Time-Series Analysis Using Third-Order Recurrence Plots

Time-Series Analysis Using Third-Order Recurrence Plots

Martijn Boussé, Philippe Dreesen, Pietro Bonizzi, Joël Karel, Ralf Peeters
Department of Advanced Computing Sciences
Maastricht University, Maastricht, The Netherlands
Email: m.bousse@maastrichtuniversity.nl

1 Introduction

Higher-order recurrence plots may enable us to reveal more structure than what is possible with traditional recurrence plots (RPs). While RPs attempt to detect recurrence relations by pair-wise comparison of time-delayed embeddings, given a time series, higher-order RPs may detect recurrences by comparing *multiple* time-delayed embeddings simultaneously. In this work, we limit ourselves to third-order recurrence plots (TORPs) for time series analysis, as they can still be graphed straightforwardly, and propose future directions.

2 Recurrence Plots & Existing Extensions

Recurrence plots (RPs) allow us to visualize recurrences in state-space trajectories, enabling us to deduce certain properties of the underlying system [1]. A recurrence is a point in the state space where the trajectory passes through at a time i and returns to at a later time j (up to some small error, say ε), i.e., recurrences are points where $\mathbf{x}^{(i)} \approx \mathbf{x}^{(j)}$ with $\mathbf{x}^{(i)}$ the state of the system at time i. More formally, thresholded RPs can be represented by an $N \times N$ matrix $\mathbf{R}$ defined element-wise as

$$r_{ij} = \begin{cases} 1 & \text{if dist}\left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}\right) \leq \varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i, j \leq N$, where dist$(\cdot)$ is some distance metric — usually the Euclidean distance — and ε is the recurrence threshold. Given a time series $\mathbf{u}$, the state can be reconstructed via a time-delayed embedding defined as $\mathbf{x}^{(i)} = \begin{bmatrix} u_i & u_{i+\tau} & \cdots & u_{i+\tau(M-1)} \end{bmatrix}^{\mathrm{T}}$ with embedding dimension M and time delay τ. The matrix $\mathbf{R}$ can then be visualized in a 2D graph by plotting a black dot at coordinate (i, j) if $r_{ij} = 1$.

Higher-order extensions to RPs have been proposed, such as 'generalized' RPs — a.k.a. "recurrence hypercubes" — and 'generalized' recurrence networks [2, 3], that are tailored to spatial data. Recurrence hypercubes are constructed by pairwise comparison (i.e., two orders) of spatial data (i.e., two orders), leading to a fourth-order tensor. A recurrence network represents the states by nodes and the recurrence relations by edges between nodes so that the network topology characterizes the recurrence patterns. Alternatively, multiple traditional RPs, obtained under varying parameter settings, can be 'stacked' into a third-order tensor as proposed in [4].

3 Third-Order Recurrence Plots & Time Series

We propose a straightforward generalization of traditional RPs as follows. We define a *third-order* recurrence plot, or TORP, that can be represented by an $N \times N \times N$ third-order tensor $\mathscr{R}$ defined element-wise, for $1 \leq i, j, k \leq N$, as

$$r_{ijk} = \begin{cases} 1 & \text{if dist}\left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}, \mathbf{x}^{(k)}\right) \leq \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

In contrast to traditional RPs, we require a distance metric that allows us to compare *three* time-delayed embeddings simultaneously instead of just two. There are many ways to do this. A TORP can then be visualized by plotting a colored dot at coordinate (i, j, k) if $r_{ijk} = 1$ in a 3D graph.

TORPs can be used for time series analysis. In Figure 1, we illustrate the unthresholded and thresholded TORP for a simple univariate and noisy time series that features three events. One can deduce the event times from the image. In this example, we used a generalization of the cosine similarity for the distance metric. Our approach can be extended easily to joint, cross, and fuzzy RPs, as well as recurrence quantification analysis, leading to new exciting directions for time-series analysis by means of higher-order RPs.

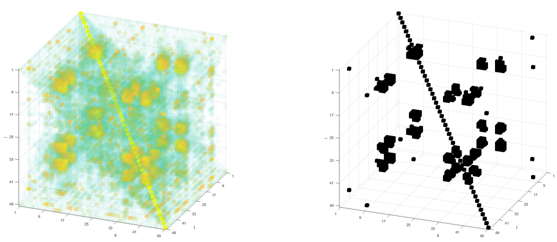

Figure 1: Unthresholded (left) and thresholded (right) TORP.

References

[1] J.-P. Eckmann, et al., "Recurrence Plots of Dynamical Systems," Europhysics Letters 4(9), pp. 973–977, 1987.

[2] N. Marwan, et al., "Generalised recurrence plot analysis for spatial data," Physics Letters, pp. 545–551, 2007.

[3] C.-B. Chen, et al., "Recurrence network modeling and analysis of spatial data," Chaos 28(8), 085714, 2018.

[4] T.D. Pham, et al., "Fuzzy cross recurrence analysis and tensor decomposition of major-depression time-series data," in proc. of DATA'21 (Jordan), pp. 28–34, 2021.