2008

Visualizing a Fourth Dimension: Hypercubic Resistor Networks

Andrew J. Nelson '08
Illinois Wesleyan University

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by

Andrew J. Nelson

Submitted to the Department of Physics
in partial fulfillment of the requirements for Research Honors

at

Illinois Wesleyan University

November 2007

Abstract

A booming field in physics research today is the search for extra dimensions. This is something that has been thought about and discussed in both the scientific and non-scientific world for a long time. Many physicists are currently attempting to answer the question: "is our world *really* four dimensional?" The purpose of this research, however, is not to answer that question.

The purpose of this work is to help *reveal* four-dimensional artifacts in our perceived three-dimensional world in order to help a student, even a non-physicist, to understand and visualize how the extra spatial dimensionality, if present, might reveal itself in measurements. To that end, models of non-trivial four-dimensional objects have been constructed that have consequences large enough to be easily measured and understood in an intuitive fashion. In building and analyzing data from two, three, and four-dimensional model systems with non-trivial interactions, large and conceptually transparent consequences of extra spatial dimensions have been discovered.

Contents

I. Introduction

Various versions of String Theory have postulated that there may be nine, eleven, or twelve different spatial dimensions in the world[i]. Each of these extra dimensions was originally thought to be compact[ii]. This notion of compact dimensionality refers to the idea that dimensions past the three immediately visible around us are wrapped upon each other. This creates an extra dimensional artifact that cannot be seen[i]. This idea can be expressed through the analogy of an ant walking along a wire. To the ant, it appears that the surface is flat, but to the larger human observer it is obvious that the ant is in fact spiraling around the wire. Contrary to this conventional belief, it has been recently proposed that extra dimensions may, in fact, be very large, or possibly even of infinite size[iii]. These extra dimensions would have consequences large enough to be observed and be radically different than those discussed through string theory.

In 1884, more than 20 years before Einstein's published papers, Edwin A. Abbott wrote a short book entitled, <u>Flatland: A Romance of Many Dimensions</u>. In this book, Abbott addresses the idea of a person, A. Square, who lives in a two-dimensional world and ventures into worlds of both higher and lower dimensions. After viewing a world of three dimensions, the traveler begins to wonder if it would be possible to have a world of four dimensions. The traveler realizes that while in a world of two dimensions he believed a line was a plane, but it was in fact a three dimensional object with height. Because of this loss of perception, it is entirely possible that while in three dimensions a solid object could have a fourth, unknown dimension[iv].

When referring to four dimensions, most people would think of what physicists refer to as three plus one dimensions, indicating three spatial dimensions and a fourth dimension of time. While this idea is usually attributed to Einstein, it was first introduced to the reading public through science fiction about 10 years prior to the 1905 publishing of his paper. In his 1895 novel <u>Time Machine</u>, H.G. Wells had already put forth the idea that there were three dimensions of space while time could be considered a fourth dimension[v]. In addition to the proposition that time was a fourth dimension, he also posited that there may be a fourth spatial dimension, perpendicular, in some sense of the word, to the three that we commonly know [Wells, 7].

A large gap in time is bridged between the ideas of Abbot, Wells, and Einstein and the latest theories about dimensionality today. Lisa Randall, a theoretical physicist from Harvard, is leading the way with the most cutting edge research about dimensionality. She has developed new and groundbreaking ideas about the possibility of four or more dimensions[ii].

The type II Randall Sundrum braneworld gravity model is one of her many important accomplishments. This model proposes that our universe is simply a membrane suspended within a larger universe[vi]. A more simple way to think of this is to conceptualize our universe as a single strand of seaweed floating through the ocean. In this model, the gateway to the other dimension is found through black holes that are dispersed throughout our own universe, even within our own galaxy. Within the constraints of general relativity in three plus one dimensions, these black holes should have long since evaporated[vi]. Therefore, the theories that are being presented by Randall and her colleagues are very groundbreaking, as they do not follow some of the most basic principles of Physics. Experimental methods are now being formulated for testing the validity of this theory[vii,viii,ix].

The goal of the research presented in this paper, however, has not been to answer the question "is our world *really* four dimensional?" The aim of our project is, instead, more pedagogical in nature. We have attempted to find a way to *reveal* four-dimensional artifacts in our perceived three-dimensional world and to help a student, even a non-physicist, understand and visualize how the extra spatial dimensionality, if present, might reveal itself in measurements. To that end, we have constructed non-trivial four-dimensional model objects that have consequences large enough to be easily measured and also to be understood in an intuitive fashion. By comparing the measured properties of two, three, and four dimensional model systems with non-trivial interactions, and by analyzing that data, we have discerned what seem to us to be large and conceptually transparent consequences of extra spatial dimensions.

One of the most important concepts that must be established before the entirety of this work may understood is the representation of one, two, three, and four dimensional systems. A simple but powerful way to think of the progression from one to four dimensions is through the words of Abbott. He articulates the idea that a point of zero dimensions, when moved to the side, leave a trail of points, making a one-dimensional line. When this line is then pulled to the side and leaves a "trail" of other lines, thus creating a two-dimensional square. Furthermore, if the square is pulled upward, leaving a trail of connected squares, it creates a space-filling cube – a three-dimensional object. Finally, *if* the cube could be moved in some fourth direction, thereby leaving a "trail" of cubes, one would have created a four-dimensional object[v]. Of course, it is important to note that for a *continuous* (i.e. space filling) cube, there is no way to implement this in practice. But a slight variation of this idea is quite fruitful.

Instead of pulling, dragging or moving a point continuously along a given direction, one can think of discrete points (nodes) that are displaced and connected to neighboring discrete points (by nearest neighbor bonds) to produce a one dimensional lattice. Mathematicians have long conceptualized these structures and referred to them as hypercubic lattices. For example, a four dimensional hypercube is defined as a set of nodes identified by four indices (x,y,z,w). Each node (x,y,z,w) is then connected by a bond to its neighbor (x+1,y,z,w).

In our physical realization of this hypercube, each bond consists of a fixed linear lumped electrical element of impedance Z_o (or conductance $S_o = 1/Z_o$). Therefore, the geometric distance between two nn (nearest neighbor) nodes is completely irrelevant. The direct interaction between two nn nodes is determined entirely by the value of the fixed electrical element connecting them. This allows for the construction of models of discrete nature as opposed to continuous models. This characteristic will prove exceedingly important in higher dimensional construction.

II. Construction

The construction of two-dimensional square lattices and three-dimensional cubes was completed in order to verify the results of the work of earlier groups and also to validate our technique for fabrication and experimentation. While two-dimensional square networks have been studied both experimentally[viii] and theoretically[x,xi] and three-dimensional cubes have been theoretically explored[x,xi], to the best of our knowledge, no work has been carried out concerning four-dimensional resistive networks. It is therefore crucial that we verify our technique of construction and measurement before beginning.

It is important that all circuit elements used in the construction of the networks are uniform. A first attempt was made to construct a two dimensional model using one kilo ohm resistors with a tolerance of five percent. However, after the construction of the model and brief initial measurements, it was realized that five percent tolerance resistors yield results with too much variance. Therefore it was decided that all resistors used in testing would be of one percent tolerance and rated at one kilo ohm.

2.1 Two and Three-Dimensional Networks

In order to construct the two-dimensional square network, resistors are laid out, 220 in total, into a grid like fashion. Each resistor is soldered to its neighbor to the left or right. This creates what is essentially a mesh of resistors, as can be seen in Figure 1 on the following page. (It is important to note that in the interest of better illustrating the structure, the figure only shows a small portion of what was constructed.) The final network constructed was ten rows wide by 11 rows high, consisting of 10 resistors on each edge. This is heretofore referred to as a ten by ten lattice.

The construction of a cube of five resistors on each side begins with the fabrication of six two-dimensional resistor networks, each five resistors by five. An ordered triple, (x,y,z) is used to locate each node for simplicity. The z digit refers to which two-dimensional lattice the node lies in. In order to create the continuous cube, each node is connected to a corresponding node in a different two-dimensional lattice through a process called "connected stacking." Using an ordered triple as an example, the node at (1,1,1) would be connected to the node at (1,1,2). These connections are created by soldering resistors between the neighboring nodes. As can be seen in Figure 2 (following page), this builds up a cube of resistors that is symmetrical in every direction.

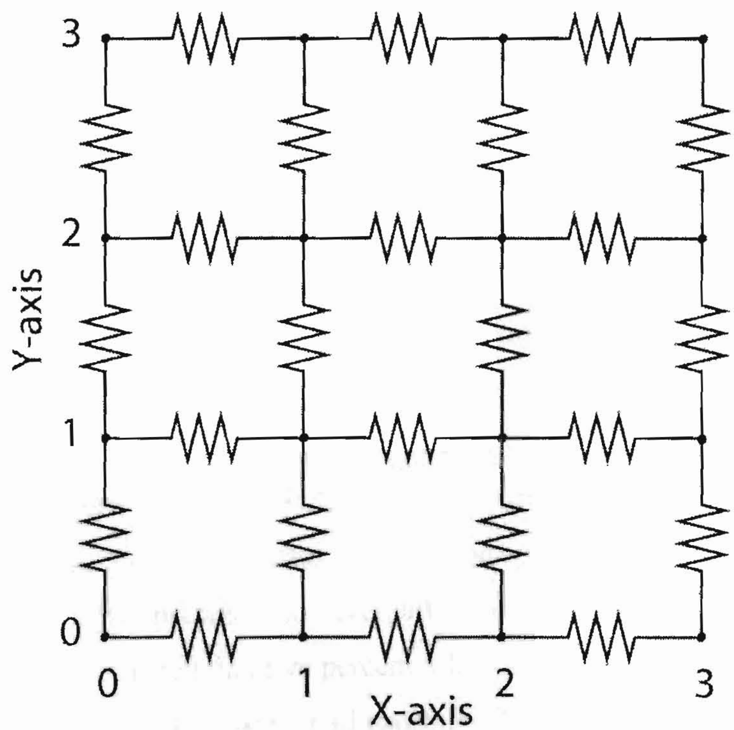

Figure 1: Illustration of three by three two-dimensional model.

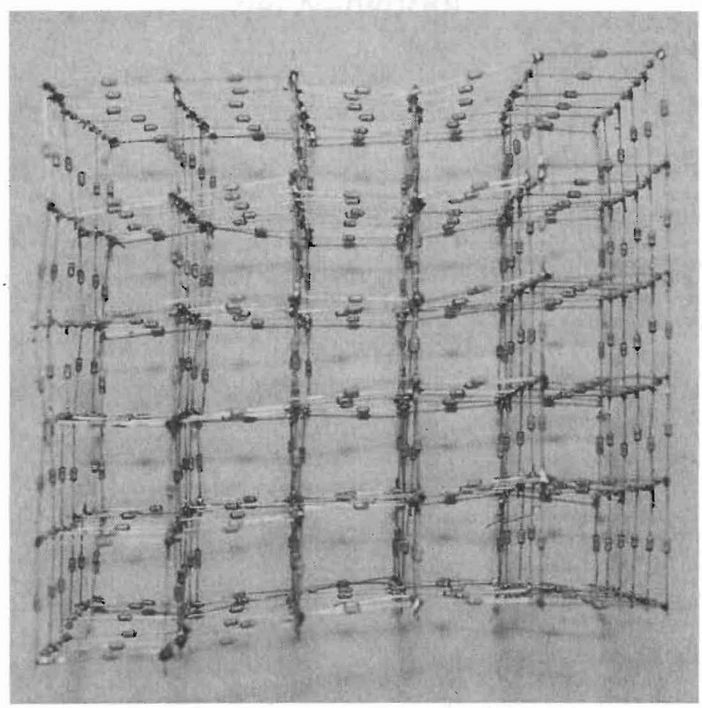

Figure 2: Three-dimensional model as viewed from above

2.2: "Flattening Out"

An important technique in the construction of devices of higher dimensionality is the ability to "flatten" higher dimensions into lower dimensional spaces. This allows the ability to create discrete objects as discussed above. Because we are using linear circuit elements that do not depend on relative distance to one another, it is possible to take a continuous structure, such as a cube of three dimensions, and make it a discrete structure with the same properties in two-dimensions. We refer to this process as "flattening out."

To create a flattened out cube, multiple square devices of five by five resistors are built. These are then laid out next to one another and the corresponding nodes are connected using resistors. As described above, a node in position (1,1,1) is connected to the neighboring node in position (1,1,2). However, because this is a discrete system, it is not necessary to create the three-dimensional model using the method of connected stacking. Instead, the square lattices are placed next to one another and the nodes are connected using lengths of 22 gauge wire with a resistor spliced in the middle, as seen in Figure 3. The resistor is soldered to a wire in order to extend the distance between the nodes. This is possible because the object is discrete, not continuous. (It is important to note that in the interest of simplicity this figure only illustrates the connections for all values of x where y=3. A similar set of connections would be made for all values of y. Also, the cube has been scaled down to three by three by three in the interest of overall size.)

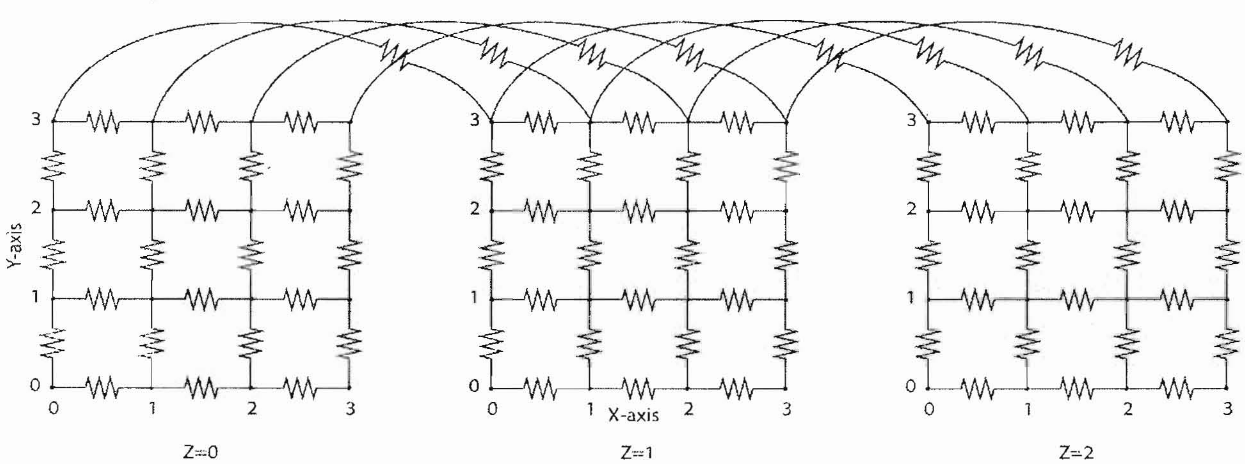

Figure 3: Illustration of a "flattened out" 3x3x3 cube used in the creation of the hypercube.

2.3: Four-dimensional network

The four-dimensional network is constructed using the flattening out method described above. Four flattened cubes, each consisting of six square lattices, six resistors by six resistors are built. The cubes are then connected together in a way similar to the connections made to construct the cube. For example, the node at (0,0,0,0) was connected, using a resistor and wire, to the node at (0,0,0,1) and so forth. The nodes are labeled (x,y,z,w) where x and y are the axes of the square lattices, z refers to the lattice number within each cube, and w refers to the cube number within the hypercube. A simplified version of the layout of the flattened hypercube can be seen in Figure 4 (page 9). (As with the illustration of the cube, connections have only been made for (x,3,z,w). The drawing is simplified to three by three squares, and only three cubes. All of this has been done to make interpretation easier.)

During initial construction of the four-dimensional hypercube, the initial goal was to build a 6x6x6x6 model. However, it is obvious that as more dimensions are added, time becomes an exponentially increasing factor. Four flattened out cubes, each 6x6x6 were constructed and connected together to create a four-dimensional model. Once this stage was reached, edge effects (discussed further below) were tested. It was determined that a device of this size was sufficiently large to model an infinite lattice, and time was therefore not invested to create the remaining two cubes.

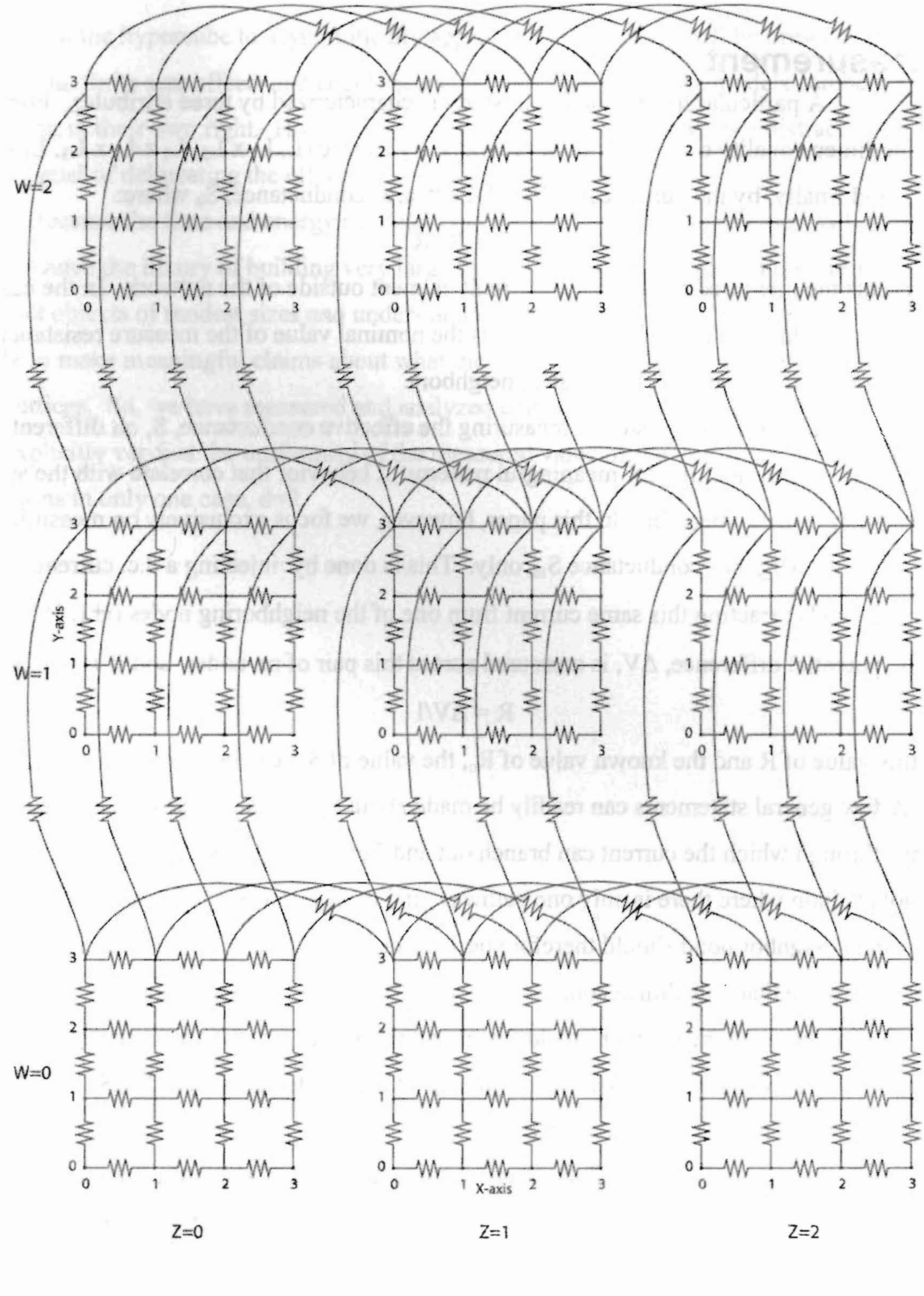

Figure 4: Illustration of a "flattened out" 3x3x3x3 hypercube.

III. Measurement

A particular hypercube in our study is characterized by three attributes. First, by its spatial dimensionality d = 1, 2, 3, or 4, secondly, by its size (L, $L_1 \times L_2$, $L_1 \times L_2 \times L_3$, $L_1 \times L_2 \times L_3 \times L_4$), and finally, by the numerical value of the "bare" conductance, S_o where:

$$S_o = R_o/R$$

R_o represents the measured resistance of a circuit element outside of the network. In the case of this experiment, this is taken to be 1Kohm. R is the nominal value of the measure resistance of the resistor used to connect any two nearest neighbors.

In general, we are interested in measuring the effective conductance, S, on different length scales and then looking for meaningful patterns of behavior that correlate with the spatial dimensionality of the hypercube. In this paper, however, we focus exclusively on measuring the effective nearest neighbor conductance S_{nn} only. This is done by injecting a d.c. current, $I_{d.c.}$ at a node (i,j,k,l) and extracting this same current from one of the neighboring nodes ($i\pm1$, $j\pm1$, $k\pm1$, $l\pm1$). The potential difference, ΔV, is measured across this pair of nn nodes, and R is computed:

$$R = \Delta V/I$$

Using this value of R and the known value of R_o, the value of S_{nn} can be determined.

A few general statements can readily be made about S_{nn}. First; because there are multiple pathways through which the current can branch out and flow, $S_{nn} > S_o$, except for the one-dimensional chain where there is only one pathway, and in that case $S_{nn} = S_o$. Measured values of the nearest neighbor bond should therefore be 1 for one dimensional and greater than 1 for systems with more than one dimension.

Second; because the number of additional pathways between any two nearest neighbor nodes increases rapidly with dimensionality, we expect the conductance ratio S_{nn}/S_o to increase monotonically with increasing dimensions.

Third; for an infinite lattice, and also for a finite lattice with a periodic boundary condition (thus simulating an infinite lattice), all nodes must be equivalent, and the conductance ratio for these cases should be the same for any nearest neighbor bond regardless of its location within the lattice. But, in general, for finite lattices with edges, the ratio is expected to be larger for bonds that are deep inside the hypercube and decrease as one approaches the edges.

For moderately sized lattices, one can expect the measured value of the conductance ratio deep within the hypercube to asymptotically approach the value that will be present in an infinite lattice. The finite size effects and edge effects that are present in finite models can be very interesting in their own right. However, in the present study they serve as a distraction from our primary goal of delineating the effects of spatial dimensionality.

Because the time and energy needed to build these hypercubes increases with L^d, one does not have the luxury of building very large hypercubes. Our approach therefore is to construct objects of modest sizes and understand these finite size and edge effects well enough to be able to make meaningful claims about what the ratio S_{nn}/S_o might be for the limiting case of finite lattices. So, we have measured and analyzed edge effects for all cases (d=2,3,4) but we have explicitly verified the uniformity of the measured value of the S_{nn}/S_o for Periodic Boundary Conditions in only one case, d=2.

IV. Effects

Before the results of the measurements are presented, there are two important characteristics of the models that must be addressed. These are the fall off of conductance near the edge, heretofore referred to as *edge effect*, and the constraints of size on objects.

4.1 Edge Effect

We expect that as measurements of conductance are taken, values will not stay consistent throughout the object. The existing theory predicts that a two-dimensional *infinite* lattice will yield a nearest neighbor bond conductance of 2[x,xi]. However, it is obviously not possible to build a lattice of infinite size. We therefore expect to see a falloff of conductance near the edge of the models to be tested. Our goal will be to build models large enough to overcome these effects.

This falloff can, in principle, be calculated using numerical techniques[viii]. However, this has not been done for the purposes of this experiment. We are seeking to build models that are sufficiently large enough to negate the edge effect and, in turn, observe values near the center of the models that are not affected by the falloff near the edge. Our concern is therefore not to eliminate the effect, but simply to make it negligible near the center.

4.2 Size constraints/periodic boundary conditions

Because the edge effect is a concern for our models, building models large enough to overcome it is important. However, it is impossible to construct a model large enough to overcome the effect entirely. Even if models of very large sizes are built, slight variations will still be seen at the innermost nearest neighbor bonds. We have therefore built a two-dimensional resistor lattice with periodic boundary conditions to simulate an infinite two-dimensional lattice. This lattice was used to demonstrate that the edge effect is not present in an infinite network.

The idea of periodic boundary conditions in computer modeling and mathematical problem solving is analogous to how it is used here. If a problem is to be solved for a particle in a box or similar situation, and one wants to ignore surface effects, the box is replicated identically throughout space to form an infinite lattice of identical boxes. Similarly, the two-dimensional lattice of resistors will be connected upon itself, essentially replicating itself in space in an infinite network of lattices.

A ten by ten two dimensional network was built. Each row was then connected from one end to the other, to create loops out of each of the rows. As can be seen in Figure 5, the mesh is constructed using resistors, ten on each side. The ends are connected back to one another in order to complete the circuit. A somewhat finer point of the construction of this model is the notion that two edges of the mesh must be left free of resistors. If all four sides have resistors, when the wire is looped over, the nodes will lie on top of one another in the discrete model. In the end, what has been built is topologically equivalent to an edge-less spherical shell of resistors. It is expected that when the conductance of this system is measured, it will be approximately 2 throughout.

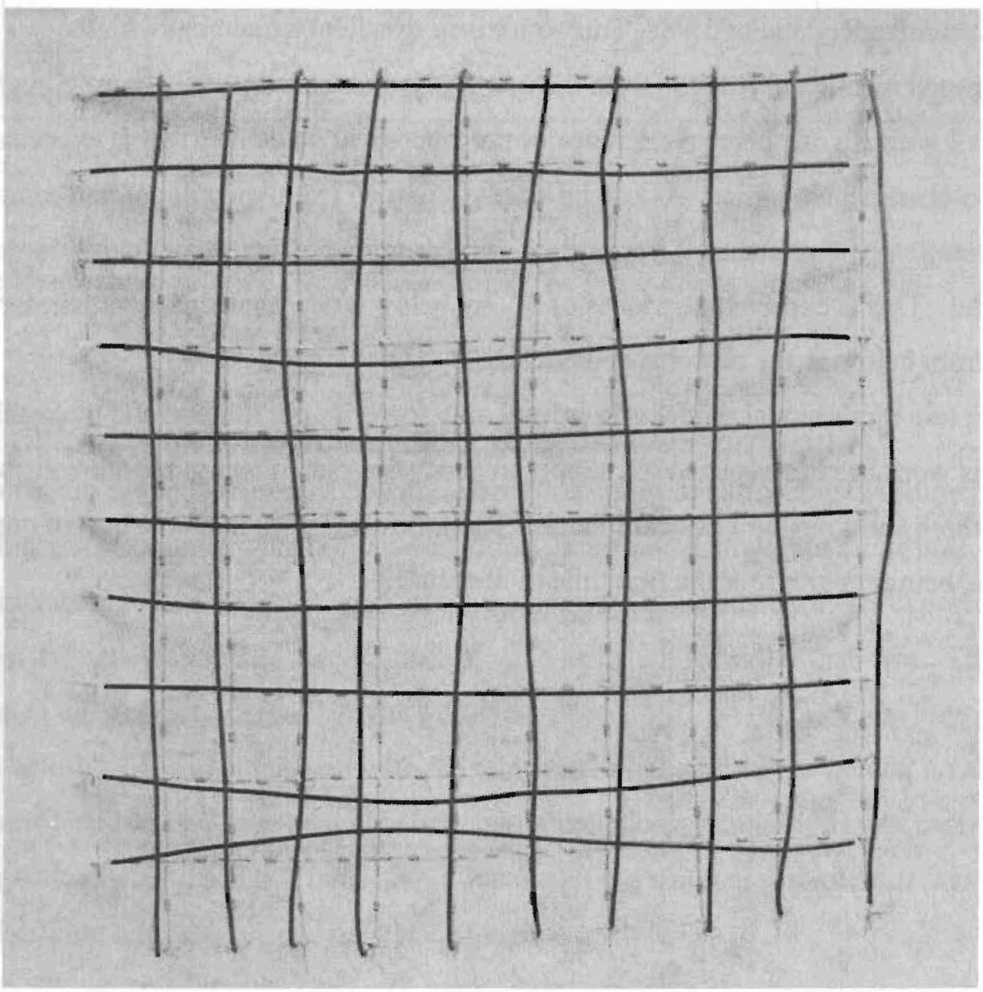

Figure 5: "Infinite" two-dimensional lattice

V. Results for Two-Dimensional Network

The two-dimensional square grid of resistors has been studied extensively, both theoretically[x,xi] and experimentally[viii]. In particular, it has been shown that computing the effective bond conductance S_{nn} across nearest neighbors of an *infinite* lattice is straightforward[viii]. The value of R_o/R of an infinite square grid is predicted to be exactly 2. Measurements have previously been made using very precise methods to determine the nearest neighbor bond conductance, S_{nn}, in a finite square lattice. These measurements were performed using 0.1 percent tolerance resistors and yielded a value of 1.990 01[viii].

5.1 Ten by ten two-dimensional lattice

The results of measurements on a ten by ten lattice of 1 percent tolerance resistors yielded a conductance ratio of 1.988. This result is in excellent agreement with the aforementioned mentioned results. Both the new result and the previously reported value of 1.990 01 fall within a one percent tolerance of the theoretical value of 2 that is expected with an infinite two-dimensional lattice. As can be seen in Figure 6 (following page), the conductance ratio asymptotically approached 2 from below as measurements were taken closer to the center of the model. This is expected, as values of S_{nn} are below 2 near the edge, and therefore approach from below as the center is approached.

The two dimensional model was created only to verify our method and the results are *not* novel. This work has been carried out simply to verify our methods for the following sections involving three and four-dimensional models. The following work is, to the best of our knowledge, being reported for the first time in literature.

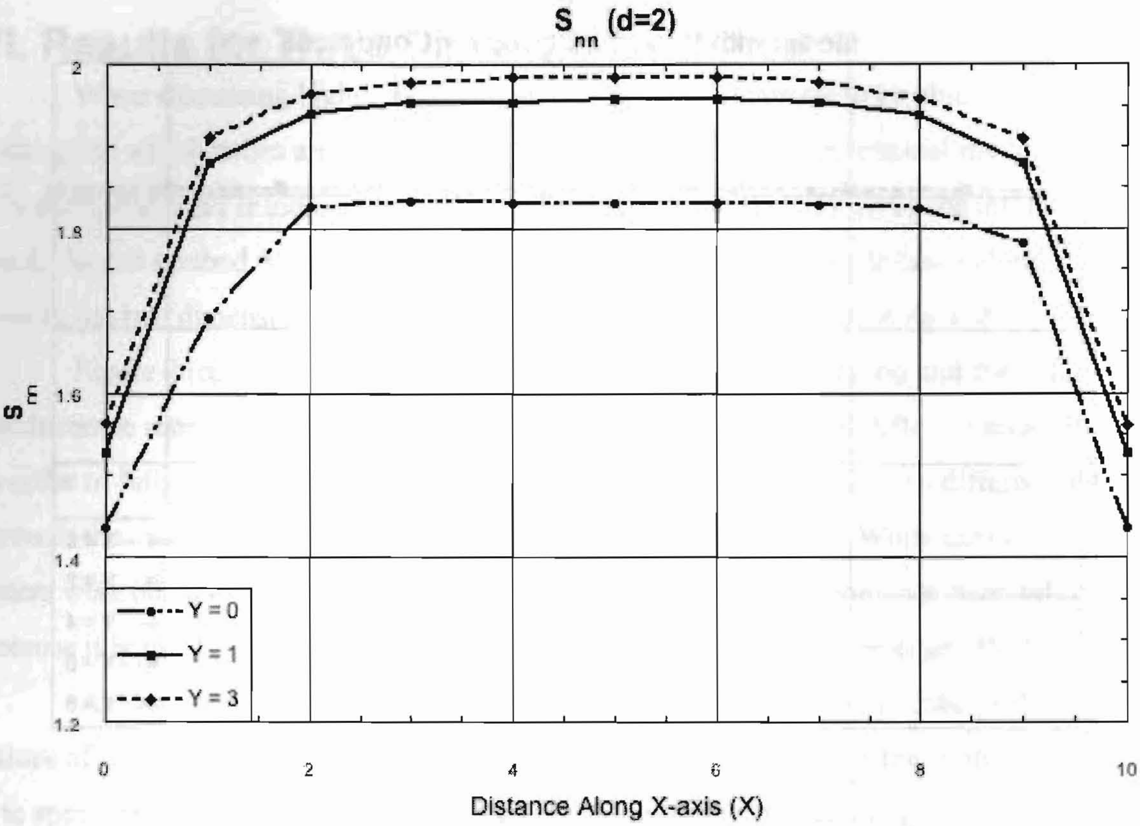

Figure 6: Bond Conductance, S_{nn} vs. location of the bond along x-axis for two-dimensional model

5.2 Ten by ten lattice with periodic boundary conditions

In order to prove that there is no edge effect in an infinite lattice, the two-dimensional lattice with periodic boundary conditions was tested. This model simulates an infinite lattice, and therefore should not exhibit the edge effect that is in the other systems. As expected, every point within the network yielded a conductance value just slightly higher than two. This result can be seen in the graph in Figure 7 on the following page.

As before, this is in agreement with the numerical methods used to predict a value of 2 in a two-dimensional lattice[viii]. In the lattice with periodic boundary conditions, the conductance ratio asymptotically approaches 2 from above, instead of the rising asymptote that was observed earlier in the finite lattice.

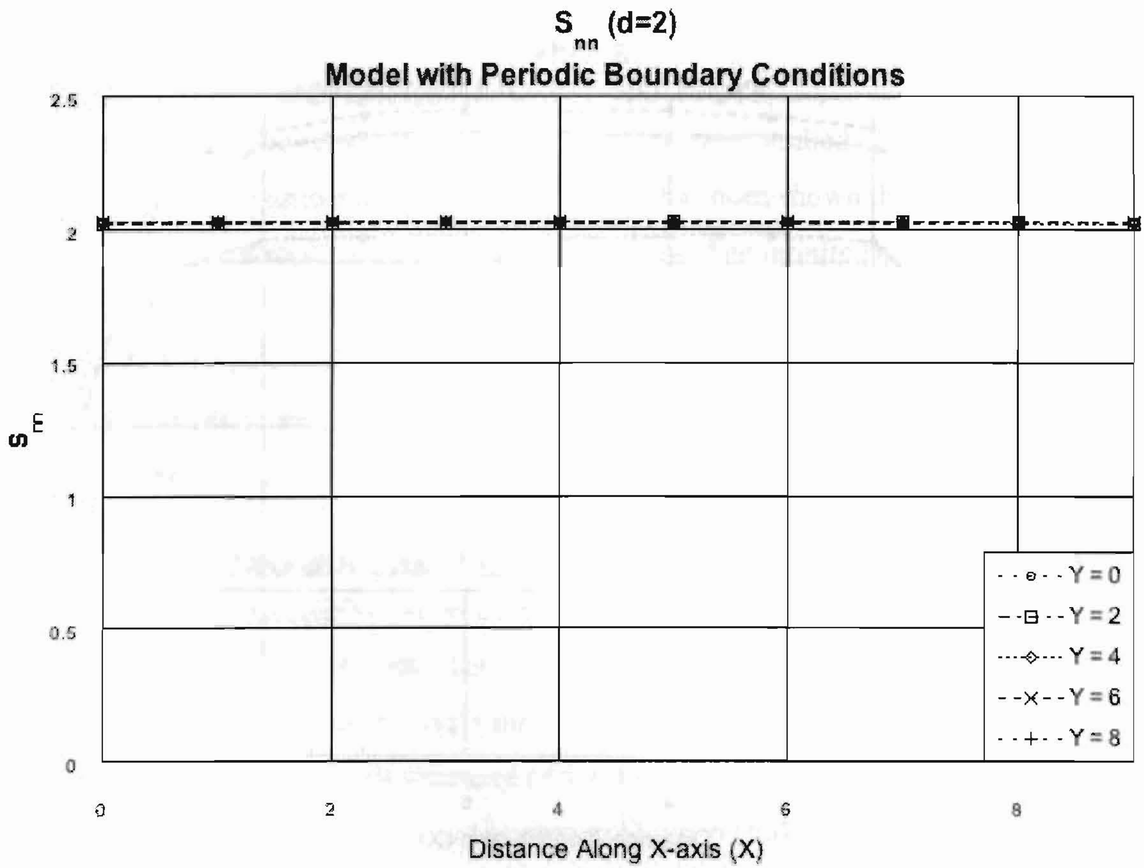

Figure 7: Bond conductance vs. location of the bond along x-axis for two-dimensional model with periodic boundary conditions

VI. Results for Three-Dimensional Lattice

When discussing higher dimensional models, it is important to be able to clearly distinguish which nodes are being referenced. Within the three-dimensional model, the location of a particular bond is indicated by referring to the coordinates (X,Y,Z) of the left terminus of the bond. In this method of reference, the values of X and Y refer to the distance along each of these axes in the two dimensional layers. The value of Z refers to the layer of the cube.

Figure 8 (page 18) depicts the relationship between bond location and the value of the conductance ratio, S_{nn}/S_o. Measurements were taken through several different areas of the cube in order to fully study the edge effects. Each line on the graph represents a different path through the cube with constant values of Y and Z while X is varied. While many more data points were taken than are represented in Figure 8, the data in the graph has been selected because it is representative of the entire cube and adequately shows the edge effect.

As expected, when measurements were taken at the edges of the cube (x=0 or x=5), values of the conductance ratio were low in comparison to values near the center (x=2 or x=3). It is apparent from the figure that the edge effect is large but decreases rapidly - with only one movement towards the center of the object.

The nearest neighbor bond at the center of the device (2,3,2) exhibited a conductance ratio of 2.95, a difference of 1.7 percent from the anticipated value of 3. At a corner of the cube (0,0,0), a nearest neighbor bond yielded a conductance ratio of 1.88. This is a 36 percent difference from the maximum measured value near the center, thus showing a significant edge effect that decreases rapidly towards the center of the object.

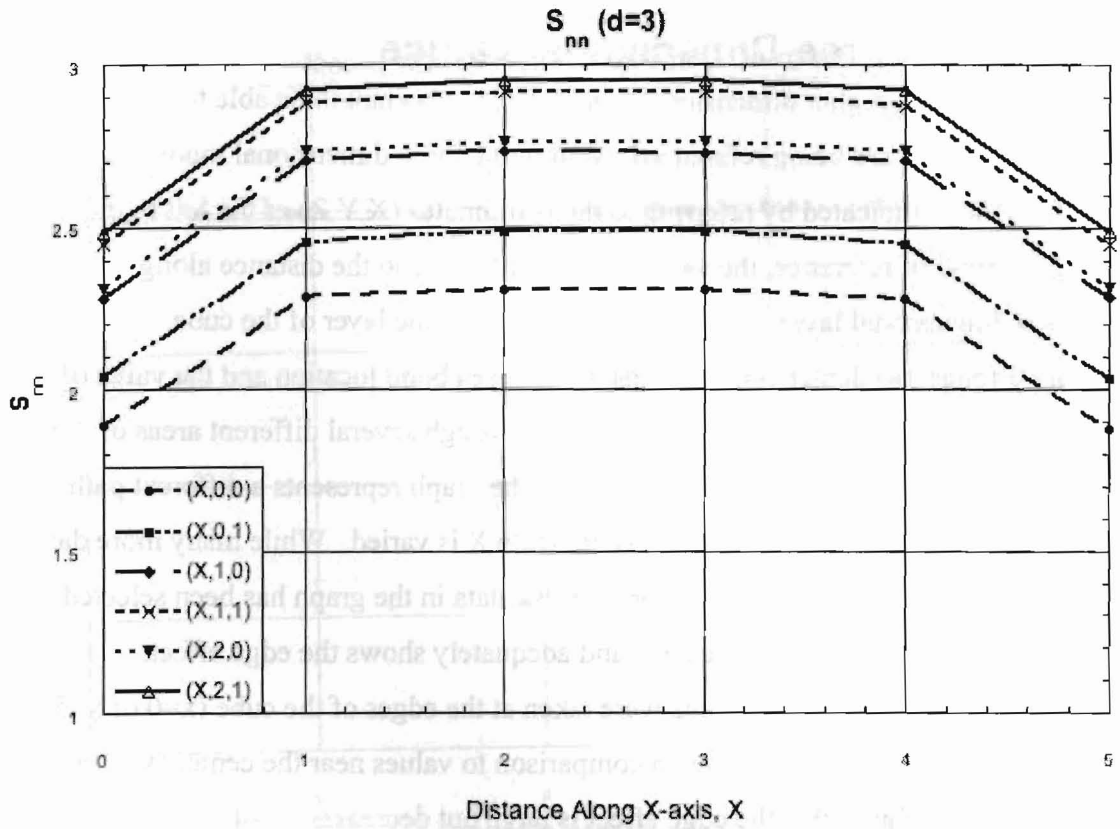

Figure 8: Bond conductance vs. location of the bond along x-axis for three-dimensional model

VII. Results for Four-Dimensional Lattice

It is important to note that due to the asymmetry in the four-dimensional model, there are various ways to graph measurements of nearest neighbor conductance. Because the model is asymmetrical, the falloff of conductance due to edge effects is, unlike in previous models, also asymmetrical. However, the values measured at the center of the hypercube, no matter the manner in which they were reached, exhibit conductances very close to 4. Therefore, the asymmetry simply allows for different ways of looking at the model.

Figure 9 (page 20), below, shows one set of paths traced through the model. In this graph, each line represents a different value of Z. The coordinates X and Y within each square lattice (Z) were kept constant, and the value of W (representing the cube number) was changed. As can be seen in the graph, when W is at 0 or 3, the edge effect is very strong, but it quickly disappears near the center of the cube.

Figure 10 (page 20) illustrates a different way that conductance was measured through the hypercube. In this example, X and Y were again kept constant within the square lattices. Each line represents a different cube number (W). Measurements were taken across a constant value of W while Z was varied. Because of the asymmetry in the model, it is clear that the edge effect is different from opposite sides of the model. However, at the center of the device the edge effect is again neutralized and a leveling off is once again seen.

From these tests, the maximum value of conductance measured was 3.95. This was measured at both (3,3,1,1) and (3,3,2,1). This yields a percent difference of 1.25% from the expected conductance of 4. This is the closest to the expected value that was obtained during experimentation. It is difficult to compare the centermost value of 3.95 to a value that is subjected to the most edge effect due to the asymmetry of the model. Due to the variations in conductance around the edge, simply choosing a value would not yield an accurate comparison. It is, however, safe to say that there is a significant and strong edge effect within the four-dimensional model that is comparable to those seen in the two and three-dimensional devices.

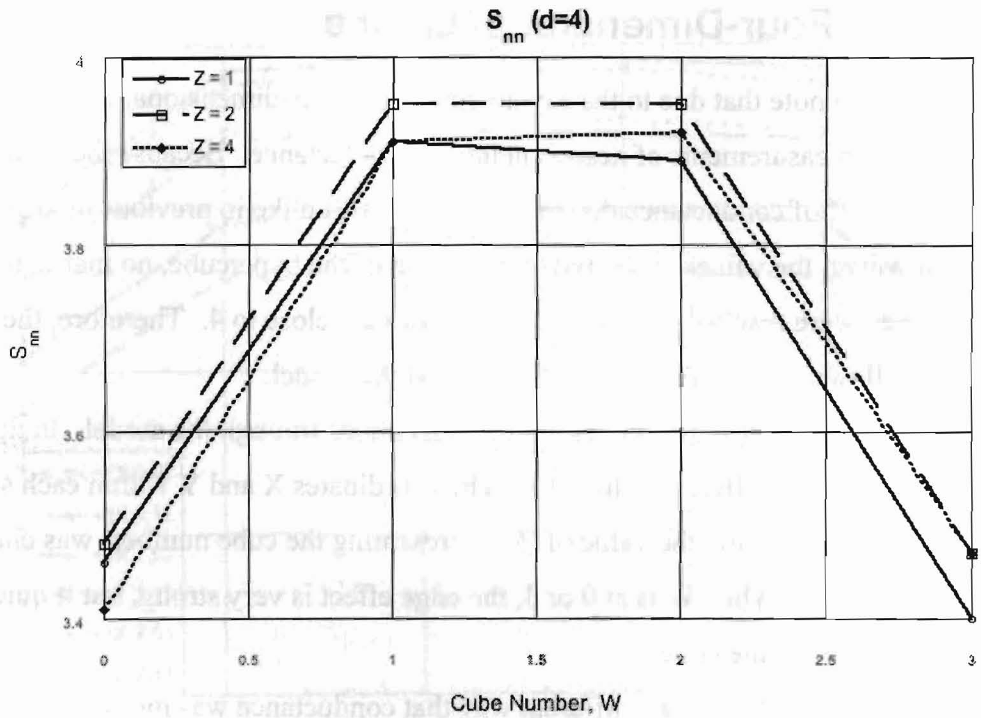

Figure 9: Conductance vs. Cube Number (W) (constant X, Y, Z)

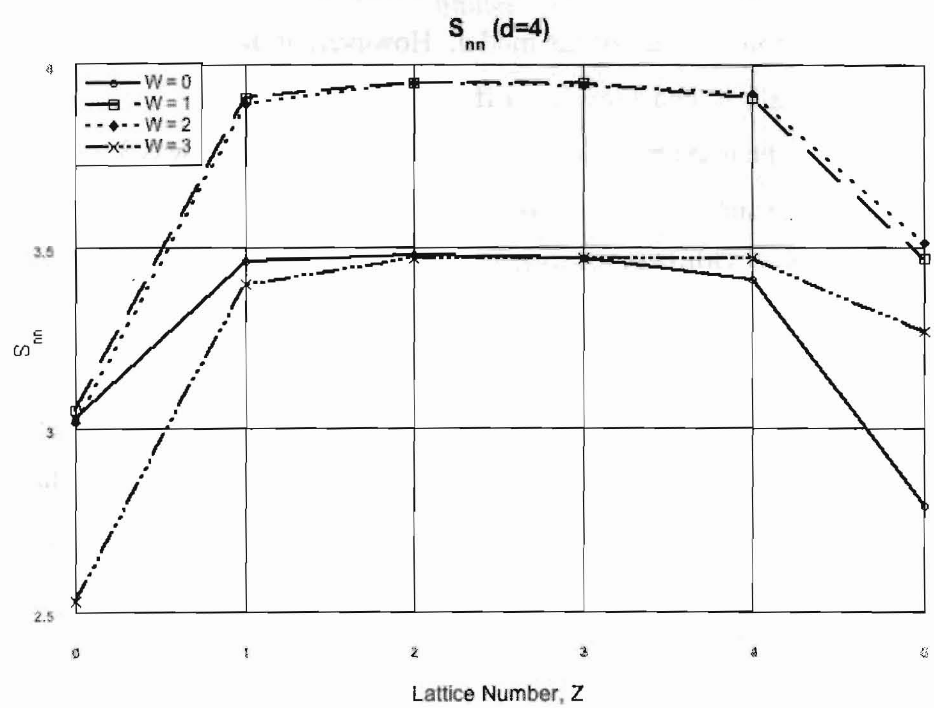

Figure 10: Conductance vs. Lattice Number (Z) (constant X, Y, W)

VIII. Dimensionality and Conductance Ratios

Perhaps the most important, yet simple concept to be taken from this research is the relationship between the dimensionality of an object and its measured conductance ratio. As can be seen below in Figure 11, these two quantities yield a linear relationship. Up to the four-dimensional devices that were tested using our methods, it was found that there was a direct linear relationship between the number of dimensions and the nearest neighbor conductance within the object. By examining our work, it is believed that this linear relationship will progress into higher dimensions.

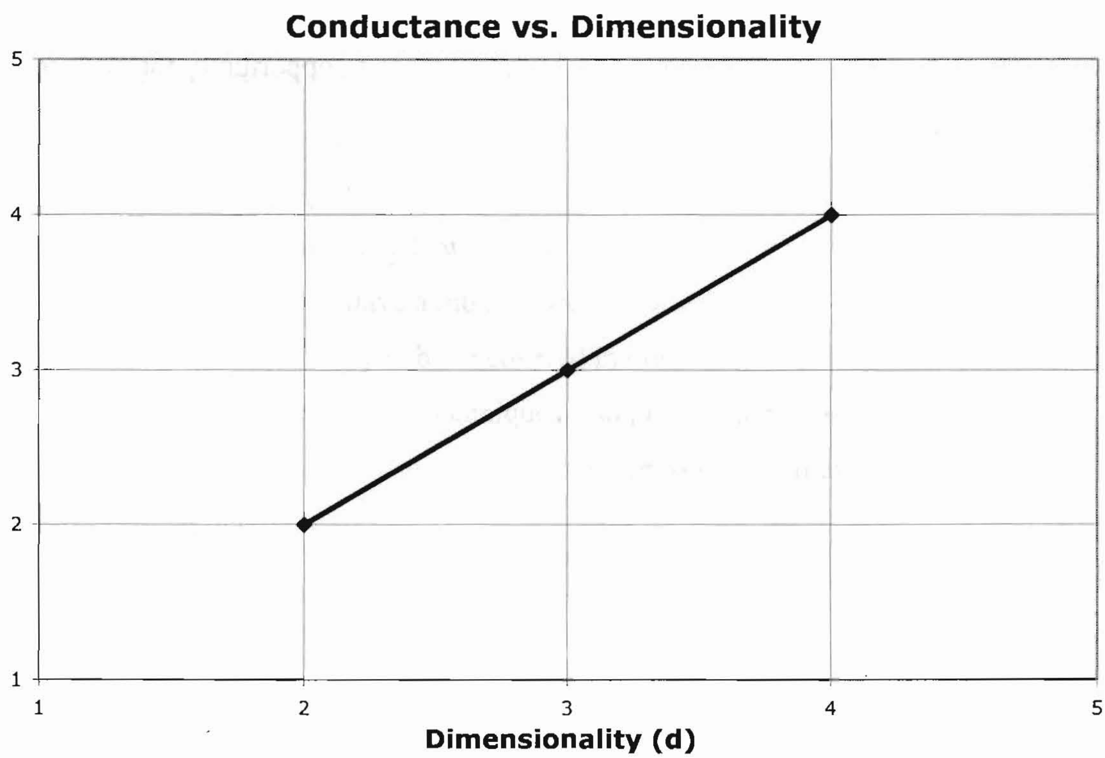

Figure 11: Graphical representation of conductance vs. dimensionality

IX. Further Work

Some data has been taken using a different method of measurement. However, not enough data has been taken to present conclusive evidence regarding this method. In this method of data collection, current is injected into the corner of a three-dimensional model, and extracted at a point moving progressively farther away. An initial measurement is taken at with only one resistor between the injection and extraction point. The second measurement has two resistors, and so forth. This pattern of movement can be described using a (n,n,n) coordinate system. The first lead was attached at the numbered point (0,0,0). The first measurement was then taken at (1,1,1), followed by (2,2,2) etc. At the time of writing, sufficient data and analysis has not been done to analyze the data collected, but it provides an opportunity for more to be learned about these systems.

X. Conclusions

As long as 125 years ago people were thinking about a fourth spatial dimension in our world. These theories have developed through Einstein's Special Relativity and the ideas of string theory introducing possibly eleven or twelve dimensions. In the past few years research has been fueled by a desire to discover a fourth spatial dimension and determine its effects on gravity and our universe.

While we recognize this research as important and innovative, we do not seek to discover a fourth dimension. Instead, our goal is the construction of simple models with non-trivial consequences of higher-dimensional spaces.

Using lumped circuit elements with linear IV characteristics we built and measured conductance of two, three, and four-dimensional models. These models are intended for pedagogical purposes, although they are not limited to that purpose. It is possible that, using these techniques of fabrication, models with non-linear IV characteristics can be constructed. The consequences of these may have a much more significant impact.

There is much more theory to be uncovered in order to explain all of the characteristics of these models. Different measurement techniques also provide promise for new results and ideas. While we have done a significant amount of work, it seems that with every turn there is something new to be uncovered.

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