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The Setting, Different Approaches, Important Properties

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be of finite Lebesgue measure and $1 < p < \infty$. The grand Lebesgue space $L_{p)}(\Omega)$ (cf. [IS92]) and the small Lebesgue space $L_{(p}(\Omega)$ (cf. [Fio00]) are rearrangement invariant Banach function spaces. The classical Lebesgue space $L_p(\Omega)$ is embedded in $L_{p)}(\Omega)$, which again is embedded in every Lebesgue space $L_{p-\varepsilon}(\Omega)$, $0 < \varepsilon < p-1$. Similarly, $L_{p+\varepsilon}(\Omega)$, $\varepsilon > 0$ is embedded in the space $L_{(p}(\Omega)$, which again is embedded in $L_p(\Omega)$.

We present a way to find their norms which are based on the decreasing rearrangement. To get there, we define specific extrapolation and interpolation constructions and use them, alone and in combination, in order to characterise the spaces. Finally, we compare them to Lorentz-Zygmund spaces.

Contents

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a set of finite Lebesgue measure, and let $1 < p < \infty$. The norm of the grand Lebesgue spaces $L_{p)}(\Omega)$ is given by

$$\|f \mid L_{p)}(\Omega)\| = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\int_\Omega |f(x)|^{p-\varepsilon} \,\mathrm{d}x \right)^{\frac{1}{p-\varepsilon}}.$$

They were introduced by Iwaniec and Sbordone, cf. [IS92], and have proved to play an important role in PDE theory (see e.g. [Sbo96], [FS98]). Their associate spaces, the small Lebesgue spaces $L_{(p'}(\Omega)$, $1/p + 1/p' = 1$, were found by Fiorenza, cf. [Fio00], and have also found several applications (see e.g. [FR03]). Their norm is given by

$$\|g \mid L_{(p'}(\Omega)\| = \inf_{g=\sum\limits_{k\in\mathbb{N}} g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k(x)|^{(p'-\varepsilon)'} \,\mathrm{d}x \right)^{\frac{1}{(p-\varepsilon)'}}.$$

In this work, we will mainly prove the norms using the decreasing rearrangement that were found by Fiorenza and Karadzhov in [FK04]. We try to present this result as directly and completely as possible. It turns out that it holds for $\mu(\Omega) = 1$

$$\|f \mid L_{p)}(\Omega)\| \sim \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 f^*(s)^p \,\mathrm{d}s \right)^{\frac{1}{p}}$$

$$\|g \mid L_{(p}(\Omega)\| \sim \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t g^*(s)^p \,\mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t}.$$

The grand and small Lebesgue spaces can in fact be seen as a model for far more general classes of spaces. In [CF05], these spaces were generalised by replacing $\varepsilon^{\frac{1}{p-\varepsilon}}$ in the definition of the grand Lebesgue spaces by $\varepsilon^{\frac{\theta}{p-\varepsilon}}$, $\theta > 0$, and the authors in [CFG13] even consider a more general function $\delta(\varepsilon)$ (analogously for the small Lebesgue spaces). It is also possible to define grand Orlicz spaces as in [CFK08]. Or, the interpolation characterisations of grand and small Lebesgue spaces could be used to define so called "abstract" grand and small spaces, as conducted in [FK04].

Disregarding the other possibilities we confine ourselves to the model case. The thesis is organised as follows: In Chapter 1, we first outline the framework of Banach function spaces in which we operate (following [BS88]), then we recall some interpolation theory basics that we will need (following [BL76]).

Chapter 2 presents a historical overview on grand and small Lebesgue spaces and the question of duality. Its statements are compiled from different papers.

An extrapolation technique taken from [CK14] is introduced in Chapter 3. It is used to characterise the grand and small Lebesgue spaces as extrapolation spaces of the classical Lebesgue spaces.

The inverse approach is presented in Chapter 4, where we combine extrapolation with interpolation techniques (following [FK04]). We prove a range of statements that the authors gathered from much more comprehensive theories of interpolation and extrapolation. Especially, two theorems on the equivalence of interpolation and extrapolation are given in the language of the extrapolation constructions of Chapter 3. This finally leads to the already referenced norms which allow to compare the spaces with Lorentz-Zygmund spaces. We show examples to prove that these embeddings are proper.

1. Preliminaries

1.1 Notation and general theory of Banach function spaces

In this work we examine Banach function spaces on subsets Ω of $\mathbb{R}^n$ of finite Lebesgue measure. Without loss of generality we can assume that $|\Omega| = \mu(\Omega) = 1$. We write e.g. L_p instead of $L_p(\Omega)$, $L_{p,q}$ for $L_{p,q}(\Omega)$, and so on. Most of the time we use this notation without recalling its meaning.

Furthermore, we consider the spaces in terms of equivalence classes, i.e. we identify f and g if $f(x) = g(x)$, $x \in \Omega$, μ-almost everywhere. Since we follow the notation of Bennett and Sharpley [BS88], all of these spaces are taken as subsets of $\mathcal{M}_0(\Omega)$, that is the space containing all measurable functions on Ω that are finite μ-a.e.

We let a Banach function space X, according to [BS88], be determined by a Banach function norm $\rho : \mathcal{M}_+ \to [0, \infty]$, where $\mathcal{M}_+$ denotes the measurable functions with values in $[0, \infty]$, through $g \in X \iff \rho(|g|) < \infty$ for $g \in \mathcal{M}_0$. We quote the axioms that the authors give for such ρ as we want to investigate specific spaces in this framework. Let χ_E denote the characteristic function of a μ-measurable subset E of Ω. It holds for all f, g, f_n $(n \in \mathbb{N})$ in $\mathcal{M}_+$:

(P1) $\rho(f) = 0 \iff f = 0$ μ-a.e.
$\rho(af) = a\rho(f)$ for any $a \geq 0$
$\rho(f + g) \leq \rho(f) + \rho(g)$

(P2) $g \leq f$ μ-a.e. $\implies \rho(g) \leq \rho(f)$

(P3) $f_n \nearrow f$ μ-a.e. $\implies \rho(f_n) \nearrow \rho(f)$

(P4) $\mu(E) < \infty \implies \rho(\chi_E) < \infty$ for any measurable $E \subset \Omega$

(P5) $\mu(E) < \infty \implies \exists c_E > 0 : \int_E f \,\mathrm{d}\mu \leq c_E \rho(f)$ for any measurable $E \subset \Omega$

We say that two expressions $s(x) \geq 0$ and $t(x) \geq 0$ are equivalent, say $s \sim t$, if there are constants $c_1, c_2 > 0$ such that for all x it holds $s(x) \leq c_1 t(x)$ and $t(x) \leq c_2 s(x)$. Sometimes, we also write $s(x) \lesssim t(x)$ (or $s(x) \gtrsim t(x)$ respectively) if only one inequality holds. By $X \hookrightarrow Y$, we denote that the Banach function space X is continuously embedded in Y, i.e. $\|f \,|\, Y\| \lesssim \|f \,|\, X\|$ and $x \subset Y$. In that sense, $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, i.e. the norms of X and Y are equivalent.

We restrict many definitions and propositions from the literature to the case $1 \le p \le \infty$, or even $1 < p < \infty$. This is done as we do not need the more general case (the grand and small Lebesgue spaces are usually defined for $1 < p < \infty$ only) and in order to avoid confusion, for example about the question on how to deal with quasi-Banach spaces.

The spaces L_p for $1 \le p \le \infty$ denote the classical Lebesgue spaces containing all measurable g with finite norm

$$\|f \mid L_p\| = \begin{cases} \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & \text{if} \quad p = \infty. \end{cases} \tag{1.1}$$

Since we assume $|\Omega| = 1 < \infty$ in our setting, we have $L_q \hookrightarrow L_p$ for $p < q$. To see this, we only need to apply Hölder's inequality with $r = q/p$, i.e.

$$\|f \mid L_p\| = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le \left(\int_\Omega 1 \,\mathrm{d}x\right)^{\frac{q-p}{p}} \left(\int_\Omega |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} = \|f \mid L_q\|. \tag{1.2}$$

This computation also shows us that the norms of the embeddings between Lebesgue spaces are uniformly bounded by 1.

We use a property shared by all the spaces considered here – that is the *rearrangement invariance*. A Banach function space is said to be rearrangement invariant if its norm is invariant under rearrangements. The decreasing rearrangement is defined as follows (cf. [BS88]): Let the *distribution function* μ_f of a function $f \in \mathcal{M}_0$ be given by

$$\mu_f(\lambda) = \mu\{x \in \Omega : |f(x)| > \lambda\} \quad \text{for} \quad \lambda \ge 0. \tag{1.3}$$

Then the *decreasing rearrangement* of f is defined to be

$$f^*(t) = \inf\{\lambda \ge 0 : \mu_f(\lambda) \le t\} \quad \text{for} \quad t \ge 0 \tag{1.4}$$

with the convention $\inf \varnothing = \infty$. Note that in our case, it holds $\mu_f(\lambda) \le |\Omega| = 1$ and $f^*(t) = 0$ for all $t \ge 1$. Now, for $1 \le p < \infty$ it holds $\int_\Omega |f(x)|^p \,\mathrm{d}x = \int_0^1 f^*(t)^p \,\mathrm{d}t$, and for $p = \infty$ it holds $\operatorname{ess\,sup}_\Omega |f(x)| = f^*(0)$ (cf. [BS88, Prop. II.1.8]).

Instead of the triangle inequality (P1) of the axioms for Banach function norms, a quasinorm ρ satisfies a c-triangle inequality $\rho(a + b) \le c[\rho(a) + \rho(b)]$ with $c > 1$. A refinement of the scale of Lebesgue spaces due to (1.1) is given by the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$, which are defined via the quasinorm

$$\|f \mid L_{p,q}\| = \begin{cases} \left(\int_0^1 \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{if} \quad 1 \le q < \infty \\ \sup_{0<t<1} t^{\frac{1}{p}} f^*(t) & \text{if} \quad q = \infty. \end{cases} \tag{1.5}$$

For $q = \infty$, these spaces are also called Marcinkiewicz spaces or weak-L_p. If the parameters p and q are in the given range, the above definition produces spaces that are rearrangement-invariant Banach spaces, cf. [BS88, Theorem IV.4.6]. In this book, the authors also give a proof for the embedding

$$L_{p,q} \hookrightarrow L_{p,r} \quad \text{for} \quad 1 \le q \le r \le \infty, \tag{1.6}$$

where the norms of inclusion are bounded by $(p/q)^{(r-q)/rq}$ if $r < \infty$, else by $(p/q)^{1/q}$. They also state that for $|\Omega| < \infty$ (a part of this fact is proven in Chapter 2, another part in Chapter 3)

$$L_{r,s} \hookrightarrow L_{p,q} \quad \text{for any} \quad 1 < p < r < \infty,\ 1 \le q, s \le \infty. \tag{1.7}$$

Another refinement of the L_p scale is given by the Zygmund spaces (cf. [BS88, Def. 6.11]) $L_p(\log L)_a$. Let $1 \le p < \infty$ and $-\infty < a < \infty$ or $p = \infty$ and $a \ge 0$. An equivalent quasinorm on this spaces is given by

$$\|f \mid L_p(\log L)_a\| = \begin{cases} \left(\int_0^1 [(1 + |\ln t|)^a f^*(t)]^p \, \mathrm{d}t \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty \\ \sup\limits_{0<t<1} (1 + |\ln t|)^a f^*(t) & \text{if} \quad p = \infty. \end{cases} \tag{1.8}$$

A third and last refinement is given by the Lorentz-Zygmund spaces, that generalise the Lorentz and Zygmund spaces. The space $L_{p,q}(\log L)_a$ is for $1 < p < \infty$, $1 \le q \le \infty$ and $-\infty < a < \infty$ determined by the quasinorm

$$\|f \mid L_{p,q}(\log L)_a\| = \begin{cases} \left(\int_0^1 \left[(1 + |\ln t|)^a \, t^{\frac{1}{p}} \, f^*(t)\right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} & \text{if} \quad 1 \le q < \infty \\ \sup\limits_{0<t<1} (1 + |\ln t|)^a \, t^{\frac{1}{p}} \, f^*(t) & \text{if} \quad q = \infty. \end{cases} \tag{1.9}$$

It holds $L_{p,q} = L_{p,q}(\log L)_0$ and $L_p(\log L)_a = L_{p,p}(\log L)_a$. Let $1 < p_1, p_2 < \infty$, $1 \le q_1, q_2 \le \infty$ and $-\infty < a_1, a_1 < \infty$. We have the embeddings (cf. [BR80])

$$L_{p_1,q_1}(\log L)_{a_1} \hookrightarrow L_{p_2,q_2}(\log L)_{a_2} \quad \text{if} \quad \begin{cases} p_2 < p_1 \\ p_1 = p_2,\ q_1 \le q_2 \text{ and } a_1 \ge a_2 \\ p_1 = p_2,\ q_1 > q_2 \text{ and } a_1 + \frac{1}{q_1} > a_2 + \frac{1}{q_2}. \end{cases}$$

We now cite the basic principles on the associate and dual space from [BS88, Chapter I]. Let X be a Banach function space. The associate space X' is given by all $g \in \mathcal{M}_0$ such that the norm

$$\|g \mid X'\| = \sup \left\{ \int_\Omega |fg| \, \mathrm{d}\mu : f \in X,\ \|f \mid X\| \le 1 \right\} \tag{1.10}$$

is finite. This space itself is a Banach function space. In this case, Hölder's inequality holds for all $f \in X$ and $g \in X'$, i.e.

$$\int_\Omega |fg| \, \mathrm{d}\mu \leq \|f \,|\, X\| \, \|g \,|\, X'\|.$$

Furthermore, it holds $X'' = X$ (cf. [BS88, Theorem 2.7]). If X, Y are Banach function spaces, then $X \hookrightarrow Y$ implies $Y' \hookrightarrow X'$. The associate space can be, roughly speaking, identified with a subspace of the dual space X^* of X (cf. [BS88, Theorem 2.9]), which is the space of all linear continuous functionals $L : X \to \mathbb{R}$ (or $\mathbb{C}$). We can compare them by the following considerations: $f \in X$ is said to have *absolutely continuous norm* in X, if for every sequence $\{E_n\}_{n=1}^\infty$ of subsets of Ω

$$\chi_{E_n} \to 0 \, \mu\text{-a.e} \implies \|f\chi_{E_n} \,|\, X\| \xrightarrow[n\to\infty]{} 0. \tag{1.11}$$

If all $f \in X$ have absolutely continuous norm, then the space X is said to have absolutely continuous norm. This is a necessary and sufficient condition such that the associate and the dual space of X can be identified. If $(X^*)^* = X$ then X is said to be reflexive, what holds if and only if both X and X' have absolutely continuous norm. As [BS88] do, we denote by X^a the set of functions that have absolutely continuous norm in X, and by X^b the closure of the set of bounded functions f with $\mu(\operatorname{supp} f) < \infty$ (i.e. L_∞) relative to the norm $\|\cdot \,|\, X\|$. It holds [BS88, Theorem I.3.11]

$$X^a \subset X^b \subset X. \tag{1.12}$$

Let $1 < p < \infty$ and $1/p + 1/p' = 1$ or $p = 1$ and $p' = \infty$. Then the spaces L_p and $L_{p'}$ are associate to each other. With the same notation and $1 < p < \infty$, $1 \leq q \leq \infty$, $-\infty < a < \infty$, it holds $(L_{p,q}(\log L)_a)' = L_{p',q'}(\log L)_{-a}$ (see e.g. [CFT04]). The quasinorms (1.5), (1.8) and (1.9) become norms if we replace f^* by $f^{**}(s) = \frac{1}{s}\int_0^s f^*(x) \, \mathrm{d}x$.

1.2 Interpolation

The grand and small Lebesgue spaces appear in the context of questions on the properties of linear operators. A powerful tool to deal with such problems has been proved to be interpolation theory. We can "interpolate" between two spaces on which we know the properties of an operator T in such a way that these properties are carried over to the interpolation space.

Here are the details: We mostly follow [BL76] unless otherwise specified. Let A_0, A_1 be two Banach spaces. We call $\{A_0, A_1\}$ an *interpolation couple* if there is a linear Hausdorff space $\mathcal{A}$ with $A_i \subset \mathcal{A}$, $i = 0, 1$. For such a couple let us define $A_0 + A_1$ as consisting of all $a \in \mathcal{A}$ that can be represented as $a = a_0 + a_1$, $a_i \in A_i$, $i = 1, 2$, with

$\|a_0 \,|\, A_0\| + \|a_1 \,|\, A_1\| < \infty$. Let its norm be given by

$$\|a \,|\, A_0 + A_1\| := \inf_{\substack{a=a_0+a_1 \\ a_i \in A_i}} \|a_0 \,|\, A_0\| + \|a_1 \,|\, A_1\|. \tag{1.13}$$

Furthermore, we equip $A_0 \cap A_1$ with the norm

$$\|a \,|\, A_0 \cap A_1\| := \max(\,\|a \,|\, A_0\|,\, \|a \,|\, A_1\|). \tag{1.14}$$

These spaces are Banach spaces too, cf. for example [BL76, Lemma 2.3.1]. Note that, if $A_0 \hookrightarrow A_1$, then $A_0 = A_0 \cap A_1$ and $A_1 = A_0 + A_1$ (in the sense of equivalent norms).

Definition 1.1. Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation couples. Then we denote by $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ the set of all linear operators

$$T : A_0 + A_1 \longrightarrow B_0 + B_1 \quad \text{with} \quad T_k := T\Big|_{A_k} \in \mathcal{L}(A_k, B_k), \quad k = 0, 1. \tag{1.15}$$

Throughout this work, we make use of the following common notations, but we do not further refer to category theory.

Notation 1.2 (cf. [Tri95]). Let $\mathscr{C}_1$ denote the category that consists of the class of all complex Banach spaces $A, B, \ldots$ as objects, and the sets $\mathcal{L}(A, B)$ as morphisms.

Let $\mathscr{C}_2$ denote the category that consists of the class of all interpolation couples $\{A_0, A_1\}, \{B_0, B_1\}, \ldots$ as objects, and the sets $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ as morphisms.

A space $A \in \mathscr{C}_1$ is called *interpolation space* between A_0 and A_1, $(A_0, A_1) \in \mathscr{C}_2$, if

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1 \qquad \text{and} \qquad T : A \to A \text{ for all } T \in \mathcal{L}(\{A_0, A_1\}).$$

An *interpolation functor* is a functor $F : \mathscr{C}_2 \to \mathscr{C}_1$, such that $F(A_0, A_1)$, $F(B_0, B_1)$ are interpolation spaces between $(A_0, A_1), (B_0, B_1) \in \mathscr{C}_2$ respectively, and $F(T) = T$ for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.

We consider two general families of interpolation functors, i.e. the $\mathcal{K}$- and the $\mathcal{J}$-method of interpolation, also known as the real interpolation method. Let $(A_0, A_1) \in \mathscr{C}_2$ and $t > 0$.

$$K(t, a) := K(t, a; A_0, A_1) := \inf_{\substack{a=a_0+a_1 \\ a_i \in A_i}} (\,\|a_0 \,|\, A_0\| + t\,\|a_1 \,|\, A_1\|), \qquad a \in A_0 + A_1 \tag{1.16}$$

$$J(t, a) := J(t, a; A_0, A_1) \;:=\; \max\;(\,\|a \,|\, A_0\|, t\,\|a \,|\, A_1\|), \qquad a \in A_0 \cap A_1 \tag{1.17}$$

If it is clear from the context w.r.t. which couple (A_0, A_1) the functionals are considered, then we frequently use the first notation that does not explicitly mention the couple. $K(t, a)$ and $J(t, a)$ are positive and increasing functions of t, while K is concave and J is convex ([BL76, Lemmas 3.1.1 & 3.2.1]). It is easy to see that for any $t > 0$ $K(t, a)$ is an equivalent norm on $A_0 + A_1$, and $J(t, a)$ is an equivalent norm on $A_0 \cap A_1$.

For $a \in A_0 \cap A_1$ and $t > 0$, $s > 0$, we have (cf. [BL76, Lemma 3.2.1])

$$J(t,a) \le \max\left(1, \frac{t}{s}\right) J(s,a) \qquad \text{and} \qquad K(t,a) \le \min\left(1, \frac{t}{s}\right) J(s,a). \tag{1.18}$$

Let now $1 \le p \le \infty$ and $0 < \theta < 1$ or $p = \infty$ and $0 \le \theta \le 1$. Then the space $(A_0, A_1)^{\mathcal{K}}_{\theta,p}$ consists of all $a \in A_0 + A_1$ such that the norm

$$\|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,p}\| := \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t,a)\right]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \\ \sup\limits_{0<t<\infty} t^{-\theta} K(t,a) & \text{if} \quad p = \infty \end{cases} \tag{1.19}$$

is finite. The space $(A_0, A_1)^{\mathcal{J}}_{\theta,p}$ is defined as consisting of all $a \in A_0 + A_1$ that can be represented as $a = \int_0^\infty u(t) \frac{\mathrm{d}t}{t}$ (convergent in $A_0 + A_1$) with $u(t) \in A_0 \cap A_1$, all $t > 0$, such that

$$\begin{cases} \left(\int_0^\infty \left[t^{-\theta} J(t, u(t))\right]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \\ \sup\limits_{0<t<\infty} t^{-\theta} J(t, u(t)) & \text{if} \quad p = \infty \end{cases} \tag{1.20}$$

is finite. Its norm is given by the infimum of (1.20) taken over all possible representations $a = \int_0^\infty u(t) \frac{\mathrm{d}t}{t}$. It is also possible to derive equivalent *discrete norms* for both spaces. For the $\mathcal{J}$-method, we have (cf. [BL76, Lemma 3.2.3])

$$\|a \,|\, (A_0, A_1)^{\mathcal{J}}_{\theta,p}\| \sim \begin{cases} \inf\limits_{a=\Sigma_\nu u_\nu} \left(\sum\limits_{\nu=1}^\infty \left[2^{-\nu\theta} J(2^\nu, a)\right]^p\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \\ \inf\limits_{a=\Sigma_\nu u_\nu} \sup\limits_\nu 2^{-\nu\theta} J(2^\nu, a) & \text{if} \quad p = \infty. \end{cases} \tag{1.21}$$

As the K-functional is a norm on $A_0 + A_1$ and $(A_0, A_1)^{\mathcal{K}}_{\theta,q} \hookrightarrow A_0 + A_1$, it is clear that it can be estimated by the norm of an interpolation space. More precisely, it holds:

Proposition 1.3 ([BL76, Theorem 3.1.2])**.** *Let $(A_0, A_1) \in \mathscr{C}_2$, $0 < \theta < 1$ and $1 \le q \le \infty$. We have (with the convention $\infty^{\frac{1}{\infty}} = 1$)*

$$K(t,a) \le (\theta q)^{\frac{1}{q}} t^\theta \, \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,q}\|. \tag{1.22}$$

Proof. We follow the computation in [Har10]. With $t^{-\theta q} = \theta q \int_t^\infty \tau^{-\theta q} \frac{\mathrm{d}\tau}{\tau}$ we have

$$\begin{aligned} t^{-\theta} K(t,a) &= (\theta q)^{\frac{1}{q}} K(t,a) \left(\int_t^\infty \tau^{-\theta q} \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}} \\ &\le (\theta q)^{\frac{1}{q}} \left(\int_t^\infty \tau^{-\theta q} K(\tau,a)^q \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}} \le (\theta q)^{\frac{1}{q}} \, \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,q}\|. \end{aligned}$$

□

It turns out that the $\mathcal{K}$- and $\mathcal{J}$-method lead to the same interpolation spaces (in the sense of equivalent norms), as Lemma 1.4 holds. Thus, it is common not to distinguish between $\mathcal{K}$- and $\mathcal{J}$-interpolation spaces and also to omit the $\mathcal{K}$ or $\mathcal{J}$ in the designation. However, many of our statements hold only for one of the methods as the norms are equivalent, but not equal. Hence, we usually explicitly denote which method is meant, except in those cases when the specific norm is not important for the statement.

Lemma 1.4 (The fundamental lemma of interpolation theory). *Let $a \in A_0 + A_1$. Assume that*

$$\lim_{t\to 0} K(t,a) = \lim_{t\to\infty} \frac{K(t,a)}{t} = 0. \tag{1.23}$$

Then, for any $\varepsilon > 0$, there is a representation $a = \sum_{\nu\in\mathbb{Z}} u_\nu$ (convergence in $A_0 + A_1$), $u_\nu \in A_0 \cap A_1$, such that

$$J(2^\nu, u_\nu) \le 3(1+\varepsilon)K(2^\nu, a), \qquad \nu \in \mathbb{Z}.$$

Example 1.5 ([Har10]). We often deal with the interpolation and extrapolation of the classical Lebesgue spaces. Here is the most basic example. Let $1 \le q \le \infty$ and $0 < \theta < 1$. Then it holds in the sense of equivalent norms

$$(L_1, L_\infty)_{\theta,q} = L_{\frac{1}{1-\theta},q}.$$

We now cite a couple of statements that we frequently use: That are a simplified reiteration theorem and a list of formulas for the K-functional.

Theorem 1.6 (Reiteration theorem [BL76, Theorem 3.5.3]). *Let $(A_0, A_1) \in \mathscr{C}_2$, $0 < \theta_i < 1$, $i = 0, 1$ and $\theta_0 \neq \theta_1$. Then it holds for arbitrary $1 \le q, q_0, q_1 \le \infty$ and $0 < \eta < 1$*

$$\Big((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1}\Big)_{\eta,q} = (A_0, A_1)_{\theta,q} \quad \textit{with} \quad \theta = (1-\eta)\theta_0 + \eta\theta_1.$$

Theorem 1.7 (The Holmstedt formula [BL76, Theorem 3.6.1]). *Let $(A_0, A_1) \in \mathscr{C}_2$. Let $0 \le \theta_0 < \theta_1 \le 1$ and $1 \le q_1, q_2 \le \infty$. Put $\lambda = \theta_1 - \theta_0$, and take $X_j = (A_0, A_1)_{\theta_j,q_j}$ for $j = 0, 1$. Then for all $a \in A_0 + A_1$ and $t > 0$ it holds (we put $K(s,a) = K(s,a;A_0,A_1)$)*

$$K(t,a;X_0,X_1) \sim \left(\int_0^{t^{\frac{1}{\lambda}}} \left[s^{-\theta_0}K(s,a)\right]^{q_0} \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q_0}} + t\left(\int_{t^{\frac{1}{\lambda}}}^{\infty} \left[s^{-\theta_1}K(s,a)\right]^{q_1} \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q_1}}. \tag{1.24}$$

Corollary 1.8 ([BL76, Corollary 3.6.2]). *In the situation of the previous theorem, it holds for all $t > 0$ and $a \in A_0 + A_1$*

$$K(t,a;X_0,A_1) \sim t\left(\int_0^{t^{\frac{1}{1-\theta_0}}} \left[s^{-\theta_0}K(s,a;A_0,A_1)\right]^{q_0} \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q_0}} \tag{1.25}$$

and

$$K(t,a;A_0,X_1) \sim t\left(\int_{t^{\frac{1}{\theta_1}}}^{\infty}\left[s^{-\theta_1}K(s,a;A_0,A_1)\right]^{q_1}\,\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q_1}}. \tag{1.26}$$

Corollary 1.9. *Let $(A_0, A_1) \in \mathscr{C}_2$. Then for all $a \in A_0 + A_1$ and $t > 0$*

$$K(t,a) \sim \int_0^t K(s,a)\,\frac{\mathrm{d}s}{s} + t\int_t^{\infty}\frac{K(s,a)}{s}\,\frac{\mathrm{d}s}{s} \tag{1.27}$$

$$\sim \int_0^t K(s,a)\,\frac{\mathrm{d}s}{s} \tag{1.28}$$

$$\sim t\int_t^{\infty}\frac{K(s,a)}{s}\,\frac{\mathrm{d}s}{s}. \tag{1.29}$$

2. Direct approaches to grand and small Lebesgue spaces

This chapter is mainly intended as a historical overview of grand and small Lebesgue spaces, restricting to those approaches that do *not explicitly* use interpolation or extrapolation theory (this motivates the chapter's title). After giving the definitions we compute some equivalent norms, state basic properties and consider the question of duality. It is due to this procedure that the statements below were compiled from different sources.

As the Lorentz spaces appear in some equivalent norms of our spaces, we start by a short lemma comparing them with Lebesgue spaces. Similar statements are given in [FK04, p. 663] and [ET96, Section 2.6.2, eq. (12)] (the latter includes a proof too).

Lemma 2.1. *For* $1 < p < \infty$ *and any* $0 < \varepsilon < p-1$ *it holds uniformly w.r.t.* ε

$$c_1 \|g \mid L_{p+\varepsilon}\| \leq \|g \mid L_{p+\varepsilon,p}\| \leq c_2 \|g \mid L_{p+2\varepsilon}\| \tag{2.1}$$

and for $0 < \varepsilon < \frac{p-1}{2}$

$$c_3 \|g \mid L_{p-2\varepsilon}\| \leq \|g \mid L_{p-\varepsilon,p}\| \leq c_4 \|g \mid L_{p-\varepsilon}\|. \tag{2.2}$$

Proof. By the monotonicity of Lorentz spaces (1.6), we know that

$$\begin{aligned} \|g \mid L_{p+\varepsilon}\| = \|g \mid L_{p+\varepsilon,p+\varepsilon}\| &\leq \left(\frac{p+\varepsilon}{p}\right)^{\frac{\varepsilon}{p(p+\varepsilon)}} \|g \mid L_{p+\varepsilon,p}\| \leq c_1^{-1} \|g \mid L_{p+\varepsilon,p}\| \\ \text{and} \quad \|g \mid L_{p-\varepsilon,p}\| &\leq \|g \mid L_{p-\varepsilon,p-\varepsilon}\| = \|g \mid L_{p-\varepsilon}\| = c_4 \|g \mid L_{p-\varepsilon}\|. \end{aligned}$$

Since we have a finite measure space, i.e. $|\Omega| = 1$, the Hölder inequality with $r = \frac{p+2\varepsilon}{p}$ and $r' = \frac{p+2\varepsilon}{2\varepsilon}$ gives

$$\begin{aligned} \|g \mid L_{p+\varepsilon,p}\| &= \left(\int_0^1 \left[t^{\frac{1}{p+\varepsilon}} g^*(t)\right]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 g^*(t)^{p+2\varepsilon}\,\mathrm{d}t\right)^{\frac{1}{p+2\varepsilon}} \underbrace{\left(\int_0^1 t^{-\frac{p+2\varepsilon}{2(p+\varepsilon)}}\,\mathrm{d}t\right)^{\frac{2\varepsilon}{p(p+2\varepsilon)}}}_{\leq c_2,\ \text{as}\ 0<\frac{p+2\varepsilon}{2(p+\varepsilon)}<\frac{3}{4}} \\ &= c_2 \|g \mid L_{p+2\varepsilon}\|. \end{aligned}$$

For $r = \frac{p}{p-2\varepsilon}$ and $r' = \frac{p}{2\varepsilon}$, it gives

$$\begin{aligned}\|g \mid L_{p-2\varepsilon}\| &= \left(\int_0^1 g^*(t)^{p-2\varepsilon}\, t^{\frac{1}{r}\frac{\varepsilon}{p-\varepsilon}} \cdot t^{-\frac{1}{r}\frac{\varepsilon}{p-\varepsilon}}\,\mathrm{d}t\right)^{\frac{1}{p-2\varepsilon}} \\ &\le \left(\int_0^1 \left[t^{\frac{1}{p-\varepsilon}} g^*(t)\right]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \underbrace{\left(\int_0^1 t^{-\frac{p-2\varepsilon}{2(p-\varepsilon)}}\,\mathrm{d}t\right)^{\frac{2\varepsilon}{p(p-2\varepsilon)}}}_{\le c_3^{-1},\text{ as } 0<\frac{p-2\varepsilon}{2(p-\varepsilon)}<\frac{1}{2}} \\ &= c_3^{-1}\,\|g \mid L_{p-\varepsilon,p}\|.\end{aligned}$$

□

2.1 Grand Lebesgue spaces

Definition 2.2 (Grand Lebesgue spaces)**.** Let for $1 < p < \infty$ the space $L_{p)}$ consist of all $g \in \mathcal{M}_0$ such that the norm

$$\|g \mid L_{p)}\| := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}}\, \|g \mid L_{p-\varepsilon}\| \tag{2.3}$$

is finite.

Theorem 2.3. *Let $1 < p < \infty$. The space $L_{p)}$ is a Banach function space.*

Proof. We only show what is not trivial. Let f, g, f_n ($n \in \mathbb{N}$) be in $\mathcal{M}_+$.

(P1) $$\begin{aligned}\|f + g \mid L_{p)}\| &\le \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\|f \mid L_{p-\varepsilon}\| + \|g \mid L_{p-\varepsilon}\|\right) \\ &\le \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}}\, \|f \mid L_{p-\varepsilon}\| + \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}}\, \|g \mid L_{p-\varepsilon}\|.\end{aligned}$$

(P2) $g \le f$ μ-a.e. $\Longrightarrow$ $\varepsilon^{\frac{1}{p-\varepsilon}}\, \|g \mid L_{p-\varepsilon}\| \le \varepsilon^{\frac{1}{p-\varepsilon}}\, \|f \mid L_{p-\varepsilon}\|$ for $0 < \varepsilon < p-1$ $\Longrightarrow$ $\|g \mid L_{p)}\| \le \|f \mid L_{p)}\|$.

(P3) Take $0 \le f_n \nearrow f$ μ-a.e. For every $\delta > 0$ there exists $0 < \varepsilon_\delta < p-1$ such that

$$\|f \mid L_{p)}\| < \varepsilon_\delta^{\frac{1}{p-\varepsilon_\delta}}\, \|f \mid L_{p-\varepsilon_\delta}\| + \delta.$$

Since the p-norms are Banach norms, $\|f_n \mid L_{p-\varepsilon_\delta}\| \nearrow \|f \mid L_{p-\varepsilon_\delta}\|$, hence there is $n_\delta \in \mathbb{N}$ with $\|f \mid L_{p-\varepsilon_\delta}\| < \|f_n \mid L_{p-\varepsilon_\delta}\| + \delta$ for all $n \ge n_\delta$. With $\varepsilon^{1/(p-\varepsilon)} < p-1$, $\varepsilon < p-1$, this yields

$$\|f \mid L_{p)}\| < \varepsilon_\delta^{\frac{1}{p-\varepsilon_\delta}}\, \|f_n \mid L_{p-\varepsilon_\delta}\| + \delta \le \|f_n \mid L_{p)}\| + \delta$$

for all $n \ge n_\delta$ and it follows $\|f_n \mid L_{p)}\| \nearrow \|f \mid L_{p)}\|$ by taking $\delta \to 0$. □

These spaces were first introduced by [IS92], where the embedding properties of Proposition 2.4 were proven and an application was given. It turned out that (roughly speaking) the Jacobian $J(x, f) = \det Df(x)$ of a function $f : \Omega \to \mathbb{R}^n$ is locally integrable if Df belongs to $L_{n)}$). This was already known for the weak-L_p and the Zygmund space in the following embeddings, but the grand Lebesgue space was shown to be the biggest space of functions having this property. The name *grand Lebesgue space* appears, to our knowledge, in [Sbo96] for the first time in literature.

Proposition 2.4 (Basic embeddings)**.** *Let* $1 < p < \infty$*. Then the following holds.*

(i) $L_p \hookrightarrow L_{p)} \hookrightarrow L_{p-\varepsilon}$ *for any* $0 < \varepsilon < p-1$

(ii) $L_p \hookrightarrow L_{p,\infty} \hookrightarrow L_{p)}$

(iii) $L_p \hookrightarrow L_p(\log L)_{-\frac{1}{p}} \hookrightarrow L_{p)}$

Proof. We do not cite the direct proofs of (ii) and (iii) that can be found in [IS92], as they are not difficult while at the same time using different approaches than we do, e.g. the Orlicz norm for the Zygmund spaces. From our later characterisations of grand and small Lebesgue spaces, the embeddings can be easily deduced, cf. Section 4.5. We only give the argument for statement (i).

(i) By (1.2) we have $\|f \mid L_{p)}\| \leq \sup\limits_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} c_\Omega \|f \mid L_p\| \leq c_{\Omega,p} \|f \mid L_p\|$. Secondly, for any $0 < \varepsilon < p-1$

$$\|f \mid L_{p-\varepsilon}\| \sim \varepsilon^{\frac{1}{p-\varepsilon}} \|f \mid L_{p-\varepsilon}\| \leq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f \mid L_{p-\varepsilon}\| = \|f \mid L_{p)}\|. \qquad \square$$

Example 2.5. We give an example for the first embedding of the previous proposition, other examples follow in Section 4.5. Let $\Omega = (0,1)$ be equipped with the Lebesgue measure, $1 < p < \infty$ and $f(t) = t^{-\frac{1}{p}}$.

(i) Then $f(t) \notin L_p$, but $f(t) \in L_{p)}$, because

$$\varepsilon^{\frac{1}{p-\varepsilon}} \|f \mid L_{p-\varepsilon}\| = \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p-\varepsilon}} < p \quad \text{for} \quad 0 < \varepsilon < p-1.$$

(ii) Take $E_n = (0, \frac{1}{n})$ and $f_n = \chi_{E_n} f$. Then

$$\|f_n \mid L_{p)}\| = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\frac{1}{n}\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} \geq \exp \lim_{\varepsilon \to 0} \frac{\ln p - \frac{\varepsilon}{p} \ln n}{p - \varepsilon} = p^{\frac{1}{p}} > 0.$$

This shows, that $f(t) = t^{-\frac{1}{p}}$ does not have absolutely continuous norm in $L_{p)}$, as is indicated in [Fio00, Prop. 3.6]. For this reason, we immediately have Proposition 2.6 below.

(iii) Take any sequence $\{f_n\}_{n=1}^{\infty}$ with $f_n \to f$ μ-a.e. and $f_n \in L_\infty$. Then, there is an interval $(0, \tau_n) \subset \Omega$ where $f_n(t) \leq \frac{1}{2} t^{-\frac{1}{p}}$, hence $f(t) - f_n(t) \geq \frac{1}{2} t^{-\frac{1}{p}}$, $0 < t < \tau_n$.

Therefore,

$$\|f - f_n \,|\, L_{p)}\| \geq \sup_{0<\varepsilon<p-1} \frac{\varepsilon^{\frac{1}{p-\varepsilon}}}{2} \left(\int_0^{\tau_n} t^{-\frac{p-\varepsilon}{p}} \,\mathrm{d}t \right)^{\frac{1}{p-\varepsilon}} \geq \frac{p^{\frac{1}{p}}}{2} > 0.$$

We conclude, that $f \notin L^b_{p)}$.

Proposition 2.6. *The space $L_{p)}$ does not have absolutely continuous norm.*

We use the following equivalent norms several times in the sequel, so we prove them now. They are given without proof in [FK04].

Lemma 2.7. *For any $0 < \varepsilon_0 < \min(1, p-1)$, it holds*

$$\|g \,|\, L_{p)}\| \overset{(2.3)}{=} \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|g \,|\, L_{p-\varepsilon}\| \sim \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon}\| \tag{2.4}$$

$$\sim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon}\| \tag{2.5}$$

$$\sim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon,p}\|. \tag{2.6}$$

Proof. Consider

$$\frac{\varepsilon^{\frac{1}{p-\varepsilon}}}{\varepsilon^{\frac{1}{p}}} = \varepsilon^{\frac{1}{p-\varepsilon}-\frac{1}{p}} = \varepsilon^{\frac{\varepsilon}{p(p-\varepsilon)}} = \exp\left(\frac{\varepsilon \ln \varepsilon}{p(p-\varepsilon)} \right). \tag{2.7}$$

With $\varepsilon \ln(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$, we see that $\varepsilon \ln \varepsilon$ is bounded from below and above on the interval $(0, p-1)$. Therefore, the expression (2.7) is bounded from below and above as, while here the bounds are bigger than 0. Thus we have $\varepsilon^{\frac{1}{p-\varepsilon}} \sim \varepsilon^{\frac{1}{p}}$. This proves the equivalent norm (2.4).

Take any $0 < \varepsilon_0 < \min(1, p-1)$ and let $q = \frac{p-1}{\varepsilon_0}$, then

$$\begin{aligned} \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon}\| &= \sup_{0<\sigma<p-1} \left(\frac{\sigma}{q}\right)^{\frac{1}{p}} \|g \,|\, L_{p-\frac{\sigma}{q}}\| \\ &\overset{p-\frac{\sigma}{q}>p-\sigma}{\geq} c\, q^{-\frac{1}{p}} \sup_{0<\sigma<p-1} \sigma^{\frac{1}{p}} \|g \,|\, L_{p-\sigma}\| \sim \|g \,|\, L_{p)}\| \end{aligned}$$

by the monotony of Lebesgue spaces (1.2). The reverse inequality is obvious and the norm (2.5) is proven.

For the replacement of $\|\cdot \,|\, L_{p-\varepsilon}\|$ by $\|\cdot \,|\, L_{p-\varepsilon,p}\|$, we use the second formula of Lemma 2.1 and compute starting from (2.5)

$$\|g \,|\, L_{p)}\| \sim \sup_{0<\varepsilon<\varepsilon_0/2} (2\varepsilon)^{\frac{1}{p}} \|g \,|\, L_{p-2\varepsilon}\| \overset{(2.2)}{\leq} c \sup_{0<\varepsilon<\varepsilon_0/2} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon,p}\| \leq c \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g \,|\, L_{p-\varepsilon,p}\|.$$

The reverse inequality follows immediately from (2.2) and the norm (2.6) is proven. □

2.2 Small Lebesgue spaces

Definition 2.8 (Small Lebesgue spaces)**.** Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The space $L_{(p}$ contains all $g \in \mathcal{M}_0$ that can be represented as $g = \sum_{k\in\mathbb{N}} g_k$, $g_k \in \mathcal{M}_0$ for $k \in \mathbb{N}$, such that the norm

$$\|g \,|\, L_{(p}\| := \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k \,|\, L_{(p'-\varepsilon)'}\|. \tag{2.8}$$

is finite.

Theorem 2.9. *Let $1 < p < \infty$. The space $L_{(p}$ is a Banach function space.*

Proof. We only show what is not trivial. Let f, g be in $\mathcal{M}_+$.

(P1) First, for $g = \sum_k g_k$ and $f = \sum_k f_k$, take $h_{2k} := g_k$ and $h_{2k-1} := f_k$ for all $k \in \mathbb{N}$, hence $h = g + f$ with $h = \sum_n h_n$. Then

$$\sum_{n=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|h_n \,|\, L_{(p'-\varepsilon)'}\| = \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k \,|\, L_{(p'-\varepsilon)'}\| + \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|f_k \,|\, L_{(p'-\varepsilon)'}\|.$$

Taking the infimum over all representations $h = \sum_n h_n$ implies

$$\|g + f \,|\, L_{(p}\| \le \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k \,|\, L_{(p'-\varepsilon)'}\| + \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|f_k \,|\, L_{(p'-\varepsilon)'}\|.$$

for any representation $g = \sum_k g_k$ and $f = \sum_k f_k$. If we now also take the infima over all those representations, it yields $\|g + f \,|\, L_{(p}\| \le \|g \,|\, L_{(p}\| + \|f \,|\, L_{(p}\|$.

Second, assume that $\|f \,|\, L_{(p}\| = 0$. Then, for any $\delta > 0$ there is a representation $f = \sum_k f_k$ with

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \underbrace{\varepsilon^{-\frac{1}{p'-\varepsilon}}}_{\ge c_1} \underbrace{\|f_k \,|\, L_{(p'-\varepsilon)'}\|}_{\ge c_2 \|f_k \,|\, L_1\|} < \delta.$$

Thus, it is $f = 0$ as L_1 is a Banach function space and we have

$$\|f \,|\, L_1\| = \|\textstyle\sum_k f_k \,|\, L_1\| \le \displaystyle\sum_{k=1}^{\infty} \|f_k \,|\, L_1\| < c_3\delta.$$

(P2) By [Fio00, Lemma 2.1], for $g \le f = \sum_k f_k$ with $f_k \ge 0$, all $k \in \mathbb{N}$, there is a decomposition $g = \sum_k g_k$ with $0 \le g_k \le f_k$, all $k \in \mathbb{N}$ (we omit the computation that proves the statement, as it is fully given in the reference). Then it holds $\|g \,|\, L_{(p}\| \le \|f \,|\, L_{(p}\|$ by

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k \,|\, L_{(p'-\varepsilon)'}\| \le \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|f_k \,|\, L_{(p'-\varepsilon)'}\|.$$

(P3) This is the proof of [Fio00, Corollary 2.8]. Consider $g_j \in \mathcal{M}_0$ with $0 \le g_j \nearrow g$ μ-a.e. and $E_n \subset \Omega$ chosen by $E_n = \{x \in \Omega : g(x) \ge n\}$. Then $\chi_{E_n} \xrightarrow[n\to\infty]{} 0$ μ-a.e. Set $F_n = \Omega \setminus E_n$, hence $\min(g,n) \ge g\chi_{F_n}$. Since $g = g\chi_{F_n} + g\chi_{E_n}$, it holds

$$\begin{aligned} \|g \,|\, L_{(p}\| &= \|g\chi_{F_n} + g\chi_{E_n} \,|\, L_{(p}\| \le \|g\chi_{F_n} \,|\, L_{(p}\| + \|g\chi_{E_n} \,|\, L_{(p}\| \\ \implies \|g \,|\, L_{(p}\| &- \|g\chi_{F_n} \,|\, L_{(p}\| \le \|g\chi_{E_n} \,|\, L_{(p}\| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

where that last limit holds due to Proposition 2.10 that is proven below. Now, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|g \,|\, L_{(p}\| - \varepsilon \le \|g\chi_{F_N} \,|\, L_{(p}\| \le \|\min(g,N) \,|\, L_{(p}\| \overset{\text{(P2)}}{\le} \|g \,|\, L_{(p}\|.$$

As $\min(g,N) \in L_\infty \hookrightarrow L_{p+1}$, it holds $\min(g_j,N) \xrightarrow[j\to\infty]{} \min(g,N)$ in L_{p+1} by the dominated convergence theorem, and hence in $L_{(p}$, as $L_{p+1} \hookrightarrow L_{(p}$. Therefore, there is $J \in \mathbb{N}$ such that for all $j \ge J$

$$\|g \,|\, L_{(p}\| - 2\varepsilon \le \|\min(g,N) \,|\, L_{(p}\| - \varepsilon \le \|\min(g_j,N) \,|\, L_{(p}\| \le \|g_j \,|\, L_{(p}\| \le \|g \,|\, L_{(p}\|. \qquad \square$$

Proposition 2.10. *The space $L_{(p}$ has absolutely continuous norm.*

Proof. We follow the proof in [Fio00, Lemma 2.6] and show that (1.11) holds. Let $E_n \subset \Omega$ with $\chi_{E_n} \xrightarrow[n\to\infty]{} 0$ μ-a.e., and let $g \in L_{(p}$. Choose any decomposition $g = \sum_k g_k$ with

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k \,|\, L_{(p'-\varepsilon)'}\| < \infty.$$

Let $a_{k,n} := \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|\chi_{E_n} g_k \,|\, L_{(p'-\varepsilon)'}\|$ for all $k,n \in \mathbb{N}$. Since $\sum_k a_{k,1} < \infty$ and $a_{k,n} \xrightarrow[n\to\infty]{} 0$, all $k \in \mathbb{N}$, it holds $\sum_k a_{k,n} \xrightarrow[n\to\infty]{} 0$. Therefore,

$$\|\chi_{E_n} g \,|\, L_{(p}\| \le \sum_{k=1}^{\infty} a_{k,n} \xrightarrow[n\to\infty]{} 0. \qquad \square$$

These spaces were first introduced in [Fio00] as associate spaces to $L_{p)}$. To be precise, in this paper the author uses the space given by Definition 2.8 as auxiliary space to define the small Lebesgue space by consisting of all measurable functions such that

$$\|g \,|\, L_{p)'}\| = \sup_{\substack{0\le\psi\le|g| \\ \psi\in L_{(p'}}} \|\psi \,|\, L_{(p'}\|, \qquad 1 < p < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is finite. In [FR03], the authors prove the monotone convergence theorem for the spaces of Definition 2.8. Making use of this fact, the equivalence of $L_{p)'}$ and $L_{(p'}$ is explicitly shown in [CF05].

Theorem 2.11 (Monotone convergence theorem for $L_{(p'}$, [FR03, Theorem 2]). *Let* $\{f_n\}_{n=1}^{\infty}$ *be a monotone nondecreasing sequence in* $L_{(p'}$ *with* $\sup_{n\in\mathbb{N}} \|f_n \,|\, L_{(p'}\| < \infty$. *For* $f = \sup_{n\in\mathbb{N}} f_n$ *it holds*

(i) $f \in L_{(p'}$

(ii) $f_n \nearrow f$ *μ-a.e.*

(iii) $f_n \to f$ *in* $L_{(p'}$.

Proof. The proof of the theorem is based on the idea that for $f_n \le f_m$ and $f_m = \sum_k f_m^{(k)}$, one can find a decomposition of f_n such that $f_n^{(k)} \le f_m^{(k)}$ for all $k \in \mathbb{N}$ (see also the proof of Theorem 2.9). This permits to estimate $\|f_m - f_n \,|\, L_{(p'}\|$ by $\|f_m \,|\, L_{(p'}\| - \|f_n \,|\, L_{(p'}\|$, what becomes small as $\|f_n \,|\, L_{(p'}\|$ is convergent. Some more precise work is needed in the process to derive something like (roughly speaking)

$$\left(\|f_m^{(k)} \,|\, L_q\|^q - \|f_n^{(k)} \,|\, L_q\|^q \right)^{\frac{1}{q}} \lesssim \|f_m^{(k)} \,|\, L_q\| - \|f_n^{(k)} \,|\, L_q\|.$$

We leave out the details that are thoroughly given in [FR03]. □

The still somewhat complicated expression (2.8) can be motivated by the later on proven Hölder inequality (2.12). Similar to Proposition 2.4, for $\varepsilon > 0$ it is easy to prove that $L_{p+\varepsilon} \hookrightarrow L_{(p} \hookrightarrow L_p$. We give more embeddings and examples in Section 4.5.

Just as for the grand Lebesgue spaces, we prove some equivalent norms that turn out to be useful in the sequel.

Lemma 2.12 (Equivalent norms). *For any* $0 < \varepsilon_0 < p' - 1$ *and* $1 < p < \infty$*, it holds*

$$\|g \,|\, L_{(p}\| \sim \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\| \tag{2.9}$$

$$\sim \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\| \tag{2.10}$$

$$\sim \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon,p}\|. \tag{2.11}$$

Proof. The main steps of the proof are taken from [FK04, Lemma 3.1].

It holds $\varepsilon^{-\frac{1}{p'-\varepsilon}} \sim \varepsilon^{-\frac{1}{p'}}$, as the boundedness of the expression (2.7) shows. Next, we define $\gamma(\varepsilon)$ by $(p'-\varepsilon)' = p + \gamma(\varepsilon)$, i.e. for $0 < \varepsilon < p'-1$ we have

$$(p'-\varepsilon)' = p + \frac{\varepsilon(p-1)^2}{1-\varepsilon(p-1)} =: p + \gamma(\varepsilon) \quad \text{with} \quad \gamma(\varepsilon) \ge \varepsilon(p-1)^2 =: c_p\varepsilon.$$

It does *not* hold $\gamma \sim \varepsilon$, as [FK04] suggest. We rather have for any $g_k \in \mathcal{M}_0$

$$\inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\gamma(\varepsilon)}\| \overset{(1.2)}{\gtrsim} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+c_p\varepsilon}\| \gtrsim \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\|.$$

The last step is clear if $c_p \geq 1$ by the embeddings of Lebesgue spaces (1.2). However, if $c_p < 1$, then we first replace the variable of the infimum by $\varepsilon' := c_p\varepsilon$, which runs from 0 to $c_p(p'-1)$, and the statement follows. On the other hand it holds with $\varepsilon \leq {c_p}^{-1}\gamma(\varepsilon)$

$$\begin{aligned}\inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\gamma(\varepsilon)}\| &\leq {c_p}^{\frac{1}{p'}} \inf_{0<\varepsilon<p'-1} \gamma(\varepsilon)^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\gamma(\varepsilon)}\| \sim \inf_{0<\gamma<\infty} \gamma^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\gamma}\| \\ &\leq \inf_{0<\gamma<p'-1} \gamma^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\gamma}\|.\end{aligned}$$

This proves the norm (2.9).

Now we show that the choice of ε_0 is arbitrary and hence that the limits of the infimum can be exchanged with any $0 < \varepsilon_0 < p' - 1$ instead of $p' - 1$. Look at $q = \frac{p'-1}{\varepsilon_0}$. Obviously, it holds

$$\inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\| \geq \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\|.$$

On the other hand,

$$\begin{aligned}\inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\| &= \inf_{0<\sigma<q\varepsilon_0} \left(\frac{\sigma}{q}\right)^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\frac{\sigma}{q}}\| \\ &\overset{(1.2)}{\leq} c_\Omega\, q^{\frac{1}{p'}} \inf_{0<\sigma<p'-1} \sigma^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\sigma}\|.\end{aligned}$$

Therefore, $\inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\| \sim \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{(p'-\varepsilon)'}\|$.

It is left to prove the norm replacement of $\|\cdot \,|\, L_{p+\varepsilon}\|$ by $\|\cdot \,|\, L_{p+\varepsilon,p}\|$. Using Lemma 2.1, (2.1) yields

$$\begin{aligned}\inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon,p}\| &\leq C \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+2\varepsilon}\| \\ &= C \cdot 2^{\frac{1}{p'}} \inf_{0<\varepsilon<\varepsilon_0} (2\varepsilon)^{-\frac{1}{p'}} \|g_k \,|\, L_{p+2\varepsilon}\| \\ &\leq C' \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, L_{p+\varepsilon}\|.\end{aligned}$$

The second inequality follows immediately from Lemma 2.1. □

2.3 Duality

Let us now come to the central statement of this chapter, that for $1/p + 1/p' = 1$ the spaces $L_{p)}$ and $L_{(p'}$ are associate to each other. The proof follows the steps of [Fio00] including the modifications indicated by [CF05]. Our aim is to show that the norm (2.3) of $L_{p)}$ is equal to the associate norm of $L_{(p'}$ given by (1.10). After showing the Hölder inequality, we prove the equality for special functions and then generalise to the whole spaces. We also try to fix a problem that occurs in Fiorenza's proof of our Lemma 2.14 when choosing the appropriate function g.

Theorem 2.13 ([Fio00, Theorem 2.5]). *Let* $1 < p < \infty$ *and* $f \in L_{p)}$. *Then the following Hölder inequality holds*

$$\int_\Omega |fg| \, \mathrm{d}\mu \leq \|f \,|\, L_{p)}\| \, \|g \,|\, L_{(p'}\| \quad \textit{for} \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.12}$$

Proof. Let $g = \sum_k g_k$ be any decomposition with measurable g_k, $k \in \mathbb{N}$. Then it holds for $f \in L_{p)}$ and $0 < \varepsilon < p - 1$

$$\begin{aligned}\int_\Omega |fg_k| \, \mathrm{d}\mu &\leq \|f \,|\, L_{p-\varepsilon}\| \, \|g_k \,|\, L_{(p-\varepsilon)'}\| \\ &= \left(\varepsilon \int_\Omega |f|^{p-\varepsilon} \, \mathrm{d}\mu\right)^{\frac{1}{p-\varepsilon}} \cdot \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'} \, \mathrm{d}\mu\right)^{\frac{1}{(p-\varepsilon)'}} \\ &\leq \|f \,|\, L_{p)}\| \cdot \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'} \, \mathrm{d}\mu\right)^{\frac{1}{(p-\varepsilon)'}} .\end{aligned}$$

Here we can take the infimum over ε. Therefore

$$\begin{aligned}\int_\Omega |fg| \, \mathrm{d}\mu &\leq \sum_{k=1}^\infty \int_\Omega |fg_k| \, \mathrm{d}\mu \\ &\leq \sum_{k=1}^\infty \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{(p-\varepsilon)}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'} \, \mathrm{d}\mu\right)^{\frac{1}{(p-\varepsilon)'}} \|f \,|\, L_{p)}\|\end{aligned}$$

and (2.12) follows by again taking the infimum over all possible decompositions $g = \sum_k g_k$. ☐

Lemma 2.14 ([Fio00, Lemma 2.9]). *Let* $f \in L_\infty$. *Then there exists* $g \in L_\infty$ *such that*

$$\int_\Omega |fg| \, \mathrm{d}\mu = \|f \,|\, L_{p)}\| \, \|g \,|\, L_{(p'}\|.$$

Proof. The inequality $\leq$ is an immediate consequence of Theorem 2.13. On the other hand, as $f \in L_\infty$, $\varepsilon^{\frac{1}{p-\varepsilon}} \|f \,|\, L_{p-\varepsilon}\| \xrightarrow[\varepsilon \to 0]{} 0$, hence there is $0 < \sigma \leq p - 1$ such that

$$\|f \,|\, L_{p)}\| = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f \,|\, L_{p-\varepsilon}\| = \sigma^{\frac{1}{p-\sigma}} \|f \,|\, L_{p-\sigma}\|.$$

If $\sigma < p - 1$, take $g = f^{\frac{p-\sigma}{(p-\sigma)'}} \in L_\infty$. Else, take $g \equiv 1$. Then in the classical Hölder inequality it holds in fact equality, i.e. (with the convention $(p - \sigma)' = \infty$ for $p - \sigma = 1$)

$$\int_\Omega |fg| \, \mathrm{d}\mu = \|f \,|\, L_{p-\sigma}\| \, \|g \,|\, L_{(p-\sigma)'}\|.$$

If $\sigma < p-1$, then

$$\begin{aligned}\|f\,|\,L_{p)}\|\,\|g\,|\,L_{(p'}\| &\leq \sigma^{\frac{1}{p-\sigma}}\,\|f\,|\,L_{p-\sigma}\|\cdot \inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\,\|g\,|\,L_{(p-\varepsilon)'}\| \\ &\leq \sigma^{\frac{1}{p-\sigma}}\,\|f\,|\,L_{p-\sigma}\|\cdot \sigma^{-\frac{1}{p-\sigma}}\,\|g\,|\,L_{(p-\sigma)'}\| = \int_\Omega |fg|\,\mathrm{d}\mu.\end{aligned}$$

Otherwise, it holds with $g\equiv 1$ and, thus, $\|g\,|\,L_{(p-\varepsilon)'}\| = 1$

$$\begin{aligned}\|f\,|\,L_{p)}\|\,\|g\,|\,L_{(p'}\| &\leq \sigma^{\frac{1}{p-\sigma}}\,\|f\,|\,L_{p-\sigma}\|\cdot \inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\,\|g\,|\,L_{(p-\varepsilon)'}\| \\ &\leq (p-1)\,\|f\,|\,L_1\|\cdot \lim_{\varepsilon\to p-1}\varepsilon^{-\frac{1}{p-\varepsilon}} \\ &= (p-1)\,\|f\,|\,L_1\|\cdot\frac{1}{p-1} = \int_\Omega |f|\,\mathrm{d}\mu = \int_\Omega |fg|\,\mathrm{d}\mu.\end{aligned}$$ □

Corollary 2.15. *Let $f\in L_{p)}$. Then*

$$\|f\,|\,L_{p)}\| = \sup\left\{\int_\Omega |fg|\,\mathrm{d}\mu : g\in L_{(p'},\ \|g\,|\,L_{(p'}\|\leq 1\right\}. \tag{2.13}$$

Proof. W.l.o.g., let $f\geq 0$ μ-a.e. Let $f_n=\min(f,n)$, hence $f_n\in L_\infty$, $0\leq f_n\nearrow f$ and $\|f_n\,|\,L_{p)}\|\nearrow\|f\,|\,L_{p)}\|$. Then by Lemma 2.14 we have $g_n\in L_\infty$ for all $n\in\mathbb{N}$, $\|g_n\,|\,L_{(p'}\|=1$ (if not then divide g_n by its norm in $L_{(p'}$), such that

$$\int_\Omega |f_ng_n|\,\mathrm{d}\mu = \|f_n\,|\,L_{p)}\|.$$

Now, for any $\delta>0$ there exists $n\in\mathbb{N}$ with $\|f_n\,|\,L_{p)}\| > \|f\,|\,L_{p)}\|-\delta$. Finally,

$$\|f\,|\,L_{p)}\|-\delta < \int_\Omega |f_ng_n|\,\mathrm{d}\mu \leq \int_\Omega |fg_n|\,\mathrm{d}\mu$$

and the statement is true as δ can be taken arbitrarily small. □

Theorem 2.16 (Associate spaces [Fio00, Theorem 3.5])**.** *The space $L_{p)}$ is associate to $L_{(p'}$ and vice versa.*

Proof. This is an immediate consequence of Corollary 2.15. □

Corollary 2.17. *The spaces $L_{p)}$ and $L_{(p}$ are rearrangement invariant Banach spaces.*

Proof. It is obvious from the definition that the grand Lebesgue spaces are rearrangement invariant. Then the small Lebesgue spaces being its associate spaces are rearrangement invariant too (cf. [BS88, Corollary II.4.4]). □

Corollary 2.18. *The spaces $L_{p)}$ and $L_{(p}$ are not reflexive.*

Proof. This is clear due to Proposition 2.6. □

3. Extrapolation approach

Extrapolation spaces are, in a certain sense, constructions inverse to interpolation spaces: whereas interpolation begins with two Banach spaces $(A_0, A_1) \in \mathscr{C}_2$ to construct a *scale* of spaces $A_\theta \in \mathscr{C}_1$, $0 < \theta < 1$, extrapolation begins with a scale to construct spaces A_0 and A_1 that can be interpreted as its endpoints. The grand Lebesgue spaces are easy to derive from such constructions, and the same is possible for the small Lebesgue spaces too. In Chapter 4, this leads to some further norms for $L_{p)}$ and $L_{(p}$ via the additional application of interpolation techniques.

There is no standard framework of extrapolation, but a number of authors have developed techniques. We mostly follow [CK14], but in the next chapter we also refer to the more general theory of [KM05] that is based on [JM91], as this is the context in which our final results appeared first in literature.

3.1 Abstract extrapolation spaces

Definition 3.1 (Compatible [CK14])**.** We call a family of Banach spaces $\{Y_\theta\}_{\theta\in\Theta}$, $\Theta \subset [0,1]$ *compatible* if there are Y_0, Y_1 with

$$Y_0 \hookrightarrow Y_\theta \hookrightarrow Y_\eta \hookrightarrow Y_1 \quad \text{for} \quad 0 < \theta < \eta < 1, \quad \theta, \eta \in \Theta \tag{3.1}$$

and the norms of inclusion between the spaces with indices in $\Theta \cap (0,1)$ are uniformly bounded.

Definition 3.2. Let $\{Y_\theta\}_{\theta\in\Theta}$ be a family of compatible Banach spaces. Let $0 \le \theta < 1$ be such that for some $\varepsilon > 0$ it holds $(\theta, \theta+\varepsilon] \subset \Theta$. Let $1 \le q \le \infty$ and φ be a function satisfying

$$\begin{aligned} &\text{(i)} \quad \varphi \text{ is monotone, positive and continuous} \\ &\text{(ii)} \quad \varphi(t) \sim \varphi(2t) \\ &\text{(iii)} \quad \left(\int_0^\varepsilon \varphi(t)^q \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty \quad \text{if } q < \infty, \text{ else} \quad \sup_{0<t<\varepsilon} \varphi(t) < \infty. \end{aligned} \tag{3.2}$$

Then we define the space $Y_\theta(\log Y)^+_{\varphi,q}$ as consisting of all $b \in \bigcap_{\eta\in(\theta,\theta+\varepsilon]} Y_\eta$ for which the following norm is finite:

$$\|b\,|\,Y_\theta(\log Y)^+_{\varphi,q}\| := \begin{cases} \left(\displaystyle\int_0^\varepsilon [\varphi(t)\,\|b\,|\,Y_{\theta+t}\|]^q \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{if} \quad 1 \le q < \infty \\ \displaystyle\sup_{0<t<\varepsilon} \varphi(t)\,\|b\,|\,Y_{\theta+t}\| & \text{if} \quad q = \infty \end{cases} \tag{3.3}$$

Proposition 3.3 (Basic properties)**.** *In the situation of Definition 3.2 the following statements concerning $Y_\theta(\log Y)^+_{\varphi,q}$ hold:*

(i) The choice of ε with $(\theta,\theta+\varepsilon] \subset \Theta$ is arbitrary.

(ii) For any $\theta < \eta \le \theta+\varepsilon$ it holds $Y_\theta(\log Y)^+_{\varphi,q} \hookrightarrow Y_\eta$.

(iii) If $\theta \in \Theta$, $\theta > 0$, then $Y_\theta \hookrightarrow Y_\theta(\log Y)^+_{\varphi,q}$.

(iv) Take $J \in \mathbb{N}$ such that $(\theta,\theta+2^{-J}] \subset \Theta$. Then an equivalent norm on $Y_\theta(\log Y)^+_{\varphi,q}$ is given by

$$\|b\,|\,Y_\theta(\log Y)^+_{\varphi,q}\| \sim \left(\sum_{n=J}^\infty \Big[\varphi(2^{-n})\,\|b\,|\,Y_{\sigma_n}\|\Big]^q\right)^{\frac{1}{q}} \quad \textit{with} \quad \sigma_n := \theta + 2^{-n}. \tag{3.4}$$

(v) $Y_\theta(\log Y)^+_{\varphi,q}$ is a Banach space.

Proof. In [CK14], (ii) and (iv) are proven. Here we give the proofs for all statements.

(i) Let $\varepsilon \ne \varepsilon'$ with $(\theta,\theta+\varepsilon], (\theta,\theta+\varepsilon'] \subset \Theta$ be given. W.l.o.g. $\varepsilon < \varepsilon'$. Let φ satisfy condition (3.2) for ε'. Now prove the equivalence of the norms induced by ε and ε' i.e.

$$\left(\int_0^\varepsilon \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \sim \left(\int_0^{\varepsilon'} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

The direction "$\le$" is obvious, for the other direction it holds with c being the uniform boundary of the norms of inclusions according to Definition 3.1

$$\begin{aligned} \left(\int_\varepsilon^{\varepsilon'} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} &\overset{\substack{Y_{\eta_1}\hookrightarrow Y_{\eta_2}\\ \eta_1\le\eta_2}}{\le} c\left(\int_\varepsilon^{\varepsilon'} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+\varepsilon}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\le c\,\|b\,|\,Y_{\theta+\varepsilon}\| \left(\int_0^{\varepsilon'} \varphi(t)^q \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \sim \|b\,|\,Y_{\theta+\varepsilon}\| \left(\int_0^{\varepsilon} \varphi(t)^q \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\le c\left(\int_0^\varepsilon \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \end{aligned}$$

(ii) As the norms of inclusion in (3.1) are uniformly bounded, it holds $\|b\,|\,Y_\eta\| \le c\,\|b\,|\,Y_{\theta+t}\|$ for any t with $0 < t < \eta-\theta$. Let $b \in Y_\theta(\log Y)^+_{\varphi,q}$, then

$$\|b\,|\,Y_\eta\| \overset{(3.2)}{=} \underbrace{\left(\int_0^{\eta-\theta} \varphi(t)^q \,\frac{\mathrm{d}t}{t}\right)^{-\frac{1}{q}}}_{:=c_1>0} \left(\int_0^{\eta-\theta} \Big[\varphi(t) \underbrace{\|b\,|\,Y_\eta\|}_{\le c\,\|b\,|\,Y_{\theta+t}\|}\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

$$\le c_2 \left(\int_0^{\eta-\theta} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \le c_2\,\|b\,|\,Y_\theta(\log Y)^+_{\varphi,q}\|.$$

(iii) $$\|b\,|\,Y_\theta(\log Y)^+_{\varphi,q}\| \le c\left(\int_0^\varepsilon [\varphi(t)\,\|b\,|\,Y_\theta\|]^q\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} = c\,\|b\,|\,Y_\theta\| \underbrace{\left(\int_0^\varepsilon \varphi(t)^q\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}}_{<\infty \text{ by } (3.2)}.$$

(iv) We can assume that $2^{-J} < \varepsilon$ by (i). As $\varphi(t) \sim \varphi(2t)$, there are $c_1, c_2 > 0$ such that for $n \ge J$

$$c_1\varphi(2^{-n}) \le \varphi(t) \le c_2\varphi(2^{-n}) \quad \text{for} \quad 2^{-n-1} \le t \le 2^{-n}.$$

Thus, we compute

$$\|b\,|\,Y_\theta(\log Y)^+_{\varphi,q}\| = \left(\sum_{n=J}^\infty \int_{2^{-n-1}}^{2^{-n}} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t} + \int_{2^{-J}}^{\varepsilon} \Big[\varphi(t)\,\|b\,|\,Y_{\theta+t}\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

$$\sim \left(\sum_{n=J}^\infty \Big[\varphi(2^{-n})\,\|b\,|\,Y_{\sigma_n}\|\Big]^q\right)^{\frac{1}{q}}.$$

(v) Take a Cauchy sequence $\{b_n\}_{n\in\mathbb{N}}$ in $Y_\theta(\log Y)^+_{\varphi,q}$ and show that it converges in $Y_\theta(\log Y)^+_{\varphi,q}$ using the norm (3.4) with J as given there. Let $\delta > 0$. Choose $n_0 \in \mathbb{N}$ by $\|b_n - b_m\,|\,Y_\theta(\log Y)^+_{\varphi,q}\| < \frac{\delta}{2}$ for $n, m \ge n_0$.

Let $j \ge J$ and $\sigma_j = \theta + 2^{-j}$. By $Y_\theta(\log Y)^+_{\varphi,q} \hookrightarrow Y_{\sigma_j}$ and the completeness of Y_{σ_j} we know that there is $b^j \in Y_{\sigma_j}$ with $b_n \xrightarrow[n\to\infty]{} b^j$. As $Y_{\sigma_j} \hookrightarrow Y_{\sigma_J}$, it holds $b^j \in Y_{\sigma_J}$ and all b^j are the same because of the uniqueness of the limits, hence $b := b^j$, $j \ge J$.

For $j \ge J$, there exists $m(j) \ge n_0$ such that

$$\|b - b_m\,|\,Y_{\sigma_j}\| < \frac{\delta}{2c} \quad \text{for} \quad m \ge m(j) \quad \text{with} \quad c = \left(\sum_{j=J}^\infty \varphi(2^{-j})^q\right)^{\frac{1}{q}}. \tag{3.5}$$

The last expression is finite because of condition (3.2) and $\varphi(t) \sim \varphi(2t)$. Now for $N > J$ let $m_N = \max\limits_{J\le j\le N} m(j)$. We calculate for $n \ge n_0$:

$$\left(\sum_{j=J}^N \Big[\varphi(t)\,\|b - b_n\,|\,Y_{\sigma_j}\|\Big]^q\right)^{\frac{1}{q}} \overset{\substack{\text{Mink.}\\ q\ge 1}}{=} \left(\sum_{j=J}^N \Big[\varphi(t)\,\|b_{m_N} - b_n + b - b_{m_N}\,|\,Y_{\sigma_j}\|\Big]^q\right)^{\frac{1}{q}}$$

$$\leq \underbrace{\left(\sum_{j=J}^{N}\Big[\varphi(t)\,\|b_{m_N}-b_n\,|\,Y_{\sigma_j}\|\Big]^q\right)^{\frac{1}{q}}}_{\leq \|b_{m_N}-b_n\,|\,Y_\theta(\log Y)^+_{\varphi,q}\|} + \left(\sum_{j=J}^{N}\Big[\varphi(t)\,\underbrace{\|b-b_{m_N}\,|\,Y_{\sigma_j}\|}_{<\frac{\delta}{2c}\text{ by (3.5)}}\Big]^q\right)^{\frac{1}{q}}$$

$$< \frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

This estimate holds independent of the choice of N. If we now take $N\to\infty$ this yields $\|b-b_n\,|\,Y_\theta(\log Y)^+_{\varphi,q}\|\leq\delta$ for $n\geq n_0$. □

Definition 3.4. Let $\{Y_\theta\}_{\theta\in\Theta}$ be a family of compatible Banach spaces (cf. Definition 3.1). Let $0<\theta\leq 1$ such that for some $\varepsilon>0$ it holds $[\theta-\varepsilon,\theta)\subset\Theta$. Let $1\leq q\leq\infty$ and ψ be a function satisfying

$$\begin{aligned} &\text{(i)} \quad \psi \text{ is monotone, positive and continuous}\\ &\text{(ii)} \quad \psi(t)\sim\psi(2t)\\ &\text{(iii)} \quad \left(\int_0^\varepsilon \psi(t)^{-q'}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q'}}<\infty \quad \text{if } q>1, \text{ else} \quad \sup_{0<t<\varepsilon}\psi(t)^{-1}<\infty. \end{aligned} \tag{3.6}$$

Then, we define the space $Y_\theta(\log Y)^-_{\psi,q}$ as consisting of all $b\in Y_1$ that can be represented as $b=\int_0^\varepsilon w(t)\,\frac{\mathrm{d}t}{t}$ with $w(t)\in Y_{\theta-t}$ such that

$$\begin{cases} \left(\int_0^\varepsilon [\psi(t)\,\|w(t)\,|\,Y_{\theta-t}\|]^q\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{if} \quad 1\leq q<\infty\\ \sup_{0<t<\varepsilon}\psi(t)\,\|w(t)\,|\,Y_{\theta-t}\| & \text{if} \quad q=\infty. \end{cases} \tag{3.7}$$

is finite. The norm $\|\cdot\,|\,Y_\theta(\log Y)^-_{\psi,q}\|$ is given by the infimum of (3.7) over all possible representations of b.

Proposition 3.5 (Basic properties)**.** *In the situation of Definition 3.4 the following statements concerning $Y_\theta(\log Y)^-_{\psi,q}$ hold:*

(i) The choice of ε with $[\theta-\varepsilon,\theta)\subset\Theta$ is arbitrary.

(ii) For any $\theta-\varepsilon\leq\eta<\theta$ it holds $Y_\eta\hookrightarrow Y_\theta(\log Y)^-_{\psi,q}$.

(iii) If $\theta\in\Theta$, $\theta<1$, then $Y_\theta(\log Y)^-_{\psi,q}\hookrightarrow Y_\theta$.

(iv) Let $J\in\mathbb{N}$ with $[\theta-2^{-J},\theta)\subset\Theta$ be arbitrary. Then an equivalent norm on $Y_\theta(\log Y)^-_{\psi,q}$ is given by

$$\|b\,|\,Y_\theta(\log Y)^-_{\psi,q}\|\sim\inf_{b=\sum b_n}\left(\sum_{n=J}^{\infty}\Big[\psi(2^{-n})\,\|b_n\,|\,Y_{\lambda_n}\|\Big]^q\right)^{\frac{1}{q}} \quad \textit{with} \quad \lambda_n:=\theta-2^{-n}. \tag{3.8}$$

(v) $Y_\theta(\log Y)^-_{\psi,q}$ is a Banach space.

Proof. The proofs of (i) to (iv) are similar to Proposition 3.3, (ii), (iii) and (iv) are also proven in [CK14]. That is why we only prove (v).

(v) We use the fact that $Y_\theta(\log Y)^-_{\psi,q}$ is complete if and only if (cf. [BL76, Lemma 2.2.1])

$$\sum_{n=1}^{\infty} \|b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| < \infty$$
$$\implies \exists b \in Y_\theta(\log Y)^-_{\psi,q} : \|b - \textstyle\sum_{n\le m} b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| \xrightarrow[m\to\infty]{} 0.$$

Hence, take a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $Y_\theta(\log Y)^-_{\psi,q}$ with $\sum_n \|b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| < \infty$. Choose for $n \in \mathbb{N}$ decompositions $b_n = \sum_j b_n^j$ where $j \ge J$ and $b_n^j \in Y_{\lambda_j}$ with $\lambda_j = \theta - 2^{-j}$, such that

$$\left(\sum_{j=J}^{\infty}\Big[\psi(2^{-j})\,\|b_n^j \,|\, Y_{\lambda_j}\|\Big]^q\right)^{\frac{1}{q}} < \|b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| + 2^{-n}. \tag{3.9}$$

It follows that for any $j \ge J$

$$\sum_{n=1}^{\infty} \|b_n^j \,|\, Y_{\lambda_j}\| \le \frac{1}{\psi(2^{-j})}\left(\sum_{n=1}^{\infty} \|b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| + 2^{-n}\right) < \infty$$

and hence, as Y_{λ_j} is complete, there is $b^j \in Y_{\lambda_j}$ with $\|b^j - \sum_{n\le m} b_n^j \,|\, Y_{\lambda_j}\| \xrightarrow[m\to\infty]{} 0$. Set $b := \sum_j b^j$. As it holds by the generalised Minkowski's inequality for infinite series (cf. [HLP99, p. 123]) for any $m \in \mathbb{N}$

$$\left(\sum_{j=J}^{\infty}\Big[\psi(2^{-j})\,\|b^j - \textstyle\sum_{n\le m} b_n^j \,|\, Y_{\lambda_j}\|\Big]^q\right)^{\frac{1}{q}} \le \left(\sum_{j=J}^{\infty}\left[\psi(2^{-j})\sum_{n=m}^{\infty}\|b_n^j \,|\, Y_{\lambda_j}\|\right]^q\right)^{\frac{1}{q}}$$
$$\overset{\text{Mink.}}{\le} \sum_{n=m}^{\infty}\left(\sum_{j=J}^{\infty}\Big[\psi(2^{-j})\,\|b_n^j \,|\, Y_{\lambda_j}\|\Big]^q\right)^{\frac{1}{q}} \overset{(3.9)}{<} \infty$$

we conclude that

$$\|b - \textstyle\sum_{n\le m} b_n \,|\, Y_\theta(\log Y)^-_{\psi,q}\| \overset{\inf}{\le} \left(\sum_{j=J}^{\infty}\Big[\psi(2^{-j})\,\|b^j - \textstyle\sum_{n\le m} b_n^j \,|\, Y_{\lambda_j}\|\Big]^q\right)^{\frac{1}{q}} \xrightarrow[m\to\infty]{} 0. \qquad \square$$

The cases $\varphi(t) = t^a$ and $\psi(t) = t^{-a}$ respectively ($a > 0$) are of particular importance in the following investigations. We write in these cases

$$Y_\theta(\log Y)^+_{\varphi,q} = Y_\theta(\log Y)^+_{a,q} \qquad \text{and} \qquad Y_\theta(\log Y)^-_{\psi,q} = Y_\theta(\log Y)^-_{a,q}. \tag{3.10}$$

The nomenclature of the spaces $Y_\theta(\log Y)^+_{\varphi,q}$ and $Y_\theta(\log Y)^-_{\psi,q}$ can be seen from the Propositions 3.7 and 3.8 below, or even better from the earlier paper [CFT04] where similar constructions were used: Finding equivalent norms for the Lorentz-Zygmund spaces $L_{p,q}(\log L)_a$ based on the L_p spaces only was the motivation to introduce these spaces.

3.2 Examples

We now characterise specific spaces as abstract extrapolation spaces, these are the grand and small Lebesgue spaces, but also the Zygmund spaces. Therefore, we extrapolate the scale of Lebesgue spaces, but we also give a construction using Lorentz spaces. The former allows us to extend statements about Lebesgue spaces to grand and small Lebesgue spaces. The characterisation of Zygmund spaces is useful in order to compare them to the grand and small Lebesgue spaces.

We start by a short lemma that is repeatedly used in the current and the next chapter, whenever a supremum and a logarithm appear.

Lemma 3.6. *Let $0 < p < \infty$, $0 < \varepsilon_0 < 1$ and $0 < t < 1$. It holds with constants depending only on p and ε_0*

$$\sup_{0<\varepsilon<\varepsilon_0} t^{\varepsilon}\varepsilon^{\frac{1}{p}} \sim (1-\ln t)^{-\frac{1}{p}}. \tag{3.11}$$

Proof. We show that $\frac{1}{\sup t^{\varepsilon p}\varepsilon} \sim (1-\ln t)$ with constants independent of t. For those t with $-\frac{1}{p\ln t} < \varepsilon_0$ it holds

$$\frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} t^{\varepsilon p}\varepsilon} \leq -\frac{1}{t^{-\frac{1}{\ln t}}\frac{1}{p\ln t}} = -ep\ln t \leq ep(1-\ln t),$$

whereas for $-\frac{1}{p\ln t} \geq \varepsilon_0$, for any $0 < \varepsilon < \varepsilon_0$ we have $t \geq \exp\left(-\frac{1}{p\varepsilon}\right)$ and therefore

$$\frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} t^{\varepsilon p}\varepsilon} \leq \frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} \exp(-1)\varepsilon} = \frac{e}{\varepsilon_0} \leq \frac{e}{\varepsilon_0}(1-\ln t).$$

On the other hand, $t^c(1-\ln t) \xrightarrow[t\to 0]{} 0$ for any $c > 0$, and hence $t^c(1-\ln t)$ is bounded for $0 < t < 1$ by $c^{-1}e^{c-1}$ (this is an easy calculation). Now with $c = p\varepsilon$ this finally yields

$$1-\ln t = \frac{\sup\limits_{0<\varepsilon<\varepsilon_0} \varepsilon t^{\varepsilon p}(1-\ln t)}{\sup\limits_{0<\varepsilon<\varepsilon_0} t^{\varepsilon p}\varepsilon} \leq \frac{\sup\limits_{0<\varepsilon<\varepsilon_0} \varepsilon(\varepsilon p)^{-1}e^{\varepsilon p-1}}{\sup\limits_{0<\varepsilon<\varepsilon_0} t^{\varepsilon p}\varepsilon} \leq p^{-1}e^{\varepsilon_0 p-1}\,\frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} t^{\varepsilon p}\varepsilon}. \qquad \square$$

Extrapolation of Lebesgue spaces

In the following we set $\Theta = [0,1]$ and $Y_\theta = L_{\frac{1}{\theta}}(\Omega)$ for $\theta \in \Theta$. This is a compatible family of Banach spaces according to Definition 3.1, as we have already computed by (1.2).

Proposition 3.7. *Let $1 < p \leq \infty$ and $a > 0$. Then it holds in the sense of equivalent norms*

$$Y_{\frac{1}{p}}(\log Y)^{+}_{a,p} = L_p(\log L)_{-a}(\Omega). \tag{3.12}$$

Proof. The proof is taken from [ET96, Theorem 1, Sec. 2.6.2].

Let $0 < \varepsilon < 1 - \frac{1}{p}$ and set $\frac{1}{p^t} = \frac{1}{p} + t$. We define an auxiliary norm by

$$\|f \mid X_{p,a}\| := \begin{cases} \left(\int_0^\varepsilon \left[t^a \, \|f \mid L_{p^t,p}(\Omega)\| \right]^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}, & p < \infty \\ \sup\limits_{0<t<\varepsilon} t^a \, \|f \mid L_{t^{-1},\infty}(\Omega)\|, & p = \infty. \end{cases} \tag{3.13}$$

The idea of the proof is to show that this norm is equivalent to both sides of (3.12), beginning with $\|f \mid X_{p,a}\| \sim \|f \mid L_p(\log L)_a\|$.

We apply the definition of the Lorentz norm and Fubini to derive for $p < \infty$

$$\begin{aligned} \|f \mid X_{p,a}\|^p &\overset{\|\cdot \mid L_{p,q}\|}{=} \int_0^\varepsilon t^{ap} \int_0^1 s^{tp} f^*(s)^p \, \mathrm{d}s \, \frac{\mathrm{d}t}{t} \\ &\overset{\text{Fubini}}{=} \int_0^{\frac{1}{2}} f^*(s)^p \int_0^\varepsilon t^{ap-1} s^{tp} \, \mathrm{d}t \, \mathrm{d}s + \int_{\frac{1}{2}}^1 f^*(s)^p \underbrace{\int_0^\varepsilon t^{ap-1} s^{tp} \, \mathrm{d}t}_{\substack{\sim (1-\ln s)^{-ap}, \text{ as both} \\ \text{expressions are bounded} \\ \text{from below and above.}}} \mathrm{d}s. \end{aligned}$$

For the first part of the integral, we have

$$\int_0^{\frac{1}{2}} f^*(s)^p \int_0^\varepsilon t^{ap-1} s^{tp} \, \mathrm{d}t \, \mathrm{d}s \overset{\tau = -p\ln(s)\cdot t}{=} \int_0^{\frac{1}{2}} f^*(s)^p \, (-p \ln s)^{-ap} \int_0^{-\varepsilon p \ln s} \tau^{ap-1} e^{-\tau} \, \mathrm{d}\tau \, \mathrm{d}s.$$

As it holds $(1 - \ln s)^{-ap} \sim (-\ln s)^{-ap}$ for $0 < s < \frac{1}{2}$ and

$$0 < c_{a,p} := \int_0^{\varepsilon p \ln 2} \tau^{ap-1} e^{-\tau} \, \mathrm{d}\tau \le \int_0^{-\varepsilon p \ln s} \tau^{ap-1} e^{-\tau} \, \mathrm{d}\tau \xrightarrow[s \to 0]{} \Gamma(ap) < \infty,$$

we conclude that $\|f \mid X_{p,a}\| \sim \|f \mid L_p(\log L)_a\|$. For $p = \infty$, we use Lemma 3.6 to see that

$$\begin{aligned} \|f \mid X_{\infty,a}\| &= \sup_{0<s<1} f^*(s) \sup_{0<t<\varepsilon} t^a s^t \\ &\sim \sup_{0<s<1} f^*(s)(1 - \ln s)^{-a} = \|f \mid L_\infty(\log L)_{-a}(\Omega)\|. \end{aligned}$$

To prove that $\|f \mid X_{p,a}\| \sim \|f \mid Y_{\frac{1}{p}}(\log Y)^+_{a,p}\|$, note that $p^t = p - \gamma t$ with $p < \gamma < p^2$. Therefore, we infer from Lemma 2.1 that

$$c_1 \, \|f \mid L_{p^{2t}(\Omega)}\| \le \|f \mid L_{p^t,p}(\Omega)\| \le c_2 \, \|f \mid L_{p^t}(\Omega)\|. \qquad \square$$

We only state the extrapolation construction to get the Zygmund spaces with positive second index. A proof is given in [ET96].

Proposition 3.8. *Let* $1 \leq p < \infty$ *and* $a > 0$*. Then it holds in the sense of equivalent norms*

$$Y_{\frac{1}{p}}(\log Y)^-_{a,p} = L_p(\log L)_a(\Omega). \tag{3.14}$$

We now come to the representation of grand and small Lebesgue spaces by the spaces $Y_\theta(\log Y)^+_{\varphi,q}$ and $Y_\theta(\log Y)^-_{\psi,q}$. Recall that we extrapolate the family $\{L_{\frac{1}{\theta}}\}_{0<\theta<1}$.

Proposition 3.9 (The grand Lebesgue space as extrapolation space)**.** *Let* $1 < p < \infty$*. It holds, in the sense of equivalent norms,*

$$L_{p)} = Y_{\frac{1}{p}}(\log Y)^+_{\frac{1}{p},\infty}. \tag{3.15}$$

Proof. This equivalence is established in [FK04] and [CK14], while we stick to the argument of the latter. We have

$$\begin{aligned}\|f \mid Y_{\frac{1}{p}}(\log Y)^+_{\frac{1}{p},\infty}\| &= \sup_{0<t<\frac{1}{p'}} t^{\frac{1}{p}} \|f \mid Y_{\frac{1}{p}+t}\| = \sup_{0<t<\frac{1}{p'}} t^{\frac{1}{p}} \|f \mid L_{\left(\frac{1}{p}+t\right)^{-1}}\| \\ &= \sup_{0<t<\frac{1}{p'}} t^{\frac{1}{p}} \|f \mid L_{p-\frac{p^2 t}{1+pt}}\|.\end{aligned}$$

Set $\varepsilon(t) := \frac{p^2 t}{1+pt}$, then $\varepsilon : (0, p'^{-1}) \to (0, p-1)$ and $t(\varepsilon) = \frac{\varepsilon}{p(p-\varepsilon)}$. By $p - \varepsilon > 1$ it holds $p < p(p-\varepsilon) < p^2$, therefore $t^{\frac{1}{p}} \sim \varepsilon^{\frac{1}{p}}$. We replace the parameter and the limits and get

$$\begin{aligned}\|f \mid Y_{\frac{1}{p}}(\log Y)^+_{\frac{1}{p},\infty}\| &= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{p(p-\varepsilon)}\right)^{\frac{1}{p}} \|f \mid L_{p-\varepsilon}\| \\ &\sim \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p}} \|f \mid L_{p-\varepsilon}\| \sim \|f \mid L_{p)}\|\end{aligned}$$

as this is the norm (2.4) from Lemma 2.7. □

Proposition 3.10 (The small Lebesgue space as extrapolation space)**.** *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*. It holds, in the sense of equivalent norms,*

$$L_{(p} = Y_{\frac{1}{p}}(\log Y)^-_{\frac{1}{p'},1}. \tag{3.16}$$

Proof. Again, we follow the proof of [CK14]. We compare the norm (3.8) of $Y_{\frac{1}{p}}(\log Y)^-_{\frac{1}{p'},1}$ with the norm (2.9) of $L_{(p}$. By

$$\lambda_n = \frac{1}{p} - 2^{-n} \implies \frac{1}{\lambda_n} = \frac{p}{1-2^{-n}p}$$

we have

$$\|f \mid Y_{\frac{1}{p}}(\log Y)^-_{\frac{1}{p'},1}\| \overset{(3.8)}{\sim} \inf_{f=\sum f_n} \sum_{n=J}^{\infty} 2^{\frac{n}{p'}} \|f_n \mid Y_{\lambda_n}\| = \inf_{f=\sum f_n} \sum_{n=J}^{\infty} 2^{\frac{n}{p'}} \|f_n \mid L_{\frac{p}{1-2^{-n}p}}\|. \tag{3.17}$$

Here, we let ε be defined by $\frac{1}{\lambda_n} = p + \varepsilon$. We assume that $J \in \mathbb{N}$ is sufficiently large so that

$$\varepsilon = \frac{2^{-n}p^2}{1 - 2^{-n}p} < p' - 1 \quad \text{and} \quad 2^{-J} < \frac{1}{p} \quad \text{for all } n \geq J. \tag{3.18}$$

This is allowed, as it was stated in Lemma 3.5 that different J lead to equivalent norms, given that J is big enough.

Let f be in $Y_{\frac{1}{p}}(\log Y)^{-}_{\frac{1}{p'},1}$. It holds $(2^n)^{1/p'} \geq (p^2\varepsilon^{-1})^{1/p'}$, $n \geq J$, by (3.18). Therefore, we have for $f_n \in L_{p+\varepsilon}$

$$2^{\frac{n}{p'}} \, \|f_n \,|\, L_{\frac{p}{1-2^{-n}p}}\| \geq p^{\frac{2}{p'}} \varepsilon^{-\frac{1}{p'}} \, \|f_n \,|\, L_{p+\varepsilon}\| \geq c_p \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \, \|f_n \,|\, L_{p+\varepsilon}\|$$

and by shifting the summation index in (3.17)

$$\|f \,|\, Y_{\frac{1}{p}}(\log Y)^{-}_{\frac{1}{p'},1}\| \overset{(3.17)}{\geq} c_p \inf_{f=\sum f_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \, \|f_k \,|\, L_{p+\varepsilon}\| \overset{(2.9)}{\sim} \|f \,|\, L_{(p}\|.$$

Conversely, let f be in $L_{(p}$. We can find a representation $f = \sum_{k\in\mathbb{N}} g_k$ and numbers $0 < \varepsilon_k < p' - 1$ with

$$\sum_{k=1}^{\infty} \varepsilon_k^{-\frac{1}{p'}} \, \|g_k \,|\, L_{p+\varepsilon_k}\| \leq 2 \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \, \|g_k \,|\, L_{p+\varepsilon}\|. \tag{3.19}$$

Now we assign the g_k to the smallest spaces Y_{λ_n}, $n \geq J$, they belong to. Let I_n for $n \geq J$ be the set of indices k such that

$$\begin{cases} \lambda_{n-1} < \frac{1}{p+\varepsilon_k} \leq \lambda_n \implies Y_{\lambda_{n-1}} \hookrightarrow L_{p+\varepsilon_k} \hookrightarrow Y_{\lambda_n} & \text{if} \quad n > J \\ \frac{1}{p+p'-1} < \frac{1}{p+\varepsilon_k} \leq \lambda_J \implies L_{p+\varepsilon_k} \hookrightarrow Y_{\lambda_J} & \text{if} \quad n = J. \end{cases} \tag{3.20}$$

Then $f_n := \sum_{k\in I_n} g_k \in Y_{\lambda_n}$, $n \geq J$, because using the uniformly bounded (by c) inclusions of the Lebesgue spaces and the fact that the $\varepsilon^{-1/p'}$ are bounded from below yields

$$\|f_n \,|\, Y_{\lambda_n}\| \leq \sum_{k\in I_n} \|g_k \,|\, Y_{\lambda_n}\| \leq c \sum_{k\in I_n} \|g_k \,|\, L_{p+\varepsilon_k}\| \leq c' \, \|f \,|\, L_{(p}\| < \infty. \tag{3.21}$$

Then it holds $f = \sum_{n\geq J} f_n$. From (3.20) we see that for $k \in I_n$, $n > J$

$$\varepsilon_k < \frac{1}{\lambda_{n-1}} - p = \frac{2^{-n+1}p^2}{1 - 2^{-n+1}p} \leq \underbrace{\frac{2p^2}{1 - 2^{-J}p}}_{=:c_{p,J}^{-p'} < \infty \text{ by } (3.18)} \cdot 2^{-n} \implies \varepsilon_k^{-\frac{1}{p'}} > c_{p,J} 2^{\frac{n}{p'}}$$

and ${\varepsilon_k}^{-1/p'} > (p'-1)^{-1/p'}$ for $k \in I_J$. This finally yields

$$\|f \mid L_{(p}\| \overset{(3.19)}{\gtrsim} \sum_{n=J}^{\infty} 2^{\frac{n}{p'}} \sum_{k\in I_n} \|g_k \mid L_{p+\varepsilon_k}\| \overset{(3.21)}{\gtrsim} \sum_{n=J}^{\infty} 2^{\frac{n}{p'}} \|f_n \mid Y_{\lambda_n}\| \gtrsim \|f \mid Y_{\frac{1}{p}}(\log Y)^{-}_{\frac{1}{p'},1}\|. \quad \square$$

Extrapolation of Lorentz spaces

Now, in a last step, we extrapolate for $1 < p < \infty$ the scale $\{Y_\theta\}_{\theta\in\Theta}$, $\Theta = [0,1]$, that is given by $Y_\theta = L_{\frac{1}{\theta},p}$ for $0 < \theta < 1$ and $Y_0 = L_\infty$, $Y_1 = L_1$. Then all Y_θ are rearrangement invariant Banach spaces according to [BS88, Theorem IV.4.6] (we avoid the spaces $L_{\infty,p}$ and $L_{1,p}$ as they do not have this property). At the same place it is claimed without proof that $L_{r,p} \hookrightarrow L_{q,p}$ for $1 \le q < r \le \infty$ if the underlying measure space is finite. We give a proof here to guarantee that we can use the scale $\{Y_\theta\}_{\theta\in\Theta}$ for extrapolation. We have to admit that the norms of the inclusions $Y_0 \hookrightarrow Y_\theta$ and $Y_\theta \hookrightarrow Y_1$, $\theta \in \Theta$, are not uniformly bounded. This could be of importance if we wanted to extrapolate at $\theta = 0, 1$, as the statements (iii) of Prop. 3.3 and 3.5 do not hold at the endpoints of the scale. In [CK14], the stronger requirements needed to fix this problem are specified, while we omitted them for the sake of simplicity. We do not want to extrapolate the scale of Lorentz spaces at $\theta = 0, 1$ and can ignore the upcoming problems.

Lemma 3.11. *Let $1 < p < \infty$ and the scale $\{Y_\theta\}_{\theta\in\Theta}$, $\Theta = [0,1]$ be defined as given above, i.e. $Y_\theta = L_{\frac{1}{\theta},p}$ for $0 < \theta < 1$ and $Y_0 = L_\infty$, $Y_1 = L_1$. This scale is a compatible family of Banach spaces.*

Proof. As the other inclusions are readily seen by (1.6) and (1.7), let us only show that $L_{r,p} \hookrightarrow L_{q,p}$ for $1 < q < r < \infty$ with norms of inclusions bounded uniformly w.r.t. q, r. It holds $0 < \frac{1}{q} - \frac{1}{r} < 1$, hence for $t \le 1$ we have $t^{1/q} = \left(t^{1/q-1/r}\right) t^{1/r} \le t^{1/r}$. This yields

$$\|f \mid L_{q,p}\| \le \left(\int_0^1 \left[t^{\frac{1}{r}} f^*(t)\right]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} = \|f \mid L_{r,p}\|. \quad \square$$

Proposition 3.12 ([FK04]). *Let $1 < p < \infty$ and the scale $\{Y_\theta\}_{\theta\in\Theta}$ be as in Lemma 3.11. It holds, in the sense of equivalent norms,*

$$L_{p)} = Y_{\frac{1}{p}}(\log Y)^{+}_{\frac{1}{p},\infty}. \tag{3.22}$$

Proof. This follows from the same argument as in the proof of Proposition 3.9, using the norm (2.6) from Lemma 2.7. $\square$

3.3 Compactness

The characterisation of grand and small Lebesgue spaces as extrapolation spaces allows us to extend statements on compact embeddings of Lebesgue spaces. This shall be proven in the following two theorems and applied afterwards to an example from [CK14].

By U_X, we denote the closed unit ball in the Banach space X.

Theorem 3.13. *Let X be a Banach space and $\{Y_\theta\}_{\theta\in\Theta}$ a compatible family of Banach spaces. Let φ be satisfying condition (3.2) and θ, ε such that $(\theta, \theta+\varepsilon] \subset \Theta$. If the linear operator*

$$T : X \longrightarrow \bigcap_{\theta<\eta\le\theta+\varepsilon} Y_\eta$$

is compact if taken as $T : X \longrightarrow Y_\eta$ for all $\theta < \eta \le \theta+\varepsilon$ and satisfies

$$\begin{cases} \left(\sum\limits_{n=J}^{\infty} [\varphi(2^{-n})\, \|T\,|\, X \to Y_{\sigma_n}\|]^q\right)^{\frac{1}{q}} < \infty & \textit{if} \quad 1 \le q < \infty \\ \lim\limits_{n\to\infty} \varphi(2^{-n})\, \|T\,|\, X \to Y_{\sigma_n}\| = 0 & \textit{if} \quad q = \infty \end{cases} \tag{3.23}$$

with $\sigma_n = \theta + 2^{-n}$, then $T : X \longrightarrow Y_\theta(\log Y)^+_{\varphi,q}$ is compact.

Proof. The following proof is taken from [CK14]. For $\delta > 0$ arbitrary, we show that there is a δ-net for $T(U_X)$ in $Y_\theta(\log Y)^+_{\varphi,q}$. As (3.23) holds, there is $N \ge J$ with

$$\begin{cases} \left(\sum\limits_{n=N+1}^{\infty} [\varphi(2^{-n})\, \|T\,|\, X \to Y_{\sigma_n}\|]^q\right)^{\frac{1}{q}} \le \frac{\delta}{4} & \text{if} \quad 1 \le q < \infty \\ \varphi(2^{-n})\, \|T\,|\, X \to Y_{\sigma_n}\| \le \frac{\delta}{4},\ n \ge N & \text{if} \quad q = \infty. \end{cases} \tag{3.24}$$

As $T : X \longrightarrow Y_{\sigma_n}, n \ge J$ is compact by assumption, we can find for arbitrary $\varepsilon(\delta, n) > 0$ an $\varepsilon(\delta, n)$-net for $T(U_X)$ in Y_{σ_n} for all $J \le n \le N$, i.e. there exist finite sets $W(\delta, n) \subset Y_{\sigma_n}$ with

$$T(U_X) \subset \bigcup_{y\in W(\delta,n)} \varepsilon(\delta,n) U_{Y_{\sigma_n}} + y \qquad \text{with} \qquad \begin{cases} \varepsilon(\delta,n) = \frac{\delta}{4\varphi(2^{-n})\, N^{1/q}} & \text{if} \quad 1 \le q < \infty \\ \varepsilon(\delta,n) = \frac{\delta}{4\varphi(2^{-n})} & \text{if} \quad q = \infty. \end{cases}$$

As all $W(\delta, n)$ are finite, there are finitely many possible choices $C_i = \{y_n\}_{n=J}^N$, $y_n \in W(\delta, n)$. Now for every C_i, look at the set

$$D_i = T(U_X)\ \cap \bigcap_{\substack{J\le n\le N \\ y_n\in C_i}} \varepsilon(\delta,n) U_{Y_{\sigma_n}} + y_n.$$

As for all $x \in U_X$ and $J \le n \le N$, there is a $y_n \in W(\delta, n)$ such that $Tx \in \varepsilon(\delta, n)U_X + y_n$, at least one D_i contains Tx. Hence, not all of the D_i are empty. If it is not empty, then choose one argument $w_i \in U_X$ such that $Tw_i \in D_i$ and let W be the (finite) collection of all those w_i.

Now look at the D_i containing Tx. As it is non-empty, there is $w_i \in W$ with $w_i \in D_i$. Hence, Tw_i and Tx are both contained in $\varepsilon(\delta, n)U_X + y_n$ with $y_n \in C_i$, $J \leq n \leq N$, for one specific choice C_i. Therefore, it holds

$$\|Tx - Tw_i \,|\, Y_{\sigma_n}\| \leq 2\varepsilon(\delta, n), \qquad J \leq n \leq N. \tag{3.25}$$

Therefore, putting Minkowski's inequality, (3.24) and (3.25) together yields that W is the wanted δ-net for $T(U_X)$, as for $1 \leq q < \infty$

$$\begin{aligned}
\|Tx - Tw_i \,|\, Y_\theta(\log Y)^+_{\varphi,q}\| &= \left(\sum_{n=J}^{\infty} \left[\varphi(2^{-n})\,\|Tx - Tw_i \,|\, Y_{\sigma_n}\|\right]^q\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{n=J}^{N} \left[\varphi(2^{-n})\,\|Tx - Tw_i \,|\, Y_{\sigma_n}\|\right]^q\right)^{\frac{1}{q}} + \left(\sum_{n=N+1}^{\infty} \left[\varphi(2^{-n}) \underbrace{\|Tx - Tw_i \,|\, Y_{\sigma_n}\|}_{\substack{\leq \|Tx \,|\, Y_{\sigma_n}\| + \|Tw_i \,|\, Y_{\sigma_n}\| \\ \leq \|T \,|\, X \to Y_{\sigma_n}\| + \|T \,|\, X \to Y_{\sigma_n}\|}}\right]^q\right)^{\frac{1}{q}} \\
&\leq \underbrace{\left(\sum_{n=J}^{N} \left[\varphi(2^{-n})\,\|Tx - Tw_i \,|\, Y_{\sigma_n}\|\right]^q\right)^{\frac{1}{q}}}_{\leq \left(\sum_{n=J}^{N}\left[\frac{\delta}{2} N^{-1/q}\right]^q\right)^{1/q} = \frac{\delta}{2}} + 2\underbrace{\left(\sum_{n=N+1}^{\infty} \left[\varphi(2^{-n})\,\|T \,|\, X \to Y_{\sigma_n}\|\right]^q\right)^{\frac{1}{q}}}_{\leq \frac{\delta}{4}} \\
&\leq \delta
\end{aligned}$$

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and for $q = \infty$

$$\begin{aligned}
\|Tx - Tw_i \,|\, Y_\theta(\log Y)^+_{\varphi,q}\| &= \sup_{n \geq J} \varphi(2^{-n})\,\|Tx - Tw_i \,|\, Y_{\sigma_n}\| \\
&\leq \sup_{J \leq n \leq N} \underbrace{\varphi(2^{-n})\,\|Tx - Tw_i \,|\, Y_{\sigma_n}\|}_{\leq \frac{\delta}{2}} + \sup_{n > N} \varphi(2^{-n})\,\underbrace{\|Tx - Tw_i \,|\, Y_{\sigma_n}\|}_{\leq 2\,\|T \,|\, X \to Y_{\sigma_n}\|} \leq \delta \qquad \square
\end{aligned}$$

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Theorem 3.14. *Let X be a Banach space and $\{Y_\theta\}_{\theta \in \Theta}$ a compatible family of Banach spaces. Let ψ be satisfying condition* (3.6) *and θ, ε such that $[\theta - \varepsilon, \theta) \subset \Theta$. If*

$$T : Y_1 \longrightarrow X$$

is compact if restricted to $T : Y_\eta \longrightarrow X$ for all $\theta - \varepsilon \leq \eta < \theta$ and satisfies

$$\begin{cases} \left(\sum_{n=J}^{\infty} [\psi^{-1}(2^{-n})\,\|T \,|\, Y_{\lambda_n} \to X\|]^{q'}\right)^{\frac{1}{q'}} < \infty & \text{if} \quad 1 < q \leq \infty \\ \lim_{n \to \infty} \psi^{-1}(2^{-n})\,\|T \,|\, Y_{\lambda_n} \to X\| = 0 & \text{if} \quad q = 1 \end{cases} \tag{3.26}$$

with $\lambda_n = \theta - 2^{-n}$, then $T : Y_\theta(\log Y)^-_{\psi,q} \longrightarrow X$ is compact.

Proof. This proof is taken from [CK14] too. As (3.26) holds, for any $\delta > 0$ there is $N \geq J$ with

$$\begin{cases} \left(\sum\limits_{n=N+1}^{\infty} [\psi^{-1}(2^{-n}) \, \|T \,|\, Y_{\lambda_n} \to X\|]^{q'}\right)^{\frac{1}{q'}} \leq \frac{\delta}{4} & \text{if} \quad 1 < q \leq \infty \\ \psi^{-1}(2^{-n}) \, \|T \,|\, Y_{\sigma_n} \to X\| \leq \frac{\delta}{4}, \; n \geq N & \text{if} \quad q = 1. \end{cases} \tag{3.27}$$

Now choose arbitrary

$$\{\delta_n\}_{n=J}^{N}, \; \delta_n > 0 \qquad \text{with} \qquad \sum_{n=J}^{N} \delta_n = \frac{\delta}{2}. \tag{3.28}$$

As $T : Y_{\lambda_n} \longrightarrow X$ is compact, let V_n for $J \leq n \leq N$ be a finite δ_n-net for $T(2\psi^{-1}(2^{-n})U_{Y_{\lambda_n}})$ and set $V = \sum_{n=J}^{N} V_n$ (which is finite as all V_n are finite).

Let now $y \in Y_\theta(\log Y)^{-}_{\psi,q}$ and $\|y \,|\, Y_\theta(\log Y)^{-}_{\psi,q}\| \leq 1$. Using the equivalent norm (3.8) for $Y_\theta(\log Y)^{-}_{\psi,q}$ with $\lambda_n = \theta - 2^{-n}$, there is a representation (with the obvious modification for $q = \infty$)

$$y = \sum_{n=J}^{\infty} y_n, \; y_n \in Y_{\lambda_n} \qquad \text{with} \qquad \left(\sum_{n=J}^{\infty} \left[\psi(2^{-n}) \, \|y_n \,|\, Y_{\lambda_n}\|\right]^q\right)^{\frac{1}{q}} \leq 2.$$

By this choice it holds $\|y_n \,|\, Y_{\lambda_n}\| \leq 2\psi^{-1}(2^{-n})$, all $n \geq J$, hence $y_n \in 2\psi^{-1}(2^{-n}) \, U_{Y_{\lambda_n}}$. Therefore, for $J \leq n \leq N$ there is $v_n \in V_n$ with $\|Ty_n - v_n \,|\, X\| \leq \delta_n$ by the choice of the δ_n-net V_n. Now choose $v \in V$ with $v = \sum_{n=J}^{N} v_n$.

Therefore, putting Minkowski's inequality, (3.27), (3.28) and Hölder's inequality together yields that V is the wanted δ-net for $T(U_{Y_\theta(\log Y)^{-}_{\psi,q}})$, as for $1 < q \leq \infty$ it holds (with the obvious modification for $q = \infty$)

$$\begin{aligned} \|Ty - v \,|\, X\| &= \|\sum_{n=J}^{\infty} Ty_n - \sum_{n=J}^{N} v_n \,|\, X\| \leq \|\sum_{n=J}^{N} Ty_n - v_n \,|\, X\| + \|\sum_{n>N} Ty_n \,|\, X\| \\ &\leq \sum_{n=J}^{N} \underbrace{\|Ty_n - v_n \,|\, X\|}_{\leq \delta_n} + \sum_{n>N} \underbrace{\|Ty_n \,|\, X\|}_{\leq \|T \,|\, Y_{\lambda_n} \to X\| \, \|y_n \,|\, Y_{\lambda_n}\|} \quad \psi^{-1}(2^{-n}) \cdot \psi(2^{-n}) \\ &\leq \frac{\delta}{2} + \left(\sum_{n>N} \left[\psi(2^{-n}) \, \|y_n \,|\, Y_{\lambda_n}\|\right]^q\right)^{\frac{1}{q}} \left(\sum_{n>N} \left[\psi^{-1}(2^{-n}) \, \|T \,|\, Y_{\lambda_n} \to X\|\right]^{q'}\right)^{\frac{1}{q'}} \\ &\leq \frac{\delta}{2} + 2 \cdot \frac{\delta}{4} = \delta \end{aligned}$$

and for $q = 1$

$$\begin{aligned} \|Ty - v \,|\, X\| &= \frac{\delta}{2} + \left(\sum_{n>N} \psi(2^{-n}) \, \|y_n \,|\, Y_{\lambda_n}\|\right) \left(\sup_{n>N} \psi^{-1}(2^{-n}) \, \|T \,|\, Y_{\lambda_n} \to X\|\right) \\ &\leq \frac{\delta}{2} + 2 \cdot \frac{\delta}{4} = \delta. \end{aligned}$$

□

Remark 3.15. If we take a bounded $\Omega \subset \mathbb{R}^n$, $1 \leq p < n/k$, $k \in \mathbb{N}$, and set $p^* = np/(n-kp)$, then for $p \leq q < p^*$ the Sobolev space $W_p^k(\Omega)$ is compactly embedded in L_q. This is shown for example in [EE87]. From Theorem 3.13 and Proposition 3.9 we immediately conclude that the embedding $W_p^k(\Omega) \hookrightarrow L_{p^*)}(\Omega)$ is compact (cf. [CK14, Corollary 4.3]).

4. Interpolation approach

The aim of this chapter is to characterise the grand and small Lebesgue spaces as interpolation spaces. This requires some preparations which are made in Sections 4.1 and 4.2. We start by introducing weighted interpolation spaces that are more general than the spaces defined in Chapter 1.

4.1 Interpolation with weights

Definition 4.1 (Weighted interpolation spaces)**.** Let $(A_0, A_1) \in \mathscr{C}_2$ and $1 \leq p \leq \infty$. Let $w : (0, \infty) \to [0, \infty)$ be a continuous weight with $w \neq 0$. The space $(A_0, A_1)^{\mathcal{K}}_{w,p}$ consists of all $a \in A_0 + A_1$ such that the norm

$$\|a \,|\, (A_0, A_1)^{\mathcal{K}}_{w,p}\| := \begin{cases} \left(\displaystyle\int_0^\infty [w(t)\, K(t, a; A_0, A_1)]^p \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty \\ \displaystyle\sup_{0<t<\infty} \; w(t)\, K(t, a; A_0, A_1) & \text{if} \quad p = \infty \end{cases}$$

is finite. The space $(A_0, A_1)^{\mathcal{J}}_{w,p}$ consists of all $a \in A_0 + A_1$ that can be represented by $a = \int_0^\infty u(t) \, \frac{\mathrm{d}t}{t}$, $u(t) \in A_0 \cap A_1$, such that

$$\begin{cases} \left(\displaystyle\int_0^\infty [w(t)\, J(t, u(t); A_0, A_1)]^p \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty \\ \displaystyle\sup_{0<t<\infty} \; w(t)\, J(t, u(t); A_0, A_1) & \text{if} \quad p = \infty \end{cases}$$

is finite. The norm $\|\cdot \,|\, (A_0, A_1)^{\mathcal{J}}_{w,p}\|$ is given by the infimum of this expression taken over all possible representations.

Obviously, for $w(t) = t^{-\theta}$, $0 < \theta < 1$, we have $(A_0, A_1)_{w,q} = (A_0, A_1)_{\theta,q}$.

The weighted interpolation spaces have a couple of properties in common with the classical Lions-Peetre spaces. Different approaches can be found in literature, an extensive theory is given e.g. by [BMR01] or [Gus78], but even the question of equivalence between the $\mathcal{K}$ and $\mathcal{J}$ method is more complex than what is dealt with here. We largely leave open the question of what can be generally assumed about the weight w so that the corresponding spaces have "good" properties. We state only some lemmas that we need in the sequel.

Lemma 4.2. *Let $(A_0, A_1) \in \mathscr{C}_2$ and $1 \le q \le \infty$. Let $w : (0,\infty) \to [0,\infty)$ be a continuous weight.*

(i) It holds for $v : (0,\infty) \to [0,\infty)$ with $v(t) = t^{-1}w(t^{-1})$

$$(A_0, A_1)^{\mathcal{K}}_{w,q} = (A_1, A_0)^{\mathcal{K}}_{v,q} \qquad \text{and} \qquad (A_0, A_1)^{\mathcal{J}}_{w,q} = (A_1, A_0)^{\mathcal{J}}_{v,q}.$$

(ii) Let $A_1 \hookrightarrow A_0$ and

$$\left(\int_0^\infty \Big[\min(1,t)w(t)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty. \tag{4.1}$$

Then for all $f \in (A_0, A_1)^{\mathcal{K}}_{w,q}$, it holds

$$\|f \,|\, (A_0, A_1)^{\mathcal{K}}_{w,q}\| \sim \left(\int_0^1 \Big[w(t)\, K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \tag{4.2}$$

Proof. (i) It holds $K(t,a;A_0,A_1) = tK(t^{-1},a;A_1,A_0)$ and $J(t,a;A_0,A_1) = tJ(t^{-1},a;A_1,A_0)$, hence we compute for the $\mathcal{K}$-method similarly to [BL76] (the proof for the $\mathcal{J}$-method is practically the same)

$$\begin{aligned}\|f \,|\, (A_0, A_1)^{\mathcal{K}}_{w,q}\| &= \left(\int_0^\infty \Big[w(t)tK(t^{-1},a;A_1,A_0)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\overset{s=t^{-1}}{=} \left(\int_0^\infty \Big[w(s^{-1})s^{-1}K(s,a;A_1,A_0)\Big]^q \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}} = \|f \,|\, (A_1, A_0)^{\mathcal{K}}_{v,q}\|.\end{aligned}$$

(ii) The fact is stated in [FK04], a proof is outlined there as well. The first inequality is obvious and holds with constant 1. For $t \ge 1$, it holds $K(t,f) \sim \|f \,|\, A_0\|$ uniformly w.r.t. t as $A_1 \hookrightarrow A_0$. Thereby

$$\begin{aligned}\|f \,|\, (A_0, A_1)^{\mathcal{K}}_{w,q}\| &\overset{\text{Mink.}}{\le} \left(\int_0^1 \Big[w(t)\,K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + \left(\int_1^\infty \Big[w(t)\,K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\overset{(4.1)}{\le} \left(\int_0^1 \Big[w(t)\,K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + c_1\,\|f \,|\, A_0\| \\ &\overset{(4.1)}{=} \left(\int_0^1 \Big[w(t)\,K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + c_2\left(\int_0^1 \Big[t\,w(t)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \|f \,|\, A_0\| \\ &= \left(\int_0^1 \Big[w(t)\,K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + c_2\left(\int_0^1 \Big[t\,w(t)\,\|f \,|\, A_0\|\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \end{aligned} \tag{4.3}$$

Again, since $A_1 \hookrightarrow A_0$, hence $A_0 = A_0 + A_1$ (in the sense of equivalent norms), it holds for $t < 1$

$$\|f \,|\, A_0\| \leq c \inf_{\substack{f=f_0+f_1 \\ f_i \in A_i}} \|f_0 \,|\, A_0\| + \|f_1 \,|\, A_1\| \overset{t<1}{\leq} \frac{c}{t} \inf_{\substack{f=f_0+f_1 \\ f_i \in A_i}} \|f_0 \,|\, A_0\| + t\,\|f_1 \,|\, A_1\| = \frac{c}{t} K(t,f). \tag{4.4}$$

We conclude that

$$\begin{aligned} \|f \,|\, (A_0,A_1)^{\mathcal{K}}_{w,q}\| &\overset{(4.3)\ \&\ (4.4)}{\leq} \left(\int_0^1 \Big[w(t)\, K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + c_3 \left(\int_0^1 \Big[t\, w(t)\, t^{-1} K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\leq c_4 \left(\int_0^1 \Big[w(t)\, K(t,f)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \end{aligned}$$

□

Lemma 4.3 (K-J equivalence: special case)**.** *Let (A_0, A_1) be a Banach pair. Let the weight $w : (0,\infty) \to [0,\infty)$ be such that*

$$w^*(s) := \int_0^\infty \min\left(1, \frac{t}{s}\right) w(t) \, \frac{\mathrm{d}t}{t} < \infty \quad \textit{for} \quad s > 0. \tag{4.5}$$

Let the weighted interpolation spaces be given as from Definition 4.1, then it holds, in the sense of equivalent norms,

$$(A_0, A_1)^{\mathcal{K}}_{w,1} = (A_0, A_1)^{\mathcal{J}}_{w^*,1}.$$

Proof. The fact is stated in [FK04, p. 659] without a proof.

Let $a \in (A_0, A_1)^{\mathcal{J}}_{w^*,1}$ with $a = \int_0^\infty u(t) \, \frac{\mathrm{d}t}{t}$, $u(t) \in A_0 \cap A_1$.

$$\begin{aligned} \|a \,|\, (A_0,A_1)^{\mathcal{K}}_{w,1}\| &= \int_0^\infty K(t,a)\, w(t)\, \frac{\mathrm{d}t}{t} \\ &\overset{\text{K is norm}}{\leq} \int_0^\infty w(t) \int_0^\infty K(s, u(s)) \, \frac{\mathrm{d}s}{s}\, \frac{\mathrm{d}t}{t} \\ &\overset{(1.18)}{\leq} \int_0^\infty w(t) \int_0^\infty \min\left(1, \frac{t}{s}\right) J(s,u(s)) \, \frac{\mathrm{d}s}{s}\, \frac{\mathrm{d}t}{t} \\ &\overset{\text{Fubini}}{=} \int_0^\infty J(s,u(s)) \underbrace{\int_0^\infty \min\left(1, \frac{t}{s}\right) w(t)\, \frac{\mathrm{d}t}{t}}_{=w^*(s)} \, \frac{\mathrm{d}s}{s} = \|a \,|\, (A_0,A_1)^{\mathcal{J}}_{w^*,1}\| \end{aligned}$$

Now let $a \in (A_0, A_1)^{\mathcal{K}}_{w,1}$. By the Holmstedt formula and the dominated convergence theorem, it holds

$$K(t,a) \overset{(1.28)}{\sim} \int_0^t K(s,a) \, \frac{\mathrm{d}s}{s} \xrightarrow[t\to 0]{} 0 \quad \text{and} \quad \frac{K(t,a)}{t} \overset{(1.29)}{\sim} \int_t^\infty \frac{K(s,a)}{s} \, \frac{\mathrm{d}s}{s} \xrightarrow[t\to\infty]{} 0.$$

Therefore, we can apply the fundamental lemma of interpolation theory, Lemma 1.4, and obtain a representation $a = \sum_{j\in\mathbb{Z}} u_j$ such that for any ε (note that w^* from (4.5) is monotonically decreasing)

$$\begin{aligned}
\|a\,|\,(A_0,A_1)^{\mathcal{J}}_{w^*,1}\| &= \int_0^\infty J(s,u(s))\,w^*(s)\,\frac{\mathrm{d}s}{s} = \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} \underbrace{J(s,u(s))}_{\le c_1\,J(2^j,u_j)}\,w^*(s)\,\frac{\mathrm{d}s}{s}\\
&\overset{\text{Fund. Lemma}}{\le} c_3(1+\varepsilon)\sum_{j=-\infty}^{\infty} K(2^j,a)\,w^*(2^j)\\
&\le c_4(1+\varepsilon)\int_0^\infty\int_0^\infty K(s,a)\,\min\Big(1,\frac{t}{s}\Big)\,w(t)\,\frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}s}{s}\\
&\overset{\text{Fubini}}{=} c_4(1+\varepsilon)\int_0^\infty\left(\int_0^t K(s,a)\,\frac{\mathrm{d}s}{s}+t\int_t^\infty s^{-1}K(s,a)\,\frac{\mathrm{d}s}{s}\right)w(t)\,\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Here we apply the Holmstedt formula (1.27), that is

$$\int_0^t K(s,a)\,\frac{\mathrm{d}s}{s}+t\int_t^\infty s^{-1}K(s,a)\,\frac{\mathrm{d}s}{s}\sim K(t,a),$$

and derive $\|a\,|\,(A_0,A_1)^{\mathcal{J}}_{w^*,1}\|\le c_5\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{w,1}\|$. □

In the subsequent characterisations of grand and small Lebesgue spaces, we use the commonly known embeddings of interpolation spaces as they are given in [BL76]. However, since we need the dependence on the parameter θ, we prove a couple of embeddings that take this into account.

Lemma 4.4 ([BL76, Theorem 3.4.1]). *Let $(A_0,A_1)\in\mathscr{C}_2$ and $0<\theta<1$. The constants in the following inequalities are independent of θ.*

(i) $\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,\infty}\|\le c_p\theta^{\frac{1}{p}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|$ *for* $1\le p<\infty$.

(ii) $\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,q}\|\le c_{p,q}\theta^{\frac{1}{p}-\frac{1}{q}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|$ *for* $1\le p<q<\infty$.

Now, suppose that $A_1\hookrightarrow A_0$.

(iii) $\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta/2,1}\|\le c_p\theta^{\frac{1}{p}-1}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|$ *for* $1<p<\infty$.

(iv) $\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{1-\theta,p}\|\le c_p\theta^{-\frac{1}{p}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{1-\theta/2,\infty}\|$ *for* $1<p<\infty$ *and* $0<\theta<\frac{1}{2}$.

Proof. In [BL76], the statements are proven without explicit constants. We modify the proofs accordingly.

(i) $$\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,\infty}\| = \sup_{0<t<\infty} t^{-\theta}K(t,a) \overset{(1.22)}{\le} (\theta p)^{\frac{1}{p}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|$$

(ii) $$\begin{aligned}
\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,q}\| &= \left(\int_0^\infty\Big[t^{-\theta}K(t,a)\Big]^q\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty\Big[t^{-\theta}K(t,a)\Big]^p\Big[\underbrace{t^{-\theta}K(t,a)}_{\overset{(1.22)}{\le}(\theta p)^{\frac{1}{p}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|}\Big]^{q-p}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\\
&\le (\theta p)^{\frac{1}{p}-\frac{1}{q}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|^{1-\frac{p}{q}}\,\|a\,|\,(A_0,A_1)^{\mathcal{K}}_{\theta,p}\|^{\frac{p}{q}}.
\end{aligned}$$

(iii) By $A_1 \hookrightarrow A_0$ and the definition of the K functional, we have $K(t,a) \leq c_1 \|a \,|\, A_0\|$. As $A_0 = A_0 + A_1$ for any $t > 0$ (in the sense of equivalent norms) and $K(1,a)$ is an equivalent norm on $A_0 + A_1$, it holds $\|a \,|\, A_0\| \leq c_2 \cdot K(1,a)$, and by (1.22) it holds $K(1,a) \leq (\theta p)^{\frac{1}{p}} \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,p}\|$. Hence,

$$\begin{aligned}
\|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta/2,1}\| &= \int_0^1 t^{\frac{\theta}{2}} \underbrace{t^{-\theta} K(t,a)}_{\overset{(1.22)}{\leq} (\theta p)^{\frac{1}{p}} \|a \,|\, (A_0,A_1)^{\mathcal{K}}_{\theta,p}\|} \frac{\mathrm{d}t}{t} + \int_1^\infty t^{-\frac{\theta}{2}} \underbrace{K(t,a)}_{\leq c_p \theta^{\frac{1}{p}} \|a \,|\, (A_0,A_1)^{\mathcal{K}}_{\theta,p}\|} \frac{\mathrm{d}t}{t} \\
&\leq c_p' \, \theta^{\frac{1}{p}} \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,p}\| \Bigg(\underbrace{\int_0^1 t^{\frac{\theta}{2}} \frac{\mathrm{d}t}{t}}_{=\frac{2}{\theta}} + \underbrace{\int_1^\infty t^{-\frac{\theta}{2}} \frac{\mathrm{d}t}{t}}_{=\frac{2}{\theta}} \Bigg) \\
&= c_p'' \, \theta^{\frac{1}{p}-1} \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta,p}\|.
\end{aligned}$$

(iv) Similarly to (iii), with c independent of θ we have $K(t,a) \leq c \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{1-\theta/2,\infty}\|$ and compute

$$\begin{aligned}
\|a \,|\, (A_0, A_1)^{\mathcal{K}}_{1-\theta,p}\| &= \Bigg(\int_0^1 \Big[t^{\frac{\theta}{2}} \underbrace{t^{\frac{\theta}{2}-1} K(t,a)}_{\leq \|a \,|\, (A_0,A_1)^{\mathcal{K}}_{1-\theta/2,\infty}\|} \Big]^p \frac{\mathrm{d}t}{t} + \int_1^\infty \Big[t^{\theta-1} \underbrace{K(t,a)}_{\leq c \|a \,|\, (A_0,A_1)^{\mathcal{K}}_{1-\theta/2,\infty}\|} \Big]^p \frac{\mathrm{d}t}{t} \Bigg)^{\frac{1}{p}} \\
&\leq c' \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{1-\theta/2,\infty}\| \Bigg(\underbrace{\int_0^1 t^{\frac{\theta}{2}p} \frac{\mathrm{d}t}{t} + \int_1^\infty t^{(\theta-1)p} \frac{\mathrm{d}t}{t}}_{=\frac{1}{p}\left(\frac{2}{\theta}+\frac{1}{1-\theta}\right)} \Bigg)^{\frac{1}{p}}.
\end{aligned}$$

Since $2 < \frac{2-\theta}{1-\theta} < 3$ for $0 < \theta < \frac{1}{2}$, the statement is proven as we have

$$\left(\frac{2}{\theta} + \frac{1}{1-\theta} \right)^{\frac{1}{p}} = \left(\frac{1}{\theta} \frac{2-\theta}{1-\theta} \right)^{\frac{1}{p}} \sim \theta^{-\frac{1}{p}}. \qquad \square$$

4.2 Two theorems on the equivalence of interpolation and extrapolation

In [KM05], the authors take effort to prove certain equivalence statements not only for interpolation and extrapolation of Banach spaces, but of quasi-Banach spaces as well by appending certain conditions. We omit all specifications needed for this purpose and limit ourselves to a less general study that suffices for the spaces of Chapter 3.

Now that we want to extrapolate scales of interpolation spaces, we need to make sure that the constructions $Y_\theta(\log Y)^+_{\varphi,q}$ and $Y_\theta(\log Y)^-_{\psi,q}$ are well-defined in such a case – this corresponds to an adequate choice of the interval Θ such that the scales are compatible in the sense of Definition 3.1.

Lemma 4.5 (Compatibility of the $\mathcal{K}$-interpolation scale)**.** *Let* $(A_0, A_1) \in \mathscr{C}_2$ *with* $A_0 \hookrightarrow A_1$*, and let* $1 \leq q \leq \infty$*. Let* $\Theta = [0, \eta) \cup \{1\}$ *with* $0 < \eta < 1$*. The scale* $\{Y_\theta\}_{\theta \in \Theta}$*,* $Y_\theta = (A_0, A_1)^{\mathcal{K}}_{\theta,q}$ *for* $0 < \theta < \eta$*,* $Y_0 = A_0$*,* $Y_1 = A_1$ *is a compatible family of Banach spaces.*

Proof. It is clear that for any $\theta_0 < \theta_1$ in Θ we have (cf. [BL76] for this fact)

$$A_0 \hookrightarrow (A_0, A_1)^{\mathcal{K}}_{\theta_0,q} \hookrightarrow (A_0, A_1)^{\mathcal{K}}_{\theta_1,q} \hookrightarrow A_1.$$

Our aim is to prove that the embeddings between the interpolation spaces are uniformly bounded. Let $0 < \theta_0 < \theta_1 < \eta$. As $A_0 \hookrightarrow A_1$, we have $K(t, a) \sim t \, \|a \,|\, A_1\|$ for $t \leq 1$. Therefore it holds with c independent of θ_0, θ_1

$$\begin{aligned}
\|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta_1,q}\|^q &= \int_0^1 \Big[t^{-\theta_1} \underbrace{K(t, a)}_{\lesssim t\, \|a \,|\, A_1\|}\Big]^q \frac{\mathrm{d}t}{t} + \int_1^\infty \Big[\underbrace{t^{-\theta_1}}_{= t^{-\theta_0} \cdot t^{\theta_0 - \theta_1}} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} \\
&\leq c \, \|a \,|\, A_1\|^q \int_0^1 t^{q(1-\theta_1)} \frac{\mathrm{d}t}{t} + \int_1^\infty \Big[t^{-\theta_0} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} \\
&= c \, \frac{1 - \theta_0}{1 - \theta_1} \cdot \|a \,|\, A_1\|^q \int_0^1 t^{q(1-\theta_0)} \frac{\mathrm{d}t}{t} + \int_1^\infty \Big[t^{-\theta_0} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} \\
&\leq c \, \frac{1}{1 - \eta} \int_0^1 \Big[t^{-\theta_0} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} + \int_1^\infty \Big[t^{-\theta_0} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} \\
&\leq c \, \frac{1}{1 - \eta} \int_0^\infty \Big[t^{-\theta_0} K(t, a)\Big]^q \frac{\mathrm{d}t}{t} = c \, \frac{1}{1 - \eta} \|a \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta_0,q}\|^q. \qquad \square
\end{aligned}$$

Theorem 4.6 (Equivalence of abstract spaces and weighted interpolation)**.** *Let* $(A_0, A_1) \in \mathscr{C}_2$ *with* $A_0 \hookrightarrow A_1$ *and* $1 \leq q \leq \infty$*, and let* $0 \leq \theta < 1$*. Choose* η *such that* $\theta < \eta < 1$*. Let* $\{Y_t\}_{t \in \Theta}$*,* $\Theta = [0, \eta) \cup \{1\}$ *be given by* $Y_t = (A_0, A_1)^{\mathcal{K}}_{t,q}$ *for* $0 < t < \eta$ *and* $Y_0 = A_0$*,* $Y_1 = A_1$*.*

Let φ *be a function satisfying condition* (3.2)*, and let* $\varepsilon > 0$ *be such that* $(\theta, \theta + \varepsilon] \subset \Theta$*. Define the weight* w *by*

$$w(t) = \begin{cases} \left(\displaystyle\int_0^\varepsilon \Big[t^{-(\theta+s)} \varphi(s)\Big]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} & \textit{if} \quad q < \infty \\ \displaystyle\sup_{0 < s < \varepsilon} t^{-(\theta+s)} \varphi(s) & \textit{if} \quad q = \infty \end{cases} \qquad \textit{for} \quad t > 0. \tag{4.6}$$

Then it holds $w(t) < \infty$ *and, in the sense of equivalent norms,*

$$Y_\theta (\log Y)^+_{\varphi,q} = (A_0, A_1)^{\mathcal{K}}_{w,q}.$$

Proof. The fact is stated in [KM05, p. 74] without proof. We prove the statement for $q < \infty$, the other case follows by the same arguments. The weight w is finite, as φ satisfies (3.2). It holds

$$\|f \,|\, Y_\theta (\log Y)^+_{\varphi,q}\| \overset{\text{Def. 3.2}}{=} \left(\int_0^\varepsilon \Big[\varphi(s) \, \|f \,|\, (A_0, A_1)^{\mathcal{K}}_{\theta+s,q}\|\Big]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}$$

$$\overset{\text{K-method}}{=} \left(\int_0^{\varepsilon} \varphi(s)^q \int_0^{\infty} \left[t^{-(\theta+s)} K(t,f) \right]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}$$

$$\overset{\text{Fubini}}{=} \left(\int_0^{\infty} K(t,f)^q \int_0^{\varepsilon} \left[t^{-(\theta+s)} \varphi(s) \right]^q \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$\overset{(4.6)}{=} \left(\int_0^{\infty} [w(t)\, K(t,f)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}. \qquad \square$$

Lemma 4.7 (Compatibility of the $\mathcal{J}$-interpolation scale)**.** *Let $(A_0, A_1) \in \mathscr{C}_2$ with $A_0 \hookrightarrow A_1$, and let $1 \le q \le \infty$. Let $\Theta = \{0\} \cup (\eta, 1]$ with $0 < \eta < 1$. The scale $\{Y_\theta\}_{\theta \in \Theta}$, $Y_\theta = (A_0, A_1)^{\mathcal{J}}_{\theta,q}$ for $\eta < \theta < 1$, $Y_0 = A_0$, $Y_1 = A_1$ is a compatible family of Banach spaces.*

Proof. As for the K scale, we only need to show that the embeddings are uniformly bounded. It is readily seen that the discrete norm of $(A_0, A_1)^{\mathcal{J}}_{\theta,q}$, $0 < \theta < 1$, is equivalent to the continuous norm with constants independent of θ (cf. [BL76, Lemma 3.2.3]).

Let $\eta < \theta_0 < \theta_1 < 1$. As $A_0 \hookrightarrow A_1$, we have $J(t,a) \sim \|a \,|\, A_0\|$ for $t \le 1$ uniformly w.r.t. t. Take any decomposition $a = \sum_{k \in \mathbb{Z}} u_k$, $u_k \in A_0$, and choose another decomposition $a = \sum_{j \in \mathbb{Z}} v_j$ by

$$v_j = \sum_{k=-\infty}^{0} u_k \text{ if } j = 0, \qquad v_j = 0 \text{ if } j < 0, \qquad v_j = u_j \text{ if } j > 0.$$

Then we have with c independent of θ_0, θ_1

$$\|a \,|\, (A_0, A_1)^{\mathcal{J}}_{\theta_1,q}\|^q \overset{\inf}{\le} \sum_{j=0}^{\infty} 2^{-j\theta_1 q} J(2^j, v_j)^q \overset{J \text{ is norm}}{\le} \Bigg(\sum_{k=-\infty}^{0} \underbrace{J(1, u_k)}_{\le c\, \|a \,|\, A_0\|} \Bigg)^q + \sum_{k=1}^{\infty} \underbrace{2^{-k\theta_1 q}}_{\le 2^{-k\theta_0 q}} J(2^k, u_k)^q$$

$$\overset{A_0 \hookrightarrow A_1}{\le} c \left(\sum_{k=-\infty}^{0} J(2^k, u_k) 2^{-\theta_0 k} \cdot 2^{\theta_0 k} \right)^q + \sum_{k=1}^{\infty} 2^{-k\theta_0 q} J(2^k, u_k)^q$$

$$\overset{\text{Hölder}}{\le} c \left(\sum_{k=-\infty}^{0} 2^{-k\theta_0 q} J(2^k, u_k)^q \right) \underbrace{\left(\sum_{k=-\infty}^{0} 2^{k\theta_0 q'} \right)^{\frac{q}{q'}}}_{= \left(\frac{1}{1 - 2^{-\theta_0 q'}} \right)^{\frac{q}{q'}}} + \sum_{k=1}^{\infty} 2^{-k\theta_0 q} J(2^k, u_k)^q$$

$$\le c \left(\frac{1}{1 - 2^{-\theta_0 q'}} \right)^{\frac{q}{q'}} \sum_{k=-\infty}^{\infty} 2^{-k\theta_0 q} J(2^k, u_k)^q.$$

Notice that the decomposition $a = \sum_j v_j$ is appropriate, i.e. $v_0 \in A_0$, if the above right hand side is finite. By taking the infimum over all possible decompositions $a = \sum_k u_k$ such that this is the case, we conclude that

$$\|a \,|\, (A_0, A_1)^{\mathcal{J}}_{\theta_1,q}\| \le c' \left(\frac{1}{1 - 2^{-\eta q'}} \right)^{\frac{1}{q'}} \|a \,|\, (A_0, A_1)^{\mathcal{J}}_{\theta_0,q}\|. \qquad \square$$

Theorem 4.8 (Equivalence of abstract spaces and weighted interpolation). *Let* $(A_0, A_1) \in \mathscr{C}_2$ *with* $A_0 \hookrightarrow A_1$ *and* $1 \le q \le \infty$, *and let* $0 < \theta \le 1$. *Choose* η *such that* $0 < \eta < \theta$. *Let* $\{Y_t\}_{t\in\Theta}$, $\Theta = \{0\} \cup (\eta, 1]$ *be given by* $Y_t = (A_0, A_1)^{\mathcal{J}}_{t,q}$ *for* $\eta < t < 1$ *and* $Y_0 = A_0$, $Y_1 = A_1$.

Let ψ *be a function satisfying condition* (3.6), *and let* $\varepsilon > 0$ *be such that* $[\theta - \varepsilon, \theta) \subset \Theta$. *Define the weight* w^* *by (set* $\frac{1}{q} + \frac{1}{q'} = 1$*)*

$$\frac{1}{w^*(s)} = \begin{cases} \left(\int_0^\varepsilon \left[\dfrac{s^{\theta-t}}{\psi(t)} \right]^{q'} \dfrac{\mathrm{d}t}{t} \right)^{\frac{1}{q'}} & \text{if} \quad q > 1 \\ \sup\limits_{0<t<\varepsilon} \dfrac{s^{\theta-t}}{\psi(t)} & \text{if} \quad q = 1 \end{cases} \qquad \text{for} \qquad s > 0. \tag{4.7}$$

Then it holds $w^*(t) < \infty$ *and, in the sense of equivalent norms,*

$$Y_\theta(\log Y)^-_{\psi,q} = (A_0, A_1)^{\mathcal{J}}_{w^*,q}.$$

Proof. We modify the proofs of Theorems 2.1, 2.2 and 2.3 in [KM05].

Let $f \in Y_\theta(\log Y)^-_{\psi,q}$. Select a decomposition $f = \int_0^\varepsilon g(t) \, \frac{\mathrm{d}t}{t}$ with $g(t) \in Y_{\theta-t}$ such that

$$\left(\int_0^\varepsilon \Big[\psi(t) \, \| g(t) \,|\, Y_{\theta-t} \| \Big]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \le 2 \, \| f \,|\, Y_\theta(\log Y)^-_{\psi,q} \|.$$

Similarly, for any $g(t) \in Y_{\theta-t}$ take $v(t,s) \in A_0 \cap A_1$ with $g(t) = \int_0^\infty v(t,s) \, \frac{\mathrm{d}s}{s}$ such that

$$\left(\int_0^\infty \Big[s^{t-\theta} J(s, v(t,s)) \Big]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} \le 2 \, \| g(t) \,|\, (A_0, A_1)^{\mathcal{J}}_{\theta-t,q} \|.$$

Now let $u(s) = \int_0^\varepsilon v(t,s) \, \frac{\mathrm{d}t}{t}$, hence by the definitions and Fubini it holds $f = \int_0^\infty u(s) \, \frac{\mathrm{d}s}{s}$. It is seen below that $J(s, u(s)) < \infty$ and hence $u(s) \in A_0 \cap A_1$ as $J(s, \cdot)$ is an equivalent norm on $A_0 \cap A_1$. For the same reason we know that the J-functional satisfies the triangle inequality. Making use of this fact and then applying Hölder's inequality yields

$$\begin{aligned} J(s, u(s)) &\le \int_0^\varepsilon J(s, v(t,s)) \, \frac{\mathrm{d}t}{t} = \int_0^\varepsilon \frac{\psi(t)}{s^{\theta-t}} J(s, v(t,s)) \cdot \frac{s^{\theta-t}}{\psi(t)} \, \frac{\mathrm{d}t}{t} \\ &\overset{\text{Hölder}}{\le} \left(\int_0^\varepsilon \Big[\psi(t) s^{t-\theta} J(s, v(t,s)) \Big]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \underbrace{\left(\int_0^\varepsilon \left[\frac{s^{\theta-t}}{\psi(t)} \right]^{q'} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q'}}}_{= \frac{1}{w^*(s)}}. \end{aligned}$$

We conclude that

$$\| f \,|\, (A_0, A_1)^{\mathcal{J}}_{w^*,q} \| \le \left(\int_0^\infty \int_0^\varepsilon \Big[\psi(t) s^{t-\theta} J(s, v(t,s)) \Big]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}$$

$$\overset{\text{Fubini}}{=} \left(\int_0^\varepsilon \psi(t)^q \underbrace{\left(\int_0^\infty \left[s^{t-\theta} J(s, v(t,s)) \right]^q \frac{\mathrm{d}s}{s} \right)}_{\leq 2 \, \| g(t) \, | \, (A_0,A_1)^{\mathcal{J}}_{\theta-t,q} \|^q} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \leq 4 \, \| f \, | \, Y_\theta(\log Y)^-_{\psi,q} \|.$$

As all of the above expressions are finite, we have $J(s, u(s)) < \infty$ and the decomposition of f is chosen appropriately.

Now let $f \in (A_0, A_1)^{\mathcal{J}}_{w^*,q}$. Select a decomposition $f = \int_0^\infty u(s) \frac{\mathrm{d}s}{s}$ with $u(s) \in A_0 \cap A_1$ such that

$$\left(\int_0^\infty \left[w^*(s) J(s, u(s)) \right]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} \leq 2 \, \| f \, | \, (A_0, A_1)^{\mathcal{J}}_{w^*,q} \|.$$

Let

$$\phi(t,s) = \frac{w^*(s) s^{\theta - t}}{\psi(t)} \quad \text{for} \quad 0 < t < \varepsilon \tag{4.8}$$

$$\overset{q>1}{\Longrightarrow} \int_0^\varepsilon \phi(t,s)^{q'} \frac{\mathrm{d}t}{t} = \left(\int_0^\varepsilon \left[\frac{s^{\theta-t}}{\psi(t)} \right]^{q'} \frac{\mathrm{d}t}{t} \right)^{-1} \left(\int_0^\varepsilon \left[\frac{s^{\theta-t}}{\psi(t)} \right]^{q'} \frac{\mathrm{d}t}{t} \right) = 1. \tag{4.9}$$

For $q = 1$, note that ψ is continuous by (3.6) and therefore w^* and ϕ are. Hence for $q = 1$ and $0 < c_1 < 1$ there is an interval $I(s) \subset (0, \varepsilon)$, $0 < s < \infty$, such that $c_1 \leq \phi(t,s) \leq 1$ for $t \in I(s)$. Let $|I(s)|$ be the length of the interval $I(s)$, then we define a partition of the unity by

$$\Phi(t,s) := \begin{cases} \phi(t,s)^{q'} & \text{if} \quad q > 1 \\ \frac{1}{|I(s)|} \chi_{I(s)}(t) & \text{if} \quad q = 1 \end{cases} \qquad \text{for} \quad (t,s) \in (0,\varepsilon) \times (0, \infty). \tag{4.10}$$

Let $g(t) = \int_0^\infty \Phi(t,s) \, u(s) \, \frac{\mathrm{d}s}{s}$, then

$$f = \int_0^\infty u(s) \frac{\mathrm{d}s}{s} \overset{(4.9)}{=} \int_0^\infty \int_0^\varepsilon \Phi(t,s) \, u(s) \, \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s} \overset{\text{Fubini}}{=} \int_0^\varepsilon g(t) \frac{\mathrm{d}t}{t}.$$

In the course of the following computations it is seen that $\| g(t) \, | \, (A_0, A_1)^{\mathcal{J}}_{\theta-t,q} \| < \infty$, therefore $g(t) \in Y_{\theta - t}$. Now we have for $1 \leq q < \infty$ (the other case works the same way)

$$\begin{aligned} \| f \, | \, Y_\theta(\log Y)^-_{\psi,q} \|^q &\overset{\inf}{\leq} \int_0^\varepsilon \left[\psi(t) \, \| g(t) \, | \, (A_0, A_1)^{\mathcal{J}}_{\theta-t,q} \| \right]^q \frac{\mathrm{d}t}{t} \\ &\overset{\inf}{\leq} \int_0^\varepsilon \psi(t)^q \int_0^\infty \Big[s^{t-\theta} \underbrace{J(s, \Phi(t,s) \, u(s))}_{= \Phi(t,s) \, J(s, u(s))} \Big]^q \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \\ &\overset{\text{Fubini}}{=} \int_0^\infty J(s, u(s))^q \int_0^\varepsilon \Big[\underbrace{\psi(t) s^{t-\theta}}_{= \frac{w^*(s)}{\phi(t,s)} \text{ by (4.8)}} \Phi(t,s) \Big]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s} \\ &= \int_0^\infty \left[w^*(s) J(s, u(s)) \right]^q \int_0^\varepsilon \left[\frac{\Phi(t,s)}{\phi(t,s)} \right]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s}. \end{aligned} \tag{4.11}$$

Now for $q > 1$ we recall the formula $(q'-1)q = q'$, hence by (4.10) and (4.9) it holds

$$\int_0^\varepsilon \left[\frac{\Phi(t,s)}{\phi(t,s)}\right]^q \frac{\mathrm{d}t}{t} = \int_0^\varepsilon \phi(t,s)^{(q'-1)q} \frac{\mathrm{d}t}{t} = \int_0^\varepsilon \phi(t,s)^{q'} \frac{\mathrm{d}t}{t} = 1.$$

For $q = 1$, it holds

$$\int_0^\varepsilon \left[\frac{\Phi(t,s)}{\phi(t,s)}\right]^q \frac{\mathrm{d}t}{t} = \frac{1}{|I(s)|}\int_{I(s)} \frac{1}{\phi(t,s)} \frac{\mathrm{d}t}{t} < \frac{1}{c_1}.$$

This yields

$$\|f \mid Y_\theta(\log Y)^{-}_{\psi,q}\| \leq \left(\frac{1}{c_1}\int_0^\infty [w^*(s)J(s,u(s))]^q \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}} \leq c_2 \|f \mid (A_0,A_1)^{\mathcal{J}}_{w^*,q}\|. \qquad \square$$

4.3 Grand Lebesgue spaces

We want to replace the scale used to obtain the grand Lebesgue space through extrapolation by interpolation spaces, such that the assumptions of Theorem 4.6 are satisfied and we can derive a norm using the non decreasing rearrangement f^*. We need the following lemmas.

Lemma 4.9. *It holds with constants independent of θ for $0 < \theta < 1$ and $1 \leq q < p < \infty$*

$$(1-\theta)^{\frac{1}{p}} \|f \mid (L_q,L_p)^{\mathcal{K}}_{\theta,p}\| \sim \|f \mid L_{r,p}\| \quad \textit{for} \quad \frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p}. \tag{4.12}$$

Proof. The proof is taken from [FK04].

Let η be given by $\frac{1}{p} = \frac{1-\eta}{q}$, then by Example 1.5 we have $L_p = (L_q, L_\infty)^{\mathcal{K}}_{\eta,p}$. Hence, we can apply the Holmstedt formula (1.26) with $q_1 = p$ and $\theta_1 = \eta$,

$$K(t,a;L_q,L_p) \sim t\left(\int_{t^{\frac{1}{\eta}}}^\infty \left[s^{-\eta}K(s,a;L_q,L_\infty)\right]^p \frac{\mathrm{d}s}{s}\right)^{\frac{1}{p}}.$$

Then

$$\begin{aligned}(1-\theta)\|f \mid (L_q,L_p)^{\mathcal{K}}_{\theta,p}\|^p &\sim (1-\theta)\int_0^\infty t^{(1-\theta)p}\int_{t^{\frac{1}{\eta}}}^\infty \left[s^{-\eta}K(s,f;L_q,L_\infty)\right]^p \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\ &\overset{\text{Fubini}}{=} (1-\theta)\int_0^\infty \left[s^{-\eta}K(s,f;L_q,L_\infty)\right]^p \underbrace{\int_0^{s^\eta} t^{(1-\theta)p}\frac{\mathrm{d}t}{t}}_{=\frac{1}{p(1-\theta)}s^{\eta p(1-\theta)}} \frac{\mathrm{d}s}{s} \\ &= \frac{1}{p}\int_0^\infty \left[s^{-\eta\theta}K(s,f;L_q,L_\infty)\right]^p \frac{\mathrm{d}s}{s} \sim \|f \mid (L_q,L_\infty)^{\mathcal{K}}_{\eta\theta,p}\|^p.\end{aligned}$$

For r given by $\frac{1}{r} = \frac{1-\eta\theta}{q}$ we have $(L_q, L_\infty)^{\mathcal{K}}_{\eta\theta,p} = L_{r,p}$, and we see that

$$\frac{1}{r} = \frac{1 - \left(1 - \frac{q}{p}\right)\theta}{q} = \frac{1-\theta}{q} + \frac{\theta}{p}. \qquad \square$$

Lemma 4.10. *For any* $0 < \varepsilon_0 < \min(1, p-1)$ *and* $1 < p < \infty$, *it holds*

$$\|f \,|\, L_{p)}\| \sim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \, \|f \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\varepsilon,\infty}\|. \tag{4.13}$$

Proof. We follow the proof of [FK04].

Take Lemma 4.9 with $1 \le q < p$, $r = p - \varepsilon$, $0 < \varepsilon < 1$. Then we determine θ as in (4.12) by

$$\frac{1}{p-\varepsilon} = \frac{1-\theta}{q} + \frac{\theta}{p} \implies \theta = \left(\frac{1}{p-\varepsilon} - \frac{1}{q}\right)\left(\frac{pq}{q-p}\right).$$

If we let $0 < \eta < 1$ be given by $\eta = 1 - \frac{q}{p}$, as it is done in the proof of Lemma 4.9, we derive

$$\theta = \frac{\varepsilon - \eta p}{(\varepsilon - p)\eta} = 1 - \frac{1-\eta}{(\varepsilon-p)\eta}\,\varepsilon =: 1 - \alpha\varepsilon$$

where $\frac{1-\eta}{p\eta} \le \alpha \le \frac{1-\eta}{(p-1)\eta}$ and hence $\alpha \sim 1$ uniformly w.r.t. ε. Now, we use the norm (2.6) and see by Lemma 4.9 that

$$\|g \,|\, L_{p)}\| \sim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \, \|g \,|\, L_{p-\varepsilon,p}\| \sim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{2}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,p}\|.$$

Lemma 4.4 (i) states that $\varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,p}\| \ge c_{p,\alpha} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,\infty}\|$, hence

$$\|g \,|\, L_{p)}\| \gtrsim \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,\infty}\|.$$

We know from Lemma 4.4 (iv) that $\varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,p}\| \le c_{p,\alpha} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon/2,\infty}\|$, hence on the other hand

$$\begin{aligned}
\|g \,|\, L_{p)}\| &\overset{\text{Lemma 4.4}}{\le} c_{p,\alpha} \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon/2,\infty}\| \\
&= 2^{\frac{1}{p}} c_{p,\alpha} \sup_{0<\varepsilon<\varepsilon_0/2} \varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\alpha\varepsilon,\infty}\| \\
&\le c'_{p,\alpha} \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \, \|g \,|\, (L_q, L_p)^{\mathcal{K}}_{1-\varepsilon,\infty}\|. \qquad \square
\end{aligned}$$

This norm is essentially the norm of an extrapolation space where we have extrapolated a scale of interpolation spaces. Therefore, we can now formulate the interpolation characterisation for the grand Lebesgue spaces as follows using the equivalence of interpolation and extrapolation.

Theorem 4.11. *Let $1 < p < \infty$ and $1 \le q < p$. Further, let $0 < \varepsilon_0 < \min(1, p-1)$, $\varepsilon_0 < \eta < 1$ and $Y_\theta = (L_p, L_q)^{\mathcal{K}}_{\theta,\infty}$ for $0 < \theta < \eta$, $Y_0 = L_p$ and $Y_1 = L_q$. Thus, $\{Y_\theta\}_{\theta\in\Theta}$ with $\Theta = [0, \eta) \cup \{1\}$ is a compatible family of Banach spaces by Lemma 4.5. It holds*

$$L_{p)} = (L_q, L_p)^{\mathcal{K}}_{w,\infty} = Y_0(\log Y)^{+}_{\frac{1}{p},\infty} \tag{4.14}$$

(in the sense of equivalent norms) with

$$w(t) = \sup_{0<\varepsilon<\varepsilon_0} t^{\varepsilon-1}\varepsilon^{\frac{1}{p}}. \tag{4.15}$$

Proof. By Theorem 4.6 it holds

$$(L_p, L_q)^{\mathcal{K}}_{v,\infty} = Y_0(\log Y)^{+}_{\frac{1}{p},\infty} \quad \text{for} \quad v(s) = \sup_{0<\varepsilon<\varepsilon_0} t^{-\varepsilon}\varepsilon^{\frac{1}{p}}.$$

From Lemma 4.2 (i) we know that $(L_q, L_p)^{\mathcal{K}}_{w,\infty} = (L_p, L_q)^{\mathcal{K}}_{v,\infty}$ with

$$w(t) = t^{-1}v(t^{-1}) = \sup_{0<\varepsilon<\varepsilon_0} t^{\varepsilon-1}\varepsilon^{\frac{1}{p}}$$

and the second equality in (4.14) is proven. We immediately get the first equality by

$$\|f \,|\, Y_0(\log Y)^{+}_{\frac{1}{p},\infty}\| \overset{\text{Def.}}{=} \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \underbrace{\|f \,|\, Y_\varepsilon\|}_{\substack{=\|f \,|\, (L_p,L_q)^{\mathcal{K}}_{\varepsilon,\infty}\| \\ =\|f \,|\, (L_q,L_p)^{\mathcal{K}}_{1-\varepsilon,\infty}\|}} \overset{(4.13)}{=} \|f \,|\, L_{p)}\|. \qquad \square$$

Corollary 4.12. *Let $1 < p < \infty$ and $1 \le q < p$.*

(i) It holds in the sense of equivalent norms $L_{p)} = (L_q, L_p)^{\mathcal{K}}_{w,\infty}$ for $w(t) = t^{-1}(1-\ln t)^{-\frac{1}{p}}$, $0 < t < 1$.

(ii) It holds

$$\|f \,|\, L_{p)}\| \sim \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 f^*(s)^p \, \mathrm{d}s \right)^{\frac{1}{p}}. \tag{4.16}$$

(iii) For $f^{**}(s) := \frac{1}{s}\int_0^s f^*(x)\,\mathrm{d}x$

$$\|f \,|\, L_{p)}\| \sim \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 f^{**}(s)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t}. \tag{4.17}$$

Proof. (i) Take $w(t)$ from (4.15). It holds by Lemma 3.6

$$w(t) = t^{-1} \sup_{0<\varepsilon<\varepsilon_0} t^{\varepsilon}\varepsilon^{\frac{1}{p}} \sim t^{-1}(1 - \ln t)^{-\frac{1}{p}} \quad \text{for} \quad 0 < t < 1.$$

Since we have

$$\int_0^\infty \min(1,t)\, w(t)\, \frac{\mathrm{d}t}{t} \sim \int_0^1 (1-\ln t)^{-\frac{1}{p}}\, \frac{\mathrm{d}t}{t} + \int_1^\infty t^{\varepsilon_0 - 1}\, \frac{\mathrm{d}t}{t} < \infty,$$

we derive from Lemma 4.2 that we need to consider $w(t)$ only for $0 < t < 1$, hence it holds

$$\begin{aligned}\|f \mid L_{p)}\| &\sim \sup_{0<t<\infty} w(t)K(t,f;L_q,L_p) \sim \sup_{0<t<1} w(t)K(t,f;L_q,L_p)\\ &\sim \sup_{0<t<1} t^{-1}(1-\ln t)^{-\frac{1}{p}} K(t,f;L_q,L_p). \end{aligned} \tag{4.18}$$

(ii) [FK04] give an argument that we present in detail. We take the following formula that has been proven by Holmstedt, cf. [Hol70, Theorem 4.1]: Let $1 \le q < p \le \infty$ and $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}$ (note that $\alpha > 1$). Then (the upper border of the second integral is 1 as $|\Omega| = 1$)

$$K(t,f;L_q,L_p) \sim \left(\int_0^{t^\alpha} f^*(s)^q\, \mathrm{d}s\right)^{\frac{1}{q}} + t\left(\int_{t^\alpha}^1 f^*(s)^p\, \mathrm{d}s\right)^{\frac{1}{p}} \tag{4.19}$$

for $0 < t < 1$. Write $\beta = \alpha^{-1} < 1$. Take any $0 < \varepsilon < \beta(1 - 1/p)$. As in that case, $t^{-\beta+\varepsilon}(1-\ln t^\beta)^{-\frac{1}{p}}$ is monotonically decreasing for $0 < t < 1$, we have

$$\begin{aligned} t^{-\beta}(1-\ln t^\beta)^{-\frac{1}{p}} \left(\int_0^t f^*(s)^q\, \mathrm{d}s\right)^{\frac{1}{q}} &= t^{-\varepsilon}\left(\int_0^t \left[t^{-\beta+\varepsilon}(1-\ln t^\beta)^{-\frac{1}{p}} f^*(s)\right]^q \mathrm{d}s\right)^{\frac{1}{q}}\\ &\le t^{-\varepsilon}\left(\int_0^t \left[s^{-\frac{1}{q}+\frac{1}{p}+\varepsilon}(1-\beta\ln s)^{-\frac{1}{p}} f^*(s)\right]^q \mathrm{d}s\right)^{\frac{1}{q}}\\ &= t^{-\varepsilon}\left(\int_0^t \left[s^{\frac{1}{p}}(1-\beta\ln s)^{-\frac{1}{p}} f^*(s)\right]^q s^{q\varepsilon}\, \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}}\\ &\overset{\text{Hölder}}{\le} t^{-\varepsilon}\left(\sup_{0<s<t} s^{\frac{1}{p}}(1-\beta\ln s)^{-\frac{1}{p}} f^*(s)\right) \underbrace{\left(\int_0^t s^{q\varepsilon}\, \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}}}_{=\left(\frac{1}{q\varepsilon}\right)^{\frac{1}{q}} t^\varepsilon}\\ &\le c_{q,\varepsilon} \sup_{0<s<1} s^{\frac{1}{p}}(1-\beta\ln s)^{-\frac{1}{p}} f^*(s) \qquad (4.20)\\ &= c_{q,\varepsilon} \sup_{0<s<\frac{1}{2}} (2s)^{\frac{1}{p}}(1-\beta\ln 2s)^{-\frac{1}{p}} f^*(2s). \end{aligned}$$

Note that the expression (4.20) is in fact an equivalent quasinorm of the Lorentz-Zygmund space $L_{p,\infty}(\log L)_{-\frac{1}{p}}$. We use this later on, cf. Section 4.5. Now, by the monotonicity of f^*, it holds

$$f^*(2s) = s^{-\frac{1}{p}}\left(\int_s^{2s} f^*(2s)^p\, \mathrm{d}x\right)^{\frac{1}{p}} \overset{|\Omega|=1}{\le} s^{-\frac{1}{p}}\left(\int_s^1 f^*(x)^p\, \mathrm{d}x\right)^{\frac{1}{p}}. \tag{4.21}$$

Therefore, by taking the supremum over the previous inequality and changing variables $t^\beta \mapsto t$ and $s^\beta \mapsto s$ respectively, it holds

$$\sup_{0<t<1} t^{-1}(1-\ln t)^{-\frac{1}{p}} \left(\int_0^{t^\alpha} f^*(s)^q \,\mathrm{d}s\right)^{\frac{1}{q}} \le c_{p,q,\varepsilon} \sup_{0<s<1} (1-\ln s)^{-\frac{1}{p}} \left(\int_{s^\alpha}^1 f^*(x)^p \,\mathrm{d}x\right)^{\frac{1}{p}} . \quad (4.22)$$

Finally, we conclude by (4.18) and (4.19) that

$$\begin{aligned} \|f \mid L_{p)}\| &\sim \sup_{0<t<1} t^{-1}(1-\ln t)^{-\frac{1}{p}} \left[\left(\int_0^{t^\alpha} f^*(s)^q \,\mathrm{d}s\right)^{\frac{1}{q}} + t\left(\int_{t^\alpha}^1 f^*(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}}\right] \\ &\ge \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_{t^\alpha}^1 f^*(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} \|f \mid L_{p)}\| &\lesssim \sup_{0<t<1} t^{-1}(1-\ln t)^{-\frac{1}{p}} \left(\int_0^{t^\alpha} f^*(s)^q \,\mathrm{d}s\right)^{\frac{1}{q}} + \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_{t^\alpha}^1 f^*(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\overset{(4.22)}{\lesssim} \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_{t^\alpha}^1 f^*(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}} . \end{aligned}$$

(4.16) now follows, if we again change the variable in the supremum $t \mapsto t^\beta$, as with constants depending only on β it holds $(1-\beta\ln t)^{-\frac{1}{p}} \sim (1-\ln t)^{-\frac{1}{p}}$, $0<t<1$.

(iii) We take the Holmstedt formula (1.24) for $(A_0, A_1) = (L_1, L_\infty)$ and $X_0 = L_q = (L_1, L_\infty)^{\mathcal{K}}_{1/q',q}$ and $X_1 = L_p = (L_1, L_\infty)^{\mathcal{K}}_{1/p',p}$. Then, we apply the formula (cf. [BL76, Theorem 5.2.1])

$$K(t, f; L_r, L_\infty) \sim \left(\int_0^{t^r} f^*(s)^r \,\mathrm{d}s\right)^{\frac{1}{r}} \quad (4.23)$$

and derive the following formula that Holmstedt also used in his proof of (4.19) (cf. [Hol70, eq. (4.5)]):

$$K(t, f; L_q, L_p) \sim \left(\int_0^{t^\alpha} f^{**}(s)^q \,\mathrm{d}s\right)^{\frac{1}{q}} + t\left(\int_{t^\alpha}^1 f^{**}(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}} .$$

This is the same formula as (4.19), except that f^{**} is now in the place of f^*. As f^{**} is monotonically decreasing, just as f^* is, we can use the same reasoning as in (ii) to prove the remaining norm (4.17). □

4.4 Small Lebesgue spaces

Before we can establish the main theorem of this section, which provides an interpolation characterisation of small Lebesgue spaces, we need the following two lemmas, which give us another norm for $L_{(p}$.

Lemma 4.13. *It holds with constants independent of* $0 < \theta < 1$

$$\theta^{\frac{1}{r}} \, \|f \,|\, (L_r, L_\infty)^{\mathcal{K}}_{\theta,r}\| \sim \|f \,|\, L_{q,r}\| \quad \textit{for} \quad \frac{1}{q} = \frac{1-\theta}{r}. \tag{4.24}$$

Proof. We again use formula (4.23). Now, by $\frac{1}{q} = \frac{1-\theta}{r}$, we calculate directly:

$$\begin{aligned}
\theta \cdot \|f \,|\, (L_r, L_\infty)^{\mathcal{K}}_{\theta,r}\|^r &\overset{\text{Def.}}{=} \theta \cdot \int_0^\infty \Big[t^{-\theta} K(t, f; L_r, L_\infty)\Big]^r \, \frac{\mathrm{d}t}{t} \\
&\overset{(4.23)}{\sim} \theta \cdot \int_0^\infty t^{-\theta r} \int_0^{t^r} f^*(s)^r \, \mathrm{d}s \, \frac{\mathrm{d}t}{t} = \int_0^\infty \int_0^{t^r} \theta t^{-\theta r} f^*(s)^r \, \mathrm{d}s \, \frac{\mathrm{d}t}{t} \\
&\overset{t^r = \tau}{\sim} \int_0^\infty \int_0^\tau \theta \tau^{-\theta} f^*(s)^r \, \mathrm{d}s \, \frac{\mathrm{d}\tau}{\tau} \\
&\overset{\text{Fubini}}{=} \int_0^\infty f^*(s)^r \int_s^\infty \theta \tau^{-\theta-1} \, \mathrm{d}\tau \, \mathrm{d}s \\
&= \int_0^\infty s^{-\theta} f^*(s)^r \, \mathrm{d}s = \int_0^\infty s^{1-\theta} f^*(s)^r \, \frac{\mathrm{d}s}{s} \\
&\overset{\frac{1}{q} = \frac{1-\theta}{r}}{=} \int_0^\infty \Big[s^{\frac{1}{q}} f^*(s)\Big]^r \, \frac{\mathrm{d}s}{s} = \|f \,|\, L_{q,r}\|^r
\end{aligned}$$

□

Lemma 4.14. *For any* $0 < \varepsilon_0 < \min(1, p' - 1)$ *and* $1 < p < \infty$*, it holds*

$$\|g \,|\, L_{(p}\| \sim \inf_{g = \sum g_k} \sum_{k=1}^\infty \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \, \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon,1}\|. \tag{4.25}$$

Proof. We follow the proof of [FK04].

Take Lemma 4.13 with $r = p$, $q = p + \varepsilon$, $0 < \varepsilon < 1$ and $\theta = 1 - \frac{p}{p+\varepsilon} = \gamma\varepsilon$ with $\gamma \sim 1$. It follows $\|g_k \,|\, L_{p+\varepsilon,p}\| \sim \varepsilon^{\frac{1}{p}} \, \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\|$. Recalling the norm (2.11), we see that

$$\|g \,|\, L_{(p}\| \sim \inf_{g = \sum g_k} \sum_{k=1}^\infty \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \, \|g_k \,|\, L_{p+\varepsilon,p}\| \sim \inf_{g = \sum g_k} \sum_{k=1}^\infty \inf_{0<\varepsilon<\varepsilon_0} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\|.$$

This needs to be compared with the norm (4.25).

For the first inequality, we show that with constants independent of ε it holds

$$\varepsilon^{\frac{1}{p}} \, \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\| \lesssim \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\gamma\varepsilon,1}\|. \tag{4.26}$$

Let us start by

$$\varepsilon^{\frac{1}{p}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\| \overset{\substack{\text{Lemma}\\ 4.4(ii)}}{\leq} c_\gamma \varepsilon^{\frac{1}{p}} \varepsilon^{1-\frac{1}{p}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,1}\| = c_\gamma \varepsilon \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,1}\|. \tag{4.27}$$

We notice that in the proof of Lemma 4.3, none of the constants of the estimates depends on the weight w. If we take $w(t) = t^{-\gamma\varepsilon}$, then it holds according to (4.1)

$$w^*(s) = \frac{1}{s}\int_0^s t^{-\gamma\varepsilon+1} \frac{\mathrm{d}t}{t} + \int_s^\infty t^{-\gamma\varepsilon} \frac{\mathrm{d}t}{t} = s^{-\gamma\varepsilon}\left(\frac{1}{\gamma\varepsilon} + \frac{1}{1-\gamma\varepsilon}\right) < \infty.$$

Note that $0 < \gamma\varepsilon < \frac{1}{2}$, therefore we have with constants independent of ε

$$\frac{1}{\gamma\varepsilon} < \frac{1}{\gamma\varepsilon(1-\gamma\varepsilon)} < \frac{2}{\gamma\varepsilon} \implies w^*(s) \sim \varepsilon^{-1} t^{-\gamma\varepsilon}.$$

We can apply Lemma 4.3 and conclude that with constants independent of ε

$$\|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,1}\| \sim \varepsilon^{-1} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\gamma\varepsilon,1}\|. \tag{4.28}$$

Putting (4.27) and (4.28) together, we get (4.26). We come to the second inequality. For $L_\infty \hookrightarrow L_p$ on finite measure spaces, it holds by Lemma 4.4 (iii)

$$\|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon/2,1}\| \leq c_{p,\gamma}\, \varepsilon^{\frac{1}{p}-1} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\|.$$

Combining this with (4.28) yields

$$\begin{aligned}\inf_{0<\varepsilon<\varepsilon_0} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\gamma\varepsilon,1}\| &\leq \inf_{0<\varepsilon<\varepsilon_0/2} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\gamma\varepsilon,1}\| \\ &= \inf_{0<\varepsilon<\varepsilon_0} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\gamma\varepsilon/2,1}\| \\ &\overset{(4.28)}{\leq} \inf_{0<\varepsilon<\varepsilon_0} c_{p,\gamma}\, \varepsilon^{\frac{1}{p}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{K}}_{\gamma\varepsilon,p}\|. \qquad \square\end{aligned}$$

Theorem 4.15. *For $1 < p < \infty$, let $0 < \varepsilon_0 < \min(1, p'-1)$, $0 < \eta < 1-\varepsilon_0$ and $Y_\theta = (L_\infty, L_p)^{\mathcal{J}}_{\theta,1}$ for $\eta < \theta < 1$, $Y_0 = L_\infty$ and $Y_1 = L_p$. Thus, $\{Y_\theta\}_{\theta\in\Theta}$ with $\Theta = \{0\}\cup(\eta, 1]$ is a compatible family of Banach spaces by Lemma 4.7. It holds*

$$L_{(p} = (L_p, L_\infty)^{\mathcal{J}}_{w^*,1} = Y_1(\log Y)^{-}_{\frac{1}{p'},1} \tag{4.29}$$

(in the sense of equivalent norms) with

$$\frac{1}{w^*(t)} = \sup_{0<\varepsilon<\varepsilon_0} t^{\varepsilon} \varepsilon^{\frac{1}{p'}}. \tag{4.30}$$

Proof. From Theorem 4.8 we have

$$(L_\infty, L_p)^{\mathcal{J}}_{v^*,1} = Y_1(\log Y)^{-}_{\frac{1}{p'},1} \quad \text{with} \quad \frac{1}{v^*(s)} \overset{(4.7)}{=} \sup_{0<\varepsilon<\varepsilon_0} \frac{s^{1-\varepsilon}}{\varepsilon^{-\frac{1}{p'}}}.$$

By Lemma 4.2 (i) it holds $(L_p, L_\infty)^{\mathcal{J}}_{w^*,q} = (L_\infty, L_p)^{\mathcal{J}}_{v^*,q}$ with

$$\frac{1}{w^*(t)} = \frac{t}{v^*(t^{-1})} = \sup_{0<\varepsilon<\varepsilon_0} \frac{t}{t^{1-\varepsilon}\varepsilon^{-\frac{1}{p'}}} = \sup_{0<\varepsilon<\varepsilon_0} t^\varepsilon \varepsilon^{\frac{1}{p'}}$$

and the second equality of (4.29) is clear.

From now on we follow [FK04]. We have proven another norm in the previous Lemma 4.14. The remaining arguments are split it into two steps: First, we show $L_{(p} \subset (L_p, L_\infty)^{\mathcal{J}}_{w^*,1}$, and second, $Y_1(\log Y)^{-}_{\frac{1}{p'},1} \subset L_{(p}$.

Let $g \in L_{(p}$. We choose $g_k \in (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon_k,1}$, $k \in \mathbb{N}$, with $g = \sum_k g_k$ and, thereby, ε_k such that

$$\|g \,|\, L_{(p}\| \gtrsim \frac{1}{2} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon,1}\| \geq \frac{1}{4} \sum_{k=1}^{\infty} \varepsilon_k^{-\frac{1}{p'}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon_k,1}\|. \tag{4.31}$$

Then take $u_{\nu k} \in L_p \cap L_\infty$, $\nu \in \mathbb{Z}$, with $g_k = \sum_\nu u_{\nu k}$ for all k and

$$\sum_{\nu=-\infty}^{\infty} 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}) \leq 2\,\|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon_k,1}\|. \tag{4.32}$$

Let $u_\nu = \sum_k u_{\nu k}$. Then we compute

$$\begin{aligned}
\|g \,|\, (L_p, L_\infty)^{\mathcal{J}}_{w^*,1}\| &\leq \sum_{\nu=-\infty}^{\infty} w^*(2^\nu) J(2^\nu u_\nu) \overset{J \text{ is norm}}{\leq} \sum_{\nu=-\infty}^{\infty} w^*(2^\nu) \sum_{k=1}^{\infty} J(2^\nu, u_{\nu k}) \\
&\leq \sum_{\nu=-\infty}^{\infty} w^*(2^\nu) \sum_{k=1}^{\infty} J(2^\nu, u_{\nu k}) \left(\varepsilon_k^{-\frac{1}{p'}} 2^{-\nu\varepsilon_k} \sup_{1\leq m\leq\infty} \varepsilon_m^{\frac{1}{p'}} 2^{\nu\varepsilon_m} \right) \\
&\leq \sum_{\nu=-\infty}^{\infty} \underbrace{w^*(2^\nu) \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p'}} 2^{\nu\varepsilon}}_{=1} \sum_{k=1}^{\infty} \varepsilon_k^{-\frac{1}{p'}} 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}) \\
&= \sum_{k=1}^{\infty} \varepsilon_k^{-\frac{1}{p'}} \sum_{\nu=-\infty}^{\infty} 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}) \\
&\overset{(4.32)}{\leq} 2 \sum_{k=1}^{\infty} \varepsilon_k^{-\frac{1}{p'}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon_k,1}\| \overset{(4.31)}{\lesssim} 8\,\|g \,|\, L_{(p}\|.
\end{aligned}$$

Let $g \in Y_1(\log Y)^{-}_{\frac{1}{p'},1}$. We use the norm (3.8) and have for any J with $2^{-J} < \varepsilon_0$ (the last step is only an index shift)

$$\|g \,|\, Y_1(\log Y)^{-}_{\frac{1}{p'},1}\| \sim \inf_{g=\sum g_j} \sum_{j=J}^{\infty} 2^{\frac{j}{p'}} \|g_j \,|\, \underbrace{(L_\infty, L_p)^{\mathcal{J}}_{1-2^{-j},1}}_{=(L_p,L_\infty)^{\mathcal{J}}_{2^{-j},1}}\|$$

$$\geq \inf_{g=\sum g_j} \sum_{j=J}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_j \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon,1}\|$$

$$= \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k \,|\, (L_p, L_\infty)^{\mathcal{J}}_{\varepsilon,1}\| \overset{(4.25)}{\sim} \|g \,|\, L_{(p}\|. \qquad \square$$

Corollary 4.16. *Let* $1 < p < \infty$.

(i) It holds $L_{(p} = (L_p, L_\infty)^{\mathcal{K}}_{w,1}$ *for* $w(t) = (1 - \ln t)^{-\frac{1}{p}}$, $0 < t < 1$.

(ii) It holds

$$\|f \,|\, L_{(p}\| \sim \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t f^*(s)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t}. \tag{4.33}$$

(iii) For $f^{**}(s) := \frac{1}{s}\int_0^s f^*(x)\,\mathrm{d}x$ *it holds also*

$$\|f \,|\, L_{(p}\| \sim \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t f^{**}(s)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t}. \tag{4.34}$$

Proof. (i) As we stated in Lemma 4.2, we need to consider $w(t)$ only for $0 < t < 1$ as $L_\infty \hookrightarrow L_p$ and can choose the weight for $t \geq 1$ arbitrarily provided that it satisfies (4.1). Hence, assume throughout the proof that for $0 < \varepsilon_0 < \min(1, p' - 1)$ as in Theorem 4.15

$$w(t) = \begin{cases} (1 - \ln t)^{-\frac{1}{p}} & \text{if} \quad 0 < t < 1 \\ t^{-\varepsilon_0} & \text{if} \quad t \geq 1. \end{cases} \tag{4.35}$$

Then (4.1) holds as

$$\int_0^\infty \min(1, t)\, w(t) \frac{\mathrm{d}t}{t} = \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \,\mathrm{d}t + \int_1^\infty t^{-\varepsilon_0} \frac{\mathrm{d}t}{t} < \infty.$$

By Theorem 4.3 it holds $(L_p, L_\infty)^{\mathcal{K}}_{w,1} = (L_p, L_\infty)^{\mathcal{J}}_{w^*,1}$ with $w^*(t)$ derived as in (4.5). We compute $w^*(s)$ starting from w of (4.35). First, consider the case $0 < s < 1$.

$$\begin{aligned} w^*(s) &= \int_0^\infty \min\left(1, \frac{t}{s}\right) w(t) \frac{\mathrm{d}t}{t} \\ &= \int_0^1 \min\left(1, \frac{t}{s}\right) (1 - \ln t)^{-\frac{1}{p}} \frac{\mathrm{d}t}{t} + \int_1^\infty t^{-\varepsilon_0} \frac{\mathrm{d}t}{t} \\ &\overset{x=1-\ln t}{=} \int_1^\infty \min\left(1, \frac{e^{1-x}}{s}\right) x^{-\frac{1}{p}} \,\mathrm{d}x + \frac{1}{\varepsilon_0} \\ &= \int_1^{1-\ln s} x^{-\frac{1}{p}} \,\mathrm{d}x + \frac{1}{s}\int_{1-\ln s}^\infty e^{1-x} x^{-\frac{1}{p}} \,\mathrm{d}x + \frac{1}{\varepsilon_0} \\ &= p'(1 - \ln s)^{\frac{1}{p'}} - p' + \frac{1}{s}\int_{1-\ln s}^\infty e^{1-x} x^{-\frac{1}{p}} \,\mathrm{d}x + \frac{1}{\varepsilon_0}. \end{aligned} \tag{4.36}$$

As

$$\begin{aligned} \int_{1-\ln s}^{\infty} e^{1-2x}\,\mathrm{d}x \quad &\lesssim \quad \int_{1-\ln s}^{\infty} e^{1-x}x^{-\frac{1}{p}}\,\mathrm{d}x \quad \lesssim \quad \int_{1-\ln s}^{\infty} e^{1-x}\,\mathrm{d}x \\ \iff \quad \frac{s^2}{2} \quad &\lesssim \quad \int_{1-\ln s}^{\infty} e^{1-x}x^{-\frac{1}{p}}\,\mathrm{d}x \quad \lesssim \quad s, \end{aligned}$$

we can divide (4.36) by p' and have

$$(1-\ln s)^{\frac{1}{p'}} - \left(1 - \frac{s}{2p'} - \frac{1}{\varepsilon_0}\right) \lesssim w^*(s) \lesssim (1-\ln s)^{\frac{1}{p'}} - \left(1 - \frac{1}{p'} - \frac{1}{\varepsilon_0}\right)$$

and therefore $w^*(s) \sim (1-\ln s)^{\frac{1}{p'}}$ for $0 < s < 1$. Now, if we replace p by p' in Lemma 3.6, we derive

$$w^*(s) \sim \frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} s^{\varepsilon}\,\varepsilon^{\frac{1}{p'}}}, \quad 0 < s < 1.$$

On the other hand, for $s \geq 1$ we have

$$\begin{aligned} w^*(s) &= \frac{1}{s}\int_0^1 (1-\ln t)^{-\frac{1}{p}}\,\mathrm{d}t + \frac{1}{s}\int_1^s t^{-\varepsilon_0}\,\mathrm{d}t + \int_s^{\infty} t^{-\varepsilon_0}\,\frac{\mathrm{d}t}{t} \\ &= \frac{c_p}{s} + \frac{1}{1-\varepsilon_0}s^{-\varepsilon_0} - \frac{1}{1-\varepsilon_0}\frac{1}{s} + \frac{1}{\varepsilon_0}s^{-\varepsilon_0} \\ &= \left(c - \frac{1}{1-\varepsilon_0}\right)s^{-1} + \frac{1}{\varepsilon_0(1-\varepsilon_0)}s^{-\varepsilon_0} \sim s^{-\varepsilon_0} \sim \frac{1}{\sup\limits_{0<\varepsilon<\varepsilon_0} s^{\varepsilon}\,\varepsilon^{\frac{1}{p'}}}. \end{aligned}$$

We conclude that w^* is essentially the weight from Theorem 4.15, hence it holds in the sense of equivalent norms $L_{(p} = (L_p, L_\infty)^{\mathcal{J}}_{w^*,1} = (L_p, L_\infty)^{\mathcal{K}}_{w,1}$.

(ii) Starting from the previous statement and the definition of interpolation spaces, we see that

$$\begin{aligned} \|f \mid L_{(p}\| &\sim \int_0^{\infty} K(t,f;L_p,L_\infty)\,w(t)\,\frac{\mathrm{d}t}{t} \overset{\text{Lemma 4.2}}{\sim} \int_0^1 K(t,f;L_p,L_\infty)\,w(t)\,\frac{\mathrm{d}t}{t} \\ &\overset{(4.23)}{\sim} \int_0^1 (1-\ln t)^{-\frac{1}{p}} \left(\int_0^{t^p} f^*(s)^p\,\mathrm{d}s\right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t} \\ &\overset{t^p=\tau}{\sim} \int_0^1 (1-\ln \tau)^{-\frac{1}{p}} \left(\int_0^{\tau} f^*(s)^p\,\mathrm{d}s\right)^{\frac{1}{p}} \frac{\mathrm{d}\tau}{\tau}. \end{aligned}$$

(iii) We first use the monotonicity of $f^*(t)$ and estimate:

$$\int_0^t f^{**}(s)^p\,\mathrm{d}s = \int_0^t \left[\frac{1}{s}\int_0^s \underbrace{f^*(u)}_{\geq f^*(s)}\,\mathrm{d}u\right]^p \mathrm{d}s \geq \int_0^t \left[f^*(s)\,\frac{1}{s}\int_0^s \mathrm{d}u\right]^p \mathrm{d}s = \int_0^t f^*(s)^p\,\mathrm{d}s.$$

For the other direction, we slightly modify the proof of Hardy´s inequality in [HLP99, Theorem 327a]. Let $n > 0$, $f_n^*(s) := \min(f^*(s), n)$ and $F_n(x) = \int_0^x f_n^*(s)\,\mathrm{d}s$, therefore $F_n(x) = \mathcal{O}(x)$ for $x \to 0$. We have

$$\begin{aligned}\int_0^t \left(\frac{F_n(x)}{x}\right)^p \mathrm{d}x &= -\frac{1}{p-1}\int_0^t F_n(x)^p \frac{\mathrm{d}}{\mathrm{d}x}(x^{1-p})\,\mathrm{d}x \\ &= \underbrace{-\frac{x^{1-p}F_n(x)^p}{p-1}\Bigg|_0^t}_{\leq 0 \text{ as } F_n(x)\sim x,\ x\to 0} + \frac{p}{p-1}\int_0^t \left(\frac{F_n(x)}{x}\right)^{p-1} f_n^*(x)\,\mathrm{d}x \\ &\leq \frac{p}{p-1}\int_0^t \left(\frac{F_n(x)}{x}\right)^{p-1} f_n^*(x)\,\mathrm{d}x \\ &\overset{\text{Hölder}}{\leq} \frac{p}{p-1}\left(\int_0^t \left[\frac{F_n(x)}{x}\right]^p \mathrm{d}x\right)^{\frac{1}{p'}} \left(\int_0^t f_n^*(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

Dividing by the second factor of the right hand side and raising the result to the power of p yields

$$\int_0^t \left(\frac{F_n(x)}{x}\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^t f_n^*(x)^p\,\mathrm{d}x.$$

If we take $n \to \infty$ here, we derive

$$\int_0^t f^{**}(x)^p\,\mathrm{d}x = \int_0^t \left(\frac{1}{x}\int_0^x f^*(s)\,\mathrm{d}s\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^t f^*(x)^p\,\mathrm{d}x. \qquad \square$$

Remark 4.17. As it has been indicated for the Lorentz-Zygmund spaces in Chapter 1, the expressions (4.16) and (4.33) that use the decreasing rearrangement f^* are in fact quasinorms. However, the expressions where f^* is replaced by f^{**} are norms.

4.5 Embeddings

As it has been indicated in Chapter 2, we use the now obtained quasinorms (4.16) and (4.33) to give examples for the strict embeddings between the known spaces that are intermediate spaces of L_p and $L_{p-\varepsilon}$ or $L_{p+\varepsilon}$ respectively, $\varepsilon > 0$ arbitrary.

Corollary 4.18 ([FK04, Remark 4.3])**.** *Let $1 < p < \infty$ and $\delta > 0$. Then it holds*

$$L_{p,\infty} \hookrightarrow L_{p)} \hookrightarrow L_{p,\infty}(\log L)_{-\frac{1}{p}} \tag{4.37}$$

$$L_p(\log L)_{-\frac{1}{p}} \hookrightarrow L_{p)} \hookrightarrow L_p(\log L)_{-\frac{1}{p}-\delta}. \tag{4.38}$$

All of these spaces are embedded in $L_{p-\varepsilon}$ for $0 < \varepsilon < p-1$, and L_p is embedded in each one of them.

Proof. (4.37) and (4.38) are stated without a proof (except for the fourth and last embedding) in [FK04]. The embeddings between Lorentz-Zygmund spaces and Lebesgue spaces have already been proven or can be seen easily. We refer to [BR80] for the details, e.g..

The second embedding of (4.37) is already shown in the proof of Corollary 4.12, as we have estimated expression (4.20) via (4.21) by the norm (4.16). The first embedding is the one that has already been stated in Proposition 2.4, we prove it as follows: For any $0 < t < 1$ it holds

$$\begin{aligned}(1-\ln t)^{-\frac{1}{p}}\left(\int_t^1 f^*(s)^p \,\mathrm{d}s\right)^{\frac{1}{p}} &= (1-\ln t)^{-\frac{1}{p}}\left(\int_t^1 \left[f^*(s)s^{\frac{1}{p}}\right]^p s^{-1}\,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\overset{\text{Hölder}}{\leq} (1-\ln t)^{-\frac{1}{p}} \sup_{t<s<1} f^*(s)s^{\frac{1}{p}} \underbrace{\left(\int_t^1 s^{-1}\,\mathrm{d}s\right)^{\frac{1}{p}}}_{=(-\ln t)^{\frac{1}{p}}} \\ &= \underbrace{\left(\frac{-\ln t}{1-\ln t}\right)^{\frac{1}{p}}}_{<1} \sup_{0<s<1} f^*(s)s^{\frac{1}{p}},\end{aligned}$$

and by taking the supremum over all t we conclude that $\|f \mid L_{p)}\| \leq \|f \mid L_{p,\infty}\|$.

For (4.38), consider first

$$\begin{aligned}\|f \mid L_{p)}\| &\sim \sup_{0<t<1}\left(\int_t^1 \left[(1-\ln t)^{-\frac{1}{p}} f^*(s)\right]^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\leq \sup_{0<t<1}\left(\int_t^1 \left[(1-\ln s)^{-\frac{1}{p}} f^*(s)\right]^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \left[(1-\ln s)^{-\frac{1}{p}} f^*(s)\right]^p \,\mathrm{d}s\right)^{\frac{1}{p}} = \|f \mid L_p(\log L)_{-\frac{1}{p}}\|.\end{aligned}$$

Second, we use the norm (2.6) and interpret $\|f \mid L_{p-\varepsilon,p}\|$ as a function of ε in $L_{p,\infty}((0,\varepsilon_0))$, which is continuously embedded in $L_{p/(1+p\delta),p}((0,\varepsilon))$ for $\delta > 0$ by (1.7). Finally, we apply the equivalent norm (3.13) for the Zygmund spaces that we have already used in the proof of Proposition 3.7. With $\frac{1}{p^\varepsilon} = \frac{1}{p} + \varepsilon$, hence $p^\varepsilon = p - \gamma\varepsilon$ with $\gamma \sim 1$, this reads as

$$\begin{aligned}\|f \mid L_{p)}\| &\overset{(2.6)}{\sim} \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|f \mid L_{p-\varepsilon,p}\| \\ &\overset{(1.7)}{\geq} c\left(\int_0^{\varepsilon_0} \left[\varepsilon^{\frac{1}{p}+\delta} \|f \mid L_{p-\varepsilon,p}\|\right]^p \frac{\mathrm{d}\varepsilon}{\varepsilon}\right)^{\frac{1}{p}} \sim \left(\int_0^{\varepsilon_0} \left[\varepsilon^{\frac{1}{p}+\delta} \|f \mid L_{p^\varepsilon,p}\|\right]^p \frac{\mathrm{d}\varepsilon}{\varepsilon}\right)^{\frac{1}{p}} \\ &\overset{(3.13)}{\sim} \|f \mid L_p(\log L)_{-\frac{1}{p}-\delta}\|.\end{aligned}$$

□

Example 4.19. Let $\Omega = (0,1)$ and $1 < p < \infty$. The inclusions in Corollary 4.18 are sharp as we show by the following examples. The first example is taken (with modifications) from [Gre93, Example 2.3]. The examples (ii) to (iv) are taken from [FK04, Remark 4.4].

(i) We are looking for $f_1 \in L_p(\log L)_{-\frac{1}{p}}$ with $f_1 \notin L_{p,\infty}$. By (4.38) we then know that $f_1 \in L_{p)}$. Let us assume that for $0 < t < 1$

$$f_1(t) = \sum_{k=1}^{\infty} a_k \chi_{E_k} \quad \text{with} \quad E_k = [m_{k+1}, m_k), \quad a_k > 0$$

where $\{m_k\}_{k=1}^{\infty}$ is a monotone sequence of numbers in $(0,1)$ with $m_k \xrightarrow[k\to\infty]{} 0$. We choose m_k and a_k for $k \geq 1$ by

$$m_k = e^{1-k^3} \qquad \text{and} \qquad a_k = \left(\frac{k}{m_k}\right)^{\frac{1}{p}}.$$

Then

$$\| f_1 \,|\, L_{p,\infty} \| = \sup_{0<t<1} t^{\frac{1}{p}} f_1(t) \geq \lim_{k\to\infty} {m_k}^{\frac{1}{p}} a_k = \lim_{k\to\infty} k^{\frac{1}{p}} = \infty$$

and on the other hand, as $(1-\ln t)^{-1}$ is monotonically increasing, it holds

$$\begin{aligned} \| f_1 \,|\, L_p(\log L)_{-\frac{1}{p}} \| &= \left(\int_0^1 \frac{f_1^*(t)^p}{1-\ln t} \,\mathrm{d}t \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} \int_{m_{k+1}}^{m_k} \frac{f_1^*(t)^p}{1-\ln t} \,\mathrm{d}t \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{k=1}^{\infty} (m_k - m_{k+1}) \cdot \frac{{a_k}^p}{1-\ln m_k} \right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{\infty} (1 - e^{-3k^2-3k-1}) \cdot \frac{k}{k^3} \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

We could also compute directly that

$$\begin{aligned} \| f_1 \,|\, L_{p)} \| &= \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_t^1 f_1^*(s)^p \,\mathrm{d}s \right)^{\frac{1}{p}} \\ &\leq \sup_{k\in\mathbb{N}} (1-\ln m_k)^{-\frac{1}{p}} \left(\int_{m_{k+1}}^1 f_1^*(s)^p \,\mathrm{d}s \right)^{\frac{1}{p}} \\ &= \sup_{k\in\mathbb{N}} \left(\frac{1}{k^3}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{k} \frac{m_n - m_{n+1}}{m_n} \cdot n \right)^{\frac{1}{p}} \leq \sup_{k\in\mathbb{N}} \left(\frac{1}{k^3}\right)^{\frac{1}{p}} \left(\frac{k(k+1)}{2} \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

(ii) $f_2(t) = t^{-\frac{1}{p}} (1-\ln t)^{\frac{1}{p}} \in L_{p,\infty}(\log L)_{-\frac{1}{p}}$, but $f_2 \notin L_{p)}$.

$$\| f_2 \,|\, L_{p,\infty}(\log L)_{-\frac{1}{p}} \| = \sup_{0<t<1} t^{\frac{1}{p}} (1-\ln t)^{-\frac{1}{p}} t^{-\frac{1}{p}} (1-\ln t)^{\frac{1}{p}} = 1 < \infty$$

$$\|f_2 \,|\, L_{p)}\| \quad \geq \lim_{t\to 0}(1-\ln t)^{-\frac{1}{p}} \left(\int_t^1 (1-\ln s)\,\frac{\mathrm{d}s}{s}\right)^{\frac{1}{p}} = \lim_{t\to 0}(1-\ln t)^{\frac{1}{p}} = \infty$$

(iii) $f_3(t) = t^{-\frac{1}{p}} \in L_{p)}$, but $f_3 \notin L_p(\log L)_{-\frac{1}{p}}$ as

$$\|f_3 \,|\, L_p(\log L)_{-\frac{1}{p}}\| = \left(\int_0^1 (1-\ln t)^{-1}\, t^{-1}\,\mathrm{d}t\right)^{\frac{1}{p}} \overset{x=1-\ln t}{=} \left(\int_1^\infty \frac{1}{x}\,\mathrm{d}x\right)^{\frac{1}{p}} = \infty.$$

(iv) $f_4(t) = t^{-\frac{1}{p}}(1-\ln t)^\alpha$, $0<\alpha<\delta$ is in $L_p(\log L)_{-\frac{1}{p}-\delta}$ for $\delta>0$, but $f_4 \notin L_{p)}$.

$$\|f_4 \,|\, L_p(\log L)_{-\frac{1}{p}-\delta}\| = \left(\int_0^1 (1-\ln t)^{-1-(\delta-\alpha)p}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \overset{x=1-\ln t}{=} \left(\int_1^\infty x^{-(\delta-\alpha)p}\,\frac{\mathrm{d}x}{x}\right)^{\frac{1}{p}} < \infty$$

$$\|f_4 \,|\, L_{p)}\| \quad \geq \lim_{t\to 0}(1-\ln t)^{-\frac{1}{p}} \left(\int_t^1 (1-\ln s)^{\alpha p}\,\frac{\mathrm{d}s}{s}\right)^{\frac{1}{p}} \sim \lim_{t\to 0}(1-\ln t)^{\alpha} = \infty$$

Corollary 4.20 ([CK14] and [CF05]). *Let* $1<p<\infty$ *and* $0<\delta<\min(1,p-1)$. *Then it holds*

$$L_{p,1}(\log L)_{\frac{1}{p'}} \hookrightarrow L_{(p} \hookrightarrow L_{p,1} \tag{4.39}$$

$$L_p(\log L)_{\frac{1}{p'}+\delta} \hookrightarrow L_{(p} \hookrightarrow L_p(\log L)_{\frac{1}{p'}} \tag{4.40}$$

All of these spaces are embedded in L_p, *whereas* $L_{p+\varepsilon}$ *is embedded in each one of them for* $\varepsilon>0$.

Proof. This can be seen from Corollary 4.18 by taking associate spaces and replacing p' by p. The associate spaces of Lorentz-Zygmund spaces can be derived e.g. as the authors in [CFT04] do. □

Example 4.21. Let $\Omega=(0,1)$ and $1<p<\infty$. We only give one simple example that does not cover all embeddings in Corollary 4.20. For $\gamma\geq 0$ we consider $f_\gamma(t) = t^{-\frac{1}{p}}\,(1-\ln t)^{-1-\gamma}$ (cf. [CFG17, p. 678]). Then $f_\gamma \in L_{(p}$ if and only if $\gamma>0$, as

$$\begin{aligned}
\|f_\gamma \,|\, L_{(p}\| &= \int_0^1 (1-\ln t)^{-\frac{1}{p}} \left(\int_0^t s^{-1}\,(1-\ln s)^{-p(1+\gamma)}\,\mathrm{d}s\right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t} \\
&\overset{x=1-\ln s}{=} \int_0^1 (1-\ln t)^{-\frac{1}{p}} \left(\int_{1-\ln t}^\infty x^{-p(1+\gamma)}\,\mathrm{d}x\right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t} \\
&\sim \int_0^1 (1-\ln t)^{-\frac{1}{p}}(1-\ln t)^{-(1+\gamma)+\frac{1}{p}}\,\frac{\mathrm{d}t}{t} \overset{\tau=1-\ln t}{=} \int_1^\infty \tau^{-\gamma}\,\frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

At the same time $f_\gamma \in L_p(\log L)_{\frac{1}{p'}+\delta}$, $\delta > 0$, if and only if $\gamma > \delta$, as

$$\|f_\gamma \,|\, L_p(\log L)_{\frac{1}{p'}+\delta}\| = \left(\int_0^1 t^{-1}(1-\ln t)^{-p+\frac{p}{p'}+p(\delta-\gamma)}\,\mathrm{d}t\right)^{\frac{1}{p}}$$
$$= \left(\int_0^1 (1-\ln t)^{-1+p(\delta-\gamma)}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \overset{\tau=1-\ln t}{=} \left(\int_1^\infty \tau^{p(\delta-\gamma)}\,\frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}}$$

Furthermore, $f_0 \in L_p$, and $f_\gamma \in L_{p,1}(\log L)_{\frac{1}{p'}}$ if and only if $\gamma > \frac{1}{p'}$. For $L_p(\log L)_{\frac{1}{p'}}$ and $L_{p,1}$, we are in the same situation as for $L_{(p}$, i.e. f_γ is contained in these spaces if and only if $\gamma > 0$ (we omit the computations, that are similar to the previous ones).

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