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Tung Nguyen
Illinois Wesleyan University

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Tung Nguyen
Research Advisor: Prof. Tian-Xiao He
Department of Mathematics
Student Research Honor
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

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Abstract

In this paper, we present a method to construct orthogonal spline-type scaling functions by using B-spline functions. B-splines have many useful properties such as compactly supported and refinable properties. However, except for the case of order one, B-splines of order greater than one are not orthogonal. To induce the orthogonality while keeping the above properties of B-splines, we multiply a class of polynomial function factors to the masks of the B-splines so that they become the masks of a spline-type orthogonal compactly-supported and refinable scaling functions in L_2. In this paper we establish the existence of this class of polynomial factors and their construction. Hence, the corresponding spline-type wavelets and the decomposition and reconstruction formulas for their Multiresolution Analysis (MRA) are obtained accordingly.

AMS Subject Classification: 42C40, 41A30, 39A70, 65T60

Key Words and Phrases: B-spline, wavelet, MRA.

1 Introduction

Wavelet Analysis is a powerful tool for compressing, processing and analyzing data. It can be applied to extract useful information from numerous types of data, including images and audio signals in Physics, Chemistry and Biology, and high-frequency time series in Economics and Finance.

The history of Wavelet Analysis can be traced back to several school of thoughts that were in isolation originally but then converged into a complete field as of now. Although modern Wavelet Analysis has been around for only 30 years, the earliest work related to Wavelet Analysis is from Alfred Haar in the beginning of 20th century. He found an orthogonal system of functions on [0, 1], which is known nowadays as the simplest basis of the family of wavelet and named after him.

Since the appearance of Haar's work, many other important contributions have been made in the field of Wavelet Analysis. Some of them are the discovery of continuous wavelet transform (CWT) in 1975 by Zweig followed by a more detailed formulation by Goupillaud, Grossmann and Morlet in 1982; the construction orthogonal wavelets with compact support by Daubechies in 1988; the introduction of multiresolution framework by Mallat in 1989; the time-frequency intepretation of CWT by Delprat in 1991 and many others.

In this research, we focus on the construction of compact-support orthogonal wavelets and scaling functions of high orders. The rationale behind this construction is that orthogonality gives the wavelets and scaling functions certain advantages. Among these the most desirable benefit of orthogonality is that it allows for a fast and efficient way to decompose the signals into coefficients as well as to reconstruct the signal from its coefficients. As a result, this property can help speed up and reduce the cost of data processing.

With this motivation, we set out to construct orthogonal wavelets systems that are build upon B-spline, a well-known class of functions. In order to set up the theoretical background, we start with the definitions of Multiresolution Analysis (MRA) and scaling functions.

Definition 1.1 *A Multiresolutional Analysis (MRA) generated by function ϕ consists of a sequence of closed subspaces $V_j, j \in \mathbb{Z}$, of $L_2(\mathbb{R})$ satisfying*

(i) *(nested) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;*

(ii) *(density) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$;*

(iii) *(separation) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;*

(iv) *(scaling) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;*

(v) *(Basis) There exists a function $\phi \in V_0$ such that $\{\phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis or a Riesz basis for V_0.*

The function whose existence asserted in (v) is called a scaling function of the MRA.

A scaling function ϕ must be a function in $L_2(\mathbb{R})$ with $\int \phi \neq 0$. Also, since $\phi \in V_0$ is also in V_1 and $\{\phi_{1,k} := 2^{j/2}\phi(2x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_1, there exists a unique sequence $\{p_k\}_{k=-\infty}^{\infty} \in l_2(\mathbb{Z})$ that describe the two-scale relation of the scaling function

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k), \tag{1.1}$$

i.e., ϕ is of a two-scale refinable property. By taking a Fourier transformation on both sides of (1.1) and denoting the Fourier transformation of ϕ by $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x)e^{-i\xi x}dx$, we have

$$\hat{\phi}(\xi) = P(z)\hat{\phi}(\frac{\xi}{2}), \tag{1.2}$$

where

$$P(z) = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k z^k \text{ and } z = e^{-i\xi/2} \tag{1.3}$$

Here, $P(z)$ is called the mask of the scaling function. Now, regarding the property that $\{\phi(x-k)\}$ must be an orthonormal basis, we have the following characterization theorem (see, for example, Chs. 2, 5 and 7 of [4])

Theorem 1.2 *Suppose the function ϕ satisfies the refinement relation $\phi(x) = \sum_{-\infty}^{\infty} p_k \phi(2x-k)$. Then we have the following necessary and sufficient conditions*

(i) Necessary condition: ϕ forms an orthonormal basis only if $|P(z)|^2 + |P(-z)|^2 = 1$ for $z \in \mathbb{C}$ with $|z| = 1$.
(ii) Sufficient condition: Suppose $P(z)$ satisfies

1. *$P(z) \in C^1$ and is 2π-periodic*
2. *$|P(z)|^2 + |P(-z)|^2 = 1$*
3. *$P(1) = 1$*
4. *$P(z) \neq 0$ for all $\xi \in [-\pi, \pi]$*

Then ϕ forms an orthonormal basis.

Finally, from the scaling function ϕ, we can construct a corresponding wavelet function ψ by the following theorem (see, for example, [1, 7, 9])

Theorem 1.3 *Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA with scaling function ϕ, and ϕ satisfies* *refinement relation in (1.1). Then we can construct a corresponding wavelet function as*

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{p_{1-k}} \phi(2x-k) \tag{1.4}$$

and denote $W_j = span\{\psi(2^j x - k) : k \in \mathbb{Z}\}$.

Furthermore, $W_j \subset V_{j+1}$ *is the orthogonal complement of* V_j *in* V_{j+1}*, and* $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z}\}$ *is an orthonormal basis of* W_j.

In the next sections, we will examine the B-spline functions as a scaling functions and construct an orthogonal spline type scaling functions from them, based on the necessary condition stated in Theorem 1.2. The proof of the main result shown in Section 2 and some properties of the constructed spline type scaling functions are presented in Section 3. The corresponding spline type wavelets and their regularities as well as the decomposition and reconstruction formulas are also given. Finally, we illustrate our construction by using some examples in Section 4.

2 Construction of orthogonal scaling functions from B-splines

In this paper we are interested in a family of B-spline functions, $B_n(x)$, the uniform B-spline with integer knots 0, 1, ..., n+1 defined as follows (see [2, 3]).

Definition 2.1 *The cardinal B-splines with integer knots in* $\mathbb{N}_0$*, denoted by* $B_n(x)$*, is defined inductively by*

$$B_1(x) := \begin{cases} 1 & if\ x \in [0,1] \\ 0 & otherwise, \end{cases}$$

$$\text{and } B_n(x) := (B_{n-1} * B_1)(x) = \int_{-\infty}^{\infty} B_{n-1}(x-t) B_0(t) dt \tag{2.1}$$

From the definition, it is easy to verify that $B_n(x)$ is compactly supported and in $L_2(\mathbb{R})$, which satisfies $\int B_n \neq 0$. By using Fourier transformation, we also have the refinement relation of $B_n(x)$

$$B_n(x) = \sum_{j=0}^{n} \frac{1}{2^{n-1}} \begin{pmatrix} n \\ j \end{pmatrix} B_n(2x-j) \tag{2.2}$$

Next, we would like to see if $B_n(x)$ forms an orthornormal basis or not. We examine the mask $P_n(z)$ of $B_n(x)$. We have

$$P_n(z) = \frac{1}{2}\sum_{j=0}^{n}\frac{1}{2^{n-1}}\left(\begin{array}{c} n \\ j \end{array}\right) z^j = \frac{(1+z)^n}{2^n} = \left(\frac{1+z}{2}\right)^n \qquad (2.3)$$

Thus considering theorem 1.2 we have

$$\begin{aligned} |P_n(z)|^2 + |P_n(-z)|^2 &= \left|\frac{1+z}{2}\right|^{2n} + \left|\frac{1-z}{2}\right|^{2n} \\ &= \left|\frac{1+cos(\xi/2) - isin(\xi/2)}{2}\right|^{2n} + \left|\frac{1-cos(\xi/2)+isin(\xi/2)}{2}\right|^{2n} \\ &= cos^{2n}(\xi/4) + sin^{2n}(\xi/4) \leq cos^2(\xi/4) + sin^2(\xi/4) = 1 \end{aligned}$$

The equality happens only when n=1. Therefore, except for the case of order one (i.e., $n = 1$), $B_n(x)$ are generally not orthogonal (indeed they are Riesz basis). To induce orthogonality, we introduce a class of polynomial function factors $S(z)$. Hence, instead of $B_n(x)$, we consider a scaling function $\phi_n(x)$ with the mask $P_n(z)S_n(z)$, i.e

$$\phi_n(\xi) = P_n(z)S_n(z)\phi_n(\xi/2) \qquad (2.4)$$

where $P_n(z)$ are defined as (2.3). We want to construct $S_n(z)$ such that the shift set of the new scaling function form an orthogonal basis. In other words, we need that $S_n(z)$ satisfy the following condition

$$|P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 = 1 \qquad (2.5)$$

Now we consider $S_n(z)$ of the following type: $S_n(z) = a_1z + a_2z^2 + ... + a_nz^n$, $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$, $i = 1..n$. When $z = 1$, from equation (2.5) we have

$$\begin{aligned} 1 &= |P_n(1)S_n(1)|^2 + |P_n(-1)S_n(-1)|^2 \\ &= |P_n(1)|^2|S_n(1)|^2 + |P_n(-1)|^2|S_n(-1)|^2 \\ &= \left|\frac{1+1}{2}\right|^{2n}|S_n(1)|^2 + \left|\frac{1-1}{2}\right|^{2n}|S_n(-1)|^2 \\ &= |S_n(1)|^2 + 0 = |S_n(1)|^2 \end{aligned}$$

Thus $S_n(1) = \sum_{i=1}^{n} a_i = \pm 1$. From Theorem 1.2 (ii), we further impose a restriction that $\sum_{i=1}^{n} a_i = 1$ in order to ensure the orthogonality of the scaling function.

Next, we set out to find the expressions and constructions of S_n. We have the following Lemma.

Lemma 2.2
Let $S_n(z)$ be defined as above. Then there holds

$$\begin{aligned} &|S_n(z)|^2 \\ &= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_ia_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_ia_{i+2}cos(2\xi/2) + ... \\ &\quad + 2a_1a_ncos((n-1)\xi/2). \end{aligned}$$

Proof. We have

$$
\begin{aligned}
&|S_n(z)|^2 \\
&= |a_1(cos(\xi/2) - isin(\xi/2)) + ... + a_n(cos(n\xi/2) - isin(n\xi/2))|^2 \\
&= |(a_1cos(\xi/2) + ... + a_ncos(n\xi/2)) - i(a_1sin(\xi/2) + ... + a_nsin(n\xi/2))|^2 \\
&= (a_1cos(\xi/2) + ... + a_ncos(n\xi/2))^2 + (a_1sin(\xi/2) + ... + a_nsin(n\xi/2))^2 \\
&= a_1(cos^2(\xi/2) + sin^2(\xi/2)) + ... + a_n(cos^2(n\xi/2) + sin^2(n\xi/2)) \\
&\quad + \sum_{i\neq j} 2a_ia_j(cos(i\xi/2)cos(j\xi/2) + sin(i\xi/2)sin(j\xi/2))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} a_i^2 + \sum_{i\neq j} 2a_ia_jcos((i-j)\xi/2) \\
&= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_ia_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_ia_{i+2}cos(2\xi/2) + ... \\
&\quad + 2a_1a_ncos((n-1)\xi/2)
\end{aligned}
$$

A similar procedure can be applied to find $|S_n(-z)|^2$

$$
\begin{aligned}
&|S_n(-z)|^2 \\
&= \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n-1} a_ia_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_ia_{i+2}cos(2\xi/2) + ... \\
&\quad + (-1)^n 2a_1a_ncos((n-1)\xi/2).
\end{aligned}
$$

From Lemma 2.2, if we write each $cos(k\xi/2)$ as a polynomial of $cos(\xi/2)$, then $|S_n(z)|^2 = Q_n(x)$ where $x = cos(\xi/2)$. Obviously, $Q_n(x)$ has the degree of $n-1$. It is also easy to observe that $|S_n(-z)|^2 = Q_n(-x)$. Now equation (2.5) becomes

$$
\begin{aligned}
1 &= |P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 \\
&= cos^{2n}(\xi/4)Q_n(x) + sin^{2n}(\xi/4)Q_n(-x) \\
&= \left(\frac{1+cos(\xi/2)}{2}\right)^n Q_n(x) + \left(\frac{1-cos(\xi/2)}{2}\right)^n Q_n(-x) \\
&= \left(\frac{1+x}{2}\right)^n Q_n(x) + \left(\frac{1-x}{2}\right)^n Q_n(-x)
\end{aligned}
$$

So finally we get

$$
\left(\frac{1+x}{2}\right)^n Q_n(x) + \left(\frac{1-x}{2}\right)^n Q_n(-x) = 1 \tag{2.6}
$$

As a side note, (2.6) is equivalent to 16.1.7 of [7], but is of quite different form so that we may obtain its solution (2.7) by using Lorentz polynomials, which yields an efficient proof of sufficiency.

Next, to show the existence of $Q(x)$ in the above equation, we make use of the Polynomial extended Euclidean algorithm (see [6]).

Lemma 2.3 Polynomial extended Euclidean algorithm *If a and b are two nonzero polynomials, then the extended Euclidean algorithm produces the unique pair of polynomials (s, t) such that as+bt=gcd(a,b), where* $deg(s) < deg(b) - deg(gcd(a,b))$ *and* $deg(t) < deg(a) - deg(gcd(a,b))$.

We notice that $gcd((\frac{1+x}{2})^n, (\frac{1-x}{2})^n) = 1$, so by Lemma 2.3, there exists uniquely $Q(x)$ and $R(x)$ with degrees less than n such that $(\frac{1+x}{2})^n Q(x) + (\frac{1-x}{2})^n R(x) = 1$. If we replace x by $-x$ in the previous equation, we have $(\frac{1-x}{2})^n Q(-x) + (\frac{1+x}{2})^n R(-x) = 1$. Due to the uniqueness of the algorithm, we conclude that $R(x) = Q(-x)$. So we have showed the existence of a unique $Q(x) = Q_n(x)$ satisfying equation 2.6.

To construct $Q_n(x)$ explicitly, we use the Lorentz polynomials shown in [8, 15] and the following technique.

$$\begin{aligned} 1 &= \left(\frac{1+x}{2} + \frac{1-x}{2}\right)^{2n-1} \\ &= \sum_{i=0}^{2n-1} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \left(\frac{1+x}{2}\right)^{2n-1-i} \left(\frac{1-x}{2}\right)^{i} \\ &= \left(\frac{1+x}{2}\right)^n \left[\sum_{i=0}^{n-1} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^{i}\right] \\ &+ \left(\frac{1-x}{2}\right)^n \left[\sum_{i=0}^{n-1} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \left(\frac{1-x}{2}\right)^{n-1-i} \left(\frac{1+x}{2}\right)^{i}\right], \end{aligned}$$

where the polynomials presenting in the brackets are the Lorentz polynomials. We notice that the degrees of the two polynomials in the brackets are $n-1$, and because $Q_n(x)$ in equation 2.6 is unique, we can conclude that

$$Q_n(x) = \sum_{i=0}^{n-1} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^{i} \tag{2.7}$$

With the construction of $Q_n(x)$, we take a step further by showing the existence of $\sum a_i^2, \sum a_i a_{i+1}, ...$ in Lemma 2.2.

It is well-known that the set $\{1, cos(t), cos(2t), ..., cos((n-1)t)\}$ is linearly independent. As a result, $\{1, cos(\xi/2), cos(2\xi/2), ..., cos((n-1)\xi/2)\}$ forms a basis of the space $P_{n-1}(x) = \{P(x) : x = cos(\xi/2)$ and P is a polynomial of degree less than n. Based on this fact and the existence of $Q_n(x)$ in equation (2.6), it is obvious that the coefficients $\sum a_i^2, \sum a_i a_{i+1}, ...$ in Lemma 2.2 must exist uniquely.

We now establish the main result of this paper.

Theorem 2.1 *Let $P_n(z)$ and $S_n(z)$ be defined as above. Then for $n = 1, 2, \ldots,$ the spline type function $\phi_n(x)$ with the mask $P(z) = P_n(z)S_n(z)$ is a scaling function that generates an orthogonal basis of V_0 in its MRA.*

Proof. From theorem 1.2 (ii), the sufficient conditions for the orthogonality of the scaling function are

1. $P(z) \in C^1$ and is 2π-periodic
2. $|P(z)|^2 + |P(-z)|^2 = 1$
3. $P(1) = 1$
4. $P(z) \neq 0$ for all $\xi \in [-\pi, \pi]$

From the construction of our $P(z) = P_n(z)S_n(z)$, the first two conditions are automatically satisfied. The third condition is also obvious: $P(1) = P_n(1)S_n(1) = \left(\frac{1+1}{2}\right)^n \sum_{i=1}^n a_i = 1$ according to the construction of $S_n(z)$. Now we will prove that the final condition is fulfilled as well.

Indeed, if $\xi \in [-\pi, \pi]$, then firstly we have

$$\begin{aligned}
|P_n(z)| &= |P_n(e^{-i\xi/2})| \\
&= \left|\frac{1+e^{-i\xi/2}}{2}\right|^n = \left|\frac{1+cos(\xi/2) - isin(\xi/2)}{2}\right|^n \\
&= |cos^2(\xi/4) - isin(\xi/4)cos(\xi/4)|^n = \sqrt{cos^4(\xi/4) + cos^2(\xi/4)sin^2(\xi/4)}^n \\
&= |cos(\xi/4)|^n \geq |cos(\pi/4)|^n > 0 \ for \ \xi \in [-\pi, \pi]
\end{aligned}$$

Secondly, from equation (2.7) we have

$$\begin{aligned}
|S_n(z)|^2 = Q_n(x) &= \sum_{i=0}^{n-1} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^i \\
&\geq \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^i \\
&= \left(\frac{1+x}{2} + \frac{1-x}{2}\right)^{n-1} = 1
\end{aligned}$$

Thus $|S_n(z)| \geq 1$ and $|P(z)| = |P_n(z)||S_n(z)| \geq |cos(\pi/4)|^n > 0$. Moreover, from the construction of $\phi_n(x)$, we immediately know they are compactly supported and refinable. We will prove that they are in $L_2(\mathbb{R})$ in next section.

3 Properties of the constructed scaling function

3.1 The scaling function $\phi_n(x)$ is in $L_2(\mathbb{R})$

To show that the newly constructed scaling function $\phi_n(x)$ with mask $P_n(z)S_n(z)$ is in $L_2(\mathbb{Z})$, we make use of the following theorem

Theorem 3.1 [11, 12] *Let ϕ be a scaling function with mask $P_n(z)S_N(z)$ where $P_n(z) = (\frac{1+z}{2})^n$ and $S_n(z) = z^i \sum_{j=0}^k a_j z^j$. Then $\phi \in L_2(\mathbb{R})$ if*

$$(k+1)\sum_{j=0}^{k} a_j^2 < 2^{2n-1} \tag{3.1}$$

.

Proof. In this case, with the $S_n(z)$ we use to construct $\phi_n(x)$, the condition (3.1) becomes

$$n\sum_{j=1}^{n} a_j^2 < 2^{2n-1} \tag{3.2}$$

Recall from Lemma 2.2 that

$$\begin{aligned} Q_n(x) &= Q_n(cos(\xi/2)) \\ &= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1} cos(\xi/2) + ... + 2a_1 a_n cos((n-1)\xi/2) \end{aligned}$$

Taking the integration from 0 to 2π of both sides, we have

$$\begin{aligned} &\int_0^{2\pi} Q_n(x) d\xi \\ &= \int_0^{2\pi} \left(\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1} cos(\xi/2) + ... + 2a_1 a_n cos((n-1)\xi/2) \right) d\xi \\ &= 2\pi \sum_{i=1}^{n} a_i^2 \end{aligned}$$

On the other hands, from the expression of $Q_n(x)$ in (2.7) we have

$$\begin{aligned} \int_0^{2\pi} Q_n(x) d\xi &= \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^{i} d\xi \\ &= \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2}\right)^{i} d\xi \end{aligned}$$

Combining the two equations above we have

$$\sum_{i=1}^{n} a_i^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2}\right)^{i} d\xi \tag{3.3}$$

Now we need to show that the expression on the right hand side of (3.3) is smaller than $\frac{2^{2n-1}}{n}$.

First of all, it is easy to see that for $0 \le i \le n-1$

$$\binom{2n-1}{i} < \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$$

Applying this inequality into (3.3) yields

$$\frac{1}{2\pi}\sum_{i=0}^{n-1}\binom{2n-1}{i}\int_0^{2\pi}\left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i}\left(\frac{1-cos(\xi/2)}{2}\right)^{i}d\xi$$
$$< \frac{1}{4\pi}\binom{2n}{n}\sum_{i=0}^{n-1}\int_0^{2\pi}\left(cos\frac{\xi}{4}\right)^{2(n-1-i)}\left(sin\frac{\xi}{4}\right)^{2i}d\xi$$

$$= \frac{1}{\pi}\binom{2n}{n}\sum_{i=0}^{n-1}\int_0^{\pi/2}(cosx)^{2(n-1-i)}(sinx)^{2i}dx \tag{3.4}$$

for $x = \xi/4$.

Now let

$$A = \sum_{i=0}^{n-1}\int_0^{\pi/2}(cosx)^{2(n-1-i)}(sinx)^{2i}dx$$

We can express A as the sumation of two terms $A = A_1 + A_2$, where

$$A_1 = \sum_{i=0}^{[\frac{n-1}{2}]}\int_0^{\pi/2}(cosx)^{2(n-1-i)}(sinx)^{2i}dx$$
$$A_2 = \sum_{i=[\frac{n-1}{2}]+1}^{n-1}\int_0^{\pi/2}(cosx)^{2(n-1-i)}(sinx)^{2i}dx$$

For $0 \le i \le [\frac{n-1}{2}], 2(n-1-i) > 2i$, the term A_1 becomes

$$A_1 = \sum_{i=0}^{[\frac{n-1}{2}]}\int_0^{\pi/2}(cosx)^{2(n-1-2i)}(sinxcosx)^{2i}dx$$
$$= \sum_{i=0}^{[\frac{n-1}{2}]}\frac{1}{4^i}\int_0^{\pi/2}(cosx)^{2(n-1-2i)}(sin2x)^{2i}dx$$
$$\le \sum_{i=0}^{[\frac{n-1}{2}]}\frac{1}{4^i}\int_0^{\pi/2}(cosx)^{2(n-1-2i)}dx$$

For $[\frac{n-1}{2}]+1 \le i \le n-1, 2(n-1-i) < 2i$, the term A_2 becomes

$$\begin{aligned} A_2 &= \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \int_0^{\pi/2} (sinxcosx)^{2(n-1-i)}(sinx)^{2(2i-n+1)}dx \\ &= \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \frac{1}{4^{n-1-i}} \int_0^{\pi/2} (sinx)^{2(2i-n+1)}(sin2x)^{2(n-1-i)}dx \\ &\le \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \frac{1}{4^{n-1-i}} \int_0^{\pi/2} (sinx)^{2(2i-n+1)}dx \\ &\le \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i} \int_0^{\pi/2} (sinx)^{2(n-1-2i)}dx \end{aligned}$$

Next, we make use of the following well-known result

$$\begin{aligned} \int_0^{\pi/2} (sinx)^{2n}dx = \int_0^{\pi/2} (cosx)^{2n}dx &= \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \\ &= \frac{(2n)!}{[(2n)!!]^2}\frac{\pi}{2} = \frac{\pi}{2}\frac{(2n)!}{4^n(n!)^2} = \frac{\pi}{2}\frac{1}{4^n}\binom{2n}{n} \end{aligned}$$

Next, we try to find the upper bound for $\binom{2n}{n}$. Based on Stirling estimation in [16], we have the following inequalities

$$\binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}} \tag{3.5}$$

and

$$\binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1+\frac{1}{12n-1}\right) \tag{3.6}$$

Using (3.6) on A_1 and A_2 yields

$$\begin{aligned} A = A_1 + A_2 &\le 2\sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i}\frac{1}{\sqrt{3(n-1-2i)+1}}\frac{\pi}{2} \\ &= \pi \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{3^i}\frac{1}{(\frac{4}{3})^i\sqrt{3(n-1-2i)+1}} \end{aligned}$$

We consider the denominator of the fraction, and let

$$f(x) = (\frac{4}{3})^{2x}[3(n-1-2x)+1],\ x \in [0, \frac{n-1}{2}]$$

Surveying the function, we have f(x) attains minimum at 0, or

$$f(x) = (\frac{4}{3})^{2x}[3(n-1-2x)+1] > f(0) = 3(n-1)+1$$

Thus, we have

$$A \leq \pi \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{3^i} \frac{1}{\sqrt{3(n-1)+1}} \leq \frac{\pi}{\sqrt{3n-2}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{3\pi}{2\sqrt{3n-2}} \tag{3.7}$$

Finally, combining (3.3), (3.4), (3.6) and (3.7) we have

$$\sum_{i=1}^{n} a_i^2 \leq \frac{1}{\pi} \begin{pmatrix} 2n \\ n \end{pmatrix} A \leq \frac{1}{\pi} \frac{4^n}{\sqrt{\pi n}} \left(1 + \frac{1}{12n-1}\right) \frac{3\pi}{2\sqrt{3n-2}}$$

For $n \geq 17$, we can easily verify that the right hand side is less than $\frac{2^{2n-1}}{n}$. Using Mathematica for direct calculation of the case $n \leq 16$, we find that the inequality in theorem 3.1 holds. Thus, it holds every interger n, and we have completed the proof, showing that $\phi \in L_2(\mathbb{R})$.

■

As examples, we consider the cases of $n = 1$ and 2. It is easy to find that $S_1(z) = z$ and $\phi_n(x)$ is the Haar function. For $n = 2$,

$$S_2(z) = \frac{1+\sqrt{3}}{2} z + \frac{1-\sqrt{3}}{2} z^2$$

and the corresponding $\phi_2(x)$ is the Daubechies scaling function.

3.2 Refinement relation and corresponding wavelets

Let $M_n(z) := P_n(z)S_n(z) = \frac{1}{2}\sum_{k=0}^{2n} c_k z^k$. Then $M_n(z)$ is the mask of $\phi_n(x)$ and thus we have the refinement equation

$$\phi_n(x) = \sum_{\forall k} c_k \phi_n(2x-k) \tag{3.8}$$

We also have the corresponding wavelets

$$\psi_n(x) = \sum_{\forall k} (-1)^k \overline{c_{1-k}} \phi_n(2x-k) \tag{3.9}$$

One natural question at this point is from the refinement equation to determine explicitly the scaling function $\phi_n(x)$ and thus the wavelet functions $\psi_n(x)$. We propose a method by the iterative procedure described in Theorem 5.23 of [1].

We start with $\phi_n^{(0)}(x) = B_1(x)$. From $\phi_n^{(0)}(x)$ we can construct $\phi_n^{(1)}(x)$ by using the refinement equation (3.5)

$$\phi_n^{(1)}(x) = \sum_{\forall k} c_k \phi_n^{(0)}(2x-k)$$

Now from $\phi_n^{(1)}(x)$ we can continue to construct $\phi_n^{(2)}(x)$ again by the refinement equation. Continuing the process for a certain number of times, we can obtain an approximation of $\phi_n(x)$.

Simple as it seems, this iteration method is not a very good way to construct ϕ as we do not know how many times should we repeat the process to find a good enough approximation for ϕ. For this reason, more direct method is developed and will be addressed in future paper.

From Theorem 2 of [11], we have the following result on the regularities of spline type wavelets $\psi_n(x)$.

Theorem 3.1 *Let $\phi_n(x)$ be constructed as the previous section with the mask shown in Theorem 2.1. Then the corresponding wavelets $\psi_n(x)$ established in (3.6) are in C^{β_n}, where β_n are greater than*

$$n - \frac{1}{2}\log_2\left(n\sum_{j=1}^{n} a_j^2\right).$$

3.3 Decomposition and reconstruction formula

Finally, upon obtaining the new scaling function and corresponding wavelet function, we may establish the corresponding decomposition and reconstruction formulas by using a routine argument. Roughly speaking consider a function $f \in L_2(R)$ with f_i being an approximation in V_j, a space generated by the j^{th} order scaling function. Then

$$f_j = \sum_{k\in Z} < f, \phi_{jk} > \phi_{jk} \tag{3.10}$$

Because we have the orthogonal direct sum decomposition $V_j = V_{j-1} + W_{j-1}$, we can express f_j using bases of V_{j-1} and bases of W_{j-1} as follows

$$f_j = f_{j-1} + g_{j-1} = \sum_{k\in Z} < f, \phi_{j-1,k} > \phi_{j-1,k} + \sum_{k\in Z} < f, \psi_{j-1,k} > \psi_{j-1,k} \tag{3.11}$$

And now we have the following theorem regarding the relationship of these coefficients.

Let $\{V_j : j \in Z\}$ be an MRA with scaling function ϕ satisfying the scaling relation $\phi(x) = \sum_{k\in Z} p_k\phi(2x-k)$, and let W_j be the orthogonal complement of V_j in V_{j+1} with wavelet function ψ. Then the coefficients relative to the different bases in (3.7) and (3.8) satisfy the following decomposition formula

$$< f, \phi_{j-1,l} > = 2^{-1/2}\sum_{k\in Z}\overline{p_{k-2l}} < f, \phi_{jk} >$$
$$< f, \psi_{j-1,l} > = 2^{-1/2}\sum_{k\in Z}(-1)^k p_{1-k+2l} < f, \phi_{jk} >$$

and the reconstruction formula

$$< f, \phi_{jl} >= 2^{-1/2} \sum_{k\in Z} p_{l-2k} < f, \phi_{j-1,k} > +2^{-1/2} \sum_{k\in Z} (-1)^k \overline{p_{1-l+2k}} < f, \phi_{j-1,k} >$$

4 Example for $n = 3$

Besides the examples of the cases $n = 1$ and 2 shown at the end of Section 3.1, in this section we give an example on the case $n = 3$. According to the previous sections, we will construct an orthogonal scaling function $\phi_3(x)$ from the third order B-spline function $B_3(x)$.

In order to construct the function $\phi_3(x)$, we start with its mask $P_3(z)S_3(z)$, where $P_3(z) = (\frac{1+z}{2})^3$ is the mask of the third order B-spline. Let $S_3(z) = a_1z+a_2z^2+a_3z^3$, then by Lemma 2.2, we have

$$\begin{aligned} Q_3(x) = |S_3(z)|^2 &= (a_1^2 + a_2^2 + a_3^2) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 2a_1a_3cos(\xi) \\ &= (a_1^2 + a_2^2 + a_3^2) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 2a_1a_3(2cos^2(\xi/2) - 1) \\ &= (a_1^2 + a_2^2 + a_3^2 - 2a_1a_3) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 4a_1a_3cos^2(\xi/2) \\ &= (a_1^2 + a_2^2 + a_3^2 - 2a_1a_3) + 2(a_1a_2 + a_2a_3)x + 4a_1a_3x^2 \end{aligned}$$

where $x = cos(\xi/2)$ and $z = e^{-i\xi/2}$.

On the other hand, by equation (2.7) we have

$$\begin{aligned} Q_3(x) &= \sum_{i=0}^{2} \left(\begin{array}{c} 5 \\ i \end{array} \right) \left(\frac{1+x}{2}\right)^{2-i} \left(\frac{1-x}{2}\right)^{i} \\ &= \left(\frac{1+x}{2}\right)^2 + 5\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right) + 10\left(\frac{1-x}{2}\right)^2 \\ &= \frac{3}{2}x^2 - \frac{9}{2}x + 4 \end{aligned}$$

Thus, from the above equations, we have the following system of equations

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 - 2a_1a_3 = 4 \\ 2(a_1a_2 + a_2a_3) = -\frac{9}{2} \\ 4a_1a_3 = \frac{3}{2} \end{cases} \tag{4.1}$$

Simplify (4.1) we get

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 = \frac{19}{4} \\ a_1a_2 + a_2a_3 = -\frac{9}{4} \\ a_1a_3 = \frac{3}{8} \end{cases} \tag{4.2}$$

From this system, we have $(a_1+a_2+a_3)^2 = a_1^2+a_2^2+a_3^2+2(a_1a_2+a_2a_3+a_1a_3) = 1$. Without loss of generality, consider the case $a_1 + a_2 + a_3 = 1$. Combining this with

$a_1a_2 + a_2a_3 = -\frac{9}{4}$ and $a_1a_3 = \frac{3}{8}$ we have the following solution

$$\begin{cases} a_1 = \frac{1}{4}\left(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) \\ a_2 = \frac{1}{2}\left(1-\sqrt{10}\right) \\ a_3 = \frac{1}{4}\left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right) \end{cases} \tag{4.3}$$

We verify the condition for ϕ_3 to be in $L_2(\mathbb{R})$

$$a_1^2 + a_2^2 + a_3^2 = \frac{19}{4} < \frac{32}{3} = \frac{2^{2\cdot 3-1}}{3}$$

Thus ϕ_3 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_3 explicitly. It has the mask

$$\begin{aligned} P_3(z)S_3(z) &= \left(\frac{1+z}{2}\right)^3 (a_1z + a_2z^2 + a_3z^3) \\ &= 0.0249x - 0.0604x^2 - 0.095x^3 + 0.325x^4 + 0.571x^5 + 0.2352x^6 \end{aligned}$$

By (1.1), (1.2) and (1.3) we have the refinement equation

$$\begin{aligned} \phi_3(x) &= 0.0498\phi_3(2x-1) - 0.121\phi_3(2x-2) - 0.191\phi_3(2x-3) \\ &+ 0.650\phi_3(2x-4) + 1.141\phi_3(2x-5) + 0.4705\phi_3(2x-6) \end{aligned} \tag{4.4}$$

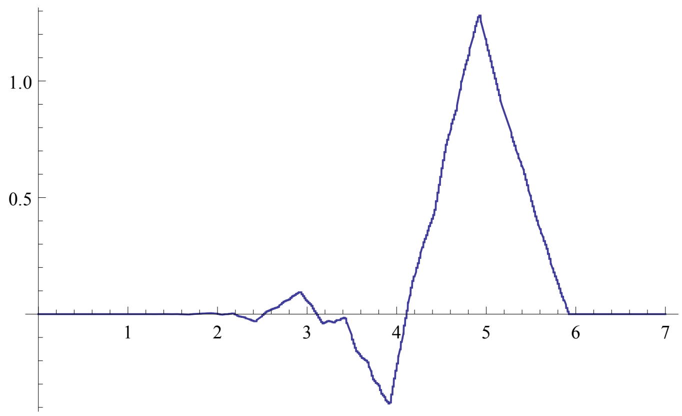
The corresponding spline type wavelet $\psi_3(x)$ can be expressed as

$$\begin{aligned} \psi_3(x) &= 0.0498\phi_3(2x) + 0.121\phi_3(2x+1) - 0.191\phi_3(2x+2) \\ &- 0.650\phi_3(2x+3) + 1.141\phi_3(2x+4) - 0.4705\phi_3(2x+5), \end{aligned}$$

which is in C^{β_3} and

$$[[\beta_3 > 3 - \frac{1}{2}\log_2(57/4).]]$$

Using the iteration technique proposed in 3.2, we have the following graph of the scaling function after 5 iterations.



5 Example for $n = 4$

In this section, we give another example on the case $n = 4$. Again, we construct an orthogonal scaling function $\phi_4(x)$ from the forth order B-spine $B_4(x)$ with the mask $P_4(z) = (\frac{1+z}{2})^4$.

Now we examine the mask $P_4(z)S_4(z)$ of $\phi_4(x)$ where we define $S_4(z) = a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$. By Lemma 2.2, we have

$$\begin{aligned}
Q_4(x) = |S_4(z)|^2 &= (a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)cos(\xi/2) \\
&+ 2(a_1a_3 + a_2a_4)cos(\xi) + 2a_1a_4cos(3\xi/2) \\
&= (a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)cos(\xi/2) \\
&+ 2(a_1a_3 + a_2a_4)(2cos^2(\xi/2) - 1) + 2a_1a_4(4cos^3(\xi/2) - 3cos(\xi/2)) \\
&= (a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4) \\
&+ (2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4)cos(\xi/2) \\
&+ (4a_1a_3 + 4a_2a_4)cos^2(\xi/2) + 8a_1a_4cos^3(\xi/2) \\
&= (a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4) \\
&+ (2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4)x \\
&+ (4a_1a_3 + 4a_2a_4)x^2 + 8a_1a_4x^3
\end{aligned}$$

where $x = cos(\xi/2)$ and $z = e^{-i\xi/2}$.

Using equation (2.7) we get another expression for $Q_4(x)$

$$\begin{aligned}
Q_4(x) &= \sum_{i=0}^{3} \binom{7}{i} \left(\frac{1+x}{2}\right)^{3-i} \left(\frac{1-x}{2}\right)^{i} \\
&= \left(\frac{1+x}{2}\right)^3 + 7\left(\frac{1+x}{2}\right)^2\left(\frac{1-x}{2}\right) + 21\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)^2 + 35\left(\frac{1-x}{2}\right)^3 \\
&= 8 - \frac{29}{2}x + 10x^2 - \frac{5}{2}x^3
\end{aligned}$$

Thus, from the above equations, we have the following system of equations

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4 = 8 \\ 2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4 = -\frac{29}{2} \\ 4a_1a_3 + 4a_2a_4 = 10 \\ 8a_1a_4 = -\frac{5}{2} \end{cases} \tag{5.1}$$

Simplify (5.1) we have the following system

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 = 13 \\ a_1a_2 + a_2a_3 + a_3a_4 = -\frac{131}{16} \\ a_1a_3 + a_2a_4 = \frac{5}{2} \\ a_1a_4 = -\frac{5}{16} \end{cases} \tag{5.2}$$

Solving for this system of equations yields 8 solutions. One of the numerical solutions is

$$\begin{cases} a_1 = 2.6064 \\ a_2 = -2.3381 \\ a_3 = 0.8516 \\ a_4 = -0.1199 \end{cases} \tag{5.3}$$

We verify the condition for ϕ_4 to be in $L_2(\mathbb{R})$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 13 < 32 = \frac{2^{2\cdot 4-1}}{4}$$

Thus ϕ_4 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_4 explicitly. It has the mask

$$\begin{aligned} P_4(z)S_4(z) &= \left(\frac{1+z}{2}\right)^4 (a_1z + a_2z^2 + a_3z^3 + a_4z^4) \\ &= 0.1629z + 0.5055z^2 + 0.4461z^3 - 0.0198z^4 - 0.1323z^5 + 0.0218z^6 \\ &+ 0.0233z^7 - 0.0075z^8 \end{aligned}$$

By (1.1), (1.2) and (1.3) we have the refinement equation

$$\begin{aligned} \phi_4(x) &= 0.3258\phi_4(2x-1) + 1.011\phi_4(2x-2) + 0.8922\phi_4(2x-3) - 0.0396\phi_4(2x-4) \\ &- 0.2646\phi_4(2x-5) + 0.0436\phi_4(2x-6) + 0.0466\phi_4(2x-7) - 0.015\phi_4(2x-8) \end{aligned} \tag{5.4}$$

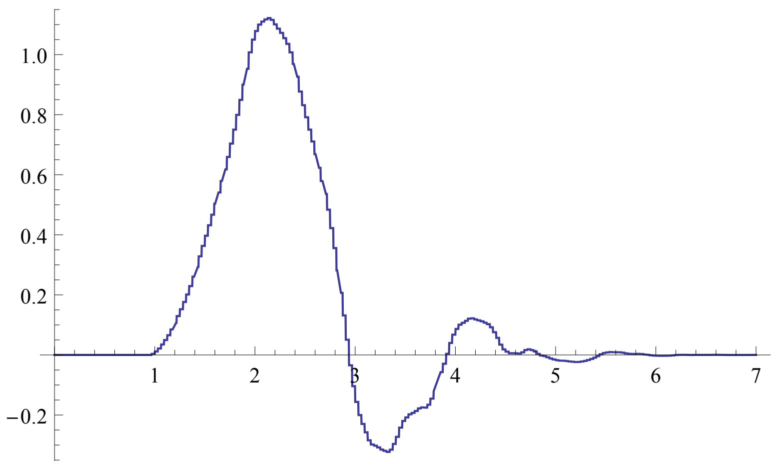
The corresponding spline type wavelet $\psi_3(x)$ can be expressed as

$$\begin{aligned} \psi_4(x) &= 0.3258\phi_4(2x) - 1.011\phi_4(2x+1) - 0.8922\phi_4(2x+2) + 0.0396\phi_4(2x+3) \\ &+ 0.2646\phi_4(2x+4) - 0.0436\phi_4(2x+5) - 0.0466\phi_4(2x+6) + 0.015\phi_4(2x+7) \end{aligned}$$

which is in C^{β_4} and

$$\beta_4 > 4 - \frac{1}{2}\log_2(52).$$

Using the iteration technique proposed in 3.2, we have the following graph of the scaling function after 5 iterations.



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