \subseteq V$ is $\mathcal{P}_3$-*convex* if $N_2(S) := S \cup \{v \in V : |N(v) \cap S| \geq 2\} \subseteq S$. In other words, $\mathcal{P}_3$-paths (i.e., paths with three vertices) starting and ending in S are included in S. The collection $\mathcal{C}$ of $\mathcal{P}_3$-convex sets in V is a *discrete convexity* in G, in the sense that $\emptyset$ and V belong to $\mathcal{C}$, and that $\mathcal{C}$ is closed under intersections (see [1]). This convexity is called $\mathcal{P}_3$-*convexity*. Several parameters related to this discrete convexity have been studied in the last few years, like the $\mathcal{P}_3$-*convexity number* (the maximum cardinality of a proper $\mathcal{P}_3$-convex subset) and the $\mathcal{P}_3$-*hull number of* G, denoted by hull(G), namely the minimum cardinality of a subset S such that its $\mathcal{P}_3$-convex hull is V, where the $\mathcal{P}_3$-*convex hull* of a subset S is the minimal $\mathcal{P}_3$-convex set that contains S.

The decision problem associated to the calculation of the $\mathcal{P}_3$-hull number of a graph is NP-complete in the general case (see [2], [3]) and polynomial for certain families of graphs like trees and cographs (see [2]).

Another interesting parameter associated to the $\mathcal{P}_3$-convexity is the $\mathcal{P}_3$-*interval number* of G, or simply $\mathcal{P}_3$-number, defined as the minimum cardinality of a subset $S \subseteq V$ such that $S \cup \{i \in V : |N(i) \cap S| \geq 2\} = V$, in other words, the minimum cardinality of a subset $S \subseteq V$ such that every vertex in V is either in S or has at least two neighbors in S. This parameter is also known in the literature as the 2-*domination number* of G and we shall denote it by $\gamma_2(G)$.

The concept of k-*domination number*, defined as the minimum cardinality of a subset $S \subseteq V$ such that $|S \cap N(v)| \geq k$ for every $v \in V \backslash S$, was introduced by Fink and Jacobson in [4]. It is a natural generalization of the well-known *domination number* of G, $\gamma(G)$, the minimum cardinality of a dominating set, i.e., a set $S \subseteq V$ such that $|N(i) \cap S| \geq 1$ for every $i \in V \backslash S$. Domination in graph theory has been a widely studied issue both from a graph theoretical point of view and from a polyhedral approach (see for example [5] and [6]). The computation of $\gamma_2(G)$ is NP-hard for a general graph G (see [2] or [7]), remains NP-hard for bipartite and chordal graphs ([8]), and is solvable in polynomial time for trees and some grids (see [2]). This parameter as many other generalizations of the domination number continue to be investigated because of their applications in diverse areas such as logistics and networks design, resource allocation, and telecommunications. However, up to our knowledge, there are very few works that resort to polyhedral approaches, even for particular cases (see for instance [9]), and none of them deal with the parameters that we are interested in.

As integer programming and cutting-plane algorithms have shown to be successful at solving many NP-hard combinatorial optimization problems, we are interested in a polyhedral study of both parameters. In a previous work, ([10]), we presented an integer linear programming formulation for the $\mathcal{P}_3$-hull number calculation and we started a study of the associated polytope. In this work we continue that study as well as its relationship with the polyhedral study of the 2-domination number calculation.

This paper is organized as follows. In Section 1 we give some basic definitions, we describe the integer programming models for the computation of both parameters, and we state some results from [10] that we shall need in the following sections. Then, we explore the polyhedral relation between both parameters (Section 2) and we study facet-defining inequalities for the associated polytopes (Section 3). Furthermore, we give complete minimal descriptions of these polytopes for paths, cycles, completes graphs, and trees (Section 4).

1. Preliminaries

Throughout this paper $G = (V, E)$ will be a simple, connected and undirected graph with vertex set V and edge set E. For every $i \in V$, $N(i)$ will be the open neighborhood of i and we will denote by $V_1 \subseteq V$ the subset of vertices with degree 1 in G. We label the vertices of G such that $V = \{1, \ldots, n\} \cup V_1$, with $\deg(i) \geq 2$ for $i = 1, \ldots, n$. Also we will denote $C_i := |N(i) \cap V_1|$, i.e., the number of neighbors of the vertex i with degree 1, for $i = 1, \dots, n$. For $j = 1, \dots, k$, we denote the j-th canonical vector in $\mathbb{R}^k$ by E_j.

Definition 1.1. *We say that $S \subseteq V$ is a* 2-dominating set *of G if every vertex $i \in V \backslash S$ has at least two neighbors in S, i.e., $|N(i) \cap S| \geq 2$. The* 2-domination number *of G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G.*

Since every vertex $i \in V_1$ should belong to any 2-dominating set, we can associate to each vertex $i \in V \backslash V_1$ a binary variable x_i, and then

$$\gamma_2(G) = |V_1| + \min \sum_{i=1}^{n} x_i$$

subject to

$$2 \leq 2x_i + \sum_{j \in N(i) \backslash V_1} x_j + C_i \quad \text{for } i = 1, \dots, n \text{ and} \tag{1}$$

$$x_i \in \{0, 1\} \quad \text{for} \quad i = 1, \dots, n. \tag{2}$$

Definition 1.2. *The* 2-domination polytope *of G is*

$$P_{2dom}(G) := \text{convex hull}\{x \in \{0,1\}^n : x \text{ verifies } (1)\}.$$

The 2-domination number of a graph is a natural generalization of the well-known domination number, but it also appears naturally in the context of discrete convexities, as we shall see bellow.

Definition 1.3. *A collection $\mathcal{C} \subseteq 2^V$ is a* discrete convexity *in G if $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$, and $C_1 \cap C_2 \in \mathcal{C}$ if C_1 and C_2 belong to $\mathcal{C}$. Every set $S \in \mathcal{C}$ is called a* convex set *in G. If $S \subseteq V$, we define the* convex hull *of S, denoted by $\text{hull}_{\mathcal{C}}(S)$, to be the minimal convex set containing S.*

Several discrete convexities can be defined using different types of paths. Formally, if $\mathcal{P} := \{p \ : \ p \text{ is a } \ u - v \text{ path, for } u \neq v \in V\}$ and $\mathcal{P}' \subseteq \mathcal{P}$ we can define a discrete convexity associated to $\mathcal{P}'$, the $\mathcal{P}'$-convexity in G, as follows.

Definition 1.4. *We say that $S \subseteq V$ is* convex *with respect to $\mathcal{P}'$ or $\mathcal{P}'$*-convex *(and we omit $\mathcal{P}'$ when it is clear by the context) if, for $u, \ v \in S$, $u \neq v$, we have that $I[u,v] := \{w \in p \ : \ p \text{ is a } \ u - v \text{ path in } \mathcal{P}'\} \subseteq S$. In other words, $I(S) := \bigcup_{u,v \in S} I[u,v] \subseteq S$.*

Definition 1.5. *If $S \subseteq V$, the $\mathcal{P}'$*-convex hull *of S, $\text{hull}_{\mathcal{P}'}(S)$, is the minimal (with respect to set inclusion) $\mathcal{P}'$-convex set that contains S.*

It is easy to see that the class of the convex sets defined by the path set $\mathcal{P}'$ is a discrete convexity in G, and we can define the following associated parameters.

Definition 1.6. *If $S \subseteq V$ and $\mathcal{P}' \subseteq \mathcal{P}$, then*

- *the* $\mathcal{P}'$-interval number *of G (or simply $\mathcal{P}'$-number) is the minimum cardinality of a subset such that $I(S) = V$.*
- *the* $\mathcal{P}'$-hull number *of G is the minimum cardinality of a subset such that its convex hull is V.*

We can find in the literature several works dedicated to the study of the convexities arising when $\mathcal{P}'$ is the subset of paths of 3 vertices (the $\mathcal{P}_3$-convexity), the induced paths of 3 vertices (the $\mathcal{P}^*$-convexity) or the shortest paths (geodetic convexity). In this work, we focus on the $\mathcal{P}_3$-convexity.

Definition 1.7. *We say that $S \subseteq V$ is* $\mathcal{P}_3$-convex *if $N_2(S) := S \cup \{i \in V : |N(i) \cap S| \geq 2\} \subseteq S$.*

This definition is equivalent to Definition 1.4 for $\mathcal{P}'$ the set of $\mathcal{P}_3$-paths in G and, furthermore, the 2-domination number $\gamma_2(G)$ is the $\mathcal{P}_3$-interval number of G. Now we shall formally define the $\mathcal{P}_3$-hull number of a graph, in order to describe an integer programming formulation to compute it.

Definition 1.8. *If $N_2^0(S) := S$ and $N_2^r(S) := N_2(N_2^{r-1}(S))$ for $r \geq 1$, then the* $\mathcal{P}_3$-convex hull *of S is $N_2^r(S)$ with $r \in \mathbb{N}$ such that $N_2^r(S) = N_2^{r+1}(S)$ (i.e., the minimal $\mathcal{P}_3$-convex set containing S). We say that $S \subseteq V$ is a* 2-conversion set *in G if its $\mathcal{P}_3$-convex hull is V. The* $\mathcal{P}_3$-hull number *of G,* hull(G), *is the minimum cardinality of a 2-conversion set.*

We can associate, to each 2-conversion set $S \subseteq V$, the parameter $\delta(S) := \min\{r \geq 0 : N_2^r(S) = N_2^{r+1}(S)\}$ (see [10]), so a 2-dominating set is a 2-conversion set with $\delta(S) \leq 1$, and then we have the obvious inequality $\text{hull}(G) \leq \gamma_2(G)$. Conversely, if $S \subseteq V$ is a 2-conversion set then $N_2^{\delta(S)-1}(S)$ is a 2-dominating set. It is easy to see that $\delta(S) \leq m := n + |V_1| - \max\{2, |V_1|\}$ for every 2-conversion set S.

We shall briefly describe the IP model introduced in [10] for the computation of the $\mathcal{P}_3$-hull number of a graph. Let x_{it} be a binary variable associated to each vertex $i \in V \backslash V_1$ (i.e., $\deg(i) \geq 2$) for $t = 0, \ldots, m-1$. Then, we have that

$$\text{hull}(G) = |V_1| + \min \sum_{i=1}^{n} x_{i0}$$

subject to

$$2x_{i(t+1)} \leq 2x_{it} + \sum_{j \in N(i) \backslash V_1} x_{jt} + C_i \text{ for } i = 1, \ldots, n,\ t = 1, \ldots, m-2 \quad (3)$$

$$2 \leq 2x_{i(m-1)} + \sum_{j \in N(i) \backslash V_1} x_{j(m-1)} + C_i \text{ for } i = 1, \ldots, n \quad (4)$$

and

$$x_{it} \in \{0,1\} \quad \text{for} \;\; i = 1,\ldots,n, \;\; t = 1,\ldots,m-1. \tag{5}$$

Definition 1.9. *We define the* $\mathcal{P}_3$-hull number polytope of G *as*

$$P_{hull}(G) := \text{convex hull}\{x \in \{0,1\}^{nm} : x \text{ verifies (3) and (4)}\}. \tag{6}$$

The following definition will be useful in the sequel.

Definition 1.10. *Let* $G = (V,E)$ *be a graph and* $x \in \{0,1\}^{nm}$ *be a feasible solution of (3) and (4). For* $t = 0,\ldots,m-1$*, we define the* support of x at t, $S_t^x \subseteq V$*, as*

$$S_t^x = \{i \in V \backslash V_1 \;:\; x_{it} = 1\} \cup V_1. \tag{7}$$

2. The polytopes

In [10], we have proved that $P_{hull}(G)$ has complete dimension. In this section we explore the relationship between the polytopes $P_{hull}(G)$ and $P_{2dom}(G)$. The following theorem allows us to lift facet-defining inequalities of $P_{2dom}(G)$ to facet-defining inequalities of $P_{hull}(G)$. More precisely, it shows that the facet-defining inequalities of $P_{hull}(G)$ that involve only the last n variables are exactly the facet-defining inequalities of $P_{2dom}(G)$. From now on, a vector $x \in \mathbb{R}^{nm}$ will be written in the form $x = (x_t)_{0 \leq t \leq m-1}$ where $x_t = (x_{it})_{i=1}^n$ for $t = 0,\ldots,m-1$.

Theorem 2.1. *Let* $\pi : \mathbb{R}^{nm} \to \mathbb{R}^n$ *be the projection map defined by*

$$\pi((x_t)_{0 \leq t \leq m-1}) = x_{m-1}.$$

Then

1. $\pi(P_{hull}(G)) = P_{2dom}(G)$ *and both polytopes are full-dimensional.*
2. *Furthermore, the inequality*

$$\sum_{i=1}^{n} \lambda_i x_{i(m-1)} \leq \lambda_0 \tag{8}$$

is a facet-defining (resp. valid) inequality for $P_{hull}(G)$ *if and only if*

$$\sum_{i=1}^{n} \lambda_i x_i \leq \lambda_0 \tag{9}$$

is a facet-defining (resp. valid) inequality for $P_{2dom}(G)$.

Proof. 1. The inclusion $\pi(P_{hull}(G)) \subseteq P_{2dom}(G)$ is obvious considering the projection of a feasible solution of (3) and (4), and then is valid for the projection of any vector of $P_{hull}(G)$, since $P_{2dom}(G)$ is convex. On the other hand, if $x_{m-1} \in \{0,1\}^n$ verifies (1), it is easy to see that the vector $\pi^{-1}(x_{m-1}) \in \mathbb{R}^{nm}$ defined as $\pi^{-1}(x_{m-1}) = (x_t)_{0 \leq t \leq m-1}$ such that $x_{it} = 1$ for $1 \leq i \leq n$, $0 \leq$

$t \leq m-2$ (i.e., the coordinates of $\pi^{-1}(x_{m-1})$ equal 1 for $t = 0, \ldots, m-2$ and it coincides with x_{m-1} for $t = m-1$), verifies (3) and (4). This implies that $\pi(P_{hull}(G)) \supseteq P_{2dom}(G)$ and then the equality holds.

Furthermore, both polytopes are full-dimensional because, if E_{it} is the it-canonical vector in $\mathbb{R}^{nm}$ for $(i,t) \in \mathcal{Z} := \{1,n\} \times \{0, m-1\}$, then the set of vectors $\{\sum_{(i,t)\in\mathcal{Z}} E_{it}\} \cup \{\sum_{(i,t)\neq(i_0,t_0)} E_{it} : (i_0,t_0) \in \mathcal{Z}\} \subset \mathbb{R}^{nm}$ and the set of their projections onto $\mathbb{R}^n$, are affinely independent in $P_{hull}(G)$ and $P_{2dom}(G)$ respectively.

2. Suppose that (8) is a valid inequality for $P_{hull}(G)$. If $x \in P_{2dom}(G)$, by *1*. there exists $(x_t)_{0\leq t\leq m-1} \in P_{hull}(G)$ such that $x_{m-1} = x$ and then x verifies (9), so this inequality is valid for $P_{2dom}(G)$. Analogously, if (9) is valid for $P_{2dom}(G)$, then (8) is valid for $P_{hull}(G)$. Now, let

$$F_{hull} := P_{hull}(G) \cap \{x \in \mathbb{R}^{nm} : \sum_{i=1}^{n} \lambda_i x_{i(m-1)} = \lambda_0\}$$

and

$$F_{2dom} := P_{2dom}(G) \cap \{x \in \mathbb{R}^{n} : \sum_{i=1}^{n} \lambda_i x_i = \lambda_0\}.$$

Suppose that F_{hull} is a facet of $P_{hull}(G)$ and $\dim(F_{2dom}) \leq n-2$, then

$$F_{2dom} \subseteq \{x \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i x_i = \alpha_0\}$$

for some $(\alpha_1, \ldots, \alpha_n, \alpha_0)$ not a multiple of $(\lambda_1, ..., \lambda_n, \lambda_0)$. This implies that

$$F_{hull} \subseteq \{x \in \mathbb{R}^{nm} : \sum_{i=1}^{n} \lambda_i x_{i(m-1)} = \lambda_0\} \cap \{x \in \mathbb{R}^{nm} : \sum_{i=1}^{n} \alpha_i x_{i(m-1)} = \alpha_0\},$$

which is not possible being F_{hull} a facet of (the full-dimensional polytope) $P_{hull}(G)$. For the converse implication, suppose that F_{2dom} is a facet of $P_{2dom}(G)$ and $F_{hull} \subseteq \mathcal{H} = \{x \in \mathbb{R}^{nm} \ : \ \sum_{(i,t)\in\mathcal{Z}} \alpha_{it} x_{it} = \alpha\}$ for an hyperplane $\mathcal{H}$ in $\mathbb{R}^{nm}$. Let x^1 be a feasible solution in F_{2dom}, then the vectors

- $\pi^{-1}(x^1)$ (defined as $\pi^{-1}(x^1) := (x_t)_{0\leq t\leq m-1}$ such that $x_{m-1} = x^1$ and $x_{it} = 1$ for $0 \leq t \leq m-2$ and $1 \leq i \leq n$), and
- $\pi^{-1}(x^1) - E_{it}$ (with E_{it} the it-canonical vector in $\mathbb{R}^{nm}$) for $1 \leq i \leq n$, $0 \leq t \leq m-2$

are feasible solutions in F_{hull}, and then they have to verify the equation defining $\mathcal{H}$. Replacing them in the mentioned equation and subtracting, we can see that $\alpha_{i_0 t_0} = 0$ for $1 \leq i_0 \leq n$ and $0 \leq t_0 \leq m-2$. So, we have

$$F_{hull} \subseteq \mathcal{H} = \{x \in \mathbb{R}^{nm} \ : \ \sum_{i=1}^{n} \alpha_{i(m-1)} x_{i(m-1)} = \alpha_0\}$$

and then

$$F_{2dom} \subseteq \{x \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} \alpha_{i(m-1)} x_i = \alpha_0\}.$$

Since we are supposing that F_{2dom} is a facet of $P_{2dom}(G)$, we have that $(\alpha_{1(m-1)}, \ldots, \alpha_{n(m-1)}, \alpha_0)$ is a multiple of $(\lambda_1, \ldots, \lambda_n, \lambda_0)$, and then the equation defining $\mathcal{H}$ is a multiple of the equation defining F_{hull}. This completes the proof. □

Theorem 2.1 can be generalized to more than one time interval. To this end, we introduce the following definition, which generalizes the parameters $\gamma_2(G)$ and $\mathrm{hull}(G)$.

Definition 2.1. *For $k = 1, ..., m$ we define $\gamma_2^k := \min\{|S| : \mathrm{hull}(S) = V \text{ and } \delta(S) \leq k\}$. This parameter can be calculated as*

$$\gamma_2^k(G) = |V_1| + \min \sum_{i=1}^{n} x_{i(m-k)}$$

s.t. constraints (3) for $t = m-k, ..., m-2$, constraints (4), and $x_{it} \in \{0,1\}$ for $i = 1, ..., n$ and $t = m-k, ..., m-1$. We define the associated polytope

$$\begin{aligned} P_k(G) \quad := \quad & \text{convex hull}\{(x_t)_{m-k \leq t \leq m-1} \in \{0,1\}^{nk} : \\ & x_t \text{ verifies (3) for } t = m-k, ..., m-1, \text{ and } x_{m-1} \text{ verifies (4)}\}. \end{aligned}$$

The following is a general version of Theorem 2.1 whose proof is completely analogous.

Theorem 2.2. *Let $\pi_k : \mathbb{R}^{nm} \to \mathbb{R}^{nk}$ be the projection map defined by $\pi((x_t)_{0 \leq t \leq m-1}) = (x_t)_{m-k \leq t \leq m-1}$ for $1 \leq k \leq m$. Then*

1. *$P_k(G) = \pi_k(P_{hull}(G))$ is a full-dimensional polytope in $\mathbb{R}^{nk}$ for $1 \leq k \leq m$,*
2. *$\sum_{t=m-k}^{m-1} \sum_{i=1}^{n} \lambda_i x_{it} \leq \lambda$ is a facet-defining (resp. valid) inequality for $P_{hull}(G)$ if and only if it is a facet-defining (resp. valid) inequality for $P_k(G)$.*

3. Valid inequalities and facets

A polyhedral study of $P_{hull}(G)$ was started in [10], establishing necessary and sufficient conditions in order that the inequalities (3), (4), and the variable bounds be facet-defining for $P_{hull}(G)$. In addition, we have introduced two families of inequalities, namely the *co-convex inequalities*

$$\sum_{i \in C} x_{it} \geq 1, \tag{10}$$

where $C \subseteq V$ is a *co-convex set* (i.e., $\overline{C}$ is $\mathcal{P}_3$-convex), and the *neighborhood inequalities*

$$x_{i(t+1)} \leq x_{it} + \sum_{j \in N(i) \setminus \{k\}} x_{jt}, \tag{11}$$

where $k \in N(i)$. Both inequalities induce facets of $P_{hull}(G)$ if appropriate hypotheses are enforced.

We now present a family of valid inequalities for $P_{hull}(G)$ and $P_{2dom}(G)$ that generalizes the co-convex inequalities (10), and we establish conditions ensuring that they are facet-defining.

Definition 3.1. *Let $k \in \{1, \dots, m-1\}$. The set $C \subseteq V \setminus V_1$ is* k-quasi-co-convex *in G if $N_2^k(\overline{C}) \neq V$. For a k-quasi-co-convex set C, the inequality*

$$\sum_{i \in C} x_{i(m-k)} \geq 1 \tag{12}$$

is called the k-quasi-co-convex inequality *associated with C.*

The k-quasi-co-convex inequalities are trivially valid for $P_{hull}(G)$. A simple example of a 1-quasi co-convex set is $N_{ij} := N[i] \setminus \{j\}$ if $i \in \overline{V_1}$ and $j \in N(i)$ or $N[i] \cap V_1 = \{j\}$. The corresponding 1-quasi co-convex inequalities coincide with the neighborhood inequalities defined in [10] for $t = m - 1$, since for this value of t, $x_{t+1} = 1$. In the next two theorems we characterize the quasi co-convex sets C such that the associated inequalities (12) are facet-defining for $P_{hull}(G)$.

Theorem 3.1. *Let $C \subseteq V \setminus V_1$ be a k-quasi-co-convex set. If the inequality (12) is facet-defining for $P_{hull}(G)$ then*

1. *C is a minimal k-quasi co-convex set in G,*
2. *for every $j \in \overline{C} \setminus V_1$ there exists $i_j \in C$ such that $N_2^k(\overline{C} \cup \{i_j\} \setminus \{j\}) = V$,*
3. *for every $j \in V \setminus V_1$ and $1 \leq r \leq k-1$ there exists $i_{jr} \in C$ such that $N_2^{k-r}(N_2^r(\overline{C} \cup \{i_{jr}\}) \setminus \{j\}) = V$.*

Proof. Suppose that $C_1 \subset C$ and C_1 is k-quasi co-convex. Then

$$F_C := \{x \in P_{hull}(G) \ : \ \sum_{i \in C} x_{i(m-k)} = 1\} \subset \{x \in P_{hull}(G) \ : \ \sum_{i \in C_1} x_{i(m-k)} = 1\}$$

which is not possible being F_C a facet of (the full-dimensional polytope) $P_{hull}(G)$.

Furthermore, if there exists $j \in \overline{C} \setminus V_1$ such that $N_2^k(\overline{C} \cup \{i\} \setminus \{j\}) \neq V$ for all $i \in C$, then

$$F_C \subset \{x \in P_{hull}(G) \ : \ x_{j(m-k)} = 1\}$$

(since otherwise any solution in F_C cannot mark all of V in k steps), contradicting again the fact that F_C is a facet of $P_{hull}(G)$. Analogously, if there exists $j \in V \setminus V_1$ and $r \in \{1, \dots, k-1\}$ such that $N_2^{k-r}(N_2^r(\overline{C} \cup \{i\}) \setminus \{j\}) \neq V$ for all $i \in C$, then a similar argument shows that

$$F_C \subset \{x \in P_{hull}(G) \ : \ x_{j(m-k+r)} = 1\},$$

again contradicting the facetness of (12). $\square$

In order to establish and prove sufficient conditions ensuring that the k-cuasi convex inequality be facet-defining, we need the following result.

Lemma 3.1. *Let C be a k-quasi co-convex set in G, then C is minimal if and only if $N_2^k(\overline{C} \cup \{i\}) = V$ for every $i \in C$.*

Proof. If there exists $i \in C$ such that $N_2^k(\overline{C} \cup \{i\}) \neq V$ then the set $C \backslash \{i\}$ is a proper subset of C and is k-quasi co-convex, contradicting the minimality of C. The same argument establishes the converse implication. $\square$

Theorem 3.2. *If $C \subseteq V$ is a k-quasi co-convex set such that C verifies the conditions* 1. *and* 2. *of Theorem 3.1 and for every $j \in V \backslash V_1$, $1 \leq r \leq k-1$ there exists $i_{jr} \in C$ such that $j \in N_2^r(\overline{C} \cup \{i_{jr}\})$ and $N_2^{k-r}(N_2^r(\overline{C} \cup \{i_{jr}\}) \backslash \{j\}) = V$, then the associated inequality (12) is facet-defining for $P_{hull}(G)$.*

Proof. Let F_C be the face of $P_{hull}(G)$ defined by the valid inequality (12). As a consequence of the previous lemma, for each $i \in C$, we can define $x^i \in \mathbb{R}^{nm}$, a feasible solution in F_C such that $S_0^{x^i} = \cdots = S_{m-k-1}^{x^i} = V$, $S_{m-k}^{x^i} = \overline{C} \cup \{i\}$, $S_{m-k+1}^{x^i} = N_2(\overline{C} \cup \{i\})$,..., $S_{m-1}^{x^i} = N_2^{k-1}(\overline{C} \cup \{i\})$. Now, suppose that

$$F_C \subseteq \mathcal{H} := \{x \in \mathbb{R}^{nm} \ : \ \sum_{(j,s) \in \mathcal{Z}} \lambda_{js} x_{js} = \lambda\}, \tag{13}$$

with $\mathcal{Z} = [1, \ldots, n] \times [0, \ldots, m-1]$. Let $1 \leq j \leq n$ and $0 \leq s \leq m-k-1$. Suppose that $i = 1 \in C$, then x^1 and $x^1 - E_{js}$, where E_{js} is the js-th canonical vector in $\mathbb{R}^{nm}$, are feasible solutions in F_C, so, replacing these solutions in the equation (13) defining the hyperplane $\mathcal{H}$ and subtracting, we have that $\lambda_{js} = 0$.

Now, take $1 \leq j \leq n$ and $r = k-1$. By hypothesis, there exists $i_{j(k-1)} \in C$ such that $j \in N_2^{k-1}(\overline{C} \cup \{i_{j(k-1)}\})$ and $N_2(N_2^{k-1}(\overline{C} \cup \{i_{j(k-1)}\}) \backslash \{j\}) = V$, then $x^{i_{j(k-1)}}$ and $x^{i_{j(k-1)}} - E_{j(m-1)}$ are feasible solutions in F_C and then $\lambda_{j(m-1)} = 0$. Repeating this argument for $r = k-2, \ldots, 1$, we prove that $\lambda_{j(m-2)} = \cdots = \lambda_{j(m-k+1)} = 0$. By the hypothesis 2. of Theorem 3.1, for $j \in \overline{C} \backslash V_1$ there exists $i_j \in C$ such that x^{i_j} and $x^{i_j} - E_{j(m-k)}$ are feasible solutions in F_C and then $\lambda_{j(m-k)} = 0$.

Finally, the hyperplane containing F_C (see (13)), is defined by the equation $\sum_{j \in C} \lambda_{j(m-k)} x_{j(m-k)} = \lambda$. Since all the feasible solutions x^i with $i \in C$ have to verify this last equation, by replacing them and subtracting we have that $\lambda_{j(m-k)} = \lambda$ for every $j \in C$, proving that this equation is a multiple of the equation that defines F_C. Then, we have proved that F_C is a facet of $P_{hull}(G)$. $\square$

Figure 1 below illustrates a tree $\mathcal{T}$ and a 2-quasi co-convex set $\{1, 2, 3\}$ satisfying the conditions of Theorem 3.2, hence the inequality $x_{12} + x_{22} + x_{32} \geq 1$ is facet-defining for $P_{hull}(\mathcal{T})$.

For the particular case of the 1-quasi co-convex sets, we have the following simpler version of the previous result.

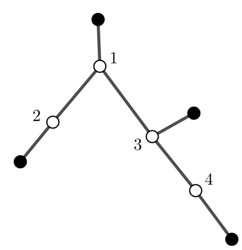

Figure 1: The k-quasi-co-convex inequality $x_{12} + x_{22} + x_{32} \geq 1$ is facet-defining for this instance.

Theorem 3.3. *Let $C \subseteq V \backslash V_1$ be a 1-quasi-co-convex set, the corresponding inequality (12) is facet-defining for $P_{hull}(G)$ if and only if*

- $N_2(\overline{C} \cup \{i\}) = V$ *for all* $i \in C$, *and*
- *for every* $j \in \overline{C} \backslash V_1$ *there exists* $i_j \in C$ *such that* $N_2(\overline{C} \cup \{i_j\} \backslash \{j\}) = V$.

By Theorem 2.1 we have the following immediate result for the 2-domination polytope.

Theorem 3.4. *Let $C \subseteq V \backslash V_1$ be a 1-quasi-co-convex set, the corresponding inequality $\sum_{i \in C} x_i \geq 1$ is facet-defining for $P_{2dom}(G)$ if and only if*

- $N_2(\overline{C} \cup \{i\}) = V$ *for all* $i \in C$, *and*
- *for every* $j \in \overline{C} \backslash V_1$ *there exists* $i_j \in C$ *such that* $N_2(\overline{C} \cup \{i_j\} \backslash \{j\}) = V$.

We finish this section defining a family of trivially valid inequalities for $P_{hull}(G)$.

Definition 3.2. *For $k = 0, ..., m - 1$, the inequality*

$$\sum_{i \in \overline{V_1}} x_{i(m-k)} \geq \gamma_2^k(G) - |V_1|. \tag{14}$$

is called the k-rank inequality *(see Definition 2.1).*

In the next section we shall resort to the rank inequalities for cycles, in this case we have the following result.

Lemma 3.2. *Let $\mathcal{C}_n$ be a cycle with n vertices and $0 \leq t \leq n - 3$. The rank inequality*

$$\sum_{i=1}^{n} x_{it} \geq \left\lceil \frac{n}{2} \right\rceil \tag{15}$$

is facet-defining for $P_{hull}(\mathcal{C}_n)$ if and only if n is odd.

Proof. It is easy to see that the subset of vertices $\{2k-1 \ : \ 1 \le k \le \lceil \frac{n}{2} \rceil\}$ is 2-dominating for $\mathcal{C}_n$, so $\gamma_{n-2-t}(\mathcal{C}_n) \le \lceil \frac{n}{2} \rceil$ for $t = 0, \dots, n-3$ (recall that $m = n-2$). Furthermore, for every 2-dominating set $S \subseteq \{1, \dots, n\}$ of $\mathcal{C}_n$ we have that $|S \cap \{2k-1, 2k\}| \ge 1$ (otherwise $N_2(\overline{S}) \subseteq \overline{S}$), then $\gamma_{n-2-t}(\mathcal{C}_n) \ge \lceil \frac{n}{2} \rceil$ for $t = 0, \dots, n-3$ and then the equality holds.

Now suppose that n is odd and fix $t_0 \in \{0, \dots, n-3\}$. Suppose also that the face

$$F_{t_0} := P(\mathcal{C}_n) \cap \{x \in \mathbb{R}^{n(n-2)} : \sum_{i=1}^{n} x_{it_0} = \frac{n+1}{2}\}$$

verifies that

$$F_{t_0} \subseteq \mathcal{B} = \{x \in \mathbb{R}^{n(n-2)} : \sum_{(i,t)\in\mathcal{Z}} \lambda_{it} x_{it} = \lambda\}.$$

For every $i = 1, \dots, n$, we define $x^i = (x^i_t)_{0 \le t \le n-2} \in \mathbb{R}^{n(n-2)}$ the feasible solution in $P_{hull}(\mathcal{C}_n)$ such that $x^i_t = \sum_{k=1}^{n} E_k$ for $t \ne t_0$, where E_k is the k-th canonical vector in $\mathbb{R}^n$, and for $t = t_0$,

$$x^i_{t_0} \ := \begin{cases} \sum_{k=\frac{i+1}{2}}^{\frac{n-1}{2}} E_{2k} + \sum_{k=0}^{\frac{i-1}{2}} E_{2k+1} & \text{if } i \text{ is odd,} \\ \\ \sum_{k=\frac{i}{2}}^{\frac{n-1}{2}} E_{2k+1} + \sum_{k=1}^{\frac{i}{2}} E_{2k} & \text{if } i \text{ is even.} \end{cases}$$

That is to say, for each odd i the active vertices of x^i in t_0 correspond to odd indices from 1 to i and even indices from $i+1$ to $n-1$, meanwhile for each even i the active vertices in t_0 correspond to the even indices from 2 to i and the odd ones from $i+1$ to n. It is easy to see that these n solutions belong to F_{t_0}, and that the same occurs for the vectors of the form $\{x^1 - E_{it} \ : \ 0 \le t \le n-3, \ t \ne t_0, \ 1 \le i \le n\}$, where E_{it} is the it-th canonical vector in $\mathbb{R}^{n(n-2)}$. So x^1 and $x^1 - E_{it}$ verify the equation defining $\mathcal{B}$ and then we have that $\lambda_{it} = 0$ for $1 \le i \le n$, $0 \le t \le n-2$, $t \ne t_0$. Then

$$\mathcal{B} = \{x \in \mathbb{R}^{n(n-2)} : \sum_{i=1}^{n} \lambda_{it_0} x_{it_0} = \lambda\}.$$

By replacing x^1 and x^3 in the equation defining $\mathcal{B}$ and subtracting the resulting equations, we obtain that $\lambda_{2t_0} = \lambda_{3t_0}$. Repeating this procedure with x^3 and x^5, it follows that $\lambda_{4t_0} = \lambda_{5t_0}$. In general, by replacing $x^1, x^3, \dots, x^n$ in the mentioned equation and subtracting, we have that $\lambda_2 = \lambda_3, \dots, \lambda_{(n-1)t_0} = \lambda_{nt_0}$. On the other hand, by replacing $x^2, x^4, \dots, x^{n-1}$ in the equation defining $\mathcal{B}$ and subtracting, we have that $\lambda_{3t_0} = \lambda_{4t_0}, \lambda_{5t_0} = \lambda_{6t_0}, \dots, \lambda_{(n-2)t_0} = \lambda_{(n-1)t_0}$. Finally, if we replace x^1 and x^{n-1} in the same equation and subtract we obtain $\lambda_{1t_0} = \lambda_{nt_0}$. Thus

$$\mathcal{B} = \{x \in \mathbb{R}^{n(n-2)} : \sum_{i=1}^{n} \lambda_{1t_0} x_{it_0} = \lambda_{1t_0} \frac{(n+1)}{2}\}.$$

That is to say, the equation defining $\mathcal{B}$ is a multiple of the equation defining F_{t_0}, this implies that F_{t_0} is a facet of $P_{hull}(\mathcal{C}_n)$ if n is odd. If n is even, it is easy to see that

$$F_{t_0} = P(\mathcal{C}_n) \cap \{x \in \mathbb{R}^{n(n-2)} : \sum_{i=1}^{n} x_{it_0} = \frac{n}{2}\} \subseteq \{x \in \mathbb{R}^{n(n-2)} : x_{1t_0} + x_{2t_0} = 1\}.$$

This inclusion is due to the fact that

$$2\sum_{i=1}^{n} x_{it_0} = x_{1t_0} + x_{nt_0} + \sum_{i=1}^{n-1}(x_{(i+1)t_0} + x_{it_0}) = n,$$

and, since $\{i, i+1\}$ and $\{n, 1\}$ are co-convex sets for $i = 1, \ldots, n-1$, we have that $x_{(i+1)t_0} + x_{it_0} \geq 1$ for all $i = 1, \ldots, n-1$, $x_{1t_0} + x_{nt_0} \geq 1$ and then each term is equal to 1. This implies that F_{t_0} is not a facet of $P_{hull}(\mathcal{C}_n)$ for an even n. □

4. Complete descriptions

In this section we present complete and minimal descriptions of $P_{hull}(G)$ and $P_{2dom}(G)$ when the graph G is a path, a cycle, or a complete graph. In the case that G is a tree, we show a complete description of $P_{2dom}(G)$ (also providing a partial description of $P_{hull}(G)$, using Theorem 2.1).

4.1. Paths

If the graph G is a path of length $n+2$, the following theorem shows that the co-convex inequalities associated to the co-convex sets $\{i, i+1\}$ and the variable bounds $x_{it} \leq 1$ are sufficient in order to provide a complete description of the associated polytope.

Theorem 4.1. *Let $\mathcal{P}_{n+2}$ be a path with $V \backslash V_1 = \{1, ..., n\}$. Then, a complete minimal description of $P_{hull}(\mathcal{P}_{n+2}) \subseteq \mathbb{R}^{n^2}$ is given by*

$$\begin{aligned} x_{it} + x_{(i+1)t} &\geq 1 \quad \text{for } 0 \leq t \leq n-1,\ 1 \leq i \leq n-1, &(16)\\ x_{it} &\leq 1 \quad \text{for } 0 \leq t \leq n-1,\ 1 \leq i \leq n. &(17)\end{aligned}$$

Proof. Let $P \subseteq \mathbb{R}^{n^2}$ be the polyhedron defined as

$$\begin{aligned} P := \{x \in \mathbb{R}^{n^2} : x_{it} + x_{(i+1)t} \geq 1 \text{ for } 0 \leq t \leq n-1, 1 \leq i \leq n-1, \quad &(18)\\ x_{it} \leq 1 \text{ for } 0 \leq t \leq n-1, 1 \leq i \leq n\}. \quad &(19)\end{aligned}$$

For $i = 1, ..., n-1$, the sets $\{i, i+1\} \subseteq V \backslash V_1$ are $\mathcal{P}_3$-co-convex and they verify the conditions of Theorem 3.2. This implies that the inequalities (16) are valid and facet-defining for $0 \leq t \leq n-1$. The inequalities $x_{it} \leq 1$ are also facet-defining for all i and t, thus $P_{hull}(\mathcal{P}_{n+2}) \subseteq P$.

We shall prove that the extreme points of P are integral and, also, they are solutions of the model constraints, which in this case are

$$\begin{aligned}
2x_{1(t+1)} &\leq 2x_{1t} + x_{2t} + 1 \text{ for } 0 \leq t \leq n-2, && (20)\\
2x_{n(t+1)} &\leq 2x_{nt} + x_{(n-1)t} + 1 \text{ for } 0 \leq t \leq n-2, && (21)\\
2x_{i(t+1)} &\leq 2x_{it} + x_{(i-1)t} + x_{(i+1)t} \text{ for } 0 \leq t \leq n-2,\ 2 \leq i \leq n-1, && (22)\\
2 &\leq 2x_{1(n-1)} + x_{2(n-1)} + 1, && (23)\\
2 &\leq 2x_{n(n-1)} + x_{(n-1)(n-1)} + 1, && (24)\\
2 &\leq 2x_{i(n-1)} + x_{(i-1)(n-1)} + x_{(i+1)(n-1)} \text{ for } 2 \leq i \leq n-1, && (25)\\
0 &\leq x_{it} \leq 1 \text{ for } 0 \leq t \leq n-1,\ 1 \leq i \leq n. && (26)
\end{aligned}$$

If $x \in P$ then $x_{it} \geq 1 - x_{(i+1)t} \geq 0$ and $x_{nt} \geq 1 - x_{(n-1)t} \geq 0$ for $1 \leq i \leq n-1$, $0 \leq t \leq n-1$, hence x verifies the variable bounds of the model. Besides, for $2 \leq i \leq n-1$, $0 \leq t \leq n-1$,

$$2 \leq (x_{it} + x_{(i+1)t}) + (x_{it} + x_{(i-1)t}) = 2x_{it} + x_{(i+1)t} + x_{(i-1)t}.$$

In particular, for $t = n-1$ we have that x verifies inequalities (25) and, using that $2 \geq 2x_{i(t+1)}$, the inequalities (22) are fulfilled too. Analogously, for $0 \leq t \leq n-1$,

$$\begin{aligned}
2 \leq (x_{1t} + x_{2t}) + (x_{1t} + 1) = 2x_{1t} + x_{2t} + 1 \text{ and}\\
2 \leq (x_{nt} + x_{(n-1)t}) + (x_{nt} + 1) = 2x_{nt} + x_{(n-1)t} + 1,
\end{aligned}$$

hence x verifies (20), (21), (23), and (24).

We shall prove now that P is integral. Let $A \in \{0,1\}^{n(n-1)\times n^2}$ be the matrix corresponding to the coefficients of the co-convex inequalities (16). Is easy to see that A is an interval matrix (the nonzero coefficients in each row are consecutive) and then it is totally unimodular. Moreover, $(A^T, -I)^T$ is totally unimodular too and then the polytope $\{x \in \mathbb{R}_+^{n^2} : Ax \geq b,\ -Ix \geq \widehat{b}\}$ where $b = (1, ..., 1)^T$ and $\widehat{b} = (-1, ..., -1)^T$, is an integral polytope (see [11]) and then it coincides with $P(\mathcal{P}_{n+2})$. □

By Theorem 2.1 we have the following direct corollary.

Corollary 4.1. *A complete minimal description of $P_{2dom}(\mathcal{P}_{n+2}) \subseteq \mathbb{R}^n$ is given by*

$$\begin{aligned}
x_i + x_{(i+1)} &\geq 1 \text{ for } 1 \leq i \leq n-1,\\
x_{i(n-1)} &\leq 1 \text{ for } 1 \leq i \leq n.
\end{aligned}$$

4.2. Cycles

In this section we present a complete description of the polytope $P_{hull}(\mathcal{C}_n)$ for $\mathcal{C}_n$ the cycle with n vertices. In this case it is necessary to distinguish between even and odd values of n.

We shall use the following result, that can be found in [11], Corolary 2.8, Section III.1.

Lemma 4.1. *Let A be a $(0, 1, -1)$ matrix with no more than two nonzero elements in each column. Then A is totally unimodular if and only if the rows of A can be partitioned into two subsets Q_1 and Q_2 such that if a column contains two nonzero elements, the following statements are true:*

a. If both nonzero elements have the same sign, then one is in a row contained in Q_1 and the other is in a row contained in Q_2

b. If the two nonzero elements have opposite sign, then both are in rows contained in the same subset.

Theorem 4.2. *Let $\mathcal{C}_n$ be a cycle with n vertices, with n even and $n \geq 4$. A complete minimal description of $P_{hull}(\mathcal{C}_n) \subseteq \mathbb{R}^{n(n-2)}$ is given by*

$$
\begin{aligned}
x_{it} + x_{(i+1)t} &\geq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n-1, && (27)\\
x_{nt} + x_{1t} &\geq 1 \quad \text{for } 0 \leq t \leq n-3, && (28)\\
x_{it} &\leq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n. && (29)
\end{aligned}
$$

Proof. Let $\widehat{P} \subset \mathbb{R}^{n(n-2)}$ be the polytope

$$
\begin{aligned}
\widehat{P} := \{x \in \mathbb{R}^{n(n-2)} : x_{it} + x_{(i+1)t} &\geq 1 \text{ for } 0 \leq t \leq n-3,\ 1 \leq i \leq n-1,\\
x_{nt} + x_{1t} &\geq 1 \text{ for } 0 \leq t \leq n-3,\\
x_{it} &\leq 1 \text{ for } 0 \leq t \leq n-3,\ 1 \leq i \leq n\}.
\end{aligned}
$$

Considering the co-convex sets $\{i, i+1\}$ and $\{1, n\}$, it is easy to see that $P(\mathcal{C}_n) \subseteq \widehat{P}$ and that the corresponding inequalities (27), (28), and (29) are facet-defining.

We shall prove that $\widehat{P}$ is an integral polytope. Let $A \in \mathbb{R}^{n(n-2)\times n(n-2)}$ be the matrix of the coefficients of the co-convex inequalities defining $\widehat{P}$, that is to say, for $i = 1, \ldots, n-1$ and $t = 0. \ldots, n-2$ the only nonzero coefficients of the it-th row of A correspond to the it-th and $(i+1)t$-th columns of A, meanwhile, for $i = n$ they correspond to the 1-th and n-th columns. Since n is even, it is easy to see that the matrix A is totally unimodular, because its rows can be partitioned into two subsets Q_1 (whose elements are the odd rows) and Q_2 (whose elements are the even rows) such that the rows corresponding to the two nonzero entries of a column are not in the same Q_i and then, by Lemma 4.1, the matrix A is totally unimodular and, in this case, $\widehat{P}$ is integral. It is easy to see that the points of $\widehat{P}$ verify the inequalities of the model (we omit the details of the proof because it is completely analogous to the proof of Theorem 4.1). This shows that $\widehat{P} = P(C_n)$. □

The argumentation used in the previous theorem to prove the unimodularity of the matrix of coefficients of the co-convex inequalities cannot be used when n is odd. Indeed, for odd cycles it is necessary to add the rank inequalities in order to get the complete description of the corresponding polytope.

Theorem 4.3. *Let $\mathcal{C}_n$ be a cycle with n vertices, with n odd and $n \geq 5$. Then, a complete and minimal description of $P_{hull}(\mathcal{C}_n) \subseteq \mathbb{R}^{n(n-2)}$ is given by*

$$\sum_{i=1}^{n} x_{it} \geq \frac{n+1}{2} \quad \text{for } 0 \leq t \leq n-3, \tag{30}$$

$$x_{it} + x_{(i+1)t} \geq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n-1, \tag{31}$$

$$x_{nt} + x_{1t} \geq 1 \quad \text{for } 0 \leq t \leq n-3, \tag{32}$$

$$x_{it} \leq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n. \tag{33}$$

Proof. Let $\widehat{P}$ be the polyhedron in $\mathbb{R}^{n(n-2)}$ defined by the above inequalities. We know that $P_{hull}(\mathcal{C}_n) \subseteq \widehat{P}$ and also that the inequalities are facet-defining for $P_{hull}(\mathcal{C}_n)$. We shall prove that $\widehat{P}$ is integral and that its points verify the inequalities defining $P_{hull}(\mathcal{C}_n)$.

Notice that $\widehat{P} = P_0 \times \cdots \times P_{n-3}$ with $P_t \subseteq \mathbb{R}^n$ the polytope in $\mathbb{R}^n$ defined by the inequalities $2\sum_{i=1}^{n} x_{it} \geq n+1$, $x_{it} + x_{(i+1)t} \geq 1$ for $1 \leq i \leq n$ $(n+1 := 1)$ and $x_{it} \leq 1$ for $1 \leq i \leq n$ and $t = 0, \ldots, n-3$ (in particular, $\dim(P_t) = n$). Let $x = (x_t)_{0 \leq t \leq n-3}$ be an extreme point of $\widehat{P}$, then each x_t is an extreme point of P_t for $0 \leq t \leq n-3$. We shall prove that every $x_t \in \mathbb{R}^n$ has integer coordinates.

Claim: If $x_t = (x_{it})_{1 \leq i \leq n}$ is an extreme point of P_t for $t = 0, \ldots, n-3$, then there exists $1 \leq i \leq n$ such that $x_{it} = 1$.

Proof of the claim. We know that x_t is the solution of a system $Ax = b$ where A is an $n \times n$ non-singular submatrix of coefficients of n inequalities defining P_t. Suppose that $x_{it} \neq 1$ for all $1 \leq i \leq n$, then we have the following possibilities.

- $x_{it} + x_{(i+1)t} = 1$ for $1 \leq i \leq n$, in this case $x_{it} = \frac{1}{2}$, but this is not possible since this vector does not verify the rank inequality because $\sum\limits_{i=1}^{n} x_{it} = \frac{n}{2} < \frac{n+1}{2}$,
- $x_{it} + x_{(i+1)t} = 1$ for $1 \leq i \leq n$, $i \neq j$ for some j, suppose $j = n$, then $x_t = (x_{1t}, 1 - x_{1t}, x_{1t}, 1 - x_{1t}, ..., x_{1t})$ and $\sum\limits_{i=1}^{n} x_{it} = \frac{n-1}{2} + x_{1t} = \frac{n+1}{2}$ then $x_{1t} = 1$. For $j \neq n$ we can repeat the previous argument and prove that $x_{(j+1)t} = 1$.

We have reached a contradiction in both cases, so we have proved that each extreme point x_t is contained in a facet of P_t of the form

$$F_i = P_t \cap \{x \in \mathbb{R}^n : x_{it} = 1\}. \ \Diamond$$

Finally, we shall prove that the extreme points of F_i are integral. Suppose that $i = 1$, then $x_t = (1, \pi(x_t))$ where $\pi(x_t) \in \mathbb{R}^{n-1}$ is an extreme point of the polyhedron

$$\begin{aligned}\pi(P_t) = \{x \in \mathbb{R}^{n-1} : \sum_{j=2}^{n} x_{jt} &\geq \frac{n-1}{2},\\ x_{it} + x_{(i+1)t} &\geq 1 \text{ for } 2 \leq i \leq n-1,\\ x_{2t} &\geq 0,\\ x_{nt} &\geq 0,\\ x_{it} &\leq 1 \text{ for } 2 \leq i \leq n\}\\ = \{x \in \mathbb{R}^{n-1} : x_{it} + x_{(i+1)t} &\geq 1 \text{ for } 2 \leq i \leq n-1,\\ x_{it} &\leq 1 \text{ for } 2 \leq i \leq n\},\end{aligned}$$

which has integral extreme points because its coefficient matrix is totally unimodular. Once again, we omit to prove that the points in $\widehat{P}$ verify the model inequalities defining $P_{hull}(\mathcal{C}_n)$ because it is analogous to the proofs of the previous theorems. □

By Theorem 2.1 we have the following immediate corollary.

Corollary 4.2. *Let $\mathcal{C}_n$ be a cycle with n vertices, for $n \geq 4$. Then, if n is even, a complete and minimal description of $P_{2dom}(\mathcal{C}_n) \subseteq \mathbb{R}^n$ is given by*

$$\begin{aligned}x_i + x_{i+1} &\geq 1 \quad \text{for } 1 \leq i \leq n-1,\\ x_n + x_1 &\geq 1,\\ x_i &\leq 1 \quad \text{for } 1 \leq i \leq n-2,\end{aligned}$$

and, if n is odd, it is necessary to add the inequality

$$\sum_{i=1}^{n} x_i \geq \frac{n+1}{2}.$$

4.3. Complete graphs

For a complete graph, the rank inequalities and the variable bounds are sufficient in order to provide a complete and minimal description of the associated polytope.

Theorem 4.4. *Let $\mathcal{K}_n$ be the complete graph with n vertices, for $n \geq 4$. Then, a complete and minimal description for $P_{hull}(\mathcal{K}_n) \subseteq \mathbb{R}^{n(n-2)}$ is given by*

$$\sum_{i=1}^{n} x_{it} \geq 2 \quad \text{for } 0 \leq t \leq n-3, \tag{34}$$

$$0 \leq x_{it} \leq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n. \tag{35}$$

Proof. The inequalities (34) are trivially valid and facet-defining for $P_{hull}(\mathcal{K}_n)$, hence this polytope is included in

$$P := \{x \in \mathbb{R}^{n(n-2)} : \sum_{i=1}^{n} x_{it} \geq 2 \quad \text{for } 0 \leq t \leq n-3, \tag{36}$$
$$0 \leq x_{it} \leq 1 \quad \text{for } 0 \leq t \leq n-3,\ 1 \leq i \leq n\}. \tag{37}$$

We shall prove that P is integral and that its points verify the model inequalities, which in this case are

$$\begin{aligned} 2x_{i(t+1)} &\leq 2x_{it} + \sum_{j \neq i} x_{jt} \quad \text{for } 1 \leq i \leq n \text{ and } 0 \leq t \leq n-3, \\ 2 &\leq 2x_{i(n-3)} + \sum_{j \neq i} x_{j(n-3)} \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

Take $x \in P$ and fix $1 \leq i \leq n$ and $0 \leq t \leq n-3$. By the inequalities (34) and (35),

$$x_{it} + \sum_{j=1}^{n} x_{jt} = 2x_{it} + \sum_{j \neq i} x_{it} \geq x_{it} + 2 \geq 2 \geq 2x_{i(t+1)},$$

hence the inequalities of the model are satisfied by x.

In order to prove the integrality of P, consider the matrix $A \in \{0,1\}^{(n-2)\times n(n-2)}$ such that $a_{\widehat{t},it} = 0$ for $\widehat{t} \neq t$ and $a_{t,it} = 1$ for $1 \leq i \leq n$ (i.e., A is the matrix of coefficients of the first $n-2$ inequalities). This matrix is an interval matrix and then is totally unimodular. So $(A^T, I, -I)^T$ is totally unimodular too and then the polytope $\{x \in \mathbb{R}_+^{n(n-2)} : Ax \geq b, Ix \geq b', -Ix \geq b''\}$ where $b = (2, \ldots, 2)^T$, $b' = (0, \ldots, 0)^T$ and $b'' = (-1, \ldots, -1)^T$, is an integral polytope that coincides with $P_{hull}(\mathcal{K}_n)$. $\square$

Once again, we have the following immediate corollary.

Corollary 4.3. *Let $\mathcal{K}_n$ be the complete graph with n vertices, for $n \geq 4$. Then, a complete and minimal description of $P_{2dom}(\mathcal{K}_n) \subseteq \mathbb{R}^n$ is given by*

$$\begin{aligned} \sum_{i=1}^{n} x_i &\geq 2, \\ 0 \leq x_i &\leq 1 \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

4.4. 2-domination polytope for trees

In this section, we shall provide a complete description of the 2-domination polytope $P_{2dom}(\mathcal{T})$ when $\mathcal{T}$ is a tree. By Theorem 2.1, the inequalities involved in this description are also facet-defining inequalities for the hull number polytope.

We first introduce some notations that we need in the sequel. Let $\mathcal{T}$ be a tree, with $V \backslash V_1 = \{1, ..., n\}$. We define the following partition of $V \backslash V_1$.

- Let $A \subseteq V \backslash V_1$ be the set of vertices of $\mathcal{T}$ of degree 2.

- Let $B \subseteq V \backslash V_1$ be the set of vertices i of $\mathcal{T}$ such that $\deg(i) \geq 3$ and there exist at least two vertices $j_1, j_2 \in N(i)$ with $\deg(j_k) \leq 2$ for $k = 1, 2$. We denote $B_1 \subseteq B$ to the set of vertices in $i \in B$ such that $N(i) \cap A = \emptyset$ (i.e., every neighbor of i is a leaf or has degree at least three).

- Let $C \subseteq V \backslash V_1$ be the set of vertices i of $\mathcal{T}$ such that $\deg(i) \geq 3$ and there exists a unique $j \in N(i)$ with $\deg(j) \leq 2$. We denote $C_1 \subseteq C$ to the set of vertices in $i \in C$ such that $N(i) \cap A = \emptyset$ (i.e., the neighbors of i are one leaf and the rest have degree at least three).

- Let $D \subseteq V \backslash V_1$ be the set of vertices i of $\mathcal{T}$ such that $\deg(j) \geq 3$ for all $j \in N[i]$.

The following lemma shows facet-defining inequalities associated to the different classes of vertices of $\mathcal{T}$.

Lemma 4.2. *Let $\mathcal{T}$ be a tree, with $V \backslash V_1 = \{1, ..., n\}$.*

1. *For each $i \in A$, i.e., $\deg(i) = 2$, the sets $R_{ij} := \{i, j\}$ are 1-quasi-co-convex for every $j \in N(i) \backslash V_1$. Furthermore, the corresponding valid inequalities $x_i + x_j \geq 1$ are facet-defining for $P_{2dom}(\mathcal{T})$.*
2. *For each $i \in D$, i.e., $\deg(i) \geq 3$ and $\deg(j) \geq 3$ for every $j \in N(i)$, the sets $Q_{ij} = N[i] \backslash \{j\}$ are 1-quasi-co-convex in V. Furthermore, the corresponding valid inequalities $\sum_{k \in N[i] \backslash \{j\}} x_k \geq 1$ are facet-defining for $P_{2dom}(\mathcal{T})$. For these vertices i, also the corresponding inequality of the model $2x_i + \sum_{j \in N(i)} x_j \geq 2$ (that we shall call M_i) and the variable bound $x_i \geq 0$ are facet-defining for $P_{2dom}(\mathcal{T})$.*
3. *For each $i \in C$, i.e., $\deg(i) \geq 3$ and i has exactly one neighbor j with degree at most 2, then the set $S_i = N[i] \backslash \{j\}$ is 1-quasi-co-convex in V. Furthermore, the corresponding valid inequality $\sum_{k \in N[i] \backslash \{j\}} x_k \geq 1$ is facet-defining for $P_{2dom}(\mathcal{T})$. If $\deg(j) = 1$ (i.e., j is a leaf and then $i \in C_1$) the variable bound $x_i \geq 0$ is a facet-defining inequality too.*

Proof. The proof is immediate and follows from Theorem 3.3. $\square$

For the sake of simplicity, and when it is clear by the context, we shall call R_{ij} to the inequality $x_i + x_j \geq 1$ associated with the set $\{i, j\}$ for $i \in A$. Analogously, if $i \in D$ and $j \in N(i)$, Q_{ij} will denote the facet-defining inequality defined in Lemma 4.2 and the same for $i \in C$ and the inequality S_i. The inequalities defined in the previous lemma together with the upper variable bound provide a complete and minimal description of the polytope $P_{2dom}(T)$.

Theorem 4.5. *Let $\mathcal{T}$ be a tree. Then, a complete and minimal description of $P_{2dom}(\mathcal{T})$ is given by the following inequalities,*

$$
\begin{aligned}
R_{ij}:\ x_i + x_j &\geq 1 \text{ for } i \in A,\ j \in N(i) \text{ or } i \in B \cup C, j \in N(i) \cap A,\\
S_i:\ x_i + \sum_{j \in N(i)\setminus(V_1 \cup A)} x_j &\geq 1 \text{ for } i \in C,\\
M_i:\ 2x_i + \sum_{j \in N(i)} x_j &\geq 2 \text{ for } i \in D,\\
Q_{ik}:\ x_i + \sum_{j \in N(i)\setminus\{k\}} x_j &\geq 1 \text{ for } i \in D,\ k \in N(i),\\
E_i:\ x_i &\geq 0 \text{ for } i \in B_1 \cup C_1 \cup D,\\
x_i &\leq 1 \text{ for } i \in V\setminus V_1.
\end{aligned}
$$

In the sequel, we shall denote the polytope defined by the above inequalities by $P_{\mathcal{T}}$. We need the following lemmas in order to prove the Theorem 4.5.

Lemma 4.3. *The inequalities $E_s : x_s \geq 0$ are valid for $P_{\mathcal{T}}$, for every $s \in V\setminus V_1$.*

Proof. For $s \in B_1 \cup C_1 \cup D$ the result is trivial. For $s \in A \cup (B\setminus B_1) \cup (C\setminus C_1)$, each $\overline{x} \in P_{\mathcal{T}}$ has to verify that $\overline{x}_s + \overline{x}_j \geq 1$ for some $j \in V\setminus V_1$, and then $\overline{x}_s \geq 0$, because $\overline{x}_j \leq 1$ is valid. □

Lemma 4.4. *If $\overline{x} \in \mathbb{R}^n$ is an extreme nonzero point of $P_{\mathcal{T}}$, then there exists a vertex i, $1 \leq i \leq n$, such that $\overline{x}_i = 1$.*

Proof. Suppose that $\overline{x}_i < 1$ for all $1 \leq i \leq n$. Let us consider an arbitrary root of $\mathcal{T}$, not a leaf, that we call vertex 1. We enumerate the rest of the vertices in $V\setminus V_1$ so that $i < j$ for every $i, j \in V$ such that $\text{dist}(i, 1) < \text{dist}(j, 1)$.

Let P be the set of all the inequalities of the form R_{ij}, S_i, M_i and Q_{ik} defined in the statement of Theorem 4.5, and $P_i \subseteq P$ be the subset corresponding to a vertex i, i.e.,

$$
\begin{aligned}
P_i &= \{R_{ij} : j \in N(i)\setminus V_1 \text{ and } i < j\} \text{ for } i \in A, && (38)\\
P_i &= \{R_{ij} : j \in N(i),\ j \in A \text{ and } i < j\} \text{ for } i \in B, && (39)\\
P_i &= S_i \cup \{R_{ij} : j \in N(i),\ j \in A \text{ and } i < j\} \text{ for } i \in C, \text{ and} && (40)\\
P_i &= M_i \cup \{Q_{ik} : j \in N(i)\} \text{ for } i \in D. && (41)
\end{aligned}
$$

Analogously, if $P^{=}$ (resp. $P^{<}$) is the subset of inequalities in the description of $P_{\mathcal{T}}$ that $\overline{x}$ verifies with equality (resp. strict inequality), we define $P_0^{=} \subseteq P^{=}$ the subset of the inequalities $E_i \ :\ x_i \geq 0$, and $P_i^{=} \subseteq P^{=}\setminus P_0^{=}$ defined as $P_i^{=} = P_i \cap P^{=}$.

We define the *support* of each subset of inequalities P_i, σ_i, as the subset of $V\setminus V_1$ whose elements are the vertices such that their corresponding variable has a nonzero coefficient in some inequality of P_i, i.e.,

- if $i \in A$: $\sigma_i = \{i\} \cup \{j \in N(i)\setminus V_1\ : j > i\}$, and $\sigma_i = \emptyset$ if $\{j \in N(i)\setminus V_1 : j > i\} = \emptyset$,

- if $i \in B$: $\sigma_i = \{i\} \cap (\{j \in N(i) \ : j > i\} \cup A)$, and $\sigma_i = \emptyset$ if $\{j \in N(i) : j > i\} \cap A = \emptyset$,
- if $i \in C$: $\sigma_i = N[i]$, and $\sigma_i = N[i] \cap \{j \geq i\}$ if the *parent* of i (neighbor of i with a smaller index than i) that we call $p(i)$, verifies that $p(i) \in A$,
- if $i \in D$: $\sigma_i = N[i]$.

We can see that, if $\mathcal{P}(i) \subseteq V$ is the subset of *predecessors* of i (vertices in the path joining i and the root 1), then

$$V \backslash V_1 = \bigcup_{1 \leq i \leq n} \sigma_i \backslash \cup_{j \in \mathcal{P}(i)} \sigma_j$$

(defining $\sigma_{\mathcal{P}(1)} = \emptyset$), and that this union is disjoint. Let F be the subset of vertices i in $V \backslash V_1$ such that the corresponding coordinate $\overline{x}_i$ is fractional, if we call $F_i = F \cap \sigma_i \backslash \cup_{j \in \mathcal{P}(i)} \sigma_j$ then $|F| = \sum_{i=1}^n |F_i|$.

As $\overline{x}$ is an extreme point of $P_{\mathcal{T}}$ then $\operatorname{rank}(P^=) = n$, so there exists a basis $\mathcal{B}$ such that $P_0^= \subseteq \mathcal{B}$ and, if $\mathcal{B}_i = \mathcal{B} \cap P_i^=$, then $|P_0^=| + \sum_{i=1}^n |\mathcal{B}_i| = n$.

Claim 1. If $j \in V$ such that $\deg(j) = 2$ or $\deg(k) = 2$ for some $k \in N(j)$ then $\overline{x}_j$ is fractional (i.e., $\overline{x}_j \neq 0$ or, equivalently, $j \in F$).

Proof of Claim 1. Since $\overline{x}_j + \overline{x}_k \geq 1$ for some $k \in N(j)$ and $\overline{x}_k < 1$ then $0 < \overline{x}_j < 1$. $\Diamond$

As a consequence of Claim 1, if a vertex $i \in \overline{F}$ (i.e., $\overline{x}_i = 0$) then $i \in B_1 \cup C_1 \cup D$, and then $x_i \geq 0$ is a facet-defining inequality, involved in the description of $P_{\mathcal{T}}$, and then we have that $E_i \in P_0^=$. As we are assuming that $n = |\{s \in V \backslash V_1 : \overline{x}_s = 0\}| + |F| = |P_0^=| + |F|$ then, if $\overline{x}$ is an extreme point, we have that

$$|F| = \sum_{i=1}^{n} |\mathcal{B}_i|. \tag{42}$$

Claim 2. If $j \in C$ (i.e., $\deg(j) \geq 3$ and $\deg(k) \leq 2$ for exactly one $k \in N(j)$) then there exist at least two fractional coordinates $\overline{x}_l$ and $\overline{x}_s$ for $l, s \in N[j] \backslash \{k\}$, i.e., $l, s \in F$.

Proof of Claim 2. Since $\overline{x}_j + \sum_{l \in N[j] \backslash \{k\}} \overline{x}_l \geq 1$ at least two terms must be nonzero, hence fractional. $\Diamond$

Claim 3. If $j \in D$ (i.e., $\deg(l) \geq 3$ for $l \in N[j]$) then

1. if $\overline{x}_j = 0$, there exist at least three fractional coordinates $\overline{x}_l$ with $l \in N(j)$. Furthermore, $Q_{jk} \in P^<$ for all $k \in N(j)$. In particular, $|\mathcal{B}_j| \leq 1$ because $P_j^= \subseteq \{M_j\}$.
2. If $\overline{x}_j$ is fractional then j has at least two neighbors with fractional coordinates. Furthermore, if Q_{jk} and Q_{jl} belong to $P^=$ then $E_s \in P_0^=$ for every $s \in N(j) \backslash \{k, l\}$. In particular $Q_{js} \in P^<$ for all $s \in N(j) \backslash \{k, l\}$ and M_j is a linear combination of the facet-defining inequalities Q_{jk}, Q_{jl} and $x_s \geq 0$ in $P^=$, thus, we have that $|\mathcal{B}_j| \leq 2$.

Proof of Claim 3.

1. Since $\overline{x}$ has to verify M_j, then $\sum_{s\in N(j)} \overline{x}_s \geq 2$, and this is only possible if at least three terms are different from zero, because $\overline{x}_i < 1$. Furthermore, if $Q_{jk} \in P^=$ for some $k \in N(j)$, then $\sum_{s\in N(j)\setminus k} \overline{x}_s = 1$ and then, by replacing in M_j, we get $\overline{x}_k = 1$, which we are supposing that is not possible.
2. As $Q_{jk} : \overline{x}_j + \sum_{l\in N(j)\setminus\{k\}} \overline{x}_l \geq 1$ is valid, there exists at least one $l \neq k$ such that $\overline{x}_l \neq 0$, but $\overline{x}$ has to verify Q_{jl} too, and then another coordinate must be fractional. Now suppose that Q_{jk} and Q_{jl} belong to $P^=$, i.e.,

$$\overline{x}_j + \sum_{s\in N(j)\setminus\{k\}} \overline{x}_s = \overline{x}_j + \sum_{s\in N(j)\setminus\{l\}} \overline{x}_s = 1.$$

   Adding the equations we have that,

$$2\overline{x}_j + 2\sum_{s\in N(j)\setminus\{l,k\}} \overline{x}_s + \overline{x}_k + \overline{x}_l = 2\overline{x}_j + \sum_{s\in N(j)} \overline{x}_s + \sum_{s\in N(j)\setminus\{l,k\}} \overline{x}_s = 2,$$

   which is only possible if $M_j \in P^=$ and $\sum_{s\in N(j)\setminus\{l,k\}} \overline{x}_s = 0$. By Claim 1, the vertices s have no neighbors of degree 2, and then $x_s \geq 0$ is facet-defining for $s \in N(j)\setminus\{l,k\}$. Furthermore, we have seen that $Q_{jk} + Q_{jl} = M_j + \sum_{s\in N(j)\setminus\{k,l\}} E_s$. $\Diamond$

Now we will use the previous claims. By grouping the vertices of $\mathcal{T}$ in an appropriate way we shall deduce that $|F| > \sum_{i=1}^{n} |\mathcal{B}_i|$, contradicting (42), and then $\overline{x}$ cannot be an extreme point. We shall analyze $|F_i|$ and $|\mathcal{B}_i|$ when $i = 1$ is the root of $\mathcal{T}$ and, on the other hand, when $i > 1$.

- If $1 \in A$, then $|\mathcal{B}_1| \leq 2 < |F_1| = 3$ (when $1 \in A\setminus A_1$) or $|\mathcal{B}_1| \leq 1 < |F_1| = 2$ if a *child* of 1 (a neighbor which is greater than 1) is a leaf (i.e., $1 \in A_1$). If $i \in A$ and $i > 1$, then $|\mathcal{B}_i| \leq 1 = |F_i| = 1$ (because $|P_i| = 1$), and then $|P_i^=| \leq 1$, and $\sigma_i = \{i, c(i)\}$ where $c(i)$ is the child of i, and $i \in \sigma_{p(i)}$ so $\sigma_i \setminus \sigma_{p(i)} = c(i)$, which belongs to F by Claim 1.

- If $1 \in B$, then $|\mathcal{B}_1| \leq |N(1) \cap A| < |N(1) \cap A| + 1 \leq |F_1|$ (because the vertex 1 and all its neighbors of degree 2 have fractional coordinates by Claim 1) or $|\mathcal{B}_1| = 0 \leq |F_1|$ if $N(1) \cap A = \emptyset$ (i.e., $1 \in B_1$). If $i \in B$ and $i > 1$, then $|\mathcal{B}_i| \leq |N(i) \cap A| - 1 \leq |F_i|$ if $p(i) \in A$, $|\mathcal{B}_i| \leq |N(i) \cap A| < |N(i) \cap A| + 1 \leq |F_i|$ if $p(i) \in B$ (because $p(i) \notin \sigma_{p(i)}$ in this case) and $|\mathcal{B}_i| \leq |N(i) \cap A| \leq |F_i|$ if $p(i) \in C \cup D$.

- If $1 \in C_1$, then $|\mathcal{B}_1| \leq 1$ and by Claim 2, $|F_1| \geq 2$. If $i \in C_1$ and $i > 1$ then $|F_i| \geq 1$.

For $i \in C\setminus C_1$ or $i \in D$ it is not true that $|\mathcal{B}_i| \leq |F_i|$. Nevertheless, the next observation will be useful in order to appropriately group vertices and obtain a similar result.

Claim 4. Let $i_1 \in C \cup D$ such that $|\mathcal{B}_{i_1}| = 2$ and $|F_{i_1}| = 1$, then

1. $i_1 > 1$,
2. $p(i_1) \in C \cup D$,
3. $|\mathcal{B}_{p(i_1)}| \leq 1$,
4. $|\mathcal{B}_{p(i_1)}| + \sum_{1 \leq k \leq r} |\mathcal{B}_{i_k}| \leq |F_{p(i_1)}| + \sum_{1 \leq k \leq r} |F_{i_k}|$ for $\{i_2, ..., i_r\}$ the subset of vertices having the same parent than i_1 (*siblings* of i_1), that belong to $C \cup D$ and verify that $|\mathcal{B}_{i_k}| = 2$ and $|F_{i_k}| = 1$. If $p(i) = 1$, the above inequality is strict.

Proof of Claim 4. If $i_1 = 1 \in (C \backslash C_1) \cup D$ then $|F_{i_1}| \geq 2$, then $i_1 > 1$. If $p(i_1) \in A$ then $i_1 \in C \backslash C_1$ and $|\mathcal{B}_{i_1}| \leq 1$, if $p(i_1) \in B$ then $|F_{i_1}| \geq 2$, then $p(i_1) \in C \cup D$. Notice that, by Claim 2 and Claim 3, $|\mathcal{B}_i| \leq 2$ and $|F_i| \geq 1$ for all $i \in C \cup D$.

Now we prove item 3. We first analyze the case i_1 and $p(i_1) \in C$. As $\overline{x}$ has to verify S_{i_1} with equality and there is only one nonzero variable in F_{i_1} corresponding to the neighbor of i_1 of degree 2, then $\overline{x}$ verifies $x_{p(i_1)} + x_{i_1} = 1$. So, if $S_{p(i_1)} \in \mathcal{B}_{p(i_1)}$, then the variables corresponding to siblings of i_1 with degree at least 3 have to be zero, and then $S_{p(i_1)}$ is a linear combination of S_{i_1} and facets $E_s \in P_0^=$, which is not possible, since $\mathcal{B}$ is a basis. Then $|\mathcal{B}_{p(i_1)}| \leq 1$. The proofs for the rest of the cases are analogous.

Finally, we prove item 4. Suppose that there exists $i_2 \in C \cup D$, a sibling of i_1 such that $|\mathcal{B}_{i_2}| = 2$ and $|F_{i_2}| = 1$. Notice that, in this case, $\overline{x}_{i_1}$ and $\overline{x}_{i_2}$ are fractional, otherwise $|F_{i_1}| > 1$.

- Let $i_1, i_2 \in D$. In this case there is only one *child* of i_1 (a vertex whose parent is i_1), $c(i_1)$, and one child of i_2, $c(i_2)$, such that the corresponding variable in $\overline{x}$ is nonzero. Then $\overline{x}$ has to verify $x_{i_1} + x_{p(i_1)} = 1$, $x_{i_1} + x_{c(i_1)} = 1$, $x_{i_2} + x_{p(i_1)} = 1$ and $x_{i_2} + x_{c(i_2)} = 1$. Then, we have the following possibilities.
  - If $p(i_1) \in D$, $\mathcal{B}_{p(i_1)} = \emptyset$, because $M_{p(i_1)}$, $Q_{p(i_1)i_1}$ and $Q_{p(i_1)i_2}$ are linear combinations of the equations above and facets $E_s \in P_0^=$, and $Q_{p(i_1)k} \notin P^=$ for $k \neq i_1, i_2$. On the other hand, $|F_{p(i_1)}| \geq 2$ (or at least 3 if $p(i_1) = 1$), and then $|\mathcal{B}_{p(i_1)}| + |\mathcal{B}_{i_1}| + |\mathcal{B}_{i_2}| = 4 \leq |F_{p(i_1)}| + |F_{i_1}| + |F_{i_2}|$. If there are more siblings of i_1, $i_j \in C \cup D$, $j = 3, \ldots, r$, such that $|\mathcal{B}_{i_j}| = 2$ and $|F_{i_j}| = 1$ then $|F_{p(i_1)}| \geq r$ (or $|F_{p(i_1)}| \geq 1 + r$ if $p(i_1) = 1$), and then $|\mathcal{B}_{p(i_1)}| + \sum_{1 \leq k \leq r} |\mathcal{B}_{i_k}| = 2r \leq |F_{p(i_1)}| + \sum_{1 \leq k \leq r} |F_{i_k}|$ and the inequality is strict if $p(i_1) = 1$.
  - If $p(i_1) \in C$, $S_{p(i_1)} \notin P^=_{p(i_1)}$ and then $|\mathcal{B}_{p(i_1)}| = 1$ and $|F_{p(i_1)}| \geq 3$ (if $R_{p(i_1)j} \in P^=_{p(i_1)}$) or $|\mathcal{B}_{p(i_1)}| = 0$ and $|F_{p(i_1)}| \geq 2$. If $p(i_1) = 1$, then $|\mathcal{B}_{p(i_1)}| \leq 1$ and $|F_{p(i_1)}| \geq 3$. Then, $|\mathcal{B}_{p(i_1)}| + |\mathcal{B}_{i_1}| + |\mathcal{B}_{i_2}| = 5 \leq |F_{p(i_1)}| + |F_{i_1}| + |F_{i_2}|$ or $|\mathcal{B}_{p(i_1)}| + |\mathcal{B}_{i_1}| + |\mathcal{B}_{i_2}| = 4 \leq |F_{p(i_1)}| + |F_{i_1}| + |F_{i_2}|$.
- Let $i_1, i_2 \in C \backslash C_1$. In this case $\overline{x}$ has to verify $x_{i_1} + x_{p(i_1)} = 1$ and $x_{i_2} + x_{p(i_1)} = 1$, because the only child with a nonzero associated variable is the one of degree 2, and then we can proceed as in the previous item.

- The proof is analogous for $i_1 \in D$ and $i_2 \in C\backslash C_1$. $\Diamond$

Now we consider the following partition of $V\backslash V_1$.

- For every $i_1 \in C \cup D$ such that $|B_{i_1}| = 2$ and $|F_{i_1}| = 1$, denote by $i = p(i_1)$, we define $\mathcal{A}_i := \{i, i_1, ...i_r\}$ where $\{i_2, ..., i_r\}$ is the subset of siblings of i_1 included in $C \cup D$ verifying that $|B_{i_k}| = 2$ and $|F_{i_k}| = 1$.
- For a vertex j such that $j \notin \mathcal{A}_i$ for none of the sets defined in the previous item, we define $\mathcal{A}_j = \{j\}$.

Let s be the number of different sets $\mathcal{A}_i$. By the previous results we have that

$$\sum_{i \in \mathcal{A}_1} |\mathcal{B}_i| < \sum_{i \in \mathcal{A}_1} |F_i| \quad \text{and} \quad \sum_{i \in \mathcal{A}_j} |\mathcal{B}_i| \leq \sum_{i \in \mathcal{A}_j} |F_i| \text{ for } j = 2, \ldots, s.$$

Then

$$|\mathcal{B}| = \sum_{j=1}^{s} \sum_{i \in \mathcal{A}_j} |\mathcal{B}_i| < \sum_{i=1}^{n} |F_i| = |F|$$

contradicting the equality (42), which must be valid if $\overline{x}$ is an extreme point. $\square$

Now we are able to prove the main result.

Proof. Proof of Theorem 4.5.

By Lemma 4.2, the inequalities R_{ij}, S_i, M_i, Q_{ik}, E_i and $x_i \leq 1$ defined in the statement of the theorem are facet-defining for $P_{2dom}(\mathcal{T})$. In order to prove that the description is complete it suffices to show that the extreme points of the polyhedron $P_{\mathcal{T}}$ defined by these inequalities are integral and satisfy the model constraints (4).

Claim 1 The points of $P_{\mathcal{T}}$ verify the inequalities (4).

Proof of Claim 1. Let $x = (x_1, \ldots, x_n) \in P_{\mathcal{T}}$. For a vertex $i \in \{1, \ldots, n\}$, we shall suppose that $C_i \leq 1$ because otherwise the inequality (4) for i is redundant. Then we have the following options,

Case 1 $\deg(i) = 2$ and $C_i = 0$ (i.e., $i \in A$ and does not have neighbors that are leafs): in this case, $N(i) = \{j, k\} \subseteq V\backslash V_1$ and then x verifies $x_i + x_j \geq 1$ and $x_i + x_k \geq 1$. Adding these inequalities we have that $2x_i + x_j + x_k \geq 2$.

Case 2 $\deg(i) = 2$ and $C_i = 1$ (i.e., $i \in A_1$ has exactly one neighbor that is a leaf): in this case, if $N(i)\backslash V_1 = \{j\}$ then x verifies $x_i + x_j \geq 1$. Then, as $x_i \geq 0$, $2x_i + x_j \geq 1$.

Case 3 $\deg(i) \geq 3$ and $\deg(j) \geq 3$ for all $j \in N(i)$ (i.e., $i \in D$): in this case the inequality (4) of the model appears in the description of $P_{\mathcal{T}}$.

Case 4 $\deg(i) \geq 3$ and $\deg(j) \geq 3$ for $j \in N(i)\backslash\{k\}$ and $\deg(k) = 2$ (i.e., $i \in C$): in this case we have that x verifies $x_i + \sum_{j \in N(i)\backslash\{k\}} x_j \geq 1$ and $x_i + x_k \geq 1$, thus $2x_i + \sum_{j \in N(i)} x_j \geq 2$.

Case 5 $\deg(i) \geq 3$ and $\deg(j) \geq 3$ for $j \in N(i)\backslash\{k\}$ and $\deg(k) = 1$ (i.e., $i \in C_1$): in this case we have that $x_i + \sum_{j\in N(i)\backslash\{k\}} x_j \geq 1$ and, as $x_i \geq 0$, $2x_i + \sum_{j\in N(i)\backslash\{k\}} x_j \geq 1$.

Case 6 $\deg(i) \geq 3$, $C_i = 0$ and i has at lest two neighbors of degree 2 (i.e., $i \in B$): in this case $x_i + x_j \geq 1$, $x_j + x_k \geq 1$, then $2x_i + x_j + x_k \geq 2$. Then, as $x_l \geq 0$ for all $l \in N(i)\backslash V_1$ and $\deg(l) \geq 3$, $2x_i + x_j + x_k + \sum_{l\in N(i)\backslash V_1\,:\,\deg(l)\geq 3} x_l \geq 2$. If $C_1 = 1$, $x_i + x_j \geq 1$ and then $2x_i + x_j + \sum_{l\in N(i)\backslash V_1\,:\,\deg(l)\geq 3} x_l \geq 1$. $\diamondsuit$

Claim 2 The polytope $P_\mathcal{T}$ is integral.

Proof of claim 2. We proceed by induction in $n = |V(\mathcal{T})\backslash V_1(\mathcal{T})|$.

1. If $n = 1$, then $\mathcal{T}$ is a star and then $P_\mathcal{T}$ is the segment $0 \leq x \leq 1$ whose extremes are $x = 0$ and $x = 1$, both integer.
2. Let $\overline{x} \in P_\mathcal{T}$ be an extreme point. By the previous lemma, there exists (at least) a vertex i_0 such that $\overline{x}_{i_0} = 1$. For $\{i_1, ..., i_k\} = N(i_0)\backslash V_1$, let $\mathcal{T}_j$ be the connected component of $V(\mathcal{T})\backslash\{i_0\}$ containing i_j and adding i_0 as a leaf. It is clear that $\mathcal{T}_j$ is a tree, and $|V(\mathcal{T}_j)\backslash V_1(\mathcal{T}_j)| < n$ for $j = 1, \cdots, k$. So, by inductive hypothesis, the extreme points of $P_{\mathcal{T}_j}$ are integer. In order to finish the proof we will show that $\pi_k(\overline{x})$ is an extreme point of $\mathcal{T}_j$ for $j = 1, \cdots, k$, where π_k is the projection map of $\mathbb{R}^n$ onto the coordinates corresponding to $V(\mathcal{T}_j)\backslash V_1(\mathcal{T}_j)$. Let $j = 1$ and suppose that $\pi_1(\overline{x}) \in \mathbb{R}^s$ is not an extreme point of $P_{\mathcal{T}_1}$, in this case there exists $0 < \alpha < 1$ and two feasible solutions $y, z \in P_{\mathcal{T}_1}$ such that $\pi_1(\overline{x}) = \alpha y + (1-\alpha)z$. We define $\overline{y} \in \mathbb{R}^n$ such that $\overline{y}_i = y_i$ for $i \in V(\mathcal{T}_1)\backslash V_1(\mathcal{T}_1)$, $\overline{y}_{i_0} = 1$ and $\overline{y}_i = \overline{x}_i$ for the rest of the vertices in $V(\mathcal{T})\backslash V_1(\mathcal{T})$, and in a similar way we define $\overline{z} \in \mathbb{R}^n$. We shall prove that they are both feasible solutions in $P_\mathcal{T}$, i.e., that $\overline{y}$, and then $\overline{z}$, verifies the inequalities defining $P_\mathcal{T}$.
   On the one hand, if I is an inequality involved in the description of $P_\mathcal{T}$ (i.e., $I \in P_\mathcal{T}$), and the support of I (vertices whose corresponding variable has a nonzero coefficient in I) is included in $\overline{V(\mathcal{T}_1)}$, then $\overline{y}$ verifies I since these variables coincide with those of $\overline{x}$, which is a feasible solution in $P_\mathcal{T}$. On the other hand, every inequality of the form $R_{i_0 j}$, S_{i_0}, M_{i_0} or $Q_{i_0 k}$ is verified by $\overline{y}$, because $\overline{y}_{i_0} = 1$, and some of them are involved in the description of $P_\mathcal{T}$ depending on $i_0 \in A, B, C$ or D.
   Finally, observe that $\deg_\mathcal{T}(i) = \deg_{\mathcal{T}_1}(i)$ for every $i \in V(\mathcal{T}_1)\backslash\{i_0\}$, and C_i (the number of degree one neighbors of i) also coincides in $\mathcal{T}$ and $\mathcal{T}_1$ except for $i = i_1$, so, an inequality of the form R_{ij}, S_i, M_i or Q_{ik} corresponding to a vertex $i \in V(\mathcal{T}_1)\backslash V_1(\mathcal{T}_1) - \{i_1\}$ belongs to $P_\mathcal{T}$ if and only if it belongs to $P_{\mathcal{T}_1}$, and therefore $\overline{y}$ verifies the corresponding ones.
   As the variable bounds are valid inequalities, by the previous observations we just have to analyze the validity of the inequalities of the form $R_{i_1 j}$, S_{i_1}, M_{i_1} or $Q_{i_1 k}$:
   (a) If $i_1 \in A$ in $\mathcal{T}$ then $R_{i_1 i_0}$ is valid for $\overline{y}$ and $R_{i_1 j} \in P_{\mathcal{T}_1}$ (and then is valid for $\overline{y}$) because $i_1 \in A$ in $\mathcal{T}_1$. The same occurs if $i_1 \in B$ in $\mathcal{T}$.

(b) If $i_1 \in C$ in $\mathcal{T}$ we have to analyze two cases.

- If $i_0 \notin A$, then $i_1 \in B$ in $\mathcal{T}_1$. The inequalities of i_1 in $P_{\mathcal{T}}$ are S_{i_1} and $R_{i_1 j}$ both are verified by $\overline{y}$ because i_0 belongs to the support of S_{i_1} (then it holds for $\overline{y}$) and, furthermore, $R_{i_1 j} \in P_{\mathcal{T}_1}$.
- If $i_0 \in A$, then $i_1 \in C$ in $\mathcal{T}_1$. The inequalities in $P_{\mathcal{T}}$ are $S_{i_1} \in P_{\mathcal{T}_1}$ and the other inequality is $R_{i_1 i_0}$ which is trivially valid for $\overline{y}$.

(c) If $i_1 \in D$ in $\mathcal{T}$ then $i_1 \in C$ in $\mathcal{T}_1$. The inequalities in $P_{\mathcal{T}}$ are M_{i_1}, $Q_{i_1 i_0}$ and $Q_{i_i j}$ for $j \neq i_0$ neighbor of i_1. Notice that $S_{i_1} = Q_{i_1 i_0} \in P_{\mathcal{T}_1}$, then $S_{i_1} + x_{i_0} + x_{i_1} = M_{i_1}$ is valid if $x_{i_0} = 1$, and then is valid for $\overline{y}$, furthermore i_0 belongs to the support of $Q_{i_i j}$ for $j \neq i_0$, and then they are valid for $\overline{y}$.

We have seen that $\pi_1(\overline{x})$ is an extreme point of $P_{\mathcal{T}_1}$ and the same argument applies for $j = 2, \ldots, k$. So, we have proved that all the coordinates of $\pi_j(\overline{x})$ are integral for $j = 1, \ldots, n$, then the same is valid for $\overline{x}$, and now the proof is complete.

□

5. Conclusions and future work

We have progressed in the study of the polyhedral counterpart of the calculation of the 2-domination and the $\mathcal{P}_3$-hull number of a graph. Our goal now is to investigate potential relationships between the facet-defining inequalities of $P_k(G)$ and those of $P_{k+1}(G)$. Although getting a complete description of $P_{hull}(G)$ is difficult even for trees, we hope that the knowledge of the polytope $P_{2dom}(G)$ and an eventual relationship between the polytopes $P_k(G)$ may allow us to advance the knowledge of the polytope $P_{hull}(G)$.

6. Bibliography

[1] M. van de Vel, Theory of Convex Structures, North-Holland, 1993.

[2] C. Centeno, M. Dourado, J. Szwarcfiter, On the convexity of paths of length two in undirected graphs, Electronic Notes in Discrete Mathematics 32 (2009) 11–18.

[3] C. Centeno, M. Dourado, L. Penso, D. Rautenbach, J. Szwarcfiter, Irreversible conversion of graphs, Theoretical Computer Science 412 (2011) 3693–3700.

[4] J. F. Fink, M. S. Jacobson, n-Domination in graphs, Graph Theory with Application to Algorithms and Computer Science, John Wiley and Sons, New York (1985) 282–300.

[5] E. J. Cockayne, Domination in undirected graphs-a survey, Theory and Applications of Graphs in Americas Bicentennial Year (1978) 141–147.

[6] A. Saxena, Polyhedral studies in domination graph theory, GSIA Working Paper.

[7] M. S. Jacobson, K. Peters, Complexity questions for n-domination and related parameters, Congr. Numer. 68 (1989) 7–22.

[8] T. J. Bean, M. A. Henning, H. C. Swart, On the integrity of distance domination in graphs, Australas. J. Combin. 10 (1994) 29–43.

[9] G. Argiroffo, M. Escalante, M. E. Ugarte, The 2-dominating set polytope of cycles and related graph classes, Electronic Notes in Discrete Mathematics 44 (2013) 269–274.

[10] M. Blaum, J. Marenco, Computing the P_3-hull number of a graph, a polyhedral approach, Discrete Applied Mathematics 255 (2019) 155–166.

[11] G. Nemhauser, L.Wolsey, Integer and combinatorial optimization, Wiley-Interscience, 1988.