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Herman rings of Blaschke products of degree higher than 3

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Abstract

Henriksen [7] investigated the Blaschke products $F_{a,\lambda} = \lambda z^2 \frac{z-a}{1-\bar{a}z}$ of degree 3. He showed that for an irrational rotation number α satisfying the Brjuno condition there exists a constant $a_0 \geqq 3$ and a continuous function $\lambda(\alpha)$ such that $F_{a,\lambda(a)}$ possesses a Herman ring and also that the modulus $M(a)$ of the Herman ring approaches 0 as a approaches a_0. We can show his results are extended for a certain class of Blaschke products of degree higher than 3. The similar question which is whether $a_0 = 3$ holds in the case of degree 3 is also discussed in the higher degree case. We show that the Herman ring disappears on the way for a certain set of irrational rotation numbers.

1 Introduction

Henriksen[7] investigated the Blaschke products of degree 3 of the form:

$$F_{a,\lambda}(z) = \lambda z^2 \frac{z-a}{1-\bar{a}z}.$$

We define a family **F**, consisting of degree 3 holomorphic mappings of the sphere by

$$\mathbf{F} = \{F_{a,\lambda} : |\lambda| = 1, 0 \leqq \arg\lambda \leqq \pi, a > 3\}.$$

For an irrational number $\theta \in (0, 1)$ let $[a_1, a_2, \cdots]$ be the continued fraction expansion of θ. Then θ is said to be of bounded type if $\{a_n\}$ is a bounded sequence. Let p_n/q_n be the n-th convergent of θ. An irrational number θ is said to be a Brjuno number if the following inequality holds ([3]):

$$\sum_n \frac{\log q_{n+1}}{q_n} < \infty.$$

We denote by $\mathcal{B}$ the set of Brjuno numbers.

Let $\theta \in (0, 1)$ denote an irrational number. Henriksen showed the following results:
(1) There exists $F \in \mathbf{F}$ which possesses a Herman ring with rotation number θ if and only if θ is a Brjuno number.

(2) Suppose θ is a Brjuno number. There exists a constant $a_0 \geqq 3$ and a continuous

$$\lambda: (a_0, \infty) \to S^1 \cap \overline{\mathbb{H}}$$

such that $F_{a', \lambda'} \in \mathbf{F}$ possesses an invariant Herman ring with rotation number θ if and only if (a', λ') lies on the graph of λ, where $\mathbb{H}$ denotes the upper half plane in $\mathbb{C}$.
(3) The modulus $M(a)$ of the invariant Herman ring possessed by $F_{a, \lambda(a)}$ is a strictly increasing function of a; $M(a) \to \infty$ as $a \to \infty$ and $M(a) \to 0$ as $a \to a_0$.
(4) If α is of bounded type then the two boundary components of a Herman ring with rotation number α possessed by a mapping $F \in \mathbf{F}$ are quasicircles each containing a critical point.

In the paper we consider Blaschke products of degree higher than 3. We introduce a family of the Blaschke products of the following form:

$$\mathcal{F}_k = \{f_{\alpha, a}^{(k)} \mid 0 \leqq \alpha \leqq 1, 0 < a < 1/(2k+1)\}.$$

where,

$$f_{\alpha, a}^{(k)}(z) = e^{2\pi i \alpha} z^{k+1} \left(\frac{1 - \bar{a} z}{z - a} \right)^k, \ k \geqq 1$$

When $k = 1$ and a is real, putting $b = \dfrac{1}{a}$ we have $f_{\alpha, a} = e^{2\pi i \alpha} z^2 \dfrac{z-b}{1-bz}$. Therefore we may consider the family $\mathcal{F}_1$ instead of the family $\mathbf{F}$. In this case we showed [5] that for a certain class of irrational numbers α, a_0 is strictly smaller than $1/3$, which corresponds to $1/a_0$ to be exact following the Henriksen's notation although we use the same terminology a_0, so that it implies a_0 is strictly larger than 3.

Since the map in $\mathcal{F}_k$ preserves the unit circle as an invariant curve, when restricted to the unit circle this map can be regarded as a circle map. If we write the lift of the map as $\hat{f}_{\alpha, a}^{(k)}(x)$ such that $\hat{f}_{\alpha, a}^{(k)}(x+1) = \hat{f}_{\alpha, a}^{(k)}(x)$, then we define the rotation number $\rho(f_{\alpha, a}^{(k)})$ of $(f_{\alpha, a}^{(k)})$ as follows:

$$\rho(f_{\alpha, a}^{(k)}) = \lim_{n \to \infty} \frac{(\hat{f}_{\alpha, a}^{(k)})^n(x) - x}{n} \text{ modulo } \mathbb{Z}.$$

We introduce the level set:

$$T_k(\theta) = \{(\alpha, a) \in [0, 1] \times (0, 1/(2k+1)) \mid \rho(f_{\alpha, a}^{(k)}) = \theta\}.$$

Then we have the following diagram of the parameter space (α, a) classified by the rotation number (Figure 1):

If the rotation number $\theta = \rho(f_{\alpha, a}^{(k)})$ is a rational number, $T_k(\theta)$ is a domain, which looks similar to an Arnold tongue as in the case of the Arnold family [2] and if it is an irrational number, $T_k(\theta)$ is just a curve.

Owing to Theorem 4.1 and the remark following the theorem [8] there exists $f_{\alpha, a}^{(k)} \in \mathcal{F}_k$ which possesses a Herman ring with rotation number θ if and only if θ is a Brjuno number (for example see Figure 2).

Under these situations we have the following theorem which corresponds to (2) and (3) of Henriksen's results.

Theorem 1 *Let θ be arbitrarily chosen in $\mathcal{B}$ and $k > 1$. Then there exists a real analytic mapping γ: $(0, \infty) \to T_k(\theta)$ defined by $\gamma(s) = (\alpha(s), a(s))$ such that the map $f^{(k)}_{\alpha(s), a(s)}$ has a Herman ring of modulus s for each $s \in (0, \infty)$ and the following conditions are satisfied:*
(1) *$a(s)$ is strictly decreasing;*
(2) *$a(s) \to 0$ as $s \to \infty$;*
(3) *there exists $a_0 \geqq 0$ such that $a(s) \to a_0$ as $s \to 0$.*
Moreover, $f^{(k)}_{\alpha, a}$ does not possess any Herman ring for any $(\alpha, a) \in T_k(\theta)$ satisfying $a \geqq a_0$.

Next we consider the problem whether $a_0 = 1/(2k+1)$ holds in Theorem 1. First we remark that Yoccoz [16] has proved that for $\theta \in \mathcal{H}$ (see [15] for the definition), every analytic circle diffeomorphism with rotation number θ is analytically linearizable. This theorem is called the Global Conjugacy Theorem (see [12]). Therefore it only remains to treat the case where the rotation number is contained in $\mathcal{B} \backslash \mathcal{H}$.

As we showed for the case $k = 1$ in [5] we can show the similar result using the idea in [4].

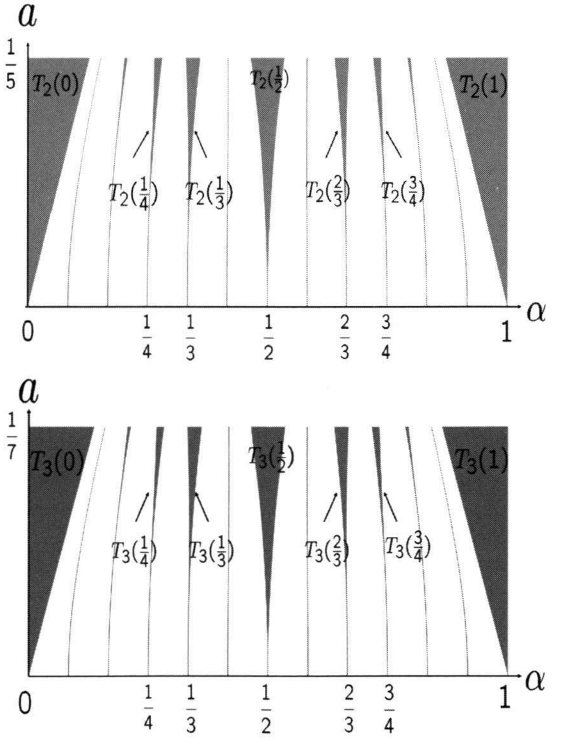

Figure 1 Parameter spaces of the mappings $f^{(2)}_{\alpha, a}$ and $f^{(3)}_{\alpha, a}$

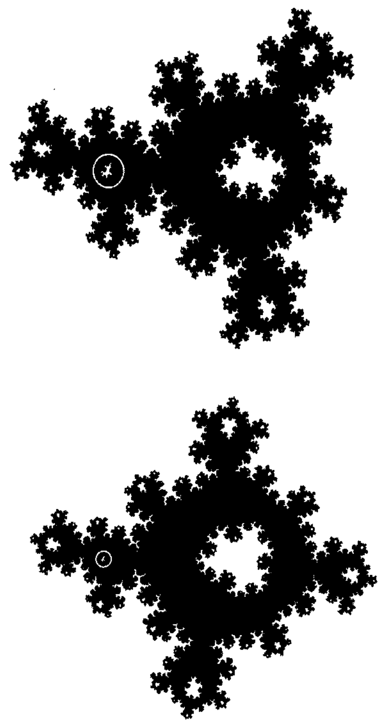

Figure 2 Herman rings of $f^{(2)}_{\sqrt{3}/3,\,0.1}$ and $f^{(3)}_{\sqrt{3}/3,\,0.055}$ with the unit circle

Theorem 2 *Let $k > 1$ be arbitrarily fixed. For any $a_0 \in (0, 1/(2k+1))$, there exists $T_k(\theta)$ for some $\theta \in \mathcal{B}\setminus\mathcal{H}$ such that along $T_k(\theta)$ $f^{(k)}_{\alpha, a}$ has a Herman ring whose boundary consists of two quasicircles with no critical point of $f^{(k)}_{\alpha, a}$ contained for $0 < a < a_0$ and no Herman ring for $a_0 \leqq a \leqq 1/(2k+1)$.*

In particular if the rotation number is of bounded type we have the following result.

Theorem 3 *Let θ be an irrational rotation number of bounded type and $k > 1$. Then for any $(\alpha, a) \in T_k(\theta)$ $f^{(k)}_{\alpha, a}$ has a Herman ring whose boundary consists of two quasicircles and each component of which has a critical point of $f^{(k)}_{\alpha, a}$.*

2 Preliminaries

2.1 Blaschke products of degree $2k+1$ as circle maps

As mentioned above, the rational map belonging to the family $\mathcal{F}_k$ preserves the unit circle as an invariant curve. Therefore, we can regard this map as a circle map.

Lemma 4 *Let $f^{(k)}_{\alpha, a} \in \mathcal{F}_k$. If we consider $f^{(k)}_{\alpha, a}$ as a mapping from the unit disk onto itself, we* *have*

$$f_{\alpha,a}^{(k)}(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha+H_a^{(k)}(\theta))},$$

where

$$H_a^{(k)}(\theta) = -\frac{k}{\pi}\sin^{-1}\left(\frac{a\sin(2\pi\theta)}{\sqrt{1+a^2-2a\cos(2\pi\theta)}}\right).$$

We put

$$\tilde{f}_{\alpha,a}^{(k)}(\theta) = \theta+\alpha+H_a^{(k)}(\theta)(\text{mod } 1).$$

Proof. We remark that the following equation holds:

$$f_{\alpha,a}^{(k)}(e^{2\pi i\theta}) = e^{2\pi i\alpha}e^{2\pi i(k+1)\theta}\left(\frac{1-ae^{2\pi i\theta}}{e^{2\pi i\theta}-a}\right)^k = e^{2\pi i(\alpha+\theta)}\left(\frac{ae^{2\pi i\theta}-1}{ae^{-2\pi i\theta}-1}\right)^k.$$

If we put

$$\frac{ae^{2\pi i\theta}-1}{ae^{-2\pi i\theta}-1} = e^{4\pi i\phi},$$

the remaining part of the proof is the same as the argument in [5].

2.2 Critical points of Blaschke products of degree $2k+1$

A rational map of degree d has $2(d-1)$ critical points.

A calculation shows that $f_{\alpha,a}^{(k)}$ has $4k$ critical points counted with multiplicity for $k \geqq 2$:

0 (of multiplicity k), ∞ (of multiplicity k),

a (of multiplicity $k-1$), $1/a$ (of multiplicity $k-1$),

$$c_\pm \equiv \frac{(2k+1)a^2+1\pm\sqrt{(a^2-1)\{(2k+1)^2a^2-1\}}}{2(k+1)a} \text{(of multiplicity 1)},$$

They satisfy for $0 < a < 1/(2k+1)$

$$0 < a < c_- < c_+ < 1/a < \infty.$$

When $k = 1$, the critical points are

0 (of multiplicity 1), ∞ (of multiplicity 1),

$$c_\pm \equiv \frac{3a^2+1\pm\sqrt{(a^2-1)(9a^2-1)}}{4a} \text{(of multiplicity 1)}.$$

2.3 Characterization of Blaschke products of degree $2k+1$

Now, we consider the following families of Blaschke products:

$$\widetilde{\mathcal{E}}_k = \{f_{\alpha,a}^{(k)} \mid 0 \leqq \alpha \leqq 1, 0 < |a| < 1/(2k+1) \text{ or } 1 < |a|\}$$

and

$$\mathcal{E}_k = \{f_{\alpha, a}^{(k)} \mid 0 \leqq \alpha \leqq 1, 0 < |a| < 1/(2k+1)\}.$$

Obviously, $\mathcal{F}_k \subset \mathcal{E}_k \subset \widetilde{\mathcal{E}_k}$ holds. Here the following characterization holds.

Lemma 5 *A rational map $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree $(2k+1)$ belongs to $\widetilde{\mathcal{E}_k}$ if and only if f has zeros of order $(k+1)$ at the origin and of order k at some point except 0 and ∞ and f maps the unit circle diffeomorphically onto itself. Every map $f \in \widetilde{\mathcal{E}_k}$ commutes with reflection in the unit circle.*

We say that f commutes with reflection in the unit circle if $\tau \circ f = f \circ \tau$ holds, where $\tau(z) = 1/\bar{z}$.

Proof. Since it is easy to show that $f \in \widetilde{\mathcal{E}_k}$ satisfies the conditions, we show the converse. Suppose that f satisfies the conditions of the lemma. We define

$$\gamma_a(z) = \frac{1-\bar{a}z}{z-a}.$$

If we put $a = 1/\bar{b}$, then it is easily seen that

$$\gamma_a(z) = e^{2\pi i\alpha}\frac{1-\bar{b}z}{z-b}$$

for some $\alpha \in [0, 1]$. We use the lemma (see Lemma 15.3 in [11]).

Lemma 6 *A rational map of degree d carries the unit circle into itself if and only if it can be written as a Blaschke product*

$$f(z) = e^{2\pi it}\gamma_{a_1}(z) \cdots \gamma_{a_d}(z) \qquad (1)$$

for some constants $e^{2\pi it} \in \partial D$ and $a_1, \cdots, a_d \in \overline{\mathbb{C}} \setminus \partial D$.

According to this lemma f is written as

$$f(z) = e^{2\pi it}\frac{1-\bar{a}_1 z}{z-a_1}\frac{1-\bar{a}_2 z}{z-a_2}\cdots\frac{1-\bar{a}_{2k+1}z}{z-a_{2k+1}},$$

for some a_i, $1 \leqq i \leqq 2k+1$. As f has a zero point of order $k+1$, taking a_{k+1} as ∞ for $1 \leqq i \leqq k+1$ it follows

$$f(z) = e^{2\pi it}z^{k+1}\frac{1-\bar{a}z}{z-a_1}\frac{1-\bar{a}_2 z}{z-a_2}\cdots\frac{1-\bar{a}_k z}{z-a_k}.$$

As f has a zero of order k at some point, which we write it as $1/\bar{a}$, then it should be $a_1 = a_2 = \cdots = a_k = a$. Thus we have

$$f(z) = e^{2\pi it}z^{k+1}\left(\frac{1-\bar{a}z}{z-a}\right)^k.$$

Since f maps the unit circle diffeomorphically onto itself, putting $z = e^{2\pi i\theta}$ it implies

$$\frac{1}{2\pi i}\frac{d}{d\theta}\log f(e^{2\pi i\theta}) > 0.$$

We have

$$\frac{1}{2\pi i}\frac{d}{d\theta}\log f(e^{2\pi i\theta}) = k+1+\frac{k(|a|^2-1)}{1-(\bar{a}z+a\bar{z})+|a|^2}.$$

Then, solving the above in equality we have $0 < |a| < 1/(2k+1)$ or $1 < |a|$.

If f is in $\widetilde{\mathcal{E}}_k$, a simple calculation shows that f commutes with reflection in the unit circle.

Proposition 7 *Suppose that $|a| > 1$. Then the following hold;*
(1) *if $|z| < 1$, then $\left|(f_{a,a}^{(k)})^n(z)\right| \to 0$ as n tends to ∞,*
(2) *if $|z| > 1$, then $\left|(f_{a,a}^{(k)})^n(z)\right| \to \infty$ as n tends to ∞,*
where $(f_{a,a}^{(k)})^n$ denotes the n-th iterate of $f_{a,a}^{(k)}$.

In this case, the Julia set $J(f_{a,a}^{(k)})$ of $f_{a,a}^{(k)}$ coinsides with the unit circle. Thus, $f_{a,a}^{(k)}$ has no Herman ring.

Proof. (1) When $|a| > 1$, $\left|\frac{1-\bar{a}z}{z-a}\right| < 1$ is equivalent to $|z| < 1$. If $|z| = R < 1$, then

$$\left|f_{a,a}^{(k)}(z)\right| = |z|^{k+1}\left|\frac{1-\bar{a}z}{z-a}\right|^k < R^2.$$

This leads that $\left|(f_{a,a}^{(k)})^n(z)\right| \to 0$ as n tends to ∞.

(2) We can show in the similar way as in the case (1).

The Fatou set of $f_{a,a}^{(k)}$ is just divided into two components, the attractive basin of 0 and that of ∞. Therefore, $J(f_{a,a}^{(k)})$ is the unit circle. Thus there is no possibility of existence of a Herman ring.

2.4 Herman's theorem

We use the following theorem ([6], see [4]):

Theorem 8 *Let f be an orientation preserving C^∞-diffeomorphism of $\mathbb{R}/\mathbb{Z}$ such that no iterate of $f+a$ lifts to the identity map, for any $a \in \mathbb{R}$. Then, there exists $a_0 \in \mathbb{R}$ such that:*
(1) *$f+a_0$ has an irrational rotation number θ;*
(2) *$f+\alpha_0 = \varphi \circ r_\theta$ where φ is a quasisymmetric map of $\mathbb{R}/\mathbb{Z}$ and $\varphi(1) = 1$; and*
(3) *φ is not C^2.*

Here, $r_\theta(x) = x+\theta \pmod 1$. Applying this theorem to the Blaschke product of degree $2k+1$, we have

Corollary 9 *Let $\tilde{f}(x)$ be $x+H_a(x) \pmod 1$ and $\tilde{f}_{a,a}^{(k)}$ as in Lemma 4. Then, for any*

$a_0 \in (0, 1/(2k+1))$ *there exists* $a_0 \in \mathbb{R}$ *such that*

(1) $\tilde{f}^{(k)}_{a_0, a_0}$ *has an irrational rotation number* θ*;*

(2) $\tilde{f}^{(k)}_{a_0, a_0} = \varphi \circ r_\theta$ *where* φ *is a quasisymmetric map of* $\mathbb{R}/\mathbb{Z}$ *and* $\varphi(1) = 1$*; and*

(3) φ *is not* C^2*.*

In particular, $\tilde{f}^{(k)}_{a_0, a_0}$ *has no Herman ring.*

For the proof see the proof of Corollary 8 in [5].

3 Proof of Theorem 1

Let $\theta \in \mathcal{B}$. According to the Theorem 4.1 and the remark in [8] we can suppose that there exists $(\alpha^*, a^*) \in T_k(\theta)$ such that $\tilde{f}^{(k)}_{\alpha^*, a^*}$ has a Herman ring W around the unit circle with the rotation number θ. For simplicity we write $\tilde{f}^{(k)}_{\alpha^*, a^*}$ as f in the sequel. As f commutes with reflection in the unit circle, $\tau W = W$ holds. By the definition of Herman rings there exists a comformal mapping

$$\varphi\colon W \to A_R,$$

where $A_R = \{z \in \mathbb{C} \mid 1/R < |z| < R\}$ for some number $R > 0$ such that

$$R_0 = \varphi \circ f \circ \varphi^{-1},$$

where $R_\theta(z) = e^{2\pi i\theta}z$. We suppose that φ maps the inner and outer boundary of A_R onto the inner and outer boundary of U, respectively and $\varphi(1) > 0$. Then the conjugacy map φ is uniquely determined. Since it is easy to see that $\tau^{-1} \circ \varphi \circ \tau$ is also a conjugacy map and also satisfies that $\tau^{-1} \circ \varphi \circ \tau(1) > 0$ and it maps the inner and outer boundary of A_R onto the inner and outer boundary of U, respectively. Therefore, $\varphi = \tau^{-1} \circ \varphi \circ \tau$ holds. Thus φ commutes with reflection in the unit circle.

The modulus of the Herman ring W is represented as $\dfrac{1}{\pi}\log R$, which we denote by M. Now, for $s \in (0, \infty)$ we put

$$\tilde{s} = s/M.$$

We define a quasiconformal homeomorphism $\phi_s\colon A_R \to A_{R^{\tilde{s}}}$ by

$$\phi_s(z) = z|z|^{\tilde{s}-1}.$$

Obviously ϕ_s preserves the unit circle as an invariant curve. It is easy to show that ϕ_s commutes with reflection in the unit circle. Writing the quasiconformal mapping

$$\Phi_s = \phi_s \circ \varphi\colon W \to A_{R^{\tilde{s}}},$$

Φ_s also commutes with reflection in the unit circle. We define the map $g_s\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as follows:

$$g_s = \begin{cases} f & \text{on } \overline{\mathbb{C}}\setminus W \\ \Phi_s^{-1} \circ R_\theta \circ \Phi_s & \text{on } W \end{cases}.$$

Then g_s commutes with reflection in the unit circle, because f and Φ_s commute with reflection in the unit circle. Since f has zeros of order $k+1$ at the origin and of order k at $1/a$, the definition shows that g_s also has zeros of order $k+1$ at the origin and of order k at some point.

Here we need the following lemma, which is a modified version in Henriksen [7] of Shishikura's fundamental lemma for quasiconformal surgery (Shishikura [13]).

Lemma 10 *Let E denote a domain. Suppose that a quasiregular mapping $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and a quasiconformal mapping $\Phi: E \to E'$ satisfies the following conditions;*

(i) $g(E) \subset E$

(ii) $\Phi \circ g \circ \Phi^{-1}$ *is analytic in E',*

(iii) $g_{\bar{z}} = 0(a.e.)$ *on $\overline{\mathbb{C}}\setminus E$.*

Then there exists a unique quasiconformal mapping ψ fixing 0, 1 and ∞ such that $\psi \circ g \circ \psi^{-1}$ is a rational function. Moreover, $\psi \circ \Phi^{-1}$ is conformal in E' and $\psi_{\bar{z}} = 0$ a.e. on $\overline{\mathbb{C}} - \cup_{n \geqq 0} g^{-n}(E)$. If g and Φ commutes with reflection in the unit circle and E is invariant with respect to reflection in the unit circle, we can suppose that ψ also commutes with reflection in the unit circle.

If we take g_s as g, W as E, $A_{R^{\tilde{s}}}$ as E' and Φ_s as Φ, it is easy to check that the conditions of Lemma 10 hold. Therefore applying the lemma we have a quasiconformal map ψ_s such that

$$G_s = \psi_s \circ g_s \circ \psi_s^{-1}$$

is a rational map and ψ commutes with reflection in the unit circle. According to this lemma,

$$\psi_s \circ \Phi_s^{-1}: A_{R_{\tilde{s}}} \to \psi_s(W)$$

is conformal. Therefore $\psi_s(W)$ is a Herman ring of G_s. Since the modulus of A_{R_s} is $\frac{1}{\pi} \log R_{\tilde{s}}$, the modulus M_s of the Herman ring $\psi_s(W)$ is

$$M_s = \frac{1}{\pi} \tilde{s} \log R = \tilde{s} M = s.$$

It is necessary to check whether G_s belongs to the family $\widetilde{\mathcal{E}_k}$. Since G_s has the same dynamics as f outside of $\psi_s(W)$, G_s also has zeros of order $k+1$ at the origin and of order k at some point. Then according to Lemma 5, G_s belongs to $\widetilde{\mathcal{E}_k}$, i.e., there exist a real number δ, $0 \leqq \delta \leqq 1$ and a complex number β satisfying $0 < |\beta| < 1/(2k+1)$ or $|\beta| > 1$ such that

$$G_s = e^{2\pi i\delta} z^{k+1}\left(\frac{1-\bar{\beta}z}{z-\beta}\right)^k.$$

The case of $|\beta| > 1$ does not occur by Proposition 7, because G_s has the Herman ring $\phi_s(W)$. Thus G_s belongs to $\widetilde{\mathcal{E}_k}$.

In order to belong to the family $\mathcal{F}_k$, β should be a positive real number. Let $R_\xi(z)$ be the rigid rotation. Consider the conjugate of G_s. Then, we have

$$\begin{aligned}\tilde{G}_s(z) &= (R_\xi^{-1} \circ G_s \circ R_\xi)(z) \\ &= e^{2\pi i\delta} z^{k+1}\left(\frac{1-\overline{\beta e^{-2\pi i\xi}}z}{z-\beta e^{-2\pi i\xi}}\right)^k.\end{aligned}$$

If we choose ξ such that $\beta e^{-2\pi i\xi}$ is a positive real number, $\tilde{G}_s$ is the required mapping. The real positive number $\beta e^{-2\pi i\xi}$ and δ depend on s, so that we write them as $a(s)$ and $\alpha(s)$, respectively. Then we can write $\tilde{G}_s$ as

$$G_s = e^{2\pi i\delta} z^{k+1}\left(\frac{1-a(s)z}{z-a(s)}\right)^k.$$

Now we show that $a(s)$ and $\alpha(s)$ are real analytic. As $a(s)$ itself is a critical point of $\tilde{G}_s$, according to the dependence of the parameter of the measurable Riemann mapping theorem ([1]) which is based on Shishikura's fundamental lemma for quasiconformal surgery $a(s) = R_\xi^{-1} \circ \phi_s(a^*)$ is analytic with respect to s, because R_ξ^{-1} carries $\phi_s(a^*)$ on the positive real half line depending analytically on s. Since

$$\tilde{G}_s(1) = (R_\xi^{-1} \circ \phi_s \circ \Phi_s^{-1}) \circ R_\theta \circ (R_\xi^{-1} \circ \phi_s \circ \Phi_s^{-1})^{-1}(1)$$

analytically depends on s,

$$\tilde{G}_s(1) = e^{2\pi i\alpha(s)}$$

shows that $\alpha(s)$ depends analytically on s.

Next we go on to show that the conditions (1), (2) and (3) are satisfied. It needs the following lemma in [9].

Lemma 11 *If the ring domain B separates the pair of points a_1, b_1 from the pair a_2, b_2 and if the spherical distance satisfies $k(a_i, b_i) \geqq \delta > 0$, $i = 1, 2$, then the modulus $M(B)$ of B satisfies*

$$M(B) \leqq \frac{\pi^2}{2\delta^2}.$$

Put $a_1 = 0$, $b_1 = a(s)$ and $a_2 = \infty$, $b_2 = 1/a(s)$ and $B = R_\xi^{-1} \circ \phi_s(W)$. By Lemma 11 if $M(B) > \frac{\pi^2}{2\delta^2}$, then

$$k(a_i, b_i) < \delta$$

This implies that if the modulus s increases then the critical point $a(s)$ tends to 0.

Suppose that $a(s_1) = a(s_2)$. Then $f^{(k)}_{\alpha(s_1), a(s_1)}$ and $f^{(k)}_{\alpha(s_2), a(s_2)}$ have Herman rings with the same rotation number θ. Therefore $(\alpha(s_1), a(s_1)), (\alpha(s_2), a(s_2)) \in T_k(\theta)$. According to Lemma 4.2 in [10], it implies that $\alpha(s_1) = \alpha(s_2)$, so that $f^{(k)}_{\alpha(s_1), a(s_1)} = f^{(k)}_{\alpha(s_2), a(s_2)}$ holds.

On the other hand the modulus of the Herman ring $f^{(k)}_{\alpha(s_i), a(s_i)}$ is s_i for $i = 1, 2$. Hence it implies that $s_1 = s_2$. This shows that $a(s)$ is injective. This completes the proof of (1) and (2).

Since $f^{(k)}_{\alpha, a}$ does not have a Herman ring for $a \geqq 1/(2k+1)$, then $a(s) < 1(2k+1)$ holds. Then owing to the continuity of $a(s)$, we have

$$\lim_{s \to 0} a(s) = a_0 \leqq 1/(2k+1).$$

This implies that (3) holds.

Finally, if there exists $a_1 > a_0$ such that $f^{(k)}_{\alpha_1, a_1}$ has a Herman ring for some α_1 satisfying $(\alpha_1, a_1) \in T_k(\theta)$, then applying the way of constructing Herman rings mentioned above to the Herman ring of $f^{(k)}_{\alpha_1, a_1}$ we can have a Herman ring of $f^{(k)}_{\alpha, a}$ for $\alpha_0 \leqq a \leqq a_1$. This contradicts that a_0 is an upper limit of $a(s)$.

4 Proof of Theorem 2

The way of the proof is almost the same except checking the form of the map as in [5],

According to Yoccoz's theorem and Theorem 4.1 and the remark in [8] we can suppose that $\theta \in \mathcal{B} \backslash \mathcal{H}$.

Let $a_0 \in (0, 1/(2k+1))$ be arbitrarily chosen and let $f^{(k)}_{\alpha_0, a_0}$ be a mapping satisfying the conditions of Corollary 9 and θ the rotation number of $f^{(k)}_{\alpha_0, a_0}$. Since θ is an irrational number, there exists a unique a_0 such that $(\alpha_0, a_0) \in T_k(\theta)$ according to Lemma 4.1 in [10].

For simplicity we put $f = f^{(k)}_{\alpha_0, a_0}$. Let $s \in (1, \infty)$ be arbitrarily fixed. Consider the annulus

$$A_s = \{1/s < |z| < s\}.$$

We denote by $C^{(s)}$ the outer component of the boundary and by $C^{(1/s)}$ the inner component of the boundary of A_s. As f maps the unit circle onto itself, we define the map on $\overline{\mathbb{C}} \backslash A_s$:

$$\tilde{g}_s(z) = \begin{cases} sf(z/s) & \text{on } |z| \geqq s \\ \dfrac{1}{s} f(sz) & \text{on } |z| \leqq \dfrac{1}{s}. \end{cases}$$

We define

$$\Phi_s^{(1/s)}: C^{(1/s)} \to C^{(1/s)}$$

by $s\varphi(z/s)$ and

$$\Phi_s^{(s)}: C^{(s)} \to C^{(s)}$$

by $\frac{1}{s}\varphi(sz)$. According to Corollary 9, $\Phi_s^{(1/s)}$ and $\Phi_s^{(s)}$ are quasisymmetrically conjugate to the rigid rotation r_θ on $C^{(1/s)}$ and $C^{(s)}$, respectively.

According to Lemma 4.1 in [4] we obtain a quasiconformal map Φ_s: $A_s \to A_s$ such that

(1)
$$\Phi_s(z) = \begin{cases} \Phi_s^{(1/s)} & \text{for } z \in C^{(1/s)} \\ \Phi_s^{(s)}(z) & \text{for } z \in C^{(s)} \end{cases},$$

(2) Φ_s commutes with reflection in the unit circle.

Then combining the maps defined on $\overline{\mathbb{C}} \setminus A_s$ and A_s, we define the map g_s: $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as follows:

$$g_s = \begin{cases} \tilde{g}_s & \text{on } \overline{\mathbb{C}} \setminus A_s \\ \Phi_s^{-1} \circ R_\theta \circ \Phi_s & \text{on } A_s \end{cases}.$$

It is easily seen that the maps coincide on the boundaries and finally g_s is continuous on $\overline{\mathbb{C}}$. We remark that g_s commutes with reflection in the unit circle by construction. The map g_s is a quasiconformal map inside the annulus and an analytic map outside the annulus. Therefore, g_s is quasiregular in $\overline{\mathbb{C}}$.

Applying Lemma 10, take g_s constructed before as g, A_s as $E = E'$ and Φ_s as Φ. Then, it is easy to check that all the conditions are satisfied. Then, there exists a quasiconformal mapping ψ_s such that

$$G_s = \psi_s \circ g_s \circ \psi_s^{-1}$$

is a rational mapping.

The construction shows that G_s has the Herman ring $\psi_s(A_s)$. In fact, suppose that G_s has a Herman ring larger than $\psi_s(A_s)$. It implies that the mapping f itself can be analytically extended to some region in either the inner or outer part of the unit circle. Thus f has a Herman ring around the unit circle, because f commutes with reflection in the unit circle, which implies that φ is analytic. This contradicts the assumption.

As the same argument as in Theorem 1 leads to the conclusion that G_s belongs to $\mathcal{E}_k$. Then some appropriate conjugacy map $R_\xi(z)$ conjugate G_s to a map $\tilde{G}_s(z) \in \mathcal{F}_k$. Then taking some $a(s) > 0$ and $0 < \alpha(s) < 1$ we can write $\tilde{G}_s$ as

$$\tilde{G}_s(z) = e^{2\pi i \alpha(s)} z^{k+1} \left(\frac{1 - a(s)z}{z - a(s)} \right)^k.$$

Finally, if there exists $a_1 > a_0$ such that $f_{\alpha_1, a_1}^{(k)}$ has a Herman ring for some α satisfying $(\alpha_1, a_1) \in T_k(\theta)$, then the way of constructing Herman rings using the Herman ring of $f_{\alpha_1, a_1}^{(k)}$ in Theorem 1 can provide a Herman ring for $f_{\alpha_0, a_0}^{(k)}$. Therefore, $f_{\alpha_0, a_0}^{(k)}$ restricted to the unit circle is analytically linearizable. This contradicts that $f_{\alpha_0, a_0}^{(k)}$ restricted to the unit circle is not analytically linearizable.

For $0 < a < 1/(2k+1)$, the critical points of $f_{\alpha, a}^{(k)}$ are 0, ∞, a, $1/a$ and

$$\frac{(2k+1)a^2+1\pm\sqrt{(a^2-1)\{(2k+1)^2a^2-1\}}}{2(k+1)a}.$$

Therefore, the critical points of $f^{(k)}_{\alpha_0, a_0}$ are not on the unit circle. Then by construction, there can be no critical points on the boundaries of the Herman ring of $\tilde{G}_s = f^{(k)}_{\alpha(s), a(s)}$.

As the boundaries are also expressed as the quasiconformal mapping, the boundaries are quasicircles. Once if we obtain a Herman ring of $f^{(k)}_{\alpha, a}$ for some $(\alpha, a) \in T_k(\theta)$, Theorem 1 shows that $f_{\alpha, a}$ has a Herman ring for $0 \leqq a < a_0$ in $T_k(\theta)$.

5 Proof of Theorem 3

If we assume that the rotation number θ is an irrational number of bounded type, then we have the following theorem due to Świątek and Herman ([6] and [14]).

Theorem 12 *Suppose $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a real analytic orientation preserving circle homeomorphism, with finitely many critical points. Then the mapping h conjugating f to a rigid rotation R_θ is quasisymmetric if and only if the rotation number θ is an irrational number of bounded type.*

We consider the Blaschke product.

$$f^{(k)}_{\alpha, 1/(2k+1)} = e^{2\pi i\alpha} z^{k+1}\left(\frac{1-\dfrac{1}{2k+1}z}{z-\dfrac{1}{2k+1}}\right)^k.$$

This map is an orientation preserving real analytic circle homeomorphism and has 1 as a critical point. Thus $f^{(k)}_{\alpha, 1/(2k+1)}$ cannot have a Herman ring around the unit circle (see Figure 3, which may help understanding, although the rotation number θ might not be irrational).

Let θ be an irrational number of bounded type. According to the continuity of the rotation number, there exists $0 \leqq \alpha_0 \leqq 1$ such that $\rho(f^{(k)}_{\alpha_0, 1/(2k+1)}) = \theta$. For simplicity we put $f = f^{(k)}_{\alpha_0, 1/(2k+1)}$. Let $s \in (1, \infty)$ be arbitrarily fixed. Repeating the argument as in the proof of Theorem 2, we have a family of rational maps in $\mathcal{F}_k$ of the following form:

$$\tilde{G}_s(z) = f^{(k)}_{\alpha(s), a(s)} = e^{2\pi i\alpha(s)} z^{k+1}\left(\frac{1-a(s)z}{z-a(s)}\right)^k.$$

For $0 < a < 1/(2k+1)$, the critical points of $f^{(k)}_{\alpha, a}$ are $0, \infty, a, 1/a$ and

$$c_\pm \equiv \frac{(2k+1)a^2+1\pm\sqrt{(a^2-1)\{(2k+1)^2a^2-1\}}}{2(k+1)a}$$

and satisfy

$$0 < a < c_- < c_+ < 1/a < \infty.$$

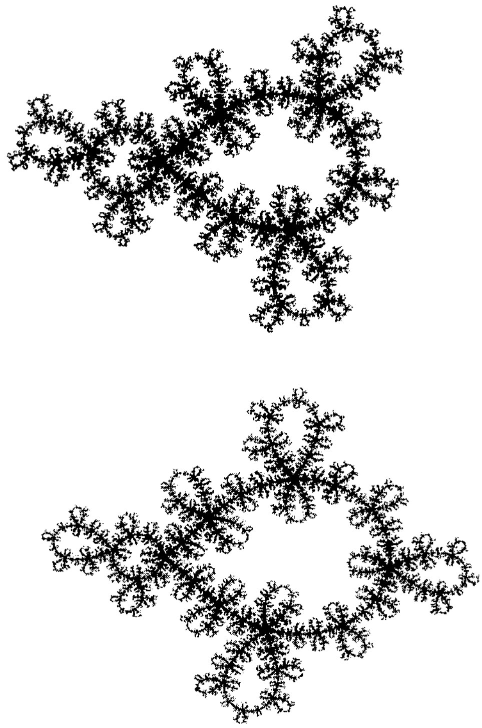

Figure 3 Julia sets of $f^{(2)}_{\sqrt{3}/3,\,1/5}$ and $f^{(3)}_{\sqrt{3}/3,\,1/7}$

As the critical point 1 of $f^{(k)}_{\alpha_0,\,1/(2k+1)}$ is on the unit circle, the construction shows that each component of the boundary of the Herman ring of $\tilde{G}_s$ has a critical point. As each component of the boundary is expressed as the quasiconformal mapping, it is a quasicircle. Since $\tilde{G}_s$ has the same rotation number θ, we have $\tilde{G}_s \in T_k(\theta)$. In this case it is easy to see that $\lim_{s\to 1} a(s) = 1/(2k+1)$. The similar argument as in Theorem 1 shows that $a(s)$ is strictly decreasing and tends to 0 as tends to ∞. Therefore $f^{(k)}_{\alpha,\,a}$ has a Herman ring with the required properties for any $(\alpha, a) \in T_k(\theta)$.

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