## Stochastic Modeling of Intraday Electricity Markets

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## Zusammenfassung

Limit-Orderbücher sind das Standardinstrument der Preisbildung in modernen Finanzmärkten. Während Strom traditionell in Auktionen gehandelt wird, gibt es IntradayStrommärkte wie beispielsweise den SIDC-Markt, in welchem Käufer und Verkäufer über Limit-Orderbücher zusammentreffen. In dieser Arbeit werden wir stochastische Modelle von Limit-Orderbüchern auf der Grundlage der zugrundeliegenden Marktmikrostruktur entwickeln. Einen besonderen Schwerpunkt legen wir dabei auf die Berücksichtigung besonderer Merkmale der Intraday-Strommärkte, die sich zum Teil deutlich von denen der Finanzmärkte unterscheiden.
Die in dieser Arbeit entwickelten Modelle beginnen mit einer realistischen und mikroskopischen (eventweisen) Beschreibung der Marktdynamik. Große Preisänderungen über kurze Zeiträume (Preissprünge) werden ebenso berücksichtigt wie begrenzte grenzüberschreitende Aktivitäten. Diese mikroskopischen Modelle sind im Allgemeinen zu rechenintensiv für praktische Anwendungen. Das Hauptziel dieser Arbeit ist es daher, geeignete Approximationen dieser mikroskopischen Modelle durch sogenannte Skalierungsgrenzprozesse herzuleiten. Zu diesem Zweck werden sorgfältig Skalierungsannahmen formuliert und in die mikroskopischen Modelle eingebaut. Diese Annahmen ermöglichen es uns, ihr Hochfrequenzverhalten zu untersuchen, vorausgesetzt, dass die Größe eines einzelnen Auftrags gegen Null konvergiert, während die Auftragseingangsrate gegen unendlich tendiert.
Aus mathematischer Sicht sind die mikroskopischen Modelle zeitdiskrete Prozesse, die durch zeitkontinuierliche Prozesse approximiert werden. Die Approximation erfolgt dabei im Sinne einer schwachen Konvergenz von Prozessen in der Skorokhod-Topologie im Raum der càdlàg-Funktionen. Auf diese Weise entwickeln wir funktionale Grenzwertsätze für (un-)endlich dimensionale Semimartingale.
Die Kalibrierung mathematischer Modelle ist aus Anwendersicht eines der Hauptanliegen. Dabei ist bekannt, dass Änderungspunkte (abrupte Schwankungen) in hochfrequenten Finanzdaten vorhanden sind. Falls sie durch endogene Effekte verursacht wurden, muss bei der Schätzung solcher Änderungspunkte die Abhängigkeit von den zugrundeliegenden Daten berücksichtigt werden. Daher erweitern wir im letzten Teil dieser Arbeit die bestehende Literatur zur Erkennung von Änderungspunkten, so dass auch zufällige, von den Daten abhängige Änderungspunkte durch bereits bekannte Teststatistiken und Schätzer für die Lage und Größe von Änderungspunkten gehandhabt werden können.


#### Abstract

Limit order books are the standard instrument for price formation in modern financial markets. While electricity has traditionally been traded through auctions, there are intraday electricity markets, such as the SIDC market, in which buyers and sellers meet via limit order books. In this thesis, stochastic models of limit order books are developed based on the underlying market microstructure. A particular focus is set on incorporating unique characteristics of intraday electricity markets, some of which are quite different from those of financial markets. The developed models in this thesis start with a realistic and microscopic (eventwise) description of the market dynamics. Large price changes over short time periods (price jumps) are taken into account, as well as limited cross-border activities. These microscopic models are generally computationally too intensive for practical applications. The main goal of this thesis is therefore to derive suitable approximations of these microscopic models by so-called scaling limits. For this purpose, appropriate scaling assumptions are carefully formulated and incorporated into the microscopic models which allow us to study their high-frequency behavior when the size of an individual order converges to zero while the order arrival rate tends to infinity. Mathematically, the microscopic models are discrete-time processes approximated by continuous-time processes. In this thesis, the approximation is in terms of weak convergence of processes in the Skorokhod topology in the space of càdlàg functions. Thus, we develop functional limit theorems toward (in-)finite dimensional semimartingales.

Calibration of mathematical models is one of the main concerns from a practitioner's point of view. It is well known that change points (abrupt variations) are present in high-frequency financial data. If they are caused by endogenous effects, the dependence on the underlying data must be considered when estimating such change points. In the final part of this thesis, we extend the existing literature on change point detection so that random change points depending on the data can also be handled by already known test statistics and estimators for the location and size of change points.


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## Introduction

In modern financial markets, buyers and sellers are typically matched using a continuous trading mechanism based on limit order books (LOBs). Limit order books are records of unexecuted buy and sell orders displayed at different price levels awaiting execution (cf. Figure A). The highest price a potential buyer is willing to pay is called the best bid price, whereas the best ask price is the smallest price of all placed sell orders. Incoming limit orders can be placed at many different price levels, while incoming market orders are matched against standing limit orders according to a set of priority rules. In most limit order books, submitted orders at more competitive prices ("price priority") and displayed orders over hidden orders at the same price level ("display priority") are executed first. Orders with the same display status and submission price are usually served on a first-come-first-serve basis.


Figure A: Illustration of the state of a limit order book model.
The development of realistic and at the same time tractable models for the dynamics of limit order books is a challenging task bearing in mind the rapid increase in trading activities. A promising approach to construct realistic model dynamics is to consider the underlying market microstructure. A realistic inclusion of the market microstructure, however, often reduces the tractability of these models. Yet, stochastic analysis provides us with powerful tools to approximate these microscopic models through tractable scaling ("high-frequency") limits.
But even beyond financial markets, there are markets based on a continuous order matching mechanism. While electricity is traditionally traded through auctions, the integrated European intraday electricity market "Single Intraday Coupling" (SIDC), launched in June 2018, is an important example of such a market. Even though order matching is handled through a limit order book, this market, however, has quite different characteristics than the purely financial markets. Those include, for example, extreme
price movements over short time periods, time-inhomogeneities, and strong dependencies on external influences such as the infeed of renewables.

The thesis on hand develops new stochastic models for limit order books that accommodate several of the characteristics of intraday electricity markets with continuous trading. In the following, we present the literature background from both mathematical and modeling perspectives. We then give overviews of the three chapters of this thesis and emphasize the mathematical hurdles and innovations. Finally, we synthesize the results of the three chapters and formulate the main contribution of this thesis.

## Background of this thesis

It is well known that the inclusion of market microstructure is essential for proper modeling of financial markets based on limit order books (cf. e.g. [6, 15, 39]). There are recent empirical studies (cf. [37, 55]) which suggest that this also applies to intraday electricity markets with a continuous matching mechanism. In mathematical finance, one research objective is to introduce a microscopic ("event-by-event") description of limit order book dynamics. Imposing suitable scaling constants to this system, one is interested in studying its scaling behavior when the number of orders gets large while each of them is of negligible size.

Mathematically, the microscopic model can be interpreted as a high- or infinite dimensional, discrete-time stochastic process. The goal is to formulate appropriate scaling assumptions that allow to establish the functional convergence of its piecewise constant interpolation toward a continuous-time stochastic limit process. To familiarize ourselves with the mathematical foundations of this thesis, let us recall the most famous example of a functional convergence theorem: Donsker's theorem, a functional extension of the central limit theorem, which shows the weak convergence of linear or piecewise constant interpolation of a scaled random walk toward a standard Brownian motion. While weak convergence of random variables is a fairly known concept in probability theory, the meaning of weak convergence of stochastic processes is a more delicate concept. In fact, the traditional mode of convergence is weak convergence of the laws of processes, considered as random elements of some functional space imposed with a complete separable metric topology. For example, if a linear interpolation of the scaled random walk is considered, Donsker's theorem is stated with respect to the topological space $\left(C([0, T], \mathbb{R}),\|\cdot\|_{\infty}\right)$, the space of continuous functions $f:[0, T] \rightarrow \mathbb{R}$ endowed with the topology induced by the sup norm $\|f\|_{\infty}:=\sup _{t \in[0, T]}|f(t)|$. If, in contrast, a piecewise constant interpolation of the scaled random walk is considered, the paths are contained in the function space $D([0, T], \mathbb{R}$ ), the space of càdlàg (an acronym for "continue à droite, limite à gauche", i.e., right-continuous with left limits) functions. However, the space $D([0, T], \mathbb{R})$ endowed with the uniform topology is not separable and hence not suitable for studying the weak convergence of stochastic processes.

A more suitable topology for the space of càdlàg functions is the $\operatorname{Skorokhod}\left(J_{1-}\right)$
topology $]$ Following its definition in Billingsley [8], let $\Lambda_{T}$ denote the class of increasing, continuous mappings of $[0, T]$ onto itself. Then, elements $f_{n}, n \in \mathbb{N}$, of $D([0, T], \mathbb{R})$ converge to a limit $f$ in the Skorokhod topology if and only if there exist functions $\lambda_{n} \in$ $\Lambda_{T}, n \in \mathbb{N}$, such that $\lim _{n \rightarrow \infty} f_{n}\left(\lambda_{n}(t)\right)=f(t)$ uniformly in $t$ and $\lim _{n \rightarrow \infty} \lambda_{n}(t)=t$ uniformly in $t$. In words, while two functions $f$ and $g$ are close to each other in the uniform topology if the graph of $f(t)$ can be carried onto the graph of $g(t)$ by a uniformly small perturbation of the ordinates, with the abscissas kept fixed, the Skorokhod topology additionally allows a uniformly small deformation of the time scale imposed by an element of $\Lambda_{T}$. Since the space of càdlàg functions endowed with the Skorokhod topology is complete and separable, the Skorokhod topology is indeed more suitable than the uniform topology to study the weak convergence of stochastic processes in the space of càdlàg functions. For limit processes with sample paths in $C([0, T], \mathbb{R})$ such as the Brownian motion, one might still establish convergence in the space of càdlàg functions with respect to the uniform topology if the concept of weak convergence is slightly changed in order to tackle the non-separability of the space ${ }^{2}$. If the limit process itself has discontinuities of the first kind (i.e. jumps), establishing functional convergence in the space $\left(D([0, T], \mathbb{R}),\|\cdot\|_{\infty}\right)$ in general fails as the topology is in many cases too strong and hence, the Skorokhod topology must be used. In this thesis, we establish the weak convergence of sequences of stochastic processes with respect to the space of càdlàg functions endowed with the Skorokhod topology.
The standard strategy to derive weak convergence of stochastic processes is to prove the following conditions: verify tightness of the sequence of stochastic processes relative to the considered topological space, establish the convergence of the finite-dimensional distributions, and identify the finite-dimensional distributions of the limit process.
The functional convergence of stochastic processes toward more general limit processes including possibly infinite dimensional semimartingales has been studied by several authors. To name just a few of them, important works in this area include Billingsley [8], Jacod and Shiryaev [50], Kurtz and Protter [57,58], and Whitt [85, 86]. From a purely mathematical point of view, this theory is the groundwork for this thesis.

The existing literature on limit order books is quite extensive (cf. e.g. Abergel et al. [1] for an overview). In economics and econometrics, limit order books have already been studied for several decades. Some early works include Stigler [75] and Garman [33. More recent empirical studies of limit order book dynamics can be found in Cont [18, Hautsch and Huang [39], and Huang et al. [45].
One approach to model the working of a limit order book is to specify the behavior and preferences of various types of agents (cf. e.g. Parlour [70], Foucault et al. [30], and Roşu (73). These models give insights into the price formation mechanism in limit order markets, but depend on unknown parameters of traders' preferences and are therefore hard to calibrate for application purposes.

[^0]Another approach is to describe the working of a limit order book by so-called zerointelligence models which are based on the notion of flows: orders are not submitted by an agent following a specific strategic behavior, but are viewed as an arriving flow whose properties can be determined through empirical observations. Some empirical studies of the order flow of limit order books can be found in Bouchaud et al. [11], Farmer et al. [29], and Mike and Farmer [65]. More recently, the development of zero-intelligence models in a mathematical rigorous manner has attracted quite a lot of researchers. One popular approach is based on a microscopic description of the order flow as done in e.g., Luckock [62], Cont et al. [22], Cont and de Larrard [19, 20], Muni Toke [79, 80], and Kelly and Yudovina [54]. Under simplifying assumptions, order events are modeled by basic Poisson processes (cf. e.g. [20, 22, 79, 80]) which allow for nice analytical results (cf. e.g. [20]). A generalization that allows the model dynamics to depend on the current state of the limit order book is studied in Muni Toke and Yoshida [82, 83 by including state dependencies to the intensities of the point processes. More generally, Hawkes processes are used to model the order events in limit order markets (cf. e.g. Muni Toke and Pomponio [81], Lallouache and Challet [59], and Lu and Abergel [61]). All these models preserve the discrete nature of the dynamics at high-frequencies but can become computationally challenging as one tries to incorporate realistic dynamics.

To overcome the drawbacks of these models, some researchers deal with continuum approximations of the order book, describing it through its time-dependent density satisfying either certain partial differential equations (cf. Lasry and Lions [60], Chayes et al. [16, Caffarelli $\sqrt{13}$, Burger et al. $\sqrt{12}$ ) or certain stochastic partial differential equations (cf. Keller-Ressel and Müller [53], Markowich [63], and Cont and Müller [21]).

Starting from a microscopic description of order book dynamics, one can introduce suitable scaling constants and study its scaling behavior when the number of orders gets large while each of them is of negligible size. The scaling limit can then either be described through a system of (partial) differential equations (in the "fluid" limit, where random fluctuations vanish), through a system of stochastic (partial) differential equations (in the "diffusion" limit, where random fluctuations dominate), or through a mixture thereof. Deriving a deterministic high-frequency limit for limit order book models guarantees that the scaling limit approximation stays tractable in view of practical applications. Such an approach is pursued by Horst and Paulsen [43], Horst and Kreher [40], and Gao and Deng [32]. The absence of arbitrage considerations, however, encourages price approximations by diffusion processes. As discussed in [19], depending on the market and/or stock of interest either a fluid or diffusive volume approximation seems to be appropriate. Horst and Kreher 42 studied the approximation of microscopic order book dynamics by both diffusive price and volume processes in the scaling limit. Their consideration of a diffusive infinite dimensional volume process is not suitable for practical applications, as e.g., the uniqueness of a solution to the established infinite dimensional stochastic differential equation is in general not guaranteed. The authors in [19] guaranteed that their diffusive volume approximation stays tractable, considering only the standing volumes at the top of the book and hence reducing the state space of the limit order book to a finite-dimensional space. Imposing suitable scaling assumptions, they prove that the bid and ask queue lengths are given in the
scaling limit by a planar Brownian motion in the first quadrant with reflections to the interior at the boundaries. In contrast to [40,42], they also determine the evolution of prices implicitly through the volume dynamics. As the tick size is constant, prices have to be approximated in the scaling limit by a pure jump process with jump times equal to those of the volume dynamics.

In terms of modeling, the papers $\sqrt[19, ~ 40, ~ 42 ~ s e r v e ~ a s ~ t h e ~ s t a r t i n g ~ p o i n t ~ o f ~ t h e ~ m o d e l s ~]{\text { s }}$ studied in this thesis.

For application purposes, the calibration of mathematical models is of great interest. It is well known (cf. e.g. 66]) that change points are present in high-frequency financial data. If they are endogenously caused, the estimation of such change points must take into account the dependence on the underlying data. Therefore, in the final part of this thesis we move away from modeling limit order books and develop a theory for detecting structural changes in the model parameters of time series data.

The existing literature on change point detection is huge and has a long history. One of the first works has been published in 1955 by Page [69]. The available literature provides statistical tests for deciding whether or not the underlying time series data contain change points. To mention only a few prominent works, see, for example, Andrews [2], Csörgő and Horváth [23], and Bai and Perron [4]. While many authors build their tests assuming that the change point occurs only in a single model parameter (typically in its mean, cf. e.g. Jiang et al. [51], or in its variance, cf. e.g. Aue et al. [3], Spokoiny [74]), Horváth [44], Gombay and Horváth [34-36], and Csörgő and Horváth [23] provide likelihood ratio-based tests that check for simultaneous changes in the parameters of quite general parametric distributions. Mathematically, the methods used to study change points usually rely on order statistics and exploit limit theorems from extreme value theory. Despite the pure testing problem, the existing literature also allows for estimation of the location and size of the change and constructs confidence intervals. Instead of considering time series data, we want to mention that some authors are concerned with the detection of change points in the parameters of continuous-time diffusions or more general Itô-semimartingales (cf. e.g. Iacus and Yoshida 47, Jiang et al. [51], Bibinger et al. [7]). In particular, [7] provide tools for the non-parametric change point detection in the volatility process of an Itô-semimartingale.

The book by Csörgő and Horváth [23 not only provides a comprehensive introduction to the theory of change point detection, it also serves as the starting point for the theory developed in the final part of this thesis.

## Chapter 1: Jump diffusion approximation for the price dynamics of a fully state dependent limit order book model

The absence of arbitrage considerations favors price approximations through diffusion processes. At the same time, large price jumps occur with positive probability even in liquid markets (cf. [9]). From an economic perspective (cf. e.g. [28,52]), such price jumps (especially large price drops) are understood as market reactions to highly unexpected, exogenous news. The empirical study in [9] suggests that most real price
jumps cannot be attributed to exogenous news, which is why they are understood as endogenous shocks. Regardless of the actual cause of large price jumps, existing price approximations typically do not account for the possibility of such extreme price movements, which can lead to poor approximations if the underlying dynamics include price jumps. Moreover, the inclusion of large price jumps is essential if one wants to approximate the price dynamics of intraday electricity markets with continuous trading. Since these markets are typically quite illiquid, large price movements over short time periods occur particularly frequently. For this reason, we establish an approximation of the prices in limit order book dynamics by jump diffusion processes in the scaling limit. In the subsequent overview, we concentrate on one-sided LOB-dynamics, i.e., the dynamics of the best bid price, the buy side volume density function relative to the best bid price, and the order arrival times. In Chapter 1, full LOB-dynamics with two price processes, two volume density functions, and a time process are studied.

For each $n \in \mathbb{N}$, we construct the microscopic (one-sided) LOB-dynamics $S^{(n)}$ as follows: let $\Delta x^{(n)}>0$ be the tick size, $\Delta v^{(n)}>0$ the individual impact of an order on the state of the book, and $T>0$ a finite time horizon. We assume that both, $\Delta x^{(n)}$ and $\Delta v^{(n)}$ tend to zero as $n \rightarrow \infty$. Moreover, we introduce the time scaling parameter $\Delta t^{(n)}>0$ with $\Delta t^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, which ensures that the time between consecutive order arrivals tends to zero. For $n \in \mathbb{N}$, we introduce a $\mathbb{Z}$-valued process $\left(J_{k}^{(n)}, k \geqslant 1\right)$ that describes the number of ticks the bid price changes due to incoming order events. Moreover, the $L^{2}(\mathbb{R})$-valued process $\left(M_{k}^{(n)}, k \geqslant 1\right)$ represents the size of an order placement/cancellation at the buy side volume density function due to an incoming order event. Finally, the $(0, \infty)$-valued process $\left(\varphi_{k}^{(n)}, k \geqslant 1\right)$ specifies the duration between consecutive order events. Then, the microscopic one-sided LOB-dynamics is given by the piecewise constant interpolation

$$
S^{(n)}(t):=S_{k}^{(n)} \quad \text { for } t \in\left[\tau_{k}^{(n)}, \tau_{k+1}^{(n)}\right) \cap[0, T]
$$

of the $E:=\mathbb{R} \times L^{2}(\mathbb{R}) \times[0, T]$-valued random variables $S_{k}^{(n)}:=\left(B_{k}^{(n)}, v_{k}^{(n)}, \tau_{k}^{(n)}\right)$, $k \in \mathbb{N}_{0}$, with deterministic initial state $S_{0}^{(n)}:=\left(B_{0}^{(n)}, v_{0}^{(n)}, 0\right)$, and

$$
\begin{aligned}
B_{k}^{(n)} & =B_{0}^{(n)}+\sum_{j=1}^{k} J_{j}^{(n)} \Delta x^{(n)} \\
v_{k}^{(n)}(x) & =v_{0}^{(n)}\left(x-B_{k}^{(n)}+B_{0}^{(n)}\right)+\sum_{j=1}^{k} M_{j}^{(n)}\left(x-B_{k}^{(n)}+B_{j-1}^{(n)}\right) \Delta v^{(n)}, \\
\tau_{k}^{(n)} & =\sum_{j=1}^{k} \varphi_{j}^{(n)} \Delta t^{(n)} .
\end{aligned}
$$

Here, $B_{k}^{(n)}$ describes the best bid price, $v_{k}^{(n)}$ the buy side volume density function relative to the best bid price, and $\tau_{k}^{(n)}$ the time after $k$ order events.

While conditions to derive limit theorems toward fluid or diffusion processes are generally well-understood (cf. e.g. [40,42]), to the best of our knowledge this is not true for a jump diffusion limit. The main novelty is therefore to come up with a new set of assumptions that allows for a jump diffusion approximation of the price dynamics in the scaling limit. In Chapter 11 we allow the jump sizes to vary across different states of the limit order book, but we require the jump intensities to be approximately the same. This indeed allows for a jump diffusion approximation of the price dynamics in the scaling limit, as the driver becomes independent of the order book dynamics (cf. Assumption 1.6. In order to ensure the tractability of our high-frequency approximation, we approximate the Hilbert-space valued volume dynamics and the $[0, T]$-valued time dynamics in the scaling limit by processes of fluid-type. The combination of the diffusive price approximation from [42 with the fluid-type approximation for the volumes from [40] requires a completely new idea for proving weak convergence. Since the state process of the limit order book takes values in an infinite dimensional Hilbert space and its limit process contains discontinuities, we have to apply many not so well-known extensions of otherwise well-known results about the convergence of probability measures in the Skorokhod space (cf. Kurtz and Protter [58] and Whitt [85, 86]).

Now, under appropriate high-frequency assumptions and after introducing the coefficient functions $p: E \rightarrow \mathbb{R}, r: E \rightarrow \mathbb{R}_{+}, \theta: E \times[-M, M] \rightarrow \mathbb{R}$, for $M>0$, $f: E \rightarrow L^{2}(\mathbb{R}), \varphi: E \rightarrow(0,1]$, the finite measure $Q$ on $\mathcal{B}(\mathbb{R})$, and the deterministic initial state of the limit dynamics $S_{0}=\left(B_{0}, v_{0}, 0\right)$, we establish the following functional convergence theorem for the microscopic order book dynamics.

Theorem (cf. Theorem 1.2.6). Under appropriate high-frequency assumptions, the microscopic LOB-dynamics $S^{(n)}$ converges weakly in the Skorokhod topology on $D([0, T], E)$ to $S=\eta \circ \zeta$, where

$$
\zeta(t):=\inf \left\{s>0: \tau^{\eta}(s)>t\right\}, \quad t \in[0, T],
$$

is a random time change and $\eta=\left(B^{\eta}, v^{\eta}, \tau^{\eta}\right)$ is the unique strong solution of the coupled diffusion-fluid system

$$
\begin{aligned}
B^{\eta}(t) & =B_{0}+\int_{0}^{t} p(\eta(u)) d u+\int_{0}^{t} r(\eta(u)) d Z(u)+\int_{0}^{T} \int_{-M}^{M} \theta(\eta(u-), y) \mu^{Q}(d u, d y), \\
v^{\eta}(t, x) & =v_{0}\left(x-B^{\eta}(t)+B_{0}\right)+\int_{0}^{t} f[\eta(u)]\left(x-B^{\eta}(t)+B^{\eta}(u)\right) d u, \\
\tau^{\eta}(t) & =\int_{0}^{t} \varphi(\eta(u)) d u,
\end{aligned}
$$

for all $t \in[0, T], x \in \mathbb{R}$, where $Z$ is a standard Brownian motion and $\mu^{Q}$ is a homogeneous Poisson random measure with intensity measure $\lambda \times Q$, independent of $Z$. Here, $\lambda$ denotes the Lebesgue measure on $[0, T]$.

We want to mention, that this limit order book approximation allows for a quite general dependence structure, where all coefficient functions are allowed to depend on current prices, volumes, and time. The usefulness of this dependence structure is illustrated in a detailed simulation study where the probabilities of different price movements as well as the limit order sizes and cancellations depend on the spread and standing volumes, especially on order imbalances (cf. Section 1.3).

The following figure depicts the evolution of the bid and ask prices and of the buy side volume density function of simulated full order book dynamics.



Figure B: The evolution of the bid and ask prices (left) and of the buy side volume density function in absolute coordinates (right).

## Chapter 2: A cross-border market model with limited transmission capacities

At the latest with the introduction of the integrated European intraday electricity market SIDC, the need has arisen to couple multiple markets with each other, i.e., to allow market participants of different countries to trade with each other on a cross-border basis. In this market, the transmission capacities that enable transactions between market participants of different countries are limited. Therefore, the ability to execute cross-border trades may be prohibited if available transmission capacities are occupied. Motivated by the SIDC market dynamics, we introduce cross-border market dynamics between two countries based on limit order books.

For each $n \in \mathbb{N}$, we construct the microscopic cross-border market dynamics $S^{(n)}$ as follows: let $F$ and $G$ be two countries that can trade with each other on a cross-border basis. Moreover, let $\delta>0$ be the tick size, $T>0$ a finite time horizon, and $\Delta v^{(n)}>0$ the average order size. Throughout, we assume the time intervals between two consecutive order arrivals to be of equal length $\Delta t^{(n)}>0$ and that both scaling parameters $\Delta t^{(n)}$ and $\Delta v^{(n)}$ tend to zero as $n \rightarrow \infty$. Moreover, let $\kappa_{-}, \kappa_{+}>0$ be the total available transmission capacities in direction $F$ to $G$ and vice versa. Then, the microscopic cross-border market dynamics is given by the piecewise constant interpolation

$$
S^{(n)}(t):=S_{k}^{(n)} \quad \text { for } t \in\left[k \Delta t^{(n)},(k+1) \Delta t^{(n)}\right) \cap[0, T],
$$

of $E:=\mathbb{R}^{2} \times \mathbb{R}_{+}^{4} \times \mathbb{R}$-valued random variables $S_{k}^{(n)}:=\left(B_{k}^{(n)}, Q_{k}^{(n)}, C_{k}^{(n)}\right), k \in \mathbb{N}_{0}$, with deterministic initial state $S_{0}^{(n)} \in E$. The process $B^{(n)}=\left(B^{F,(n)}, B^{G,(n)}\right)$ describes the dynamics of the best bid prices in countries $F$ and $G, Q^{(n)}$ describes the number of unexecuted limit orders at the best bid/ask queues in countries $F$ and $G$, and $C^{(n)}$ describes the net number of executed cross-border trades between $F$ and $G$ over time. The best ask price process $A^{I,(n)}$ of country $I=F, G$ is for simplicity modeled by $A^{I,(n)}:=B^{I,(n)}+\delta$. Hence, for each $n \in \mathbb{N}$, the microscopic cross-border market dynamics $S^{(n)}:=\left(S^{(n)}(t)\right)_{t \in[0, T]}$ are given by two bid and ask price processes, two bid and ask queue length processes, and a capacity process.

In order to describe how incoming market and limit orders at the best bid and ask prices change the state of $S^{(n)}$, it is not only important to differentiate if an incoming order effects the state of the best bid or ask queue, we also need to keep track of the origin of each incoming order. Therefore, let $(i, I) \in\{b, a\} \times\{F, G\}$ be one out of four order events that change the state of our market dynamics and let $\left(V_{k}^{i, I,(n)}, k \geqslant 1\right)$ denote the incoming order sizes of type $(i, I)$, for each $n \in \mathbb{N}$.

Our starting point is to study the evolution of the net order flow process $X^{(n)}:=$ $\left(X^{b, F,(n)}, X^{a, F,(n)}, X^{b, G,(n)}, X^{a, G,(n)}\right)$, where

$$
X^{i, I,(n)}(t):=\sum_{k=1}^{\left\lfloor T / \Delta t^{(n)}\right\rfloor} X_{k}^{i, I,(n)} \mathbb{1}_{\left[k \Delta t^{(n)},(k+1) \Delta t^{(n)}\right)}(t) \quad \text { and } \quad X_{k}^{i, I,(n)}:=\sum_{j=1}^{k} V_{j}^{i, I,(n)}
$$

Imposing appropriate high-frequency assumptions, we approximate the net order flow process by a four-dimensional linear Brownian motion $X$ in the scaling limit.

Next, we construct the so-called active (resp. inactive) order book dynamics $\widetilde{S}^{(n)}$ (resp. $\left.\widetilde{\widetilde{S}}^{(n)}\right)$ based on the order flow $X^{(n)}$ and on sequences of random variables $R^{+,(n)}:=$ $\left(R_{k}^{+,(n)}\right)_{k \geqslant 1}$ and $R^{-,(n)}:=\left(R_{k}^{-,(n)}\right)_{k \geqslant 1}$ describing the sizes of the best bid and ask queues after price changes. The active dynamics $\widetilde{S}^{(n)}$ (resp. inactive dynamics $\widetilde{\widetilde{S}}^{(n)}$ ) describes the evolution of the cross-border market if cross-border trades are allowed (resp. prohibited). The construction of the active/inactive dynamics through appropriate functions $\widetilde{\Psi}$ and $\widetilde{\Psi}$ from the underlying order flow $X^{(n)}$ and the sequences $R^{+,(n)}$ and $R^{-,(n)}$ and the analysis of the continuity sets of these functions are the major challenges in Chapter 2. In more detail, we characterize the bid/ask components of the active volume dynamics between successive price changes as a series of solutions to the one-dimensional Skorokhod problem following successive reflections from the axes. This allows us to still apply the continuous mapping approach even though the reflection matrix in the definition of the active volume dynamics does not fulfill the usual regularity conditions considered in the literature of semimartingale reflecting Brownian motions (cf. e.g. Varadhan and Williams [84], Taylor and Williams [77], and Ernst et al. 27]). In this way, we are able to identify the limit process of the volume dynamics between consecutive price changes as a solution of a reflected stochastic differential equation with absorption. Then, we can deduce a functional convergence result for the active dynamics to a continuous-time limit process $\widetilde{S}$ by an application of
the continuous mapping theorem. Similarly, we derive a limit theorem for the inactive dynamics to a continuous-time limit process $\widetilde{\widetilde{S}}$.

Finally, we construct the microscopic cross-border market dynamics from the active/inactive dynamics by introducing suitable sequences of stopping times $\left(\rho_{l}^{(n)}\right)_{l \geqslant 0}$ and $\left(\sigma_{l}^{(n)}\right)_{l \geqslant 1}$ that indicate the start of an active respectively inactive regime. Denoting by $\left(\rho_{l}\right)_{l \geqslant 0}$ and $\left(\sigma_{l}\right)_{l \geqslant 1}$ their limits, we are ready to state the main result of Chapter 2

Theorem (cf. Theorem 2.5.1). Under appropriate high-frequency assumptions, the piecewise constant interpolation of the microscopic dynamics $S^{(n)}$ converges weakly in the Skorokhod topology on $D([0, T], E)$ to a continuous-time regime switching process $S$, whose dynamics are described as follows: let $l \geqslant 1$.

- In each active regime $\left[\rho_{l-1}, \sigma_{l}\right), S$ behaves as $\widetilde{S}$ starting in $S\left(\rho_{l-1}\right)$. In words, the volume dynamics follows a four-dimensional linear Brownian motion in the positive orthant with oblique reflection at the axes. Each time two queues simultaneously hit zero, the process is reinitialized at a new value in the interior of $\mathbb{R}_{+}^{4}$. The bid price dynamics follows a two-dimensional pure jump process with jump times equal to those of the volume dynamics. In particular, its components (i.e. the bid prices of both countries) agree on $\left[\rho_{l-1}, \sigma_{l}\right)$. The dynamics of the capacity process follows a bounded continuous process of finite variation.
- In each inactive regime $\left[\sigma_{l}, \rho_{l}\right), S$ behaves as $\tilde{S}$ starting in $S\left(\sigma_{l}\right)$. In words, the volume dynamics follows a four-dimensional linear Brownian motion in the interior of $\mathbb{R}_{+}^{4}$. Each time it hits one of the axes, the two components corresponding to the origin of the depleted component are reinitialized at a new value in $(0, \infty)^{2}$ while the others stay unchanged. The bid price dynamics follows a two-dimensional pure jump process whose components jump on hits of the corresponding components of the volume process of the axes. In particular, its components (i.e. the bid prices of both countries) do almost surely not jump simultaneously. The dynamics of the capacity process stays constant and equal to either $-\kappa_{-}$or $\kappa_{+}$.

Our model is a further development of the reduced-form representation of a national limit order book in Cont and de Larrard [19]. To this end, we analyze the dynamics of two national limit order books together with a two-sided capacity process over time. Since the transmission capacities that allow for cross-border trades are limited, our model alternates between regimes in which cross-border trades are allowed (active regimes) and regimes in which market orders can only be matched with standing volumes of the same origin (inactive regimes). While the volumes and prices are approximated as in 19 in the scaling limit by a diffusion limit and a pure jump limit, respectively, our newly introduced capacity process converges to a bounded continuous process of finite variation. Instead of only inferring the individual convergence of prices and volumes as in [19], we prove weak convergence of the joint market dynamics. In doing so, we even establish weak convergence of the prices in the Skorokhod $J_{1}$-topology instead of the slightly weaker Skorokhod $M_{1}$-topology used in 19 .

The evolution of the cross-border market dynamics is depicted in Figure C.


Figure C: The cross-border market model based on two LOBs: the queue size processes at the best bid (top left) and best ask price (top right), the bid price processes (bottom left), and the capacity process (bottom right). The white and gray areas illustrate the different regimes in which cross-border trades are possible (white) or prohibited (gray).

Based on this approximation of the dynamics of a cross-border market between two countries, we can study the effect of coupling two markets on the evolution of the limit order books. First, we discuss different market situations of our model in a detailed simulation study. Second, we compare the active and inactive dynamics by simulating the mean number of price changes and the mean bid price ranges in the active and inactive dynamics. Finally, we discuss a theoretical result on the conditional distribution of the duration between price changes (cf. Proposition 2 and Remark 2 in [19]) for the shared order book. We conclude that coupling two markets always increases the standing volumes as the unexecuted limit orders from the national order books are summarized in a shared order book. This usually leads to more liquidity and thus fewer price changes and smaller bid price ranges. However, the change in the trading behavior (indicated by the drift and volatility parameters of the net order flow process) might amplify or cancel out this effect.

## Chapter 3: Parametric change point detection with random occurrence of the change point

The time of a regime switch from an active to an inactive regime in our cross-border market model in Chapter 2 is modeled by a stopping time depending on the observable order sizes and on the total available transmission capacities. The latter are not publicly available and therefore often unknown. In order to estimate the time of a regime switch, the existing literature on change point detection has to be extended to also cover
randomly occurring change points possibly depending on the data.
We consider independent observations $X_{1}, \cdots, X_{n}$ with values in $\mathbb{R}^{m}$ that are (not necessarily identical) normally distributed. Let $\theta_{j}:=\theta\left(\mu_{j}, \Sigma_{j}\right) \in \mathbb{R}^{m} \times \mathbb{R}^{m \times m}$ denote a transformation of the mean $\mu_{j} \in \mathbb{R}^{m}$ and covariance matrix $\Sigma_{j} \in \mathbb{R}^{m \times m}$ of the distribution of $X_{j}$. Moreover, we assume that the data contain at most one change point. Then, we want to test the null hypothesis "no change point"

$$
H_{0}: \quad \theta_{1}=\cdots=\theta_{n}
$$

against the alternative "there exists one change point"

$$
\begin{array}{ll}
H_{1}: & \text { There exists a } k_{n}^{*} \in\{1, \cdots, n-1\} \text { such that } \\
& \theta_{1}=\cdots=\theta_{k_{n}^{*}} \neq \theta_{k_{n}^{*}+1}=\cdots=\theta_{n} .
\end{array}
$$

In contrast to the existing literature (cf. e.g. Csörgő and Horváth [23]), we allow that the true location of the change point $k_{n}^{*}$ is random. In [23] test statistics ( $S_{n}(k), 1 \leqslant k \leqslant n$ ) based on the $\log$-likelihood ratio have been introduced, i.e., $S_{n}(k)=-\log \Lambda_{k}$, for $k=1, \cdots, n$, where

$$
\Lambda_{k}:=\frac{\sup _{\theta_{0} \in \Theta} \prod_{1 \leqslant i \leqslant n} f\left(X_{i} ; \theta_{0}\right)}{\sup _{\theta_{0}^{(1)}, \theta_{0}^{(2)} \in \Theta} \Pi_{1 \leqslant i \leqslant k} f\left(X_{i} ; \theta_{0}^{(1)}\right), \prod_{k<i \leqslant n} f\left(X_{i} ; \theta_{0}^{(2)}\right)} .
$$

Here, $f(\cdot, \theta)$ denotes the normal density function with respect to the parameter $\theta \in$ $\Theta \subset \mathbb{R}^{m} \times \mathbb{R}^{m \times m}$. They suggest to use the maximally selected log-likelihood ratio to reject $H_{0}$ if

$$
\mathcal{S}_{n}:=\max _{1 \leqslant k \leqslant n} S_{n}(k)
$$

is large. Since under the null hypothesis no change point occurs in the data, we can apply the stated limit results in $\left[23\right.$ for the test statistic $\mathcal{S}_{n}$ under the null hypothesis.

If the null hypothesis has been rejected, the fact that $k_{n}^{*}$ is random becomes a challenge. Let $\theta^{(1)}$ and $\theta^{(2)}$ denote the true values of the parameters before and after the change. Our main idea to tackle the randomness in $k_{n}^{*}$ is to study the limiting behavior of ( $S_{n}(k), 1 \leqslant k \leqslant n-1$ ) uniformly for all possible choices of the change point. Therefore, we include a second time parameter in the test statistics such that $\left(S_{n}\left(k, k^{*}\right), 1 \leqslant k, k^{*} \leqslant n-1\right)$ depends on two time parameters, namely the true and estimated location of the change point. An application of Taylor's formula of the first order allows us to rewrite

$$
S_{n}\left(k, k^{*}\right)-\mu_{n}\left(k, k^{*}\right)=Z_{n}\left(k, k^{*}\right)+R_{n}\left(k, k^{*}\right) \quad \text { for } k, k^{*} \in\{1, \cdots, n-1\},
$$

where $\mu_{n}\left(k, k^{*}\right)$ is asymptotically the mean of $S_{n}\left(k, k^{*}\right), R_{n}\left(k, k^{*}\right)$ is the remainder of Lagrange form, and $Z_{n}\left(k, k^{*}\right)$ denotes the transformed test statistic. Moreover, let $\delta$ denotes some transformation of the size of the change $\left\|\theta^{(1)}-\theta^{(2)}\right\|$. In Chapter 3 , we focus on the setting in which the size of the change in the parameter vanishes (in particular $\delta \rightarrow 0$ ), but $n \delta^{2} / \log \log (n) \rightarrow \infty$ as $n \rightarrow \infty$. This setting is crucial to describe
the minimum detectable size of a change in the parameter relative to the sample size. Moreover, we assume that the change point occurs in "the middle of the data", i.e., we assume that the change point fraction $\lambda_{n}^{*}:=k_{n}^{*} / n$ satisfies $\left|\lambda_{n}^{*}-\lambda^{*}\right|=o_{\mathbb{P}}\left(\left(n \delta^{2}\right)^{-1}\right)$ and $\lambda^{*}$ is a random variable which takes its values almost surely in a closed subset of $(0,1)$. Then, under appropriate high-frequency assumptions, our main achievement is to obtain a limit result for the piecewise constant interpolation $Z_{n}:=\left(Z_{n}(t, \lambda)\right)_{t, \lambda \in[0,1]}$ of the transformed test statistic $\left(Z_{n}\left(k, k^{*}\right): 1 \leqslant k, k^{*} \leqslant n-1\right)$.

Theorem (cf. Theorem 3.4.4). Under the alternative and appropriate high-frequency assumptions, we have

$$
\frac{1}{\sqrt{n \delta^{2}}} Z_{n} \Rightarrow Z^{*}
$$

in the Skorokhod topology on $D\left([0,1]^{2}, \mathbb{R}\right)$, where $Z^{*}$ is a Gaussian process with mean zero and covariance function

$$
c\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)=\sigma_{A}^{2} \begin{cases}(1-\lambda)\left(1-\lambda^{\prime}\right) \min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\}, & \text { if } t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime} \\ (1-\lambda) \lambda^{\prime} \min \left\{\frac{t\left(1-t^{\prime}\right)}{(1-t) t^{\prime}}, 1\right\}, & \text { if } t \leqslant \lambda, t^{\prime}>\lambda^{\prime} \\ \lambda\left(1-\lambda^{\prime}\right) \min \left\{\frac{1-t t^{\prime}}{t\left(1-t^{\prime}\right)}, 1\right\}, & \text { if } t>\lambda, t^{\prime} \leqslant \lambda^{\prime} \\ \lambda \lambda^{\prime} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t>\lambda, t^{\prime}>\lambda^{\prime}\end{cases}
$$

for $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in[0,1]^{2} \times[0,1]^{2} \backslash\{((1,1),(1,1))\}$ and $c((1,1),(1,1))=0$, and $\sigma_{A}^{2}$ depends on the limit $\theta_{A}$ of the true parameters $\theta^{(1)}, \theta^{(2)}$ before and after the change.

The hard part of the proof is to establish the convergence of the finite-dimensional distributions. This, however, can be nicely simplified by an application of the famous fourth moment theorem by Nualart and Peccati [68].

After establishing the above limit theorem, it is indeed straight-forward to derive the consistency, the convergence rate, and the limit distribution of the estimator for the fractional change point

$$
\hat{\lambda}_{n}:=\frac{1}{n} \underset{1 \leqslant k \leqslant n-1}{\arg \max } S_{n}(k)
$$

under the alternative, where the fractional change point is given by $\lambda_{n}^{*}:=k_{n}^{*} / n$. In particular, we are able to derive the convergence rate $\left(n \delta^{2}\right)^{-1}$ and the limit distribution for the deviation $\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$ stated in Csörgő and Horváth 23 for a deterministic location of a change point. The estimator $\hat{k}_{n}$ is given by $\hat{k}_{n}:=n \hat{\lambda}_{n}$. Finally, we validate our theoretical results in a simulation study, in which we also discuss important generalizations of our model framework: extensions to weakly dependent observations and non-parametric change point detection in the volatility process of an Itô-semimartingale.

## Main contribution of this thesis

The thesis on hand derives two analytically tractable descriptions of macroscopic limit order book dynamics (prices, standing volumes, and capacities) from the underlying microscopic dynamics (individual order events) that incorporate several characteristics being present in intraday electricity markets with continuous trading. In Chapter 1. we put emphasis on modeling quite general dynamics of a single contract in a national limit order book and differentiate between price changing order events of different magnitude and non-price changing order events. We allow the price and volume dynamics to depend on the current state (e.g., current spread and order imbalance) and incorporate a time-inhomogeneity. Approximating the price dynamics by a jump diffusion in the scaling limit is appealing from empirical observations and allows to study the effects of large price jumps on the evolution of the limit order book dynamics. While all quantities can be estimated from order flow data, the model does not give any insights in the price formation process. To obtain an understanding of the price evolution, prices dynamics should be implicitly determined by the volume dynamics. For this reason, in Chapter 2, we develop a simple queuing model for the limit order book dynamics of two countries in which price changes are caused due to a full depletion of the (cumulative) best bid or ask queues. This allows us to study the effect of coupling two limit order markets on price evolution and traded volume.

The two models follow fundamentally different approaches and the merging of them into a joint analytically tractable model is anything but trivial. Nevertheless, both models are interesting from a practitioner's point of view. For application purposes, neither model is generally preferable to the other, and the selection of the appropriate model depends on the underlying problem and available data.

Additionally, this thesis generalizes the existing literature on change point detection in parametric models to randomly occurring change points, where the location of the change point might depend on the underlying data. This is of great importance for the calibration of both models derived in this thesis as high-frequency data are known to contain change points. In particular, this extension allows us to estimate the time of a regime switch in the cross-border market model in Chapter 2 if only order flow data are available.

# 1 Jump diffusion approximation for the price dynamics of a fully state dependent limit order book model 

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This chapter includes an extensive version of the author's accepted manuscript (Postprint).


#### Abstract

We study a microscopic limit order book model, in which the order dynamics depends on the current best bid and ask price and the current volume density functions, simultaneously, and derive its macroscopic high-frequency dynamics. As opposed to the existing literature on scaling limits for limit order book models, we include price changes which do not scale with the tick size in our model to account for large price movement, being for example triggered by highly unforeseen events. We show that, when the size of an individual limit order and the tick size tend to zero while the order arrival rate tends to infinity, the microscopic limit order book model dynamics converges to two one-dimensional jump diffusion processes describing the prices coupled with two infinite dimensional fluid processes describing the standing volumes at the buy and sell side.


### 1.1 Introduction

Electronic limit order books are widely used tools in economics to carry out transactions in financial markets. They are records, maintained by an exchange or specialist, of unexecuted orders awaiting execution. Motivated by the tremendous increase of trading activities, stochastic analysis provides powerful tools for understanding the dynamics of a limit order book via the description of suitable scaling ("high-frequency") limits. Scaling limits allow for a tractable description of the macroscopic dynamics (prices and standing volumes) derived from the underlying microscopic dynamics (individual order arrivals). While modeling liquid stock markets, prices are typically approximated in the scaling limit by diffusion processes (cf. e.g. [5, 20, 42, 46]). At the same time, there is a broad consensus that (large) jumps may occur as responses to highly unexpected
news (cf. [9, 28, 52]). In this chapter, we specify reasonable microscopic dynamics of a limit order book which takes the possibility of such surprising news flows into account. Introducing suitable scaling constants, we derive the functional convergence of the microscopic model to a coupled system of two one-dimensional jump diffusions and two infinite dimensional stochastic fluid processes, describing the price and volume dynamics, respectively.

At any given point in time, a limit order book depicts the number of unexecuted buy and sell orders at different price levels (cf. Figure 1.1). The highest price a potential buyer is willing to pay is called the best bid price, whereas the best ask price is the smallest price of all placed sell orders. Incoming limit orders can be placed at many different price levels, while incoming market orders are matched against standing limit orders according to a set of priority rules.


Figure 1.1: Illustration of the state of a limit order book model.

One popular approach to study limit order books is based on event-by-event descriptions of the order flow as done in e.g., $20,22,32,40,43,54,62$. The derived stochastic systems typically yield realistic models as they preserve the discrete nature of the dynamics at high frequencies, but turn out to be computationally challenging. They also give only little insight into the underlying structure of the order flow. To overcome the drawbacks of these models, some researchers deal with continuum approximations of the order book, describing it through its time-dependent density satisfying either certain partial differential equations as in [12, 13, 16,60 or certain stochastic partial differential equations as in 21,63 .

Combining these two approaches, one can introduce suitable scaling constants in the microscopic order book dynamics and study its scaling behavior when the number of orders gets large while each of them is of negligible size. The scaling limit can then either be described through a system of (partial) differential equations (in the "fluid" limit, where random fluctuations vanish), through a system of stochastic (partial) differential equations (in the "diffusion" limit, where random fluctuations dominate), or through a mixture thereof. Deriving a deterministic high-frequency limit for limit order book models guarantees that the scaling limit approximation stays tractable in view of practical applications. Such an approach is pursued by Horst and Paulsen [43], Horst and Kreher [40], and Gao and Deng [32]. In contrast, a diffusion limit for the order book
dynamics can be found in Cont and de Larrard [20] or Horst and Kreher [42]. While [20] only analyzes the volumes standing at the top of the book, [42 takes the whole standing volumes into account leading to both, a diffusive price and a diffusive volume approximation. Depending on the market and/or stock of interest either a fluid or a diffusive volume approximation seems to be appropriate. However, a diffusive infinite dimensional volume process makes practical applications more difficult, as e.g., the uniqueness of a solution to the infinite dimensional stochastic differential equation need not guaranteed (cf. [42]). On the other hand, absence of arbitrage considerations encourage price approximations by diffusion processes.
Our model is a further development of the model considered in 40] and [42, where the order book dynamics are influenced by both, current bid and ask prices as well as standing volumes of the bid and sell sides. This is a reasonable starting point as there is considerable empirical evidence (cf. e.g. [6, 15, 39]) that the state of the order book, especially the order imbalance at the top of the book, has a noticeable impact on order dynamics. In [40 and [42] the authors start from an event-by-event description of a limit order market based on the submission of market orders, limit orders, and the cancellation thereof. Their description allows them to write down the evolution of the bid and ask price and the buy and sell side volume density functions, which are both denoted in relative price coordinates. Denoting volumes in relative price coordinates is appealing from a modeling point of view as the empirical distribution of limit order placements at a given distance from the best price is almost stationary (cf. [10, 22]). An important simplifying assumption made in $[40,42]$ is that all price changes are assumed to be equal to the tick size and hence become infinitely small in the limit. This seems to be appropriate in an efficient market setting with high liquidity (cf. [29]). However, if the sizes of all price changes become negligible in the limit, there is no possibility to include jumps in the macroscopic price approximation. Even in highly liquid markets, price jumps occur with positive probability (cf. [9]). As discussed in several empirical studies (cf. e.g. 28,52 ) price jumps may be caused by exogenous news. However, most of them cannot be shown to be related to unforeseen news and are understood as endogenous shocks (cf. [9]). No matter what causes large price movements, there is a need for an approximation which takes price jumps into account. In addition, such a model also allows for a reasonable approximation of intraday electricity markets with a continuous trading mechanism (such as the SIDC market), where extreme price spikes during a trading day can be observed: Figure 1.2 depicts the EPEX SPOT intraday prices from one single day. Comparing the minimum and maximum intraday prices paid for one single delivery time (products for different delivery times are carried out through different order books), huge differences occur and prices can become negative.


Figure 1.2: Extreme price differences occur between minimum and maximum intraday prices in the German intraday electricity market.

Motivated by these facts, we extend the results of 40,42 in two ways. First, we take their event-by-event description of a limit order book model and include price changes which do not scale to zero. While doing so, we allow the jump sizes to vary across different states of the LOB, but we require the jump intensities to be approximately the same. This indeed allows for a jump diffusion approximation of the price dynamics in the scaling limit, as the driver becomes independent of the order book dynamics. Second, we combine the diffusive price approximation from [42] with the fluid approximation for the relative volume density process from [40]. Therefore, we end up with two onedimensional stochastic differential equations describing the bid and ask price dynamics coupled with two infinite dimensional fluid processes approximating the relative volume dynamics of the bid and ask side in the scaling limit. Conditionally on the price movements, the latter behave like deterministic PDEs, since random fluctuations of the queue sizes vanish. However, they are still random because their coefficients depend on the whole limit order book dynamics, including prices. To give the reader an intuition for our model, we perform a simulation study of the full order book dynamics in Section 1.3, in which we allow the probabilities of different price movements as well as the limit order sizes and cancellations to depend on the spread and standing volumes, especially on order imbalances.

Mathematically, we derive a limit theorem which goes beyond the standard theory of finite-dimensional, diffusive limit processes with continuous sample paths. First, our state process takes values in an infinite dimensional Hilbert space, which requires to apply results from Kurtz and Protter [58]. Second, while conditions to derive limit theorems toward diffusion processes (in finite dimensions) are generally well-understood, to the best of our knowledge this is not true for a jump diffusion limit. Therefore, one of our main mathematical achievements is to formulate the correct assumptions, which allow to derive such a limit process. Furthermore, to deal with the possible discontinuities of the limit process we need to apply many not so well-known extensions of otherwise well-known results about the convergence of probability measures in the Skorokhod space, which can be found in [85, 86]. Last but not least, while the correct scaling assumptions to obtain a diffusive-fluid system can readily be deduced from
[40, 42], the mixture of the two results requires a totally new idea of the proof.
To this end, we first approximate the sequence of discrete-time limit order book models $S^{(n)}$ by a sequence of discrete-time processes $\widetilde{S}^{(n)}$, in which we replace the random fluctuations in the volume dynamics by their conditional expectations. This simplifies the subsequent analysis as the new infinite dimensional system $\widetilde{S}^{(n)}$ is driven by two independent one-dimensional noise processes. However, the original volume processes $v_{b}^{(n)}$ and $v_{a}^{(n)}$ describe the volume dynamics relative to the best bid and ask price, respectively, and therefore do not only depend on the previous order book dynamics, but also on the current prices. This prevents us from directly applying convergence results for infinite dimensional semimartingales ${ }^{1}$ To bypass this problem, we construct approximate order book dynamics with respect to absolute volume functions, denoted $\widetilde{S}^{(n), a b s}$, through a random shift in the location variable. This allows us to apply results of Kurtz and Protter [58] on the weak convergence of stochastic integrals in infinite dimensions and to prove that $\widetilde{S}^{(n), a b s}$ converges weakly in the Skorokhod topology to the unique solution of a coupled diffusion-fluid system. Finally, exploiting the properties of the Skorokhod topology, we can conclude the weak convergence of our original discrete order book dynamics $S^{(n)}$ to $S$ being the unique solution of a coupled diffusion-fluid system.

### 1.1.1 Jump diffusion approximation of the prices: empirical evidence

Additionally to the references about the occurrence of jumps in equity prices cited above, we provide some empirical motivation for our model based on order book data from the European intraday electricity market SIDC. On SIDC electricity contracts with different durations (hour, half-hour, quarter-hour) and delivery times are traded through different order books. In the following, we analyze data from the German market area for the hourly product with delivery time 1 pm from March 5, 2020. ${ }^{2}$ Considering the evolution of the best bid and ask prices during the last five hours before closing, we observe the occurrence of an extreme price increase over a short time period shortly after 12 pm .

[^1]

Figure 1.3: Evolution of the best bid and ask price of the hour product with delivery time 1 pm from the German SIDC market area on March 5, 2020.

We observe similar price evolutions for other durations and delivery times. This provides some first empirical evidence for the occurrence of price jumps in intraday electricity markets and suggests that any reasonable model of intraday electricity price dynamics should take (large) price jumps into account.

### 1.1.2 Outline of Chapter 1

The remainder of Chapter 1 is structured as follows: Section 1.2 describes a microscopic, stochastic model for a two-sided limit order book. Moreover, we introduce assumptions under which we are able to establish a scaling limit for the model dynamics and state our main result. In Section 1.3 we present a simulation study of the full order book dynamics. As our assumption for the large price jumps (cf. Assumption 1.6 in Section 1.2) might be rather technical at first sight, we provide three examples of jump behaviors in Section 1.4 which are supported by our model. In Section 1.5 we state a proof sketch of our main theorem, whereas the technical details are presented in Section 1.6 .

Notation. In the following, $\lambda$ denotes Lebesgue measure and $\varepsilon_{x}$ denotes Dirac measure at $x \in \mathbb{R}$, i.e., for any $A \in \mathcal{B}(\mathbb{R})$ we have $\varepsilon_{x}(A)=1$ if $x \in A$ and $\varepsilon_{x}(A)=0$ otherwise. For a discrete-time stochastic process $X:=\left\{X_{k}: k \in \mathbb{N}_{0}\right\}$ let $\delta X_{k}:=$ $X_{k}-X_{k-1}, k \in \mathbb{N}$, denote the $k$-th increment of $X$. For any continuous-time stochastic process $Y$, let $\Delta Y(t):=Y(t)-Y(t-)$ denote the jump of $Y$ at time $t>0$. Moreover, for any $\mathbb{R}$-valued function $f: E \rightarrow \mathbb{R}$, we denote by $f^{+}(x):=\max \{f(x), 0\}$ and $f^{-}(x):=-\min \{f(x), 0\}$ the positive and negative part of the function $f$, respectively. Furthermore, for any $x, y \in \mathbb{R}$ let us denote by $x \vee y:=\max \{x, y\}$ and $x \wedge y:=\min \{x, y\}$.

### 1.2 The microscopic model

In what follows, we fix some finite time horizon $T>0$ and introduce the Hilbert space

$$
\begin{aligned}
E & :=\mathbb{R} \times L^{2}(\mathbb{R}) \times \mathbb{R} \times L^{2}(\mathbb{R}) \times[0, T] \\
\|(b, v, a, w, t)\|_{E}^{2} & :=|b|^{2}+\|v\|_{L^{2}}^{2}+|a|^{2}+\|w\|_{L^{2}}^{2}+|t|^{2}
\end{aligned}
$$

We describe the random evolution of a sequence of limit order book models through a sequence of $E$-valued stochastic processes $S^{(n)}=\left(B^{(n)}, v_{b}^{(n)}, A^{(n)}, v_{a}^{(n)}, \tau^{(n)}\right)$, where for each $n \in \mathbb{N}$ the $\mathbb{R}$-valued processes $B^{(n)}, A^{(n)}$ specify the dynamics of the best bid and ask prices, the $L^{2}(\mathbb{R})$-valued processes $v_{b}^{(n)}, v_{a}^{(n)}$ specify the dynamics of the buy and ask side volume density functions relative to the best bid and ask price, and the $[0, T]$-valued process $\tau^{(n)}$ describes the dynamics of the order arrival times. For each $n \in \mathbb{N}, S^{(n)}$ is defined on a probability space $\left(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)}\right)$, where we will write $\mathbb{E}$ and $\mathbb{P}$ instead of $\mathbb{E}^{(n)}$ and $\mathbb{P}^{(n)}$.

Remark 1.2.1. In order to simplify the subsequent analysis, we include the order arrival times $\tau^{(n)}$ in the state dynamics of our LOB model. In the same manner, one can also include additional exogenous factor processes: let $Y^{(n)}$ be an $\mathbb{R}^{d}$-valued stochastic process with almost surely càdlàg sample paths. Then, the state space of the $L O B$ model $\left(S^{(n)}, Y^{(n)}\right)$ is given by

$$
E^{\prime}:=E \times \mathbb{R}^{d}, \quad\|\cdot\|_{E^{\prime}}^{2}:=\|\cdot\|_{E}^{2}+\|\cdot\|_{\mathbb{R}^{d}}^{2} .
$$

To derive a high-frequency approximation under this more general setting, additional conditions on the convergence of $Y^{(n)}$ as $n \rightarrow \infty$ have to be satisfied. The factor process $Y^{(n)}$ can be used to model external influences on the LOB-dynamics, such as the infeed of renewables in intraday electricity markets, the performance of a stock index in equity markets, or political influences on general market conditions.

The order book changes due to arriving market and limit orders and due to cancellations, where we differentiate between so-called passive limit orders that are placed on top of standing volumes and aggressive limit orders that are placed inside the spread. In the $n$-th model, the $k$-th order event occurs at a random point in time $\tau_{k}^{(n)}$. Throughout, we assume that $\tau_{0}^{(n)}=0$ for all $n \in \mathbb{N}$. The time between two consecutive order events will tend to zero as $n \rightarrow \infty$. Furthermore, we introduce the tick size $\Delta x^{(n)}$ and the average size of a passive limit order placement $\Delta v^{(n)}$, which are both assumed to tend to zero as $n \rightarrow \infty$. We put $x_{j}^{(n)}:=j \Delta x^{(n)}$ for $j \in \mathbb{Z}$ and define the interval $I^{(n)}(x)$ as

$$
\begin{equation*}
I^{(n)}(x):=\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right) \quad \text { for } x_{j}^{(n)} \leqslant x<x_{j+1}^{(n)} . \tag{1.2.1}
\end{equation*}
$$

Further, we denote by $\Delta x^{(n)} \mathbb{Z}:=\left\{x_{j}^{(n)}: j \in \mathbb{Z}\right\}$ the $\Delta x^{(n)}$-grid. In order to model placements of limit orders inside the spread, the relative volume density functions $v_{b}^{(n)}$ and $v_{a}^{(n)}$ are defined on the whole real line. We refer to the volumes standing at negative

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distances from the best bid/ask price as the shadow book (cf. Figure 1.4). The idea of the shadow book is taken from [43] and was subsequently also used in [5, 40 42$]$. It has to be understood as a technical tool to model the conditional distribution of the size of limit order placements inside the spread. Each current volume density function of the visible book is extended in a sufficiently "smooth" way to the left to obtain a well-defined scaling limit for the volume functions. The shadow book follows the same dynamics as the volumes of the visible book and becomes part of the visible book through price changes (cf. Example 1.1 below). Needless to say, the shadow book cannot be observed in real world markets, but this does not play a role for our analysis (cf. also Remark 1.2.3 below).


Figure 1.4: Ask-side volume density function in relative coordinates; green: standing volume; grey: shadow book.

### 1.2.1 The initial state

In the $n$-th model, the initial state of the limit order book is given by a (positive) best bid price $B_{0}^{(n)} \in \Delta x^{(n)} \mathbb{Z}$, a (positive) best ask price $A_{0}^{(n)} \in \Delta x^{(n)} \mathbb{Z}$ satisfying $B_{0}^{(n)}<A_{0}^{(n)}$, and non-negative buy and ask side volume density functions $v_{b, 0}^{(n)}, v_{a, 0}^{(n)} \in L^{2}(\mathbb{R})$, which are given relative to the best bid and ask price. Here, $v_{b, 0}^{(n)}$ and $v_{a, 0}^{(n)}$ are supposed to be deterministic step functions which only jump at points in $\Delta x^{(n)} \mathbb{Z}$. To be precise,

$$
\int_{x_{j}^{(n)}}^{x_{j+1}^{(n)}} v_{a, 0}^{(n)}(x) d x
$$

represents the liquidity available at time $t=0$ for buying (sell side of the book) at a price which is $j \in \mathbb{N}_{0}$ ticks above the best ask price. Similarly,

$$
\int_{x_{l}^{(n)}}^{x_{l+1}^{(n)}} v_{b, 0}^{(n)}(x) d x
$$

gives the liquidity available at time $t=0$ for selling (buy side of the book) at a price which is $l \in \mathbb{N}_{0}$ ticks below the best bid price $\cdot^{3}$

At time $t=0$ the state of the limit order book is deterministic for all $n \in \mathbb{N}$ and is denoted by

$$
S_{0}^{(n)}:=\left(B_{0}^{(n)}, v_{b, 0}^{(n)}, A_{0}^{(n)}, v_{a, 0}^{(n)}, 0\right) \in E
$$

In order to prove a convergence result for the microscopic order book sequence to a high-frequency limit, we need to state convergence assumptions on the initial values.

Assumption 1.1 (Convergence of the initial states). There exist a constant $L>0$ and non-negative functions $v_{b, 0}, v_{a, 0} \in L^{2}(\mathbb{R})$ such that for any $x, \widetilde{x} \in \mathbb{R}$ and $I=b, a$,

$$
\begin{equation*}
\left\|v_{I, 0}(\cdot+x)-v_{I, 0}(\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant L|x-\widetilde{x}| \tag{1.2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v_{I, 0}^{(n)}-v_{I, 0}\right\|_{L^{2}} \longrightarrow 0 \tag{1.2.3}
\end{equation*}
$$

Also, there exist $B_{0}, A_{0} \in \mathbb{R}_{+}$with $B_{0} \leqslant A_{0}$ such that $B_{0}^{(n)} \rightarrow B_{0}$ and $A_{0}^{(n)} \rightarrow A_{0}$. We denote $S_{0}:=\left(B_{0}, v_{b, 0}, A_{0}, v_{a, 0}, 0\right) \in E$.

If $v_{I, 0} \in L^{2}(\mathbb{R}), I=b, a$, is Lipschitz-continuous with constant $L>0$ and has compact support in $[-M, M]$ for $M>0$, then $\sqrt{1.2 .2}$ is satisfied for the constant $2 L \sqrt{M}>0$, i.e., for $x, \widetilde{x} \in \mathbb{R}$, we have

$$
\left\|v_{I, 0}(\cdot+x)-v_{I, 0}(\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant L\left\|\mathbb{1}_{[-2 M, 2 M]}(\cdot)|x-\widetilde{x}|\right\|_{L^{2}} \leqslant 2 L \sqrt{M}|x-\widetilde{x}|
$$

Furthermore, 1.2 .2 and 1.2 .3 together imply a similar Lipschitz condition for the $v_{I, 0}^{(n)}, n \in \mathbb{N}$, up to a deterministic sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ converging to zero: for any $x, \widetilde{x} \in \mathbb{R}$, it holds
$\left\|v_{I, 0}^{(n)}(\cdot+x)-v_{I, 0}^{(n)}(\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant 2\left\|v_{I, 0}^{(n)}-v_{I, 0}\right\|_{L^{2}}+\left\|v_{I, 0}(\cdot+x)-v_{I, 0}(\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant a_{n}+L|x-\widetilde{x}|$.

### 1.2.2 Event types and arrival times

The order book changes due to incoming order events. In the $n$-th model, the consecutive times of incoming order events are described by

$$
\begin{equation*}
\tau_{k}^{(n)}=\tau_{k-1}^{(n)}+\varphi_{k}^{(n)} \Delta t^{(n)}, \quad k \in \mathbb{N} \tag{1.2.4}
\end{equation*}
$$

where $\left(\varphi_{k}^{(n)}\right)_{k \geqslant 1}$ is a sequence of positive random variables that (scaled by $\left.\Delta t^{(n)}\right)$ specify the duration between two consecutive order events and $\Delta t^{(n)}$ goes to zero as $n \rightarrow \infty$. Further, we denote by $N_{t}^{(n)}$ the random number of incoming order events in $[0, t]$ for any $t \leqslant T$.

[^2]In the following, we will differentiate between four types of events that may change the state of the book at each time $\tau_{k}^{(n)}$ :
A. Either a market sell order of size equal to the first $\xi_{k}^{(n)}$ queues of the bid volumes arrives, which forces the best bid price to decrease by $\xi_{k}^{(n)}$ ticks and the relative bid side volume density function to shift $\xi_{k}^{(n)}$ ticks to the left, or an aggressive buy limit order is placed inside the spread, which forces the best bid price to increase by $\xi_{k}^{(n)}$ ticks and the relative bid side volume density function to shift $\xi_{k}^{(n)}$ ticks to the right.
B. A passive buy limit order placement of size $\frac{\Delta v^{(n)}}{\Delta x^{(n)}} \omega_{k}^{(n)}$ at price level $\pi_{k}^{(n)}$ relative to the best bid price occurs. If $\omega_{k}^{(n)}<0$, this corresponds to a cancellation of volume.
C. Either a market buy order of size equal to the first $\xi_{k}^{(n)}$ queues of the ask volumes arrives, which forces the best ask price to increase by $\xi_{k}^{(n)}$ ticks and the relative ask side volume density function to shift $\xi_{k}^{(n)}$ ticks to the left, or an aggressive sell limit order is placed inside the spread, which forces the best ask price to decrease by $\xi_{k}^{(n)}$ ticks and the relative ask side volume density function to shift $\xi_{k}^{(n)}$ ticks to the right.
D. A passive sell limit order placement of size $\frac{\Delta v^{(n)}}{\Delta x^{(n)}} \omega_{k}^{(n)}$ at price level $\pi_{k}^{(n)}$ relative to the best ask price occurs. If $\omega_{k}^{(n)}<0$, this corresponds to a cancellation of volume.

The event types $A$ and $C$ lead to price changes of the best bid respectively ask price and will be called active order events. Here, the $\mathbb{Z}$-valued random variable $\xi_{k}^{(n)}$ specifies the number of ticks the respective price process changes. In contrast, the event types $B$ and $D$ do not lead to changes in the best bid and ask price and will be referred to as passive order events. The effect of market orders that do not lead to a price change is equivalent to a cancellation of standing volume. Here, the $\mathbb{R}$-valued random variables $\omega_{k}^{(n)}$ and $\pi_{k}^{(n)}$ specify the size and the location of a placement/cancellation, which does not result in a price change. In the following, event types are determined by a field of random variables $\left(\phi_{k}^{(n)}\right)_{k, n \in \mathbb{N}}$ taking values in the set $\{A, B, C, D\}$.

### 1.2.3 State dynamics of the order book

The price dynamics of the LOB-models can be described as follows: for each $k, n \in \mathbb{N}$,

$$
\begin{align*}
B_{k}^{(n)} & =B_{k-1}^{(n)}+\mathbb{1}_{\left\{\phi_{k}^{(n)}=A\right\}} \Delta x^{(n)} \xi_{k}^{(n)}  \tag{1.2.5}\\
A_{k}^{(n)} & =A_{k-1}^{(n)}+\mathbb{1}_{\left\{\phi_{k}^{(n)}=C\right\}^{(n)}} \Delta x^{(n)} \xi_{k}^{(n)}
\end{align*}
$$

In what follows, we denote by $\delta B_{k}^{(n)}:=B_{k}^{(n)}-B_{k-1}^{(n)}$ and $\delta A_{k}^{(n)}:=A_{k}^{(n)}-A_{k-1}^{(n)}, k \in \mathbb{N}$, the bid respectively ask price change caused by the $k$-th order event. Next, we define the placement/cancellation operator in the following way:

$$
\begin{align*}
& M_{b, k}^{(n)}(x):=\frac{\omega_{k}^{(n)}}{\Delta x^{(n)}} \mathbb{1}\left\{x \in I^{(n)}\left(\pi_{k}^{(n)}\right)\right\}^{\mathbb{1}}\left\{\phi_{k}^{(n)}=B\right\},  \tag{1.2.6}\\
& M_{a, k}^{(n)}(x):=\frac{\omega_{k}^{(n)}}{\Delta x^{(n)}} \mathbb{1}\left\{x \in I^{(n)}\left(\pi_{k}^{(n)}\right)\right\}^{\mathbb{1}}\left\{\phi_{k}^{(n)}=D\right\} .
\end{align*}
$$

Then, the dynamics of the volume density function relative to the best bid ${ }^{4}$ and ask price, respectively, are described by

$$
\begin{align*}
v_{b, k}^{(n)}(x) & =v_{b, k-1}^{(n)}\left(x-\delta B_{k}^{(n)}\right)+\Delta v^{(n)} M_{b, k}^{(n)}(x), \\
v_{a, k}^{(n)}(x) & =v_{a, k-1}^{(n)}\left(x+\delta A_{k}^{(n)}\right)+\Delta v^{(n)} M_{a, k}^{(n)}(x), \tag{1.2.7}
\end{align*}
$$

for $x \in \mathbb{R}$. On a first sight, this approach could potentially lead to negative volumes. However, this can be avoided by imposing additional assumptions on the joint conditional distribution of the random variables $\phi_{k}^{(n)}, \omega_{k}^{(n)}$, and $\pi_{k}^{(n)}$ (cf. Assumption 1.8 below). For example, volume cancellations can be modeled to be proportional to the standing volume as done in our simulation study in Section 1.3. Note that this is possible since we allow the limit order book dynamics to depend on the current prices and volumes simultaneously.
The volume changes take place in the visible or shadow book, depending on the sign of $\pi_{k}^{(n)}$. If $\pi_{k}^{(n)} \geqslant 0$, then the visible book changes; if $\pi_{k}^{(n)}<0$, then the placement/cancellation takes place in the shadow book. The shadow book interacts with the visible book through price changes which shift the relative volume density functions. The following example illustrates the working of the shadow book.

Example 1.1 (The shadow book). Suppose that the $k$-th incoming order event is a limit order placement into the shadow book one tick above the current best bid price, i.e.,

$$
\phi_{k}^{(n)}=B, \quad \pi_{k}^{(n)}=-\Delta x^{(n)}, \quad \text { and } \quad \omega_{k}^{(n)}>0 .
$$

Furthermore, suppose that the $(k+1)$-th event is an aggressive buy limit order placement in the spread up to two ticks above the best bid price, i.e., $\phi_{k+1}^{(n)}=A$ and $\xi_{k+1}^{(n)}=2$. Then,

$$
B_{k+1}^{(n)}=B_{k}^{(n)}+2 \Delta x^{(n)}=B_{k-1}^{(n)}+2 \Delta x^{(n)}
$$

and for all $x \in\left[\Delta x^{(n)}, 2 \Delta x^{(n)}\right)$ corresponding to standing volumes one tick above the

[^3]current best bid price,
$$
v_{b, k+1}^{(n)}(x)=v_{b, k}^{(n)}\left(x-2 \Delta x^{(n)}\right)=v_{b, k-1}^{(n)}\left(x-2 \Delta x^{(n)}\right)+\frac{\Delta v^{(n)}}{\Delta x^{(n)}} \omega_{k}^{(n)}
$$
while for all $x \notin\left[\Delta x^{(n)}, 2 \Delta x^{(n)}\right)$,
$$
v_{b, k+1}^{(n)}(x)=v_{b, k}^{(n)}\left(x-2 \Delta x^{(n)}\right)=v_{b, k-1}^{(n)}\left(x-2 \Delta x^{(n)}\right) .
$$

For each $n \in \mathbb{N}$, the microscopic limit order book dynamics are defined through the piecewise constant interpolation

$$
\begin{equation*}
S^{(n)}(t):=S_{k}^{(n)} \quad \text { for } t \in\left[\tau_{k}^{(n)}, \tau_{k+1}^{(n)}\right) \cap[0, T] \tag{1.2.8}
\end{equation*}
$$

of the $E$-valued random variables

$$
S_{k}^{(n)}:=\left(B_{k}^{(n)}, v_{b, k}^{(n)}, A_{k}^{(n)}, v_{a, k}^{(n)}, \tau_{k}^{(n)}\right), \quad k \in \mathbb{N}_{0}
$$

Finally, we introduce the $\sigma$-field $\mathcal{F}_{0}^{(n)}:=\left\{\emptyset, \Omega^{(n)}\right\}, \mathcal{F}_{k}^{(n)}:=\sigma\left(\varphi_{j}^{(n)}, \phi_{j}^{(n)}, \omega_{j}^{(n)}, \pi_{j}^{(n)}, \xi_{j}^{(n)}\right.$ : $j \leqslant k$ ) for each $k, n \in \mathbb{N}$, and assume that $S_{k}^{(n)}$ is $\mathcal{F}_{k}^{(n)}$-measurable. In what follows, we denote $\mathcal{B}^{(n)}:=\left(\Omega^{(n)}, \mathcal{F}^{(n)},\left(\mathcal{F}_{k}^{(n)}\right)_{k \geqslant 0}, \mathbb{P}^{(n)}\right)$ for all $n \in \mathbb{N}$.

Remark 1.2.2. The state and time dynamics of our model are driven by the random variables $\left(\varphi_{k}^{(n)}\right)$ (event times), ( $\left.\phi_{k}^{(n)}\right)$ (event types), $\left(\omega_{k}^{(n)}\right)$ (sizes of passive orders), $\left(\pi_{k}^{(n)}\right)$ (relative price levels of passive orders), and $\left(\xi_{k}^{(n)}\right)$ (sizes of price changes). In particular, the process $S^{(n)}, n \in \mathbb{N}$, is adapted to the filtration $\mathbb{G}^{(n)}=\left(\mathcal{G}_{t}^{(n)}\right)_{t \in[0, T]}$, where $\mathcal{G}_{t}^{(n)}:=\mathcal{F}_{N_{t}^{(n)}}^{(n)}$ for $t \in[0, T]$.

### 1.2.4 A high-frequency approximation of the microscopic model

In this section we state our assumptions regarding the distributional properties of the arrival times, price changes, and order placement/cancellations as well as an assumption on the relation between the scaling parameters $\Delta t^{(n)}, \Delta x^{(n)}$, and $\Delta v^{(n)}$, which will allow us to derive a high-frequency limit. We then present our main result.

First, we assume that the second moment of the unscaled interarrival times is uniformly bounded, which ensures that the random fluctuations of the order arrival time dynamics will vanish in the high-frequency limit. Moreover, we assume that the conditional expectation of each interarrival time only depends on the current state of the order book.

Assumption 1.2 (Conditions on interarrival times).
i) It holds $\sup _{k, n \in \mathbb{N}} \mathbb{E}\left[\left(\varphi_{k}^{(n)}\right)^{2}\right]<\infty$.
ii) Moreover, there exist measurable, bounded functions $\varphi^{(n)}: E \rightarrow(0, \infty), n \in \mathbb{N}$, such that

$$
\mathbb{E}\left[\varphi_{k}^{(n)} \mid \mathcal{F}_{k-1}^{(n)}\right]=\varphi^{(n)}\left(S_{k-1}^{(n)}\right) \quad \text { a.s. }
$$

iii) Furthermore, there exists a Lipschitz continuous function $\varphi: E \rightarrow(0,1]$ with Lipschitz constant $L>0$ such that

$$
\sup _{s \in E}\left|\varphi^{(n)}(s)-\varphi(s)\right| \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Remark 1.2.3. We note that by definition not only the visible book, but also the shadow book is adapted to the filtration $\left(\mathcal{F}_{k}^{(n)}\right)_{k \in \mathbb{N}_{0}}$. Therefore, the filtration $\left(\mathcal{F}_{k}^{(n)}\right)_{k \in \mathbb{N}_{0}}$ (resp. $\mathbb{G}^{(n)}$ ) must not be misunderstood as the market filtration. Especially, while from a mathematical point of view Assumption 1.2 ii) is sufficient for the derivation of a Markovian high-frequency limit process (cf. Theorem 1.2.6 below), in applications the function $\varphi^{(n)}$ will only depend on the visible book as it is the case in our simulation study (cf. Section 1.3). Moreover, Assumption 1.2 ii) should be interpreted in the correct way: the conditional expectation of the interarrival times does neither depend on future spread placements through the shadow book (which would be absurd, anyway) nor on the past evolution of the order book, but only on its current state. In the same way Assumptions 1.3 ii) and 1.4 should be understood.

Next, we present our assumption on the conditional expectations of the placement/cancellation operator of the volume dynamics. It is of the same flavor as the one for the interarrival times and ensures that the random fluctuations generated by the volume dynamics will vanish in the high-frequency limit as well.

Assumption 1.3 (Conditions on placements/cancellations).
i) It holds $\sup _{k, n \in \mathbb{N}} \mathbb{E}\left[\left(\omega_{k}^{(n)}\right)^{2}\right]<\infty$.
ii) There exist measurable functions $f_{b}^{(n)}, f_{a}^{(n)}: E \rightarrow L^{2}(\mathbb{R}), n \in \mathbb{N}$, such that for all $k \in \mathbb{N}$,

$$
\begin{aligned}
& f_{b}^{(n)}\left[S_{k-1}^{(n)}\right](\cdot)=\frac{1}{\Delta x^{(n)}} \mathbb{E}\left[\omega_{k}^{(n)} \mathbb{1}_{\left\{\phi_{k}^{(n)}=B\right\}^{\mathbb{1}}\left\{\cdot \in I^{(n)}\left(\pi_{k}^{(n)}\right)\right\}} \mid \mathcal{F}_{k-1}^{(n)}\right] \text { a.s. } \\
& f_{a}^{(n)}\left[S_{k-1}^{(n)}\right](\cdot)=\frac{1}{\Delta x^{(n)}} \mathbb{E}\left[\omega_{k}^{(n)} \mathbb{1}_{\left\{\phi_{k}^{(n)}=D\right\}^{1}\left\{\cdot \in I^{(n)}\left(\pi_{k}^{(n)}\right)\right\}} \mid \mathcal{F}_{k-1}^{(n)}\right] \text { a.s. }
\end{aligned}
$$

iii) There exist bounded, Lipschitz continuous functions $f_{b}, f_{a}: E \rightarrow L^{2}(\mathbb{R})$ with Lipschitz constant $L>0$ such that as $n \rightarrow \infty$,

$$
\sup _{s \in E}\left\{\left\|f_{b}^{(n)}[s]-f_{b}[s]\right\|_{L^{2}}+\left\|f_{a}^{(n)}[s]-f_{a}[s]\right\|_{L^{2}}\right\} \rightarrow 0
$$

for any $x, \widetilde{x} \in \mathbb{R}, I=b, a$,

$$
\sup _{s \in E}\left\|f_{I}[s](\cdot+x)-f_{I}[s](\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant L|x-\widetilde{x}|
$$

and

$$
\sup _{s \in E}\left\|f_{I}[s](\cdot) \mathbb{1}_{[r, \infty)}(|\cdot|)\right\|_{L^{2}} \xrightarrow{r \rightarrow \infty} 0
$$

As we aim to derive a jump-diffusion-type limit for the price dynamics, the assumption for the price changes will be of a different form. We differentiate between so-called small and large price changes. Small price changes are assumed to become negligible as the number of orders gets large. This framework is analyzed in $\sqrt[40]{43}$, where all price changes are assumed to be equal to $\pm \Delta x^{(n)}$ and $\Delta x^{(n)} \rightarrow 0$. In order to take more extreme price movements into account, we include large price changes in our model, which do not converge to zero as $n \rightarrow \infty$. The following assumption introduces the scaling of the conditional first and second moments of the small price jumps and the scaling of the conditional probabilities of the large price jumps.

Assumption 1.4 (Conditions on the price changes). Let $\left(\delta_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}_{+}$be a null sequence that satisfies $\Delta x^{(n)} \leqslant \delta_{n}$ for all $n \in \mathbb{N}$.
i) There exist bounded, measurable functions $p_{b}^{(n)}, p_{a}^{(n)}: E \rightarrow \mathbb{R}$ and $r_{b}^{(n)}, r_{a}^{(n)}: E \rightarrow$ $\mathbb{R}_{+}$such that for all $k \in \mathbb{N}$,

$$
\begin{aligned}
& \mathbb{E}\left[\left(\xi_{k}^{(n)}\right)^{2} \mathbb{1}_{\left\{\phi_{k}^{(n)}=A\right\}^{\mathbb{1}}\left\{0<\Delta x^{(n)}\left|\xi_{k}^{(n)}\right| \leqslant \delta_{n}\right\} \mid} \mid \mathcal{F}_{k-1}^{(n)}\right]=\frac{\Delta t^{(n)}}{\left(\Delta x^{(n)}\right)^{2}}\left(r_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2} \quad \text { a.s. } \\
& \mathbb{E}\left[\left(\xi_{k}^{(n)}\right)^{2} \mathbb{1}_{\left\{\phi_{k}^{(n)}=C\right\}^{1}\left\{0<\Delta x^{(n)}\left|\xi_{k}^{(n)}\right| \leqslant \delta_{n}\right\} \mid} \mid \mathcal{F}_{k-1}^{(n)}\right]=\frac{\Delta t^{(n)}}{\left(\Delta x^{(n)}\right)^{2}}\left(r_{a}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2} \quad \text { a.s. }
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}\left[\xi_{k}^{(n)} \mathbb{1}_{\left\{\phi_{k}^{(n)}=A\right\}^{\mathbb{1}}\left\{0<\Delta x^{(n)}\left|\xi_{k}^{(n)}\right| \leqslant \delta_{n}\right\} \mid} \mid \mathcal{F}_{k-1}^{(n)}\right]=\frac{\Delta t^{(n)}}{\Delta x^{(n)}} p_{b}^{(n)}\left(S_{k-1}^{(n)}\right) \quad \text { a.s. } \\
& \mathbb{E}\left[\xi_{k}^{(n)} \mathbb{1}_{\left\{\phi_{k}^{(n)}=C\right\}^{1}\left\{0<\Delta x^{(n)}\left|\xi_{k}^{(n)}\right| \leqslant \delta_{n}\right\} \mid} \mid \mathcal{F}_{k-1}^{(n)}\right]=\frac{\Delta t^{(n)}}{\Delta x^{(n)}} p_{a}^{(n)}\left(S_{k-1}^{(n)}\right) \quad \text { a.s. }
\end{aligned}
$$

Further, there exists another null sequence $\left(\eta_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}_{+}$with $\delta_{n} / \eta_{n} \rightarrow 0$ such that for all $n \in \mathbb{N}, s \in E$, we have

$$
\min \left\{r_{b}^{(n)}(s), r_{a}^{(n)}(s)\right\}>\eta_{n}
$$

ii) For all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, there exist measurable, bounded functions $k_{b, j}^{(n)}, k_{a, j}^{(n)}$ : $E \rightarrow \mathbb{R}_{+}$with $k_{b, j}^{(n)} \equiv k_{a, j}^{(n)} \equiv 0$ whenever $\left|x_{j}^{(n)}\right| \leqslant \delta_{n}$, such that for all $j \in \mathbb{Z}$ with $\left|x_{j}^{(n)}\right|>\delta_{n}$

$$
\begin{aligned}
& \mathbb{P}\left[\Delta x^{(n)} \xi_{k}^{(n)}=x_{j}^{(n)}, \phi_{k}^{(n)}=A \mid \mathcal{F}_{k-1}^{(n)}\right]=\Delta t^{(n)} k_{b, j}^{(n)}\left(S_{k-1}^{(n)}\right), \\
& \mathbb{P}\left[\Delta x^{(n)} \xi_{k}^{(n)}=x_{j}^{(n)}, \phi_{k}^{(n)}=C \mid \mathcal{F}_{k-1}^{(n)}\right]=\Delta t^{(n)} k_{a, j}^{(n)}\left(S_{k-1}^{(n)}\right)
\end{aligned}
$$

The null sequence $\delta_{n}$ separates the price changes into two regimes. First, we have the regime of prices changes becoming negligible in the limit. The second one describes those which do not scale to zero. The null sequence $\eta_{n}$ is introduced for technical reasons only, because it guarantees that the diffusion component does not vanish in the $n$-th model, which will simplify the convergence proof of the price changes becoming negligible in the limit toward a diffusion process.

Remark 1.2.4. We note that Assumption 1.4 i) is a generalization of the assumptions made in [40-43], where a further scaling parameter $\Delta p^{(n)}=o(1)$ is introduced that controls the proportion of price changes among all events. Ensuring that this proportion relates to the other scaling parameters as $\left(\Delta x^{(n)}\right)^{2} \Delta p^{(n)} \approx \Delta t^{(n)}$, Assumption 2.1 in [42] implies indeed a scaling of order $\Delta t^{(n)}$ for the second moments of price changes as we demand in the first two equations in Assumption 1.4 i). Note however, that Assumption 1.4 does not necessarily imply that the proportion of price changing events converges to zero. Indeed, suppose that all four events happen with equal probability independently of anything else and that $\delta_{n}=\Delta x^{(n)}$. Then Assumption 1.4 i) is satisfied with $\Delta t^{(n)}=\left(\Delta x^{(n)}\right)^{2}$. Furthermore, our small price changes can be of order larger than $\Delta x^{(n)}$ if the probability of price changing events goes to zero. To see this, suppose that $A$ and $C$ events occur with equal probability $\Delta t^{(n)} / \Delta x^{(n)}$ (which goes to zero by Assumption 1.7 below) and that $\left|\xi_{k}^{(n)}\right| \approx\left(\Delta x^{(n)}\right)^{-1 / 2}$, in which case Assumption 1.4 i) also holds true.

The next assumption guarantees that the coefficient functions $p_{b}^{(n)}, p_{a}^{(n)}, r_{b}^{(n)}$, and $r_{a}^{(n)}$ satisfy the right limiting behavior.
Assumption 1.5 (Convergence assumptions corresponding to the small jumps). There exist bounded, Lipschitz continuous functions $p_{b}, p_{a}: E \rightarrow \mathbb{R}$ and $r_{b}, r_{a}: E \rightarrow \mathbb{R}_{+}$with Lipschitz constant $L>0$ such that as $n \rightarrow \infty$, it holds that

$$
\sup _{s \in E}\left\{\left|p_{b}^{(n)}(s)-p_{b}(s)\right|+\left|p_{a}^{(n)}(s)-p_{a}(s)\right|+\left|r_{b}^{(n)}(s)-r_{b}(s)\right|+\left|r_{a}^{(n)}(s)-r_{a}(s)\right|\right\} \rightarrow 0
$$

Next, we need to specify assumptions that guarantee the convergence of the large jumps. To this end, we first construct kernels $K_{b}^{(n)}, K_{a}^{(n)}: E \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}_{+}$representing the conditional distributions of the large price changes by setting

$$
\begin{equation*}
K_{b}^{(n)}(s, A):=\sum_{j \in \mathbb{Z}} \mathbb{1}_{A}\left(x_{j}^{(n)}\right) k_{b, j}^{(n)}(s), \quad K_{a}^{(n)}(s, A):=\sum_{j \in \mathbb{Z}} \mathbb{1}_{A}\left(x_{j}^{(n)}\right) k_{a, j}^{(n)}(s) \tag{1.2.9}
\end{equation*}
$$

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for $A \in \mathcal{B}(\mathbb{R})$ and $s \in E$. In particular, for $s \in E$ and $I=b, a$, we have $K_{I}^{(n)}\left(s, I^{(n)}(x)\right)=$ $k_{I, j}^{(n)}(s)$ if $x_{j}^{(n)} \leqslant x<x_{j+1}^{(n)}$. The following assumption guarantees that in the limit the driving jump measures do not depend on the order book dynamics, which is necessary to derive a jump diffusion for the prices as opposed to more general (and more complex) semimartingale dynamics in the limit. We will assume that the driving jump measures have compact support in $[-M, M]$. To define their discrete approximations later on, we introduce for all $n \in \mathbb{N}$ the set $\mathbb{Z}_{M}^{(n)}:=\left\{j \in \mathbb{Z}:-M \leqslant x_{j}^{(n)} \leqslant M\right\}$.
Assumption 1.6 (Convergence assumptions corresponding to the large jumps). There exist kernels $K_{b}, K_{a}: E \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}_{+}$satisfying $K_{b}(s,\{0\})=K_{a}(s,\{0\})=0$ for all $s \in$ $E$ as well as finite measures $Q_{b}, Q_{a}$ on $\mathcal{B}(\mathbb{R})$ with compact support in $[-M, M]$ satisfying $Q_{b}(\{0\})=Q_{a}(\{0\})=0$ and measurable, bounded functions $\theta_{b}, \theta_{a}: E \times[-M, M] \rightarrow \mathbb{R}$ such that for $I=b, a$,
i) for every $s \in E$ the map $x \mapsto \theta_{I}(s, x)$ is uniformly equicontinuous and for every $y \in[-M, M]$ either $\theta_{I}(s, y)=0$ or $x \mapsto \theta_{I}(s, x)$ is strictly increasing in an open neighborhood of y ${ }^{5}$;
ii) for all $s \in E$ and all $A \in \mathcal{B}([-M, M])$,

$$
\begin{equation*}
Q_{I}(A)=K_{I}\left(s, \theta_{I}(s, A)\right)+Q_{I}\left(\left\{x \in A: \theta_{I}(s, x)=0\right\}\right), \tag{1.2.10}
\end{equation*}
$$

where $\left.\theta_{I}(s, A):=\left\{\theta_{I}(s, x): x \in A\right\}^{6}\right]$
iii)

$$
\sup _{s \in E} \sum_{j}\left|K_{I}^{(n)}\left(s, \theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)-K_{I}\left(s, \theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right| \longrightarrow 0 .
$$

iv) There exists a constant $L>0$ such that for all $s, \widetilde{s} \in E$ and $y \in[-M, M]$,

$$
\left|\theta_{I}(s, y)-\theta_{I}(\widetilde{s}, y)\right| \leqslant L\|s-\widetilde{s}\|_{E}
$$

Moreover, setting for all $n \in \mathbb{N}$,

$$
\begin{equation*}
\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right):=\left\lceil\frac{\theta_{I}\left(s, x_{j}^{(n)}\right)}{\Delta x^{(n)}}\right\rceil \cdot \Delta x^{(n)}, \quad s \in E, j \in \mathbb{Z}_{M}^{(n)} \tag{1.2.11}
\end{equation*}
$$

we have
v) for all $s \in E$ and $i \in \mathbb{N}$ with $K_{I}^{(n)}\left(s,\left\{x_{i}^{(n)}\right\}\right)>0$ there exists a unique $j \in \mathbb{Z}_{M}^{(n)}$ such that

$$
x_{i}^{(n)}=\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right) ;
$$

[^4]vi)
$$
\left.\sup _{s \in E} \sum_{j} \int_{\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)} \mid \mathbb{1}_{\left\{\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)=0\right\}}\right\}^{\left.-\mathbb{1}_{\left\{\theta_{I}(s, x)=0\right.}\right\}}| | Q_{I}(d x) \longrightarrow 0
$$

Together, part ii) and part iii) of Assumption 1.6 require the distribution of some transformation of the large jumps to converge to a limit that is independent of the state $s \in E$. This should be compared to Assumption 1.4 i ), where we indirectly require that the distribution of the standardized small price changes converges to a standard Gaussian law.

## Remark 1.2.5.

i) We require $Q_{b}$ and $Q_{a}$ to be compactly supported on $[-M, M]$, so that we can take the separable space $C_{b}([-M, M])$ of bounded, continuous functions on $[-M, M]$ as test functions, which will be important to be able to apply the results from Kurtz and Protter 55].
ii) For $I=b, a$, Assumption 1.6 v) asks for bijectivity of the discretized coefficient function $\theta_{I}^{(n)}$ on the support of the measure $K_{I}^{(n)}(s, \cdot)$. Of course, we need surjectivity to get a representation of the sum of large jumps as a discrete stochastic integral. Injectivity will allow us to map the large jumps of the respective price process uniquely to the jumps of the corresponding integrator defined below.

The advantage in working with $Q_{b}$ and $Q_{a}$ instead of $K_{b}$ and $K_{a}$ is that they are independent of the order book dynamics. The key requirement in Assumption 1.6 is the validity of equation 1.2 .10 , which may look a little bit mysterious in the beginning. It says that the jump sizes of $K_{b}, K_{a}$ may vary across different states $s \in E$, but that the jump intensities stay the same, modulo the modification of jump sizes, as long as the jump size does not vanish to zero. In Section 1.4, we provide explicit examples of different jump behaviors satisfying Assumption 1.6.

For later use, we will extend the definition of $\theta_{I}^{n)}, I=b, a$, to the whole interval $[-M, M]$ by linear interpolation, i.e., we set for all $s \in E, x_{j}^{(n)} \leqslant x<x_{j+1}^{(n)}$,

$$
\begin{equation*}
\theta_{I}^{(n)}(s, x):=\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)+\frac{x-x_{j}^{(n)}}{\Delta x^{(n)}}\left(\theta_{I}^{(n)}\left(s, x_{j+1}^{(n)}\right)-\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)\right) \tag{1.2.12}
\end{equation*}
$$

Then, for $s \in E$, the map $x \mapsto \theta_{I}^{(n)}(s, x)$ is continuous with bounded support, hence bounded.

The next assumption introduces the crucial relation between the different scaling parameters $\Delta t^{(n)}, \Delta x^{(n)}$, and $\Delta v^{(n)}$. It is a mixture of the scaling assumption in 40 for the parameters $\Delta t^{(n)}$ and $\Delta v^{(n)}$ and the one in 42 for the parameters $\Delta t^{(n)}, \Delta x^{(n)}$, and $\Delta p^{(n)}$; the latter, however, occurs only implicitly in Assumption 1.4. Because of this assumption, an approximation of the price dynamics by a jump diffusion together with an approximation of the volume dynamics by a fluid process can be obtained in the high-frequency limit.

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Assumption 1.7 (Relation between the scaling parameters). There exists a constant $C>0$ such that

$$
\lim _{n \rightarrow \infty} \frac{\Delta t^{(n)}}{\Delta x^{(n)}}=0 \quad \text { and } \quad \lim _{n \rightarrow \infty} \frac{\Delta t^{(n)}}{\Delta v^{(n)}}=C
$$

In what follows, we will assume that $C=1$. Any other constant would require further constants in the limiting dynamics.

With all these preparation done, we can now state our main result.
Theorem 1.2.6 (Main result). Under Assumptions 1.11 .7 the microscopic LOBdynamics $S^{(n)}$ converges weakly in the Skorokhod topology to $S=\eta \circ \zeta$, where $\zeta(t):=$ $\inf \left\{s>0: \tau^{\eta}(s)>t\right\}, t \in[0, T]$, is a random time change and $\eta=\left(B^{\eta}, v_{b}^{\eta}, A^{\eta}, v_{a}^{\eta}, \tau^{\eta}\right)$ is the unique strong solution of the coupled diffusion-fluid system

$$
\begin{align*}
B^{\eta}(t)= & B_{0}+\int_{0}^{t} p_{b}(\eta(u)) d u+ \\
& \int_{0}^{t} r_{b}(\eta(u)) d Z_{b}(u) \\
& +\int_{0}^{t} \int_{[-M, M]} \theta_{b}(\eta(u-), y) \mu_{b}^{Q}(d u, d y), \\
v_{b}^{\eta}(t, x)= & v_{b, 0}\left(x-\left(B^{\eta}(t)-B_{0}\right)\right)+\int_{0}^{t} f_{b}[\eta(u)]\left(x-\left(B^{\eta}(t)-B^{\eta}(u)\right)\right) d u,  \tag{1.2.13}\\
A^{\eta}(t)= & A_{0}+\int_{0}^{t} p_{a}(\eta(u)) d u+\int_{0}^{t} r_{a}(\eta(u)) d Z_{a}(u) \\
& +\int_{0}^{t} \int_{[-M, M]} \theta_{a}(\eta(u-), y) \mu_{a}^{Q}(d u, d y), \\
v_{a}^{\eta}(t, x)= & v_{a, 0}\left(x+A^{\eta}(t)-A_{0}\right)+\int_{0}^{t} f_{a}[\eta(u)]\left(x+A^{\eta}(t)-A^{\eta}(u)\right) d u, \\
\tau^{\eta}(t)= & \int_{0}^{t} \varphi(\eta(u)) d u,
\end{align*}
$$

for all $t \in[0, T], x \in \mathbb{R}$, where $Z_{b}, Z_{a}$ are independent standard Brownian motions and $\mu_{b}^{Q}, \mu_{a}^{Q}$ are independent homogeneous Poisson random measures with intensity measures $\lambda \times Q_{b}$ and $\lambda \times Q_{a}$, independent of $Z_{b}, Z_{a}$. Here, $\lambda$ denotes the Lebesgue measure on $[0, T]$.

We present a proof sketch of our main theorem in Section 1.5 whereas the technical details are given in Section 1.6.

Remark 1.2.7. Let the assumptions of Theorem 1.2.6 be satisfied and suppose that there exist functions $h_{b}, h_{a}: E \rightarrow \mathbb{R}$ such that $\theta_{I}(s, x)=h_{I}(s) x, I=b, a$, for all $s \in E$ and $x \in[-M, M]$. Then the dynamics of the prices simplifies to

$$
\begin{aligned}
B^{\eta}(t) & =B_{0}+\int_{0}^{t} p_{b}(\eta(u)) d u+\int_{0}^{t} r_{b}(\eta(u)) d Z_{b}(u)+\int_{0}^{t} h_{b}(\eta(u-)) d L_{b}(u) \\
A^{\eta}(t) & =A_{0}+\int_{0}^{t} p_{a}(\eta(u)) d u+\int_{0}^{t} r_{a}(\eta(u)) d Z_{a}(u)+\int_{0}^{t} h_{a}(\eta(u-)) d L_{a}(u)
\end{aligned}
$$

for $t \in[0, T]$, where $L_{b}, L_{a}$ are one-dimensional, independent Lévy processes with jumps in $[-M, M]$.

Corollary 1.2.8. Let the assumptions of Theorem 1.2.6 be satisfied. Further, assume that for $I=b$, a the functions $v_{I, 0}: \mathbb{R} \rightarrow \mathbb{R}_{+}$and $f_{I}[s]: \mathbb{R} \rightarrow \mathbb{R}_{+}$, $s \in E$, are twice continuously differentiable. Then the microscopic LOB-dynamics $S^{(n)}$ converges weakly in the Skorokhod topology to $S=\left(B, v_{b}, A, v_{a}, \tau\right)$, starting in $S_{0}$, and being the unique strong solution of the following coupled $S D E-S P D E$ system: for $(t, x) \in[0, T] \times \mathbb{R}$,

$$
\begin{aligned}
d B(t)= & \frac{p_{b}(S(t))}{\varphi(S(t))} d t+r_{b}(S(t)) \zeta^{1 / 2}(t) d \widetilde{Z}_{b}(t)+\int_{-M}^{M} \theta_{b}(S(t-), y) \widetilde{\mu}_{b}^{Q}(d t, d y) \\
d v_{b}(t, x)= & \left(-\frac{\partial v_{b}}{\partial x}(t, x) p_{b}(S(t))+\frac{1}{2} \frac{\partial^{2} v_{b}}{\partial x^{2}}(t, x)\left(r_{b}(S(t))\right)^{2}+f_{b}[S(t)](x)\right) \frac{1}{\varphi(S(t))} d t \\
& -\frac{\partial v_{b}}{\partial x}(t, x) r_{b}(S(t)) \zeta^{1 / 2}(t) d \widetilde{Z}_{b}(t)+\left(v_{b}(t-, x-\Delta B(t))-v_{b}(t-, x)\right) \\
d A(t)= & \frac{p_{a}(S(t))}{\varphi(S(t))} d t+r_{a}(S(t)) \zeta^{1 / 2}(t) d \widetilde{Z}_{a}(t)+\int_{-M}^{M} \theta_{a}(S(t-), y) \widetilde{\mu}_{a}^{Q}(d t, d y) \\
d v_{a}(t, x)= & \left.\frac{\partial v_{a}}{\partial x}(t, x) p_{a}(S(t))+\frac{1}{2} \frac{\partial^{2} v_{a}}{\partial x^{2}}(t, x)\left(r_{a}(S(t))\right)^{2}+f_{a}[S(t)](x)\right) \frac{1}{\varphi(S(t))} d t \\
& \quad+\frac{\partial v_{a}}{\partial x}(t, x) r_{a}(S(t)) \zeta^{1 / 2}(t) d \widetilde{Z}_{a}(t)+\left(v_{a}(t-, x+\Delta A(t))-v_{a}(t-, x)\right)
\end{aligned}
$$

$$
d \tau(t)=d t
$$

where $\widetilde{Z}_{I}, I=b, a$, are independent Brownian motions and $\widetilde{\mu}_{b}^{Q}$ and $\widetilde{\mu}_{a}^{Q}$ are independent, integer-valued random jump measures with compensators $\widetilde{\nu}_{I}^{Q}(d t, d y)=(\varphi(S(t)))^{-1} d t \times$ $Q_{I}(d y)$ for $I=b, a$, and $\zeta(t)=\int_{0}^{t}(\varphi(S(u)))^{-1} d u$. Here, $\Delta B(t):=B(t)-B(t-)$ and $\Delta A(t):=A(t)-A(t-)$ denote the jump of the best bid and ask price at time $t>0$.

Note that the above SPDEs for the volume processes are degenerate. If we condition the volume dynamics on the price movements, they behave like deterministic PDEs, since random fluctuations of the queue sizes vanish in the high frequency limit. The proof of the above corollary is postponed to Section 1.6.3.

In order to guarantee that the bid and ask price, the spread, and the volume density functions do not become negative, certain conditions on the joint distribution of the driving variables have to be satisfied, which are specified in the following assumption.

Assumption 1.8 (Conditions to guarantee non-negative prices, spread, and volumes).
i) For all $k, n \in \mathbb{N}$ it holds

$$
\begin{aligned}
& \mathbb{P}\left[\Delta x^{(n)} \xi_{k}^{(n)} \geqslant A_{k-1}^{(n)}-B_{k-1}^{(n)}, \phi_{k}^{(n)}=A \mid \mathcal{F}_{k-1}^{(n)}\right]=0 \\
& \mathbb{P}\left[\Delta x^{(n)} \xi_{k}^{(n)} \leqslant B_{k-1}^{(n)}-A_{k-1}^{(n)}, \phi_{k}^{(n)}=C \mid \mathcal{F}_{k-1}^{(n)}\right]=0
\end{aligned}
$$

ii) For all $k, n \in \mathbb{N}$ it holds that

$$
\mathbb{P}\left[\Delta x^{(n)} \xi_{k}^{(n)} \leqslant-B_{k-1}^{(n)}, \phi_{k}^{(n)}=A \mid \mathcal{F}_{k-1}^{(n)}\right]=0
$$

iii) For all $k, n \in \mathbb{N}$ it holds that

$$
\begin{aligned}
& \mathbb{P}\left[v_{b, k-1}^{(n)}\left(\pi_{k}^{(n)}\right) \leqslant-\omega_{k}^{(n)}, \phi_{k}^{(n)}=B \mid \mathcal{F}_{k-1}^{(n)}\right]=0 \\
& \mathbb{P}\left[v_{a, k-1}^{(n)}\left(\pi_{k}^{(n)}\right) \leqslant-\omega_{k}^{(n)}, \phi_{k}^{(n)}=D \mid \mathcal{F}_{k-1}^{(n)}\right]=0
\end{aligned}
$$

The following corollary of Theorem 1.2 .6 is a direct consequence of the weak convergence result $S^{(n)} \Rightarrow S$ and the characterization of the limit $S$.

Corollary 1.2.9. Let the assumptions of Theorem 1.2.6 be satisfied. Then:
i) Under Assumption 1.8 i), we have

$$
r_{a}(s)=r_{b}(s)=0, \quad p_{a}(s) \geqslant 0 \geqslant p_{b}(s) \quad \forall s=(a, v, a, w, t) \in E
$$

and the spread stays non-negative, i.e., for all $t \in[0, T], A(t) \geqslant B(t)$ a.s. If in addition also $B_{0}^{(n)}, A_{0}^{(n)} \geqslant 0$ for all $n \in \mathbb{N}$ and Assumption 1.8 ii) holds, we have

$$
r_{b}(s)=0, \quad p_{b}(s) \geqslant 0 \quad \forall s=(0, v, a, w, t) \in E
$$

and the bid and ask prices stay non-negative, i.e., for all $t \in[0, T], B(t), A(t) \geqslant 0$ a.s.
ii) Under Assumption 1.8 iii), we have

$$
\left\|f_{b}^{-}[s](\cdot) \mathbb{1}_{\{v(\cdot)=0\}}\right\|_{L^{2}}=0, \quad\left\|f_{a}^{-}[s](\cdot) \mathbb{1}_{\{w(\cdot)=0\}}\right\|_{L^{2}}=0 \quad \forall s=(b, v, a, w, t) \in E
$$

and both volume density functions are non-negative, i.e., for all $t \in[0, T]$,

$$
\left\|v_{a}^{-}(t)\right\|_{L^{2}}=\left\|v_{b}^{-}(t)\right\|_{L^{2}}=0 \quad \text { a.s. }
$$

### 1.3 Simulation study

In this section, we present a simulation study of the order book dynamics introduced in the previous section. It demonstrates the usefulness of the general dependence structure, where all coefficient functions are allowed to depend on current prices and volumes. Such dependencies are plausible according to the observations in [6, 15, 39]. Among others, the simulation study shows the impact of endogenously and exogenously triggered large jumps in the price dynamics.

Let $\Delta p^{(n)}=o(1)$ denote a scaling parameter that controls the proportion of active order events among all events (cf. also Remark 1.2.4). Further, let us fix some $h>0$.

For each $s=\left(b, v_{b}, a, v_{a}, t\right) \in E$, we define the $\operatorname{spread} \operatorname{Sp}(s)$ and an order imbalance factor $\operatorname{Im}(s)$ via

$$
\operatorname{Sp}(s):=a-b \quad \text { and } \quad \operatorname{Im}(s):=\frac{\operatorname{VolBid}(s)}{\operatorname{VolBid}(s)+\operatorname{VolAsk}(s)}
$$

with

$$
\operatorname{VolBid}(s):=\int_{0}^{h} v_{b}(x) d x \quad \text { and } \quad \operatorname{VolAsk}(s):=\int_{0}^{h} v_{a}(x) d x
$$

In the following, we denote $\mathrm{Sp}_{k}^{(n)}:=\operatorname{Sp}\left(S_{k}^{(n)}\right)$,

$$
\operatorname{VolBid}_{k}^{(n)}:=\operatorname{VolBid}\left(S_{k}^{(n)}\right), \operatorname{VolAsk}_{k}^{(n)}:=\operatorname{VolAsk}\left(S_{k}^{(n)}\right), \operatorname{Im}_{k}^{(n)}:=\operatorname{Im}\left(S_{k}^{(n)}\right)
$$

For simplicity, let the order arrival times be equidistant and deterministic, i.e., $\tau_{k}^{(n)}=$ $t_{k}^{(n)}:=k \Delta t^{(n)}$ for all $k=0, \cdots, T_{n}:=\left\lfloor T / \Delta t^{(n)}\right\rfloor$. Moreover, we assume that all small price changes are of size $\pm \Delta x^{(n)}$, while the sizes of the large price changes may depend on the current state of the book through the spread and the cumulative volumes at the top of the book.

Let us set $\gamma_{n}(x):=\exp \left(-\gamma_{1}\left(x-\Delta x^{(n)}\right)\right)$ for some $\gamma_{1}>0$. We allow the probabilities of the small price changes to depend on the current imbalance factor and spread as

$$
\begin{aligned}
& \frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=1, \phi_{k+1}^{(n)}=A \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta p^{(n)}}=\Delta x^{(n)} \operatorname{Im}_{k}^{(n)}\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)+\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right) \\
& \frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=1, \phi_{k+1}^{(n)}=C \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta p^{(n)}}=\Delta x^{(n)} \operatorname{Im}_{k}^{(n)} \gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)+\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=-1, \phi_{k+1}^{(n)}=A \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta p^{(n)}}=\Delta x^{(n)}\left(1-\operatorname{Im}_{k}^{(n)}\right) \gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)+\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right) \\
& \frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=-1, \phi_{k+1}^{(n)}=C \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta p^{(n)}}=\Delta x^{(n)}\left(1-\operatorname{Im}_{k}^{(n)}\right)\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)+\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)
\end{aligned}
$$

The above four probabilities sum up to $4 \Delta p^{(n)}\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)+\Delta x^{(n)} \Delta p^{(n)}$. This choice of the conditional distribution for the occurrence of small price changes guarantees that the bid and ask price do not cross. If the spread is small, the first and second term of these probabilities are approximately of the same size. Moreover, the probability of an upward price change is increasing with the imbalance factor, i.e., a price increase is more likely to occur if the standing volume at the top of buy side is significantly higher than the standing volume at the top of the sell side. This behavior of the price is motivated by the empirical observations in e.g., 14, 87. Moreover, if the spread is equal
to $\Delta x^{(n)}$ the probabilities of observing an increase in the best bid price or a decrease in the best ask price are zero. For large spreads, the conditional probabilities are all dominated by their second term and hence are all of similar size. Here, the parameter $\gamma_{1}>0$ controls the influence of the spread on the order book dynamics. In particular, we obtain the following feedback functions:
$p_{b}^{(n)}\left(S_{k}^{(n)}\right)=-\left(1-\operatorname{Im}_{k}^{(n)}\right)+\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right), \quad p_{a}^{(n)}\left(S_{k}^{(n)}\right)=\operatorname{Im}_{k}^{(n)}-\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)$,
and

$$
\begin{aligned}
& \left(r_{b}^{(n)}\left(S_{k}^{(n)}\right)\right)^{2}=2\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)+\mathcal{O}\left(\Delta x^{(n)}\right) \\
& \left(r_{a}^{(n)}\left(S_{k}^{(n)}\right)\right)^{2}=2\left(1-\gamma_{n}\left(\operatorname{Sp}_{k}^{(n)}\right)\right)+\mathcal{O}\left(\Delta x^{(n)}\right)
\end{aligned}
$$

Note that for all $s \in E$ with $\operatorname{Sp}(s)=\Delta x^{(n)}$, the diffusion coefficients vanish, while the drift of the bid price is negative and the drift of the ask price is positive. This guarantees that the prices move apart if the spread equals $\Delta x^{(n)}$.

Next, let us turn to the conditional probabilities of observing large price changes. In our setting, it is important that the jump intensities are independent of the order book dynamics. Nevertheless, the jump sizes are allowed to vary across different states of the book. In order to ensure that the bid and ask prices do not cross, the jump sizes of the bid and ask must depend on the current spread. Moreover, small standing volumes at the top of the bid or ask side increase the size of a large jump. As noted in Remark 1.2.1 the jump behavior might also be influenced by external factors. To model such external influences, we take a discretized Poisson process $\left(Y_{k}^{(n)}\right)_{k \geqslant 0}$ with intensity parameter $\sigma>0$, which only jumps at times $\left\{t_{k}^{(n)}: k=1, \cdots, T_{n}\right\}$ and fix some threshold level $\kappa>0$. If $Y_{k}^{(n)}$ crosses the threshold level, jump sizes increase significantly by a factor $1+\eta_{1} \geqslant 1$, for some $\eta_{1} \in \mathbb{N}_{0}$.

Altogether, the sizes of the large price jumps depending on the current state $s \in E$ and external influence $y \in \mathbb{N}_{0}$ are modeled as follows: take $\eta_{2}>0$ and $j_{b}^{+}, j_{a}^{+}, j_{b}^{-}, j_{a}^{-} \in$ $\Delta x^{(n)} \mathbb{Z}$ with $j_{b}^{+}, j_{a}^{+}>\Delta x^{(n)}$ and $j_{b}^{-}, j_{a}^{-}<-\Delta x^{(n)}$. Then we define

$$
\begin{array}{ll}
J_{b}^{+}(s, y)=\min \left\{\rho(y) j_{b}^{+}, \operatorname{Sp}(s)-\Delta x^{(n)}\right\}, & J_{a}^{+}(s, y)=\left\lfloor\frac{\rho(y) \eta_{2} j_{a}^{+}}{\operatorname{VolAsk}(s) \Delta x^{(n)}}\right\rfloor \Delta x^{(n)}, \\
J_{a}^{-}(s, y)=\max \left\{\rho(y) j_{a}^{-},-\operatorname{Sp}(s)+\Delta x^{(n)}\right\}, & J_{b}^{-}(s, y)=\left\lfloor\frac{\rho(y) \eta_{2} j_{b}^{-}}{\operatorname{VolBid}(s) \Delta x^{(n)}}\right\rfloor \Delta x^{(n)},
\end{array}
$$

where $\rho(y):=1+\eta_{1} \mathbb{1}(y>\kappa)$ and $\eta_{1}, \eta_{2}$ control the impact of the external factor and the cumulative standing volumes, respectively, on the size of the large jumps. Now, for
non-negative $\lambda_{b}^{+}, \lambda_{b}^{-}, \lambda_{a}^{+}$, and $\lambda_{a}^{-} \in[0,1]$ with $\lambda_{b}^{+}+\lambda_{b}^{-}+\lambda_{a}^{+}+\lambda_{a}^{-}=1$, we set

$$
\begin{aligned}
& \frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=j, \phi_{k+1}^{(n)}=A \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta t^{(n)}}=\lambda_{b}^{+} \mathbb{1}_{\left\{x_{j}^{(n)}=J_{b}^{+}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)\right\}^{+\lambda_{b}^{-}} \mathbb{1}_{\left\{x_{j}^{(n)}=J_{b}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)\right\}}}^{\frac{\mathbb{P}\left[\xi_{k+1}^{(n)}=j, \phi_{k+1}^{(n)}=C \mid \mathcal{F}_{k}^{(n)}\right]}{\Delta t^{(n)}}=\lambda_{a}^{+} \mathbb{1}_{\left\{x_{j}^{(n)}=J_{a}^{+}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)\right\}^{+}+\lambda_{a}^{-} \mathbb{1}_{\left\{x_{j}^{(n)}=J_{a}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)\right\}}} .} \begin{array}{l}
\end{array},
\end{aligned}
$$

Hence, with probability $\Delta t^{(n)}$ a large price change occurs. These choices of probabilities for the large price jumps yield the following feedback functions: for $I=b, a$, we have

$$
\begin{aligned}
K_{I}\left(S_{k}^{(n)}, Y_{k}^{(n)}, d x\right) & =\lambda_{I}^{-} \varepsilon_{J_{I}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)}(d x)+\lambda_{I}^{+} \varepsilon_{J_{I}^{+}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)}(d x) \\
Q_{I}(d x) & =\lambda_{I}^{-} \varepsilon_{-1}(d x)+\lambda_{I}^{+} \varepsilon_{1}(d x)
\end{aligned}
$$

and

$$
\begin{aligned}
& \theta_{I}\left(S_{k}^{(n)}, Y_{k}^{(n)}, x\right) \\
& \quad= \begin{cases}J_{I}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right) & : x \in(-\infty,-1] \\
J_{I}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)+\frac{x+1}{2}\left\{J_{I}^{+}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)-J_{I}^{-}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right)\right\} & : x \in(-1,1) \\
J_{I}^{+}\left(S_{k}^{(n)}, Y_{k}^{(n)}\right) & : x \in[1, \infty)\end{cases}
\end{aligned}
$$

In the following, we denote by $p_{k+1}^{(n)}:=\mathbb{P}\left[\phi_{k+1}^{(n)} \in\{A, C\} \mid \mathcal{F}_{k}^{(n)}\right]$ the conditional probability of a price changing event at time $t_{k+1}^{(n)}$ which is uniquely determined by the previous equations. Now, let us turn to the limit order placements. For simplicity, we assume that they are always of size 10 and are normally distributed around the best bid respectively ask price, i.e.,

$$
\begin{aligned}
& \mathbb{P}\left[\phi_{k+1}^{(n)}=B, \omega_{k+1}^{(n)}=10, \pi_{k+1}^{(n)} \in d y \mid \mathcal{F}_{k}^{(n)}\right]=\left(1-p_{k+1}^{(n)}\right)\left(1-\operatorname{Im}_{k}^{(n)}\right) \frac{1}{2 \pi} e^{-y^{2}} d y \\
& \mathbb{P}\left[\phi_{k+1}^{(n)}=D, \omega_{k+1}^{(n)}=10, \pi_{k+1}^{(n)} \in d y \mid \mathcal{F}_{k}^{(n)}\right]=\left(1-p_{k+1}^{(n)}\right) \operatorname{Im}_{k}^{(n)} \frac{1}{2 \pi} e^{-y^{2}} d y
\end{aligned}
$$

Moreover, cancellation of volume is supposed to be proportional to the current volume: for all $x \leqslant 0$,

$$
\begin{aligned}
& \frac{\mathbb{P}\left[\phi_{k+1}^{(n)}=B, \omega_{k+1}^{(n)} \in d x, \pi_{k+1}^{(n)} \in d y \mid \mathcal{F}_{k}^{(n)}\right]}{1-p_{k+1}^{(n)}}=\frac{\operatorname{Im}_{k}^{(n)}}{v_{b, k}^{(n)}(y)} \mathbb{1}_{\left[-v_{b, k}^{(n)}(y), 0\right]}(x) \frac{1}{2 \pi} e^{-y^{2}} d x d y \\
& \left.\frac{\mathbb{P}\left[\phi_{k+1}^{(n)}=D, \omega_{k+1}^{(n)} \in d x, \pi_{k+1}^{(n)} \in d y \mid \mathcal{F}_{k}^{(n)}\right]}{1-p_{k+1}^{(n)}}=\frac{1-\operatorname{Im}_{k}^{(n)}}{v_{a, k}^{(n)}(y)} \mathbb{1}_{\left[-v_{a, k}^{(n)}(y), 0\right.}\right]^{(x)} \frac{1}{2 \pi} e^{-y^{2}} d x d y
\end{aligned}
$$

We have chosen the limit order placements and cancellations in such a way that a

### 1.3. SIMULATION STUDY

high imbalance factor results in more order placements at the ask side and more order cancellations at the bid side, while a small imbalance factor leads to more order placements at the bid side and more order cancellations at the ask side. This induces an equalizing effect. Supposing $\Delta v^{(n)}=\Delta t^{(n)}$, the coefficient functions of the relative volume densities are given by

$$
\begin{aligned}
f_{b}^{(n)}\left(S_{k}^{(n)}, x\right) & =\frac{1-p_{k+1}^{(n)}}{\Delta x^{(n)}} \frac{1}{2 \pi} \int_{I^{(n)}(x)}\left\{10 \cdot\left(1-\operatorname{Im}_{k}^{(n)}\right)-\frac{v_{b, k}^{(n)}(y)}{2} \operatorname{Im}_{k}^{(n)}\right\} e^{-y^{2}} d y, \\
f_{a}^{(n)}\left(S_{k}^{(n)}, x\right) & =\frac{1-p_{k+1}^{(n)}}{\Delta x^{(n)}} \frac{1}{2 \pi} \int_{I^{(n)}(x)}\left\{10 \cdot \operatorname{Im}_{k}^{(n)}-\frac{v_{a, k}(y)}{2}\left(1-\operatorname{Im}_{k}^{(n)}\right)\right\} e^{-y^{2}} d y .
\end{aligned}
$$

We run two different simulations of the above specified model. For both, we suppose that $\Delta x^{(n)}=n^{-1}, \Delta p^{(n)}=n^{-1 / 2}, \Delta t^{(n)}=\left(\Delta x^{(n)}\right)^{2} \Delta p^{(n)}=n^{-5 / 2}$ and choose

$$
n=100, \quad h=0.55, \quad \gamma_{1}=1, \quad \eta_{2}=100, \quad T=2 .
$$

In a first simulation we further choose $\eta_{1}=0$ and $\lambda_{b}^{-}=1, \lambda_{b}^{+}=\lambda_{a}^{-}=\lambda_{a}^{+}=0$, i.e., only downward jumps at the best bid price are possible and there is no external factor. Moreover, we start with a bid price $B_{0}=6.9$, an ask price $A_{0}=7$, and with a limit order book that has a severe imbalance at time $t=0$ : standing volumes at the bid side are much higher than standing volumes at the ask side, i.e., we choose
$v_{0, b}(x)=0.0075(x-4)^{2}(x+4)^{2} \mathbb{1}_{[-4,4]}(x), \quad v_{a, 0}(x)=0.0025(x-4)^{2}(x+4)^{2} \mathbb{1}_{[-4,4]}(x)$.
For these parameter values, Figure 1.5 shows the evolution of the best bid and ask prices over time, while Figure 1.6 depicts the evolution of the absolute volume density functions $u_{b}$ and $u_{a}$ of the visible book over time, where

$$
u_{b}(x)=v_{b}(-x+B) \mathbb{1}_{\{x \leqslant B\}}, \quad u_{a}(x)=v_{a}(x-A) \mathbb{1}_{\{x \geqslant A\}} .
$$



Figure 1.5: The evolution of the best bid (blue) and the best ask price (yellow).

The evolution of the prices is influenced by the spread as well as the imbalance factor. We start with a quite small spread and a large imbalance factor implying very small price volatilities, a slightly negative drift of the best bid price, and a positive drift of the best ask price. Hence, the prices move apart from each other. At $t \approx 0.4$, we observe a price drop in the best bid price which heavily increases the spread between bid and ask. Therefore, both prices become more volatile, but imbalances are still significant. After a second price drop in the best bid price at $t \approx 0.7$, the huge spread now dominates the price evolution and the spread decreases. In the last quarter, we observe a similar price evolution of the best bid and ask price, which is caused by the fact that the imbalance factor and the spread stabilize around 0.5 and $\ln (2)$, respectively.

In Figure 1.6 we observe that the cumulative volumes at the top of both sides of the limit order book converge. In this simulation study the sell side volume approaches the bid side volume because we have chosen the size of the order placements much greater than the size of average cancellations for the initial volume density functions. If placements would be of smaller size, the opposite effect could be observed, i.e., the buy side volumes at the top of the book would decrease to approach the sell side volumes at the top of the book. Moreover, the price drops in the bid price lead to a significant decrease of order volumes at the top of the bid side and hence to a decrease of the volume imbalance factor, which subsequently forces the spread to narrow again.


Figure 1.6: The evolution of the bid side (left) and ask side (right) volume density functions (in absolute coordinates; visible books only).

In a second simulation, starting from the same initial values, we allow jumps in all directions $\left(\lambda_{b}^{+}=\lambda_{a}^{-}=0.15, \lambda_{b}^{-}=\lambda_{a}^{+}=0.35\right)$ and assume a rather strong dependence of the jump sizes on the external factor $Y^{(n)}$ by choosing $\eta_{1}=9$ and $\kappa=10$, i.e., after $Y^{(n)}$ hits the threshold, the absolute value of the jump sizes increases by the factor 10 . We depict the corresponding bid and ask price evolution of two runs of our simulation in Figure 1.7. In both runs $Y^{(n)}$ hits the threshold shortly after $t=1$.


Figure 1.7: The evolution of the best bid (blue) and the best ask price (yellow).

### 1.4 Examples of large price jumps

In this section we provide three examples of jump distributions that satisfy the rather technical Assumption 1.6. They are toy examples and not meant to mimic asset price jumps observed in real data, but rather to illustrate the range of jump distributions supported by our model.
Example 1.2 (Jumps at the ask follow a state-dependent shifted continuous distribution). Let $M>0$ and suppose that $\delta_{n}=\Delta x^{(n)}$ and $\left(\Delta x^{(n)}\right)^{-1} \in \mathbb{N}$. Fix some continuous distribution function $F$ on $\mathcal{B}([-M, M])$ and two Lipschitz continuous, bounded functions $\mu, \sigma: E \rightarrow \mathbb{R}$ such that $\sigma(s) \geqslant 1$ for all $s \in E$. If $j \in \mathbb{Z} \backslash\{-1,0,1\}$ is such that there exists an $i \in \mathbb{Z}_{M}^{(n)}$ with

$$
\sigma(s) x_{i}^{(n)} \leqslant x_{j}^{(n)}-\mu(s)<\sigma(s) x_{i}^{(n)}+\Delta x^{(n)},
$$

the distribution of the large price jumps at the best ask is given by

$$
k_{a, j}^{(n)}(s)=F\left(\left[x_{i}^{(n)}, x_{i+1}^{(n)}\right) \cap[-M, M]\right) .
$$

If none such $i \in \mathbb{Z}$ exists, we set $k_{a, j}^{(n)}(s)=0$. Note that all $k_{a, j}^{(n)}$ 's are well-defined as the intervals $\left[\sigma(s) x_{i}^{(n)}, \sigma(s) x_{i}^{(n)}+\Delta x^{(n)}\right), i \in \mathbb{N}$, are non-overlapping due to $\sigma(s) \geqslant 1$. Then define for all $A \in \mathcal{B}(\mathbb{R})$,

$$
K_{a}(s, A):=F\left(\left\{\frac{x-\mu(s)}{\sigma(s)}: x \in A\right\} \cap[-M, M]\right), \quad Q_{a}(A):=F(A \cap[-M, M]),
$$

and for all $x \in[-M, M]$,

$$
\theta_{a}(s, x):=\sigma(s) x+\mu(s) .
$$

Then, for all $A \in \mathcal{B}([-M, M])$,

$$
K_{a}\left(s, \theta_{a}(s, A)\right)=F(A)=Q_{a}(A) .
$$

Hence, Assumption 1.6 ii) is satisfied. By the Lipschitz continuity of $\mu, \sigma: E \rightarrow \mathbb{R}$, we have for all $s, \widetilde{s} \in E$ and $x \in[-M, M]$

$$
\left|\theta_{a}(s, x)-\theta_{a}(\widetilde{s}, x)\right| \leqslant|\mu(s)-\mu(\widetilde{s})|+M|\sigma(s)-\sigma(\widetilde{s})| \leqslant L(1+M)\|s-\widetilde{s}\|_{E}
$$

and therefore, Assumption 6 iv$)$ holds true. By construction, note that also the Assumptions 1.6 i) and v) are satisfied. Hence, we can apply Lemma 1.6.1 and obtain for all $s \in E$ and $j \in \mathbb{Z}$,

$$
\begin{aligned}
& K_{a}\left(s, \theta_{a}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)-K_{a}^{(n)}\left(s, \theta_{a}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right) \\
& =F\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right) \cap[-M, M]\right)-K_{a}^{(n)}\left(s,\left[\sigma(s) x_{j}^{(n)}+\mu(s), \sigma(s) x_{j}^{(n)}+\mu(s)+\Delta x^{(n)}\right)\right) \\
& =0
\end{aligned}
$$

yielding the validity of Assumption 1.6 iii). Finally, as $n \rightarrow \infty$,

$$
\begin{aligned}
& \sup _{s \in E} \sum_{j} \int_{\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)} \mid \mathbb{1}_{\left\{\theta_{a}^{(n)}\left(s, x_{j}^{(n)}\right)=0\right\}^{-\mathbb{1}_{\left\{\theta_{a}(s, x)=0\right\}} \mid} \mid Q_{a}(d x)} \quad=\sup _{s \in E} \sum_{j} \mathbb{1}_{\left\{\theta_{a}^{(n)}\left(s, x_{j}^{(n)}\right)=0\right\}^{Q_{a}}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)} \\
& \quad=\sup _{s \in E} \sum_{j} \mathbb{1}_{\left\{\sigma(s) x_{j}^{(n)}+\mu(s) \in\left(-\Delta x^{(n)}, 0\right]\right\}} Q_{a}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right) \\
& \quad \leqslant \sup _{j} Q_{a}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right) \rightarrow 0 .
\end{aligned}
$$

Therefore, Assumption 1.6 vi) holds true as well.
Example 1.2 shows that jump intensities can follow quite general distribution functions. Moreover, observe that if $\mu(s)-M \sigma(s)>b-a$ for all $s=\left(b, v_{b}, a, v_{a}, t\right) \in E$, then the (negative) jumps of the ask price will never lead to a crossing of bid and ask prices. While $\theta_{a}(s, \cdot)$ is linear for all $s \in E$ in the above example, we note that in general also non-linear transformations are possible. In the next example we discuss a non-linear transformation of the jump sizes.

Example 1.3 (Large proportional price drops at the bid, if bid prices are greater or equal to one). Let $M>1 / 2$ and suppose that $\left(\Delta x^{(n)}\right)^{-1} \in 2 \mathbb{N}$. Further, we set $E_{b \geqslant 1}:=\left\{\left(b, v_{b}, a, v_{a}, t\right) \in E: b \geqslant 1\right\}$. Then, for all $j \in \mathbb{Z}$ and $s \in E$, we describe the distribution of the large price jumps at the best bid price by

$$
K_{b}^{(n)}\left(s,\left\{x_{j}^{(n)}\right\}\right)=\left\{\begin{array}{ll}
\mathbb{1}\left\{x_{j}^{(n)} \in\left[-\frac{b \wedge 2 M}{2},-\frac{b \wedge 2 M}{2}+\Delta x^{(n)}\right)\right\} & : s \in E_{b \geqslant 1} \\
0 & : \text { else }
\end{array} .\right.
$$

### 1.4. EXAMPLES OF LARGE PRICE JUMPS

For all $A \in \mathcal{B}(\mathbb{R})$, we set

$$
K_{b}(s, A)= \begin{cases}\varepsilon_{-1 / 2}\left(\left\{\frac{x}{b \wedge 2 M}: x \in A\right\}\right) & : s \in E_{b \geqslant 1}, \quad Q_{b}(d x)=\varepsilon_{-1 / 2}(d x), \\ 0 & : \text { else }\end{cases}
$$

and for all $x \in[-M, M]$,

$$
\theta_{b}(s, x)=\left(b \mathbb{1}_{\{b \geqslant 1\}} \wedge 2 M\right) x .
$$

Then we have for $b \geqslant 1$,

$$
\begin{aligned}
K_{b}\left(s, \theta_{b}(s, A)\right) & +Q_{b}\left(\left\{x \in A: \theta_{b}(s, x)=0\right\}\right) \\
& =K_{b}\left(s, \theta_{b}(s, A)\right)=K_{b}(s,\{(b \wedge 2 M) x: x \in A\})=\varepsilon_{-1 / 2}(A)=Q_{b}(A)
\end{aligned}
$$

and for $b<1$,

$$
K_{b}\left(s, \theta_{b}(s, A)\right)+Q_{b}\left(\left\{x \in A: \theta_{b}(s, x)=0\right\}\right)=Q_{b}\left(\left\{x \in A: \theta_{b}(s, x)=0\right\}\right)=Q_{b}(A) .
$$

Hence, Assumption 1.6 ii) is satisfied. By construction, note that also the Assumptions $(1.6 i)$, iv), and $v$ ) are fulfilled. Furthermore, by Lemma 1.6.1 we have

$$
\begin{aligned}
& \sup _{s \in E} \sum_{j}\left|K_{b}\left(s, \theta_{b}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)-K_{b}^{(n)}\left(s, \theta_{b}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right| \\
& =\sup _{s \in E_{b \geqslant 1} \cap E} \sum_{j} \left\lvert\, \mathbb{1}_{\left\{-\frac{b \wedge \wedge M}{2} \in\left[(b \wedge 2 M) x_{j}^{(n)},(b \wedge 2 M) x_{j+1}^{(n)}\right)\right\}}\right. \\
& \left.\quad-\mathbb{1}\left\{\theta_{b}^{(n)}\left(s, x_{j}^{(n)}\right) \in\left[-\frac{b \wedge 2 M}{2},-\frac{b \wedge 2 M}{2}+\Delta x^{(n)}\right)\right\} \right\rvert\, \\
& =\sup _{s \in E_{b \geqslant 1} \cap E} \sum_{j}\left|\mathbb{1}_{\left\{x_{j}^{(n)}=-\frac{1}{2}\right\}}-\mathbb{1}\left\{(b \wedge 2 M) x_{j}^{(n)} \in\left[-\frac{b \wedge 2 M}{2},-\frac{b \wedge 2 M}{2}+\Delta x^{(n)}\right)\right\}\right|=0
\end{aligned}
$$

and hence also Assumption iii) is satisfied. Finally, Assumption 1.6 vi) holds true since

$$
\begin{aligned}
& \sup _{s \in E} \sum_{j} \int_{\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)}\left|\mathbb{1}_{\left\{\theta_{b}^{(n)}\left(s, x_{j}^{(n)}\right)=0\right\}}-\mathbb{1}_{\left\{\theta_{b}(s, x)=0\right\}}\right| Q_{b}(d x) \\
& \left.\quad=\left.\sup _{s \in E}\right|^{\mathbb{1}}\left\{\theta_{b}^{(n)}\left(s,-\frac{1}{2}\right)=0\right\}^{-\mathbb{1}}\left\{\theta_{b}\left(s,-\frac{1}{2}\right)=0\right\} \right\rvert\,=0 .
\end{aligned}
$$

While in the above two examples the driving jump distribution in the pre-limit does not depend on $n$, the next example shows that this is generally possible. Indeed, even a slight dependence on $s \in E$ in the pre-limit is allowed as long as it vanishes for $n \rightarrow \infty$.

Example 1.4 (Poisson approximation to binomial-distributed jumps at the ask, bounded by state-dependent levels). Suppose that $\left(\Delta x^{(n)}\right)^{-1} \in \mathbb{N}$ and $M \in \mathbb{N}$. Let $p_{n}: E \rightarrow(0,1)$ and $m_{0}, M_{0}: E \rightarrow \mathbb{N}$ satisfy

$$
\begin{aligned}
\sup _{s \in E}\left|n p_{n}(s)-\lambda\right| \rightarrow 0, & M \geqslant M_{0}(s) \geqslant m_{0}(s) \geqslant 1 \quad \forall s \in E \\
\left|m_{0}(s)-m_{0}(\widetilde{s})\right| \leqslant L\|s-\widetilde{s}\|_{E}, & \left|M_{0}(s)-M_{0}(\widetilde{s})\right| \leqslant L\|s-\widetilde{s}\|_{E} \quad \forall s, \widetilde{s} \in E .
\end{aligned}
$$

For $s \in E$ let the distribution of the large price jumps at the ask be described by

$$
\begin{aligned}
& K_{a}^{(n)}(s, d x) \\
& :=\sum_{k=m_{0}(s)}^{M_{0}(s)}\binom{n}{k+M-M_{0}(s)}\left(p_{n}(s)\right)^{k+M-M_{0}(s)}\left(1-p_{n}(s)\right)^{n-k-M+M_{0}(s)} \varepsilon_{k}(d x) .
\end{aligned}
$$

Then,

$$
K_{a}(s, d x):=\sum_{k=m_{0}(s)}^{M_{0}(s)} e^{-\lambda} \frac{\lambda^{k+M-M_{0}(s)}}{\left(k+M-M_{0}(s)\right)!} \varepsilon_{k}(d x), \quad Q_{a}(d x):=\sum_{k=1}^{M} e^{-\lambda} \frac{\lambda^{k}}{k!} \varepsilon_{k}(d x)
$$

and for $\bar{M}_{0}(s):=M-M_{0}(s)+m_{0}(s)$,

$$
\theta_{a}(s, x):= \begin{cases}x-M+M_{0}(s) & : x \in\left[\bar{M}_{0}(s), M\right] \\ m_{0}(s) \cdot\left(x-\bar{M}_{0}(s)+1\right) & : x \in\left(\bar{M}_{0}(s)-1, \bar{M}_{0}(s)\right) \\ 0 & : x \in\left[-M, \bar{M}_{0}(s)-1\right]\end{cases}
$$

satisfy for any $A \in \mathcal{B}([-M, M])$,

$$
K_{a}\left(s, \theta_{a}(s, A)\right)=K_{a}\left(s,\left\{x-M+M_{0}(s): x \in A\right\}\right)=Q_{a}\left(A \cap\left[\bar{M}_{0}(s), M\right]\right)
$$

and
$Q_{a}\left(\left\{x \in A: \theta_{a}(s, x)=0\right\}\right)=Q_{a}\left(A \cap\left[1, \bar{M}_{0}(s)-1\right]\right)=Q_{a}(A)-Q_{a}\left(A \cap\left[\bar{M}_{0}(s), M\right]\right)$.
Hence, Assumption 1.6 ii) is satisfied. By construction also Assumptions 1.6 i) and v) hold. Moreover,

$$
\begin{aligned}
& \sup _{s \in E} \sum_{j}\left|K_{a}\left(s, \theta_{a}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)-K_{a}^{(n)}\left(s, \theta_{a}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right| \\
& \quad=\sup _{s \in E} \sum_{k=\bar{M}_{0}(s)}^{M}\left|e^{-\lambda} \frac{\lambda^{k}}{k!}-\binom{n}{k}\left(p_{n}(s)\right)^{k}\left(1-p_{n}(s)\right)^{n-k}\right| \leqslant \sup _{s \in E} 2 n\left(p_{n}(s)\right)^{2} \rightarrow 0
\end{aligned}
$$

i.e., Assumption 1.6 iii) is satisfied. If $s, \widetilde{s} \in E$ satisfy $M_{0}(s)-m_{0}(s)=M_{0}(\widetilde{s})-m_{0}(\widetilde{s})$,
then
$\left|\theta_{a}(s, x)-\theta_{a}(\widetilde{s}, x)\right| \leqslant\left|M_{0}(s)-M_{0}(\widetilde{s})\right|+\left|m_{0}(s)-m_{0}(\widetilde{s})\right| \leqslant 2 L\|s-\widetilde{s}\|_{E} \quad \forall x \in[-M, M]$.
If $s, \widetilde{s} \in E$ satisfy $M_{0}(s)-m_{0}(s)>M_{0}(\widetilde{s})-m_{0}(\widetilde{s})$, then we have for all $x \in[-M, M]$ the estimate

$$
\begin{aligned}
\left|\theta_{a}(s, x)-\theta_{a}(\widetilde{s}, x)\right| & \leqslant\left|M_{0}(s)-M_{0}(\widetilde{s})\right|+M \varepsilon_{(\bar{M}(s)-1, \bar{M}(\widetilde{s}))}(\{x\}) \\
& \leqslant\left|M_{0}(s)-M_{0}(\widetilde{s})\right|+M\left|m_{0}(\widetilde{s})-M_{0}(\widetilde{s})-m_{0}(s)+M_{0}(s)+1\right| \\
& \leqslant(1+2 M)\left|M_{0}(s)-M_{0}(\widetilde{s})\right|+2 M\left|m_{0}(\widetilde{s})-m_{0}(s)\right| \\
& \leqslant(1+4 M) L\|s-\widetilde{s}\|_{E}
\end{aligned}
$$

Hence, Assumption 1.6 iv) holds. Finally, we note that for all $x_{j}^{(n)} \in \mathbb{N}$ we have $\theta_{a}^{(n)}\left(s, x_{j}^{(n)}\right)=\theta_{a}\left(s, x_{j}^{(n)}\right)$. As $Q_{a}$ only charges $\mathbb{N}$, this shows that Assumption 1.6 vi) is also satisfied.

Example 1.4 illustrates very well the restrictions imposed on the limiting jump distribution through Assumption 1.6. while the range of jump sizes (parameterized through $m_{0}$ and $M_{0}$ ) can differ across states, the $\lambda$ determining the jump intensities has to be constant and cannot depend on the state $s \in E$. This is necessary to obtain jump diffusion dynamics - rather than more general (and even more complicated) semimartingale dynamics - in the high-frequency limit.

### 1.5 Proof sketch of the main theorem

In this section, we present a step-by-step proof sketch of Theorem 1.2.6. It should give the reader an overview about the proof strategy and the main methods. The technical details can be found in Section 1.6 .

## Step 1: State and time separation

By making use of the time change theorem, we can simplify our subsequent analysis to equidistant, deterministic order arrival times, where the time intervals between two consecutive order arrivals are of length $\Delta t^{(n)}$. To this end, we set $t_{k}^{(n)}:=k \Delta t^{(n)}$ for $k \in \mathbb{N}_{0}, T_{n}:=\left\lfloor T / \Delta t^{(n)}\right\rfloor$, and define the state process $\eta^{(n)}$ via

$$
\eta^{(n)}(t):=\sum_{k=0}^{T_{n}} S_{k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t), \quad t \in[0, T]
$$

Introducing the process

$$
\tau^{\eta,(n)}(u):=\sum_{k=0}^{T_{n}} \tau_{k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(u), \quad u \in[0, T]
$$

we define the time process $\zeta^{(n)}$ via

$$
\zeta^{(n)}(t):=\inf \left\{u>0: \tau^{\eta,(n)}(u)>t\right\} \wedge\left(T_{n}+1\right) \Delta t^{(n)}, \quad t \in[0, T]
$$

The advantage of the state and time separation is that we can focus on first analyzing the convergence of the state process $\eta^{(n)}$, for which we will prove the following convergence result (cf. Steps 2-5 below).

Proposition 1.5.1. Let the assumptions of Theorem 1.2.6 be satisfied. Then, $\eta^{(n)}$ converges weakly in the Skorokhod topology to $\eta=\left(B^{\eta}, v_{b}^{\eta}, A^{\eta}, v_{a}^{\eta}, \tau^{\eta}\right)$ being the unique strong solution of the coupled diffusion-fluid system in 1.2.13.

Let us now define the composition of the state process $\eta^{(n)}$ with the time process $\zeta^{(n)}$ as

$$
S^{(n), *}(t):=\eta^{(n)}\left(\zeta^{(n)}(t)-\Delta t^{(n)}\right), \quad t \in[0, T]
$$

Relying on a time change argument for processes with discontinuities, our main result readily follows from statements ii) and iii) of the following corollary.

Corollary 1.5.2. Let Assumptions $1.1-1.7$ be satisfied. Then,
i) $\zeta^{(n)} \Rightarrow \zeta$ in the Skorokhod topology, where $\zeta^{-1}(t)=\tau^{\eta}(t)=\int_{0}^{t} \varphi(\eta(u)) d u$,
ii) $S^{(n), *} \Rightarrow \eta \circ \zeta=: S$ in the Skorokhod topology, and
iii) as $n \rightarrow \infty$,

$$
\mathbb{P}\left[\sup _{t \in[0, T]}\left\|S^{(n), *}(t)-S^{(n)}(t)\right\|_{E}>0\right] \rightarrow 0
$$

In the subsequent steps we present a sketch of the proof of Proposition 1.5.1. To this end, let us define $\eta^{(n)}:=\left(B^{\eta,(n)}, v_{b}^{\eta,(n)}, A^{\eta,(n)}, v_{a}^{\eta,(n)}, \tau^{\eta,(n)}\right)$, where for $t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right) \cap$ $[0, T]$,

$$
\begin{align*}
B^{\eta,(n)}(t) & =B_{k}^{(n)},
\end{align*} \quad v_{b}^{\eta,(n)}(t, \cdot)=v_{b, k}^{(n)}, ~ l o v_{k}^{(n)}, \quad v_{a}^{\eta,(n)}(t, \cdot)=v_{a, k}^{(n)}, \quad \tau^{\eta,(n)}(t)=\tau_{k}^{(n)} .
$$

## Step 2: Representation of the LOB-dynamics as a stochastic difference equation and convergence of its integrators

To prove Proposition 1.5.1, we will apply results of Kurtz and Protter [58 about the convergence of stochastic differential equations in infinite dimension. To this end, we rewrite the discrete-time dynamics of $\eta^{(n)}$ in the form of a proper stochastic difference equation, whose driving processes converge to limit processes which are independent of the order book sequence. First, we decompose
$B^{\eta,(n)}(t)=B_{0}^{(n)}+B^{\eta, s,(n)}(t)+B^{\eta, \ell,(n)}(t), \quad A^{\eta,(n)}(t)=A_{0}^{(n)}+A^{\eta, s,(n)}(t)+A^{\eta, \ell,(n)}(t)$,
where $B^{\eta, s,(n)}, A^{\eta, s,(n)}$ and $B^{\eta, \ell,(n)}, A^{\eta, \ell,(n)}$ describe the price dynamics of the small and large price changes, respectively, i.e.,

$$
\begin{aligned}
& B^{\eta, s,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \delta B_{k}^{(n)} \mathbb{1}_{\left\{\left|\delta B_{k}^{(n)}\right| \leqslant \delta_{n}\right\}}, \quad A^{\eta, s,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \delta A_{k}^{(n)} \mathbb{1}_{\left\{\left|\delta A_{k}^{(n)}\right| \leqslant \delta_{n}\right\}}, \\
& B^{\eta, \ell,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \delta B_{k}^{(n)} \mathbb{1}\left\{\left|\delta B_{k}^{(n)}\right|>\delta_{n}\right\}, \quad A^{\eta, \ell,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \delta A_{k}^{(n)} \mathbb{1}\left\{\left|\delta A_{k}^{(n)}\right|>\delta_{n}\right\} .
\end{aligned}
$$

Next, observe that we can write the price dynamics of the small price changes $B^{\eta, s,(n)}$ and $A^{\eta, s,(n)}$ as

$$
\begin{align*}
& B^{\eta, s,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor}\left\{p_{b}^{(n)}\left(S_{k-1}^{(n)}\right) \Delta t^{(n)}+r_{b}^{(n)}\left(S_{k-1}^{(n)}\right) \delta Z_{b, k}^{(n)}\right\} \\
& A^{\eta, s,(n)}(t):=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor}\left\{p_{a}^{(n)}\left(S_{k-1}^{(n)}\right) \Delta t^{(n)}+r_{a}^{(n)}\left(S_{k-1}^{(n)}\right) \delta Z_{a, k}^{(n)}\right\}, \tag{1.5.2}
\end{align*}
$$

where $Z_{I, k}^{(n)}:=\sum_{j=1}^{k} \delta Z_{I, j}^{(n)}$ for $I=b, a$ and

$$
\begin{aligned}
\delta Z_{b, j}^{(n)} & :=\frac{\delta B_{j}^{(n)} \mathbb{1}_{\left\{0<\left|\delta B_{j}^{(n)}\right| \leqslant \delta_{n}\right\}}-\Delta t^{(n)} p_{b}^{(n)}\left(S_{j-1}^{(n)}\right)}{r_{b}^{(n)}\left(S_{j-1}^{(n)}\right)}, \\
\delta Z_{a, j}^{(n)} & :=\frac{\delta A_{j}^{(n)} \mathbb{1}_{\left\{0<\left|\delta A_{j}^{(n)}\right| \leqslant \delta_{n}\right\}}-\Delta t^{(n)} p_{a}^{(n)}\left(S_{j-1}^{(n)}\right)}{r_{a}^{(n)}\left(S_{j-1}^{(n)}\right)}
\end{aligned}
$$

Then, Proposition 1.5 .3 states that the processes

$$
\begin{equation*}
Z_{b}^{(n)}(t):=\sum_{k=1}^{T_{n}} Z_{b, k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right.}(t), Z_{a}^{(n)}(t):=\sum_{k=1}^{T_{n}} Z_{a, k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t) \tag{1.5.3}
\end{equation*}
$$

for $t \in[0, T]$, converge to two independent Brownian motions and therefore the integrators in 1.5 .2 converge to processes that are independent of the order book dynamics.

Proposition 1.5.3. Let Assumptions 1.4 and 1.7 be satisfied. Then, as $n \rightarrow \infty$, the process $\left(Z_{b}^{(n)}, Z_{a}^{(n)}\right)$ converges weakly in the Skorokhod topology to a standard planar Brownian motion $\left(Z_{b}, Z_{a}\right)$. In particular, $Z_{b}$ and $Z_{a}$ are independent.

Let us now turn to the processes $B^{\eta, \ell,(n)}, A^{\eta, \ell,(n)}$ corresponding to the price dynamics
of the large price changes. First, note that their joint jump measure is given by

$$
\begin{aligned}
& \mu^{\eta,(n)}([0, t], d x, d y) \\
&:=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor}\left[\mathbb{1}_{\left\{\left|\delta B_{k}^{(n)}\right|>\delta_{n}\right\}^{\varepsilon}}^{\left(\delta B_{k}^{(n)}, 0\right)}\right. \\
&(d x, d y)+\mathbb{1}_{\left\{\left|\delta A_{k}^{(n)}\right|>\delta_{n}\right\}^{\varepsilon}}^{\left(0, \delta A_{k}^{(n)}\right)} \\
&(d x, d y)]
\end{aligned}
$$

for all $t \in[0, T]$, since $A$ and $C$ events do not occur simultaneously. Setting for all $t \in[0, T]$,

$$
\mu_{b}^{\eta,(n)}([0, t], d x):=\mu^{\eta,(n)}([0, t], d x,\{0\}), \quad \mu_{a}^{\eta,(n)}([0, t], d y):=\mu^{\eta,(n)}([0, t],\{0\}, d y)
$$

we can rewrite $B^{\eta, \ell,(n)}$ and $A^{\eta, \ell,(n)}$ as

$$
B^{\eta, \ell,(n)}(t)=\int_{\mathbb{R}} x \mu_{b}^{\eta,(n)}([0, t], d x) \quad \text { and } \quad A^{\eta, \ell,(n)}(t)=\int_{\mathbb{R}} y \mu_{a}^{\eta,(n)}([0, t], d y)
$$

for all $t \in[0, T]$. Then the compensator $\nu^{\eta,(n)}$ of $\mu^{\eta,(n)}$ is given by

$$
\begin{aligned}
\nu^{\eta,(n)} & ([0, t], d x, d y) \\
& :=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor}\left[\Delta t^{(n)} K_{b}^{(n)}\left(S_{k-1}^{(n)}, d x\right) \varepsilon_{0}(d y)+\Delta t^{(n)} K_{a}^{(n)}\left(S_{k-1}^{(n)}, d y\right) \varepsilon_{0}(d x)\right]
\end{aligned}
$$

for $t \in[0, T]$. Similarly equation 1.2 .10 in Assumption 1.6 will be our starting point to construct a sequence of discrete-time integrators, which converges to a limit independent of the order book dynamics. Given the finite measures $Q_{b}, Q_{a}$ introduced in Assumption 1.6 for technical convenience we assume for all $n \in \mathbb{N}$ the existence of two independent, homogeneous Poisson random measures $\mu_{I}^{Q,(n)}$, on $\mathcal{B}^{(n)}$, for $I=b$, $a$, with intensity measures $Q_{I}, I=b, a$. For all $n \in \mathbb{N}$ we now consider the random jump measure

$$
\mu^{J^{(n)}}(d t, d x, d y):=\mu_{b}^{J^{(n)}}(d t, d x) \varepsilon_{0}(d y)+\mu_{a}^{J^{(n)}}(d t, d y) \varepsilon_{0}(d x)
$$

where for $I=b, a, t \in[0, T]$, and $A \in \mathcal{B}([-M, M])$ we define

$$
\begin{align*}
\mu_{I}^{J^{(n)}}([0, t], A):= & \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \sum_{j: x_{j}^{(n)} \in A} \mu_{I}^{\eta,(n)}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)\right\}\right) \\
& +\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \sum_{j: x_{j}^{(n)} \in A} \mu_{I}^{Q,(n)}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right) \mathbb{1}_{\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)=0\right\}} \tag{1.5.4}
\end{align*}
$$

Then the compensator of $\mu^{J^{(n)}}$ is given by

$$
\nu^{J^{(n)}}(d t, d x, d y):=\nu_{b}^{J^{(n)}}(d t, d x) \varepsilon_{0}(d y)+\nu_{a}^{J^{(n)}}(d t, d y) \varepsilon_{0}(d x)
$$

with

$$
\begin{align*}
\nu_{I}^{J(n)}([0, t], A) & :=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \sum_{j: x_{j}^{(n)} \in A} \nu_{I}^{\eta,(n)}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)\right\}\right)  \tag{1.5.5}\\
& +\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j} \mathbb{1}_{\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)=0\right.} Q_{I}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)
\end{align*}
$$

being the compensator of $\mu_{I}^{J^{(n)}}$ for $I=b, a$ and $(t, A) \in[0, T] \times \mathcal{B}([-M, M])$. Here, $\nu_{I}^{\eta,(n)}$ denotes the compensator of $\mu_{I}^{\eta,(n)}$ for $I=b, a$. Now, as stated in Lemma 1.5.4 below, we found a representation of $B^{\eta, \ell,(n)}, A^{\eta, \ell,(n)}$ in terms of the coefficient functions $\theta_{b}^{(n)}, \theta_{a}^{(n)}$ introduced in Assumption 1.6 and the random jump measures $\mu_{b}^{J^{(n)}}, \mu_{a}^{J^{(n)}}$ introduced in (1.5.4).

Lemma 1.5.4. Let Assumptions 1.4 and 1.6 hold. Then,

$$
\begin{align*}
B^{\eta, \ell,(n)}(t) & =\int_{0}^{t} \int_{[-M, M]} \theta_{b}^{(n)}\left(\eta^{(n)}(u-), y\right) \mu_{b}^{J^{(n)}}(d u, d y) \quad \text { a.s. } \\
A^{\eta, \ell,(n)}(t) & =\int_{0}^{t} \int_{[-M, M]} \theta_{a}^{(n)}\left(\eta^{(n)}(u-), y\right) \mu_{a}^{J^{(n)}}(d u, d y) \quad \text { a.s. } \tag{1.5.6}
\end{align*}
$$

For all $n \in \mathbb{N}$, we construct a stochastic process $X^{(n)}$, indexed by $[0, T] \times C_{b}\left([-M, M]^{2}\right)$, in the following way: for all $t \in[0, T], g \in C_{b}\left([-M, M]^{2}\right), I=b$, a, we set

$$
\begin{equation*}
X^{(n)}(t, g):=\int_{[-M, M]^{2}} g(x, y) \mu^{J^{(n)}}([0, t], d x, d y) \tag{1.5.7}
\end{equation*}
$$

The following theorem proves the convergence of the $\left(X^{(n)}\right)_{n \in \mathbb{N}}$ and thereby shows that the sequences $\left(\mu_{I}^{J^{(n)}}\right)_{n \in \mathbb{N}}, I=b, a$, converge to two independent, homogeneous Poisson random measures.

Theorem 1.5.5. Assume that Assumptions 1.4 and 1.6 are satisfied. Then for any $m \in \mathbb{N}$ and any $g_{1}, \cdots, g_{m} \in C_{b}\left([-M, M]^{2}\right)$ it holds that

$$
\begin{equation*}
\left(X^{(n)}\left(\cdot, g_{1}\right), \cdots, X^{(n)}\left(\cdot, g_{m}\right)\right) \Rightarrow\left(X\left(\cdot, g_{1}\right), \cdots, X\left(\cdot, g_{m}\right)\right) \tag{1.5.8}
\end{equation*}
$$

in $D\left(\mathbb{R}^{m} ;[0, T]\right)$ with

$$
X(t, g):=\int_{[-M, M]^{2}} g(x, 0) \mu_{b}^{Q}([0, t], d x)+g(0, y) \mu_{a}^{Q}([0, t], d y)
$$

for $g \in C_{b}\left([-M, M]^{2}\right), t \in[0, T]$, where $\mu_{I}^{Q}, I=b$, a, are independent, homogeneous Poisson random measures with intensity measures given by $\lambda \times Q_{I}, I=b, a$, respectively.

Remark 1.5.6. The processes $X^{(n)}, n \in \mathbb{N}$, and $X$ can be interpreted as semimartingale random measures: we can define for any $A \in \mathcal{B}\left([-M, M]^{2}\right)$ and $t \in[0, T], X(t, A):=$ $X\left(t, \mathbb{1}_{A}\right)=\mu^{Q}([0, t], A)$. Then $X$ is a semimartingale random measure in the sense of Kurtz and Protter (cf. Section 2 in $[58]$ ), indexed by $[0, T] \times \mathcal{B}\left([-M, M]^{2}\right)$.

Now, let us turn to the time and volume dynamics $\tau^{\eta,(n)}, v_{b}^{\eta,(n)}$, and $v_{a}^{\eta,(n)}$. First note, that they can be rewritten for $k=0, \ldots, T_{n}$ and $x \in \mathbb{R}$ as follows:

$$
\begin{aligned}
\tau_{k}^{(n)}= & \Delta t^{(n)} \sum_{j=1}^{k} \varphi^{(n)}\left(S_{j-1}^{(n)}\right)+R_{\varphi, k}^{(n)}, \\
v_{b, k}^{(n)}(x)= & v_{b, 0}^{(n)}\left(x-\left(B_{k}^{(n)}-B_{0}^{(n)}\right)\right) \\
& +\Delta v^{(n)} \sum_{j=1}^{k} f_{b}^{(n)}\left[S_{j-1}^{(n)}\right]\left(x-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)+R_{b, k}^{(n)}(x), \\
v_{a, k}^{(n)}(x)= & v_{a, 0}^{(n)}\left(x+A_{k}^{(n)}-A_{0}^{(n)}\right) \\
& +\Delta v^{(n)} \sum_{j=1}^{k} f_{a}^{(n)}\left[S_{j-1}^{(n)}\right]\left(x+A_{k}^{(n)}-A_{j-1}^{(n)}\right)+R_{a, k}^{(n)}(x),
\end{aligned}
$$

where

$$
\begin{align*}
& R_{b, k}^{(n)}(x):=\Delta v^{(n)} \sum_{j=1}^{k}\left(M_{b, j}^{(n)}-f_{b}^{(n)}\left[S_{j-1}^{(n)}\right]\right)\left(x-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right),  \tag{1.5.9}\\
& R_{a, k}^{(n)}(x):=\Delta v^{(n)} \sum_{j=1}^{k}\left(M_{a, j}^{(n)}-f_{a}^{(n)}\left[S_{j-1}^{(n)}\right]\right)\left(x+A_{k}^{(n)}-A_{j-1}^{(n)}\right),
\end{align*}
$$

and

$$
\begin{equation*}
R_{\varphi, k}^{(n)}:=\Delta t^{(n)} \sum_{j=1}^{k}\left(\varphi_{j}^{(n)}-\varphi^{(n)}\left(S_{j-1}^{(n)}\right)\right) . \tag{1.5.10}
\end{equation*}
$$

Let us denote $R_{I}^{(n)}(t):=R_{I, k}^{(n)}, I=b, a, \varphi$, if $t_{k}^{(n)} \leqslant t<t_{k+1}^{(n)}$. According to the next proposition, the random fluctuations of the time and volume dynamics vanish in the high-frequency limit.

Proposition 1.5.7. Under Assumptions 1.2, 1.3, and 1.7, we have

$$
\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left|R_{\varphi, k}^{(n)}\right|^{2}\right] \rightarrow 0 \quad \text { and } \quad \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{I, k}^{(n)}\right\|_{L^{2}}^{2}\right] \rightarrow 0 \quad \text { for } I=b, a .
$$

In particular, the $L^{2}(\mathbb{R})$-valued processes $R_{I}^{(n)}=\left(R_{I}^{(n)}(t)\right)_{t \in[0, T]}, I=b$, a, and the $[0, T]$-valued process $R_{\varphi}^{(n)}=\left(R_{\varphi}^{(n)}(t)\right)_{t \in[0, T]}$ converge weakly in the Skorokhod topology to the zero process.

Altogether, we can write the microscopic LOB-dynamics as a stochastic difference equation, i.e., for all $x \in \mathbb{R}$ and $k=0, \ldots, T_{n}$,

$$
\begin{aligned}
& B_{k}^{(n)}=B_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{b}^{(n)}\left(S_{j-1}^{(n)}\right) \Delta t^{(n)}+r_{b}^{(n)}\left(S_{j-1}^{(n)}\right) \delta Z_{b, j}^{(n)}\right. \\
& \left.+\int_{-M}^{M} \theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right) \mu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right) \\
& v_{b, k}^{(n)}(x)=v_{b, 0}^{(n)}\left(x-\left(B_{k}^{(n)}-B_{0}^{(n)}\right)\right) \\
& +\Delta v^{(n)} \sum_{j=1}^{k} f_{b}^{(n)}\left[S_{j-1}^{(n)}\right]\left(x-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)+R_{b, k}^{(n)}(x) \\
& A_{k}^{(n)}=A_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{a}^{(n)}\left(S_{j-1}^{(n)}\right) \Delta t^{(n)}+r_{a}^{(n)}\left(S_{j-1}^{(n)}\right) \delta Z_{a, j}^{(n)}\right. \\
& \left.+\int_{-M}^{M} \theta_{a}^{(n)}\left(S_{j-1}^{(n)}, y\right) \mu_{a}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
& v_{a, k}^{(n)}(x)=v_{a, 0}^{(n)}\left(x+A_{k}^{(n)}-A_{0}^{(n)}\right) \\
& +\Delta v^{(n)} \sum_{j=1}^{k} f_{a}^{(n)}\left[S_{j-1}^{(n)}\right]\left(x+A_{k}^{(n)}-A_{j-1}^{(n)}\right)+R_{a, k}^{(n)}(x), \\
& \tau_{k}^{(n)}=\Delta t^{(n)} \sum_{j=1}^{k} \varphi^{(n)}\left(S_{j-1}^{(n)}\right)+R_{\varphi, k}^{(n)},
\end{aligned}
$$

and we have shown in Proposition 1.5.3. Proposition 1.5.5. and Proposition 1.5.7 that its integrators converge to limit processes that do not depend on the order book dynamics.

## Step 3: A first approximation of the microscopic state process $\eta^{(n)}$

Proposition 1.5 .7 suggests, that the fluctuations of the time and volume dynamics vanish in the high-frequency limit. Based on this observation, we introduce a new sequence $\left(\widetilde{S}_{k}^{(n)}\right)_{k \geqslant 0}$ of order book models in which the random innovations $M_{I, k}^{(n)}, k \in \mathbb{N}$, are replaced by the approximations $f_{I}\left[\widetilde{S}_{k-1}^{(n)}\right], k \in \mathbb{N}$, and $v_{I, 0}^{(n)}$ is replaced by its limit $v_{I, 0}$, for both $I=b, a$. Moreover, each $\varphi_{k}^{(n)}$ is replaced by the approximation $\varphi\left(\widetilde{S}_{k-1}^{(n)}\right)$ and we replace $\Delta v^{(n)}$ by $\Delta t^{(n)}$ as they are of the same order by Assumption 1.7. Therefore,
we define $\widetilde{S}_{k}^{(n)}=\left(\widetilde{B}_{k}^{(n)}, \widetilde{v}_{b, k}^{(n)}, \widetilde{A}_{k}^{(n)}, \widetilde{v}_{a, k}^{(n)}, \widetilde{\tau}_{k}^{(n)}\right)$ for $k=0, \ldots, T_{n}$ as

$$
\begin{aligned}
& \widetilde{B}_{k}^{(n)}= B_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \Delta t^{(n)}\right. \\
&+r_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \delta Z_{b, j}^{(n)} \\
&\left.\quad+\int_{-M}^{M} \theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right) \mu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
& \widetilde{v}_{b, k}^{(n)}(x)= v_{b, 0}\left(x-\left(\widetilde{B}_{k}^{(n)}-B_{0}^{(n)}\right)\right)+\sum_{j=1}^{k} f_{b}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(x-\left(\widetilde{B}_{k}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right)\right) \Delta t^{(n)}, \\
& \widetilde{A}_{k}^{(n)}= A_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{a}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \Delta t^{(n)}+r_{a}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \delta Z_{a, j}^{(n)}\right. \\
&\left.\quad+\int_{-M}^{M} \theta_{a}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right) \mu_{a}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
& \widetilde{v}_{a, k}^{(n)}(x)= v_{a, 0}\left(x+\widetilde{A}_{k}^{(n)}-A_{0}^{(n)}\right)+\sum_{j=1}^{k} f_{a}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(x+\widetilde{A}_{k}^{(n)}-\widetilde{A}_{j-1}^{(n)}\right) \Delta t^{(n)}, \\
& \widetilde{\tau}_{k}^{(n)}= \sum_{j=1}^{k} \varphi\left(\widetilde{S}_{j-1}^{(n)}\right) \Delta t^{(n)} .
\end{aligned}
$$

For all $n \in \mathbb{N}$, we denote $\widetilde{\eta}^{(n)}(t):=\widetilde{S}_{k}^{(n)}$, if $t_{k}^{(n)} \leqslant t<t_{k+1}^{(n)}$. The next proposition states that the interpolated state process $\eta^{(n)}$ is approximately equal to $\widetilde{\eta}^{(n)}$ as $n \rightarrow \infty$.

Proposition 1.5.8. If Assumptions 1.11 .7 are satisfied, then

$$
\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|S_{k}^{(n)}-\widetilde{S}_{k}^{(n)}\right\|_{E}^{2}\right] \rightarrow 0
$$

In particular, the process $\eta^{(n)}-\widetilde{\eta}^{(n)}$ converges weakly in the Skorokhod topology to the zero process.

## Step 4: Limit theorem for the state process with respect to absolute volumes

Note that the dynamics of $\widetilde{v}_{b}^{(n)}$ and $\widetilde{v}_{a}^{(n)}$ defined above are not given in standard semimartingale form due to the time dependent shift in the $x$-variable. This prevents us from directly applying convergence results for infinite dimensional semimartingales. To overcome this issue, we first prove an intermediate convergence result for the discretetime order book sequence with respect to absolute volumes. Therefore, let us define the approximated absolute volume dynamics

$$
\begin{equation*}
\widetilde{u}_{b, k}^{(n)}(x):=\widetilde{v}_{b, k}^{(n)}\left(-x+\widetilde{B}_{k}^{(n)}\right), \quad \widetilde{u}_{a, k}^{(n)}(x):=\widetilde{v}_{a, k}^{(n)}\left(x-\widetilde{A}_{k}^{(n)}\right), \quad k=0, \cdots, T_{n} . \tag{1.5.11}
\end{equation*}
$$

Then we introduce $\widetilde{S}_{k}^{(n), a b s}=\left(\widetilde{B}_{k}^{(n)}, \widetilde{u}_{b, k}^{(n)}, \widetilde{A}_{k}^{(n)}, \widetilde{u}_{a, k}^{(n)}, \widetilde{\tau}_{k}^{(n)}\right), k=0, \cdots, T_{n}$, and note that a priori the coefficient functions in the dynamics of $\widetilde{S}^{(n), a b s}$ are still functions of $\widetilde{S}^{(n)}$ and hence in particular of the approximated relative volume dynamics $\widetilde{v}_{b}^{(n)}$ and $\widetilde{v}_{a}^{(n)}$. For this reason, we introduce a shift operator $\psi: E \rightarrow E$ such that for all $s=(b, v, a, w, t) \in E$,

$$
\psi(s):=(b, v(-(\cdot-b)), a, w(\cdot-a), t)
$$

Then, $\psi\left(\widetilde{S}_{k}^{(n), a b s}\right)=\widetilde{S}_{k}^{(n)}$ for all $k=0, \cdots, T_{n}$ and we can rewrite the dynamics of $\widetilde{S}^{(n), a b s}$ in such a way that all coefficient functions directly depend on $\widetilde{S}^{(n), a b s}$ :

$$
\begin{gathered}
\widetilde{B}_{k}^{(n)}=B_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{b}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right) \Delta t^{(n)}+r_{b}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right) \delta Z_{b, j}^{(n)}\right. \\
\\
\left.\quad+\int_{[-M, M]} \theta_{b}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right), y\right) \mu_{b}^{J(n)}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
\widetilde{u}_{b, k}^{(n)}(x)=v_{0}\left(-x+B_{0}^{(n)}\right)+\sum_{j=1}^{k} f_{b}\left[\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right]\left(-x+\widetilde{B}_{j-1}^{(n)}\right) \Delta t^{(n)}, \\
\widetilde{A}_{k}^{(n)}=A_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{a}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right) \Delta t^{(n)}+r_{a}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right) \delta Z_{a, j}^{(n)}\right. \\
\left.\quad+\int_{[-M, M]} \theta_{a}^{(n)}\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right), y\right) \mu_{a}^{J(n)}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
\widetilde{u}_{a, k}^{(n)}(x)=v_{0}\left(x-A_{0}^{(n)}\right)+\sum_{j=1}^{k} f_{a}\left[\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right]\left(x-\widetilde{A}_{j-1}^{(n)}\right) \Delta t^{(n)}, \\
\widetilde{\tau}_{k}^{(n)}=\sum_{j=1}^{k} \varphi\left(\psi\left(\widetilde{S}_{j-1}^{(n), a b s}\right)\right) \Delta t^{(n)} .
\end{gathered}
$$

For all $n \in \mathbb{N}$, we define its piecewise constant interpolation as $\widetilde{\eta}^{(n), a b s}(t):=\widetilde{S}_{k}^{(n), a b s}$, if $t_{k}^{(n)} \leqslant t<t_{k+1}^{(n)}$, for $t \in[0, T]$. Now the following theorem shows the convergence of $\widetilde{\eta}^{(n), a b s}$. The proof is an application of the results in Kurtz and Protter 58.

Theorem 1.5.9. Let Assumptions 1.1 1.7 be satisfied. Then the interpolation of the approximated LOB-dynamics with respect to the absolute volume density function $\widetilde{\eta}^{(n), a b s}$ converges weakly in the Skorokhod topology to $\eta^{a b s}=\left(B^{\eta}, u_{b}^{\eta}, A^{\eta}, u_{a}^{\eta}, \tau^{\eta}\right)$ being the unique strong solution to the coupled SDE system:

$$
\begin{align*}
& B^{\eta}(t)= B_{0}+\int_{0}^{t} p_{b}\left(\psi\left(\eta^{a b s}(u)\right)\right) d u+\int_{0}^{t} r_{b}\left(\psi\left(\eta^{a b s}(u)\right)\right) d Z_{b}(u) \\
&+\int_{0}^{t} \int_{[-M, M]} \theta_{b}\left(\psi\left(\eta^{a b s}(u-)\right), y\right) \mu_{b}^{Q}(d u, d y), \\
& u_{b}^{\eta}(t, x)= v_{b, 0}\left(-x+B_{0}\right)+ \\
& \int_{0}^{t} f_{b}\left[\psi\left(\eta^{a b s}(u)\right)\right]\left(-x+B^{\eta}(u)\right) d u  \tag{1.5.12}\\
& A^{\eta}(t)= A_{0}+\int_{0}^{t} p_{a}\left(\psi\left(\eta^{a b s}(u)\right)\right) d u+\int_{0}^{t} r_{a}\left(\psi\left(\eta^{a b s}(u)\right)\right) d Z_{a}(u) \\
&+\int_{0}^{t} \int_{[-M, M]} \theta_{a}\left(\psi\left(\eta^{a b s}(u-)\right), y\right) \mu_{a}^{Q}(d u, d y) \\
& u_{a}^{\eta}(t, x)= v_{a, 0}\left(x-A_{0}\right)+\int_{0}^{t} f_{a}\left[\psi\left(\eta^{a b s}(u)\right)\right]\left(x-A^{\eta}(u)\right) d u \\
& \tau^{\eta}(t):= \int_{0}^{t} \varphi\left(\psi\left(\eta^{a b s}(u)\right)\right) d u
\end{align*}
$$

for all $(t, x) \in[0, T] \times \mathbb{R}$, where $Z_{b}, Z_{a}$ are two independent, standard Brownian motions and $\mu_{b}^{Q}, \mu_{a}^{Q}$ are two independent, homogeneous Poisson random measures with intensity measures $\lambda \times Q_{b}$ and $\lambda \times Q_{a}$, respectively, independent of $Z_{b}$ and $Z_{a}$.

## Step 5: End of the proof

With a slight abuse of notation, by the Skorokhod representation theorem we may assume that $\widetilde{\eta}^{(n), a b s}$ converges almost surely in the Skorokhod topology to $\eta^{a b s}$. Hence, there exists a sequence of continuous, strictly increasing functions $\gamma_{n}:[0, T] \rightarrow[0, T]$ with $\sup _{t \in[0, T]}\left|\gamma_{n}(t)-t\right| \rightarrow 0$ such that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\|\widetilde{\eta}^{(n), a b s}\left(\gamma_{n}(t)\right)-\eta^{a b s}(t)\right\|_{E} \rightarrow 0 \quad \text { a.s. } \tag{1.5.13}
\end{equation*}
$$

Then, we have

$$
\begin{align*}
& \sup _{t \in[0, T]}\left\|\psi\left(\widetilde{\eta}^{(n), a b s}\left(\gamma_{n}(t)\right)\right)-\psi\left(\eta^{a b s}(t)\right)\right\|_{E} \\
& \leqslant \sup _{t \in[0, T]}\left\{\left|\widetilde{B}^{(n)}\left(\gamma_{n}(t)\right)-B^{\eta}(t)\right|+\left|\widetilde{A}^{(n)}\left(\gamma_{n}(t)\right)-A^{\eta}(t)\right|\right. \\
& +\left\|\widetilde{u}_{b}^{(n)}\left(\gamma_{n}(t),-\left(\cdot-\widetilde{B}^{(n)}\left(\gamma_{n}(t)\right)\right)\right)-u_{b}^{\eta}\left(t,-\left(\cdot-B^{\eta}(t)\right)\right)\right\|_{L^{2}} \\
& \left.+\left\|\widetilde{u}_{a}^{(n)}\left(\gamma_{n}(t), \cdot-\widetilde{A}^{(n)}\left(\gamma_{n}(t)\right)\right)-u_{a}^{\eta}\left(t, \cdot-A^{\eta}(t)\right)\right\|_{L^{2}}+\left|\widetilde{\tau}^{(n)}\left(\gamma_{n}(t)\right)-\tau^{\eta}(t)\right|\right\} . \tag{1.5.14}
\end{align*}
$$

Applying Assumptions 1.1 and 1.3 iii), observe that for any $x, \widetilde{x}$ and $I=b, a$, we have

$$
\begin{align*}
& \left\|u_{I}^{\eta}(t, \cdot+x)-u_{I}^{\eta}(t, \cdot+\widetilde{x})\right\|_{L^{2}} \\
& \leqslant \\
& \quad\left\|v_{I, 0}(\cdot+x)-v_{I, 0}(\cdot+\widetilde{x})\right\|_{L^{2}}  \tag{1.5.15}\\
& \quad \quad+\int_{0}^{t}\left\|f_{I}\left[\psi\left(\eta^{a b s}(u)\right)\right](\cdot+x)-f_{I}\left[\psi\left(\eta^{a b s}(u)\right)\right](\cdot+\widetilde{x})\right\|_{L^{2}} d u \\
& \leqslant \\
& \leqslant|x-\widetilde{x}|+T \sup _{s \in E}\left\|f_{I}[s](\cdot+x)-f_{I}[s](\cdot+\widetilde{x})\right\|_{L^{2}} \\
& \leqslant \\
& L(1+T)|x-\widetilde{x}|
\end{align*}
$$

and note that a similar estimate also holds true for $\widetilde{u}_{I}^{(n)}, I=b, a$. Hence, we can bound

$$
\begin{aligned}
& \left\|\widetilde{u}_{b}^{(n)}\left(\gamma_{n}(t),-\left(\cdot-\widetilde{B}^{(n)}\left(\gamma_{n}(t)\right)\right)\right)-u_{b}^{\eta}\left(t,-\left(\cdot-B^{\eta}(t)\right)\right)\right\|_{L^{2}} \\
& \quad \leqslant\left\|\widetilde{u}_{b}^{(n)}\left(\gamma_{n}(t), \cdot\right)-u_{b}^{\eta}(t, \cdot)\right\|_{L^{2}}+\left\|u_{b}^{\eta}\left(t,-\left(\cdot-\widetilde{B}^{(n)}\left(\gamma_{n}(t)\right)\right)\right)-u_{b}^{\eta}\left(t,-\left(\cdot-B^{\eta}(t)\right)\right)\right\|_{L^{2}} \\
& \quad \leqslant\left\|\widetilde{u}_{b}^{(n)}\left(\gamma_{n}(t), \cdot\right)-u_{b}^{\eta}(t, \cdot)\right\|_{L^{2}}+L(1+T)\left|\widetilde{B}^{(n)}\left(\gamma_{n}(t)\right)-B^{\eta}(t)\right|
\end{aligned}
$$

An analogous estimate holds for the ask side. Plugging these bounds into equation (1.5.14) and applying (1.5.13), it follows that

$$
\sup _{t \in[0, T]}\left\|\psi\left(\tilde{\eta}^{(n), a b s}\left(\gamma_{n}(t)\right)\right)-\psi\left(\eta^{a b s}(t)\right)\right\|_{E} \rightarrow 0 \quad \text { a.s. }
$$

Hence, $\widetilde{\eta}^{(n)}=\psi\left(\widetilde{\eta}^{(n), a b s}\right) \Rightarrow \psi\left(\eta^{a b s}\right)=: \eta$ which solves (1.2.13). Therefore, Proposition 1.5.1 holds true. An application of Corollary 1.5 .2 finishes the proof of Theorem 1.2.6

### 1.6 Technical details

Proof of Corollary 1.5.2. According to Proposition 1.5.1, $\eta^{(n)}$ converges weakly in the Skorokhod topology to $\eta:=\left(B^{\eta}, v_{b}^{\eta}, A^{\eta}, v_{a}^{\eta}, \tau^{\eta}\right)$ being the unique strong solution of the coupled diffusion-fluid system in (1.2.13). Since $\varphi^{(n)}, \varphi$ are strictly positive, $\tau^{\eta,(n)}, \tau^{\eta}$ are increasing (resp. strictly increasing) and therefore,

$$
\zeta^{(n)}(t)=\left(\tau^{\eta,(n)}\right)^{-1}(t)=\inf \left\{u>0: \tau^{\eta,(n)}(u)>t\right\} \quad \text { and } \quad \zeta(t)=\left(\tau^{\eta}\right)^{-1}(t)
$$

exist and are increasing. Moreover, $\zeta$ is continuous and even strictly increasing. By Corollary 13.6.4 in Whitt [86], the inverse map is continuous at strictly increasing functions. By the continuous mapping theorem we therefore conclude that $\left(\zeta^{(n)}, \eta^{(n)}\right) \Rightarrow$ $(\zeta, \eta)$ in the Skorokhod topology. Especially, this proves i).

Since $\zeta$ is continuous and strictly increasing almost surely and since $\eta$ is continuous at time $\zeta(T)$ almost surely, Theorem 3.1 in Whitt 85 yields the continuity of the composition map in $(\zeta, \eta)$ in the Skorokhod topology. Hence, we can apply the continuous
mapping theorem to conclude that

$$
S^{(n), *}=\eta^{(n)} \circ\left(\zeta^{(n)}-\Delta t^{(n)}\right) \Rightarrow \eta \circ \zeta=: S
$$

in the Skorokhod topology, which proves ii).
Finally, it follows from Theorem 12.5 in that $\tau^{\eta,(n)}(T)$ converges weakly to $\tau^{\eta}(T)$ and hence

$$
\mathbb{P}\left[\sup _{t \in[0, T]}\left\|S^{(n), *}(t)-S^{(n)}(t)\right\|_{E}>0\right]=\mathbb{P}\left[\tau^{\eta,(n)}(T)>T\right] \rightarrow \mathbb{P}\left[\tau^{\eta}(T)>T\right]=0,
$$

since by Assumption 1.2 iii),

$$
\tau^{\eta}(T)=\int_{0}^{T} \varphi(\eta(t)) d t \leqslant T
$$

This proves iii).
Proof of Proposition 1.5.3. First note that for all $n \in \mathbb{N}, k \leqslant T_{n}$,

$$
\frac{\Delta x^{(n)}\left|p_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right|}{\left(r_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}}=\left.\frac{\Delta x^{(n)} \mid \mathbb{E}\left[\delta B_{k}^{(n)} \mathbb{1}\left\{0<\left|\delta B_{k}^{(n)}\right| \leqslant \delta_{n}\right\}^{\left.\mid \mathcal{F}_{k-1}^{(n)}\right] \mid}\right.}{\mathbb{E}\left[\left(\delta B_{k}^{(n)}\right)^{2} \mathbb{1}\left\{0<\left|\delta B_{k}^{(n)}\right| \leqslant \delta_{n}\right\}\right.}\right|^{\left.\mathcal{F}_{k-1}^{(n)}\right]} \leqslant 1 \text { a.s. }
$$

Together with Assumption 1.4, this gives the bound

$$
\frac{\Delta t^{(n)}\left(p_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}}{\left(r_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}} \leqslant \frac{\Delta t^{(n)}}{\Delta x^{(n)}}\left|p_{b}^{(n)}\left(S_{k-1}^{(n)}\right)\right|,
$$

which converges to zero by Assumption 1.7. A similar bound holds for $p_{a}^{(n)}$ and $r_{a}^{(n)}$. Hence, for all $t \in[0, T]$ and $I=b, a$,

$$
\begin{aligned}
\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left[\left(\delta Z_{k, I}^{(n)}\right)^{2} \mid \mathcal{F}_{k-1}^{(n)}\right] & =\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \frac{\Delta t^{(n)}\left(r_{I}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}-\left(\Delta t^{(n)} p_{I}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}}{\left(r_{I}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}} \\
& =\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)}\left(1-\frac{\Delta t^{(n)}\left(p_{I}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}}{\left(r_{I}^{(n)}\left(S_{k-1}^{(n)}\right)\right)^{2}}\right) \rightarrow t .
\end{aligned}
$$

Moreover, as $A$ and $C$ events do not occur simultaneously, we have for all $t \in[0, T]$,

$$
\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left[\delta Z_{b, k}^{(n)} \delta Z_{a, k}^{(n)} \mid \mathcal{F}_{k-1}^{(n)}\right]=-\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \frac{\left(\Delta t^{(n)}\right)^{2} p_{b}^{(n)}\left(S_{k-1}^{(n)}\right) p_{a}^{(n)}\left(S_{k-1}^{(n)}\right)}{r_{b}^{(n)}\left(S_{k-1}^{(n)}\right) r_{a}^{(n)}\left(S_{k-1}^{(n)}\right)} \rightarrow 0
$$

Finally, we observe that for all $\varepsilon>0$ and $I=b, a$,

$$
\begin{aligned}
\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left[\left|\delta Z_{I, k}^{(n)}\right|^{2} \mathbb{1}_{\left\{\left|\delta Z_{I, k}^{(n)}\right|>\varepsilon\right\}}\right] & \leqslant \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left[\left|\delta Z_{I, k}^{(n)}\right|^{2} \mathbb{1}_{\left\{\frac{\delta_{n}}{\eta_{n}}>\varepsilon\right\}}\right] \\
& \leqslant \frac{\delta_{n}}{\varepsilon \eta_{n}} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left|\delta Z_{I, k}^{(n)}\right|^{2} \leqslant \frac{t \delta_{n}}{\varepsilon \eta_{n}}
\end{aligned}
$$

which converges to zero by Assumption 1.3 i). As $\left(\delta Z_{b, k}^{(n)}, \delta Z_{a, k}^{(n)}\right)_{k, n}$ is a triangular martingale difference array, we may conclude by the functional limit theorem that $\left(Z_{b}^{(n)}, Z_{a}^{(n)}\right) \Rightarrow\left(Z_{b}, Z_{a}\right)$ in the Skorokhod topology, where $Z_{b}$ and $Z_{a}$ are independent Brownian motions.

Proof of Lemma 1.5.4. We state the proof for $B^{\eta, \ell,(n)}$. The definition of $\mu_{b}^{J^{(n)}}$ yields

$$
\begin{aligned}
\int_{0}^{t} & \int_{[-M, M]} \theta_{b}^{(n)}\left(\eta^{(n)}(u-), y\right) \mu_{b}^{J^{(n)}}(d u, d y) \\
& =\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \int_{[-M, M]} \theta_{b}^{(n)}\left(S_{k-1}^{(n)}, y\right) \mu_{b}^{J^{(n)}}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right), d y\right) \\
& \stackrel{(1)}{=} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \sum_{j \in \mathbb{Z}_{M}^{(n)}} \theta_{b}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right) \mu_{b}^{\eta,(n)}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),\left\{\theta^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)\right\}\right) \\
& \stackrel{(2)}{=} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \sum_{j \in \mathbb{Z}} x_{j}^{(n)} \mu_{b}^{\eta,(n)}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),\left\{x_{j}^{(n)}\right\}\right)=\int_{\mathbb{R}} y \mu_{b}^{\eta,(n)}([0, t], d y)=B^{\eta, \ell,(n)}(t),
\end{aligned}
$$

where in (1) we used the definition of $\mu_{b}^{J^{(n)}}$ noting that it only charges the grid points $x_{j}^{(n)}, j \in \mathbb{Z}_{M}^{(n)}$, and in (2) we used Assumption 1.6 vi).

In order to prove the convergence of the driving jump measures to a limit independent of the order book dynamics, we first need a technical result regarding the coefficient function $\theta$ and its approximations $\theta^{(n)}$. Note that this result has already been applied in Section 1.4

Lemma 1.6.1. Let Assumptions 1.6 i) and v) be satisfied. Then for all $j \in \mathbb{Z}_{M}^{(n)}$, $I=b, a$, and $s \in E$,

$$
K_{I}^{(n)}\left(s,\left\{\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)\right\}\right)=K_{I}^{(n)}\left(s, \theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)
$$

Proof. Let $I=b, a$. First, suppose that $K_{I}^{(n)}\left(s,\left\{\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)\right\}\right)>0$. We want to show that in this case,

$$
\begin{equation*}
\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right) \in \theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right) \tag{1.6.1}
\end{equation*}
$$

In fact, by definition

$$
\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right) \in\left[\theta_{I}\left(s, x_{j}^{(n)}\right), \theta_{I}\left(s, x_{j}^{(n)}\right)+\Delta x^{(n)}\right) .
$$

Suppose that $\theta_{I}\left(s, x_{j+1}^{(n)}\right)<\theta_{I}\left(s, x_{j}^{(n)}\right)+\Delta x^{(n)}$ (otherwise, the relation in 1.6.1) is already satisfied). Then,
$\theta_{I}^{(n)}\left(s, x_{j+1}^{(n)}\right)=\left\lceil\frac{\theta_{I}\left(s, x_{j+1}^{(n)}\right)}{\Delta x^{(n)}}\right\rceil \Delta x^{(n)} \leqslant\left\lceil\frac{\theta_{I}\left(s, x_{j}^{(n)}\right)+\Delta x^{(n)}}{\Delta x^{(n)}}\right\rceil \Delta x^{(n)}=\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)+\Delta x^{(n)}$.
By Assumption 1.6 v ), we must have equality in the above line, which implies that

$$
\theta_{I}\left(s, x_{j+1}^{(n)}\right)>\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right) .
$$

Hence, (1.6.1) is satisfied. Second, suppose that there exists $x \in\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)$ with $\theta_{I}(s, x) \in \Delta x^{(n)} \mathbb{Z}$ and $K_{I}^{(n)}(s, \theta(s, x))>0$. Then by Assumption 1.6 v$)$ there exists $i \in \mathbb{N}$ with $\theta_{I}(s, x)=\theta_{I}^{(n)}\left(s, x_{i}^{(n)}\right)$. By strict monotonicity of $\theta_{I}$ (cf. Assumption 1.6 i)) we have

$$
\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)=\left\lceil\frac{\theta_{I}\left(s, x_{j}^{(n)}\right)}{\Delta x^{(n)}}\right\rceil \Delta x^{(n)} \leqslant\left\lceil\frac{\theta_{I}(s, x)}{\Delta x^{(n)}}\right\rceil \Delta x^{(n)}=\theta_{I}(s, x)
$$

and

$$
\theta(s, x)=\left\lceil\frac{\theta_{I}(s, x)}{\Delta x^{(n)}}\right\rceil \cdot \Delta x^{(n)}<\left\lceil\frac{\theta_{I}\left(s, x_{j+1}^{(n)}\right)}{\Delta x^{(n)}}\right\rceil \cdot \Delta x^{(n)}=\theta_{I}^{(n)}\left(s, x_{j+1}^{(n)}\right) .
$$

Hence, we must have $i=j$, i.e., $\theta_{I}(s, x)=\theta_{I}^{(n)}\left(s, x_{j}^{(n)}\right)$.
Proof of Theorem 1.5.5. From 1.5.5) we have for all $g \in C_{b}([-M, M]), t \in[0, T]$, and $I=b, a$,

$$
\begin{aligned}
& \int_{[-M, M]} g(y) \nu_{I}^{J^{(n)}}([0, t], d y)= \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j \in \mathbb{Z}_{M}^{(n)}} g\left(x_{j}^{(n)}\right) K_{I}^{(n)}\left(S_{k-1}^{(n)},\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)\right\}\right) \\
&+\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j \in \mathbb{Z}_{M}^{(n)}} g\left(x_{j}^{(n)}\right) \mathbb{1}\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)=0\right\} \\
& Q_{I}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right) .
\end{aligned}
$$

First, from Lemma 1.6.1 we see that

$$
\begin{aligned}
& \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j \in \mathbb{Z}_{M}^{(n)}} g\left(x_{j}^{(n)}\right) K_{I}^{(n)}\left(S_{k-1}^{(n)},\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)\right\}\right) \\
& \quad=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j} g\left(x_{j}^{(n)}\right) K_{I}^{(n)}\left(S_{k-1}^{(n)}, \theta_{I}\left(S_{k-1}^{(n)},\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right) .
\end{aligned}
$$

Next, we bound

$$
\begin{aligned}
& I_{n}^{(1)}:=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j} g\left(x_{j}^{(n)}\right)\left[K_{I}^{(n)}\left(S_{k-1}^{(n)}, \theta_{I}\left(S_{k-1}^{(n)},\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right. \\
& \left.-K_{I}\left(S_{k-1}^{(n)}, \theta_{I}\left(S_{k-1}^{(n)},\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right]
\end{aligned}
$$

by

$$
\left|I_{n}^{(1)}\right| \leqslant t \cdot\|g\|_{\infty} \cdot \sup _{s \in E} \sum_{j}\left|K_{I}^{(n)}\left(s, \theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)-K_{I}\left(s, \theta_{I}\left(s,\left[x_{j-1}^{(n)}, x_{j}^{(n)}\right)\right)\right)\right|
$$

which converges to zero by Assumption 1.6 iii). Moreover, we note that also

$$
I_{n}^{(2)}:=\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j \in \mathbb{Z}_{M}^{(n)}} g\left(x_{j}^{(n)}\right) \int_{x_{j}^{(n)}}^{x_{j+1}^{(n)}}\left(\mathbb{1}_{\left\{\theta_{I}^{(n)}\left(S_{k-1}^{(n)}, x_{j}^{(n)}\right)=0\right\}} \mathbb{1}_{\left\{\theta_{I}\left(S_{k-1}^{(n)}, x\right)=0\right\}}\right) Q_{I}(d x)
$$

goes to zero by Assumption 1.6 vi). Therefore,

$$
\begin{aligned}
& \int_{[-M, M]} g(y) \nu_{I}^{J(n)}([0, t], d y) \\
& =\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j} g\left(x_{j}^{(n)}\right)\left[K_{I}\left(S_{k-1}^{(n)}, \theta_{I}\left(S_{k-1}^{(n)},\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right)\right. \\
& \left.\quad+Q_{I}\left(\left\{x \in\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right): \theta_{I}\left(S_{k-1}^{(n)}, x\right)=0\right\}\right)\right]+I_{n}^{(1)}+I_{n}^{(2)} \\
& \stackrel{(1)}{=} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \sum_{j} g\left(x_{j}^{(n)}\right) Q_{I}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)+I_{n}^{(1)}+I_{n}^{(2)} \rightarrow t \int_{[-M, M]} g(y) Q_{I}(d y),
\end{aligned}
$$

where in (1) we used Assumption 1.6 ii . By the definition of the measure $\mu^{J^{(n)}}$ we then also have

$$
\int_{[-M, M]^{2}} g(x, y) \nu^{J^{(n)}}([0, t], d x, d y) \rightarrow t \int_{[-M, M]^{2}} g(x, y)\left[Q_{b}(d x) \varepsilon_{0}(d y)+Q_{a}(d y) \varepsilon_{0}(d x)\right] .
$$

As a Poisson random measure is uniquely determined by its intensity measure we conclude by Theorem 2.6 in that for any $g_{1}, \cdots, g_{m} \in C_{b}\left([-M, M]^{2}\right)$,

$$
\left(X^{(n)}\left(\cdot, g_{1}\right), \cdots, X^{(n)}\left(\cdot, g_{m}\right)\right) \Rightarrow\left(X\left(\cdot, g_{1}\right), \cdots, X\left(\cdot, g_{m}\right)\right)
$$

in $D\left(\mathbb{R}^{m} ;[0, T]\right)$ with

$$
X(t, g):=\int_{[-M, M]^{2}} g(x, y)\left[\mu_{b}^{Q}([0, t], d x) \varepsilon_{0}(d y)+\mu_{a}^{Q}([0, t], d y) \varepsilon_{0}(d x)\right]
$$

where $\mu_{I}^{Q}, I=b, a$, are two independent Poisson random measures with intensity measure $\lambda \times Q_{I}$ for $I=b, a$, respectively.
Remark 1.6.2. Carefully inspecting the above proof, we see that we have actually shown for $I=b, a$ the almost sure estimate
$\left|\int_{-M}^{M} g(y) \nu_{I}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)-\Delta t^{(n)} \sum_{j \in \mathbb{Z}_{M}^{(n)}} g\left(x_{j}^{(n)}\right) Q_{I}\left(\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right)\right|=o\left(\|g\|_{\infty} \Delta t^{(n)}\right)$,
which will be used later in the proof of Proposition 1.5.8 (cf. Section 1.6.1) and Section 1.6.4.

The following technical lemma is used in the subsequent proof of Proposition 1.5.7 and provides an estimate for Hilbert space valued, discrete-time martingales. It is a direct consequence of Theorem 6.1 in 71]:

Lemma 1.6.3. Let $H$ be a Hilbert space. Then there exists a constant $C>0$ such that for every $H$-valued, discrete-time martingale $X$ with $X_{0}=0$, we have

$$
\mathbb{E}\left(\sup _{i \geqslant 1}\left\|X_{i}\right\|_{H}^{2}\right) \leqslant C \mathbb{E}\left[\sum_{i=1}^{\infty}\left\|X_{i}-X_{i-1}\right\|_{H}^{2}\right] .
$$

Proof of Proposition 1.5.7. First, we observe that

$$
\left\|R_{b, k}^{(n)}\right\|_{L^{2}}=\left\|\Delta v^{(n)} \sum_{j=1}^{k}\left(M_{b, j}^{(n)}-f_{b}^{(n)}\left[S_{j-1}^{(n)}\right]\right)\left(\cdot+B_{j-1}^{(n)}\right)\right\|_{L^{2}}=:\left\|\widetilde{R}_{b, k}^{(n)}\right\|_{L^{2}} .
$$

Therefore, it is enough to prove the result for $\left(\widetilde{R}_{b, k}^{(n)}\right)_{k \geqslant 0}$. By Assumption 1.3 ii , this sequence is a martingale with values in $L^{2}(\mathbb{R})$, since for all $x \in \mathbb{R}$,
$\mathbb{E}\left[\left(\widetilde{R}_{b, k}^{(n)}-\widetilde{R}_{b, k-1}^{(n)}\right)(x) \mid \mathcal{F}_{k-1}^{(n)}\right]=\mathbb{E}\left[\Delta v^{(n)}\left(M_{b, k}^{(n)}-f_{b}^{(n)}\left[S_{k-1}^{(n)}\right]\right)\left(x+B_{k-1}^{(n)}\right) \mid \mathcal{F}_{k-1}^{(n)}\right]=0 \quad$ a.s.
Applying Lemma 1.6 .3 as well as Assumptions 1.3 i) and 1.7 , there exists $C>0$ such
that

$$
\begin{aligned}
\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|\widetilde{R}_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right] & \leqslant C \mathbb{E}\left[\sum_{k=1}^{T_{n}}\left\|\widetilde{R}_{b, k}^{(n)}-\widetilde{R}_{b, k-1}^{(n)}\right\|_{L^{2}}^{2}\right] \\
& =C \mathbb{E}\left[\sum_{k=1}^{T_{n}}\left\|\Delta v^{(n)}\left(M_{b, k}^{(n)}-f_{b}^{(n)}\left[S_{k-1}^{(n)}\right]\right)\right\|_{L^{2}}^{2}\right] \\
& \leqslant C T_{n}\left(\Delta v^{(n)}\right)^{2} \sup _{k \leqslant T_{n}} \mathbb{E}\left[\left\|\left(M_{b, k}^{(n)}\right)^{2}\right\|_{L^{1}}\right] \\
& \leqslant C T \frac{\left(\Delta v^{(n)}\right)^{2}}{\left(\Delta t^{(n)}\right)^{2}} \frac{\Delta t^{(n)}}{\Delta x^{(n)}} \sup _{k \leqslant T_{n}} \mathbb{E}\left[\left(\omega_{k}^{(n)}\right)^{2}\right] \rightarrow 0
\end{aligned}
$$

Hence, $\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right] \rightarrow 0$. Analogously, one can show that $\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{a, k}^{(n)}\right\|_{L^{2}}^{2}\right]$ $\rightarrow 0$. Next, observe that also

$$
R_{\varphi, k}^{(n)}:=\Delta t^{(n)} \sum_{j=1}^{k}\left(\varphi_{j}^{(n)}-\varphi^{(n)}\left(S_{j-1}^{(n)}\right)\right)
$$

defines a martingale and by Lemma 1.6 .3 we have

$$
\begin{aligned}
\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left|R_{\varphi, k}^{(n)}\right|^{2}\right] & \leqslant C \mathbb{E}\left[\sum_{k=1}^{T_{n}}\left|R_{\varphi, k}^{(n)}-R_{\varphi, k-1}^{(n)}\right|^{2}\right] \\
& =C \mathbb{E}\left[\sum_{k=1}^{T_{n}}\left|\Delta t^{(n)}\left(\varphi_{k}^{(n)}-\varphi^{(n)}\left(S_{k-1}^{(n)}\right)\right)\right|^{2}\right] \\
& \leqslant C T \Delta t^{(n)} \sup _{k \leqslant T_{n}} \mathbb{E}\left[\left(\varphi_{k}^{(n)}\right)^{2}\right] \rightarrow 0
\end{aligned}
$$

where we applied Assumption 1.2 i).

### 1.6.1 Proof of Proposition 1.5 .8

In order to prove the claim, we will introduce two further candidates for approximations of $\left(S_{k}^{(n)}\right)_{k \geqslant 0}$ - namely $\left(\bar{S}_{k}^{(n)}\right)_{k \geqslant 0}$ and $\left(\hat{S}_{k}^{(n)}\right)_{k \geqslant 0}$. We define $\bar{S}_{k}^{(n)}=\left(\bar{B}_{k}^{(n)}, \bar{v}_{b, k}^{(n)}, \bar{A}_{k}^{(n)}, \bar{v}_{a, k}^{(n)}\right.$, $\left.\bar{\tau}_{k}^{(n)}\right)$ for $k=0, \ldots, T_{n}$ as

$$
\bar{B}_{k}^{(n)}:=B_{k}^{(n)}, \quad \bar{A}_{k}^{(n)}:=A_{k}^{(n)}, \quad \bar{\tau}_{k}^{(n)}:=\sum_{j=1}^{k} \varphi\left(S_{j-1}^{(n)}\right) \Delta t^{(n)}
$$

$$
\begin{aligned}
& \bar{v}_{b, k}^{(n)}(x):=v_{b, 0}\left(x-\left(B_{k}^{(n)}-B_{0}^{(n)}\right)\right)+\sum_{j=1}^{k} f_{b}\left[S_{j-1}^{(n)}\right]\left(x-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right) \Delta t^{(n)}, \\
& \bar{v}_{a, k}^{(n)}(x):=v_{a, 0}\left(x+\left(A_{k}^{(n)}-A_{0}^{(n)}\right)\right)+\sum_{j=1}^{k} f_{a}\left[S_{j-1}^{(n)}\right]\left(x+\left(A_{k}^{(n)}-A_{j-1}^{(n)}\right)\right) \Delta t^{(n)} .
\end{aligned}
$$

Note that the coefficient functions in the dynamics of $\bar{S}_{k}^{(n)}$ still depend on the original LOB sequence. In the second approximation $\hat{S}_{k}^{(n)}=\left(\hat{B}_{k}^{(n)}, \hat{v}_{b, k}^{(n)}, \hat{A}_{k}^{(n)}, \hat{v}_{a, k}^{(n)}, \hat{\tau}_{k}^{(n)}\right), k=$ $0, \cdots, T_{n}$, the diffusion coefficient and the coefficient of the compensated jumps depend on the approximated LOB-dynamics $\left(\widetilde{S}_{k}^{(n)}\right)_{k \geqslant 0}$, whereas all other coefficients still depend on the original LOB sequence, i.e., for $k=0, \ldots, T_{n}$,

$$
\begin{aligned}
& \hat{B}_{k}^{(n)}:=B_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{b}^{(n)}\left(S_{j-1}^{(n)}\right) \Delta t^{(n)}+r_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \delta Z_{b, j}^{(n)}\right. \\
&+\int_{[-M, M]} \theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\left(\mu_{b}^{J^{(n)}}-\nu_{b}^{J^{(n)}}\right)\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right) \\
&\left.+\int_{[-M, M]} \theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right) \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
& \hat{v}_{b, k}^{(n)}(x):=v_{b, 0}\left(x-\left(\hat{B}_{k}^{(n)}-B_{0}^{(n)}\right)\right)+\sum_{j=1}^{k} f_{b}\left[S_{j-1}^{(n)}\right]\left(x-\left(\hat{B}_{k}^{(n)}-\hat{B}_{j-1}^{(n)}\right)\right) \Delta t^{(n)}, \\
& \hat{A}_{k}^{(n)}:=A_{0}^{(n)}+\sum_{j=1}^{k}\left(p_{a}^{(n)}\left(S_{j-1}^{(n)}\right) \Delta t^{(n)}+r_{a}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right) \delta Z_{a, j}^{(n)}\right. \\
& \quad+\int_{[-M, M]} \theta_{a}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\left(\mu_{a}^{J^{(n)}}-\nu_{a}^{J J^{(n)}}\right)\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right) \\
&\left.\quad+\int_{[-M, M]} \theta_{a}^{(n)}\left(S_{j-1}^{(n)}, y\right) \nu_{a}^{J_{a}^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right), \\
& \hat{v}_{a, k}^{(n)}(x):= v_{a, 0}\left(x+\left(\hat{A}_{k}^{(n)}-A_{0}^{(n)}\right)\right)+\sum_{j=1}^{k} f_{a}\left[S_{j-1}^{(n)}\right]\left(x+\left(\hat{A}_{k}^{(n)}-\hat{A}_{j-1}^{(n)}\right)\right) \Delta t^{(n)}, \\
& \hat{\tau}_{k}^{(n)}:=\bar{\tau}_{k}^{(n)} .
\end{aligned}
$$

In what follows, let $\left(a_{n}\right)_{n \in \mathbb{N}}$ be a deterministic null sequence possibly changing from line to line. Further, we write $A \lesssim B$ if $A \leqslant C B$ for some deterministic constant $C>0$.

Step 1: We prove that $\left(\bar{S}_{k}^{(n)}\right)_{k \geqslant 0}$ is indeed an approximation for $\left(S_{k}^{(n)}\right)_{k \geqslant 0}$. Applying Proposition 1.5.7 as well as Assumptions 1.1, 1.2, 1.3 iii), and 1.7, we see that

$$
\begin{aligned}
& \mathbb{E} {\left[\sup _{k \leqslant T_{n}}\left\|S_{k}^{(n)}-\bar{S}_{k}^{(n)}\right\|_{E}^{2}\right] } \\
& \leqslant 4\left\|v_{b, 0}^{(n)}-v_{b, 0}\right\|_{L^{2}}^{2} \\
&+4 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|\sum_{j=1}^{k}\left(f_{b}\left[S_{j-1}^{(n)}\right]-f_{b}^{(n)}\left[S_{j-1}^{(n)}\right]\right) \Delta v^{(n)}\right\|_{L^{2}}^{2}\right]+4 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right] \\
&+4\left\|v_{a, 0}^{(n)}-v_{a, 0}\right\|_{L^{2}}^{2} \\
&+4 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|\sum_{j=1}^{k}\left(f_{a}\left[S_{j-1}^{(n)}\right]-f_{a}^{(n)}\left[S_{j-1}^{(n)}\right]\right) \Delta v^{(n)}\right\|_{L^{2}}^{2}\right]+4 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{a, k}^{(n)}\right\|_{L^{2}}^{2}\right] \\
&+ 2 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left|\sum_{j=1}^{k}\left(\varphi\left(S_{j-1}^{(n)}\right)-\varphi^{(n)}\left(S_{j-1}^{(n)}\right)\right) \Delta t^{(n)}\right|^{2}\right]+2 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left|R_{\varphi, k}^{(n)}\right|^{2}\right]+a_{n} \\
& \lesssim\left\|v_{b, 0}^{(n)}-v_{b, 0}\right\|_{L^{2}}^{2}+\left\|v_{a, 0}^{(n)}-v_{a, 0}\right\|_{L^{2}}^{2}+\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|R_{a, k}^{(n)}\right\|_{L^{2}}^{2}\right] \\
&+ \frac{\left(\Delta v^{(n)}\right)^{2}}{\left(\Delta t^{(n)}\right)^{2}}\left[\sup _{s \in E}\left\|f_{b}[s]-f_{b}^{(n)}[s]\right\|_{L^{2}}^{2}+\sup _{s \in E}\left\|f_{a}[s]-f_{a}^{(n)}[s]\right\|_{L^{2}}^{2}\right] \\
&+ \sup _{s \in E}\left|\varphi(s)-\varphi^{(n)}(s)\right|^{2}+\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left|R_{\varphi, k}^{(n)}\right|^{2}\right]+a_{n} \rightarrow 0 .
\end{aligned}
$$

Step 2: We prove upper bounds for the pathwise $L^{2}(\mathbb{R})$-errors of the approximated volume functions. Let $\left(S_{k}^{(n)}\right)_{k \geqslant 0}$ and $\left(\widetilde{S}_{k}^{(n)}\right)_{k \geqslant 0}$ be given as above. For all $k \leqslant T_{n}$, applying equation (1.2.2) in Assumption 1.1, we have

$$
\left\|v_{b, 0}\left(\cdot-\left(B_{k}^{(n)}-B_{0}^{(n)}\right)\right)-v_{b, 0}\left(\cdot-\left(\widetilde{B}_{k}^{(n)}-B_{0}^{(n)}\right)\right)\right\|_{L^{2}} \leqslant L\left|B_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right| .
$$

Further, using the Lipschitz-continuity of $f_{b}: E \times \mathbb{R} \rightarrow \mathbb{R}$ in both variables, cf. Assumption 1.3 iii), we have

$$
\begin{aligned}
& \left\|\sum_{j=1}^{k}\left(f_{b}\left[S_{j-1}^{(n)}\right]\left(\cdot-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)-f_{b}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(\cdot-\left(\widetilde{B}_{k}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right)\right)\right) \Delta t^{(n)}\right\|_{L^{2}} \\
& \leqslant \Delta t^{(n)} \sum_{j=1}^{k}\left\{\left\|f_{b}\left[S_{j-1}^{(n)}\right]\left(\cdot-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)-f_{b}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(\cdot-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)\right\|_{L^{2}}\right. \\
& \left.\quad+\left\|f_{b}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(\cdot-\left(B_{k}^{(n)}-B_{j-1}^{(n)}\right)\right)-f_{b}\left[\widetilde{S}_{j-1}^{(n)}\right]\left(\cdot-\left(\widetilde{B}_{k}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right)\right)\right\|_{L^{2}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \leqslant \Delta t^{(n)} \sum_{j=1}^{k} L\left\{\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+\left|B_{k}^{(n)}-B_{j-1}^{(n)}-\left(\widetilde{B}_{k}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right)\right|\right\} \\
& \leqslant \Delta t^{(n)} \sum_{j=1}^{k} 2 L\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+L T\left|B_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right|
\end{aligned}
$$

Combining both bounds, we conclude that

$$
\begin{equation*}
\left\|\bar{v}_{b, k}^{(n)}-\widetilde{v}_{b, k}^{(n)}\right\|_{L^{2}} \lesssim\left|B_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E} \tag{1.6.2}
\end{equation*}
$$

Similarly, one can show that

$$
\begin{align*}
\| \hat{v}_{b, k}^{(n)} & -\widetilde{v}_{b, k}^{(n)} \|_{L^{2}} \\
& \lesssim\left|\hat{B}_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left\{\left|\hat{B}_{j-1}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right|+\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}\right\} \tag{1.6.3}
\end{align*}
$$

and

$$
\begin{align*}
\| \bar{v}_{b, k}^{(n)}- & \hat{v}_{b, k}^{(n)} \|_{L^{2}} \\
& \lesssim\left|B_{k}^{(n)}-\hat{B}_{k}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left|B_{j-1}^{(n)}-\hat{B}_{j-1}^{(n)}\right| \lesssim \sup _{j \leqslant k}\left|B_{j}^{(n)}-\hat{B}_{j}^{(n)}\right| \tag{1.6.4}
\end{align*}
$$

Analogous estimates hold for the approximating processes on the ask side.
Step 3: We prove upper bounds for the differences of the price and time coefficient functions. Let $i \leqslant T_{n}$. Since $\left(\delta Z_{b, j}^{(n)}\right)_{j \geqslant 0}$ defines a martingale difference array, we can apply Doob's inequality for $p=2$ and Assumption 1.5 to conclude

$$
\begin{gather*}
\mathbb{E}\left[\sup _{k \leqslant i}\left(\sum_{j=1}^{k}\left(r_{b}^{(n)}\left(S_{j-1}^{(n)}\right)-r_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right)\right) \delta Z_{b, j}^{(n)}\right)^{2}\right] \\
\lesssim \mathbb{E}\left[\sum_{j=1}^{i}\left(r_{b}^{(n)}\left(S_{j-1}^{(n)}\right)-r_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right)\right)^{2} \Delta t^{(n)}\right] \\
\lesssim \Delta t^{(n)} \sum_{j=1}^{i} \mathbb{E}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}+a_{n} \tag{1.6.5}
\end{gather*}
$$

For the drift component, using the Lipschitz-continuity of $p_{b}$, we have

$$
\begin{align*}
\sup _{k \leqslant i}\left|\sum_{j=1}^{k}\left(p_{b}^{(n)}\left(S_{j-1}^{(n)}\right)-p_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right)\right) \Delta t^{(n)}\right| & \leqslant \Delta t^{(n)} \sum_{j=1}^{i}\left|p_{b}^{(n)}\left(S_{j-1}^{(n)}\right)-p_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}\right)\right| \\
& \leqslant L \Delta t^{(n)} \sum_{j=1}^{i}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+a_{n} . \tag{1.6.6}
\end{align*}
$$

Analogously for the time component and using the Lipschitz-continuity of $\varphi$, we conclude

$$
\begin{equation*}
\sup _{k \leqslant i}\left|\sum_{j=1}^{k}\left(\varphi\left(S_{j-1}^{(n)}\right)-\varphi\left(\widetilde{S}_{j-1}^{(n)}\right)\right) \Delta t^{(n)}\right| \leqslant L \Delta t^{(n)} \sum_{j=1}^{i}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+a_{n} . \tag{1.6.7}
\end{equation*}
$$

For the compensated jumps, we can argue as follows:

$$
\begin{align*}
& \mathbb{E} {\left[\sup _{k \leqslant i}\left(\sum_{j=1}^{k} \int_{-M}^{M}\left(\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right)\left(\mu_{b}^{J^{(n)}}-\nu_{b}^{J^{(n)}}\right)\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right)^{2}\right] } \\
& \stackrel{(1)}{\lesssim} \sum_{j=1}^{i} \mathbb{E}\left[\left(\int_{-M}^{M}\left(\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right)\left(\mu_{b}^{J^{(n)}}-\nu_{b}^{J^{(n)}}\right)\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right)^{2}\right] \\
& \lesssim \sum_{j=1}^{i} \mathbb{E}\left[\left(\int_{-M}^{M}\left|\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right| \mu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right)^{2}\right] \\
&+\sum_{j=1}^{i} \mathbb{E}\left[\left(\int_{-M}^{M}\left|\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right| \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right)^{2}\right] \\
& \stackrel{(2)}{=} \sum_{j=1}^{i} \mathbb{E}\left[\int_{-M}^{M}\left|\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right|^{2} \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right] \\
&+\sum_{j=1}^{i} \mathbb{E}\left[\left(\int_{-M}^{M}\left|\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right| \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right)^{2}\right] \\
& \stackrel{(3)}{\lesssim} \sum_{j=1}^{i} \mathbb{E}\left[\left(\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}+\left(\Delta x^{(n)}\right)^{2}\right) \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right),[-M, M]\right)\right] \\
&+\sum_{j=1}^{i} \mathbb{E}\left[\left(\left(\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\| \|_{E}+\Delta x^{(n)}\right) \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right),[-M, M]\right)\right)^{2}\right] \\
& \stackrel{(4)}{\lesssim} \Delta t^{(n)}\left(1+\Delta t^{(n)}\right) \sum_{j=1}^{i} \mathbb{E}\left[\left(\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}+\left(\Delta x^{(n)}\right)^{2}\right)\left(Q_{b}([-M, M])+a_{n}\right)\right] \\
& \lesssim \Delta t^{(n)} \sum_{j=1}^{i} \mathbb{E}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}+a_{n} . \tag{1.6.8}
\end{align*}
$$

In (1) we applied the Burkholder-Davis-Gundy inequality for $p=2$ (cf. Theorem IV.4.48 in $72 \mid$ ). In (2) we used the fact that $\mu^{J^{(n)}}$ is an integer valued random measure, which only increases at times $t_{k}^{(n)}, k=0, \ldots, T_{n}$, and that $\nu^{J^{(n)}}$ is the compensator of $\mu^{J^{(n)}}$. Finally, in (3) we applied the uniform Lipschitz-estimate for $\theta$ (cf. Assumption $1.6 \mathrm{v})$ ) and in (4) we used the estimate in Remark 1.6 .2 with $g \equiv 1$.

Finally, applying Assumption 1.6 iv) and the estimate in Remark 1.6.2, we have

$$
\begin{align*}
& \sup _{k \leqslant i}\left|\sum_{j=1}^{k} \int_{-M}^{M}\left(\theta_{b}^{(n)}\left(S_{j-1}^{(n)}, y\right)-\theta_{b}^{(n)}\left(\widetilde{S}_{j-1}^{(n)}, y\right)\right) \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right), d y\right)\right| \\
& \quad \lesssim \sum_{j=1}^{i}\left(\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+\Delta x^{(n)}\right) \nu_{b}^{J^{(n)}}\left(\left[t_{j}^{(n)}, t_{j+1}^{(n)}\right),[-M, M]\right) \\
& \quad \lesssim \Delta t^{(n)} \sum_{j=1}^{i}\left(\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+\Delta x^{(n)}\right)\left(Q_{b}([-M, M])+a_{n}\right) \\
& \quad \lesssim \Delta t^{(n)} \sum_{j=1}^{i}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+a_{n} \tag{1.6.9}
\end{align*}
$$

Again, analogous estimates hold for the respective processes on the ask side.
Step 4: We prove that $\left(\hat{S}_{k}^{(n)}\right)_{k \geqslant 0}$ is indeed an approximation for $\left(S_{k}^{(n)}\right)_{k \geqslant 0}$. Applying equations (1.6.3), 1.6.6), 1.6.7), and (1.6.9, we conclude

$$
\begin{aligned}
& \left\|\hat{S}_{k}^{(n)}-\widetilde{S}_{k}^{(n)}\right\|_{E} \\
& \leqslant\left|\hat{B}_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right|+\left\|\hat{v}_{b, k}^{(n)}-\widetilde{v}_{b, k}^{(n)}\right\|_{L^{2}}+\left|\hat{A}_{k}^{(n)}-\widetilde{A}_{k}^{(n)}\right|+\left\|\hat{v}_{a, k}^{(n)}-\widetilde{v}_{a, k}^{(n)}\right\|_{L^{2}}+\left|\hat{\tau}_{k}^{(n)}-\widetilde{\tau}_{k}^{(n)}\right| \\
& \lesssim\left|\hat{B}_{k}^{(n)}-\widetilde{B}_{k}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left|\hat{B}_{j-1}^{(n)}-\widetilde{B}_{j-1}^{(n)}\right|+\left|\hat{A}_{k}^{(n)}-\widetilde{A}_{k}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left|\hat{A}_{j-1}^{(n)}-\widetilde{A}_{j-1}^{(n)}\right| \\
& \quad+\Delta t^{(n)} \sum_{j=1}^{k}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+\left|\hat{\tau}_{k}^{(n)}-\widetilde{\tau}_{k}^{(n)}\right| \\
& \quad \begin{array}{l}
\lesssim \sup _{j \leqslant k}\left|\hat{B}_{j}^{(n)}-\widetilde{B}_{j}^{(n)}\right|+\sup _{j \leqslant k}\left|\hat{A}_{j}^{(n)}-\widetilde{A}_{j}^{(n)}\right|+\Delta t^{(n)} \sum_{j=1}^{k}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+\sup _{j \leqslant k}\left|\hat{\tau}_{j}^{(n)}-\widetilde{\tau}_{j}^{(n)}\right| \\
\lesssim \Delta t^{(n)} \sum_{j=1}^{k}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}+a_{n} \\
\lesssim \Delta t^{(n)} \sum_{j=1}^{k}\left(\left\|S_{j-1}^{(n)}-\hat{S}_{j-1}^{(n)}\right\|_{E}+\left\|\hat{S}_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}\right)+a_{n} .
\end{array} \$ .
\end{aligned}
$$

Applying a discrete version of the Gronwall Lemma (cf. e.g., Lemma 4.34 in 25 ), we
have for all $k \leqslant T_{n}$,

$$
\begin{equation*}
\left\|\hat{S}_{k}^{(n)}-\widetilde{S}_{k}^{(n)}\right\|_{E} \lesssim \Delta t^{(n)} \sum_{j=1}^{k}\left\|S_{j-1}^{(n)}-\hat{S}_{j-1}^{(n)}\right\|_{E}+a_{n} \lesssim \sup _{j \leqslant k-1}\left\|S_{j}^{(n)}-\hat{S}_{j}^{(n)}\right\|_{E}+a_{n} \tag{1.6.10}
\end{equation*}
$$

Now equations 1.6.5, 1.6.8, and 1.6.10 yield for $i \leqslant T_{n}$,

$$
\begin{align*}
\mathbb{E}\left[\sup _{k \leqslant i}\left(\hat{B}_{k}^{(n)}-B_{k}^{(n)}\right)^{2}\right] & \lesssim \Delta t^{(n)} \sum_{j=1}^{i} \mathbb{E}\left\|S_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}+a_{n} \\
& \lesssim \Delta t^{(n)} \sum_{j=1}^{i}\left(\mathbb{E}\left\|S_{j-1}^{(n)}-\hat{S}_{j-1}^{(n)}\right\|_{E}^{2}+\mathbb{E}\left\|\hat{S}_{j-1}^{(n)}-\widetilde{S}_{j-1}^{(n)}\right\|_{E}^{2}\right)+a_{n} \\
& \lesssim \Delta t^{(n)} \sum_{j=1}^{i} \mathbb{E}\left[\sup _{l \leqslant j-1}\left\|S_{l}^{(n)}-\hat{S}_{l}^{(n)}\right\|_{E}^{2}\right]+a_{n} \tag{1.6.11}
\end{align*}
$$

and a similar estimate holds for the best ask price and its approximation. Hence, we can conclude for all $i \leqslant T_{n}$ that

$$
\begin{aligned}
& \mathbb{E}\left[\sup _{k \leqslant i}\left\|\hat{S}_{k}^{(n)}-S_{k}^{(n)}\right\|_{E}^{2}\right] \\
& \leqslant \mathbb{E}\left[\sup _{k \leqslant i}\left(\hat{B}_{k}^{(n)}-B_{k}^{(n)}\right)^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant i}\left(\hat{A}_{k}^{(n)}-A_{k}^{(n)}\right)^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant i}\left|\bar{\tau}_{k}^{(n)}-\tau_{k}^{(n)}\right|^{2}\right] \\
& +2\left(\mathbb{E}\left[\sup _{k \leqslant i}\left\|\hat{v}_{b, k}^{(n)}-\bar{v}_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant i}\left\|\bar{v}_{b, k}^{(n)}-v_{b, k}^{(n)}\right\|_{L^{2}}^{2}\right]\right) \\
& \quad+2\left(\mathbb{E}\left[\sup _{k \leqslant i}\left\|\hat{v}_{a, k}^{(n)}-\bar{v}_{a, k}^{(n)}\right\|_{L^{2}}^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant i}\left\|\bar{v}_{a, k}^{(n)}-v_{a, k}^{(n)}\right\|_{L^{2}}^{2}\right]\right) \\
& \\
& \stackrel{(1)}{\vdots} \mathbb{E}\left[\sup _{k \leqslant i}\left(\hat{B}_{k}^{(n)}-B_{k}^{(n)}\right)^{2}\right]+\mathbb{E}\left[\sup _{k \leqslant i}\left(\hat{A}_{k}^{(n)}-A_{k}^{(n)}\right)^{2}\right]+a_{n} \\
& \quad(2) \\
& \stackrel{(2)}{\lesssim} \Delta t^{(n)} \sum_{j=1}^{i} \mathbb{E}\left[\sup _{l \leqslant j-1}\left\|S_{l}^{(n)}-\hat{S}_{l}^{(n)}\right\|_{E}^{2}\right]+a_{n} .
\end{aligned}
$$

In (1) we plugged in the estimate in 1.6.4 and used step 1 , while in (2) we used 1.6.11. With the discrete Gronwall Lemma (cf. Lemma 4.34 in 25) we conclude

$$
\mathbb{E}\left[\sup _{k \leqslant i}\left\|S_{k}^{(n)}-\hat{S}_{k}^{(n)}\right\|_{E}^{2}\right] \lesssim a_{n}\left(1+\Delta t^{(n)} \sum_{j=1}^{i} e^{\sum_{m=j+1}^{i} C \Delta t^{(n)}}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Step 5: The result now follows from

$$
\begin{aligned}
\mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|S_{k}^{(n)}-\widetilde{S}_{k}^{(n)}\right\|_{E}^{2}\right] & \leqslant 2 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|S_{k}^{(n)}-\hat{S}_{k}^{(n)}\right\|_{E}^{2}\right]+2 \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|\hat{S}_{k}^{(n)}-\widetilde{S}_{k}^{(n)}\right\|_{E}^{2}\right] \\
& \lesssim \mathbb{E}\left[\sup _{k \leqslant T_{n}}\left\|S_{k}^{(n)}-\hat{S}_{k}^{(n)}\right\|_{E}^{2}\right]+a_{n} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$, where (1.6.10) is used in (1). Then, step 4 yields the final convergence.

### 1.6.2 Proof of Theorem 1.5 .9

In this section we give the proof of Theorem 1.5 .9 applying results of Kurtz and Protter 58 about the convergence of infinite dimensional SDEs. Therefore, let us first construct the stochastic integral such that we can directly apply their results. In terms of the stochastic processes $Z_{I}^{(n)}, I=b, a$ introduced in 1.5.3) and $X_{I}^{(n)}(t, g):=$ $\int_{[-M, M]} g(y) \mu_{I}^{J^{(n)}}([0, t], d y), t \in[0, T], g \in C_{b}([-M, M])$, for $I=b, a$, we define, for any $n \in \mathbb{N}, t \in[0, T]$, and $g_{1}, g_{2} \in C_{b}([-M, M])$,

$$
\begin{equation*}
Y^{(n)}\left(t, g_{1}, g_{2}\right):=\left(t_{k}^{(n)}, Z_{b, k}^{(n)}, Z_{a, k}^{(n)}, X_{b}^{(n)}\left(t, g_{1}\right), X_{a}^{(n)}\left(t, g_{2}\right)\right), \quad t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right) . \tag{1.6.12}
\end{equation*}
$$

We formalize the stochastic integral with respect to $Y^{(n)}$ in Section 1.6.4.
According to the estimate in 1.5.15, it is enough to define our coefficient functions $G^{(n)}$ on the subspace $\widetilde{E} \subset E$, where

$$
\begin{aligned}
& \widetilde{E}:=\left\{\left(b, v_{b}, a, v_{a}, t\right) \in E:\right. \\
&\left.\left\|v_{I}(\cdot+x)-v_{I}(\cdot+\widetilde{x})\right\|_{L^{2}} \leqslant L(1+T)|x-\widetilde{x}|, \forall x, \widetilde{x} \in \mathbb{R}, I=b, a\right\} .
\end{aligned}
$$

The space $\widetilde{E}$ endowed with the norm $\|\cdot\|_{E}$ is a closed subspace of $E$ and hence again a Banach space. We define the coefficient functions $G^{(n)}: \widetilde{E} \rightarrow \hat{E}$ (see Section 1.6.4 for the definition of the space $\hat{E}$ ) via

$$
G^{(n)}:=\left(G_{b}^{(n)}, G_{a}^{(n)}, G_{t}^{(n)}\right)
$$

where

$$
\begin{aligned}
G_{b}^{(n)} & :=\left(G_{b}^{(n), 1}, G_{b}^{(n), 2}, 0, G_{b}^{(n), 4}, 0 ; G_{b}^{(n), 6}, 0,0,0,0\right) \\
G_{a}^{(n)} & :=\left(G_{a}^{(n), 1}, 0, G_{a}^{(n), 3}, 0, G_{a}^{(n), 5} ; G_{a}^{(n), 6}, 0,0,0,0\right) \\
G_{t}^{(n)} & :=\left(G_{t}^{(n), 1}, 0,0,0,0\right)
\end{aligned}
$$

and for $s=\left(b, v_{b}, a, v_{a}, t\right) \in \widetilde{E}, x \in \mathbb{R}$, and $y \in[-M, M]$, we define

$$
\begin{aligned}
G_{b}^{(n), 1}(s) & :=p_{b}^{(n)}(\psi(s)), & G_{a}^{(n), 1}(s) & :=p_{a}^{(n)}(\psi(s)), \\
G_{b}^{(n), 2}(s) & :=r_{b}^{(n)}(\psi(s)), & G_{a}^{(n), 3}(s) & :=r_{a}^{(n)}(\psi(s)), \\
G_{b}^{(n), 4}(s, y) & :=\theta_{b}^{(n)}(\psi(s), y), & G_{a}^{(n), 5}(s, y) & :=\theta_{a}^{(n)}(\psi(s), y),
\end{aligned}
$$

and
$G_{b}^{(n), 6}(s, x):=f_{b}[\psi(s)](-(x-b)), G_{a}^{(n), 6}(s, x):=f_{a}[\psi(s)](x-a), G_{t}^{(n), 1}(s):=\varphi(\psi(s))$.
The first five components of $G_{b}^{(n)}$ and $G_{a}^{(n)}$ describe the coefficient functions corresponding to the price, whereas their last five components describe the coefficient functions of the volume dynamics. Note that $G_{I}^{(n), 6}, I=b, a$, and $G_{t}^{(n), 1}$ are independent of $n \in \mathbb{N}$, which results from our construction of the approximated limit order book dynamics in step 3 of Section 1.5. Then the microscopic state dynamics can be represented as

$$
\widetilde{\eta}^{(n), a b s}(t)=S_{0}^{(n), a b s}+\int_{0}^{t} \int_{[-M, M]} G^{(n)}\left(\widetilde{\eta}^{(n), a b s}(u-), y\right) Y^{(n)}(d u, d y)
$$

for $t \in[0, T]$, where $S_{0}^{(n), a b s}:=\left(B_{0}^{(n)}, v_{b, 0}\left(-x+B_{0}^{(n)}\right), A_{0}^{(n)}, v_{a, 0}\left(x-A_{0}^{(n)}\right), 0\right)$.
Further, the candidate for the limit of the coefficient functions $G^{(n)}$ is given by

$$
G:=\left(G_{b}, G_{a}, G_{t}\right): \widetilde{E} \rightarrow \hat{E}
$$

where $G_{b}, G_{a}$ and $G_{t}$ are defined in the same way as $G_{b}^{(n)}, G_{a}^{(n)}$ and $G_{t}^{(n)}$ but their components are defined for $s=\left(b, v_{b}, a, v_{a}, t\right) \in \widetilde{E}, x \in \mathbb{R}$ and $y \in[-M, M]$ by

$$
\begin{aligned}
G_{b}^{1}(s) & :=p_{b}(\psi(s)), & G_{a}^{1}(s) & :=p_{a}(\psi(s)), \\
G_{b}^{2}(s) & :=r_{b}(\psi(s)), & G_{a}^{3}(s) & :=r_{a}(\psi(s)), \\
G_{b}^{4}(s, y) & :=\theta_{b}(\psi(s), y), & G_{a}^{5}(s, y) & :=\theta_{a}(\psi(s), y)
\end{aligned}
$$

and

$$
G_{b}^{6}(s, x):=f_{b}[\psi(s)](-(x-b)), \quad G_{a}^{6}(s, x):=f_{a}[\psi(s)](x-a), \quad G_{t}^{1}(s)=\varphi(\psi(s)) .
$$

The next lemma proves that $G$ is indeed the limit of the coefficient functions $G^{(n)}$.
Lemma 1.6.4. Let Assumptions 1.2, 1.3, 1.5, and 1.6 be satisfied. Then, as $n \rightarrow \infty$,

$$
\sup _{s \in \widetilde{E}}\left\|G^{(n)}(s)-G(s)\right\|_{\hat{E}} \rightarrow 0
$$

Proof. By the definition of the $\hat{E}$-norm, for all $s \in \tilde{E}$, we have

$$
\begin{aligned}
& \left\|G^{(n)}(s)-G(s)\right\|_{\hat{E}}^{2} \\
& \lesssim|\varphi(\psi(s))-\varphi(\psi(s))|^{2}+\sup _{I=b, a}\left|p_{I}^{(n)}(\psi(s))-p_{I}(\psi(s))\right|^{2}+\sup _{I=b, a}\left|r_{I}^{(n)}(\psi(s))-r_{I}(\psi(s))\right|^{2} \\
& \quad+\sup _{I=b, a}\left(\sup _{y \in[-M, M]}\left|\theta_{I}^{(n)}(\psi(s), y)-\theta_{I}(\psi(s), y)\right|\right)^{2}+\sup _{I=b, a}\left\|f_{I}[\psi(s)]-f_{I}[\psi(s)]\right\|_{L^{2}}^{2} .
\end{aligned}
$$

As the shift operator $\psi$ maps elements of $\tilde{E}$ to $E$, we can apply Assumptions 1.2, 1.5. and 1.6 to conclude that each summand converges to zero. This finishes the proof.

Considering the coefficient functions only on the subspace $\widetilde{E}$ ensures that their limits are still Lipschitz continuous as compositions of the Lipschitz-continuous functions $p_{b}$, $p_{a}, r_{b}, r_{a}, \theta_{b}, \theta_{a}, \varphi$ and (shifted) $f_{b}, f_{a}$ with $\psi$. This will be necessary to prove the relative compactness of the stochastic integral later on.

Lemma 1.6.5. Let Assumptions 1.2, 1.3, 1.5 , and 1.6 be satisfied. Then $G: \widetilde{E} \rightarrow \hat{E}$ defined above is Lipschitz-continuous. Especially, if $\left(s_{n}\right), s \subset D(\widetilde{E} ;[0, T])$ are such that $\sup _{u \leqslant T}\left\|s_{n}(u)-s(u)\right\|_{E} \rightarrow 0$, then also $\sup _{u \leqslant T}\left\|G\left(s_{n}(u)\right)-G(s(u))\right\|_{\hat{E}} \rightarrow 0$.

Proof. By Assumptions $1.2,1.3,1.5$ and 1.6 the functions $p_{I}, r_{I}, \theta_{I}, f_{I}$, for $I=b, a$, and $\varphi$ are Lipschitz continuous. Hence, for any $s=\left(b, v_{b}, a, v_{a}, t\right), \widetilde{s}=\left(\widetilde{b}, \widetilde{v}_{b}, \widetilde{a}, \widetilde{v}_{a}, \widetilde{t}\right) \in \widetilde{E}$,

$$
\begin{aligned}
& \|G(\widetilde{s})-G(s)\|_{\tilde{E}}^{2} \leqslant L^{2}\|\psi(\widetilde{s})-\psi(s)\|_{E}^{2} \\
& \leqslant L^{2}\left\{|\widetilde{b}-b|^{2}+\left\|\widetilde{v}_{b}(-(\cdot-\widetilde{b}))-v_{b}(-(\cdot-b))\right\|_{L^{2}}^{2}\right. \\
& \left.+|\widetilde{a}-a|^{2}+\left\|\widetilde{v}_{a}(\cdot-\widetilde{a})-v_{a}(\cdot-a)\right\|_{L^{2}}^{2}+|\widetilde{t}-t|^{2}\right\} \\
& \leqslant L^{2}\left\{2\|\widetilde{s}-s\|_{E}^{2}+2\left\|v_{b}(-(\cdot-\widetilde{b}))-v_{b}(-(\cdot-b))\right\|_{L^{2}}^{2}+2\left\|v_{a}(\cdot-\widetilde{a})-v_{a}(\cdot-a)\right\|_{L^{2}}^{2}\right\} \\
& \leqslant L^{2}\left\{2\|\widetilde{s}-s\|_{E}^{2}+2\left(L^{2}(1+T)^{2}\right)\left(|\widetilde{b}-b|^{2}+|\widetilde{a}-a|^{2}\right)\right\} \\
& \leqslant 2 L^{2}\left(1+L^{2}(1+T)^{2}\right)\|\widetilde{s}-s\|_{E}^{2} .
\end{aligned}
$$

Moreover, this implies that

$$
\sup _{u \leqslant T}\left\|G\left(s_{n}(u)\right)-G(s(u))\right\|_{\hat{E}} \leqslant C \sup _{u \leqslant T}\left\|s_{n}(u)-s(u)\right\|_{E} \rightarrow 0,
$$

if $s_{n}, s \in D(\widetilde{E} ;[0, T])$ satisfy $\sup _{u \leqslant T}\left\|s_{n}(u)-s(u)\right\|_{E} \rightarrow 0$ as $n \rightarrow \infty$.
With this preparation done, we are ready to present the proof of Theorem 1.5.9.
Proof of Theorem 1.5.9. We first note that $\widetilde{\eta}^{(n), a b s}$ takes values in the Banach space $\left(\widetilde{E},\|\cdot\|_{E}\right)$. Hence, we can restrict ourselves to this space and prove the statement in

### 1.6. TECHNICAL DETAILS

Theorem 1.5 .9 only in the closed subspace $D(\widetilde{E} ;[0, T])$. We need to show three things: the sequence $\left(\widetilde{\eta}^{(n), a b s}\right)_{n \in \mathbb{N}}$ is relative compact, any limit point satisfies

$$
\begin{align*}
B^{\eta}(t)= & B_{0}+\int_{0}^{t} p_{b}\left(\psi\left(\eta^{a b s}(u-)\right)\right) d u+\int_{0}^{t} r_{b}\left(\psi\left(\eta^{a b s}(u-)\right)\right) d Z_{b}(u) \\
& \quad+\int_{0}^{t} \int_{[-M, M]} \theta_{b}\left(\psi\left(\eta^{a b s}(u-), y\right) \mu_{b}^{Q}(d u, d y),\right. \\
u_{b}^{\eta}(t, x)= & v_{b, 0}\left(-x+B_{0}\right)+\int_{0}^{t} f_{b}\left[\psi\left(\eta^{a b s}(u-)\right)\right]\left(-\left(x-B^{\eta}(u-)\right)\right) d u, \\
A^{\eta}(t)= & A_{0}+\int_{0}^{t} p_{a}\left(\psi\left(\eta^{a b s}(u-)\right)\right) d u+\int_{0}^{t} r_{a}\left(\psi\left(\eta^{a b s}(u-)\right)\right) d Z_{a}(u)  \tag{1.6.13}\\
& \quad+\int_{0}^{t} \int_{[-M, M]} \theta_{a}\left(\psi\left(\eta^{a b s}(u-), y\right) \mu_{a}^{Q}(d u, d y),\right. \\
u_{a}^{\eta}(t, x)= & v_{a, 0}\left(x-A_{0}\right)+\int_{0}^{t} f_{a}\left[\psi\left(\eta^{a b s}(u-)\right)\right]\left(x-A^{\eta}(u-)\right) d u \\
\tau^{\eta}(t)= & \int_{0}^{t} \varphi\left(\psi\left(\eta^{a b s}(u-)\right)\right) d u
\end{align*}
$$

for $(t, x) \in[0, T] \times \mathbb{R}$, and there exists at most one solution $\eta^{a b s}=\left(B^{\eta}, u_{b}^{\eta}, A^{\eta}, u_{a}^{\eta}, \tau^{\eta}\right)$ of (1.6.13).

In order to prove the first two things, we will apply Theorem 7.6 in $[58]$. Let us verify its conditions: thanks to Proposition 1.5 .3 and Proposition 1.5.5, we have for any $m \in \mathbb{N}$ and $g_{1}, \cdots, g_{m} \in C_{b}\left([-M, M]^{2}\right)$,

$$
\left(Z_{b}^{(n)}, Z_{a}^{(n)}\right) \Rightarrow\left(Z_{b}, Z_{a}\right), \quad\left(X^{(n)}\left(\cdot, g_{1}\right), \cdots, X^{(n)}\left(\cdot, g_{m}\right)\right) \Rightarrow\left(X\left(\cdot, g_{1}\right), \cdots, X\left(\cdot, g_{m}\right)\right)
$$

in $D\left(\mathbb{R}^{2} ;[0, T]\right)$ and $D\left(\mathbb{R}^{m} ;[0, T]\right)$, respectively. Since $\left(Z_{b}, Z_{a}\right)$ is a standard planar Brownian motion, its paths are almost surely continuous. Hence, Corollary 3.33 in [50] implies the joint convergence of the integrators $\left(Z_{b}^{(n)}, Z_{a}^{(n)}\right)$ and $\left(X^{(n)}\left(\cdot, g_{1}\right), \cdots, X^{(n)}\left(\cdot, g_{m}\right)\right)$ for any $m \in \mathbb{N}$ and $g_{1}, \cdots, g_{m} \in C_{b}\left([-M, M]^{2}\right)$. By Theorem 1.5.5, $X(\cdot, g)$ is a pure jump Lévy process for all $g \in C_{b}\left([-M, M]^{2}\right)$. Hence, the quadratic covariation of $X(\cdot, g)$ with $Z_{I}, I=b, a$, is equal to zero almost surely and $X$ is independent of $\left(Z_{b}, Z_{a}\right)$. Moreover, the sequence of integrators is uniformly tight by Lemma 1.6.6. By Assumption 1.1, we have

$$
S_{0}^{(n), a b s} \rightarrow\left(B_{0}, v_{b, 0}\left(-\left(\cdot-B_{0}\right)\right), A_{0}, v_{a, 0}\left(\cdot-A_{0}\right), 0\right)=S_{0}^{a b s}
$$

Since $S_{0}^{a b s}$ is deterministic, we conclude that $\left(S_{0}^{(n), a b s}, Y^{(n)}\right) \Rightarrow\left(S_{0}^{a b s}, Y\right)$ in the Skorokhod topology. Lemma 1.6 .4 and Lemma 1.6 .5 imply that $G^{(n)}, n \in \mathbb{N}$, and $G$ satisfy Condition C. 2 of 58 . Moreover, as the shift operator $\psi$ maps elements of $\widetilde{E}$ to $E$, it is sufficient to prove the boundedness and compactness requirements with respect to the unshifted coefficient functions considered on the larger space $E$. To this end,

Assumptions 1.2 iii), 1.3 iii), 1.4 and 1.6 imply that for $I=b, a$,

$$
\sup _{n \in \mathbb{N}} \sup _{s \in E}\left\{\left|p_{I}^{(n)}(s)\right|+\left|r_{I}^{(n)}(s)\right|+\left\|\theta_{I}^{(n)}(s, \cdot)\right\|_{L^{\infty}}+\left\|f_{I}^{(n)}[s]\right\|_{L^{2}}+\left|\varphi^{(n)}(s)\right|\right\}<\infty
$$

The required compactness condition of the sequence of coefficient functions $\left(G^{(n)}\right)_{n \in \mathbb{N}}$ can be show as follows: applying the Heine-Borel theorem, we conclude the compactness condition for the coefficient functions $\left(p_{I}^{(n)}\right)_{n \in \mathbb{N}},\left(r_{I}^{(n)}\right)_{n \in \mathbb{N}}$, for $I=b$, a, and $\varphi$. Next, a combination of the Fréchet-Kolmogorov theorem with the equicontinuity and equitightness of $f_{I}, I=b, a$, cf. Assumption 1.3 iii), yields the compactness condition for the coefficient functions $f_{I}, I=b, a$. Last, the Arzelà-Ascoli theorem in combination with Assumption 1.6 i) ensures that the compactness condition is satisfied for the coefficient functions $\left(\theta_{I}^{(n)}\right)_{n \in \mathbb{N}}, I=b, a$. Hence, the requirements of Theorem 7.6 in 58] are satisfied and we may conclude that the sequence $\left(S_{0}^{(n), a b s}, \widetilde{\eta}^{(n), a b s}, Y^{(n)}\right)_{n \in \mathbb{N}}$ is relatively compact and any limit point satisfies 1.6 .13 ).

Next, we will show uniqueness of a strong solution to (1.6.13) by a standard Gronwall argument. Therefore, suppose $\eta=\left(B, v_{b}, A, v_{a}, \tau\right), \widetilde{\eta}=\left(B, \widetilde{v}_{b}, \widetilde{A}, \widetilde{v}_{a}, \widetilde{\tau}\right)$ are two strong solutions to 1.6 .13 . Then the Lipschitz-continuity of $G$ by Lemma 1.6.5 implies:

$$
\begin{aligned}
& \mathbb{E}\left[\sup _{s \leqslant t}|B(s)-\widetilde{B}(s)|^{2}\right] \\
& \leqslant 4 \mathbb{E}\left[\operatorname { s u p } _ { s \leqslant t } \left\{\left|\int_{0}^{s}\left(p_{b}(\psi(\eta(u-)))-p_{b}(\psi(\widetilde{\eta}(u-)))\right) d u\right|^{2}\right.\right. \\
& +\left|\int_{0}^{s}\left(r_{b}(\psi(\eta(u-)))-r(\psi(\widetilde{\eta}(u-)))\right) d Z_{b}(u)\right|^{2} \\
& +\left|\int_{0}^{s} \int_{[-M, M]}\left(\theta_{b}(\psi(\eta(u-)), y)-\theta_{b}(\psi(\widetilde{\eta}(u-)), y)\right)\left(\mu_{b}^{Q}-\nu_{b}^{Q}\right)(d u, d y)\right|^{2} \\
& \left.\left.+\left|\int_{0}^{s} \int_{[-M, M]}\left(\theta_{b}(\psi(\eta(u-)), y)-\theta_{b}(\psi(\widetilde{\eta}(u-)), y)\right) \nu_{b}^{Q}(d u, d y)\right|^{2}\right\}\right] \\
& \stackrel{(1)}{\lesssim} T \int_{0}^{t} \mathbb{E}\left|p_{n}(\psi(\eta(u-)))-p_{b}(\psi(\widetilde{\eta}(u-)))\right|^{2} d u+\int_{0}^{t} \mathbb{E}\left|r_{b}(\psi(\eta(u-)))-r_{b}(\psi(\widetilde{\eta}(u-)))\right|^{2} d u \\
& +T Q_{b}([-M, M]) \int_{0}^{t} \int_{[-M, M]} \mathbb{E}\left|\theta_{b}(\psi(\eta(u-)), y)-\theta_{b}(\psi(\widetilde{\eta}(u-)), y)\right|^{2} Q_{b}(d y) d u \\
& +\int_{0}^{t} \int_{[-M, M]} \mathbb{E}\left|\theta_{b}(\psi(\eta(u-)), y)-\theta_{b}(\psi(\widetilde{\eta}(u-)), y)\right|^{2} Q_{b}(d y) d u \\
& \stackrel{(2)}{\lesssim} \int_{0}^{t} \mathbb{E}\|\eta(u-)-\widetilde{\eta}(u-)\|_{E}^{2} d u .
\end{aligned}
$$

Here, in (1) we applied the Burkholder-Davis-Gundy inequality for $p=2$ and that $\nu_{b}^{Q}=\lambda \times Q_{b}$ is the compensator of the Poisson jump measure $\mu_{b}^{Q}$. In (2) we applied the Lipschitz-continuity of the coefficient function $G$. Similarly, it can be shown that
$\mathbb{E}\left[\sup _{s \leqslant t}|A(s)-\widetilde{A}(s)|^{2}\right] \lesssim \int_{0}^{t} \mathbb{E}\|\eta(u-)-\widetilde{\eta}(u-)\|_{E}^{2} d u$ and $\mathbb{E}\left[\sup _{s \leqslant t}\left\|v_{I}(s)-\widetilde{v}_{I}(s)\right\|_{L^{2}}^{2}\right] \lesssim$ $\int_{0}^{t} \mathbb{E}\|\eta(u-)-\widetilde{\eta}(u-)\|_{E}^{2} d u$ for $I=b, a$, and $\mathbb{E}\left[\sup _{s \leqslant t}|\tau(s)-\widetilde{\tau}(s)|^{2}\right] \lesssim \int_{0}^{t} \mathbb{E} \| \eta(u-)-$ $\widetilde{\eta}(u-) \|_{E}^{2} d u$. An application of Gronwall's Lemma (cf. e.g., Lemma 2.7 in 78 ) yields that $\eta=\widetilde{\eta}$ almost surely. Now, Corollary 7.8 in $[58]$ yields the existence of a unique strong solution of 1.6 .13 ) in the space $\left(\widetilde{E},\|\cdot\|_{E}\right)$.

Hence, $\widetilde{\eta}^{(n), a b s} \Rightarrow \eta^{a b s}$ in $D(\widetilde{E} ;[0, T])$, where $\eta^{a b s}$ is the unique solution to (1.6.13). Since $\widetilde{E} \subset E$ is closed, we moreover deduce the weak convergence in $D(E ;[0, T])$.

Finally, denote by $D$ the random set of times of discontinuities of $\eta^{a b s}$. Since all discontinuities of $\eta^{a b s}$ are generated by two independent Poisson random measure $\mu_{b}^{Q}$ and $\mu_{a}^{Q}$, which have no fixed times of discontinuity, the set $D$ is at most countable almost surely. Hence, 1.6 .13 is almost surely equivalent to 1.5 .12 .

### 1.6.3 Proof of Corollary 1.2 .8

Proof of Corollary 1.2.8. Thanks to Theorem 1.2.6. $S^{(n)}$ converges weakly in the Skorokhod topology to $S=\eta \circ \zeta$, where $\eta=\left(B^{\eta}, v_{b}^{\eta}, A^{\eta}, v_{a}^{\eta}, \tau^{\eta}\right)$ is the unique strong solution to the system given in 1.2 .13 and $\zeta(t):=\inf \left\{s>0: \tau^{\eta}(s)>t\right\}$.

First, we will show that $\eta$ solves the following coupled SDE-SPDE system: for $(t, x) \in[0, T] \times \mathbb{R}$,

$$
\begin{aligned}
d B^{\eta}(t)= & p_{b}(\eta(t)) d t+r_{b}(\eta(t)) d Z_{b}(t)+\int_{[-M, M]} \theta_{b}(\eta(t-), y) \mu_{b}^{Q}(d t, d y) \\
d v_{b}^{\eta}(t, x)= & \left(-\frac{\partial v_{b}^{\eta}}{\partial x}(t, x) p_{b}(\eta(t))+\frac{1}{2} \frac{\partial^{2} v_{b}^{\eta}}{\partial x^{2}}(t, x)\left(r_{b}(\eta(t))\right)^{2}+f_{b}[\eta(t)](x)\right) d t \\
& -\frac{\partial v_{b}^{\eta}}{\partial x}(t, x) r_{b}(\eta(t)) d Z_{b}(t)+\left(v_{b}^{\eta}\left(t-, x-\Delta B^{\eta}(t)\right)-v_{b}^{\eta}(t-, x)\right), \\
d A^{\eta}(t)= & p_{a}(\eta(t)) d t+r_{a}(\eta(t)) d Z_{a}(t)+\int_{[-M, M]} \theta_{a}(\eta(t-), y) \mu_{a}^{Q}(d t, d y), \\
d v_{a}^{\eta}(t, x)= & \left(\frac{\partial v_{a}^{\eta}}{\partial x}(t, x) p_{a}(\eta(t))+\frac{1}{2} \frac{\partial^{2} v_{a}^{\eta}}{\partial x^{2}}(t, x)\left(r_{a}(\eta(t))\right)^{2}+f_{a}[\eta(t)](x)\right) d t \\
& +\frac{\partial v_{a}^{\eta}}{\partial x}(t, x) r_{a}(\eta(t)) d Z_{a}(t)+\left(v_{a}^{\eta}\left(t-, x+\Delta A^{\eta}(t)\right)-v_{a}^{\eta}(t-, x)\right), \\
d \tau^{\eta}(t)= & \varphi(\eta(t)) d t
\end{aligned}
$$

Since both, $v_{b, 0}$ and $f_{b}[s]$, are twice continuously differentiable, we can apply Itô's formula for semimartingales with jumps (cf. Theorem II.7.32 in $[72]$ ) and obtain for
any $x \in \mathbb{R}$,

$$
\begin{aligned}
& v_{b, 0}\left(x-\left(B^{\eta}(t)-B_{0}\right)\right) \\
&= v_{b, 0}(x)-\int_{0}^{t} v_{b, 0}^{\prime}\left(x-\left(B^{\eta}(s-)-B_{0}\right)\right) d B^{\eta}(s) \\
&+\frac{1}{2} \int_{0}^{t} v_{b, 0}^{\prime \prime}\left(x-\left(B^{\eta}(s)-B_{0}\right)\right)\left(r_{b}(\eta(s))\right)^{2} d s+\sum_{0<s \leqslant t}\left\{v_{b, 0}\left(x-\left(B^{\eta}(s)-B_{0}\right)\right)\right. \\
&\left.-v_{b, 0}\left(x-\left(B^{\eta}(s-)-B_{0}\right)\right)+v_{b, 0}^{\prime}\left(x-\left(B^{\eta}(s-)-B_{0}\right)\right) \Delta B^{\eta}(s)\right\}
\end{aligned}
$$

as well as

$$
\begin{aligned}
& \int_{0}^{t} f_{b}[\eta(u)]\left(x-\left(B^{\eta}(t)-B^{\eta}(u)\right)\right) d u \\
& =\int_{0}^{t} f_{b}[\eta(u)](x) d u-\int_{0}^{t}\left(\int_{0}^{s} f_{b}^{\prime}[\eta(u)]\left(x-\left(B^{\eta}(s-)-B^{\eta}(u)\right)\right) d u\right) d B^{\eta}(s) \\
& +\frac{1}{2} \int_{0}^{t}\left(\int_{0}^{s} f_{b}^{\prime \prime}[\eta(u)]\left(x-\left(B^{\eta}(s)-B^{\eta}(u)\right)\right) d u\right)\left(r_{b}(\eta(s))\right)^{2} d s \\
& +\sum_{0<s \leqslant t}\left\{\int_{0}^{s}\left(f_{b}[\eta(u)]\left(x-\left(B^{\eta}(s)-B^{\eta}(u)\right)\right)-f_{b}[\eta(u)]\left(x-\left(B^{\eta}(s-)-B^{\eta}(u)\right)\right)\right) d u\right. \\
& \left.\quad+\left(\int_{0}^{s} f_{b}^{\prime}[\eta(u)]\left(x-\left(B^{\eta}(s-)-B^{\eta}(u)\right)\right) d u\right) \Delta B^{\eta}(s)\right\} .
\end{aligned}
$$

Combining both equations and using that $v_{b}^{\eta}(t, x)=v_{b}^{\eta}\left(t-, x-\Delta B^{\eta}(t)\right)$ for all $t \in[0, T]$, we conclude that

$$
\begin{aligned}
v_{b}^{\eta}(t, x)= & v_{b, 0}(x)-\int_{0}^{t} \frac{\partial v_{b}^{\eta}}{\partial x}(s-, x) d B^{\eta}(s)+\frac{1}{2} \int_{0}^{t} \frac{\partial^{2} v_{b}^{\eta}}{\partial x^{2}}(s, x)\left(r_{b}(\eta(s))\right)^{2} d s \\
& +\int_{0}^{t} f_{b}[\eta(s)](x) d s+\sum_{0<s \leqslant t} \frac{\partial v_{b}^{\eta}}{\partial x}(s-, x) \Delta B^{\eta}(s)+\sum_{0<s \leqslant t}\left(v_{b}^{\eta}(s, x)-v_{b}^{\eta}(s-, x)\right) \\
= & v_{b, 0}(x)+\int_{0}^{t}\left(-\frac{\partial v_{b}^{\eta}}{\partial x}(s, x) p_{b}(\eta(s))+\frac{1}{2} \frac{\partial^{2} v_{b}^{\eta}}{\partial x^{2}}(s, x)\left(r_{b}(\eta(s))\right)^{2}+f_{b}[\eta(s)](x)\right) d s \\
& -\int_{0}^{t} \frac{\partial v_{b}^{\eta}}{\partial x}(s, x) r_{b}(\eta(s)) d Z_{b}(s)+\sum_{0<s \leqslant t}\left(v_{b}^{\eta}\left(s-, x-\Delta B^{\eta}(s)\right)-v_{b}^{\eta}(s-, x)\right)
\end{aligned}
$$

Similarly, we can show that $v_{a}^{\eta}$ solves the SPDE $d v_{a}^{\eta}$ and also $B^{\eta}, A^{\eta}$, and $\tau^{\eta}$ solve the SDEs $d B^{\eta}, d A^{\eta}$, and $d \tau^{\eta}$. Hence, the solution $\eta$ of the system in 1.2 .13 indeed solves the stated SDE-SPDE system in (1.6.14).

By the definition of $\zeta$, observe that $\zeta=\left(\tau^{\eta}\right)^{-1}$. Hence,

$$
\zeta^{\prime}(t)=\left(\left(\tau^{\eta}\right)^{-1}\right)^{\prime}=\frac{1}{\left(\tau^{\eta}\right)^{\prime}\left(\left(\tau^{\eta}\right)^{-1}(t)\right)}=\frac{1}{\varphi(\eta \circ \zeta(t))}=\frac{1}{\varphi(S(t))}
$$

Since $\zeta$ is a continuous time change, the processes $S, Z_{I}, I=b, a$, and $X(\cdot, g)$ for
$g \in C_{b}\left([-M, M]^{2}\right)$ are adapted to $\zeta$ in the sense of Definition 10.13 in 49. In particular, for $I=b, a$, we have by Theorem 10.17 in [49] that

$$
\left\langle Z_{I} \circ \zeta, Z_{I} \circ \zeta\right\rangle_{t}=\left\langle Z_{I}, Z_{I}\right\rangle_{\zeta(t)}=\zeta(t)
$$

since $Z_{I}, I=b, a$, are standard Brownian motions. Moreover, by Theorem 10.27 in 49, there exists an integer-valued random jump measure $\widetilde{\mu}_{I}^{Q}$ such that for all $t \in[0, T]$,

$$
X_{I}(\zeta(t), g):=\int_{[-M, M]} g(y) \mu_{I}^{Q}([0, \zeta(t)], d y)=\int_{[-M, M]} g(y) \widetilde{\mu}_{I}^{Q}([0, t], d y)
$$

whose compensator is given by $\widetilde{\nu}_{I}^{Q}(d t, d y)=(\varphi(S(t)))^{-1} d t \times Q_{I}(d y)$. Finally, we can apply Proposition 10.21 and Theorem 10.27 in 49 and conclude for all bounded, continuous functions $g_{1}: E \rightarrow \mathbb{R}, g_{2}: E \times[-M, M] \rightarrow \mathbb{R}$, and $t \in[0, T]$,

$$
\begin{aligned}
\int_{0}^{\zeta(t)} g_{1}(\eta(u)) d u & =\int_{0}^{t} g_{1}(S(u))(\varphi(S(u)))^{-1} d u \\
\int_{0}^{\zeta(t)} g_{1}(\eta(u)) d Z_{I}(u) & =\int_{0}^{t} g_{1}(S(u)) \zeta^{1 / 2}(u) d \widetilde{Z}_{I}(u) \\
\int_{0}^{\zeta(t)} \int_{[-M, M]} g_{2}(\eta(u), y) \mu_{I}^{Q}(d u, d y) & =\int_{0}^{t} \int_{[-M, M]} g_{2}(S(u), y) \widetilde{\mu}_{I}^{Q}(d u, d y)
\end{aligned}
$$

where $\widetilde{Z}_{b}, \widetilde{Z}_{a}$ are again two independent standard Brownian motions, independent of $\widetilde{\mu}_{b}^{Q}, \widetilde{\mu}_{a}^{Q}$. Combining these observations with the the fact that $\eta$ solves the the system in (1.6.14), we conclude that $S$, starting in $S_{0}$, indeed solves the stated SDE-SPDE system.

### 1.6.4 Construction of the stochastic integral

In this section we introduce the stochastic integrals with respect to $Y^{(n)}$ and $Y$. The concept of integration follows [58]. Let $Z_{b}^{(n)}, Z_{a}^{(n)}$, and $X^{(n)}$ be the processes introduced in 1.5.3 and (1.5.7), respectively. Further, let us introduce $X_{I}^{(n)}(t, g):=$ $\int_{[-M, M]} g(y) \mu_{I}^{J^{(n)}}([0, t], d y)$ for $t \in[0, T], g \in C_{b}([-M, M])$, and $I=b, a$. Then we define the sequence of integrators $Y^{(n)}$ as in 1.6 .12 by putting for any $n \in \mathbb{N}, t \in[0, T]$, and $g_{1}, g_{2} \in C_{b}([-M, M])$,

$$
\begin{equation*}
Y^{(n)}\left(t, g_{1}, g_{2}\right):=\left(t_{k}^{(n)}, Z_{b, k}^{(n)}, Z_{a, k}^{(n)}, X_{b}^{(n)}\left(t, g_{1}\right), X_{a}^{(n)}\left(t, g_{2}\right)\right), \quad t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right) \tag{1.6.15}
\end{equation*}
$$

Since the processes $Z_{b}^{(n)}, Z_{a}^{(n)}$, and $X^{(n)}$ are semimartingales on the stochastic basis $\widetilde{B}^{(n)}=\left(\Omega^{(n)}, \mathcal{F}^{(n)}, \widetilde{\mathbb{G}}^{(n)}=\left(\widetilde{\mathcal{G}}_{t}^{(n)}\right)_{t \geqslant 0}, \mathbb{P}^{(n)}\right)$, where $\widetilde{\mathcal{G}}_{t}^{(n)}=\mathcal{F}_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{(n)}$, we conclude that $Y^{(n)}$ defines a semimartingale on $\widetilde{\mathcal{B}}^{(n)}$. As integrands for $Y^{(n)}$ we consider càdlàg,
$\left(\widetilde{\mathcal{G}}_{t}^{(n)}\right)$-adapted processes, which take their values in the space

$$
\hat{E}:=\hat{E}_{b} \times \hat{E}_{a} \times \hat{E}_{t} \quad \text { with }\left\|\left(X_{1}, X_{2}, X_{3}\right)\right\|_{\hat{E}}^{2}:=\left\|X_{1}\right\|_{\hat{E}_{b}}^{2}+\left\|X_{2}\right\|_{\hat{E}_{a}}^{2}+\left\|X_{3}\right\|_{\hat{E}_{t}}^{2}
$$

where

$$
\begin{aligned}
& \hat{E}_{b}:=\left(\mathbb{R} \times \mathbb{R} \times\{0\} \times C_{b}([-M, M]) \times\{0\}\right) \times\left(L^{2}(\mathbb{R}) \times\{0\} \times\{0\} \times\{0\} \times\{0\}\right), \\
& \hat{E}_{a}:=\left(\mathbb{R} \times\{0\} \times \mathbb{R} \times\{0\} \times C_{b}([-M, M])\right) \times\left(L^{2}(\mathbb{R}) \times\{0\} \times\{0\} \times\{0\} \times\{0\}\right), \\
& \hat{E}_{t}:=\mathbb{R}_{+} \times\{0\} \times\{0\} \times\{0\} \times\{0\},
\end{aligned}
$$

endowed with the norms

$$
\begin{align*}
\left\|\left(\left(a_{1}, a_{2}, 0, a_{4}, 0\right),\left(a_{6}, 0,0,0,0\right)\right)\right\|_{\hat{E}_{b}}^{2}:=\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}+\left\|a_{4}\right\|_{\infty}^{2}+\left\|a_{6}\right\|_{L^{2}(\mathbb{R})}^{2}, \\
\left\|\left(\left(a_{1}, 0, a_{3}, 0, a_{5}\right),\left(a_{6}, 0,0,0,0\right)\right)\right\|_{\hat{E}_{a}}^{2}:=\left|a_{1}\right|^{2}+\left|a_{3}\right|^{2}+\left\|a_{5}\right\|_{\infty}^{2}+\left\|a_{6}\right\|_{L^{2}(\mathbb{R})}^{2},  \tag{1.6.16}\\
\left\|\left(a_{1}, 0,0,0,0\right)\right\|_{\hat{E}_{t}}:=\left|a_{1}\right|^{2} .
\end{align*}
$$

In Kurtz and Protter [58], the integrands are allowed to take their values in more general spaces where all components are allowed to be unequal to zero. Since the discrete volume dynamics have a much easier structure than the integrals considered in [58], it is simply not necessary to consider e.g., $L^{2}(\mathbb{R})$-valued integrals whose integrators are indexed by $[0, T] \times C_{b}\left([-M, M]^{2}\right)$. Therefore, we reduce our considerations to this much easier setting. We define $\mathcal{S}_{\hat{E}}^{(n)}$ as the set of processes $a^{(n)}:=\left(a_{b}^{(n)}, a_{a}^{(n)}, a_{t}^{(n)}\right):$ $\Omega \times[0, T] \times \mathbb{R} \times[-M, M] \rightarrow \hat{E}:=\hat{E}_{b} \times \hat{E}_{a} \times \hat{E}_{t}$ that are of the form

$$
\begin{align*}
& a_{b}^{(n)}(t, x, y):=\left(\left(a_{b}^{(n), 1}(t), a_{b}^{(n), 2}(t), 0, a_{b}^{(n), 4}(t, y), 0\right),\left(a_{b}^{(n), 6}(t, x), 0,0,0,0\right)\right), \\
& a_{a}^{(n)}(t, x, y):=\left(\left(a_{a}^{(n), 1}(t), 0, a_{a}^{(n), 3}(t), 0, a_{a}^{(n), 5}(t, y)\right),\left(a_{a}^{(n), 6}(t, x), 0,0,0,0\right)\right), \\
& a_{t}^{(n)}(t, x, y):=\left(a_{t}^{(n), 1}(t), 0,0,0,0\right) \tag{1.6.17}
\end{align*}
$$

for càdlàg and $\left(\widetilde{\mathcal{G}}_{t}^{(n)}\right)$-adapted processes $a_{b}^{(n), 1}, a_{a}^{(n), 1}, a_{b}^{(n), 2}, a_{a}^{(n), 3}, a_{b}^{(n), 4}, a_{a}^{(n), 5}, a_{b}^{(n), 6}$, $a_{a}^{(n), 6}$ and $a_{t}^{(n), 1}$. For $a^{(n)} \in S_{\tilde{E}}^{(n)}$ with the representation as above, the integral with respect to $Y^{(n)}$ (which we will interpret as semimartingale random measure as mentioned
in Remark 1.5.6 is given as

$$
\begin{aligned}
& \int_{0}^{t} \int_{[-M, M]} a^{(n)}(u-, x, y) Y^{(n)}(d u, d y) \\
& :=\left(\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 1}\left(t_{k}^{(n)}-\right) \Delta t^{(n)}+\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 2}\left(t_{k}^{(n)}-\right) \delta Z_{b, k}^{(n)}\right. \\
& \\
& \quad+\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)\rfloor}\right\rfloor} \int_{[-M, M]} a_{b}^{(n), 4}\left(t_{k}^{(n)}-, y\right) \mu_{b}^{J^{(n)}}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right), d y\right), \\
& \\
& \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 6}\left(t_{k}^{(n)}-, x\right) \Delta t^{(n)}, \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{a}^{(n), 1}\left(t_{k}^{(n)}-\right) \Delta t^{(n)}+\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{a}^{(n), 3}\left(t_{k}^{(n)}-\right) \delta Z_{a, k}^{(n)} \\
& \\
& \quad+\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)\rfloor} \int_{[-M, M]} a_{a}^{(n), 5}\left(t_{k}^{(n)}-, y\right) \mu_{a}^{J^{(n)}}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right), d y\right),\right.} \\
& \left.\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{a}^{(n), 6}\left(t_{k}^{(n)}-, x\right) \Delta t^{(n)}, \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{t}^{(n), 1}\left(t_{k}^{(n)}-\right) \Delta t^{(n)}\right) .
\end{aligned}
$$

Similarly, let $Z_{b}, Z_{a}$ be two independent standard Brownian motions and let $X_{b}, X_{a}$ be given by $X_{I}(t, g):=\int_{[-M, M]} g(y) \mu_{I}^{Q}([0, t], d y)$ for $t \in[0, T], g \in C_{b}([-M, M])$, and $I=b, a$. Further, let $\left(\mathcal{F}_{t}\right)_{t \in[0, T]}$ be any filtration to which

$$
\begin{equation*}
Y\left(t, g_{1}, g_{2}\right)=\left(t, Z_{b}(t), Z_{a}(t), X_{b}\left(t, g_{1}\right), X_{a}\left(t, g_{2}\right)\right), \quad g_{1}, g_{2} \in C_{b}([-M, M]) \tag{1.6.18}
\end{equation*}
$$

is adapted. We will denote by $\mathcal{S}_{\hat{E}}$ the set of $\hat{E}$-valued processes $a$ of the form 1.6.17), for which, $a_{b}^{1}, a_{a}^{1}, a_{b}^{2}, a_{a}^{3}, a_{b}^{4}, a_{a}^{5}, a_{b}^{6}, a_{a}^{6}$ and $a_{t}^{1}$ are càdlàg, $\left(\mathcal{F}_{t}\right)$-adapted processes. The integral of $a \in \mathcal{S}_{\hat{E}}$ with respect to $Y$ is then defined by

$$
\begin{aligned}
& \quad \int_{0}^{t} \int_{[-M, M]} a(u-, x, y) Y(d u, d y) \\
& :=\left(\int_{0}^{t} a_{b}^{1}(u-) d u+\int_{0}^{t} a_{b}^{2}(u-) d Z_{b}(u)+\int_{0}^{t} \int_{[-M, M]} a_{b}^{4}(u-, y) \mu_{b}^{Q}(d u, d y),\right. \\
& \quad \int_{0}^{t} a_{b}^{6}(u-, x) d u, \int_{0}^{t} a_{a}^{1}(u-) d u+\int_{0}^{t} a_{a}^{3}(u-) d Z_{a}(u)
\end{aligned} \quad \begin{aligned}
& \left.\quad \int_{0}^{t} \int_{[-M, M]} a_{a}^{5}(u-, y) \mu_{a}^{Q}(d u, d y), \int_{0}^{t} a_{a}^{5}(u-, x) d u, \int_{0}^{t} a_{t}^{1}(u-) d u\right) .
\end{aligned}
$$

In order to prove the relative compactness of our processes in the proof of Theorem 1.5.9. we need to show that $Y^{(n)}$ is uniformly tight in the sense of 58 .

Theorem 1.6.6. Suppose Assumptions 1.4, 1.6, and 1.7 are satisfied. Then the sequence $\left(Y^{(n)}\right)_{n \in \mathbb{N}}$ is uniformly tight, i.e.,

$$
\mathcal{H}_{t}=\bigcup_{n}\left\{\left\|\int_{0}^{t} \int_{-M}^{M} a^{(n)}(u-, x, y) Y^{(n)}(d u, d y)\right\|_{E}: a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)}, \sup _{s \leqslant t}\left\|a^{(n)}(u)\right\|_{\hat{E}} \leqslant 1 \quad a . s .\right\}
$$

is stochastically bounded for all $t \in[0, T]$.
Proof. To prove that the sequence $\left(Y^{(n)}\right)_{n \in \mathbb{N}}$ is indeed uniformly tight, it suffices to show that for any $t \in[0, T]$ there exists a constant $C(t)$ such that for all $n \in \mathbb{N}$ and $a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)}$ with $\sup _{u \leqslant t}\left\|a^{(n)}(u)\right\|_{\hat{E}} \leqslant 1$,

$$
\begin{equation*}
\mathbb{E}\left\|\int_{0}^{t} \int_{[-M, M]} a^{(n)}(u-, x, y) Y^{(n)}(d u, d y)\right\|_{E} \leqslant C(t) \tag{1.6.19}
\end{equation*}
$$

Therefore, let $a^{(n)} \in \mathcal{S}_{\hat{E}}^{(n)}$ satisfying $\sup _{u \leqslant t}\left\|a^{(n)}(u)\right\|_{\hat{E}} \leqslant 1$. This gives us the following estimates. First,

$$
\mathbb{E}\left|\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 1}\left(t_{k}^{(n)}-\right) \Delta t^{(n)}\right| \leqslant \Delta t^{(n)} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left|a_{b}^{(n), 1}\left(t_{k}^{(n)}-\right)\right| \leqslant t .
$$

Second, using the triangular inequality we have

$$
\mathbb{E}\left\|\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 6}\left(t_{k}^{(n)}-, \cdot\right) \Delta t^{(n)}\right\|_{L^{2}} \leqslant \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \mathbb{E}\left\|a_{b}^{(n), 6}\left(t_{k}^{(n)}-, \cdot\right)\right\|_{L^{2}} \leqslant t .
$$

Third, we recall that $a^{(n)}(t)$ is $\left(\widetilde{\mathcal{G}}_{t}^{(n)}\right)$-adapted. Thus $a^{(n)}\left(t_{k}^{(n)}-\right) \in \mathcal{F}_{k-1}^{(n)}$ for all $k \leqslant T_{n}$ and we have

$$
\mathbb{E}\left(\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} a_{b}^{(n), 2}\left(t_{k}^{(n)}-\right) \delta Z_{b, k}^{(n)}\right)^{2}=\Delta t^{(n)} \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left(a_{b}^{(n), 2}\left(t_{k}^{(n)}-\right)\right)^{2} \leqslant t .
$$

Fourth, since $\nu_{b}^{J^{(n)}}$ is the compensator of the random jump measure $\mu_{b}^{J^{(n)}}$ and applying
the estimate in Remark 1.6.2, we have

$$
\begin{aligned}
& \mathbb{E}\left|\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \int_{[-M, M]} a_{b}^{(n), 4}\left(t_{k}^{(n)}-, y\right) \mu_{b}^{J^{(n)}}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right), d y\right)\right| \\
& \leqslant \mathbb{E}\left[\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \int_{[-M, M]}\left|a_{b}^{(n), 4}\left(t_{k}^{(n)}-, y\right)\right| \nu_{b}^{J^{(n)}}\left(\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right), d y\right)\right] \\
& \leqslant \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)}\left(\int_{[-M, M]} \mathbb{E}\left|a_{b}^{(n), 4}\left(t_{k}^{(n)}-, y\right)\right| Q_{b}(d y)+a_{n}\right) \\
& \leqslant Q_{b}([-M, M]) \sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \Delta t^{(n)} \mathbb{E}\left[\sup _{y \in[-M, M]}\left|a_{b}^{(n), 4}\left(t_{k}^{(n)}-, y\right)\right|\right]+a_{n} t \\
& \leqslant\left(Q_{b}([-M, M])+a_{n}\right) t \leqslant C t
\end{aligned}
$$

for some $C>0$, independent of $n$, since $Q_{b}$ is a finite measure by Assumption 1.6 and $\left(a_{n}\right)_{n \in \mathbb{N}}$ is a deterministic null-sequence. The remaining terms can be bounded in the same way. Combining all these upper bounds, we have shown that 1.6 .19 holds, yielding the claim.

# 2 A cross-border market model with limited transmission capacities 

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We develop a cross-border market model between two countries in which the transmission capacities that enable transactions between market participants of different countries are limited. Starting from two so-called reduced-form representations of national limit order book dynamics, we allow incoming market orders to be matched with standing volumes of the foreign market, resulting in cross-border trades. We introduce a microscopic model that consists of two bid and ask price processes, four queue length processes that describe the number of unexecuted limit orders at the best bid and ask prices, and a capacity process. The latter counts the net number of executed cross-border trades over time. Since the transmission capacities in our model are limited, our model alternates between regimes in which cross-border trades are possible and regimes in which incoming market orders can only be matched against limit orders of the same origin. If the size of an individual order converges to zero while the order arrival rate tends to infinity, we derive a continuous-time limit approximation of our microscopic market dynamics. If transmission capacities are available, the limit process behaves as follows: the volume dynamics is a four-dimensional linear Brownian motion in the positive orthant with oblique reflection at the axes. Each time two queues simultaneously hit zero, the process is reinitialized at a new value in the interior of $\mathbb{R}_{+}^{4}$. The capacity dynamics turns out to be a bounded continuous process of finite variation. Since the tick size in our model is constant, the price dynamics follows a two-dimensional pure jump process with jump times equal to those of the volume approximation. The usefulness of the ability to transact across borders is illustrated through a simulation study.

### 2.1 Introduction

Limit order books (LOBs) are a standard tool for price formation in modern financial markets. They are records of unexecuted buy and sell orders awaiting execution. While the recent financial mathematical literature is concerned with limit order book models for a single market, real-world opportunities to trade on multiple markets simultaneously require more complex model structures. For example, the introduction of the integrated European intraday electricity market "Single Intraday Coupling" (SIDC) has created the need to describe the effects of coupling multiple markets, i.e., when cross-border interactions are allowed between market participants of different countries. Due to limited transmission capacities in the SIDC, market participants could be temporarily prohibited from trading across borders. In this work, we introduce microscopic crossborder market dynamics between two countries which are based on limit order books. In particular, we limit the total number of cross-border trades. Unfortunately, the resulting microscopic market dynamics are too complex to get a good understanding of the effects of coupling multiple markets. For this reason, we introduce suitable scaling constants to our microscopic model. If the size of an individual order converges to zero while the order arrival rate tends to infinity, we show that our model can be approximated by a tractable, continuous-time regime switching process. Using this continuous-time approximation, we investigate the impacts of coupling two markets on price evolution in a detailed simulation study.

At any given point in time, a limit order book depicts the number of unexecuted buy and sell orders at different price levels (cf. Figure 2.1). The highest price a potential buyer is willing to pay is called the best bid price, whereas the best ask price is the smallest price of all placed sell orders. Incoming limit orders can be placed at many different price levels, while incoming market orders are matched against standing limit orders according to a set of priority rules.


Figure 2.1: Illustration of the state of a limit order book model.
For purely financial markets the financial mathematical literature already provides powerful tools for analyzing limit order books. One approach to study limit order books is based on an event-by-event description of the order flow as done in $20,22,32,40-43,54.62$. The derived stochastic systems typically yield realistic models as they preserve the discrete nature of the dynamics at high frequencies but turn out to be computationally
challenging. To overcome the drawbacks of these models, some researchers deal with continuum approximations of the order book, describing it through its time-dependent density satisfying either certain partial differential equations (cf. $[12,13,16,60]$ ) or certain stochastic partial differential equations (cf. [21,63]).

Combining these two approaches, one can introduce suitable scaling constants to the microscopic order book dynamics and study its scaling behavior when the number of orders gets large while each of them is of negligible size. The scaling limit can then either be described through a system of (partial) differential equations (in the "fluid" limit, where random fluctuations vanish), through a system of stochastic (partial) differential equations (in the "diffusion" limit, where random fluctuations dominate), or through a mixture thereof. Deriving a deterministic high-frequency limit for microscopic limit order book models guarantees that the scaling limit approximation stays tractable in view of practical applications. Such an approach is pursued by Horst and Paulsen [43], Horst and Kreher 40], and Gao and Deng [32]. However, the absence of arbitrage considerations encourages price approximations by diffusion processes. As discussed in Cont and de Larrard [19], depending on the market and/or stock of interest either a fluid or a diffusive volume approximation seems to be appropriated. Horst and Kreher 42 studied the approximation of microscopic order book dynamics by both diffusive price and volume processes in the scaling limit. However, their consideration of a diffusive infinite dimensional volume process is not suitable for practical applications, as e.g., the uniqueness of a solution to the established infinite dimensional stochastic differential equation is in general not guaranteed. For this reason, in Chapter 1 we have studied diffusive price approximations coupled with infinite dimensional fluid type volume approximations. This model yields realistic price approximations while the approximations of the infinite dimensional volumes are still tractable. The authors in [19] guaranteed that their diffusive volume approximation stays tractable considering only the standing volumes at the top of the book and hence reducing the state space of the limit order book to a finite-dimensional space. Moreover, the price dynamics is implicitly determined by the volume dynamics. As the tick size is constant, prices in the high-frequency limit must be approximated by pure jump processes and not by diffusion type processes.

In our work, we introduce a first model of a cross-border market between two countries based on limit order books. Our model is a further development of the one considered in Cont and de Larrard [19] in which the order flow directly effects the price evolution. The authors studied a reduced-form representation of a limit order book, i.e., in which the order book dynamics is given by the best bid and ask prices as well as the number of standing limit orders at the best bid respectively ask price (cf. Figure 2.2). With this reduction of the state space, a diffusion type limit for the queue lengths has been derived. In more detail, under heavy traffic conditions the authors prove that the bid and ask queue lengths are given in the high-frequency limit by a planar Brownian motion in the first quadrant with inward jumps at hits of the boundaries. At the same time, the price processes are implicitly determined by the volume dynamics and turn out to be pure jump processes whose jump times equal to those of the corresponding volume processes.


Figure 2.2: Illustration of a limit order book model with a reduced-state space.
We extend their model in multiple ways. First, we analyze the reduced-form representations of two limit order book models over time. Second, we allow market orders to be matched with standing volumes of the limit order book of the foreign country which leads to cross-border trades. Finally, motivated by the limitation of transmission capacities in the SIDC, we limit the total number of cross-border trades. This might lead to structural changes in the trading behavior and market matching mechanism as market orders can only be matched with domestic standing volumes if all capacities are occupied. For this reason, we need to keep track of the origin of each incoming order and introduce a capacity process that counts the net number of executed cross-border trades over time. Our microscopic model is therefore described by two best bid and ask queue size processes, two best bid and ask price processes, and a two-sided capacity process. Both, the price and capacity dynamics are determined implicitly by the queue size dynamics as follows: if one queue would become negative due to an incoming market order, the corresponding queue is set to zero and the remaining order size is depleted from the corresponding queue in the foreign market as long as both, enough standing volume and transmission capacity remain. This leads to a cross-border trade. If the cumulative best bid or ask queue would be depleted by an incoming market order, all queues are reinitialized by random variables (representing the depth of the books) and the price processes change by one tick. Each time a cross-border trade has been executed, the capacity process is updated. Now, starting in a so-called active regime in which cross-border trading is possible, we switch to a so-called inactive regime if the capacity process hits one of its boundaries (and hence the total number of cross-border trades has been executed). Then, in the inactive regime, market participants can only execute market orders against limit orders of the same origin. While in the active regime the best bid and ask prices of both national limit order books coincide, they become different in the inactive regime. With a simple trick of an efficient allocation of capacities, it is possible to switch back to an active regime. Based on this microscopic cross-border market model, by introducing appropriate scaling constants, we establish a high-frequency approximation in which the limit approximation is given by a continuous-time regime switching processes.

The limit process behaves during active regimes as follows: the volume dynamics is a four-dimensional linear Brownian motion in the positive orthant with oblique reflection at the axes. Each time two queues simultaneously hit zero, the process is reinitialized at a new value in the interior of $\mathbb{R}_{+}^{4}$. The bid (resp. ask) price dynamics
behaves as a two-dimensional pure jump process with jump times equal to those of the volume process. The capacity dynamics turns out to be a bounded continuous process of finite variation that is constructed from the local times at zero of the volume process components. In contrast, during inactive regimes, the volume dynamics behaves like a four-dimensional linear Brownian motion in the interior of $\mathbb{R}_{+}^{4}$. Each time it hits one of the axes, the two components corresponding to the origin of the depleted component are reinitialized at a new value in $(0, \infty)^{2}$ while the others stay unchanged. The bid (resp. ask) price dynamics follows a two-dimensional pure jump process whose components jump at hitting times of the corresponding components of the volume process of the axes. In particular, they follow two different one-dimensional pure jump processes which do almost surely not jump simultaneously.

The high-frequency approximation during inactive regimes can be deduced from the results in 19 . To study the scaling behavior of the cross-border market dynamics during active regimes, we characterize the bid/ask components of the volume dynamics between successive price changes as a series of solutions to the one-dimensional Skorokhod problem following successive reflections from the axes. This allows us to still apply the continuous mapping approach even though our reflection matrix in the definition of the active volume dynamics does not fulfill the usual regularity conditions considered in the literature of semimartingale reflecting Brownian motions, cf. e.g., [27, 77, 84. In this way, we are able to identify the limit process of the volume dynamics between consecutive price changes as a solution of a reflected stochastic differential equation with absorption. Thereafter, we can derive limit results for the price and capacity dynamics during active regimes.


Figure 2.3: The cross-border market model based on two LOBs: the queue size processes at the best bid (top left) and best ask price (top right), the bid price processes (bottom left), and the capacity process (bottom right). The white areas represent the active regimes whereas the gray ones represent the inactive regimes.

### 2.1. INTRODUCTION

The evolution of our cross-border market dynamics is depicted in Figure 2.3. We discuss the behavior of our cross-border market model in different market situations through a detailed simulation study. Moreover, we study the effects of coupling two markets on price evolution by comparing the mean number of price changes and the mean bid price ranges in simulated active and simulated inactive dynamics.

### 2.1.1 Model dynamics: empirical evidence

In this subsection, we motivate that the order flow, the trading behavior in neighboring countries, and the limitation of transmission capacities all have a considerable effect on the evolution of limit order books in a cross-border market and hence, should be included to any reasonable model. We analyze order book data ${ }^{11}$ (from March 05, 2020), trade book data ${ }^{11}$ (from June 30, 2020), and capacity flow data ${ }^{2}$ (from January 2021).

Incorporation of the order flow and rise in trading liquidity: empirical studies (cf. e.g. 37,55$]$ ) suggest that the incorporation of the market microstructure is crucial in any realistic model of intraday electricity markets with continuous trading. In 55 updated weather forecasts, trade volume, and the demand quote are identified as the main price-driving factors in the German intraday market. Similarly, it is shown in 37 that not only fundamentals, but also trading behavior are important determinants of the liquidity available in intraday electricity markets.

Moreover, the coupling of multiple intraday electricity markets has increased trading volumes compared to the time before the introduction of the SIDC (cf. Figure 2.4a). At the same time, the overall number of trades in the SIDC is steadily growing and has increased more than fivefold (from around 3.5 million to 18 million) since the launch of the SIDC in June 2018 (cf. Figure 2.4b). This might be explained, on the one hand, by the 2nd and 3rd go-live waves in November 2019 and September 2021, and, on the other hand, by the growing acceptance of the SIDC which is supported by e.g., the rising share of renewable energy in the European energy mix. We note that the 4 th and 5 th go-live waves are already planed coupling Greece and Slovakia with the other SIDC countries. Going forward, we expect this trend toward more liquid intraday electricity markets to continue as we anticipate a rising share of renewables in the European energy mix, an expansion of grid capacities, and a merging of products with different delivery durations into a single market.

[^5]
(a) Trading volume (in thousand MWh) before and after the 2nd go-live wave.

(b) Overall number of trades (in million) per quarter of the SIDC.

Figure 2.4: Rise in trading liquidity due to the launch of the SIDC.
At the same time, the main share of order and trading volume over the trading session appears in the last few hours before closing. This can be seen in Figure 2.5a in which we depict the cumulative trading volumes and execution prices of the trade book (from March 05, 2020) for expiry 2 pm over time. The increase of liquidity also yields relatively small bid-ask spreads compared to other electricity markets with continuous trading. In Figure 2.5b, we depict the best bid and ask prices (from June 30, 2020) for expiry 2 pm over the last three hours before closing. It shows that bid-ask spreads are most of the time smaller than one euro.


Figure 2.5: Executed trades and bid-ask spreads in the German continuous intraday electricity market for expiry 2 pm .

We expect the spreads to become smaller if the liquidity in the SIDC increases. In summary, even if these markets are still very illiquid compared to many financial markets, the development of the past years shows a clear trend toward significantly more liquid intraday electricity markets.

In the recent financial mathematical literature, a standard approach is to approximate the typically intractable microscopic limit order book dynamics by well-studied, tractable continuous-time scaling limits if the number of order events is large while

### 2.1. INTRODUCTION

each individual order is of negligible size. This might be a reasonable modeling assumption for highly liquid financial markets. Models for intraday electricity markets with continuous trading should include the market microstructure, i.e., each individual market and limit order. To obtain a good understanding of these markets, we will also study its high-frequency behavior after carefully formulating and incorporating scaling assumptions. The continuous increase in trading liquidity in the SIDC gives hope that the resulting limit approximations will gain plausibility in the future.

Influence of the trading behavior in the neighboring countries: coupling multiple national limit order books and the resulting possibility to match incoming market orders with standing volumes of foreign limit order books influence the trading behaviors. To see that, we compare the transaction flow between the German and Austrian market area. In Figure 2.6, we depict the number of transactions in the German respectively Austrian market area with focus on the buy and sell areas of transactions in form of two Sankey diagrams. In each diagram, the buy areas are depicted on the left and the sell areas on the right axis. "XBID" refers to products in which one part of a trade was made by a trading member of a different exchange (as opposed to EPEX SPOT). We observe that for both, the German and Austrian market area, the other country is the main importer of electricity in comparison to other countries. At the same time, Austria exports four times more electricity to Germany than it domestically transacts. We therefore expect Austria's trading activities to be heavily dependent on the German trading behavior. In contrast, Germany exports comparably few electricity to other countries and most of its transactions are made domestically.


Figure 2.6: Sankey diagram depicting the number of SIDC trades for delivery time 7 pm relative to the different buy and sell market areas. Buy areas are depicted on the left and sell areas on the right axis.

Studying the transaction flow over time reveals the same picture. In Figure 2.7, we
depict the cumulative trading volumes and execution prices relative to the execution time. We observe that the cumulative quantity in the German market area is much higher (over five times higher) than in the Austrian market area. The dashed gray lines in Figure 2.7 denote the last transaction between Austria and Germany which coincides with the last cross-border trade in both market areas.
In both pictures, we observe that the execution prices rapidly change after the last transaction between Germany and Austria. The behavior of the prices in both market areas indicates that traders from Germany could buy electricity for a smaller price while traders from Austria could sell electricity for a higher price due to the coupling. Additionally, in both pictures the execution prices recover after short time to their old levels. The rapid change of the execution prices in both market areas indicates that the coupling has a high influence on the trading behavior in this market situation. We further observe a decrease in the number of transactions in the Austrian market area after the last trade with Germany. This effect is hardly observable in the German market area. The higher liquidity in the German intraday market may be a reasonable explanation for that.


Figure 2.7: Transactions for the German and Austrian market area for delivery time 7 pm . The $y$-axis displays the executed trades: prices (left) and accumulated volumes (right). The time stamps are presented on the $x$-axis. The dotted gray line depicts the last executed trade between Germany and Austria.

Occupation of transmission capacities: to move toward a fully integrated European electricity market, the major obstacle to overcome is the limitation of transmission capacities. If national markets decouple during a trading session due to the full occupation of capacity, cross-border trading between them is prohibited. Different energy generation in different European countries, such as e.g., the utilization of nuclear or renewable energy resources, yields huge price differences in the national order books and hence motivate participants to buy or sell energy on the foreign market for the best price. The rapid change of the prices in Figure 2.7 after the last cross-border trade between Germany and Austria indicates a full occupation of the transmission

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capacity in our considered data set. Moreover, the available transmission capacities are typically occupied at the end of a trading session: Figure 2.8 shows that in most continuous intraday electricity markets in January 2021, the transmission capacities between France and Spain are occupied at the end of the trading session.

Transmission system operators expand the existing grid capacities. However, the different utilization of resources and the rising share of renewables in the European energy mix will probably still yield an occupation of transmission capacities in the future. Hence, incorporating a limitation of cross-border trading is necessary to develop an understanding of the SIDC market dynamics.


Figure 2.8: Available transmission capacity (yellow) and its occupation in direction Spain to France and vice versa in January 2021.

### 2.1.2 Outline of Chapter 2

The remainder of chapter is structured as follows: in Section 2.2 we specify the discretetime cross-border market dynamics $S^{(n)}$ and state conditions that guarantee its convergence to a continuous-time limit. Moreover, we introduce the active market dynamics $\widetilde{S}^{(n)}$ (resp. the inactive market dynamics $\widetilde{\widetilde{S}}^{(n)}$ ) describing the evolution of the crossborder market dynamics when cross-border trades are possible (resp. prohibited). Thereafter, in Section 2.3, we analyze the active dynamics and derive a functional convergence result for it. In Section 2.4, we provide an overview for analyzing the inactive market dynamics and state the corresponding limit result. Finally, in Section 2.5 with help of the convergence results for $\widetilde{S}^{(n)}$ and $\widetilde{S^{(n)}}$, we present our main result and prove that the cross-border market dynamics $S^{(n)}$ converges weakly in the Skorokhod topology to a continuous-time regime switching process. In Section 2.6, we discuss different market situations of our model in simulations and study the impact of
coupling two markets on price evolution. Many technical proofs are stated in Section 2.7.
Notation. We denote by $\mathbb{R}_{+}:=[0, \infty)$ and $\mathbb{R}_{-}:=(-\infty, 0]$ the positive and negative real half-line, respectively. Moreover, for each $x \in \mathbb{R}^{d}, d \geqslant 1$, we denote by $\|x\|^{2}:=$ $\sum_{i=1}^{d} x_{i}^{2}$ the euclidean norm in $\mathbb{R}^{d}$, and for $\omega \in D\left([0, T], \mathbb{R}^{d}\right),\|\omega\|_{\infty}:=\sup _{t \in[0, T]}\|\omega(t)\|$ the sup norm. For $[a, b] \subset[0, T]$, we further denote $\|\omega\|_{[a, b]}:=\left\|\left.\omega\right|_{[a, b]}\right\|_{\infty}$. Furthermore, for stochastic processes $X$ and $Y$ we write $X \simeq Y$ if they have the same finitedimensional distributions. In what follows, we encounter projections onto a single or onto multiple coordinates of a function $\omega \in D\left([0, T], \mathbb{R}^{k}\right)$ or a vector $x \in \mathbb{R}^{k}$. Therefore, let $\pi_{j}^{(k)}: D\left([0, T], \mathbb{R}^{k}\right) \rightarrow D([0, T], \mathbb{R}), k \geqslant 1$, denote the $j$-th projection map, i.e., for $\omega=\left(\omega_{1}, \cdots, \omega_{k}\right) \in D\left([0, T], \mathbb{R}^{k}\right)$, we have that

$$
\pi_{j}^{(k)} \omega=\omega_{j} \in D([0, T], \mathbb{R}), \quad \text { for } 1 \leqslant j \leqslant k
$$

With a little abuse of notation, we also write $\pi_{j}^{(k)} x=x_{j}$, for $x=\left(x_{1}, \cdots, x_{k}\right) \in \mathbb{R}^{k}$, $1 \leqslant j \leqslant k$. Further, let us denote by $\pi_{i, j}^{(k)}: D\left([0, T], \mathbb{R}^{k}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right), k \geqslant 2$, the $(i, j)$-th projection map, i.e., $\pi_{i, j}^{(k)} \omega=\left(\omega_{i}, \omega_{j}\right) \in D\left([0, T], \mathbb{R}^{2}\right), 1 \leqslant i, j \leqslant k$. Again, for $x \in \mathbb{R}^{k}$, we write $\pi_{i, j}^{(k)} x=\left(x_{i}, x_{j}\right) \in \mathbb{R}^{2}$. Finally, we introduce the following short hand notations: $\pi_{j}:=\pi_{j}^{(4)}$ and $\pi_{i, j}:=\pi_{i, j}^{(4)}$ for $1 \leqslant i, j \leqslant 4$. Moreover, $\pi_{F}:=\pi_{1,2}, \pi_{G}:=\pi_{3,4}$, $\pi_{b}:=\pi_{1,3}$, and $\pi_{a}:=\pi_{2,4}$. These projections will be used to determine the standing volumes at the bid and ask price of one country or at the bid/ask price of both countries from the four-dimensional queue size dynamics.

### 2.2 The microscopic market dynamics

Let us fix some finite time horizon $T>0$, the tick size $\delta>0$, and the space $E:=$ $\mathbb{R}^{2} \times \mathbb{R}_{+}^{4} \times \mathbb{R}$. Let us consider two neighboring countries $F$ ("France") and $G$ ("Germany"). Each of these countries has a national limit order book through which it can trade its goods domestically. Moreover, as long as enough transmission capacities remain, market orders can also be matched against the standing volumes of the foreign limit order book ( $G$ is the foreign country for orders with origin $F$ and vise versa). In the following, we describe the cross-border market dynamics by an extension of the reduced-form representation of a limit order book model introduced by Cont and de Larrard [19] to two possibly interacting limit order books. In more detail, we describe the random evolution of a sequence of cross-border market models through a sequence of $E$-valued stochastic processes $S^{(n)}:=\left(S^{(n)}(t)\right)_{t \in[0, T]}$ with

$$
S^{(n)}(t)=\left(B^{(n)}(t), Q^{(n)}(t), C^{(n)}(t)\right)
$$

where for each $n \in \mathbb{N}$, the $\mathbb{R}^{2}$-valued process $B^{(n)}:=\left(B^{F,(n)}, B^{G,(n)}\right)$ specifies the dynamics of the best bid prices in $F$ and $G$, the $\mathbb{R}_{+}^{4}$-valued process $Q^{(n)}:=\left(Q^{b, F,(n)}, Q^{a, F,(n)}\right.$, $\left.Q^{b, G,(n)}, Q^{a, G,(n)}\right)$ specifies the dynamics of standing volumes at the best bid/ask price
in $F$ and $G$, and the $\mathbb{R}$-valued process $C^{(n)}$ describes the (scaled) net number of crossborder trades. We will refer to $C^{(n)}$ as the capacity process of the cross-border market model $S^{(n)}$. In the following, for all $n \in \mathbb{N}$, the bid price process $B^{(n)}$ takes values in the subspace $(\delta \mathbb{Z})^{2} \subset \mathbb{R}^{2}$, where $\delta \mathbb{Z}:=\{j \delta: j \in \mathbb{Z}\}$ denotes the $\delta$-grid. Moreover, the best ask price process $A^{(n)}:=\left(A^{F,(n)}, A^{G,(n)}\right)$ is implicitly given by the best bid price process as we set $A^{I,(n)}=B^{I,(n)}+\delta$ for $I=F, G$ and all $n \in \mathbb{N}$. Throughout, we assume that all random variables are defined on some common, complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 2.2.1 (Constant spread condition in a reduced-form model). Motivated by the reduced-form representation of a limit order book model (cf. Cont and de Larrard [19, 20]), we assume that the spread is of fixed size $\delta>0$ for all $t \in[0, T]$ and $n \in \mathbb{N}$. Keeping the spread fixed over time is crucial as we only concentrate on modeling limit order placements outside the spread. However, this is not so unrealistic as a closing of a spread greater than one tick happens very quickly in liquid markets (cf. e.g. the empirical study in $\sqrt{199]}$ ). A frequently used assumption in recent literature dealing with high-frequency approximations of microscopic limit order book dynamics is that the tick size converges to zero as $n \rightarrow \infty$ (cf. e.g. [40, 42] and the model introduced in Chapter 11). However, for the reduced-form model, this would yield a zero spread as well as constant price processes in the high-frequency limit. For this reason, we also fix the tick size $\delta>0$ for all $n \in \mathbb{N}$.

The cross-border market dynamics change due to arriving market orders and limit orders at the best bid and ask queues in $F$ respectively $G$. For simplicity, we assume the time intervals between two consecutive order arrivals to be of equal length $\Delta t^{(n)}>0$. This assumption is also frequently used in other recent literature on limit order book models (cf. e.g. $[40,42])$. Note that extensions to randomly spaced arrival times are possible as has been done in [5, 19, 43] but this will not be in the focus of our analysis. Then, there are $T_{n}:=\left\lfloor T / \Delta t^{(n)}\right\rfloor$ such events taking place at times

$$
t_{k}^{(n)}:=k \Delta t^{(n)}, \quad k=1, \cdots, T_{n}
$$

We assume that $\Delta t^{(n)}$ goes to zero as $n \rightarrow \infty$, i.e., the number of orders increases as $n \rightarrow \infty$. Furthermore, we introduce the average size of a limit order placement, which we denote by $\Delta v^{(n)}>0$. As $\Delta t^{(n)}$, it is assumed to tend to zero as $n \rightarrow \infty$. In the following, we specify the sequence of order book models for which we establish a scaling limit when the average limit order size tends to zero while the number of order events tends to infinity.

### 2.2.1 The initial state

In the $n$-th model, the initial state of the cross-border market model is given by (positive) best bid prices $B_{0}^{F,(n)}, B_{0}^{G,(n)} \in \delta \mathbb{Z}$, non-negative queue sizes $Q_{0}^{(n)}:=\left(Q_{0}^{b, F,(n)}\right.$, $\left.Q_{0}^{a, F,(n)}, Q_{0}^{b, G,(n)}, Q_{0}^{a, G,(n)}\right) \in\left(\Delta v^{(n)} \mathbb{N}\right)^{4}$, and initially occupied capacity $C_{0}^{(n)} \in \Delta v^{(n)} \mathbb{Z}$. For simplicity, we choose $C_{0}^{(n)}=0$ for all $n \in \mathbb{N}$. Since we are interested in studying
the cross-border market dynamics when the transmission capacities are limited, we introduce $\kappa_{-}>0$ denoting the total quantity of transmission capacity in direction $F$ to $G$ (i.e. exports from $F$ ) and $\kappa_{+}>0$ denoting the total quantity in direction $G$ to $F$ (i.e. imports to $F$ ). Since $C_{0}^{(n)}=0$, we therefore assume that at time $t=0$ cross-border trading is possible. Moreover, if cross-border trading is allowed, it is natural to assume that the bid prices of $F$ and $G$ coincide, so we further assume that $B_{0}^{F,(n)}=B_{0}^{G,(n)}$ for all $n \in \mathbb{N}$. Hence, at time $t=0$, the state of the cross-border market is deterministic for all $n \in \mathbb{N}$ and is denoted by

$$
S_{0}^{(n)}:=\left(\left(B_{0}^{F,(n)}, B_{0}^{F,(n)}\right),\left(Q_{0}^{b, F,(n)}, Q_{0}^{a, F,(n)}, Q_{0}^{b, G,(n)}, Q_{0}^{a, G,(n)}\right), 0\right) \in E .
$$

In order to prove a convergence result for the microscopic model to a high-frequency limit, we need to state the following convergence assumptions on the initial values.

Assumption 2.1 (Convergence of the initial state). There exist $B_{0}^{F} \in \delta \mathbb{Z}$ and $Q_{0}:=$ $\left(Q_{0}^{b, F}, Q_{0}^{a, F}, Q_{0}^{b, G}, Q_{0}^{a, G}\right) \in(0, \infty)^{4}$ such that $B_{0}^{F,(n)} \rightarrow B_{0}^{F}$ and $Q_{0}^{(n)} \rightarrow Q_{0}$ as $n \rightarrow \infty$. In the following, we denote $S_{0}:=\left(\left(B_{0}^{F}, B_{0}^{F}\right),\left(Q_{0}^{b, F}, Q_{0}^{a, F}, Q_{0}^{b, G}, Q_{0}^{a, G}\right), 0\right) \in E$.

Note that under this assumption, neither the initial cumulative best bid queue nor the initial cumulative best ask queue is zero. This prevents an occurrence of a price changing event at time $t=0$.

### 2.2.2 Event types, order sizes, and the depth of the limit order books

The cross-border market dynamics change by incoming order events. In order to determine their effects on the state of the cross-border market dynamics $S^{(n)}$, we introduce two sequences of random variables $\left(\phi_{k}^{(n)}\right)_{k=1, \cdots, T_{n}}$ and $\left(\psi_{k}^{(n)}\right)_{k=1, \cdots, T_{n}}$ determining the type and the origin of an incoming order, i.e., for all $k=1, \cdots, T_{n}$,

$$
\phi_{k}^{(n)} \in\{b, a\}, \quad \psi_{k}^{(n)} \in\{F, G\},
$$

and $\phi_{k}^{(n)}=b$ (resp. $\phi_{k}^{(n)}=a$ ) if the $k$-th incoming order event affects the bid side (resp. the ask side) of a limit order book and $\psi_{k}^{(n)}=F$ (resp. $\psi_{k}^{(n)}=G$ ) if the $k$-th incoming order has origin $F$ (resp. origin $G$ ). Dependent on the evaluation of the random vector $\left(\phi_{k}^{(n)}, \psi_{k}^{(n)}\right) \in\{b, a\} \times\{F, G\}$, we observe four different order events which change the state of $S^{(n)}$. For $k=1, \cdots, T_{n}$ the following two order events correspond to possible transactions in direction $F$ to $G$ (i.e. exports from $F$ ):
(b,F) A market sell / limit buy order with origin $F$ arrives.
( $\mathbf{a}, \mathbf{G}$ ) A market buy / limit sell order with origin $G$ arrives.
The next two order events correspond to possible transactions in direction $G$ to $F$ (i.e. imports to $F$ ):
$(\mathbf{b}, \mathbf{G})$ A market sell / limit buy order with origin $G$ arrives.
( $\mathbf{a}, \mathbf{F}$ ) A market buy / limit sell order with origin $F$ arrives.
In the following, we refer to the evaluation of the random vector $\left(\phi_{k}^{(n)}, \psi_{k}^{(n)}\right)$ as the type of the $k$-th order. Further, we denote by $\mathcal{I}^{E x}:=\{(b, F),(a, G)\}$ and $\mathcal{I}^{I m}:=\{(a, F),(b, G)\}$ the order types corresponding to possible exports from $F$ or imports to $F$, respectively.

Next, let us introduce the random sequence $\left(V_{k}^{(n)}\right)_{k=1, \cdots, T_{n}}$ representing the sizes of incoming orders and let us denote by

$$
V_{k}^{i, I,(n)}:=V_{k}^{(n)} \mathbb{1}_{\left\{\left(\phi_{k}^{(n)}, \psi_{k}^{(n)}\right)=(i, I)\right\}}
$$

the order sizes of incoming orders of type $(i, I) \in\{b, a\} \times\{F, G\}, k=1, \cdots, T_{n}$. Throughout, we assume that $V_{k}^{(n)} \in\left\{-\Delta v^{(n)}, \Delta v^{(n)}\right\}$ for all $k=1, \cdots, T_{n}$ and $n \in \mathbb{N}$. Note that $V_{k}^{(n)}=\Delta v^{(n)}$ if the $k$-th order is a limit order placement at the best bid or ask price and $V_{k}^{(n)}=-\Delta v^{(n)}$ if the $k$-th order is a market order.

In order to derive a heavy traffic approximation for the cross-border market model $S^{(n)}$, we need to state further assumptions to our model. First, we present an assumption on the mean and covariance structure of $\left(V_{k}^{i, I,(n)},(i, I) \in\{b, a\} \times\{F, G\}\right)_{k=1, \cdots, T_{n}}$. In particular, we allow a dependence structure corresponding to strong (or uniform) mixing of the incoming orders. Combined with the right scaling relation between $\Delta v^{(n)}$ and $\Delta t^{(n)}$, this assumption guarantees that the partial sums of the order sizes verify a certain version of Donsker's theorem for a dependent sequence of random variables (cf. [8, Theorem 19.1]). Note, that this dependence condition can be replaced by a $\rho$-mixing condition as discussed in [8, Theorem 19.2] and is therefore stronger than the $\alpha$-mixing condition.
Assumption 2.2 (Sequence of order sizes). For all $n \in \mathbb{N},\left(V_{k}^{i, I,(n)},(i, I) \in\{b, a\} \times\right.$ $\{F, G\})_{k=1, \cdots, T_{n}}$ is a stationary, uniform mixing array of random variables. Moreover,
i) there exist $\mu^{i, I,(n)} \in \mathbb{R}, \sigma^{i, I,(n)}>0$ for all $(i, I) \in\{b, a\} \times\{F, G\}$ such that

$$
\begin{gathered}
\mathbb{E}\left[V_{1}^{i, I,(n)}\right]=\left(\Delta v^{(n)}\right)^{2} \mu^{i, I,(n)} \\
\operatorname{Var}\left[V_{1}^{i, I,(n)}\right]+2 \sum_{k=2}^{T_{n}} \operatorname{Cov}\left[V_{1}^{i, I,(n)}, V_{k}^{i, I,(n)}\right]=\left(\Delta v^{(n)}\right)^{2}\left(\sigma^{i, I,(n)}\right)^{2}
\end{gathered}
$$

as well as $\sigma^{(i, I),(j, J),(n)} \in \mathbb{R}$ for all $(i, I),(j, J) \in\{b, a\} \times\{F, G\}$ with $(i, I) \neq(j, J)$ such that

$$
\begin{aligned}
& 2 \operatorname{Cov}\left[V_{1}^{i, I,(n)}, V_{1}^{j, J,(n)}\right] \\
+ & 2 \sum_{k=2}^{T_{n}}\left(\operatorname{Cov}\left[V_{1}^{i, I,(n)}, V_{k}^{j, J,(n)}\right]+\operatorname{Cov}\left[V_{k}^{i, I,(n)}, V_{1}^{j, J,(n)}\right]\right)=\left(\Delta v^{(n)}\right)^{2} \sigma^{(i, I),(j, J),(n)}
\end{aligned}
$$

ii) For all $(i, I) \in\{b, a\} \times\{F, G\}$, there exist $\mu^{i, I} \in \mathbb{R}$, $\sigma^{i, I}>0$ and for all $(i, I),(j, J) \in\{b, a\} \times\{F, G\}$ with $(i, I) \neq(j, J)$ there exist $\sigma^{(i, I),(j, J)} \in \mathbb{R}$ such
that as $n \rightarrow \infty$, we have

$$
\left(\mu^{i, I,(n)}, \sigma^{i, I,(n)}\right) \rightarrow\left(\mu^{i, I}, \sigma^{i, I}\right) \quad \text { and } \quad \sigma^{(i, I),(j, J),(n)} \rightarrow \sigma^{(i, I),(j, J)} .
$$

iii) For all $(i, I),(j, J) \in\{b, a\} \times\{F, G\}$ with $(i, I) \neq(j, J)$, the corresponding correlation coefficient

$$
\rho^{(i, I),(j, J)}:=\frac{\sigma^{(i, I),(j, J)}}{\sigma^{i, I} \sigma^{j, J}} \quad \text { satisfies } \quad\left|\rho^{(i, I),(j, J)}\right|<1 .
$$

The dependence structure given in the above assumption either requires that the sequence of order sizes becomes independent as $n \rightarrow \infty$ or asks for $m$-dependence of the sequence of order sizes for all $n \in \mathbb{N}$ large enough, for any $m \in \mathbb{N}$ fixed, i.e., $\left(V_{1}^{(n)}, \cdots, V_{k}^{(n)}\right)$ and $\left(V_{k+j}^{(n)}, \cdots, V_{k+j+l}^{(n)}\right)$ are independent for all $k, l \in \mathbb{N}$ for large $n$ whenever $j>m$. This dependence structure is frequently referred in the literature as weak dependence. The third statement in the above assumption requires that the partial sums of the order sizes corresponding to different order types converge to limits with different underlying Brownian motions, i.e., a perfect correlation between their scaling limits is prohibited.
The next assumption introduces the critical scaling assumption under which the partial sums of the order sizes verify a functional central limit theorem. A similar assumption is stated in e.g. [20,42].

Assumption 2.3 (Relation between the scaling parameters). There exists a constant $C>0$ such that

$$
\lim _{n \rightarrow \infty} \frac{\Delta t^{(n)}}{\left(\Delta v^{(n)}\right)^{2}}=C .
$$

In what follows, we assume that $C=1$. Note, that each other constant would simply lead to further constants in the scaling limit.

In order to describe the dynamics of the queue size process $Q^{(n)}$, we need to characterize its value after price changes. In the following, we allow the sizes of the queue lengths after a price change to depend on the current state of the queue length process. This is motivated by pegged limit orders which are typically observed in electronic trading platforms. Let $\left(\tau_{k}^{(n)}\right)_{k \geqslant 1}$ be the sequence of stopping times at which we observe price changes in our cross-border market dynamics $S^{(n)}$.

Assumption 2.4 (Size of the order queues after price changes).
i) For all $n \in \mathbb{N}$, there exist independent sequences of iid random variables $\left(\epsilon_{k}^{+,(n)}\right)_{k \geqslant 1}$ and $\left(\epsilon_{k}^{-,(n)}\right)_{k \geqslant 1}$ with values in $(0, \infty)^{4}$, where $\epsilon_{1}^{+,(n)} \sim f_{n}^{+}$and $\epsilon_{1}^{-,(n)} \sim f_{n}^{-}$for some probability distributions $\left(f_{n}^{+}\right)_{n \in \mathbb{N}},\left(f_{n}^{-}\right)_{n \in \mathbb{N}}$. Moreover, there exists a function $\Phi^{(n)}: \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4} \rightarrow\left(\Delta v^{(n)} \mathbb{N}\right)^{4}$ such that

$$
\exists \alpha>0, \forall(x, y) \in \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4}, \forall j=1,2,3,4, \quad \pi_{j} \Phi^{(n)}(x, y) \geqslant \alpha \pi_{j} y
$$

Furthermore, the queue sizes after the $k$-th price change are equal to $R_{k}^{+,(n)}$ or $R_{k}^{-,(n)}$ depending on whether the $k$-th price change is a price increase or decrease, where for all $k \geqslant 1$, we set

$$
\begin{align*}
R_{k}^{+,(n)} & :=\Phi^{(n)}\left(Q^{(n)}\left(\tau_{k}^{(n)}-\right), \epsilon_{k}^{+,(n)}\right) \\
R_{k}^{-,(n)} & :=\Phi^{(n)}\left(Q^{(n)}\left(\tau_{k}^{(n)}-\right), \epsilon_{k}^{-,(n)}\right) \tag{2.2.1}
\end{align*}
$$

ii) Additionally, there exist probability distributions $f^{+}$and $f^{-}$on $(0, \infty)^{4}$ and $\Phi \in C^{2}\left(\mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4},(0, \infty)^{4}\right)$, such that

$$
\left(f_{n}^{+}, f_{n}^{-}\right) \Rightarrow\left(f^{+}, f^{-}\right) \quad \text { and } \quad\left\|\Phi^{(n)}-\Phi\right\|_{\infty} \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

Remark 2.2.2. With minor modifications, we could also allow in the subsequent analysis that the distributions determining the queue sizes after price changes depend on the origin of the queue whose depletion caused the price change. For notational reasons, we will not include this to our model.

In the next subsections, for all $n \in \mathbb{N}$, we describe the dynamics of the cross-border market model $S^{(n)}$ as follows: if there are enough transmission capacities remain, the national LOBs are coupled, i.e., incoming market orders can be matched against the standing volumes of the national and foreign limit order book. As the capacities in both directions are limited by $\kappa_{-}, \kappa_{+}>0$, it may happen that the national LOBs decouple, i.e., market orders can only be matched against limit orders with the same origin. Hence, our cross-border market model switches between the following two regimes:

- the active regime in which the LOBs of $F$ and $G$ are coupled, and
- the inactive regime in which the LOBs of $F$ and $G$ are decoupled.

In order to describe how $S^{(n)}$ behaves during its different regimes, we introduce the active dynamics $\widetilde{S}^{(n)}$ and inactive dynamics $\widetilde{S}^{(n)}$ describing the evolution of the two national LOBs as if we were in the active respectively inactive regime for the whole trading period.

### 2.2.3 Description of the active dynamics

In this subsection, we introduce for each $n \in \mathbb{N}$ the active dynamics given by the piecewise constant interpolation

$$
\widetilde{S}^{(n)}(t)=\sum_{k=0}^{T_{n}} \widetilde{S}_{k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t), \quad t \in[0, T]
$$

of the $E$-valued random variables

$$
\widetilde{S}_{k}^{(n)}:=\left(\widetilde{B}_{k}^{(n)}, \widetilde{Q}_{k}^{(n)}, \widetilde{C}_{k}^{(n)}\right), \quad k \in \mathbb{N}_{0}
$$

where $\widetilde{B}_{k}^{(n)}$ denotes the bid prices of $F$ and $G, \widetilde{Q}_{k}^{(n)}$ denotes the sizes of the best bid respectively ask queues in $F$ and $G$, and $\widetilde{C}_{k}^{(n)}$ denotes the net number of cross-border trades after $k$ order events. Note that $\left(\widetilde{C}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ describes net number of crossborder trades in an unlimited setting, i.e., as if $\kappa_{-}=\kappa_{+}=\infty$. Since the national LOBs are coupled in the active dynamics, we may suppose that $\widetilde{B}_{k}^{F,(n)}=\widetilde{B}_{k}^{G,(n)}$ for all $k=0, \cdots, T_{n}$ and $n \in \mathbb{N}$. For the same reason, we have chosen the initial best bid prices to coincide in Assumption 2.1. Moreover, we allow a national order queue to be equal to zero as long as the corresponding order queue in the foreign market is strictly larger than zero. Keeping this in mind, we can summarize the national order books in a shared order book (cf. Figure 2.9).


Figure 2.9: Summary of the national order books in a shared order book, provided the national markets are coupled.

Next, we specify how incoming orders change the state of the active dynamics. Therefore, let us denote by $\left(\widetilde{\tau}_{l}^{(n)}\right)_{l \geqslant 1}$ the sequences of stopping times at which we observe a price change in $\left(\widetilde{S}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$. Further, we introduce the sequences of random variables representing the order sizes after a price change, for $l \geqslant 1$, by

$$
\widetilde{R}_{l}^{+,(n)}:=\Phi^{(n)}\left(\widetilde{Q}_{\left[\widetilde{\tau}_{l}^{(n)}-/ \Delta t^{(n)}\right\rfloor}^{(n)}, \epsilon_{l}^{+,(n)}\right), \quad \widetilde{R}_{l}^{-,(n)}:=\Phi^{(n)}\left(\widetilde{Q}_{\left\lfloor\widetilde{\tau}_{l}^{(n)}-/ \Delta t^{(n)}\right\rfloor}^{(n)}, \epsilon_{l}^{-,(n)}\right) .
$$

Let $l(k)$ denote the number of price changes after $k$ order events in $\left(\widetilde{S}_{k}^{(n)}\right)_{k=0, \ldots, T_{n}}$. Then, $\left(\widetilde{S}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ evolves as follows: let $\widetilde{S}_{0}^{(n)} \in(\delta \mathbb{Z})^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z} \subset E$ with $\widetilde{B}_{0}^{F,(n)}=\widetilde{B}_{0}^{G,(n)}$ be the deterministic initial state and denote by $\widetilde{Q}_{k}^{i,(n)}:=\widetilde{Q}_{k}^{i, F,(n)}+$ $\widetilde{Q}_{k}^{i, G,(n)}$ the cumulative queue of type $i=b, a$ after $k \geqslant 1$ order events. If the $k$-th incoming order is of type $(b, F)$, then

$$
\begin{align*}
& \widetilde{S}_{k}^{(n)}=\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(V_{k}^{(n)}, 0,0,0\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, F,(n)} \geqslant-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0,0, V_{k}^{(n)}, 0\right), V_{k}^{(n)}\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, F(n)}<-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{B}_{k-1}^{(n)}-(\delta, \delta), \widetilde{R}_{l(k),(n)}^{-,(n)} \widetilde{C}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, F,(n)}=-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b, G,(n)}=0\right\}} \\
& \left.+\left(\widetilde{B}_{k-1}^{(n)}-(\delta, \delta), \widetilde{R}_{l(k)}^{-,(n)}, \widetilde{C}_{k-1}^{(n)}+V_{k}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, F,(n)}=0\right.}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b, G,(n)}=-V_{k}^{(n)}\right\} . \tag{2.2.2}
\end{align*}
$$

If the $k$-th incoming order is of type $(a, F)$, then

$$
\begin{align*}
& \widetilde{S}_{k}^{(n)}=\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0, V_{k}^{(n)}, 0,0\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, F,(n)} \geqslant-V_{k}^{(n)}\right\}^{1}\left\{\widetilde{Q}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0,0,0, V_{k}^{(n)}\right),-V_{k}^{(n)}\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, F,(n)}<-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{B}_{k-1}^{(n)}+(\delta, \delta), \widetilde{R}_{l(k)}^{+,(n)}, \widetilde{C}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, F,(n)}=-V_{k}^{(n)}\right\}^{1}\left\{\widetilde{Q}_{k-1}^{a, G,(n)}=0\right\}} \\
& +\left(\widetilde{B}_{k-1}^{(n)}+(\delta, \delta), \widetilde{R}_{l(k)}^{+,(n)}, \widetilde{C}_{k-1}^{(n)}-V_{k}^{(n)}\right) \mathbb{1}\left\{\widetilde{Q}_{k-1}^{a, F,(n)}=0\right\}^{1}\left\{\widetilde{Q}_{k-1}^{a, G,(n)}=-V_{k}^{(n)}\right\} \tag{2.2.3}
\end{align*}
$$

If the $k$-th incoming order is of type $(b, G)$, then

$$
\begin{align*}
& \widetilde{S}_{k}^{(n)}=\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0,0, V_{k}^{(n)}, 0\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, G,(n)} \geqslant-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(V_{k}^{(n)}, 0,0,0\right),-V_{k}^{(n)}\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, G,(n)}<-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{B}_{k-1}^{(n)}-(\delta, \delta), \widetilde{R}_{l(k)}^{-,(n)}, \widetilde{C}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, G,(n)}=-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b, F,(n)}=0\right\}} \\
& \left.\quad+\left(\widetilde{B}_{k-1}^{(n)}-(\delta, \delta), \widetilde{R}_{l(k)}^{-,(n)}, \widetilde{C}_{k-1}^{(n)}-V_{k}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, G,(n)}=0\right.}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{b, F,(n)}=-V_{k}^{(n)}\right\} \tag{2.2.4}
\end{align*}
$$

If the $k$-th incoming order is of type $(a, G)$, then

$$
\begin{align*}
& \widetilde{S}_{k}^{(n)}=\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0,0,0, V_{k}^{(n)}\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, G,(n)} \geqslant-V_{k}^{(n)}\right\}^{1}\left\{\widetilde{Q}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0, V_{k}^{(n)}, 0,0\right), V_{k}^{(n)}\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, G,(n)}<-V_{k}^{(n)}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}+\left(\widetilde{B}_{k-1}^{(n)}+(\delta, \delta), \widetilde{R}_{l(k)}^{+,(n)}, \widetilde{C}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, G,(n)}=-V_{k}^{(n)}\right\}^{1}\left\{\widetilde{Q}_{k-1}^{a, F,(n)}=0\right\}} \\
& \left.+\left(\widetilde{B}_{k-1}^{(n)}+(\delta, \delta), \widetilde{R}_{l(k)}^{+,(n)}, \widetilde{C}_{k-1}^{(n)}+V_{k}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a, G,(n)}=0\right.}\right\}^{\mathbb{1}}\left\{\widetilde{Q}_{k-1}^{a, F,(n)}=-V_{k}^{(n)}\right\}
\end{align*}
$$

Remark 2.2.3. Let the $k$-th incoming order be of type $(i, I) \in\{b, a\} \times\{F, G\}$. Let $J=\{F, G\} \backslash I$ be the index corresponding to the neighboring country of the $k$-th order. As described in (2.2.2)-(2.2.5), the active dynamics change as follows:
i) Let the $k$-th incoming order be a limit order, i.e., $V_{k}^{i, I,(n)}=\Delta v^{(n)}$. Then, its size is added to the national queue $\widetilde{Q}_{k-1}^{i, I,(n)}$, i.e., $\widetilde{Q}_{k}^{i, I,(n)}=\widetilde{Q}_{k-1}^{i, I,(n)}+V_{k}^{i, I,(n)}$.
ii) Let the $k$-th incoming order be a market order, i.e., $V_{k}^{i, I,(n)}=-\Delta v^{(n)}$ :

- If the incoming order can be matched against the national queue $\widetilde{Q}_{k-1}^{i, I,(n)}$ while its cumulative queue $\widetilde{Q}_{k-1}^{i,(n)}$ is not depleted by the size of the incoming order, we reduce the national queue $\widetilde{Q}_{k-1}^{i, I,(n)}$ by the size of the incoming market order, i.e., $\widetilde{Q}_{k}^{i, I,(n)}=\widetilde{Q}_{k-1}^{i, I,(n)}+V_{k}^{i, I,(n)}$. In particular, we allow that $\widetilde{Q}_{k-1}^{i, I,(n)}+V_{k}^{i, I,(n)}=0$ provided that the corresponding cumulative queue is strictly greater than zero.
- If the national queue satisfies $\widetilde{Q}_{k-1}^{i, I,(n)}=0$ while its cumulative queue $\widetilde{Q}_{k-1}^{i,(n)}$ is not depleted by the size of the incoming order, we reduce the foreign queue $\widetilde{Q}_{k-1}^{i, J,(n)}$ by the size of the incoming market order, i.e., $\widetilde{Q}_{k}^{i, J,(n)}=$ $\widetilde{Q}_{k-1}^{i, J,(n)}+V_{k}^{i, I,(n)}$. This yields a cross-border trade and changes the state of the capacity process by $\Delta v^{(n)}$.
- If its cumulative queue $\widetilde{Q}_{k-1}^{i,(n)}$ is depleted by the size of the incoming order, all order queues are reinitialized by either the random variable $\widetilde{R}_{l(k)}^{+,(n)}$ or $\widetilde{R}_{l(k)}^{-,(n)}$ and both national bid price processes change by one tick in the same direction. Depending on whether the national queue $\widetilde{Q}_{k-1}^{i, I,(n)}=0$ or not, we also change the state of the capacity process by $\Delta v^{(n)}$.

The price processes in $2.2 .2-2.2 .5$ have increments of maximum length $\delta>0$ and hence describe the evolution of the prices of limit order books which contain no gaps (empty levels). If there were gaps in the order books, this would result in jumps of more than one tick in the price dynamics. For convenience, we ignore the feature of price jumps larger than one tick. Note that the assumption that price changing events increase respectively decrease the prices only by a single tick is not unrealistic. It has been shown in an empirical study (cf. e.g. 29$]$ ), that around $85 \%$ of the sell market orders which lead to price changes match exactly the size of the standing volumes at the best ask price.

Moreover, as described in (2.2.2)-(2.2.5), the dynamics of the two-sided capacity process over time equals

$$
\begin{equation*}
\widetilde{C}_{k}^{(n)}=\widetilde{C}_{0}^{(n)}+\left\{M_{k}^{b, G,(n)}+M_{k}^{a, F,(n)}-M_{k}^{b, F,(n)}-M_{k}^{a, G,(n)}\right\} \tag{2.2.6}
\end{equation*}
$$

where for each $(i, I) \in\{b, a\} \times\{F, G\}$ and $M_{0}^{i, I,(n)}=0$ we define by

$$
\begin{equation*}
\left.M_{k}^{i, I,(n)}:=\Delta v^{(n)} \sum_{j=1}^{k} \mathbb{1}_{\left\{\widetilde{Q}_{j-1}^{i, I,(n)}=0\right.}\right\}^{\mathbb{1}}\left\{V_{j}^{i, I,(n)}=-\Delta v^{(n)}\right\}, \quad k=1, \cdots, T_{n} \tag{2.2.7}
\end{equation*}
$$

the (scaled) number of cross-border trades triggered by events of type $(i, I)$. In the following, we set $M_{k}^{(n)}:=\left(M_{k}^{b, F,(n)}, M_{k}^{a, F,(n)}, M_{k}^{b, G,(n)}, M_{k}^{a, G,(n)}\right)$ for $k=0, \cdots, T_{n}$.

Remark 2.2.4. The definition of the capacity process corresponds to an efficient cross-border trading assumption and therefore describes the net number of cross-border trades. In more detail, we understand cross-border trading to be efficient if the following is satisfied:
i) The goods are actual transmitted at the end of the trading session.
ii) Only the difference of imports and exports are transmitted cross-border while all other goods are distributed through domestic trading.
The idea of efficient cross-border trading is presented in Figure 2.10.


Figure 2.10: Efficient cross-border trading between different traders $F_{1}, F_{2}$ from $F$ and $G_{1}, G_{2}$ from $G$ and order sizes $x, y>0$ with $x>y$. Only the difference $x-y$ is actually transmitted cross-border.

### 2.2.4 Description of the inactive dynamics

In this subsection, we introduce for each $n \in \mathbb{N}$ the inactive dynamics given by the piecewise constant interpolation

$$
\widetilde{S}^{(n)}(t)=\sum_{k=0}^{T_{n}} \widetilde{S}_{k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t), \quad t \in[0, T],
$$

of the $E$-valued random variables

$$
\widetilde{S}_{k}^{(n)}:=\left(\widetilde{\widetilde{B}}_{k}^{(n)}, \widetilde{Q}_{k}^{(n)}, \widetilde{C}_{k}^{(n)}\right), \quad k \in \mathbb{N}_{0}
$$

where $\widetilde{B}_{k}^{(n)}$ denotes the bid prices of $F$ and $G, \widetilde{Q}_{k}^{(n)}$ denotes the sizes of the best bid respectively ask queues in $F$ and $G$, and $\widetilde{\widetilde{C}}_{k}^{(n)}$ denotes the net number of cross-border trades after $k$ order events. Since the national order books are decoupled in the inactive dynamics, we generally differentiate the best bid prices of $F$ and $G$. Hence, we cannot summarize the national LOBs in a shared order book and the depletion of a single national order queue already causes a price change in the corresponding national LOB. Finally, since no cross-border trades are possible due to the fact that the order books are decoupled, the capacity process $\left(\widetilde{\widetilde{C}}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ is constant for the whole trading period, i.e., $\widetilde{\widetilde{C}}_{k}^{(n)}=\widetilde{\widetilde{C}}_{0}^{(n)}$ for all $k=0, \cdots, T_{n}$ and $n \in \mathbb{N}$.

Next, let us specify how incoming order events change the state of the inactive dynamics. Therefore, let us denote by $\left(\widetilde{\widetilde{\tau}}_{l}^{(n)}\right)_{l \geqslant 1}$ the sequences of stopping times at which we observe a price change at either the best bid price of $F$ or $G$ in $\left(\widetilde{\widetilde{S}}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$. Further, we introduce the sequences of random variables representing the order sizes after a price change, for $l \geqslant 1$, by

$$
\widetilde{\widetilde{R}}_{l}^{+,(n)}:=\Phi^{(n)}\left(\widetilde{\widetilde{Q}}_{\left[\widetilde{\tau}_{l}^{(n)}-/ \Delta t^{(n)}\right\rfloor}^{(n)}, \epsilon_{l}^{+,(n)}\right), \quad \widetilde{\widetilde{R}}_{l}^{-,(n)}:=\Phi^{(n)}\left(\widetilde{\widetilde{Q}}_{\left\lfloor\widetilde{\tau}_{l}^{(n)}-/ \Delta t^{(n)}\right\rfloor}^{(n)}, \epsilon_{l}^{-,(n)}\right) .
$$

Again, let $l(k)$ denote the number of price changes after $k$ order events in $\left(\widetilde{\widetilde{S}}_{k}^{(n)}\right)_{k=0, \ldots, T_{n}}$. Then, $\left(\widetilde{\widetilde{S}}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ evolves as follows: let $\widetilde{\widetilde{S}}_{0}^{(n)} \in(\delta Z)^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z} \subset E$ be the deterministic initial state. If the $k$-th incoming order is of type $(b, F)$, then

$$
\begin{align*}
\widetilde{\widetilde{S}}_{k}^{(n)}= & \left(\widetilde{\widetilde{S}}_{k-1}^{(n)}+\left((0,0),\left(V_{k}^{(n)}, 0,0,0\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{b, F(n)}>-V_{k}^{(n)}\right\}}  \tag{2.2.8}\\
& +\left(\widetilde{\widetilde{B}}_{k-1}^{(n)}-(\delta, 0), \pi_{F} \widetilde{\widetilde{R}}_{l(k)}^{-,(n)}, \pi_{G} \widetilde{Q}_{k-1}^{(n)}, \widetilde{\widetilde{C}}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{\widetilde{Q}}_{k-1}^{b, F,(n)} \leqslant-V_{k}^{(n)}\right\}} .
\end{align*}
$$

If the $k$-th incoming order is of type $(a, F)$, then

$$
\begin{align*}
\widetilde{\widetilde{S}}_{k}^{(n)}= & \left(\widetilde{\widetilde{S}}_{k-1}^{(n)}+\left((0,0),\left(0, V_{k}^{(n)}, 0,0\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{\widetilde{Q}}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}  \tag{2.2.9}\\
& +\left(\widetilde{\widetilde{B}}_{k-1}^{(n)}+(\delta, 0), \pi_{F} \widetilde{\widetilde{R}}_{l(k)}^{+,(n)}, \pi_{G} \widetilde{\widetilde{Q}}_{k-1}^{(n)}, \widetilde{\widetilde{C}}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{\widetilde{Q}}_{k-1}^{a, F,(n)} \leqslant-V_{k}^{(n)}\right\}}
\end{align*}
$$

If the $k$-th incoming order is of type $(b, G)$, then

$$
\begin{align*}
\widetilde{\widetilde{S}}_{k}^{(n)}= & \left(\widetilde{\widetilde{S}}_{k-1}^{(n)}+\left((0,0),\left(0,0, V_{k}^{(n)}, 0\right), 0\right)\right) \mathbb{1}_{\{ }\left\{\widetilde{\widetilde{Q}}_{k-1}^{b, G(n)}>-V_{k}^{(n)}\right\}  \tag{2.2.10}\\
& +\left(\widetilde{\widetilde{B}}_{k-1}^{(n)}-(0, \delta), \pi_{F} \widetilde{Q}_{k-1}^{(n)}, \pi_{G} \widetilde{\widetilde{R}}_{l(k)}^{-,(n)}, \widetilde{\widetilde{C}}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{\widetilde{Q}}_{k-1}^{b, G,(n)} \leqslant-V_{k}^{(n)}\right\}} .
\end{align*}
$$

If the $k$-th incoming order is of type $(a, G)$, then

$$
\begin{align*}
\widetilde{\widetilde{S}}_{k}^{(n)}= & \left(\widetilde{S}_{k-1}^{(n)}+\left((0,0),\left(0,0,0, V_{k}^{(n)}\right), 0\right)\right) \mathbb{1}_{\left\{\widetilde{Q}_{k-1}^{a,(n)}>-V_{k}^{(n)}\right\}}  \tag{2.2.11}\\
& +\left(\widetilde{\widetilde{B}}_{k-1}^{(n)}+(0, \delta), \pi_{F} \widetilde{Q}_{k-1}^{(n)}, \pi_{G} \widetilde{\widetilde{R}}_{l(k)}^{+,(n)}, \widetilde{\widetilde{C}}_{k-1}^{(n)}\right) \mathbb{1}_{\left\{\widetilde{\widetilde{Q}}_{k-1}^{a, G,(n)} \leqslant-V_{k}^{(n)}\right\}} .
\end{align*}
$$

Remark 2.2.5. The above description of inactive dynamics is a straight-forward extension of the discrete-time dynamics in Cont and de Larrard [19] to two noninteracting LOBs.

### 2.2.5 The cross-border market dynamics as a regime switching process

In this subsection, we finally introduce the microscopic dynamics of our cross-border market model $S^{(n)}$ which can be interpreted as a regime switching process switching between the active and inactive regimes.

Remark 2.2.6 (Preliminary considerations).
i) While the capacity process $\widetilde{C}^{(n)}$ is unbounded and changes over time, $\widetilde{\widetilde{C}}^{(n)}$ stays constant. The bounded capacity process $\widetilde{C}^{(n)}$ will take its values in $\left[-\kappa_{-}, \kappa_{+}\right]$and its dynamics will develop similarly to $\widetilde{C}^{(n)}$ during active regimes and will stay constant and equals either $\kappa_{-}$or $\kappa_{+}$during inactive regimes. Moreover, we will use the unbounded capacity process $\widetilde{C}^{(n)}$ as an indicator to switch from an active to an inactive regime.
ii) The bid price processes of the national LOBs of the active dynamics $\widetilde{S}^{(n)}$ coincide. In contrast, the price processes of the inactive dynamics $\widetilde{\widetilde{S}}^{(n)}$ develop as nonidentical pure jump processes. In order to switch back to an active regime, the price processes during inactive regimes have to coincide.
iii) It is not clear what happens if the capacity process $C^{(n)}$ is equal to one of its boundary values $\left\{-\kappa_{-}, \kappa_{+}\right\}$while the best bid prices coincide as the limit order books might be coupled or decoupled. This situation can be interpreted as a partially active regime in which cross-border trades are possible but only in one direction. Instead of modeling a third regime, we include the partially active regimes into the active or inactive regimes. This is possible since the active respectively inactive dynamics develop equally as long as no order queue has been depleted. In a partially active regime, depending on the type of the incoming market order, we might stay in the current regime or switch to the next regime. Note that before switching from an inactive to an active regime, we always are in a partially active regime (which will be included to the inactive regimes). In contrast, we might directly switch from an active to an inactive regime without previously being in a partially active regime (cf. Figure 2.11).


Figure 2.11: Incorporation of the partially active regime. The market dynamics develop in the partially active regime as in the active (patterned, turquoise) or inactive regime (patterned, orange).

A short consideration of the definition of the active dynamics in 2.2.2)-2.2.5 reveals, that a single LOB is allowed to be "empty", i.e., might have no standing volumes at the best bid and ask queues, provided that the order queues of the foreign LOB are both strictly larger than zero. Then, we may run into problems if we switch to the next inactive regime as we cannot uniquely determine the direction of the next price change for the empty LOB. In order to bypass this issue, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to support a sequence of iid Bernoulli random variables $\left(U_{k}^{(n)}\right)_{k, n \geqslant 1}$, where $U_{k}^{(n)} \in\{-1,1\}$.

Next, let us introduce short-hand notations for the order type indicator random variables at time $t \in[0, T]$ by
$\mathbb{1}_{(i, I)}^{(n)}(t):=\mathbb{1}_{(i, I)}\left(\phi_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{(n)}, \psi \psi_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{(n)}\right) \quad$ and $\quad \mathbb{1}^{(n)}(t):=\left(\mathbb{1}_{(b, F)}^{(n)}, \mathbb{1}_{(a, F)}^{(n)}, \mathbb{1}_{(b, G)}^{(n)}, \mathbb{1}_{(a, G)}^{(n)}\right)(t)$.
In order to describe the state of the cross-border market model at the start of an inactive regime, we introduce a sequence of processes $\widetilde{Z}_{k}^{(n)}:=\left(\widetilde{Z}_{k}^{F,(n)}, \widetilde{Z}_{k}^{G,(n)}\right)$ taking values in $\{-1,0,1\}^{2}$ by

for all $t \in[0, T]$ and $I=F, G$. Moreover, we introduce the function

$$
h: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right), \quad h: \omega \mapsto\left(\pi_{1} \omega+\pi_{3} \omega, \pi_{2} \omega+\pi_{4} \omega\right)
$$

Note that for all $t \in[0, T]$ and $Q^{(n)}(t)$ denoting the sizes of the best bid and ask queues of $F$ respectively $G$ at time $\left\lfloor t / \Delta t^{(n)}\right\rfloor, h\left(Q^{(n)}\right)(t)$ describes the sizes of the cumulative best bid and ask queues at time $\left\lfloor t / \Delta t^{(n)}\right\rfloor$.

Recall that we denote by $\left(R_{l}^{+,(n)}\right)_{l \geqslant 1},\left(R_{l}^{-,(n)}\right)_{l \geqslant 1}$ introduced in 2.2.1 the order sizes after a price increase respectively decrease, by $\left(\tau_{l}^{(n)}\right)_{l \geqslant 1}$ the sequence of stopping times at which we observe a price change in $S^{(n)}$, and by $l^{(n)}(t)$ the number of price changes in $S^{(n)}$ in $[0, t]$, for all $t \in[0, T]$. Last, for all $s \in[0, T]$, let us introduce the short-hand notations $\widetilde{S}^{(n), s}$ and $\widetilde{\widetilde{S}}(n), s$ denoting the active respectively inactive dynamics starting in the state $S^{(n)}(s) \in E$.

With all this preparation done, we are finally able to introduce the cross-border market dynamics as a regime switching process.

Definition 2.2.7 (Cross-border market dynamics $S^{(n)}$ ). Let $n \in \mathbb{N}, S_{0}^{(n)} \in(\delta \mathbb{Z})^{2} \times$ $\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times\{0\}$ with $B_{0}^{F,(n)}=B_{0}^{G,(n)}, \kappa_{+}, \kappa_{-}>0, \rho_{0}^{(n)}:=0$, and $k \geqslant 1$. The crossborder market dynamics are given as follows:

- The active regime: for $0 \leqslant t<\sigma_{k}^{(n)}-\rho_{k-1}^{(n)}$, we set

$$
S^{(n)}\left(t+\rho_{k-1}^{(n)}\right):=\widetilde{S}^{(n), \rho_{k-1}^{(n)}}(t)
$$

where $\sigma_{k}^{(n)}:=\sigma_{k}^{I m,(n)} \wedge \sigma_{k}^{E x,(n)}$ determines the start of the next inactive regime and

$$
\begin{aligned}
\sigma_{k}^{I m,(n)} & :=\inf \left\{t \geqslant \rho_{k-1}^{(n)}: C^{(n)}(t) \geqslant \kappa_{+} \text {and } \exists(i, I) \in \mathcal{I}^{I m} \text { with } h\left(Q^{(n)}\right)\left(t-\Delta t^{(n)}\right)\right. \\
& \left.+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{(n)} h\left(\mathbb{1}^{(n)}\right)(t) \in(0, \infty)^{2} \text { and } Q^{i, I,(n)}\left(t-\Delta t^{(n)}\right)+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{i, I,(n)} \leqslant 0\right\} \wedge T, \\
\sigma_{k}^{E x,(n)} & :=\inf \left\{t \geqslant \rho_{k-1}^{(n)}: C^{(n)}(t) \leqslant-\kappa_{-} \text {and } \exists(i, I) \in \mathcal{I}^{E x} \text { with } h\left(Q^{(n)}\right)\left(t-\Delta t^{(n)}\right)\right. \\
& \left.+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{(n)} h\left(\mathbb{1}^{(n)}\right)(t) \in(0, \infty)^{2} \text { and } Q^{i, I,(n)}\left(t-\Delta t^{(n)}\right)+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{i, I,(n)} \leqslant 0\right\} \wedge T .
\end{aligned}
$$

- Starting value of the next inactive regime: we set $C^{(n)}\left(\sigma_{k}^{(n)}\right):=\widetilde{C}^{(n), \rho_{k-1}^{(n)}}\left(\sigma_{k}^{(n)}\right)$. Further, for $I=F, G$ and $\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right) \in\{-1,0,1\}$ being introduced in 2.2.12), we set

$$
\begin{aligned}
\pi_{I} Q^{(n)}\left(\sigma_{k}^{(n)}\right):= & \pi_{I} \widetilde{Q}^{(n), \rho_{k-1}^{(n)}}\left(\sigma_{k}^{(n)}\right) \mathbb{1}\left\{\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right)=0\right\} \\
& +\pi_{I} R_{l^{(n)}\left(\sigma_{k}^{(n)}-\right)+1}^{+,(n)} R_{l^{(n)}\left(\sigma_{k}^{(n)}-\right)+1}^{-,(n)}\left\{\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right)=1\right\} \\
& \left.\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right)=-1\right\}
\end{aligned}
$$

and

$$
B^{I,(n)}\left(\sigma_{k}^{(n)}\right):=B^{I,(n)}\left(\sigma_{k}^{(n)}-\right)+\delta\left(\mathbb{1}_{\left.\left\{\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right)=1\right\}^{-\mathbb{1}_{\{ }\left\{\widetilde{Z}_{k}^{I,(n)}\left(\sigma_{k}^{(n)}\right)=-1\right\}}\right) . . . . . .}\right.
$$

- The inactive regime: for $0 \leqslant t<\rho_{k}^{(n)}-\sigma_{k}^{(n)}$, we set

$$
S^{(n)}\left(t+\sigma_{k}^{(n)}\right):=\widetilde{\widetilde{S}}^{(n), \sigma_{k}^{(n)}}(t)
$$

where $\rho_{k}^{(n)}:=\rho_{k}^{\text {Im, }(n)} \mathbb{1}_{\left\{C^{(n)}\left(\sigma_{k}^{(n)}\right)=-\kappa_{-}\right\}}+\rho_{k}^{\text {Ex,(n) }} \mathbb{1}_{\left\{C^{(n)}\left(\sigma_{k}^{(n)}\right)=\kappa_{+}\right\}}$determines the start of the next active regime and

$$
\begin{aligned}
\rho_{k}^{I m,(n)}:=\inf \left\{t \geqslant \sigma_{k}^{(n)}:\right. & \left|B^{F,(n)}(t)-B^{G,(n)}(t)\right|=0 \text { and } \\
& \left.\exists(i, I) \in \mathcal{I}^{I m} \text { with } Q^{i, I,(n)}\left(t-\Delta t^{(n)}\right)+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{i, I,(n)} \leqslant 0\right\} \wedge T,
\end{aligned}
$$

$$
\begin{aligned}
\rho_{k}^{E x,(n)}:=\inf \left\{t \geqslant \sigma_{k}^{(n)}:\right. & \left|B^{F,(n)}(t)-B^{G,(n)}(t)\right|=0 \text { and } \\
& \left.\exists(i, I) \in \mathcal{I}^{E x} \text { with } Q^{i, I,(n)}\left(t-\Delta t^{(n)}\right)+V_{\left\lfloor t / \Delta t^{(n)}\right\rfloor}^{i, I,(n)} \leqslant 0\right\} \wedge T .
\end{aligned}
$$

- Starting value of the next active regime: we set $C^{(n)}\left(\rho_{k}^{(n)}\right):=C^{(n)}\left(\rho_{k}^{(n)}-\right), B^{(n)}\left(\rho_{k}^{(n)}\right):=$ $B^{(n)}\left(\rho_{k}^{(n)}-\right)$, and for $(i, I) \in\{b, a\} \times\{F, G\}$,

$$
Q^{i, I,(n)}\left(\rho_{k}^{(n)}\right):=Q^{i, I,(n)}\left(\rho_{k}^{(n)}-\right)+V_{\left\lfloor\rho_{k}^{(n)} / \Delta t^{(n)}\right\rfloor}^{i, I,(n)}
$$

## Remark 2.2.8.

i) The starting times of the inactive regimes being determined by the stopping times $\left(\sigma_{k}^{(n)}\right)_{k \geqslant 1}$ may look surprisingly complex. We wish to switch to an inactive regime if the capacity process hits one of its boundary values. Since we also include the partially active regime partly into the active regime (cf. Remark 2.2.6), the stopping times have to be modified appropriately. Note, that these stopping times simplify in the high-frequency limit (cf. Theorem 2.5.1 below) in which the partially active regimes in the active regimes disappear.
ii) The start of next active regime depends on the direction in which the transmission capacity has been occupied. After the national best bid prices coincide again and a national order queue would be depleted by an incoming market order allowing for cross-border trades in the non-occupied direction, we immediately switch back to an active regime and couple the national LOBs. We note, that this indeed prevents that the national bid prices diverge again (cf. the definition of the starting value of the next active regime).

Remark 2.2.9 (Distribution of order sizes might depend on the type of the regime). In our model, a regime switch only changes the order matching mechanism (cf. Definition 2.2.7). However, it might be of interest to also allow a change in the trading behavior. This can be incorporated to our model by slight generalizations of Assumptions 2.2 and 2.4: one might assume that the distribution of the order sizes as well as of the queue sizes after price changes depend on the type of the current regime. For sake of notation, we will not include this to our model.

### 2.3 Analysis of the active dynamics

In this section, we analyze the active market dynamics $\widetilde{S}^{(n)}=\left(\widetilde{S}^{(n)}(t)\right)_{t \in[0, T]}$ and derive its heavy traffic approximation. Recall, that

$$
\widetilde{S}^{(n)}(t)=\widetilde{S}_{k}^{(n)} \quad \text { for } t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right),
$$

where the discrete-time dynamics are defined in equations 2.2 .2$)-(2.2 .5)$ in Section 2.2.3 Let $\widetilde{S}_{0}^{(n)}:=\left(\widetilde{B}_{0}^{(n)}, \widetilde{Q}_{0}^{(n)}, \widetilde{C}_{0}^{(n)}\right) \in(\delta \mathbb{Z})^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z}$ be the initial value
of $\widetilde{S}^{(n)}$ with $\widetilde{B}_{0}^{F,(n)}=\widetilde{B}_{0}^{G,(n)}$. In order to derive a limit theorem for the active dynamics, we need to introduce the so-called net order flow process $X^{(n)}$ by

$$
\begin{equation*}
X^{(n)}:=\left(X^{b, F,(n)}, X^{a, F,(n)}, X^{b, G,(n)}, X^{a, G,(n)}\right), \tag{2.3.1}
\end{equation*}
$$

where for $(i, I) \in\{b, a\} \times\{F, G\}$ and $t \in[0, T]$ we have

$$
X^{i, I,(n)}(t):=\sum_{k=1}^{T_{n}} X_{k}^{i, I,(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t) \quad \text { and } \quad X_{k}^{i, I,(n)}:=\sum_{j=1}^{k} V_{j}^{i, I,(n)} .
$$

Proposition 2.3.1 (Functional central limit theorem for the net order flow process). Let Assumptions 2.2 and 2.3 be satisfied. Then, the net order flow process $X^{(n)}$ converges weakly in the Skorokhod topology on $D\left([0, T], \mathbb{R}^{4}\right)$ to a four-dimensional linear Brownian motion, i.e.,

$$
\begin{equation*}
X^{(n)} \Rightarrow X:=\left(\Sigma^{1 / 2} B(t)+t \mu\right)_{t \geqslant 0}, \tag{2.3.2}
\end{equation*}
$$

where $B$ is a standard four-dimensional Brownian motion and

$$
\mu:=\left(\begin{array}{c}
\mu^{b, F} \\
\mu^{a, F} \\
\mu^{b, G} \\
\mu^{a, G}
\end{array}\right), \quad \Sigma:=\left(\begin{array}{cccc}
\left(\sigma^{b, F}\right)^{2} & \sigma^{(b, F),(a, F)} & \sigma^{(b, F),(b, G)} & \sigma^{(b, F),(a, G)} \\
\sigma^{(b, F),(a, F)} & \left(\sigma^{a, F}\right)^{2} & \sigma^{(a, F),(b, G)} & \sigma^{(a, F),(a, G)} \\
\sigma^{(b, F),(b, G)} & \sigma^{(a, F),(b, G)} & \left(\sigma^{b, G}\right)^{2} & \sigma^{(b, G),(a, G)} \\
\sigma^{(b, F),(a, G)} & \sigma^{(a, F),(a, G)} & \sigma^{(b, G),(a, G)} & \left(\sigma^{a, G}\right)^{2}
\end{array}\right) .
$$

The proof is given in Section 2.7.1. Based on this limit theorem, we will derive convergence results for the queue size process $\widetilde{Q}^{(n)}$, the capacity process $\widetilde{C}^{(n)}$, and the price process $\widetilde{B}^{(n)}$.

### 2.3.1 The queue size process as a regulated process

In this subsection, we will rewrite the queue size process $\widetilde{Q}^{(n)}$ in terms of transformations of the net order flow $X^{(n)}$ and the sequences of random variables $\widetilde{R}^{+,(n)}:=\left(\widetilde{R}_{k}^{+,(n)}\right)_{k \geqslant 1}$ and $\widetilde{R}^{-,(n)}:=\left(\widetilde{R}_{k}^{-,(n)}\right)_{k \geqslant 1}$ corresponding to the order sizes after price changes.

To ease notation, we introduce by $t \mapsto \ell_{t}^{(2)}(\omega)$ the component-wise reflection at zero of some $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$, i.e.,

$$
\ell_{t}^{(2)}(\omega):=\left(\sup _{s \leqslant t}\left(-\pi_{1}^{(2)} \omega(s)\right)^{+}, \sup _{s \leqslant t}\left(-\pi_{2}^{(2)} \omega(s)\right)^{+}\right) \quad \text { for } t \in[0, T],
$$

where $x^{+}:=\max \{x, 0\}$.

Let us introduce a function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ which describes the evolution of the queue size process $\pi_{i} \widetilde{Q}, i=b, a$, corresponding to one side of the shared order book provided that the cumulative queue size process stays strictly larger than zero (i.e. no price change has been observed).

Definition 2.3.2. Let $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$. We inductively define for $k \in \mathbb{N}$ functions $\widetilde{g}_{k}(\omega), g_{k}(\omega) \in D\left([0, T], \mathbb{R}^{2}\right)$, and $g(\omega) \in D\left([0, T], \mathbb{R}_{+}^{2}\right)$ as follows:

- Set $g_{1}(\omega)=\widetilde{g}_{1}(\omega)=\omega$ and $\hat{\tau}_{1}:=\hat{\tau}_{1}(\omega):=\inf \left\{t \geqslant 0: \exists i \in\{1,2\}\right.$ with $\pi_{i}^{(2)} \omega(t) \leqslant$ $0\} \wedge T$.
- For $k \geqslant 2$, set $g_{k}(\omega)(t)=\widetilde{g}_{k}(\omega)(t)=g_{k-1}(\omega)(t)$ for $t \in\left[0, \hat{\tau}_{k-1}\right)$. If $\widetilde{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \in \mathbb{R}_{-}^{2}$, set $g_{k}(\omega)(t)=\widetilde{g}_{k}(\omega)(t)=(0,0)$ for all $t \in\left[\hat{\tau}_{k-1}, T\right]$. Otherwise, for $t \geqslant \hat{\tau}_{k-1}$ first define

$$
\widetilde{g}_{k}(\omega)(t):=g_{k-1}(\omega)(t)+\ell_{t}^{(2)}\left(g_{k-1}(\omega)\right) R
$$

with reflection matrix $R:=\left(\begin{array}{rr}1 & -1 \\ -1 & 1\end{array}\right)$ and

$$
\begin{aligned}
\hat{\tau}_{k}:=\hat{\tau}_{k}(\omega):=\inf \left\{t \geqslant \hat{\tau}_{k-1}:\right. & \pi_{1}^{(2)} \widetilde{g}_{k}(\omega)(t) \mathbb{1}_{\left\{\pi_{2}^{(2)} g_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \leqslant 0\right\}} \\
& \left.+\pi_{2}^{(2)} \widetilde{g}_{k}(\omega)(t) \mathbb{1}_{\left\{\pi_{1}^{(2)} g_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \leqslant 0\right\}} \leqslant 0\right\} \wedge T
\end{aligned}
$$

Then set $g_{k}(\omega)(t)=\widetilde{g}_{k}(\omega)\left(t \wedge \hat{\tau}_{k}-\right)+\omega(t)-\omega\left(t \wedge \hat{\tau}_{k}-\right)$ for all $t \in[0, T]$.

- Set $\hat{\tau}_{0}:=0$ and

$$
g(\omega)(t):=\sum_{k=1}^{\infty} g_{k}(\omega)(t) \mathbb{1}_{\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)}(t)=\sum_{k=1}^{\infty} \widetilde{g}_{k}(\omega)(t) \mathbb{1}_{\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)}(t) \quad \text { for } t \in[0, T)
$$

Moreover, if there exists a finite $\kappa \in \mathbb{N}$ such that $\hat{\tau}_{\infty}(\omega):=\lim _{k \rightarrow \infty} \hat{\tau}_{k}(\omega)=\hat{\tau}_{\kappa}(\omega)$, then set $g(\omega)(T)=\widetilde{g}_{\kappa+1}(\omega)(T)$. Otherwise, set $g(\omega)(T)=(0,0)$.

Remark 2.3.3 (Modifications for smaller time domains). We also want to apply the function $g$ to an element $\omega \in D\left([0, t], \mathbb{R}^{2}\right)$, where $t<T$. With a little abuse of notation, we write $g(\omega) \in D\left([0, t], \mathbb{R}_{+}^{2}\right)$ without further comment. Here, $g(\omega) \in D\left([0, t], \mathbb{R}^{2}\right)$ is defined as in Definition 2.3.2, where $T$ is replaced by $t$. In the same way, we apply the subsequently defined functions to elements of $D\left([0, t], \mathbb{R}^{2}\right)$ and $D\left([0, t], \mathbb{R}^{4}\right)$, respectively, for $t<T$.

As desired, the function $g$ describes the dynamics of the queue size process $\pi_{i} \widetilde{Q}^{(n)}$, $i=b, a$, corresponding to one side of the shared order book between consecutive price changes if we plug in the corresponding components of the net order flow process
$\pi_{i} X^{(n)}$, i.e., on the event that no price change appears during $[0, t]$ for some $t \in[0, T]$, we can write

$$
\pi_{i} \widetilde{Q}^{(n)}(s)=g\left(\pi_{i} \widetilde{Q}_{0}^{(n)}+\pi_{i} X^{(n)}\right)(s) \quad \text { for all } s \in[0, t]
$$

Let us now introduce the function $h_{1}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D([0, T], \mathbb{R})$ given by

$$
\begin{equation*}
h_{1}(\omega)=\pi_{1}^{(2)} \omega+\pi_{2}^{(2)} \omega \tag{2.3.3}
\end{equation*}
$$

With a little abuse of notation, we will write $h_{1}(x)=x_{1}+x_{2} \in \mathbb{R}$ for $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$. The next lemma states that the sum of the components of $g(\omega)$ can be identified with the sum of the components of $\omega$ as long as $g(\omega)$ does not hit the origin.

Lemma 2.3.4. Let $\omega \in D\left([0, T], \mathbb{R}^{2}\right), \tau(\omega):=\inf \{t \geqslant 0: g(\omega)(t)=(0,0)\} \wedge T$, and assume that $\hat{\tau}_{\infty}(\omega)=\tau(\omega)$. Then,

$$
\left(h_{1} \circ g\right)(\omega)(t)=h_{1}(\omega)(t)+\sup _{s \leqslant t}\left(-h_{1}(\omega)(s)\right)^{+} \quad \text { for } t \in[0, \tau(\omega)]
$$

Moreover, we have $\tau(\omega)=\inf \left\{t \geqslant 0: h_{1}(\omega)(t) \leqslant 0\right\} \wedge T$.
The proof is postponed to Section 2.7 .3 . For some $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$, we will frequently assume that the function $g(\omega) \in D\left([0, T], \mathbb{R}_{+}^{2}\right)$ has only two behaviors to approach $\hat{\tau}_{\infty}(\omega)$ : if $\hat{\tau}_{\infty}<T$, then $g(\omega)$ undergoes an infinitely number of successive reflections from the two axes. Otherwise, $g(\omega)$ is only finitely often reflected and satisfies $g(\omega)(T) \in$ $(0, \infty)^{2}$. This behavior is described in the next condition.

Condition (I). We say that $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$ satisfies condition (I) if one of the two mutually exclusive conditions holds true:
(Ia) There exists a finite $\kappa \in \mathbb{N}$ such that $\hat{\tau}_{\infty}(\omega)=\hat{\tau}_{\kappa}(\omega)=T$ and $g(\omega)(T) \in(0, \infty)^{2}$.
(Ib) For all $k \in \mathbb{N}$, it holds that $\hat{\tau}_{k}(\omega)<\hat{\tau}_{\infty}(\omega)<T$.
In order to characterize the continuity set of the function $g$, we endow the space $D\left([0, T], \mathbb{R}^{2}\right)$ with the Skorokhod topology (cf. e.g. Billingsley 88$)$. Moreover, let us introduce the function space $C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ containing all continuous functions $\omega \in C\left([0, T], \mathbb{R}^{2}\right)$ avoiding the origin and whose components cross the axes each time they touch them (cf. equation $(2.7 .4$ for details). Then, the following lemma characterizes the continuity set of $g$.

Lemma 2.3.5 (Continuity of $g$ ). Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ satisfy condition (I), $h_{1}\left(\omega_{0}\right) \in C_{0}^{\prime}([0, T], \mathbb{R})$, and assume that $\hat{\tau}_{\infty}\left(\omega_{0}\right)=\tau\left(\omega_{0}\right)$. Then, the function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is continuous at $\omega_{0}$.

The proof is stated in Section 2.7.2.2. Let us introduce a function that maps càdlàg, $\mathbb{R}^{4}$-valued functions onto the space of càdlàg functions with values in $\mathbb{R}_{+}^{4}$. For $\omega \in$
$D\left([0, T], \mathbb{R}^{4}\right)$, let

$$
\begin{align*}
& G: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right) \\
G(\omega)= & \left(\pi_{1}^{(2)} g\left(\pi_{b} \omega\right), \pi_{1}^{(2)} g\left(\pi_{a} \omega\right), \pi_{2}^{(2)} g\left(\pi_{b} \omega\right), \pi_{2}^{(2)} g\left(\pi_{a} \omega\right)\right) \tag{2.3.4}
\end{align*}
$$

We observe that the function $G$ describes the dynamics of the queue size process $\widetilde{Q}^{(n)}$ corresponding to the shared order book between consecutive price changes if we plug in the net order flow process $X^{(n)}$, i.e., on the event that no price change appears during $[0, t]$ for some $t \in[0, T]$, we can write

$$
\widetilde{Q}^{(n)}(s)=G\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}\right)(s) \quad \text { for all } s \in[0, t]
$$

We note that the continuity set of $G$ can be directly deduced from Lemma 2.3.5.
Corollary 2.3.6 (Continuity of $G)$. Let $\omega_{0} \in D\left([0, T], \mathbb{R}^{4}\right)$ be such that its projections $\pi_{b} \omega_{0}$ and $\pi_{a} \omega_{0}$ satisfy the assumptions of Lemma 2.3.5. Then, the function $G: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is continuous at $\omega_{0}$.

We recall the definition of the function $h: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ given by

$$
\begin{equation*}
h(\omega)=\left(h_{1}\left(\pi_{b} \omega\right), h_{1}\left(\pi_{a} \omega\right)\right)=\left(\pi_{1} \omega+\pi_{3} \omega, \pi_{2} \omega+\pi_{4} \omega\right) \tag{2.3.5}
\end{equation*}
$$

and note that the process $h\left(\widetilde{Q}^{(n)}\right)$ describes the evolution of the cumulative queue size process over time. Next, observe that the first hitting time map $\widetilde{\tau}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ defined by

$$
\begin{equation*}
\widetilde{\tau}(\omega):=\inf \left\{t \geqslant 0: \exists i \in\{1,2\} \text { with }\left(\pi_{i}^{(2)} \circ h\right)(\omega)(t) \leqslant 0\right\} \wedge T \tag{2.3.6}
\end{equation*}
$$

equals $\widetilde{\tau}_{1}(\omega):=\inf \left\{t \geqslant 0: \exists i \in\{1,2\}\right.$ with $\left.\left(\pi_{i}^{(2)} \circ h \circ G\right)(\omega)(t)=0\right\} \wedge T$ (cf. Lemma 2.3.4). Moreover, introducing the first hitting time maps $\widetilde{\tau}_{b}, \widetilde{\tau}_{a}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ by

$$
\begin{align*}
& \widetilde{\tau}_{b}(\omega):=\inf \left\{t \geqslant 0:\left(\pi_{1}^{(2)} \circ h\right)(\omega)(t) \leqslant 0\right\} \wedge T \\
& \widetilde{\tau}_{a}(\omega):=\inf \left\{t \geqslant 0:\left(\pi_{2}^{(2)} \circ h\right)(\omega)(t) \leqslant 0\right\} \wedge T \tag{2.3.7}
\end{align*}
$$

we can rewrite $\widetilde{\tau}(\omega)=\widetilde{\tau}_{b}(\omega) \wedge \widetilde{\tau}_{a}(\omega)$. In particular, $\widetilde{\tau}(\omega)$ reveals the first hitting time before $T$ of $h(\omega)$ of the axes $\{(0, y): y>0\} \cup\{(x, 0): x \geqslant 0\}$.

Next, based on the definition of the functions $G$ and $\widetilde{\tau}$, we introduce another function that can be used to construct the queue size process $\widetilde{Q}^{(n)}$ from the net order flow process $X^{(n)}$ and the random sequences $\widetilde{R}^{+,(n)}$ and $\widetilde{R}^{-,(n)}$. In particular, this function takes the possibility of price changes into account leading to reinitializations of the queue size process $\widetilde{Q}^{(n)}$ at new positions inside $\mathbb{R}_{+}^{4}$.

Definition 2.3.7. Let $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ and let $R=\left(R_{n}\right)_{n \geqslant 1}, \widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1} \in\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$. For $k \in \mathbb{N}_{0}$, we define $\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R}), \widetilde{\Psi}^{Q}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}_{+}^{4}\right)$ as follows:

- Set $\widetilde{\Psi}_{0}^{Q}(\omega, R, \widetilde{R}):=G(\omega)$.
- Let $k \geqslant 1$ and $\widetilde{\Psi}_{k-1}^{Q}:=\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})$. If $\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)=T$, then $\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R})=$ $\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})$. Otherwise, we define

$$
\begin{aligned}
& \widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R}):=\widetilde{\Psi}_{k-1}^{Q} \mathbb{1}_{\left[0, \widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right)} \\
& \quad+\mathbb{1}_{\left[\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right), T\right]}\left\{\mathbb{1}_{\left\{\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)=\widetilde{\tau}_{a}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right\}} G\left(R_{k}+\omega-\omega\left(\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right)\right)\right. \\
& \left.\quad+\mathbb{1}_{\left.\left\{\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)=\widetilde{\tau}_{b} \widetilde{\Psi}_{k-1}^{Q}\right)\right\}} G\left(\widetilde{R}_{k}+\omega-\omega\left(\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right)\right)\right\} .
\end{aligned}
$$

- Finally, we set $\widetilde{\tau}_{0}:=0, \widetilde{\tau}_{k}:=\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})\right)$ for $k \geqslant 1$, and

$$
\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(t)=\sum_{k=1}^{\infty} \widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})(t) \mathbb{1}_{\left[\widetilde{\tau}_{k-1}, \widetilde{\tau}_{k}\right)}(t) \quad \text { for } t \in[0, T) .
$$

Moreover, if there exists a finite $N_{T}$ such that $\widetilde{\tau}_{N_{T}}<T$ and $\widetilde{\tau}_{N_{T}+1}=T$, then set $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(T)=\widetilde{\Psi}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(T)$. Otherwise, set $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(T)=(0,0,0,0)$.
The above definition states, that the path of $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ is obtained by "regulating" the path of $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ according to the function $G$ and the sequences $\left(R_{n}\right)_{n \geqslant 1}$ and $\left(\widetilde{R}_{n}\right)_{n \geqslant 1}$ : between two hitting times $\widetilde{\tau}_{k}$ and $\widetilde{\tau}_{k+1}$, the function $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ behaves as $G\left(R_{k}+\omega-\omega\left(\widetilde{\tau}_{k}\right)\right)$ or $G\left(\widetilde{R}_{k}+\omega-\omega\left(\widetilde{\tau}_{k}\right)\right)$ depending on whether $\left(h \circ \widetilde{\Psi}_{k-1}^{Q}\right)(\omega, R, \widetilde{R})$ first hits the $y$ - or $x$-axis. Moreover, at the times $\widetilde{\tau}_{1}, \widetilde{\tau}_{2}, \cdots$, the process jumps to a new position inside $\mathbb{R}_{+}^{4}$ taken from the sequence $\left(R_{n}\right)_{n \geqslant 1}$ or $\left(\widetilde{R}_{n}\right)_{n \geqslant 1}$, respectively.
Remark 2.3.8. If there exists a finite $N_{T}$ such that $\widetilde{\tau}_{N_{T}}<T$ and $\widetilde{\tau}_{N_{T}+1}=T$ then by construction of $\widetilde{\Psi}^{Q}$, we have

$$
\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(t)=\widetilde{\Psi}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(t) \quad \text { for all } t \in[0, T] .
$$

Let us now study the continuity set of the function $\widetilde{\Psi}^{Q}$. Therefore, we endow the space $D\left([0, T], \mathbb{R}^{4}\right)$ also with the Skorokhod topology. Moreover, we endow the set $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ with the topology induced by cylindrical semi-norms, defined as follows: for a sequence $\left(R^{n}\right)_{n \geqslant 1} \subset\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$,

$$
R^{n} \rightarrow R \in\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \Leftrightarrow\left(\forall k \geqslant 1, \sup \left\{\left\|R_{1}^{n}-R_{1}\right\|, \cdots,\left\|R_{k}^{n}-R_{k}\right\|\right\} \rightarrow 0\right) .
$$

The space $D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ is then endowed with the corresponding product topology.

Theorem 2.3.9 (Continuity of $\left.\widetilde{\Psi}^{Q}\right) . \operatorname{Let}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfy the following four conditions:
i) $\omega(0), R_{k}$, and $\widetilde{R}_{k} \in(0, \infty)^{4}$ for all $k \geqslant 1$.
ii) There exists a finite, $\mathbb{N}$-valued $N_{T}$ such that $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(t)=\widetilde{\Psi}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(t)$ for $t \in[0, T]$ and $\left(h \circ \widetilde{\Psi}_{N_{T}}^{Q}\right)(\omega, R, \widetilde{R})(T) \in(0, \infty)^{2}$.
iii) Let $\widetilde{\varphi}_{k}(\omega, R, \widetilde{R}):=\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega\left(\cdot+\widetilde{\tau}_{k}\right)-\omega\left(\widetilde{\tau}_{k}\right)$. The functions $\omega$ and $\widetilde{\varphi}_{k}(\omega, R, \widetilde{R}), 1 \leqslant k \leqslant N_{T}$, satisfy the conditions of Corollary 2.3.6.
iv) Finally, $(0,0) \notin\left(h \circ \widetilde{\Psi}^{Q}\right)(\omega, R, \widetilde{R})([0, T])$.

Then, the function $\widetilde{\Psi}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ introduced in Definition 2.3.7 is continuous at $(\omega, R, \widetilde{R})$.

The proof of Theorem 2.3.9 is postponed to Section 2.7.2.3. The conditions in the above theorem are not very instructive at first glance. Note that condition i) ensures that $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ starts and is reinitialized at points in the interior of $\mathbb{R}_{+}^{4}$. Moreover, condition ii) ensures that the function $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ is reinitialized only $N_{T}<\infty$ often in $[0, T]$, but not at time $T$. Furthermore, conditions iii) ensures that $G$ is continuous at $\omega$ and at the random shift $\widetilde{\varphi}_{k}(\omega, R, \widetilde{R})$ of $\omega$ for all $1 \leqslant k \leqslant N_{T}$. Finally, condition iv) guarantees that the reinitialization of $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ at $\widetilde{\tau}_{k}$ by either $R_{k}$ or $\widetilde{R}_{k}$ is continuous at $(\omega, R, \widetilde{R})$. Later, we will apply the map $\widetilde{\Psi}^{Q}$ to a four-dimensional linear Brownian motion $X$ starting in the interior of $\mathbb{R}_{+}^{4}$ and random sequences $R^{+}$and $R^{-}$ with almost surely values in $(0, \infty)^{4}$. We show in the proof of Theorem 2.3.19 below, that ( $X, R^{+}, R^{-}$) fulfills conditions i)-iv) with probability one implying that it lies with probability one in the continuity set of $\widetilde{\Psi}^{Q}$.

Now, the function $\widetilde{\Psi}^{Q}$ may be applied to any càdlàg stochastic process: given a càdlàg process $X$ with values in $\mathbb{R}^{4}$ and random sequences $R=\left(R_{n}\right)_{n \geqslant 1}$ and $\widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1}$ with values in $(0, \infty)^{4}$, the process $\widetilde{\Psi}^{Q}(X, R, \widetilde{R})$ is a càdlàg process with values in $\mathbb{R}_{+}^{4}$. Finally, we deduce that the queue size process $\widetilde{Q}^{(n)}$ may be constructed by this procedure.

Theorem 2.3.10. For each $n \in \mathbb{N}$, the queue size process $\widetilde{Q}^{(n)}$ is given by

$$
\begin{equation*}
\widetilde{Q}^{(n)}=\widetilde{\Psi}^{Q}\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \tag{2.3.8}
\end{equation*}
$$

where

- $\widetilde{Q}_{0}^{(n)} \in(0, \infty)^{4}$ are the initial queue sizes of the bid/ask queues,
- $X^{(n)}$ is the piecewise constant interpolation of the net order flow process,
- $\widetilde{R}^{+,(n)} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$ is the sequence of queue sizes after price increases in $\widetilde{S}^{(n)}$,
- $\widetilde{R}^{-,(n)} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$ is the sequence of queue sizes after price decreases in $\widetilde{S}^{(n)}$.

Proof. Each component of the net order flow process $X^{(n)}$ represents the sum of incoming order sizes of a specific type $(i, I) \in\{b, a\} \times\{F, G\}$ over time. Then, a comparison of the described dynamics of the individual queue size processes (see the definition in $(2.2 .2)-(2.2 .5)$ ) with the construction procedure in Definitions 2.3 .2 and Definition 2.3.7 yields the stated equality (2.3.8).

### 2.3.2 The capacity process as a regulated process

Let us introduce the piecewise constant interpolation of the discrete-time processes $\left(\widetilde{C}_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ and $\left(M_{k}^{(n)}\right)_{k=0, \cdots, T_{n}}$ introduced in 2.2.6) and 2.2.7) by

$$
\widetilde{C}^{(n)}(t)=\widetilde{C}_{k}^{(n)} \quad \text { and } \quad M^{(n)}(t)=M_{k}^{(n)} \quad \text { for } t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)
$$

In this subsection, we want to derive a representation of the unrestricted capacity process $\widetilde{C}^{(n)}$ as a regulated process of the net order flow process $X^{(n)}$ and the sequences of random variables $\widetilde{R}^{+,(n)}$ and $\widetilde{R}^{-,(n)}$. To this end, let us introduce the function $\bar{g}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ which describes the dynamics of the number of crossborder trades triggered by events affecting one side of the shared order book provided that the cumulative queue size process of the corresponding side stays strictly larger than zero (i.e. no price change has been observed).

Definition 2.3.11. Let $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$. Moreover, for $k \in \mathbb{N}$, let $\hat{\tau}_{k}:=\hat{\tau}_{k}(\omega), \widetilde{g}_{k}(\omega)$, and $g_{k}(\omega)$ be as in Definition 2.3.2. We inductively define for $k \in \mathbb{N}$ the functions $\bar{g}_{k}(\omega)$ and $\bar{g}(\omega) \in D\left([0, T], \mathbb{R}_{+}^{2}\right)$ as follows: set $\bar{g}_{1}(\omega):=(0,0)$. For $k \geqslant 2$, we set $\bar{g}_{k}(\omega)(t)=\bar{g}_{k-1}(\omega)(t)$ for $t \in\left[0, \hat{\tau}_{k-1}\right)$.

- If $\widetilde{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \notin \mathbb{R}_{-}^{2}$, we set for $t \geqslant \hat{\tau}_{k-1}$

$$
\bar{g}_{k}(\omega)(t):=\bar{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}-\right)+\ell_{t}^{(2)}\left(g_{k-1}(\omega)\right)
$$

- In contrast, if $\widetilde{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \in \mathbb{R}_{-}^{2}$, we set for $t \geqslant \hat{\tau}_{k-1}$

$$
\bar{g}_{k}(\omega)(t):=\bar{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}-\right)+\ell_{\hat{\tau}_{k-1}}^{(2)}\left(\widetilde{g}_{k-1}(\omega)\right) .
$$

- Finally, we set

$$
\bar{g}(\omega)(t):=\sum_{k=1}^{\infty} \bar{g}_{k}(\omega)(t) \mathbb{1}_{\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)}(t) \quad \text { for } t \in\left[0, \hat{\tau}_{\infty}\right) .
$$

If there exists a finite $\kappa \in \mathbb{N}$ with $\hat{\tau}_{\infty}=\hat{\tau}_{\kappa}$ then set $\bar{g}(\omega)(t)=\bar{g}_{\kappa+1}(\omega)(t)$ for all $t \geqslant \hat{\tau}_{\kappa}$. Otherwise, set $\bar{g}(\omega)\left(\hat{\tau}_{\infty}\right)=\lim _{t \rightarrow \hat{\tau}_{\infty}} \bar{g}(\omega)(t)$ and $\bar{g}(\omega)(t)=\bar{g}(\omega)\left(\hat{\tau}_{\infty}\right)$ for all $t>\hat{\tau}_{\infty}$.
As desired, the function $\bar{g}$ describes the dynamics of the number of cross-border trades $\pi_{i} M^{(n)}$ triggered by events of type $i=b, a$ if we plug in the corresponding
components of the net order flow process $\pi_{i} X^{(n)}$, i.e., on the event that no price change appears during $[0, t]$ for some $t \in[0, T]$, it holds

$$
\pi_{i} M^{(n)}(s)=\bar{g}\left(\pi_{i} \widetilde{Q}_{0}^{(n)}+\pi_{i} X^{(n)}\right)(s) \quad \text { for all } s \in[0, t] .
$$

Moreover, if $\widetilde{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}\right) \notin \mathbb{R}_{-}^{2}$ but $\widetilde{g}_{k}(\omega)\left(\hat{\tau}_{k}\right) \in \mathbb{R}_{-}^{2}$, our definition of $\bar{g}(\omega)$ on $\left[\hat{\tau}_{k}, T\right]$ ensures that the possibly last cross-border trade has been counted and extends $\bar{g}(\omega)$ from $\left[0, \hat{\tau}_{k}\right)$ to the whole time interval $[0, T]$. Comparing the definition of $\bar{g}$ with that of $g$, we obtain for $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$ the relation

$$
g(\omega)=\omega+\bar{g}(\omega) R .
$$

Hence, we can directly deduce the continuity of the function $\hat{g}:=\pi_{2}^{(2)} \bar{g}-\pi_{1}^{(2)} \bar{g}$ from the characterization of the continuity set of $g$ derived in Lemma 2.3.5

Lemma 2.3.12. Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ satisfy condition $(\boldsymbol{I}), h_{1}\left(\omega_{0}\right) \in$ $C_{0}^{\prime}([0, T], \mathbb{R})$, and assume that $\hat{\tau}_{\infty}\left(\omega_{0}\right)=\tau\left(\omega_{0}\right)$. Then, the function $\hat{g}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow$ $D([0, T], \mathbb{R})$ given by $\hat{g}:=\pi_{2}^{(2)} \bar{g}-\pi_{1}^{(2)} \bar{g}$ is continuous at $\omega_{0}$.
Proof. Thanks to Lemma 2.3.5, the function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is continuous at $\omega_{0}$. Then, the continuity of $\hat{g}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D([0, T], \mathbb{R})$ at $\omega_{0}$ directly follows from the relation $\hat{g}(\omega)=\pi_{1}^{(2)} \omega-\pi_{1}^{(2)} g(\omega)$, for $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$.

Let us introduce another function $\hat{G}$ that maps càdlàg processes with values in $\mathbb{R}^{4}$ to the space of càdlàg processes with values in $\mathbb{R}$. For $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$, let

$$
\begin{equation*}
\hat{G}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D([0, T], \mathbb{R}), \quad \hat{G}(\omega)=\hat{g}\left(\pi_{b} \omega\right)-\hat{g}\left(\pi_{a} \omega\right) \tag{2.3.9}
\end{equation*}
$$

We observe that the function $\hat{G}$ describes the dynamics of the capacity process $\widetilde{C}^{(n)}$ if we plug in the net order flow process $X^{(n)}$, i.e., on the event that no price change appears during $[0, t]$ for some $t \in[0, T]$, we have

$$
\widetilde{C}^{(n)}(s)=\widetilde{C}_{0}^{(n)}+\hat{G}\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}\right)(s) \quad \text { for all } s \in[0, t] .
$$

Now, the continuity set of $\hat{G}$ can be directly deduced from Lemma 2.3.12.
Corollary 2.3.13. (Continuity of $\hat{G})$ Let $\omega_{0} \in D\left([0, T], \mathbb{R}^{4}\right)$ be such that its projections $\pi_{b} \omega_{0}$ and $\pi_{a} \omega_{0}$ satisfy the assumptions of Lemma 2.3.12. Then, the function $\hat{G}$ : $D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D([0, T], \mathbb{R})$ is continuous at $\omega_{0}$.

Next, we introduce a function $\widetilde{\Psi}^{C}$ that will be used to describe the evolution of the unrestricted capacity process $\widetilde{C}^{(n)}$ over time. Note that also $\widetilde{C}^{(n)}$ is affected by the reinitialization of the queue size process after each price change.

Definition 2.3.14. Let $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ and $R=\left(R_{n}\right)_{n \geqslant 1}, \widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$. For $k \in \mathbb{N}$ let the functions $\widetilde{\tau}_{k}:=\widetilde{\tau}_{k}(\omega, R, \widetilde{R})$ and $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})$ be as in Definition 2.3.7. We define $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R}) \in D([0, T], \mathbb{R})$ as follows:

- For $t \leqslant \widetilde{\tau}_{1}$, let $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})(t)=\hat{G}(\omega)(t)$.
- For $k \geqslant 1$ and $\widetilde{\tau}_{k}<t \leqslant \widetilde{\tau}_{k+1}$, we set

$$
\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})(t)=\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\hat{G}\left(\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega-\omega\left(\widetilde{\tau}_{k}\right)\right)(t)
$$

- Moreover, for $t>\widetilde{\tau}_{\infty}:=\lim _{k \rightarrow \infty} \widetilde{\tau}_{k}$, we set $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})(t)=0$.

Since $R, \widetilde{R} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$, we have $\hat{G}\left(\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega-\omega\left(\widetilde{\tau}_{k}\right)\right)\left(\widetilde{\tau}_{k}\right)=0$ for all $k \in \mathbb{N}$ and hence $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)=\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}+\right)$. We conclude that $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})$ indeed takes values in the space $D([0, T], \mathbb{R})$. For $k \geqslant 1$, the function $\widetilde{\Psi}^{C}(\omega, R, \widetilde{R})$ is not reinitialized at $\widetilde{\tau}_{k}$ but still depends on the reinitialization of $\Psi^{Q}(\omega, R, \widetilde{R})$ at $\widetilde{\tau}_{k}$. Thanks to the characterization of the continuity set of $\widetilde{\Psi}^{Q}$ in Theorem 2.3.9, we immediately obtain an understanding of the continuity points of $\widetilde{\Psi}^{C}$.

Theorem 2.3.15 (Continuity of $\left.\widetilde{\Psi}^{C}\right)$. Let $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left((0, \infty)^{4}\right)^{\mathbb{N}} \times$ $\left((0, \infty)^{4}\right)^{\mathbb{N}}$ satisfy the conditions i)-iv) in Theorem 2.3.9. Then the function $\widetilde{\Psi}^{C}$ : $D\left([0, T], \mathbb{R}^{4}\right) \times\left((0, \infty)^{4}\right)^{\mathbb{N}} \times\left((0, \infty)^{4}\right)^{\mathbb{N}} \rightarrow D([0, T], \mathbb{R})$ introduced in Definition 2.3.14 is continuous at $(\omega, R, \widetilde{R})$.

Proof. Since $(\omega, R, \widetilde{R})$ satisfies the conditions of Theorem 2.3.9, we know that $\widetilde{\Psi}^{Q}$ is continuous at $(\omega, R, \widetilde{R})$. Since condition ii) holds, the map $N_{T}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times$ $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow \mathbb{N}_{0} \cup\{+\infty\}$ is continuous at $(\omega, R, \widetilde{R})$ and $N_{T}(\omega, R, \widetilde{R})<\infty$. Moreover, condition iii) together with Corollary 2.3 .13 implies that $\hat{G}$ is continuous at $\omega$ and at $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega\left(\cdot+\widetilde{\tau}_{k}\right)-\omega\left(\widetilde{\tau}_{k}\right)$ for all $1 \leqslant k \leqslant N_{T}$. Hence, $\widetilde{\Psi}^{C}$ is the sum of finitely many functions that are continuous at $(\omega, R, \widetilde{R})$ and hence must be itself continuous at $(\omega, R, \widetilde{R})$.

Finally, we can construct the dynamics of the unrestricted capacity process $\widetilde{C}^{(n)}$ with the procedure introduced in Definition 2.3.14.
Theorem 2.3.16. For each $n \in \mathbb{N}$, the process $\widetilde{C}^{(n)}$ is given by

$$
\begin{equation*}
\widetilde{C}^{(n)}=\widetilde{C}_{0}^{(n)}+\widetilde{\Psi}^{C}\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \tag{2.3.10}
\end{equation*}
$$

where

- $\left(\widetilde{Q}_{0}^{(n)}, \widetilde{C}_{0}^{(n)}\right) \in(0, \infty)^{4} \times \mathbb{R}$ are the initial bid/ask queues and capacity dynamics,
- $X^{(n)}$ is the piecewise constant interpolation of the net order flow process,
- $\widetilde{R}^{+,(n)} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$ is the sequence of queue sizes after a price increase in $\widetilde{S}^{(n)}$,
- $\widetilde{R}^{-,(n)} \in\left((0, \infty)^{4}\right)^{\mathbb{N}}$ is the sequence of queue sizes after a price decrease in $\widetilde{S}^{(n)}$.

Proof. Each component of the net order flow process $X^{(n)}$ represents the sum of incoming order sizes of a specific type $(i, I) \in\{b, a\} \times\{F, G\}$ over time. Then, a comparison of the described dynamics of the number of cross-border trades and of the capacity process (see the definition in $(2.2 .6)$ and $(2.2 .7)$ ) with the construction procedure in Definition 2.3 .11 and Definition 2.3 .14 yields the stated equality 2.3 .10 .

### 2.3.3 The price process as a regulated process

In this subsection, we introduce a new representation of the bid price process $\widetilde{B}^{(n)}$ in terms of transformations of the net order flow process $X^{(n)}$ and the sequences of random variables $\widetilde{R}^{+,(n)}$ and $\widetilde{R}^{-,(n)}$ corresponding to the order sizes after price changes.

Definition 2.3.17. Let $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ and $R=\left(R_{n}\right)_{n \geqslant 1}, \widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1} \in\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$. For $k \in \mathbb{N}$, let $\widetilde{\tau}_{k}:=\widetilde{\tau}_{k}(\omega, R, \widetilde{R})$ and $\widetilde{\Psi}_{k-1}^{Q}:=\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})$ be as in Definition 2.3.7. Then, we define the functions $N_{b}(\omega, R, \widetilde{R}), N_{a}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{N}_{0} \cup\{+\infty\}\right)$, and $\widetilde{\Psi}^{B}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{2}\right) b y$
$N_{a}(\omega, R, \widetilde{R}):=\sum_{\widetilde{\tau}_{k} \leqslant \cdot} \mathbb{1}_{\left\{\left(\pi_{2}^{(2)} \circ h \circ \widetilde{\Psi}_{k-1}^{Q}\right)\left(\widetilde{\tau}_{k}\right) \leqslant 0\right\}, \quad N_{b}(\omega, R, \widetilde{R}):=\sum_{\widetilde{\tau}_{k} \leqslant \cdot} \mathbb{1}\left\{\left(\pi_{1}^{(2)} \circ h \circ \widetilde{\Psi}_{k-1}^{Q}\right)\left(\widetilde{\tau}_{k}\right) \leqslant 0\right\}, ~, ~, ~}$,
and

$$
\widetilde{\Psi}^{B}(\omega, R, \widetilde{R}):=\delta\left(N_{a}(\omega, R, \widetilde{R})-N_{b}(\omega, R, \widetilde{R})\right)(1,1)
$$

It can be easily deduced from the described bid price dynamics in equations (2.2.2)2.2.5 that

$$
\begin{equation*}
\widetilde{B}^{(n)}=\widetilde{B}_{0}^{(n)}+\widetilde{\Psi}^{B}\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \tag{2.3.11}
\end{equation*}
$$

where $\widetilde{B}_{0}^{(n)}=\left(\widetilde{B}_{0}^{F,(n)}, \widetilde{B}_{0}^{F,(n)}\right) \in(\delta \mathbb{Z})^{2}$ by the assumptions on the initial states. The following theorem characterizes the continuity set of the function $\widetilde{\Psi}^{B}$.
Theorem 2.3.18 (Continuity of $\left.\widetilde{\Psi}^{B}\right)$. Let $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfy the conditions $i$ )-iv) in Theorem 2.3.9. Moreover, assume that

$$
\begin{equation*}
\operatorname{Disc}\left(N_{a}(\omega, R, \widetilde{R})\right) \cap \operatorname{Disc}\left(N_{b}(\omega, R, \widetilde{R})\right)=\varnothing \tag{2.3.12}
\end{equation*}
$$

where $\operatorname{Disc}\left(N_{i}(\omega, R, \widetilde{R})\right):=\left\{t \in[0, T]: N_{i}(\omega, R, \widetilde{R})(t-) \neq N_{i}(\omega, R, \widetilde{R})(t)\right\}$ for $i=b, a$. Then, the map $\widetilde{\Psi}^{B}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ is continuous at $(\omega, R, \widetilde{R})$.

Proof. Since the conditions i)-iv) of Theorem 2.3 .9 hold for $(\omega, R, \widetilde{R})$, we know by Corollary 2.7 .13 that the maps $N_{b}, N_{a}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{N}_{0}\right)$ are continuous at $(\omega, R, \widetilde{R})$. Since also the condition in 2.3 .12 is satisfied, we can apply Theorem 4.1 in Whitt 85$]$ and conclude that also the map $\widetilde{\Psi}^{B}: D\left([0, T], \mathbb{R}^{4}\right) \times$ $\left(\mathbb{R}_{+}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ is continuous at $(\omega, R, \widetilde{R})$. This finishes the proof.

### 2.3.4 A limit theorem for the active dynamics

In this subsection, we derive a convergence theorem for the active market dynamics $\widetilde{S}^{(n)}:=\left(\widetilde{B}^{(n)}, \widetilde{Q}^{(n)}, \widetilde{C}^{(n)}\right)$. The proof is based on the functional central limit theorem for $X^{(n)}$ (cf. Proposition 2.3.1 combined with the continuous mapping theorem. In the following, let us introduce the sequence of stopping times corresponding to the prices changes in $\widetilde{S}^{(n)}$ by $\widetilde{\tau}_{0}^{(n)}:=0$ and

$$
\begin{equation*}
\widetilde{\tau}_{k}^{(n)}:=\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\left(\widetilde{Q}_{0}^{(n)}+X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right)\right) \quad \text { for } k \geqslant 1 \tag{2.3.13}
\end{equation*}
$$

where the first hitting time map $\widetilde{\tau}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ is defined in 2.3.6. Next, for $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left((0, \infty)^{4}\right)^{\mathbb{N}} \times\left((0, \infty)^{4}\right)^{\mathbb{N}}$ and $s_{0}:=\left(b_{0}, q_{0}, c_{0}\right) \in \mathbb{R}^{2} \times$ $(0, \infty)^{4} \times \mathbb{R}$ let us introduce the function

$$
\widetilde{\Psi}\left(s_{0} ; \omega, R, \widetilde{R}\right):=\left(b_{0}+\widetilde{\Psi}^{B}\left(q_{0}+\omega, R, \widetilde{R}\right), \widetilde{\Psi}^{Q}\left(q_{0}+\omega, R, \widetilde{R}\right), c_{0}+\widetilde{\Psi}^{C}\left(q_{0}+\omega, R, \widetilde{R}\right)\right)
$$

and observe that by Theorem 2.3.10, Theorem 2.3.16 and equation 2.3.11) the process $\widetilde{S}^{(n)}:=\left(\widetilde{B}^{(n)}, \widetilde{Q}^{(n)}, \widetilde{C}^{(n)}\right)=\widetilde{\Psi}\left(\widetilde{S}_{0}^{(n)} ; X^{(n)}, R^{+,(n)}, R^{-,(n)}\right)$.

Theorem 2.3.19 (Limit theorem for the active dynamics $\widetilde{S}^{(n)}$ ). Let Assumptions 2.2, 2.3. and 2.4 be satisfied and let $\widetilde{S}_{0}^{(n)} \in(\delta \mathbb{Z})^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z}$ with $\widetilde{B}_{0}^{F,(n)}=\widetilde{B}_{0}^{G,(n)}$ for all $n \in \mathbb{N}$ be such that $\widetilde{S}_{0}^{(n)} \rightarrow \widetilde{S}_{0} \in(\delta \mathbb{Z})^{2} \times(0, \infty)^{4} \times \mathbb{R}$. Then,

$$
\widetilde{S}^{(n)}=\widetilde{\Psi}\left(\widetilde{S}_{0}^{(n)} ; X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \Rightarrow \widetilde{\Psi}\left(\widetilde{S}_{0} ; X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)=:(\widetilde{B}, \widetilde{Q}, \widetilde{C})=: \widetilde{S}
$$

in the Skorokhod topology on the space $D([0, T], E)$, where the sequences $\widetilde{R}^{+}:=\left(\widetilde{R}_{k}^{+}\right)_{k \geqslant 1}$ and $\widetilde{R}^{-}:=\left(\widetilde{R}_{k}^{-}\right)_{k \geqslant 1}$ are defined in equation 2.3.17 below.

Proof. Proposition 2.3.1 combined with the assumption on the initial state shows, that the net order flow process $Y^{(n)}:=\widetilde{Q}_{0}^{(n)}+X^{(n)}$ satisfies a functional central limit theorem, i.e.,

$$
\begin{equation*}
Y^{(n)} \Rightarrow Y:=\left(\widetilde{Q}_{0}+\Sigma^{1 / 2} B(t)+t \mu\right)_{t \geqslant 0} \tag{2.3.14}
\end{equation*}
$$

in the Skorokhod topology on $D\left([0, T], \mathbb{R}^{4}\right)$, where $B$ defines a standard, four-dimensional Brownian motion and the drift $\mu$ and the covariance matrix $\Sigma$ are defined in Proposition 2.3 .1 . With a slight abuse of notion, by the Skorokhod representation theorem and Assumption 2.4 we may assume the existence of independent sequences of iid random variables $\left(\epsilon_{k}^{+}\right)_{k \geqslant 1},\left(\epsilon_{k}^{-}\right)_{k \geqslant 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\epsilon_{1}^{+} \sim f^{+}, \epsilon_{1}^{-} \sim f^{-}$, and

$$
\mathbb{P}\left[Y^{(n)} \rightarrow Y, \text { and for all } k \geqslant 1, \epsilon_{k}^{+,(n)} \rightarrow \epsilon_{k}^{+}, \epsilon_{k}^{-,(n)} \rightarrow \epsilon_{k}^{-}\right]=1
$$

Applying Theorem 2.3.10, Theorem 2.3.16, and equation 2.3.11), we have $\widetilde{S}^{(n)}:=$ $\left(\widetilde{B}^{(n)}, \widetilde{Q}^{(n)}, \widetilde{C}^{(n)}\right)=\widetilde{\Psi}\left(\widetilde{S}_{Q}^{(n)} ; X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right)$. In the following, we will show that $\left(\widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \rightarrow\left(\widetilde{R}^{+}, \widetilde{R}^{-}\right) \mathbb{P}$-almost surely for appropriately constructed sequences
$\widetilde{R}^{+}$and $\widetilde{R}^{-}$, and $\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$lies, with probability one, in the continuity set of $\widetilde{\Psi}^{I}, I=$ $B, Q, C$. Then, the assumption on the initial states, equation (2.3.14), and the continuous mapping theorem yield the stated result. Let us first prove that $\left(\widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \rightarrow$ $\left(\widetilde{R}^{+}, \widetilde{R}^{-}\right) \mathbb{P}$-almost surely.
For $i=b, a$, the process $\pi_{i} Y$ defines a planar Brownian motion starting in $\pi_{i} \widetilde{Q}_{0} \in$ $(0, \infty)^{2}$. Applying Lemma 2.7.7 and Corollary 2.3.6 we conclude that $G: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow$ $D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is $\mathbb{P}$-almost surely continuous at $Y$. Now, Proposition 2.3.1 and the continuous mapping theorem (cf. Billingsley [8, Theorem 2.7]) imply

$$
G\left(Y^{(n)}\right) \rightarrow G(Y) \quad \mathbb{P} \text {-a.s. }
$$

in the Skorokhod topology. Next, we construct the limit process $\widetilde{Q}$ of $\widetilde{Q}^{(n)}$ by induction: let $\widetilde{\tau}_{1}^{*}=\widetilde{\tau}(G(Y))$ be the first hitting time of $(h \circ G)(Y)$ of the axes $\{(0, y): y>$ $0\} \cup\{(x, 0): x>0\}$ and let us set $\widetilde{Q}(t)=G(Y)(t)$ for $t<\widetilde{\tau}_{1}^{*}$. Since $(h \circ G)(Y)=h(Y)$ on $\left[0, \widetilde{\tau}_{1}^{*}\right)$ (cf. Lemma 2.3.4), $\widetilde{\tau}_{1}^{*}$ equals the first hitting time of $h(Y)$ of the axes and $h(Y)$ crosses the axes at $\widetilde{\tau}_{1}^{*}$ with probability one. Since $Y$ is a four-dimensional linear Brownian motion, $h(Y)$ is again a planar Brownian motion and has therefore almost surely sample paths in $C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$. Now, the continuity of the first hitting time and last value map (cf. Lemma 2.7 .2 ) and the continuous mapping theorem imply

$$
\left(\widetilde{\tau}_{1}^{(n)}, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{1}^{(n)}-\right)\right) \rightarrow\left(\widetilde{\tau}_{1}^{*}, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}-\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Let us set

$$
\widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right):=\Phi\left(G(Y)\left(\widetilde{\tau}_{1}^{*}-\right), \epsilon_{1}^{+}\right) \mathbb{1}_{\left\{\widetilde{\tau}_{a, 1}=\widetilde{\tau}_{1}^{*}\right\}}+\Phi\left(G(Y)\left(\widetilde{\tau}_{1}^{*}-\right), \epsilon_{1}^{-}\right) \mathbb{1}_{\left\{\tilde{\tau}_{b, 1}=\tilde{\tau}_{1}^{*}\right\}},
$$

where, for the first hitting time maps $\widetilde{\tau}_{b}, \widetilde{\tau}_{a}$ introduced in equation 2.3.7), we set

$$
\widetilde{\tau}_{a, 1}:=\widetilde{\tau}_{a}(G(Y)) \quad \text { and } \quad \widetilde{\tau}_{b, 1}:=\widetilde{\tau}_{b}(G(Y)) .
$$

Again, since we can relate $\widetilde{\tau}_{1}^{*}$ (and also $\widetilde{\tau}_{b, 1}, \widetilde{\tau}_{a, 1}$ ) to the first hitting time of $h(Y)$ of the axes, and $h(Y)$ has, with probability one, sample paths in $C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, we can apply Lemma 2.7 .3 to conclude that $G(Y)$ lies, with probability one, in the continuity set of $H_{i}: \omega \mapsto \mathbb{1}_{\left\{\tilde{\tau}(\omega)=\tilde{\tau}_{i}(\omega)\right\}}, i=b, a$. So, using Assumption 2.4 (i.e. $\left\|\Phi^{(n)}-\Phi\right\|_{\infty} \rightarrow 0$ and the continuity of $\Phi(\cdot, \cdot)$ ), the continuity of the last value map, and the continuous mapping theorem, we conclude

$$
\widetilde{Q}^{(n)}\left(\widetilde{\tau}_{1}^{(n)}\right) \rightarrow \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right) \quad \mathbb{P} \text {-a.s. }
$$

Let us now show the induction step: assume that we have defined $\widetilde{Q}$ on $\left[0, \widetilde{\tau}_{k}^{*}\right]$ and have shown that

$$
\left(\widetilde{\tau}_{1}^{(n)}, \cdots, \widetilde{\tau}_{k}^{(n)}, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{1}^{(n)}\right), \cdots, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{k}^{(n)}\right)\right) \rightarrow\left(\widetilde{\tau}_{1}^{*}, \cdots, \widetilde{\tau}_{k}^{*}, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Next, we define the stopping time $\widetilde{\tau}_{k+1}^{*}:=\widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)\right)$, i.e., $\widetilde{\tau}_{k+1}^{*}$ denotes
the first hitting time of $\left(h \circ \widetilde{\Psi}_{k}^{Q}\right)\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)$ of the axes $\{(0, y): y>0\} \cup$ $\{(x, 0): x>0\}$. Now, we want to extend the definition of $\widetilde{Q}$ to $\left[0, \widetilde{\tau}_{k+1}^{*}\right]$ by setting

$$
\begin{aligned}
\widetilde{Q}(t) & :=\widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)(t), \quad \text { for } t<\widetilde{\tau}_{k+1}^{*} \\
\widetilde{Q}\left(\widetilde{\tau}_{k+1}^{*}\right) & \left.:=\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k+1}^{*}-\right), \epsilon_{k}^{+}\right) \mathbb{1}_{\left\{\widetilde{\tau}_{a, k+1}=\widetilde{\tau}_{k+1}^{*}\right\}}+\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k+1}^{*}-\right), \epsilon_{k+1}^{-}\right) \mathbb{1}_{\left\{\widetilde{\tau}_{b, k+1}=\widetilde{\tau}_{k+1}^{*}\right\}}\right\}
\end{aligned}
$$

where

$$
\widetilde{\tau}_{a, k+1}:=\widetilde{\tau}_{a}\left(\widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)\right), \quad \widetilde{\tau}_{b, k+1}:=\widetilde{\tau}_{b}\left(\widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)\right)
$$

Let us denote $\widetilde{B}_{k}:=\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)+Y\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-Y\left(\widetilde{\tau}_{k}^{*}\right)$ and observe that $\widetilde{B}_{k}$ defines again a four-dimensional linear Brownian motion. Applying again Lemma 2.7.7 and Corollary 2.3.6, we conclude that $G$ is $\mathbb{P}$-almost surely continuous at $\widetilde{B}_{k}$. Moreover, we observe that $(h \circ G)\left(\widetilde{B}_{k}\right)=h\left(\widetilde{B}_{k}\right)$ on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right)$. Therefore, $\widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}$ equals the first hitting time of $h\left(\widetilde{B}_{k}\right)$ of the axes. Since $Y$ is a four-dimensional linear Brownian motion, we conclude that $h\left(\widetilde{B}_{k}\right)$ is a planar Brownian motion and has therefore, almost surely, sample paths in $C_{0}^{\prime}\left(\left[0, T-\widetilde{\tau}_{k}^{*}\right], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$. Further, applying Assumption 2.4, we have that $\widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \ldots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right) \in(0, \infty)^{4}$ with probability one. Now, by the definition of $\widetilde{\Psi}_{k}^{Q}$, we conclude that $(0,0) \notin\left(h \circ \widetilde{\Psi}_{k}^{Q}\right)\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)\left(\left[0, \widetilde{\tau}_{k+1}^{*}\right]\right)$ almost surely. Hence, by Lemma 2.7.9, $\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)$ lies with probability one in the continuity set of $\widetilde{\Psi}_{k}^{Q}$. So, by the continuous mapping theorem, we have

$$
\widetilde{\Psi}_{k}^{Q}\left(Y^{(n)}, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{1}^{(n)}\right), \cdots, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{k}^{(n)}\right)\right) \rightarrow \widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

in the Skorokhod topology. Again, combining the continuity of the first hitting time and last value map (cf. Lemma 2.7.2) and the continuous mapping theorem, we conclude that

$$
\left(\widetilde{\tau}_{k+1}^{(n)}, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{k+1}^{(n)}-\right)\right) \rightarrow\left(\widetilde{\tau}_{k+1}^{*}, \widetilde{Q}\left(\widetilde{\tau}_{k+1}^{*}-\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Again, since we can relate $\widetilde{\tau}_{k+1}^{*}$ (and also $\left.\widetilde{\tau}_{b, k+1}, \widetilde{\tau}_{a, k+1}\right)$ to the first hitting time of $h\left(\widetilde{B}_{k}\right)$ of the axes and $h\left(\widetilde{B}_{k}\right)$ has, with probability one, sample paths in $C_{0}^{\prime}([0, T-$ $\left.\left.\widetilde{\tau}_{k}^{*}\right], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, we can apply Lemma 2.7 .3 to conclude that $\widetilde{\Psi}_{k}^{Q}\left(Y, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right)$ lies, with probability one, in the continuity set of $H_{i}: \omega \mapsto \mathbb{1}_{\left\{\widetilde{\tau}(\omega)=\widetilde{\tau}_{i}(\omega)\right\}}, i=b, a$. So, using again Assumption 2.4 , the continuity of the last value map, and the continuous mapping theorem, we conclude that $\widetilde{Q}^{(n)}\left(\widetilde{\tau}_{k+1}^{(n)}\right) \rightarrow \widetilde{Q}\left(\widetilde{\tau}_{k+1}^{*}\right) \mathbb{P}$-almost surely. So finally, we have shown for all $k \geqslant 1$, that

$$
\left(\widetilde{\tau}_{1}^{(n)}, \cdots, \widetilde{\tau}_{k}^{(n)}, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{1}^{(n)}\right), \cdots, \widetilde{Q}^{(n)}\left(\widetilde{\tau}_{k}^{(n)}\right)\right) \rightarrow\left(\widetilde{\tau}_{1}^{*}, \cdots, \widetilde{\tau}_{k}^{*}, \widetilde{Q}\left(\widetilde{\tau}_{1}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Now, we construct the sequences $\widetilde{R}^{+}, \widetilde{R}^{-}$by setting $\widetilde{R}_{k}^{+}:=\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}-\right), \epsilon_{k}^{+}\right)$and $\widetilde{R}_{k}^{-}:=$ $\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}-\right), \epsilon_{k}^{-}\right)$, for $k \geqslant 1$. Then, we have

$$
\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)=\widetilde{R}_{k}^{+} \text {if } \widetilde{\tau}_{a, k}=\widetilde{\tau}_{k}^{*}, \quad \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)=\widetilde{R}_{k}^{-} \text {if } \widetilde{\tau}_{b, k}=\widetilde{\tau}_{k}^{*}
$$

and $\left(\widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \rightarrow\left(\widetilde{R}^{+}, \widetilde{R}^{-}\right) \mathbb{P}$-almost surely.
By Assumption 2.4 we have $\widetilde{Q}\left(\widetilde{\tau}_{\sim}^{*}\right), \cdots, \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right) \in(0, \infty)^{4}$ with probability one. Applying Lemma 2.3.4 and Theorem 2.3.1, we conclude that $h\left(\widetilde{B}_{k}\right), k \geqslant 0$, are planar Brownian motions, whose increments are independent over $k$, each starting in the interior of the positive orthant. Note that the distribution of the first hitting time of a planar Brownian motion is well studied (cf. e.g. [19, 48, 64, 90] or the conditional distribution in (2.6.1). Hence, for any $t>0$ and $y \in(0, \infty)^{4}, k \geqslant 0$, we conclude that

$$
\begin{equation*}
\mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{k}\right) \leqslant t \mid \widetilde{B}_{k}(0)=y\right]=: \delta(t, y) \in(0,1) \tag{2.3.15}
\end{equation*}
$$

and further note that this probability is decreasing in $y$ (component-wise). Next, for each $\varepsilon>0$, let us denote by $U(\varepsilon):=\left\{x \in \mathbb{R}_{+}^{4}: \pi_{j} x \leqslant \varepsilon \quad \forall j=1,2,3,4\right\}$ the $\varepsilon$-ball with respect to the sup-norm and $\epsilon_{k}:=\epsilon_{k}^{+} \mathbb{1}_{\left\{\tilde{\tau}_{a, k}=\tilde{\tau}_{k}^{*}\right\}}+\epsilon_{k}^{-} \mathbb{1}_{\left\{\tilde{\tau}_{b}, k=\widetilde{\tau}_{k}^{*}\right\}}, k \in \mathbb{N}$. Now for any $\varepsilon>0$ and $m \geqslant 0$ we have thanks to Assumption 2.4 i),

$$
\begin{aligned}
& \mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{m}\right) \leqslant t \mid \widetilde{S}(t), t \leqslant \widetilde{\tau}_{m}^{*}\right] \\
& \quad \leqslant \mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{m}\right) \leqslant t \mid \widetilde{B}_{m}(0) \notin U(\varepsilon)\right] \mathbb{1}_{\left\{\widetilde{B}_{m}(0) \notin U(\varepsilon)\right\}}+\mathbb{1}_{\left\{\widetilde{B}_{m}(0) \in U(\varepsilon)\right\}} \\
& \quad \leqslant \delta(t,(\varepsilon, \varepsilon, \varepsilon, \varepsilon)) \mathbb{1}_{\left\{\widetilde{B}_{m}(0) \notin U(\varepsilon)\right\}}+\mathbb{1}_{\left\{\widetilde{B}_{m}(0) \in U(\varepsilon)\right\}} \\
& \quad \leqslant(1-\delta(t,(\varepsilon, \varepsilon, \varepsilon, \varepsilon))) \mathbb{1}_{\left\{\alpha \epsilon_{m} \in U(\varepsilon)\right\}}+\delta(t,(\varepsilon, \varepsilon, \varepsilon, \varepsilon))=: U_{m}(t, \varepsilon),
\end{aligned}
$$

where $U_{m}(t, \varepsilon)$ is independent of $\left\{\widetilde{\tau}\left(\widetilde{B}_{k}\right) \leqslant t \forall k \leqslant m-1\right\}$. As $\eta(t, \varepsilon):=\mathbb{E}\left[U_{m}(t, \varepsilon)\right]$ is independent of $m$ and strictly smaller than 1 for $\varepsilon$ small enough by Assumption 2.4 i ), we may conclude that

$$
\begin{aligned}
\mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{k}\right) \leqslant t \forall k \leqslant m\right] & =\mathbb{E}\left[\prod_{k \leqslant m-1} \mathbb{1}_{\left\{\widetilde{\tau}\left(\widetilde{B}_{k}\right) \leqslant t\right\}} \cdot \mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{m}\right) \leqslant t \mid \widetilde{S}(t), t \leqslant \widetilde{\tau}_{m}^{*}\right]\right] \\
& \leqslant \mathbb{P}\left[\widetilde{\tau}\left(\widetilde{B}_{k}\right) \leqslant t \forall k \leqslant m-1\right] \cdot \eta(t, \varepsilon) \leqslant(\eta(t, \varepsilon))^{m} \xrightarrow{m \rightarrow \infty} 0 .
\end{aligned}
$$

Hence, with probability one, there exists a finite, $\mathbb{N}$-valued random variable $N_{T}$ such that $\widetilde{\tau}_{N_{T}}^{*}<T, \widetilde{\tau}_{N_{T}+1}^{*}=T$, and

$$
\widetilde{Q}:=\widetilde{\Psi}_{N_{T}}^{Q}\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right)=\widetilde{\Psi}^{Q}\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right) \quad \mathbb{P} \text {-a.s. }
$$

Let us now show that ( $Y, \widetilde{R}^{+}, \widetilde{R}^{-}$) lies with probability one in the continuity set of $\widetilde{\Psi}^{I}$, for $I=B, Q, C$. Therefore, we verify the conditions i)-iv) of Theorem 2.3.9. First, $Y(0)=\widetilde{Q}_{0} \in(0, \infty)^{4}$ and by Assumption 2.4 the elements of $\widetilde{R}^{+}$and $\widetilde{R}^{-}$have values in $(0, \infty)^{4}$ with probability one. Hence, condition i) holds with probability one. As argued above, we can relate the stopping times $\widetilde{\tau}_{k}^{*}, k \geqslant 1$, to the first hitting times of $h\left(\widetilde{B}_{k-1}\right), k \geqslant 1$, of the axes and as we already have shown, with probability one, there exists a finite, $\mathbb{N}$-valued random variable $N_{T}$ such that $\widetilde{\tau}_{N_{T}}^{*}<T, \widetilde{\tau}_{N_{T}+1}^{*}=T$, and either $\widetilde{\tau}_{k}^{*}=\widetilde{\tau}_{a, k}$ or $\widetilde{\tau}_{k}^{*}=\widetilde{\tau}_{b, k}$ for all $1 \leqslant k \leqslant N_{T}$. Since $h\left(\widetilde{B}_{N_{T}}\right)$ is again a planar Brownian
motion, $\left(h \circ \widetilde{\Psi}_{N_{T}}^{Q}\right)\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right)(T) \in(0, \infty)^{2}$ with probability one. Hence, also condition ii) and iv) are satisfied. Moreover, the processes $\widetilde{B}_{k}, k \geqslant 0$, are four-dimensional linear Brownian motions. Applying again Lemma 2.7.7 and Corollary 2.3.6, we conclude that also condition iii) is satisfied with probability one for all $1 \leqslant k \leqslant N_{T}$. An application of Theorem 2.3 .9 and Theorem 2.3 .15 yields that $\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$lies with probability one in the continuity set of $\widetilde{\Psi}^{Q}$ and $\Psi^{C}$. Again since $h\left(\widetilde{B}_{k}\right), k \geqslant 0$, are planar Brownian motions, we conclude that also condition 2.3 .12 in Theorem 2.3.18 is satisfied implying that $\left(Y, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$lies with probability one in the continuity set of $\Psi^{B}$. We can therefore apply the continuous mapping theorem and the assumption on the initial state to conclude that

$$
\widetilde{\Psi}\left(\widetilde{S}_{0}^{(n)} ; X^{(n)}, \widetilde{R}^{+,(n)}, \widetilde{R}^{-,(n)}\right) \rightarrow \widetilde{\Psi}\left(\widetilde{S}_{0} ; X, \widetilde{R}^{+}, \widetilde{R}^{-}\right) \quad \mathbb{P} \text {-a.s. }
$$

in the Skorokhod topology as $n \rightarrow \infty$. This finishes the proof.
In the following, we denote by $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 1}$ the stopping times corresponding to the times of price changes in the limit $\widetilde{S}$ of $\widetilde{S}^{(n)}$, i.e., by Theorem 2.3.19 we have $\widetilde{\tau}_{0}^{*}:=0$ and

$$
\begin{equation*}
\widetilde{\tau}_{k}^{*}:=\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\left(\widetilde{Q}_{0}+X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)\right) \quad \text { for } k \geqslant 1 \tag{2.3.16}
\end{equation*}
$$

where the random sequences $\widetilde{R}^{+}, \widetilde{R}^{-}$describe the queue sizes after price changes in $\widetilde{S}$ with

$$
\begin{equation*}
\widetilde{R}_{k}^{+}:=\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}-\right), \epsilon_{k}^{+}\right), \quad \widetilde{R}_{k}^{-}:=\Phi\left(\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}-\right), \epsilon_{k}^{-}\right) \tag{2.3.17}
\end{equation*}
$$

for independent sequences of iid random variables $\left(\epsilon_{k}^{+}\right)_{k \geqslant 1}$ and $\left(\epsilon_{k}^{-}\right)_{k \geqslant 1}$ with $\epsilon_{1}^{+} \sim f^{+}$ and $\epsilon_{1}^{-} \sim f^{-}$, and the distributions $f^{+}, f^{-}$in Assumption 2.4. Analogously to the proof of Theorem 2.3.19, we define the stopping times $\left(\widetilde{\tau}_{b, k}, \widetilde{\tau}_{a, k}\right)_{k \geqslant 1}$.

### 2.3.5 Identification of the distribution of the limit dynamics

Before studying the distribution of the limit dynamics, for $W$ being a planar Brownian motion, we identify the process $g(W)$ with a special case of a two-dimensional semimartingale reflecting Brownian motion (SRBM) absorbed at the origin, which we will call a sum-preserving $S R B M$ absorbed at the origin.

Definition 2.3.20 (Sum-preserving SRBM absorbed at the origin.). A sum-preserving $S R B M$ associated with $(x, \mu, \Sigma) \in \mathbb{R}_{+}^{2} \times \mathbb{R}^{2} \times \mathbb{R}^{2 \times 2}$ that starts from $x$ and is absorbed at the origin, is a triple of continuous, adapted, two-dimensional processes $(Z, W, l)$ on some filtered probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geqslant 0}, \mathbb{P}_{x}\right)$ such that for $\tau:=\inf \{t \geqslant 0: Z(t)=0\} \wedge T$, under $\mathbb{P}_{x}$,

$$
Z(t)= \begin{cases}W(t)+l(t) R & \text { for all } t \leqslant \tau  \tag{2.3.18}\\ 0 & \text { for all } t \geqslant \tau\end{cases}
$$

with reflection matrix

$$
R:=\left(\begin{array}{rr}
1 & -1  \tag{2.3.19}\\
-1 & 1
\end{array}\right)
$$

where
i) $W$ is an adapted, two-dimensional process such that $B:=\{W(t \wedge \tau)-\mu(t \wedge$ $\left.\tau), \mathcal{F}_{t}, t \geqslant 0\right\}$ is a martingale with variation process $\Sigma(t \wedge \tau)$ for all $t \geqslant 0$, and $W(0)=x \mathbb{P}_{x}$-a.s.,
ii) $l$ is an adapted, two-dimensional process such that $\mathbb{P}_{x^{-}}$-a.s. for each $i \in\{1,2\}$, the $i$-th component $l_{i}$ of $l$ satisfies
a) $l_{i}(0)=0$,
b) $l_{i}$ is non-decreasing,
c) $\int_{0}^{t} Z_{i}(t) d l_{i}(t)=0$ for all $t \in[0, T]$, and
d) $l_{i}(t)=l_{i}(\tau)$ for all $t \geqslant \tau$.

Such a process has been extensively studied in the literature, cf. e.g. [27, 31, 77, 84]. Note that a SRBM has also been introduced for other choices of reflection matrices, often assuming that the matrix $R$ is completely- $\mathscr{S}$ (cf. e.g. [24, 38, 77]). The matrix $R$ in the above definition, however, does not satisfy the completely- $\mathscr{S}$ property. For this reason, it is indeed important to set $Z(t)=(0,0)$ for all $t>\tau$ as the process $Z$ cannot be described anymore by the SDE in (2.3.18) on ( $\tau, T]$ as it is trap at the origin. The weak existence and uniqueness in law of this process has been derived in e.g., 77 , Section 4]. In particular, the process $(Z, W, l)$ is a continuous semimartingale.

For a planar Brownian motion $W$ starting in $x \in \mathbb{R}_{+}^{2}$ with mean $\mu \in \mathbb{R}^{2}$ and covariance matrix $\Sigma \in \mathbb{R}^{2 \times 2}$, we deduce in the following proposition that $g(W)$ is the unique strong solution of the SDE in (2.3.18).

Proposition 2.3.21. Let $(Z, W, l)$ be a sum-preserving SRBM associated with $(x, \mu, \Sigma) \in$ $\mathbb{R}_{+}^{2} \times \mathbb{R}^{2} \times \mathbb{R}^{2 \times 2}$ that is absorbed at the origin. Then, $Z=g(W)$ almost surely. In particular, $g(W)$ is the unique strong solution of the SDE in (2.3.18).

Proof. Let $W=\left(W_{1}, W_{2}\right)$ be a planar Brownian motion starting in $x \in \mathbb{R}_{+}^{2}$ with mean $\mu \in \mathbb{R}^{2}$ and covariance matrix $\Sigma \in \mathbb{R}^{2 \times 2}$. Moreover, let $(Z, W, l)$ be a SRBM associated with $(x, \mu, \Sigma)$ as introduced in Definition 2.3 .20 and let $\tau:=\inf \{t \geqslant 0$ : $Z(t)=0\} \wedge T=\inf \left\{t \geqslant 0: h_{1}(W)(t) \leqslant 0\right\} \wedge T$. If $\tau<T$, by the definition of $g(W)$ we have $g(W)(t)=(0,0)$ for all $t \geqslant \tau$. This proves the stated result on $[\tau, T]$. For $k \in \mathbb{N}$, let $\hat{\tau}_{k}:=\hat{\tau}_{k}(W)$ be as in Definition 2.3.2. To prove the stated result on $[0, \tau)$, we will inductively show for each $k \in \mathbb{N}$, that $g(W)$ is the unique strong solution of the SDE in (2.3.18) on each sub-interval $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$.
Induction start: Let us consider the sub-interval $\left[0, \hat{\tau}_{1}\right)$. By definition of $\hat{\tau}_{1}$, we have $W_{1}, W_{2}>0$ on $\left[0, \hat{\tau}_{1}\right)$. Hence, $l_{1}(t)=l_{2}(t)=0$ for all $t \in\left[0, \hat{\tau}_{1}\right)$. We conclude that

$$
g(W)=W=Z \quad \text { on }\left[0, \hat{\tau}_{1}\right) .
$$

This proves the induction start.
Induction hypothesis: For all $j=1, \cdots, k-1$, let us assume that $g(W)=Z$ almost surely on $\left[0, \hat{\tau}_{j}\right)$.

Induction step: Let us consider the sub-interval $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$. Without loss of generality, let consider the event $\left\{\hat{\tau}_{k-1}=\hat{\tau}_{k-1,1}(W)\right\}$, i.e., $g(W)$ hits the $x$-axis at $\hat{\tau}_{k-1}$. On this event, we have $\sup _{s \leqslant \cdot}\left(-\pi_{2}^{(2)} g_{k-1}(W)(s)\right)^{+}=0$ on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$. Applying Definition 2.3.2. we conclude on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$ that

$$
\begin{aligned}
& \pi_{1}^{(2)} g(W)=\pi_{1}^{(2)} g_{k-1}(W)+\sup _{s \leqslant}\left(-\pi_{1}^{(2)} g_{k-1}(W)(s)\right)^{+} \\
& \pi_{2}^{(2)} g(W)=\pi_{2}^{(2)} g_{k-1}(W)-\sup _{s \leqslant}\left(-\pi_{1}^{(2)} g_{k-1}(W)(s)\right)^{+} .
\end{aligned}
$$

Now, let us define for $t \geqslant 0$

$$
Y_{k-1}^{(1)}(t):=\pi_{1}^{(2)} g_{k-1}(W)\left(t+\hat{\tau}_{k-1}\right)+l_{k-1}^{(1)}(t) \text { and } l_{k-1}^{(1)}(t):=\sup _{s \leqslant t}\left(-\pi_{1}^{(2)} g_{k-1}\left(s+\hat{\tau}_{k-1}\right)\right)^{+} .
$$

By construction, $Y_{k-1}^{(1)}$ and $l_{k-1}^{(1)}$ are continuous processes satisfying $Y_{k-1}^{(1)} \geqslant 0, l_{k-1}^{(1)}(0)=$ $0, l_{k-1}^{(1)}$ is monotonically non-decreasing, and $l_{k-1}^{(1)}$ increases only on the set $\{t \geqslant$ $\left.0: Y_{k-1}^{(1)}(t)=0\right\}$. Hence, $\left(Y_{k-1}^{(1)}, l_{k-1}^{(1)}\right)$ solves the one-dimensional Skorokhod problem starting in $t=0$ (cf. e.g. [89, Lemma 2.1]). By construction, we further conclude that $\left(\pi_{1}^{(2)} g(W), \sup _{s \leq} .\left(-\pi_{1}^{(2)} g_{k-1}(W)(s)\right)^{+}\right)$solves the one-dimensional Skorokhod problem on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$. Next, since $g(W)=Z$ on $\left[0, \hat{\tau}_{k-1}\right)$ by the induction hypothesis and $Z$ is a continuous process, we have

$$
\begin{aligned}
& Z_{1}(t)=W_{1}(t)+l_{1}(t)-l_{2}\left(\hat{\tau}_{k-1}\right) \\
& Z_{2}(t)=W_{2}(t)-l_{1}(t)+l_{2}\left(\hat{\tau}_{k-1}\right)
\end{aligned}
$$

for $t \in\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}^{*}\right)$, where $\hat{\tau}_{k}^{*}$ denotes the first time after $\hat{\tau}_{k-1}$ at which $Z_{2}$ equals zero. By definition of ( $Z, W, l$ ), we conclude that $\left(Z_{1}, l_{1}-l_{1}\left(\hat{\tau}_{k-1}\right)\right)$ solves the onedimensional Skorokhod problem on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}^{*}\right)$. By the uniqueness of the solution of the one-dimensional Skorokhod problem, we must have $\pi_{1}^{(2)} g(W)=Z_{1}$ almost surely and $\sup _{s \leq .}\left(-\pi_{1}^{(2)} g_{k-1}(W)(s)\right)^{+}=l_{1}-l_{1}\left(\hat{\tau}_{k-1}\right)$ almost surely on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k} \wedge \hat{\tau}_{k}^{*}\right)$. Now, applying the definition of $g_{k-1}$ and the induction hypothesis, we get for $t \in\left[\hat{\tau}_{k-1}, \hat{\tau}_{k} \wedge \hat{\tau}_{k}^{*}\right)$

$$
\begin{aligned}
\pi_{2}^{(2)} g(W)(t) & =\pi_{2}^{(2)} g_{k-1}(W)(t)-\sup _{s \leqslant t}\left(-\pi_{1}^{(2)} g_{k-1}(W)(s)\right)^{+} \\
& =\pi_{2}^{(2)} g_{k-1}(W)\left(\hat{\tau}_{k-1}\right)+W_{2}(t)-W_{2}\left(\hat{\tau}_{k-1}\right)-l_{1}(t)+l_{1}\left(\hat{\tau}_{k-1}\right) \\
& =W_{2}(t)-l_{1}(t)+l_{2}\left(\hat{\tau}_{k-1}\right)=Z_{2}(t) .
\end{aligned}
$$

In particular, we must have $\hat{\tau}_{k}=\hat{\tau}_{k}^{*}$. Hence, $g(W)=Z$ almost surely on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$. Together with the induction hypothesis, we finally conclude that $g(W)=Z$ almost
surely on $\left[0, \hat{\tau}_{k}\right)$. This finishes our induction and proves the stated result.
Next, we will slightly reformulate the SDE system in 2.3.18. Therefore, let us first introduce the concept of a local time of a continuous semimartingale $X$.

Definition 2.3.22 (Local time of a continuous semimartingale). For an $\mathbb{R}$-valued continuous semimartingale $\left(X_{t}\right)_{t \geqslant 0}$ with quadratic variation process $\left(\langle X\rangle_{t}\right)_{t \geqslant 0}$, there exits a measurable process $\left(L_{t}^{a}(X)\right)_{t \geqslant 0, a \in \mathbb{R}}$ such that almost surely for all bounded $\varphi: \mathbb{R} \rightarrow \mathbb{R}$

$$
\int_{0}^{t} \varphi\left(X_{s}\right) d\langle X\rangle_{s}=\int_{\mathbb{R}} \varphi(a) L_{t}^{a}(X) d a, \quad \text { for } t \geqslant 0
$$

The process $\left(L_{t}^{a}(X)\right)_{t \geqslant 0, a \in \mathbb{R}}$ is called the local time of the process $X$ and has a modification such that almost surely $(t, a) \mapsto L_{t}^{a}(X)$ is continuous in $t$ and càdlàg in $a$. Without loss of generality, we will always work with this continuous modification without further comments. Moreover, for all $a \in \mathbb{R}$ the process $\left(L_{t}^{a}(X)\right)_{t \geqslant 0}$ is monotone non-decreasing, hence of finite variation. For more details, see e.g. [88, 89]. In the following, we write $\left(L_{t}(X)\right)_{t \geqslant 0}:=\left(L_{t}^{0}(X)\right)_{t \geqslant 0}$. Moreover, for any $\mathbb{R}^{2}$-valued continuous semimartingale $X$, we introduce the component-wise local time of $X$ at zero by

$$
\begin{equation*}
L_{t}^{(2)}(X):=\left(L_{t}\left(\pi_{1}^{(2)} X\right), L_{t}\left(\pi_{2}^{(2)} X\right)\right) \quad \text { for } t \geqslant 0 \tag{2.3.20}
\end{equation*}
$$

Proposition 2.3.23. Let $(Z, W, l)$ be the unique strong solution of the coupled SDE in (2.3.18) on $[0, \tau]$. Then, $Z$ satisfies

$$
\begin{equation*}
Z(t)=W(t)+\frac{1}{2} L_{t}^{(2)}(Z) R \tag{2.3.21}
\end{equation*}
$$

for $t \leqslant \tau$ almost surely.
Remark 2.3.24. Note that the scaling factor $\frac{1}{2}$ in front of the component-wise local time of $Z$ comes from the almost sure identity between reflection and local time at zero given in, e.g., Lemma 2.12 in 〔89]: let $(\rho, \ell)$ be the solution of the one-dimensional Skorokhod problem (with respect to a one-dimensional Brownian motion B) and $\left(L_{t}\right)_{t \geqslant 0}:=\left(L_{t}(\rho)\right)_{t \geqslant 0}$ be its local time at zero. Then

$$
\ell(t)=\frac{1}{2} L_{t} \quad \text { and } \quad \rho(t)=B(t)+\frac{1}{2} L_{t} \quad t \geqslant 0, \text { a.s. }
$$

If, moreover $\mu=0$, it can be shown that $L_{t}(|B|)=2 L_{t}(B), t \geqslant 0$, and by an application of Tanaka's formula that $|B(t)|=x+\int_{0}^{t} \operatorname{sign}(B(s)) d B(s)+L_{t}(B)$ which is equivalent in law to $\rho$.

Proof of Proposition 2.3.23. Let $\hat{\tau}_{k}:=\hat{\tau}_{k}(W), k \in \mathbb{N}$, where $\hat{\tau}_{k}$ is introduced in Definition 2.3.2. We argue analogously as in the proof of Proposition 2.3.21 and consider $Z$ on each sub-interval $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right), k \in \mathbb{N}$. Therefore, we only present a proof sketch. On the sub-interval $\left[0, \hat{\tau}_{1}\right)$, we have $Z_{1}, Z_{2}>0$ implying that $Z$ satisfies 2.3 .21 on $\left[0, \hat{\tau}_{1}\right)$.

For each $k \geqslant 2$ we inductively argue as follows: note that either $\hat{\tau}_{k-1}=\hat{\tau}_{k-1,1}(W)$ or $\hat{\tau}_{k-1}=\hat{\tau}_{k-1,2}(W)$ implying that either $\left(Z_{1}, l_{1}-l_{1}\left(\hat{\tau}_{k-1}\right)\right)$ or $\left(Z_{2}, l_{2}-l_{2}\left(\hat{\tau}_{k-1}\right)\right)$ solves the one-dimensional Skorokhod problem on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$. Without loss of generality, let us consider the event $\left\{\hat{\tau}_{k-1}=\hat{\tau}_{k-1,1}(W)\right\}$. On this event, since $\left(Z_{1}, l_{1}-l_{1}\left(\hat{\tau}_{k-1}\right)\right)$ solves the one-dimensional Skorokhod problem and applying Lemma 2.12 in [89], we conclude that

$$
l_{1}(t)-l_{1}\left(\hat{\tau}_{k-1}\right)=\frac{1}{2}\left(L_{t}\left(Z_{1}\right)-L_{\hat{\tau}_{k-1}}\left(Z_{1}\right)\right)
$$

for $t \in\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$ almost surely. Moreover, since $Z_{2}>0$ on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$ by construction, both $l_{2}$ and $t \mapsto L_{t}\left(Z_{2}\right)$ do not increase on $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$ almost surely. Hence, for $t \in$ $\left[\hat{\tau}_{k-1}, \hat{\tau}_{k}\right)$, we have

$$
l_{2}(t)-l_{2}\left(\hat{\tau}_{k-1}\right)=0=\frac{1}{2}\left(L_{t}\left(Z_{2}\right)-L_{\hat{\tau}_{k-1}}\left(Z_{2}\right)\right) .
$$

Now, combining the definition of $Z$ with the induction hypothesis yields that $Z$ satisfies the SDE in (2.3.21) almost surely on $\left[0, \hat{\tau}_{k}\right)$. This finishes the proof.

Combining Proposition 2.3.21 and Proposition 2.3.23, we obtain that the process $g(W)$ fulfills the SDE

$$
\begin{equation*}
g(W)(t)=W(t)+\frac{1}{2} L_{t}^{(2)}(g(W)) R \tag{2.3.22}
\end{equation*}
$$

for $t \in[0, \tau(W)]$, where $R$ is the reflection matrix defined in (2.3.19) and $L_{t}^{(2)}(g(W))$ denotes the component-wise local time of $g(W)$. For $t>\tau(W)$, we have $g(W)(t)=$ $(0,0)$. With this new representation of the process $g(W)$, we can now characterize the distribution of $\widetilde{Q}$.

Theorem 2.3.25 (Limit distribution of the queue size process in active regimes). Let the assumptions of Theorem 2.3 .19 be satisfied and let $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 0}$ be the sequences of stopping times introduced in 2.3.16). Let us introduce the sequence $\left(\widetilde{B}_{k}\right)_{k \in \mathbb{N}_{0}}$ of four-dimensional linear Brownian motions, each with mean $\mu$ and covariance matrix $\Sigma$ (cf. Theorem 2.3.1), given by

$$
\begin{equation*}
\widetilde{B}_{k}:=\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)+X\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-X\left(\widetilde{\tau}_{k}^{*}\right) . \tag{2.3.23}
\end{equation*}
$$

For the limit $\widetilde{Q}=\left(\widetilde{Q}^{b, F}, \widetilde{Q}^{a, F}, \widetilde{Q}^{b, G}, \widetilde{Q}^{a, G}\right)$ of the queue size process $\widetilde{Q}^{(n)}$, it holds:
i) On each interval $\left[\widetilde{\tau}_{k}^{*}, \widetilde{\tau}_{k+1}^{*}\right), k \geqslant 0$, the process $\widetilde{Q}$ is distributed as a continuous semimartingale, i.e., for $i=b, a$, it holds

$$
\left(\widetilde{Q}^{i, F}, \widetilde{Q}^{i, G}\right)\left(\cdot+\hat{\tau}_{k}\right)=\pi_{i} \widetilde{B}_{k}+\frac{1}{2} L_{\cdot}^{(2)}\left(g\left(\pi_{i} \widetilde{B}_{k}\right)\right) R \quad \text { on }\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right),
$$

where $R$ is the reflection matrix introduced in 2.3.19), $\left(L_{t}^{(2)}(Y)\right)_{t \geqslant 0}$ defines the component-wise local time of a $\mathbb{R}^{2}$-valued process $Y$, and the function $g$ is intro-
duced in Definition 2.3.2. In particular, for $i=b, a$, the process $\left(\widetilde{Q}^{i, F}, \widetilde{Q}^{i, G}\right)\left(\cdot+\widetilde{\tau}_{k}^{*}\right)$ behaves as a sum-preserving $S R B M$ driven by $\pi_{i} \widetilde{B}_{k}$ on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right)$.
ii) At each time $t=\widetilde{\tau}_{k}^{*}, k \geqslant 1$, the process $\widetilde{Q}$ is reinitialized by the random variable $\widetilde{R}_{k}^{+}$or $\widetilde{R}_{k}^{-}$, i.e.,

$$
\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)=\widetilde{R}_{k}^{+} \text {if } \widetilde{\tau}_{a, k}=\widetilde{\tau}_{k}^{*} \quad \text { or } \quad \widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)=\widetilde{R}_{k}^{-} \text {if } \widetilde{\tau}_{b, k}=\widetilde{\tau}_{k}^{*}
$$

iii) Finally, the process $\widetilde{Q}$ has only finitely many discontinuities and is therefore $a$ semimartingale on the whole interval $[0, T]$.
Proof. The relation $\widetilde{Q}=\widetilde{\Psi}^{Q}\left(\widetilde{Q}_{0}+X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$gives us a construction of our process $\widetilde{Q}$, where $\widetilde{Q}_{0} \in(0, \infty)^{4}$ is its initial state, $X$ is introduced in Proposition 2.3.1 and defines a four-dimensional linear Brownian motion with drift $\mu$ and covariance matrix $\Sigma$, and $\widetilde{R}^{+}, \widetilde{R}^{-}$are the sequences of random variables introduced in 2.3.17) with values in $\left((0, \infty)^{4}\right)^{\mathbb{N}} \mathbb{P}$-almost surely. Now, for all $k \geqslant 0, \widetilde{B}_{k}$ introduced in Theorem 2.3 .25 is a four-dimensional linear Brownian motions starting at $\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)$ with drift $\mu$ and covariance matrix $\Sigma$. By the definition of $\widetilde{\Psi}^{Q}$, for $k \geqslant 0$, we have

$$
\widetilde{Q}\left(\cdot+\widetilde{\tau}_{k}^{*}\right)=\widetilde{\Psi}^{Q}\left(\widetilde{Q}_{0}+X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)\left(\cdot+\widetilde{\tau}_{k}^{*}\right)=G\left(\widetilde{B}_{k}\right) \quad \text { on }\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right)
$$

An application of the definition of $G$ and the representation in 2.3 .22 proves part i).
In order to prove part ii) and iii), we note that ii) and the fact that $\Psi^{Q}\left(\widetilde{Q}_{0}+X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$ has almost surely only finitely many discontinuities are already shown in the proof of Theorem 2.3.19. Together with part i) we conclude that $\widetilde{Q}=\widetilde{\Psi}^{Q}\left(\widetilde{Q}_{0}+X, \widetilde{R}^{+}, \widetilde{R}^{-}\right)$is a semimartingale on the whole interval $[0, T]$ which finally yields the statement iii).

Moreover, it turns out that the process $\widetilde{Q}$ is a Markov process.
Theorem 2.3.26 (Identification of $\widetilde{Q}$ as a Markov process). The process $\widetilde{Q}$ is a Markov process with values almost surely in $\mathbb{R}_{+}^{4} \backslash\{(0,0,0,0)\}$, initial value $\widetilde{Q}_{0} \in(0, \infty)^{4}$, and infinitesimal generator $\mathcal{A}$ given on $\widetilde{\mathbb{R}}_{+}^{4}:=\mathbb{R}_{+}^{4} \backslash\left\{x \in \mathbb{R}_{+}^{4}: \pi_{1} x=\pi_{3} x=0\right.$ or $\pi_{2} x=$ $\left.\pi_{4} x=0\right\}$ by

$$
\begin{align*}
\mathcal{A} h= & \mu^{b, F} \frac{\partial h}{\partial x_{1}}+\mu^{a, F} \frac{\partial h}{\partial x_{2}}+\mu^{b, G} \frac{\partial h}{\partial x_{3}}+\mu^{a, G} \frac{\partial h}{\partial x_{4}} \\
& +\frac{\left(\sigma^{b, F}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{1}^{2}}+\frac{\left(\sigma^{a, F}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{2}^{2}}+\frac{\left(\sigma^{b, G}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{3}^{2}}+\frac{\left(\sigma^{a, G}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{4}^{2}} \\
& +\sigma^{(b, F),(a, F)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{2}}+\sigma^{(b, F),(b, G)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{3}}+\sigma^{(b, F),(a, G)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{4}}  \tag{2.3.24}\\
& +\sigma^{(a, F),(b, G)} \frac{\partial^{2} h}{\partial x_{2} \partial x_{3}}+\sigma^{(a, F),(a, G)} \frac{\partial^{2} h}{\partial x_{2} \partial x_{4}}+\sigma^{(b, G),(a, G)} \frac{\partial^{2} h}{\partial x_{3} \partial x_{4}}
\end{align*}
$$

and for $y=\left(y_{1}, y_{2}, y_{3}, y_{4}\right) \in \widetilde{\mathbb{R}}_{+}^{4}$

$$
\begin{align*}
& \mathcal{A} h\left(y_{1}, 0, y_{3}, 0\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\Phi\left(\left(y_{1}, 0, y_{3}, 0\right), u\right)\right) f^{+}(d u),  \tag{2.3.25}\\
& \mathcal{A} h\left(0, y_{2}, 0, y_{4}\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\Phi\left(\left(0, y_{2}, 0, y_{4}\right), u\right)\right) f^{-}(d u),
\end{align*}
$$

and whose domain is the set $\operatorname{dom}(\mathcal{A})$ of functions $h \in C^{2}\left(\widetilde{\mathbb{R}}_{+}^{4}, \mathbb{R}\right) \cap C^{0}\left(\mathbb{R}_{+}^{4}, \mathbb{R}\right)$ verifying, for all $x_{1}, x_{2}, x_{3}, x_{4}>0$ and all $y=\left(y_{1}, y_{2}, y_{3}, y_{4}\right) \in \widetilde{\mathbb{R}}_{+}^{4}$, the boundary conditions:

1. Reflecting boundary condition in the sense that the function $h$ satisfies

$$
\begin{array}{ll}
\left(\frac{\partial h}{\partial x_{1}}-\frac{\partial h}{\partial x_{3}}\right)\left(0, y_{2}, x_{3}, y_{4}\right)=0 & \left(\frac{\partial h}{\partial x_{3}}-\frac{\partial h}{\partial x_{1}}\right)\left(x_{1}, y_{2}, 0, y_{4}\right)=0 \\
\left(\frac{\partial h}{\partial x_{2}}-\frac{\partial h}{\partial x_{4}}\right)\left(y_{1}, 0, y_{3}, x_{4}\right)=0 & \left(\frac{\partial h}{\partial x_{4}}-\frac{\partial h}{\partial x_{2}}\right)\left(y_{1}, x_{2}, y_{3}, 0\right)=0
\end{array}
$$

2. Inward jump boundary condition in the sense that the function $h$ satisfies

$$
\begin{aligned}
& h\left(y_{1}, 0, y_{3}, 0\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\Phi\left(\left(y_{1}, 0, y_{3}, 0\right), u\right)\right) f^{+}(d u) \\
& h\left(0, y_{2}, 0, y_{4}\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\Phi\left(\left(0, y_{2}, 0, y_{4}\right), u\right)\right) f^{-}(d u)
\end{aligned}
$$

Remark 2.3.27 (The cumulative queue size dynamics). Studying the above result, we conclude that the limit $h(\widetilde{Q})$ of the cumulative queue size dynamics is again a Markov process with values almost surely in $\mathbb{R}_{+}^{2} \backslash\{(0,0)\}$. In the interior of $\mathbb{R}_{+}^{2}$, the process $h(\widetilde{Q})$ behaves as planar Brownian motion starting in $h\left(\widetilde{Q}_{0}\right) \in(0, \infty)^{2}$ with mean $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ (cf. Corollary 2.7.1). Moreover, it is instantaneously reinitialized at a new value in the interior of $\mathbb{R}_{+}^{2}$ each time one of its components hits the axes $\{(0, y): y>0\} \cup\{(x, 0): x>0\}$. In particular, the cumulative queue size dynamics behaves like the bid and ask queues in a single-country LOB-model, whose limit behavior is described by Cont and de Larrard in [19, Theorem 2].

The proof of Theorem 2.3 .26 is given in Section 2.7.3. Let $W$ be again the planar Brownian motion as defined above. Then, by definition of the function $\hat{g}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow$ $D([0, T], \mathbb{R})$ and studying the proof of Proposition 2.3.21, we deduce that

$$
\begin{equation*}
\hat{g}(W)(t)=\frac{1}{2}\left\{L_{t}\left(\pi_{2}^{(2)} g(W)\right)-L_{t}\left(\pi_{1}^{(2)} g(W)\right)\right\}, \quad \text { for } t \in[0, \tau(W)], \tag{2.3.26}
\end{equation*}
$$

and for $t>\tau(W)$, we have $\hat{g}(W)(t)=\hat{g}(W)(\tau(W))$.

Theorem 2.3.28 (Limit distribution of the capacity process in active regimes). Let the assumptions of Theorem 2.3.19 be satisfied and let $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 0}$ be the sequence of stopping times introduced in 2.3.16). The limit $\widetilde{C}$ of the capacity process $\widetilde{C}^{(n)}$ is a continuous process of finite variation. On each interval $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right], k \geqslant 0$, we have

$$
\begin{aligned}
& \widetilde{C}\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-\widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right) \\
& =\frac{1}{2}\left\{L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}_{k}\right)\right)-L \cdot\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}_{k}\right)\right)-L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}_{k}\right)\right)+L \cdot\left(\pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}_{k}\right)\right)\right\}
\end{aligned}
$$

where the function $g$ is introduced in Definition 2.3.2 and $\widetilde{B}_{k}$ is the four-dimensional linear Brownian motion defined in 2.3.23.
Proof. By the definition of $\widetilde{\Psi}^{C}$, for each $k \geqslant 0$, we have $\widetilde{C}\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-\widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right)=\hat{G}\left(\widetilde{B}_{k}\right)$ on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right]$, where $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 0}$ are the stopping times introduced in 2.3.16 and $\hat{G}\left(\widetilde{B}_{k}\right)(0) \equiv 0$. An application of the definition of $\hat{G}$ and the representation in (2.3.26) identifies $\widetilde{C}\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-\widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right)$ on each interval $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right], k \geqslant 0$, as a continuous process of finite variation. Arguing again as in the proof of Theorem 2.3.19, there exists a finite, $\mathbb{N}$-valued $N_{T}$ such that $\widetilde{\tau}_{N_{T}}^{*}<T$ and $\widetilde{\tau}_{N_{T}+1}^{*}=T$ with probability one. Hence, $\widetilde{C}$ is almost surely the sum of $N_{T}+1$ continuous processes of finite variation and thus itself a continuous process of finite variation.

Remark 2.3.29 (Queue size and capacity process live on different time scales). By construction, both the queue size process $\widetilde{Q}^{(n)}$ and the capacity process $\widetilde{C}^{(n)}$ change by incoming orders of size $\pm \Delta v^{(n)}$. However, the above limit results yield under the chosen scaling of $\left(\Delta v^{(n)}\right)^{2} \approx \Delta t^{(n)}$ that $\widetilde{Q}^{(n)}$ is approximated in the limit by a semimartingale with a non-trivial martingale part whereas the capacity process is approximated by a continuous process of finite variation. Hence, $\widetilde{Q}^{(n)}$ lives on a finer time scale than the capacity process. This can already be deduced from the microscopic construction, as the queue size process changes by each incoming market and limit order but the capacity process only changes by market orders leading to cross-border trades.

With help of Theorem 2.3.26, we obtain a detailed description of the limit distribution of the price dynamics in the active regimes. The limit approximation of the bid price process is entirely characterized by hitting times of the axes of the two-dimensional process $h(\widetilde{Q})$. The result follows directly from the definition of the function $\widetilde{\Psi}^{B}$.

Theorem 2.3.30 (Limit distribution of the prices process in active regimes). Let the assumptions of Theorem 2.3.19 be satisfied. Then, the limit $\widetilde{B}=\left(\widetilde{B}^{F}, \widetilde{B}^{F}\right)$ satisfies

$$
\left.\widetilde{B}^{F}(t)=\widetilde{B}_{0}^{F}+\delta \sum_{0 \leqslant s \leqslant t}\left(\mathbb{1}_{\left\{\left(\pi_{2}^{(2)} \circ h\right)(\widetilde{Q})(s-)=0\right.}\right\}^{\left.-\mathbb{1}^{\{ }\left(\pi_{1}^{(2)} \circ h\right)(\widetilde{Q})(s-)=0\right\}}\right)
$$

and is a piecewise constant càdlàg process which

- increases by one tick every time the process $h(\widetilde{Q})$ hits the horizontal axis $\{y=0\}$,
- decreases by one tick every time the process $h(\widetilde{Q})$ this the vertical axis $\{x=0\}$.


### 2.3.6 Probabilistic results of the active limit dynamics

We finish this section by presenting important properties of the process $\widetilde{S}$. They become important when deriving a limit theorem for the cross-border market model $S^{(n)}$.

Lemma 2.3.31. Let us introduce $\sigma:=\inf \left\{t \geqslant 0: \widetilde{C}(t) \leqslant-\kappa_{-}\right.$or $\left.\widetilde{C}(t) \geqslant \kappa_{+}\right\} \wedge T$, the first time at which the process $\widetilde{C}$ hits the boundary of $\left[-\kappa_{-}, \kappa_{+}\right]$. Then, with probability one, the limit $h(\widetilde{Q})$ of the cumulative queue size process satisfies

$$
\begin{equation*}
\mathbb{P}\left[h(\widetilde{Q})(\sigma) \in(0, \infty)^{2}\right]=1 \tag{2.3.27}
\end{equation*}
$$

In words, with probability one, the hitting time $\sigma$ does not coincide with the random time of a price change in $\widetilde{S}$.

Note that thanks to the above result, with probability one, a regime switch from an active to an inactive regime in $\widetilde{S}$ already occurs if $\widetilde{C}$ hits the boundary of $\left[-\kappa_{-}, \kappa_{+}\right]$. Recall that $\widetilde{Q}:=\left(\widetilde{Q}^{b, F}, \widetilde{Q}^{a, F}, \widetilde{Q}^{b, G}, \widetilde{Q}^{a, G}\right)$. The following result shows that, with probability one, no combination of bid and ask queue is simultaneous zero.

Lemma 2.3.32. Let $I, J \in\{F, G\}$. With probability one, we have

$$
\mathbb{P}\left[\exists t \in[0, T]:\left(\widetilde{Q}^{b, I}, \widetilde{Q}^{a, J}\right)(t)=(0,0)\right]=0
$$

In particular, if $I=J$, we conclude, with probability one, that no national limit order book in $\widetilde{S}$ is empty.

The proofs are omitted at this point and are presented in Section 2.7.3.

### 2.4 The inactive dynamics

In this section, we introduce a heavy traffic approximation for the inactive dynamics. Note that the subsequent analysis and approximation of the inactive dynamics is, with one exception, a straight-forward extension of the reduced-form representation of a LOB introduced in Cont and de Larrard [19] to two non-interacting LOBs. In the following, we only present the ideas and results and refer to Section 2.3 and $\sqrt{19}$ for the technical details of the proofs.

Remark 2.4.1. Note that in Cont and de Larrard 19] the weak convergence of prices is established in the slightly weaker and not so standard Skorokhod $M_{1}$-topology (cf. e.g. Whitt [86] for an introduction to the Skorokhod $M_{1}$-topology). However, by replicating our strategy for the analysis of the active dynamics in Section 2.3. we are able to prove weak convergence of the prices in the stronger Skorokhod $J_{1}$-topology.

Let us recall that the inactive dynamics $\widetilde{\widetilde{S}}^{(n)}=\left(\widetilde{\widetilde{S}}^{(n)}(t)\right)_{t \in[0, T]}$ is given by

$$
\widetilde{\widetilde{S}}^{(n)}(t)=\widetilde{\widetilde{S}}_{k}^{(n)} \quad \text { for } t \in\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)
$$

where its discrete-time dynamics is defined in the equations 2.2.8-2.2.11) in Section 2.2.4 Let $\widetilde{\widetilde{S}}_{0}^{(n)}:=\left(\widetilde{\widetilde{B}}_{0}^{(n)}, \widetilde{\widetilde{Q}}_{0}^{(n)}, \widetilde{C}_{0}^{(n)}\right) \in(\delta \mathbb{Z})^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z}$ be the initial state of $\widetilde{S}^{(n)}$. Again, the limit results for the queue size process $\widetilde{\widetilde{Q}}^{(n)}$ and the price process $\widetilde{\widetilde{B}}^{(n)}$ are based on a limit result for the net order flow process $X^{(n)}$ (cf. Proposition 2.3.1) and Assumption 2.4 guaranteeing the convergence of the sequences $\widetilde{\widetilde{R}}^{+,(n)}$, $\widetilde{\widetilde{R}}^{-,(n)}$. We introduce for $i=1,2,3,4$ the first hitting time maps $\widetilde{\widetilde{\tau}}_{i}, \widetilde{\widetilde{\tau}}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ by

$$
\widetilde{\widetilde{\tau}}_{i}(\omega):=\inf \left\{t \geqslant 0: \pi_{i} \omega(t) \leqslant 0\right\} \wedge T
$$

and $\widetilde{\tau}(\omega):=\widetilde{\tau}_{1}(\omega) \wedge \widetilde{\tau}_{2}(\omega) \wedge \widetilde{\tau}_{3}(\omega) \wedge \widetilde{\tau}_{4}(\omega)$. Next, let us introduce a function $\widetilde{\widetilde{\Psi}}^{Q}$ that can be used to describe the dynamics of the queue size process $\widetilde{\widetilde{Q}}^{(n)}$ over time.
Definition 2.4.2. Let $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ and $R=\left(R_{n}\right)_{n \geqslant 1}, \widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1} \in\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$. For $k \in \mathbb{N}_{0}$, we define functions $\widetilde{\widetilde{\Psi}}_{k}^{Q}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right)$ and $\widetilde{\widetilde{\Psi}}^{Q}(\omega, R, \widetilde{R}) \in$ $D\left([0, T], \mathbb{R}_{+}^{4}\right)$ as follows:

- Set $\widetilde{\widetilde{\Psi}}_{0}^{Q}(\omega, R, \widetilde{R}):=\omega$.
- Let $k \geqslant 1$ and $\widetilde{\tilde{\Psi}}_{k-1}^{Q}:=\widetilde{\widetilde{\Psi}}_{k-1}^{Q}(\omega, R, \widetilde{R})$. If $\widetilde{\widetilde{\tau}}\left(\widetilde{\widetilde{\Psi}}_{k-1}^{Q}\right)=T$, then $\widetilde{\widetilde{\Psi}}_{k}^{Q}(\omega, R, \widetilde{R})=\widetilde{\tilde{\Psi}}_{k-1}^{Q}$. Otherwise, we set $\widetilde{\widetilde{\Psi}}_{k}^{Q}(\omega, R, \widetilde{R})(t)=\widetilde{\widetilde{\Psi}}_{k-1}^{Q}(t)$ for $t<\widetilde{\widetilde{\tau}}\left(\widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)$ and for $t \geqslant \widetilde{\tau}\left(\widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)$, we define

$$
\begin{aligned}
& \pi_{F} \widetilde{\widetilde{\Psi}}_{k}^{Q}(\omega, R, \widetilde{R})(t):=\pi_{F} \widetilde{\widetilde{\Psi}}_{k-1}^{Q}(t)+\mathbb{1}_{\left\{\widetilde{\widetilde{\tau}}^{\left(\tilde{\Psi}_{k-1}^{Q}\right)=} \tilde{\widetilde{\tau}}_{2}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right\}} \pi_{F}\left(R_{k}-\widetilde{\widetilde{\Psi}}_{k-1}^{Q}\left(\widetilde{\widetilde{\tau}}^{\left.\left.\left(\tilde{\tilde{\Psi}}_{k-1}^{Q}\right)\right)\right)}\right.\right. \\
& +\mathbb{1}_{\left\{\widetilde{\widetilde{\tau}}^{\left(\widetilde{\Psi}_{k-1}^{Q}\right)} \tilde{\widetilde{\tau}}_{1}\left(\widetilde{\widetilde{\Psi}}_{k-1}^{Q}\right)\right\}} \pi_{F}\left(\widetilde{R}_{k}-\widetilde{\tilde{\Psi}}_{k-1}^{Q}\left(\widetilde{\widetilde{\tau}}\left(\widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)\right)\right), \\
& \pi_{G} \widetilde{\widetilde{\Psi}}_{k}^{Q}(\omega, R, \widetilde{R})(t):=\pi_{G} \widetilde{\widetilde{\Psi}}_{k-1}^{Q}(t)+\mathbb{1}_{\left.\left\{\widetilde{\tau}^{( } \widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)=\widetilde{\widetilde{\tau}}_{4}\left(\widetilde{\widetilde{\Psi}}_{k-1}^{Q}\right)\right\}} \pi_{G}\left(R_{k}-\widetilde{\tilde{\Psi}}_{k-1}^{Q}\left(\widetilde{\widetilde{\tau}}^{( }\left(\widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)\right)\right) \\
& +\mathbb{1}_{\left\{\widetilde{\widetilde{\tau}}^{\left(\widetilde{\Psi}_{k-1}^{Q}\right)}{ }^{Q} \widetilde{\widetilde{\tau}}_{3}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right\}} \pi_{G}\left(\widetilde{R}_{k}-\widetilde{\tilde{\Psi}}_{k-1}^{Q}\left(\widetilde{\tau}\left(\widetilde{\tilde{\Psi}}_{k-1}^{Q}\right)\right)\right) .
\end{aligned}
$$

- Finally, we set $\widetilde{\tau}_{0}:=0, \widetilde{\tau}_{k}:=\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})\right)$ for $k \geqslant 1$, and

$$
\widetilde{\widetilde{\Psi}}^{Q}(\omega, R, \widetilde{R})(t)=\sum_{k=1}^{\infty} \widetilde{\widetilde{\Psi}}_{k-1}^{Q}(\omega, R, \widetilde{R})(t) \mathbb{1}_{\left[\widetilde{\tau}_{k-1}, \widetilde{\tau}_{k}\right)}(t) \quad \text { for } t \in[0, T)
$$

Moreover, if there exists a finite $N_{T}$ such that $\widetilde{\tau}_{N_{T}}<T$ and $\widetilde{\widetilde{\tau}}_{N_{T}+1}=T$, then set $\widetilde{\tilde{\Psi}}^{Q}(\omega, R, \widetilde{R})(T)=\widetilde{\widetilde{\Psi}}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(T)$. Otherwise, set $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(T)=(0,0,0,0)$.
The function $\widetilde{\widetilde{\Psi}}^{Q}$ is obtained by regulating the path of $\omega$ with help of the sequences $R$ and $\widetilde{R}$ : between two consecutive times $\widetilde{\tau}_{k}$ and $\widetilde{\widetilde{\tau}}_{k+1}$ the function $\widetilde{\widetilde{\Psi}}^{Q}(\omega, R, \widetilde{R})$ equals $\widetilde{\tilde{\Psi}}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega-\omega\left(\widetilde{\tau}_{k}\right)$. The first time one of its components hits the axes, the components corresponding to the origin of the incoming order are reinitialized according
to $R_{k+1}$ or $\widetilde{R}_{k+1}$, whereas the components corresponding to the foreign order book stay unchanged.

In the next theorem, we characterize the continuity set of $\widetilde{\Psi} Q$. Therefore, let us introduce the functions space $C_{0}^{\prime}\left([0, T],\left(\mathbb{R}^{2} \backslash\{(0,0)\}\right) \times\left(\mathbb{R}^{2} \backslash\{(0,0)\}\right)\right)$ that contains all continuous functions $\omega \in C\left([0, T], \mathbb{R}^{4}\right)$ whose components cross the axes each time they touch them and whose projections $\pi_{F} \omega, \pi_{G} \omega$ avoid the origin ( 0,0 ).

Theorem 2.4.3 (Continuity of $\widetilde{\Psi} Q)$. Let $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ be such that
i) $\omega(0), R_{k}$, and $\widetilde{R}_{k} \in(0, \infty)^{4}$ for all $k \geqslant 1$.
ii) There exists a finite, $\mathbb{N}$-valued $N_{T}$ such that $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})(t)=\widetilde{\widetilde{\Psi}}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(t)$ for $t \in[0, T]$ and $\widetilde{\Psi}_{N_{T}}^{Q}(\omega, R, \widetilde{R})(T) \in(0, \infty)^{4}$.
iii) Let $\widetilde{\varphi}_{k}(\omega, R, \widetilde{R}):=\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega\left(\cdot+\widetilde{\tau}_{k}\right)-\omega\left(\widetilde{\tau}_{k}\right)$. For $1 \leqslant k \leqslant N_{T}$ we have $\omega, \widetilde{\varphi}_{k}(\omega, R, \widetilde{R}) \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\} \times \mathbb{R}^{2} \backslash\{(0,0)\}\right)$.
Then, the function $\tilde{\Psi}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ introduced in Definition 2.4.2 is continuous at $(\omega, R, \widetilde{R})$.

Using the procedure introduced by the function $\tilde{\widetilde{\Psi}}$, we found a new representation of the queue size process $\widetilde{\widetilde{Q}}^{(n)}$, i.e., for all $n \in \mathbb{N}$, we have $\widetilde{\widetilde{Q}}^{(n)}=\widetilde{\Psi}^{Q}\left(\widetilde{\widetilde{Q}}_{0}^{(n)}+\right.$ $\left.X^{(n)}, \widetilde{\widetilde{R}}^{+,(n)}, \widetilde{\widetilde{R}}^{-,(n)}\right)$, where $\widetilde{Q}_{0}^{(n)}$ are the initial queue sizes, $X^{(n)}$ is introduced in (2.3.1), and $\widetilde{R}^{+,(n)}$ and $\widetilde{R}^{-,(n)}$ describe the queue sizes after price increases/decreases in $\widetilde{S}^{(n)}$. Next, let us introduce a function $\widetilde{\widetilde{\Psi}}^{B}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ that can be used to describe the dynamics of the prices in the inactive regimes.
Definition 2.4.4. Let $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$ and $R=\left(R_{n}\right)_{n \geqslant 1}, \widetilde{R}=\left(\widetilde{R}_{n}\right)_{n \geqslant 1} \in\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$. For $k \in \mathbb{N}$, let $\widetilde{\tau}_{k}:=\widetilde{\tau}_{k}(\omega, R, \widetilde{R})$ and $\widetilde{\Psi}_{k-1}^{Q}:=\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})$ be as in Definition 2.4.2. Then, we define the functions $N_{i}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{N}_{0}\right), i=1,2,3,4$, and $\widetilde{\Psi}^{B}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{2}\right)$ by

$$
N_{i}(\omega, R, \widetilde{R}):=\sum_{\widetilde{\tau}_{k} \leqslant} \mathbb{1}_{\left\{\pi_{i} \widetilde{\Psi}_{k-1}^{Q}\left(\widetilde{\tau}_{k}\right) \leqslant 0\right\}} \quad \text { for } i=1,2,3,4
$$

and

$$
\widetilde{\Psi}^{B}(\omega, R, \widetilde{R}):=\delta\left(N_{2}(\omega, R, \widetilde{R})-N_{1}(\omega, R, \widetilde{R}), N_{4}(\omega, R, \widetilde{R})-N_{3}(\omega, R, \widetilde{R})\right) .
$$

Theorem 2.4.5 (Continuity of $\left.\widetilde{\widetilde{\Psi}}^{B}\right) . \operatorname{Let}(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfy the conditions i)-iii) in Theorem 2.4.3. Moreover, for each $1 \leqslant i, j \leqslant 4$ with $i \neq j$, assume that

$$
\begin{equation*}
\operatorname{Disc}\left(N_{i}(\omega, R, \widetilde{R})\right) \cap \operatorname{Disc}\left(N_{j}(\omega, R, \widetilde{R})\right)=\varnothing \tag{2.4.1}
\end{equation*}
$$

where $\operatorname{Disc}\left(N_{i}(\omega, R, \widetilde{R})\right):=\left\{t \in[0, T]: N_{i}(\omega, R, \widetilde{R})(t-) \neq N_{i}(\omega, R, \widetilde{R})(t)\right\}$ for $i=$ $1,2,3,4$. Then, the map $\tilde{\Psi}^{B}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ is continuous at $(\omega, R, \widetilde{R})$.

Again, with the procedure introduced by the function $\widetilde{\widetilde{\Psi}}^{B}$, we can rewrite the bid price process $\widetilde{\widetilde{B}}^{(n)}$, i.e., for all $n \in \mathbb{N}$, we have $\widetilde{\widetilde{B}}^{(n)}=\widetilde{\widetilde{B}}_{0}^{(n)}+\widetilde{\widetilde{\Psi}}^{B}\left(\widetilde{\widetilde{Q}}_{0}^{(n)}+X^{(n)}, \widetilde{\widetilde{R}}^{+,(n)}, \widetilde{\widetilde{R}}^{-,(n)}\right)$. Finally, we define for $s_{0}:=\left(b_{0}, q_{0}, c_{0}\right) \in E$ the function

$$
\begin{gathered}
\widetilde{\widetilde{\Psi}}: E \times D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D([0, T], E) \\
\widetilde{\widetilde{\Psi}}\left(s_{0} ; \omega, R, \widetilde{R}\right):=\left(b_{0}+\widetilde{\widetilde{\Psi}}^{B}\left(q_{0}+\omega, R, \widetilde{R}\right), \widetilde{\Psi}^{Q}\left(q_{0}+\omega, R, \widetilde{R}\right), c_{0}\right)
\end{gathered}
$$

Replicating the arguments of Theorem 2.3.19, we are able to derive a joint limit theorem for the inactive dynamics $\widetilde{\widetilde{S}}^{(n)}:=\left(\widetilde{\widetilde{B}}^{(n)}, \widetilde{Q}^{(n)}, \widetilde{\widetilde{C}}^{(n)}\right)$.

Theorem 2.4.6 (Limit theorem for the inactive dynamics $\left.\widetilde{\widetilde{S}}^{(n)}\right)$. Let Assumptions 2.2, 2.3, and 2.4 be satisfied. Assume that $\widetilde{\widetilde{S}}_{0}^{(n)} \in(\delta \mathbb{Z})^{2} \times\left(\Delta v^{(n)} \mathbb{N}\right)^{4} \times \Delta v^{(n)} \mathbb{Z}$ with $\widetilde{S}_{0}^{(n)} \rightarrow \widetilde{S}_{0} \in(\delta \mathbb{Z})^{2} \times(0, \infty)^{4} \times \mathbb{R}$. Then,

$$
\widetilde{\widetilde{S}}^{(n)}=\widetilde{\widetilde{\Psi}}\left(S_{0}^{(n)} ; X^{(n)}, \widetilde{\widetilde{R}}^{+,(n)}, \widetilde{\widetilde{R}}^{-,(n)}\right) \Rightarrow \widetilde{\widetilde{\Psi}}\left(\widetilde{\widetilde{S}}_{0} ; X, \widetilde{\widetilde{R}}^{+}, \widetilde{\widetilde{R}}^{-}\right)=:(\widetilde{\widetilde{B}}, \widetilde{\widetilde{Q}}, \widetilde{\widetilde{C}})=: \widetilde{\widetilde{S}}
$$

in the Skorokhod topology on the space $D([0, T], E)$.
Let us characterize the distribution of the processes $\widetilde{\widetilde{B}}$ and $\widetilde{\widetilde{Q}}:$ let $\widetilde{\tau}_{0}^{*}:=0$ and

$$
\begin{equation*}
\widetilde{\widetilde{\tau}}_{k}^{*}:=\widetilde{\widetilde{\tau}}\left(\widetilde{\widetilde{\Psi}}_{k-1}^{Q}\left(\widetilde{\widetilde{Q}}_{0}+X, \widetilde{\widetilde{R}}^{+}, \widetilde{\widetilde{R}}^{-}\right)\right) \text {for } k \geqslant 1 \tag{2.4.2}
\end{equation*}
$$

where the random sequences $\widetilde{\widetilde{R}}^{+}, \widetilde{\widetilde{R}}^{-}$describe the queue sizes after price changes in $\widetilde{\widetilde{S}}$ with

$$
\tilde{\widetilde{R}}_{k}^{+}:=\Phi\left(\widetilde{\widetilde{Q}}\left(\widetilde{\widetilde{\tau}}_{k}^{*}-\right), \epsilon_{k}^{+}\right) \quad \text { and } \quad \tilde{\widetilde{R}}_{k}^{-}:=\Phi\left(\widetilde{\widetilde{Q}}\left(\widetilde{\tau}_{k}^{*}-\right), \epsilon_{k}^{-}\right)
$$

for independent sequences of iid random variables $\left(\epsilon_{k}^{+}\right)_{k \geqslant 1}$ and $\left(\epsilon_{k}^{-}\right)_{k \geqslant 1}$ with $\epsilon_{1}^{+} \sim f^{+}$ and $\epsilon_{1}^{-} \sim f^{-}$, and the distributions $f^{+}, f^{-}$in Assumption 2.4. In the next proposition, we identify the limit process $\widetilde{\widetilde{Q}}$ as a Markov process similarly to Lemma 5 in 19.

Proposition 2.4.7 (Limit distribution of the queue size process in inactive regimes). Let the assumptions of Theorem 2.4 .6 be satisfied and let $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 0}$ be the sequence of stopping times introduced in 2.4.2). The limit $\widetilde{\tilde{Q}}$ is a four-dimensional Markov process with values almost surely in $\mathbb{R}_{+}^{4} \backslash\left\{\left(x_{1}, x_{2}, x_{3}, x_{4}\right): \exists i, j, i \neq j\right.$ with $\left.x_{i}=x_{j}=0\right\}$ satisfying the following:
i) Inbetween two consecutive stopping times $\widetilde{\tau}_{k}^{*}$ and $\widetilde{\tau}_{k+1}^{*}$ the process $\widetilde{Q}$ follows a four-dimensional linear Brownian motion starting at $\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)$ with drift $\mu$ and covariance matrix $\Sigma$ (see Proposition 2.3.1 for the definitions of $\mu$ and $\Sigma$ ).
ii) The infinitesimal generator $\mathcal{A}$ of $\widetilde{\widetilde{Q}}$ is given on $(0, \infty)^{4}$ by

$$
\begin{align*}
\mathcal{A} h= & \mu^{b, F} \frac{\partial h}{\partial x_{1}}+\mu^{a, F} \frac{\partial h}{\partial x_{2}}+\mu^{b, G} \frac{\partial h}{\partial x_{3}}+\mu^{a, G} \frac{\partial h}{\partial x_{4}} \\
& +\frac{\left(\sigma^{b, F}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{1}^{2}}+\frac{\left(\sigma^{a, F}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{2}^{2}}+\frac{\left(\sigma^{b, G}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{3}^{2}}+\frac{\left(\sigma^{a, G}\right)^{2}}{2} \frac{\partial^{2} h}{\partial x_{4}^{2}} \\
& +\sigma^{(b, F),(a, F)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{2}}+\sigma^{(b, F),(b, G)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{3}}+\sigma^{(b, F),(a, G)} \frac{\partial^{2} h}{\partial x_{1} \partial x_{4}}  \tag{2.4.3}\\
& +\sigma^{(a, F),(b, G)} \frac{\partial^{2} h}{\partial x_{2} \partial x_{3}}+\sigma^{(a, F),(a, G)} \frac{\partial^{2} h}{\partial x_{2} \partial x_{4}}+\sigma^{(b, G),(a, G)} \frac{\partial^{2} h}{\partial x_{3} \partial x_{4}}
\end{align*}
$$

and for $x_{1}, x_{2}, x_{3}, x_{4}>0$

$$
\begin{align*}
& \mathcal{A} h\left(x_{1}, 0, x_{3}, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\pi_{F} \Phi\left(\left(x_{1}, 0, x_{3}, x_{4}\right), u\right), x_{3}, x_{4}\right) f^{+}(d u), \\
& \mathcal{A} h\left(0, x_{2}, x_{3}, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\pi_{F} \Phi\left(\left(0, x_{2}, x_{3}, x_{4}\right), u\right), x_{3}, x_{4}\right) f^{-}(d u), \\
& \mathcal{A} h\left(x_{1}, x_{2}, x_{3}, 0\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(x_{1}, x_{2}, \pi_{G} \Phi\left(\left(x_{1}, x_{2}, x_{3}, 0\right), u\right)\right) f^{+}(d u),  \tag{2.4.4}\\
& \mathcal{A} h\left(x_{1}, x_{2}, 0, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(x_{1}, x_{2}, \pi_{G} \Phi\left(\left(x_{1}, x_{2}, 0, x_{4}\right), u\right)\right) f^{-}(d u),
\end{align*}
$$

and whose domain is the set $\operatorname{dom}(\mathcal{A})$ of functions $C^{2}\left((0, \infty)^{4}, \mathbb{R}\right) \cap C^{0}\left(\mathbb{R}_{+}^{4}, \mathbb{R}\right)$ verifying for all $x_{1}, x_{2}, x_{3}, x_{4}>0$ the inward jump boundary conditions

$$
\begin{aligned}
& h\left(x_{1}, 0, x_{3}, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\pi_{F} \Phi\left(\left(x_{1}, 0, x_{3}, x_{4}\right), u\right), x_{3}, x_{4}\right) f^{+}(d u), \\
& h\left(0, x_{2}, x_{3}, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\pi_{F} \Phi\left(\left(0, x_{2}, x_{3}, x_{4}\right), u\right), x_{3}, x_{4}\right) f^{-}(d u), \\
& h\left(x_{1}, x_{2}, x_{3}, 0\right)=\int_{\mathbb{R}_{+}^{4}} h\left(x_{1}, x_{2}, \pi_{G} \Phi\left(\left(x_{1}, x_{2}, x_{3}, 0\right), u\right)\right) f^{+}(d u), \\
& h\left(x_{1}, x_{2}, 0, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} h\left(x_{1}, x_{2}, \pi_{G} \Phi\left(\left(x_{1}, x_{2}, 0, x_{4}\right), u\right)\right) f^{-}(d u) .
\end{aligned}
$$

The next result follows directly from the definition of the function $\tilde{\widetilde{\Psi}}^{B}$ introduced in Definition 2.4.4 and identifies the distribution of the limit process $\widetilde{B}$.

Theorem 2.4.8 (Limit distribution of the price process in inactive regimes). Let the assumptions of Theorem 2.4.6 be satisfied. Then, the limit $\widetilde{B}=\left(\widetilde{B}^{F}, \widetilde{B}^{G}\right)$ satisfies

$$
\left.\widetilde{\widetilde{B}}(t)=\widetilde{\widetilde{B}}_{0}+\delta \sum_{0 \leqslant s \leqslant t}\left(\mathbb{1}_{\left\{\pi_{2} \widetilde{\widetilde{Q}}^{(s-)=0}\right\}^{-\mathbb{1}}\left\{\pi_{1} \tilde{\widetilde{Q}}(s-)=0\right\}}, \mathbb{1}_{\left\{\pi_{4} \widetilde{\widetilde{Q}}(s-)=0\right.}\right\}^{-\mathbb{1}}\left\{\pi_{3} \tilde{\widetilde{Q}}(s-)=0\right\}\right)
$$

which is a piecewise constant càdlàg process whose components do almost surely not jump simultaneously.

### 2.5 Analysis of the cross-border market dynamics

In this section, we finally derive the weak convergence of the sequence of the crossborder market models $S^{(n)}$ being introduced in Definition 2.2.7. For that, we need the convergence results corresponding to the active dynamics $\widetilde{S}^{(n)}$ and inactive dynamics $\tilde{S}^{(n)}$ derived in Theorem 2.3.19 and Theorem 2.4.6. respectively. Before stating our main theorem, let us introduce the limits $\left(\rho_{k}\right)_{k \geqslant 0}$ and $\left(\sigma_{k}\right)_{k \geqslant 1}$ of the discrete-time stopping times $\left(\rho_{k}^{(n)}\right)_{k \geqslant 0}$ and $\left(\sigma_{k}^{(n)}\right)_{k \geqslant 1}$. Therefore, let $S:=(B, Q, C)$ denote the limit process of $S^{(n)}$ (defined below). Again, let $\rho_{0}:=0$, and for $k \geqslant 1$, we set

$$
\sigma_{k}:=\sigma_{k}^{I m} \wedge \sigma_{k}^{E x}
$$

where

$$
\sigma_{k}^{I m}:=\inf \left\{t \geqslant \rho_{k-1}: C(t) \geqslant \kappa_{+}\right\} \wedge T, \quad \sigma_{k}^{E x}:=\inf \left\{t \geqslant \rho_{k-1}: C(t) \leqslant-\kappa_{-}\right\} \wedge T,
$$

and

$$
\rho_{k}:=\rho_{k}^{I m} \mathbb{1}_{\left\{C\left(\sigma_{k}\right)=-\kappa_{-}\right\}}+\rho_{k}^{E x} \mathbb{1}_{\left\{C\left(\sigma_{k}\right)=\kappa_{+}\right\}},
$$

where
$\rho_{k}^{I m}:=\inf \left\{t \geqslant \sigma_{k}:\left|B^{F}(t)-B^{G}(t)\right|=0\right.$ and $\exists(i, I) \in \mathcal{I}^{I m}$ with $\left.Q^{i, I}(t-) \leqslant 0\right\} \wedge T$, $\rho_{k}^{E x}:=\inf \left\{t \geqslant \sigma_{k}:\left|B^{F}(t)-B^{G}(t)\right|=0\right.$ and $\exists(i, I) \in \mathcal{I}^{E x}$ with $\left.Q^{i, I}(t-) \leqslant 0\right\} \wedge T$.

We observe that the limits $\sigma_{k}, k \geqslant 1$, have a much easier representation as its discretetime versions $\sigma_{k}^{(n)}, k \geqslant 1$, as we directly switch in the high-frequency limit from an active to an inactive regime if all capacities in one direction are occupied (cf. Lemma 2.3.31. In contrast, $\rho_{k}, k \geqslant 0$, are the canonical high-frequency versions of $\rho_{k}^{(n)}, k \geqslant 0$.

Similarly to the discrete-time setting, we denote $\left(\tau_{k}\right)_{k \geqslant 1}$ the sequence of stopping times at which we observe a price change in $S, l(t) \in \mathbb{N}_{0}$ the number of price changes of $S$ in $[0, t], t \in[0, T]$, and $R_{k}^{+}:=\Phi\left(Q\left(\tau_{k}-\right), \epsilon_{k}^{+}\right), R_{k}^{-}:=\Phi\left(Q\left(\tau_{k}-\right), \epsilon_{k}^{-}\right), k \geqslant 1$, the
queue sizes after price changes. Finally, we denote for all $s \in[0, T]$

$$
\begin{aligned}
& \widetilde{S}^{s}:=\widetilde{\Psi}\left(S(s) ; X(\cdot+s)-X(s),\left(R_{l(s)+j}^{+}\right)_{j \geqslant 1},\left(R_{l(s)+j}^{-}\right)_{j \geqslant 1}\right), \\
& \widetilde{\widetilde{S}}^{s}:=\widetilde{\Psi}\left(S(s) ; X(\cdot+s)-X(s),\left(R_{l(s)+j}^{+}\right)_{j \geqslant 1},\left(R_{l(s)+j}^{-}\right)_{j \geqslant 1}\right),
\end{aligned}
$$

the active and inactive dynamics starting in $S(s) \in E$, respectively.
Theorem 2.5.1 (Main result). Let Assumptions 2.1 2.4 be satisfied. Then, the microscopic dynamics $S^{(n)}=\left(S^{(n)}(t)\right)_{t \in[0, T]}$ of the cross-border market model converges weakly in the Skorokhod topology on $D([0, T], E)$ to a continuous-time regime switching process $S$, whose dynamics is described as follows: for all $k \geqslant 0$, we have

- $S \simeq \widetilde{S}^{\rho_{k}}$ on $\left[\rho_{k}, \sigma_{k+1}\right)$, i.e., on the interval $\left[\rho_{k}, \sigma_{k+1}\right)$ the volume dynamics $Q$ is a four-dimensional linear Brownian motion in the positive orthant with oblique reflection at the axes. Each time two queues simultaneously hit zero, the process is reinitialized at a new value in the interior of $\mathbb{R}_{+}^{4}$. The bid price dynamics $B$ is a two-dimensional pure jump process with jump times equal to those of the volume dynamics. In particular, we have that $B^{F} \equiv B^{G}$. The capacity dynamics $C$ is a continuous process of finite variation with values in $\left[-\kappa_{-}, \kappa_{+}\right]$.
- $S \simeq \widetilde{\widetilde{S}}^{\sigma_{k+1}}$ on $\left[\sigma_{k+1}, \rho_{k+1}\right)$, i.e., on the interval $\left[\sigma_{k+1}, \rho_{k+1}\right)$ the volume dynamics $Q$ is a four-dimensional linear Brownian motion in the interior of $\mathbb{R}_{+}^{4}$. Each time it hits one of the axes, the two components corresponding to the origin of the depleted component are reinitialized at a new value in $(0, \infty)^{2}$ while the others stay unchanged. The price dynamics $B$ is a two-dimensional pure jump process whose components jump at hitting times of the corresponding components of the volume process of the axes. In particular, $B^{F}$ and $B^{G}$ follow different one-dimensional pure jump processes which do almost surely not jump simultaneously. The capacity dynamics $C$ stays constant and equal to either $-\kappa_{-}$or $\kappa_{+}$.

We inductively construct a candidate for the limit process $S=(S(t))_{t \in[0, T]}$ and show that $S^{(n)} \Rightarrow S$ in the Skorokhod topology by applying Theorem 2.3.19 and Theorem 2.4.6 together with the continuity of the first hitting time and last value map (see Theorem 13.6.4 in [86] and Lemma 2.7.2].

Proof. We apply Skorokhod representation in order to be able to refer to arguments of almost sure convergence instead of weak convergence. Hence, with a slight abuse of notation, applying Proposition 2.3.1 and Assumption 2.4 we may assume

$$
\mathbb{P}\left[Y^{(n)} \rightarrow Y, \quad \forall k \geqslant 1, \epsilon_{k}^{+,(n)} \rightarrow \epsilon_{k}^{+}, \epsilon_{k}^{-,(n)} \rightarrow \epsilon_{k}^{-}\right]=1
$$

where $Y^{(n)}:=Q_{0}^{(n)}+X^{(n)}, Y:=Q_{0}+X, \epsilon_{k}^{+} \sim f^{+}$, and $\epsilon_{k}^{-} \sim f^{-}$for the distributions $f^{+}, f^{-}$in Assumption 2.4. Recall, that $\left(\tau_{k}^{(n)}\right)_{k \geqslant 1}$ denotes the sequence of stopping times at which we observe a price change in $S^{(n)}, R_{k}^{+,(n)}:=\Phi^{(n)}\left(Q^{(n)}\left(\tau_{k}^{(n)}-\right), \epsilon_{k}^{+,(n)}\right)$ and
$R_{k}^{-,(n)}:=\Phi^{(n)}\left(Q^{(n)}\left(\tau_{k}^{(n)}-\right), \epsilon_{k}^{-,(n)}\right), k \geqslant 1$, denote the queue sizes after a price increase respectively decrease in $S^{(n)}$, and $l^{(n)}(t) \in \mathbb{N}_{0}$ denotes the number of price changes in $S^{(n)}$ in $[0, t], t \in[0, T]$. Then, we define for $s \in[0, T]$

$$
\begin{aligned}
\widetilde{S}^{(n), s} & :=\widetilde{\Psi}\left(S^{(n)}(s) ; X^{(n)}(\cdot+s)-X^{(n)}(s),\left(R_{l(n)(s)+j}^{+,(n)}\right)_{j \geqslant 1},\left(R_{l l^{(n)}(s)+j}^{-,(n)}\right)_{j \geqslant 1}\right), \\
\widetilde{S}^{(n), s} & :=\widetilde{\Psi}\left(S^{(n)}(s) ; X^{(n)}(\cdot+s)-X^{(n)}(s),\left(R_{l(n)(s)+j}^{+,(n)}\right)_{j \geqslant 1},\left(R_{l l^{(n)}(s)+j}^{-,(n)}\right)_{j \geqslant 1}\right)
\end{aligned}
$$

the microscopic active and inactive dynamics starting in $S^{(n)}(s) \in E$. Similarly, we introduce their components $\widetilde{B}^{(n), s}, \widetilde{Q}^{(n), s} \widetilde{C}^{(n), s}, \widetilde{B}^{(n), s}$, and $\widetilde{Q}^{(n), s}$. In the following, we will prove the stated convergence result by induction.

Induction start: Let us introduce the functions $\sigma^{I m}, \sigma^{E x}, \sigma: D([0, T], \mathbb{R}) \rightarrow[0, T]$ by

$$
\sigma^{I m}(\omega):=\inf \left\{t \geqslant 0: \omega(t) \geqslant \kappa_{+}\right\} \wedge T, \quad \sigma^{E x}(\omega):=\inf \left\{t \geqslant 0: \omega(t) \leqslant-\kappa_{-}\right\} \wedge T,
$$

and $\sigma(\omega):=\sigma^{I m}(\omega) \wedge \sigma^{E x}(\omega)$. Note that for the stopping time $\sigma_{1}^{(n)}$ introduced in Definition 2.2.7, there exists a function $\widetilde{\sigma}: D([0, T], E) \rightarrow[0, T]$ satisfying $\sigma_{1}^{(n)}=$ $\widetilde{\sigma}\left(\widetilde{S}^{(n), 0}\right) \geqslant \sigma\left(\widetilde{C}^{(n), 0}\right)=: \sigma_{1,1}^{(n)}$. Then, we set $S(t)=\widetilde{S}^{0}(t)$ for $0 \leqslant t<\sigma_{1}$, where $\sigma_{1}:=\sigma\left(\widetilde{C}^{0}\right)$. By Assumption 2.1 and Theorem 2.3.19. we conclude that

$$
\begin{equation*}
\widetilde{S}^{(n), 0} \rightarrow \widetilde{S}^{0} \quad \mathbb{P} \text {-a.s. } \tag{2.5.1}
\end{equation*}
$$

in the Skorokhod topology. In particular, we have that $\widetilde{C}^{(n), 0} \rightarrow \widetilde{C}^{0} \mathbb{P}$-almost surely in the Skorokhod topology. By Theorem 2.3.28, the paths of $\widetilde{C}^{0}$ take with probability one their values in $C_{\kappa_{+}}^{\prime}([0, T], \mathbb{R}) \cap C_{-\kappa_{-}}^{\prime}([0, T], \mathbb{R})$. Hence, we can apply the continuity of the first hitting time and last value map (cf. Lemma 2.7.2) to conclude that

$$
\left(\sigma_{1,1}^{(n)}, S^{(n)}\left(\sigma_{1,1}^{(n)}-\right)\right) \rightarrow\left(\sigma_{1}, S\left(\sigma_{1}-\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Now, by Lemma 2.3.31, we have that

$$
\mathbb{P}\left[h\left(\widetilde{Q}^{0}\right)\left(\sigma_{1}\right) \notin(0, \infty)^{2}\right]=0,
$$

i.e., with probability one, no price changing event occurs at $\sigma_{1}$ in $\widetilde{S}^{0}$. Hence, $\sigma_{1}=\widetilde{\sigma}\left(\widetilde{S}^{0}\right)$ $\mathbb{P}$-almost surely. Now, together with 2.5.1 , we conclude that also $\left(\sigma_{1}^{(n)}, S^{(n)}\left(\sigma_{1}^{(n)}-\right)\right) \rightarrow$ $\left(\sigma_{1}, S\left(\sigma_{1}-\right)\right) \mathbb{P}$-almost surely.

Next, let us introduce the function $Z: \mathbb{R}^{2} \rightarrow\{-1,0,1\}$ by

$$
Z(x):= \begin{cases}0 & : x \in(0, \infty)^{2}  \tag{2.5.2}\\ -1 & : x \in \mathbb{R}_{-} \times(0, \infty) \\ 1 & : x \in(0, \infty) \times \mathbb{R}_{-}\end{cases}
$$

and set $\widetilde{Z}_{1}^{I}:=Z\left(\pi_{I} \widetilde{Q}^{0}\left(\sigma_{1}\right)\right)$ for $I=F, G$. Then, we define $S\left(\sigma_{1}\right)=\left(B\left(\sigma_{1}\right), Q\left(\sigma_{1}\right), C\left(\sigma_{1}\right)\right)$ as follows: let $C\left(\sigma_{1}\right)=C\left(\sigma_{1}-\right)$,

$$
\pi_{I} Q\left(\sigma_{1}\right)=\pi_{I} Q\left(\sigma_{1}-\right) \mathbb{1}_{\left\{\widetilde{Z}_{1}^{I}=0\right\}}+\pi_{I} R_{l\left(\sigma_{1}-\right)+1}^{+} \mathbb{1}_{\left\{\widetilde{Z}_{1}^{I}=1\right\}}+\pi_{I} R_{l\left(\sigma_{1}-\right)+1}^{-} \mathbb{1}_{\left\{\widetilde{Z}_{1}^{I}=-1\right\}}
$$

and

$$
B^{I}\left(\sigma_{1}\right)=B^{I}\left(\sigma_{1}-\right)+\delta\left(\mathbb{1}_{\left\{\widetilde{Z}_{1}^{I}=1\right\}}-\mathbb{1}_{\left\{\widetilde{Z}_{1}^{I}=-1\right\}}\right)
$$

Let us show that $\widetilde{Z}_{1}^{I,(n)}\left(\sigma_{1}^{(n)}\right) \rightarrow \widetilde{Z}_{1}^{I} \mathbb{P}$-almost surely for $I=F, G$, where the random process $\widetilde{Z}_{1}^{I,(n)}, I=F, G$, is introduced in 2.2 .12 . First, we define $\widetilde{Z}_{1}^{I,(n), *}:=$ $Z\left(\pi_{I} \widetilde{Q}^{(n), 0}\left(\sigma_{1}^{(n)}\right)\right)$ for $I=F, G$. Since almost surely no price changing event occurs at $\sigma_{1}$ and no national limit order book is empty in $\widetilde{S}^{0}$ (cf. Lemma 2.3.31 and Lemma 2.3.32, we conclude that

$$
\begin{equation*}
\left|\widetilde{Z}_{1}^{I,(n)}\left(\sigma_{1}^{(n)}\right)-\widetilde{Z}_{1}^{I,(n), *}\right| \rightarrow 0 \quad \mathbb{P} \text {-a.s. } \tag{2.5.3}
\end{equation*}
$$

By Lemma 2.3.31, Lemma 2.3.32, and the construction of $\widetilde{C}^{0}$, we conclude with probability one, that exactly one component of $\widetilde{Q}^{0}\left(\sigma_{1}\right)$ is in $\mathbb{R}$ _ while the remaining components are in $(0, \infty)$. Because of (2.5.1), we conclude for all $n$ large enough, that the same component of $\widetilde{Q}^{(n), 0}\left(\sigma_{1}^{(n)}\right)$ is in $\mathbb{R}_{-}$while the others are in $(0, \infty)$. Hence, for $I=F, G, \pi_{I} \widetilde{Q}^{(n), 0}\left(\sigma_{1}^{(n)}\right)$ and $\pi_{I} \widetilde{Q}^{0}\left(\sigma_{1}\right)$ are both either in $(0, \infty)^{2}, \mathbb{R}_{-} \times$ $(0, \infty)$, or $(0, \infty) \times \mathbb{R}_{-}$with probability one for all $n$ large enough. We conclude that $Z\left(\pi_{I} \widetilde{Q}^{(n), 0}\left(\sigma_{1}^{(n)}\right)\right) \rightarrow Z\left(\pi_{I} \widetilde{Q}^{0}\left(\sigma_{1}\right)\right) \mathbb{P}$-almost surely. Thus, together with 2.5.3), $\widetilde{Z}_{1}^{I,(n)}\left(\sigma_{1}^{(n)}\right) \rightarrow \widetilde{Z}_{1}^{I} \mathbb{P}$-almost surely. Applying (2.5.1), Lemma 2.7.10, and the fact that $\mathbb{P}$-almost surely no price changing event occurs at $\sigma_{1}$ in $\widetilde{S}^{0}$ (cf. Lemma 2.3.31, we further conclude that $l^{(n)}\left(\sigma_{1}^{(n)}-\right) \rightarrow l\left(\sigma_{1}-\right) \mathbb{P}$-almost surely. Hence, together with the continuity of the last value map (cf. Lemma 2.7.2), Assumption 2.4, and equation (2.5.1), we get

$$
\begin{equation*}
S^{(n)}\left(\sigma_{1}^{(n)}\right) \rightarrow S\left(\sigma_{1}\right) \quad \mathbb{P} \text {-a.s. } \tag{2.5.4}
\end{equation*}
$$

In order to extend the definition of $S$ beyond $\left[0, \sigma_{1}\right]$, we introduce a function $\bar{\rho}$ : $D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ by

$$
\bar{\rho}(\omega):=\inf \left\{t \geqslant 0:\left|\pi_{1}^{(2)} \omega(t)-\pi_{2}^{(2)} \omega(t)\right|<\frac{\delta}{2}\right\} \wedge T .
$$

Furthermore, we define the functions $\bar{\tau}^{I m}, \bar{\tau}^{E x}, \bar{\tau}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ by

$$
\begin{aligned}
& \bar{\tau}^{I m}(\omega):=\inf \left\{t \geqslant 0: \exists i \in\{2,3\} \text { with } \pi_{i} \omega(t) \leqslant 0\right\} \wedge T \\
& \bar{\tau}^{E x}(\omega):=\inf \left\{t \geqslant 0: \exists i \in\{1,4\} \text { with } \pi_{i} \omega(t) \leqslant 0\right\} \wedge T
\end{aligned}
$$

and $\bar{\tau}(\omega):=\bar{\tau}^{I m}(\omega) \wedge \bar{\tau}^{E x}(\omega)$. Then, for $l, j \geqslant 1$ and $\bar{\tau}_{l, 0}^{(n)}:=\sigma_{l}^{(n)}$, we define

$$
\begin{align*}
& \bar{\rho}_{l, j}^{(n)}:=\bar{\tau}_{l, j-1}^{(n)}+\bar{\rho}\left(\widetilde{\widetilde{B}}^{(n), \bar{\tau}_{l, j-1}^{(n)}}\right), \text { and }  \tag{2.5.5}\\
& \bar{\tau}_{l, j}^{(n)}:=\bar{\rho}_{l, j}^{(n)}+\bar{\tau}\left(Q^{(n)}\left(\bar{\rho}_{l, j}^{(n)}\right)+X^{(n)}\left(\cdot+\bar{\rho}_{l, j}^{(n)}\right)-X^{(n)}\left(\bar{\rho}_{l, j}^{(n)}\right)\right)
\end{align*}
$$

Now, we define for $t \in\left[0, \bar{\tau}_{1,1}-\sigma_{1}\right)$,

$$
S\left(t+\sigma_{1}\right):=\widetilde{\widetilde{S}}^{\sigma_{1}}(t)
$$

where $\bar{\tau}_{1,1}:=\bar{\rho}_{1,1}+\bar{\tau}\left(\widetilde{\widetilde{Q}}^{\sigma_{1}}\left(\bar{\rho}_{1,1}\right)+X\left(\cdot+\bar{\rho}_{1,1}\right)-X\left(\bar{\rho}_{1,1}\right)\right)$ and $\bar{\rho}_{1,1}:=\sigma_{1}+\bar{\rho}\left(\widetilde{\widetilde{B}}^{\sigma_{1}}\right)$. By Theorem 2.4.6 together with 2.5 .4 , we conclude that $\widetilde{\widetilde{S}}^{(n), \sigma_{1}} \rightarrow \widetilde{\widetilde{S}}^{\sigma_{1}} \mathbb{P}$-almost surely in the Skorokhod topology. In particular, $\widetilde{\mathbb{B}^{(n),} \sigma_{1}} \rightarrow \widetilde{\widetilde{B}}^{\sigma_{1}} \mathbb{P}$-almost surely in the Skorkhod topology. Note, that $\bar{\rho}_{1,1}^{(n)} \leqslant \rho_{1}^{(n)}$ and by Assumption 2.4. we have that $\bar{\rho}_{1,1}$ does not indicate the start of the next inactive regime. Hence, together with the continuity of the first exit time and last value map (cf. Theorem 13.6.4 in Whitt 86), we conclude that

$$
\left(\bar{\rho}_{1,1}^{(n)}, S^{(n)}\left(\bar{\rho}_{1,1}^{(n)}-\right), S^{(n)}\left(\bar{\rho}_{1,1}^{(n)}\right)\right) \rightarrow\left(\bar{\rho}_{1,1}, S\left(\bar{\rho}_{1,1}-\right), S\left(\bar{\rho}_{1,1}\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

In particular, $Q\left(\bar{\rho}_{1,1}\right)=\widetilde{\widetilde{Q}}^{\sigma_{1}}\left(\bar{\rho}_{1,1}\right)$. By Proposition 2.3.1. with probability one, the paths of $Q\left(\bar{\rho}_{1,1}\right)+X\left(\cdot+\bar{\rho}_{1,1}\right)-X\left(\bar{\rho}_{1,1}\right)$ are in $C_{0}^{\prime}\left(\left[0, T-\bar{\rho}_{1,1}\right], \mathbb{R}^{4}\right)$. Hence, applying the continuity of the first hitting time and last value map (cf. Lemma 2.7.2), we conclude that

$$
\left(\bar{\tau}_{1,1}^{(n)}, S^{(n)}\left(\bar{\tau}_{1,1}^{(n)}-\right)\right) \rightarrow\left(\bar{\tau}_{1,1}, S\left(\bar{\tau}_{1,1}-\right)\right) \quad \mathbb{P} \text {-a.s. }
$$

Now, depending on the first component of the queue size process being depleted, we either switch to the next active regime or stay in the current inactive regime, i.e., we set

$$
\begin{aligned}
S\left(\bar{\tau}_{1,1}\right):= & S\left(\bar{\tau}_{1,1}-\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1,1}=\bar{\tau}_{1,1}^{I m}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=\kappa_{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1,1}=\bar{\tau}_{1,1}^{E x}\right\}}\right\} \\
& +\widetilde{\widetilde{S}}^{\sigma_{1}}\left(\bar{\tau}_{1,1}-\sigma_{1}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1,1}=\bar{\tau}_{1,1}^{E x}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=\kappa_{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1,1}=\bar{\tau}_{1,1}^{I m}\right\}}\right\} .
\end{aligned}
$$

The above assignment is well-defined, since the queue size process of $\widetilde{\widetilde{S}}^{\sigma_{1}}$ follows between two consecutive price changes the distribution of a four-dimensional linear Brownian motion (cf. Theorem 2.4.6). Hence, there are almost surely no two queues hitting zero at $\bar{\tau}_{1,1}$ and therefore, the cumulative best bid and ask queues are almost surely strictly greater than zero. Therefore, we do not observe a price change when transmitting from the current inactive to the next active regime. By Lemma 2.7.3, we conclude the continuity of the maps $H^{I m}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{I m}(\omega)=\bar{\tau}(\omega)\right\}}$ and $H^{E x}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{E x}(\omega)=\bar{\tau}(\omega)\right\}}$ and therefore,

$$
S^{(n)}\left(\bar{\tau}_{1,1}^{(n)}\right) \rightarrow S\left(\bar{\tau}_{1,1}\right) \quad \text { P-a.s. }
$$

Now, let us assume that $S\left(\bar{\tau}_{1,1}\right)=\widetilde{\widetilde{S}}^{\sigma_{1}}\left(\bar{\tau}_{1,1}-\sigma_{1}\right)$, i.e., we stay in the current inactive regime. Moreover, assume for $j \geqslant 2$ and $0 \leqslant t \leqslant \bar{\tau}_{1, j-1}-\sigma_{1}$, we have inductively constructed $S\left(t+\sigma_{1}\right)=\widetilde{\widetilde{S}}^{\sigma_{1}}(t)$ provided we did not switch the regime in $\left[0, \bar{\tau}_{1, j-1}-\sigma_{1}\right]$. Then, we extend the definition of $S$ to $\left[\bar{\tau}_{1, j-1}, \bar{\tau}_{1, j}\right)$ by setting $S\left(t+\sigma_{1}\right)=\widetilde{S}^{\sigma_{1}}(t)$ for $0 \leqslant t<\bar{\tau}_{1, j}-\sigma_{1}$, where $\bar{\tau}_{1, j}:=\bar{\rho}_{1, j}+\bar{\tau}\left(\widetilde{\widetilde{Q}}^{\sigma_{1}}\left(\bar{\rho}_{1, j}\right)+X\left(\cdot+\bar{\rho}_{1, j}\right)-X\left(\bar{\rho}_{1, j}\right)\right)$ and $\bar{\rho}_{1, j}:=\bar{\tau}_{1, j-1}+\bar{\rho}\left(\widetilde{\widetilde{B}}^{\bar{\tau}_{1, j-1}}\right)$. Then, we can first argue as for $\bar{\rho}_{1,1}$ and therefore conclude that

$$
\left(\bar{\rho}_{1, j}^{(n)}, S^{(n)}\left(\bar{\rho}_{1, j}^{(n)}-\right), S^{(n)}\left(\bar{\rho}_{1, j}^{(n)}\right)\right) \rightarrow\left(\bar{\rho}_{1, j}, S\left(\bar{\rho}_{1, j}-\right), S\left(\bar{\rho}_{1, j}\right)\right) \quad \text { P-a.s. }
$$

In particular, $Q\left(\bar{\rho}_{1, j}\right)=\tilde{\widetilde{Q}}^{\sigma_{1}}\left(\bar{\rho}_{1, j}\right)$. Furthermore, by Proposition 2.3.1 with probability one the path of $Q\left(\bar{\rho}_{1, j}\right)+X\left(\cdot+\bar{\rho}_{1, j}\right)-X\left(\bar{\rho}_{1, j}\right)$ is in $C_{0}^{\prime}\left(\left[0, T-\bar{\rho}_{1, j}\right], \mathbb{R}^{4}\right)$. Hence, applying the continuity of the first hitting time and last value map (cf. Lemma 2.7.2), we conclude that $\left(\bar{\tau}_{1, j}^{(n)}, S^{(n)}\left(\bar{\tau}_{1, j}^{(n)}-\right)\right) \rightarrow\left(\bar{\tau}_{1, j}, S\left(\bar{\tau}_{1, j}-\right)\right) \mathbb{P}$-almost surely. Again, depending on which component of the queue size process hits zero first, we either switch to the next active regime or stay in the current inactive regime, i.e., we define

$$
\begin{aligned}
S\left(\bar{\tau}_{1, j}\right):= & S\left(\bar{\tau}_{1, j}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1, j}=\bar{\tau}_{1, j}^{I m}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=\kappa_{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1, j}=\bar{\tau}_{1, j}^{E x}\right\}}\right\} \\
& +\widetilde{\widetilde{S}}^{\sigma_{1}}\left(\bar{\tau}_{1, j}-\sigma_{1}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1, j}=\bar{\tau}_{1, j}^{E x}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{1}\right)=\kappa_{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{1, j}=\bar{\tau}_{1, j}^{I m}\right\}}\right\}
\end{aligned}
$$

Again, the above assignment is well-defined. By Lemma 2.7.3, we conclude the continuity of the map $H^{I m}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{I m}(\omega)=\bar{\tau}(\omega)\right\}}$ and $H^{E x}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{E x}(\omega)=\bar{\tau}(\omega)\right\}}$. Hence, $S^{(n)}\left(\bar{\tau}_{1, j}^{(n)}\right) \rightarrow S\left(\bar{\tau}_{1, j}\right) \mathbb{P}$-almost surely. In contrast, let us assume that there exists a $j \geqslant 1$ such that $S\left(\bar{\tau}_{1, j}\right)=S\left(\bar{\tau}_{1, j}-\right)$. By construction, we have that $\rho_{1}^{(n)}=\bar{\tau}_{1, j}^{(n)} \rightarrow \bar{\tau}_{1, j}=\rho_{1}$ $\underset{\sim}{\mathbb{P}}$-almost surely. In particular, we start with the next active regime. Note that $C\left(\rho_{1}\right)=$ $\widetilde{C}^{\rho_{1}}\left(\rho_{1}\right)$ and since the paths of $\widetilde{C}^{\rho_{1}}$ are elements in $C\left(\left[0, T-\rho_{1}\right], \mathbb{R}\right)$ and a single component of $Q$ hits zero at $\rho_{1}$, we almost surely do not switch back to the inactive regime at $\rho_{1}$.

Induction hypothesis: Assume that for all $k \geqslant 2$, we have already shown that

$$
\begin{align*}
\left(\sigma_{1}^{(n)}, \rho_{1}^{(n)}, \cdots\right. & \left., \sigma_{k-1}^{(n)}, \rho_{k-1}^{(n)}, S^{(n)}\left(\sigma_{1}^{(n)}\right), S^{(n)}\left(\rho_{1}^{(n)}\right), \cdots, S^{(n)}\left(\sigma_{k-1}^{(n)}\right), S^{(n)}\left(\rho_{k-1}^{(n)}\right)\right)  \tag{2.5.6}\\
& \rightarrow\left(\sigma_{1}, \rho_{1}, \cdots, \sigma_{k-1}, \rho_{k-1}, S\left(\sigma_{1}\right), S\left(\rho_{1}\right), \cdots, S\left(\sigma_{k-1}\right), S\left(\rho_{k-1}\right)\right)
\end{align*}
$$

$\mathbb{P}$-almost surely and that we have constructed our candidate $S$ on the whole interval $\left[0, \rho_{k-1}\right]$.

Induction step: We aim to extend the definition of $S$ to the interval $\left[\rho_{k-1}, \rho_{k}\right]$. Again, we have $\sigma_{k}^{(n)}=\widetilde{\sigma}\left(\widetilde{S}^{(n), \rho_{k-1}^{(n)}}\right) \geqslant \sigma\left(\widetilde{C}^{(n), \rho_{k-1}^{(n)}}\right)=: \sigma_{k, 1}^{(n)}$. Then, for $0 \leqslant t<\sigma_{k}-\rho_{k-1}$, we set

$$
S\left(t+\rho_{k-1}\right)=\widetilde{S}^{\rho_{k-1}}(t)
$$

where $\sigma_{k}:=\sigma\left(\widetilde{C}^{\rho_{k-1}}\right)$. Again, by Theorem 2.3.19 and since $S^{(n)}\left(\rho_{k-1}^{(n)}\right) \rightarrow S\left(\rho_{k-1}\right)$ $\mathbb{P}$-almost surely by (2.5.6), we have that

$$
\widetilde{S}^{(n), \rho_{k-1}^{(n)}} \rightarrow \widetilde{S}^{\rho_{k-1}} \quad \mathbb{P} \text {-a.s. }
$$

in the Skorokhod topology on the space $D([0, T], E)$. Again, by Theorem 2.3 .28 , the path of $\widetilde{C}^{\rho_{k-1}}$ takes with probability one its values in $C_{\kappa_{+}}^{\prime}\left(\left[0, T-\rho_{k-1}\right], \mathbb{R}\right) \cap C_{-\kappa_{-}}^{\prime}([0, T-$ $\left.\left.\rho_{k-1}\right], \mathbb{R}\right)$. Hence, we can apply the continuity of the first hitting time and last value map (cf. Lemma 2.7.2), to conclude that $\left(\sigma_{k, 1}^{(n)}, S^{(n)}\left(\sigma_{k, 1}^{(n)}-\right)\right) \rightarrow\left(\sigma_{k}, S\left(\sigma_{k}-\right)\right) \mathbb{P}$-almost surely. Now, again by Lemma 2.3.31 with probability one, we have that $h\left(\widetilde{Q}^{\rho_{k-1}}\right)\left(\sigma_{k}\right) \in(0, \infty)^{2}$. Hence, $\sigma_{k}=\widetilde{\sigma}\left(\widetilde{S}^{\rho_{k-1}}\right)$ and therefore, $\left(\sigma_{k}^{(n)}, S^{(n)}\left(\sigma_{k}^{(n)}-\right)\right) \rightarrow\left(\sigma_{k}, S\left(\sigma_{k}-\right)\right) \mathbb{P}$-almost surely. Now, we define $S\left(\sigma_{k}\right)=\left(B\left(\sigma_{k}\right), Q\left(\sigma_{k}\right), C\left(\sigma_{k}\right)\right)$ as follows: let $\widetilde{Z}_{k}^{I}:=Z\left(\pi_{I} \widetilde{Q}^{\rho_{k-1}}\left(\sigma_{k}\right)\right)$, where the function $Z$ is introduced in (2.5.2). Moreover, we set $C\left(\sigma_{k}\right)=C\left(\sigma_{k}-\right)$, for $I=F, G$,

$$
\pi_{I} Q\left(\sigma_{k}\right)=\pi_{I} Q\left(\sigma_{k}-\right) \mathbb{1}_{\left\{\widetilde{Z}_{k}^{I}=0\right\}}+\pi_{I} R_{l\left(\sigma_{k}-\right)+1}^{+} \mathbb{1}_{\left\{\widetilde{Z}_{k}^{I}=1\right\}}+\pi_{I} R_{l\left(\sigma_{k}-\right)+1}^{-} \mathbb{1}_{\left\{\widetilde{Z}_{k}^{I}=-1\right\}},
$$

and

$$
B^{I}\left(\sigma_{k}\right)=B^{I}\left(\sigma_{k}-\right)+\delta\left(\mathbb{1}_{\left\{\widetilde{Z}_{k}^{I}=1\right\}}-\mathbb{1}_{\left\{\widetilde{Z}_{k}^{I}=-1\right\}}\right)
$$

Now, we can argue similarly as for $S\left(\sigma_{1}\right)$ and obtain that

$$
\begin{align*}
\left(l^{(n)}\left(\sigma_{k}^{(n)}-\right), \widetilde{Z}_{k}^{F}\left(\sigma_{k}^{(n)}\right), \widetilde{Z}_{k}^{G}\right. & \left.\left(\sigma_{k}^{(n)}\right), S^{(n)}\left(\sigma_{k}^{(n)}\right)\right)  \tag{2.5.7}\\
& \rightarrow\left(l\left(\sigma_{k}-\right), \widetilde{Z}_{k}^{F}, \widetilde{Z}_{k}^{G}, S\left(\sigma_{k}\right)\right) \quad \text { P-a.s. }
\end{align*}
$$

Now, we extend the definition of $S$ such that for all $0 \leqslant t<\bar{\tau}_{k, 1}-\sigma_{k}$,

$$
S\left(t+\sigma_{k}\right)=\widetilde{\widetilde{S}}^{\sigma_{k}}(t)
$$

where $\bar{\tau}_{k, 1}:=\bar{\rho}_{k, 1}+\bar{\tau}\left(\widetilde{\widetilde{Q}}^{\sigma_{k}}\left(\bar{\rho}_{k, 1}\right)+X\left(\cdot+\bar{\rho}_{k, 1}\right)-X\left(\bar{\rho}_{k, 1}\right)\right)$ and $\bar{\rho}_{k, 1}:=\sigma_{k}+\bar{\rho}\left(\widetilde{\widetilde{B}}^{\sigma_{k}}\right)$. By Theorem 2.4.6 and (2.5.7), we conclude that $\widetilde{S}^{(n), \sigma_{k}^{(n)}} \rightarrow \widetilde{S}^{\sigma_{k}} \mathbb{P}^{\text {-almost surely in the Sko- }}$ rokhod topology. Together with the continuity of the first hitting time and last value map (cf. Theorem 13.6.4 in Whitt $86 \mid$ ), we conclude that $\left(\bar{\rho}_{k, 1}^{(n)}, S^{(n)}\left(\bar{\rho}_{k, 1}^{(n)}-\right), S^{(n)}\left(\bar{\rho}_{k, 1}^{(n)}\right)\right) \rightarrow$ $\left(\bar{\rho}_{k, 1}, S\left(\bar{\rho}_{k, 1}-\right), S\left(\bar{\rho}_{k, 1}\right)\right) \mathbb{P}$-almost surely. In particular, $\widetilde{\widetilde{Q}}^{\sigma_{k}}\left(\bar{\rho}_{k, 1}\right)=Q\left(\bar{\rho}_{k, 1}\right)$. By Proposition 2.3.1. the process $Q\left(\bar{\rho}_{k, 1}\right)+X\left(\cdot+\bar{\rho}_{k, 1}\right)-X\left(\bar{\rho}_{k, 1}\right)$ takes almost surely its path in $C_{0}^{\prime}\left(\left[0, T-\bar{\rho}_{k, 1}\right], \mathbb{R}^{4}\right)$. Hence, we can apply the continuity of the first hitting time and last value map (cf. Lemma 2.7.2 to conclude that $\left(\bar{\tau}_{k, 1}^{(n)}, S^{(n)}\left(\bar{\tau}_{k, 1}^{(n)}-\right)\right) \rightarrow\left(\bar{\tau}_{k, 1}, S\left(\bar{\tau}_{k, 1}-\right)\right)$ $\mathbb{P}$-almost surely. Depending on which component of the queue size process has been depleted first, we either switch to the next active regime or stay in the current inactive
regime, i.e., we set

$$
\begin{aligned}
S\left(\bar{\tau}_{k, 1}\right):= & S\left(\bar{\tau}_{k, 1}-\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, 1}=\bar{\tau}_{k, 1}^{I m}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=\kappa^{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, 1}=\bar{\tau}_{k, 1}^{E x}\right\}}\right\} \\
& +\widetilde{\widetilde{S}}^{\sigma_{k}}\left(\bar{\tau}_{k, 1}-\sigma_{k}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, 1}=\bar{\tau}_{k, 1}^{E x}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=\kappa_{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, 1}=\bar{\tau}_{k, 1}^{I m}\right\}}\right\} .
\end{aligned}
$$

Again, the above assignment is well-defined. By Lemma 2.7.3, we conclude the continuity of the map $H^{I m}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{I m}(\omega)=\bar{\tau}(\omega)\right\}}$ and $H^{E x}: \omega \mapsto \mathbb{1}_{\left\{\bar{\tau}^{E x}(\omega)=\bar{\tau}(\omega)\right\}}$. Hence, $S^{(n)}\left(\bar{\tau}_{k, 1}^{(n)}\right) \rightarrow S\left(\bar{\tau}_{k, 1}\right) \mathbb{P}$-almost surely. Now, assume that $S\left(\bar{\tau}_{k, 1}\right)=\widetilde{\widetilde{S}}^{\sigma_{k}}\left(\bar{\tau}_{k, 1}-\sigma_{k}\right)$. Further, assume for $j \geqslant 2$ and $0 \leqslant t \leqslant \bar{\tau}_{k, j-1}-\sigma_{k}$, that we have constructed $S\left(t+\sigma_{k}\right)=\widetilde{\widetilde{S}}^{\sigma_{k}}(t)$ provided we did not switch the regime in $\left[0, \bar{\tau}_{k, j-1}-\sigma_{k}\right]$. Again, we extend the definition of $S$ to $\left[\bar{\tau}_{k, j-1}, \bar{\tau}_{k, j}\right)$ by setting $S\left(t+\sigma_{k}\right)=\widetilde{\widetilde{S}}^{\sigma_{k}}(t)$ for $0 \leqslant t<\bar{\tau}_{k, j}-\sigma_{k}$, where $\bar{\tau}_{k, j}:=\bar{\rho}_{k, j}+\bar{\tau}\left(\widetilde{Q}^{\sigma_{k}}\left(\bar{\rho}_{k, j}\right)+X\left(\cdot+\bar{\rho}_{k, j}\right)-X\left(\bar{\rho}_{k, j}\right)\right)$ and $\bar{\rho}_{k, j}:=\bar{\tau}_{k, j-1}+\bar{\rho}\left(\widetilde{B}^{\bar{\tau}_{k, j-1}}\right)$. Then, we can argue as for $\bar{\rho}_{k, 1}$ to conclude that $\left(\bar{\rho}_{k, j}^{(n)}, S^{(n)}\left(\bar{\rho}_{k, j}^{(n)}-\right), S^{(n)}\left(\bar{\rho}_{k, j}^{(n)}\right)\right) \rightarrow$ $\left(\bar{\rho}_{k, j}, S\left(\bar{\rho}_{k, j}-\right), S\left(\bar{\rho}_{k, j}\right)\right) \mathbb{P}$-almost surely. In particular $\widetilde{\widetilde{Q}}^{\sigma_{k}}\left(\bar{\rho}_{k, j}\right)=Q\left(\bar{\rho}_{k, j}\right)$. Moreover, again by Proposition 2.3.1, the process $Q\left(\bar{\rho}_{k, j}\right)+X\left(\cdot+\bar{\rho}_{k, j}\right)-X\left(\bar{\rho}_{k, j}\right)$ has almost surely paths in $C_{0}^{\prime}\left(\left[0, T-\bar{\rho}_{k, j}\right], \mathbb{R}^{4}\right)$. Hence, we can apply the continuity of the first hitting time and last value map (cf. Theorem 2.7.2 to conclude that $\left(\bar{\tau}_{k, j}^{(n)}, S^{(n)}\left(\bar{\tau}_{k, j}^{(n)}-\right)\right) \rightarrow$ $\left(\bar{\tau}_{k, j}, S\left(\bar{\tau}_{k, j}-\right)\right) \mathbb{P}$-almost surely. Again, depending on which component of the queue size process has been depleted first, we either switch to the next active regime or stay in the current inactive regime, i.e., we define

$$
\begin{aligned}
S\left(\bar{\tau}_{k, j}\right):= & S\left(\bar{\tau}_{k, j}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, j}=\bar{\tau}_{k, j}^{I m}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=\kappa^{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, j}=\bar{\tau}_{k, j}^{E x}\right\}}\right\} \\
& +\widetilde{\widetilde{S}}^{\sigma_{k}}\left(\bar{\tau}_{k, j}-\sigma_{k}\right)\left\{\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=-\kappa_{-}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, j}=\bar{\tau}_{k, j}^{E x}\right\}}+\mathbb{1}_{\left\{C\left(\sigma_{k}\right)=\kappa^{+}\right\}} \mathbb{1}_{\left\{\bar{\tau}_{k, j}=\bar{\tau}_{k, j}^{I m}\right\}}\right\} .
\end{aligned}
$$

Arguing as above, we can conclude that $S^{(n)}\left(\bar{\tau}_{k, j}^{(n)}\right) \rightarrow S\left(\bar{\tau}_{k, j}\right) \mathbb{P}$-almost surely. In contrast, let us assume that there exists a $j \geqslant 1$ such that $S\left(\bar{\tau}_{k, j}\right)=S\left(\bar{\tau}_{k, j}-\right)$. Then $\bar{\tau}_{k, j}=\rho_{k}$ and we conclude that

$$
\rho_{k}^{(n)}=\bar{\tau}_{k, j}^{(n)} \rightarrow \rho_{k} \quad \mathbb{P} \text {-a.s. }
$$

i.e., we switch to the next active regime. Note again that $C\left(\rho_{k}\right)=\widetilde{C}^{\rho_{k}}\left(\rho_{k}\right)$ and since almost surely the paths of $\widetilde{C}^{\rho_{k}}$ are elements of $C\left(\left[0, T-\rho_{k}\right], \mathbb{R}\right)$ and a single component of $Q$ hits zero at $\rho_{k}$, we almost surely do not switch back to the inactive regime at $\rho_{k}$. This finishes our induction.

In summary, we conclude for our candidate $S$ and all $k \geqslant 1$,

$$
\begin{gathered}
\left(\sigma_{1}^{(n)}, \rho_{1}^{(n)}, \cdots, \sigma_{k}^{(n)}, \rho_{k}^{(n)}, S^{(n)}\left(\sigma_{1}^{(n)}\right), S^{(n)}\left(\rho_{1}^{(n)}\right), \cdots, S^{(n)}\left(\sigma_{k}^{(n)}\right), S^{(n)}\left(\rho_{k}^{(n)}\right)\right) \\
\rightarrow\left(\sigma_{1}, \rho_{1}, \cdots, \sigma_{k}, \rho_{k}, S\left(\sigma_{1}\right), S\left(\rho_{1}\right), \cdots, S\left(\sigma_{k}\right), S\left(\rho_{k}\right)\right)
\end{gathered}
$$

$\mathbb{P}$-almost surely. It remains to show that $\mathbb{P}\left[\rho_{k}=T\right] \rightarrow 1$ as $k \rightarrow \infty$. For this note that a switch from an active to an inactive regime causes the bid prices of both countries to move apart from each other with probability one. Before the time of the next regime switch to an active regime, we must therefore have at least one price change in one of the countries and at least two queues belonging to that country get reinitialized. Hence, at time $\rho_{2 k}$ the queues of one of the countries have been reinitialized at least $k$ times at either $\pi_{I} R_{k}^{+}$or $\pi_{I} R_{k}^{-}$for some $k \in \mathbb{N}$ and either $I=F$ or $I=G$. By Assumption 2.4 we know that each $R_{k}^{ \pm}$is component-wise bounded from below by $\alpha \epsilon_{k}^{ \pm}$, where the ( $\epsilon_{k}^{ \pm}$) are iid. As the starting time $\rho_{k}$ of an active regime can be bounded from below by the hitting time at zero of one of the components of $Q$, we can thus bound $\rho_{2 k}$ from below by the sum of the first hitting times at the axes of $k$ independent planar Brownian motions, each started from $\min \left\{\pi_{F} \epsilon_{k}^{+}, \pi_{F} \epsilon_{k}^{-}, \pi_{G} \epsilon_{k}^{+}, \pi_{G} \epsilon_{k}^{-}\right\}$. But on each compact interval $[0, T]$ only finitely many independent planar Brownian motions will hit the axes and hence we must have $\mathbb{P}\left[\rho_{k}=T\right] \rightarrow 1$. Therefore, we obtain from Theorem 2.3.19 and Theorem 2.4.6 that

$$
S^{(n)} \rightarrow S \quad \mathbb{P} \text {-a.s. }
$$

in the Skorokhod topology on the space $D([0, T], E)$. This proves the stated result.

### 2.6 Simulation study

We discuss the evolution of our market dynamics through simulation studies of different model calibrations. Moreover, we investigate the impact of coupling two markets on price evolution. Therefore, we perform a comparative statics of the active and inactive market dynamics and discuss a theoretical result on the conditional distribution of the duration between price changes.

### 2.6.1 Simulation of different market situations

Throughout this section, we choose $\Delta t^{(n)}=n^{-1}, \Delta v^{(n)}=n^{-1 / 2}$, and simulate $n=$ 10,000 time steps. For simplicity, we assume that the order sizes are independent and simulate the queue sizes after price changes from independent uniform distributions on $\left\{j \Delta v^{(n)}: j=10, \cdots, 20\right\}$. Moreover, we choose $\kappa_{+}=\kappa_{-}=0.5$ and $\delta=0.1$.
In a first simulation, we consider a cross-border market which is balanced between cross-border trades in direction $F$ to $G$ and vice versa. In addition, we assume that the occurrence of different order types is equally likely and choose the frequency of order cancellations higher than the frequency of order placements. The latter is not unrealistic, since in liquid markets the frequency of market orders and cancellations at
the best bid and ask prices is known to be much higher than the frequency of order placements. In more detail, for all $(i, I) \in\{b, a\} \times\{F, G\}$, we choose

$$
\mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(i, I)\right]=0.25 \quad \text { and } \quad \mathbb{P}^{(i, I)}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)}\right]=\frac{1}{2}+5 \Delta v^{(n)}
$$

where $\mathbb{P}^{(i, I)}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)}\right]:=\mathbb{P}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)} \mid\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(i, I)\right]$. This choice yields the model parameters $\mu^{i, I,(n)}=-2.5$ and $\left(\sigma^{i, I,(n)}\right)^{2}=0.25-n^{-1}\left(\mu^{i, I,(n)}\right)^{2}$ for all $(i, I) \in\{b, a\} \times\{F, G\}$. We present one realization of this balanced cross-border market model in Figure 2.12 .


Figure 2.12: Simulation of bid queues, ask queues, and bid prices of $F$ (orange) and $G$ (turquoise) as well as the capacity process for $n=10,000$ (balanced setting). The white areas represent the active regimes whereas the gray ones represent the inactive regimes.

We observe three regime switches in Figure 2.12; the first around 4,500, the second around 7,000 , and the last around 8,000 time steps. During the active regimes (white areas) the capacity process moves up and down. This is because the frequency of market orders that leads to an increase of the capacity process (i.e. imports to $F$ ) and the frequency of orders that leads to a decrease of the capacity process (i.e. exports from $F$ ) are of comparable size. Moreover, by replicating the above simulation 1, 000 times, we obtain an empirical probability of around 0.53 to observe at least one regime switch for the chosen model parameters. Since we have chosen the frequency of order cancellations higher than the frequency of order placements, we observe several price changes and a comparable high fluctuation of the capacity process. If we change this relation in favor of order placements, we would observe less price changes and fewer fluctuations of the capacity process. This would also reduce the empirical probability to observe a regime switch. The cross-border market model in this simulation might replicate a market situation in which the actual national best bid prices are on a comparable level.

In this setting, the main advantage of the market coupling is therefore the increase of standing volumes which typically leads to fewer price changes.
In a second simulation, we analyze a so-called imbalanced cross-border market in which we choose the frequency of cross-border trades in direction $G$ to $F$ (imports to $F$ ) smaller than the frequency of cross-border trades in direction $F$ to $G$ (exports from $F$ ). In more detail, for all $(i, I) \in\{b, a\} \times\{F, G\}$, we choose $\mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(i, I)\right]=0.25$. Moreover, let $\mathbb{P}^{(i, I)}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)}\right]=0.5+5 \Delta v^{(n)}$ for $(i, I) \in\{(b, F),(a, G)\}$ and $\mathbb{P}^{(i, I)}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)}\right]=0.5$ for $(i, I) \in\{(a, F),(b, G)\}$. This yields the model parameters $\mu^{b, F,(n)}=\mu^{a, G,(n)}=-2.5, \mu^{a, F,(n)}=\mu^{b, G,(n)}=0$, and $\left(\sigma^{i, I,(n)}\right)^{2}=0.25-$ $n^{-1}\left(\mu^{i, I,(n)}\right)^{2}$ for all $(i, I) \in\{b, a\} \times\{F, G\}$. We present one realization of this imbalanced cross-border market model in Figure 2.13.

We observe a single regime switch from the active to the inactive regime around 2,500 time steps. Since we have chosen the frequency of market orders corresponding to possible exports from $F$ higher than the frequency of those corresponding to possible imports to $F$, the dynamics of the capacity process has a higher probability to move downward. Since this imbalance is maintained after the regime switch, the best bid price of $F$ decreases whereas the price of $G$ increases. Note that the bid price of $G$ increases after the regime switch since market orders at the best ask price are more likely than market orders at the best bid price. For a similar reason, the bid price of $F$ decreases after the regime switch. Our chosen model assumptions might replicate a market in which the actual national price in $G$ is much higher than in $F$. Therefore, market participants in $F$ are motivated to sell orders to $G$ for better prices.


Figure 2.13: Simulation of bid and ask queues, bid prices of $F$ (orange) and $G$ (turquoise), and the capacity process for $n=10,000$ (imbalanced setting). The white areas represent the active regimes whereas the gray ones represent the inactive regimes.

Motivated by empirical observations in real-world markets, not only the market
matching mechanism but also the model parameters might change because of the occurrence of a regime switch. Therefore, let us choose in a third simulation the model parameters during the active regimes similarly as before: $\mu^{b, F,(n)}=\mu^{a, G,(n)}=-2.5$, $\mu^{a, F,(n)}=\mu^{b, G,(n)}=0$, and $\left(\sigma^{i, I,(n)}\right)^{2}=0.25-n^{-1}\left(\mu^{i, I,(n)}\right)^{2}$ for all $(i, I) \in\{b, a\} \times$ $\{F, G\}$. In contrast, during the inactive regimes, let us choose $\mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(i, F)\right]=$ 0.1 for $i=b, a, \mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(b, G)\right]=0.3$, and $\mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(a, G)\right]=0.5$. Moreover, we assume $\mathbb{P}^{(i, I)}\left[V_{1}^{i, I,(n)}=-\Delta v^{(n)}\right]=0.5$ for all $(i, I) \in\{(b, F),(a, F),(b, G)\}$ and $\mathbb{P}^{(a, G)}\left[V_{1}^{a, G,(n)}=-\Delta v^{(n)}\right]=0.5+2.5 \Delta v^{(n)}$. This yields the following model parameters during the inactive regimes: $\mu^{b, F,(n)}=\mu^{a, F,(n)}=\mu^{b, G,(n)}=0, \mu^{a, G,(n)}=$ -2.5 , and $\left(\sigma^{b, F,(n)}\right)^{2}=\left(\sigma^{a, F,(n)}\right)^{2}=0.1,\left(\sigma^{b, G,(n)}\right)^{2}=0.3$, and $\left(\sigma^{a, G,(n)}\right)^{2}=0.5-$ $n^{-1}\left(\mu^{a, G,(n)}\right)^{2}$. We present one realization of this imbalanced cross-border market model with a change of the model parameters in Figure 2.14 .

In this last simulation, we observe a single regime switch around 3,000 time steps. While we have chosen the same parameters of the underlying order flow for the active regime as in the second simulation, we changed them in the inactive regime. In particular, the model parameters of the queue size process of $F$ are changed in favor of less order arrivals and price changes during the inactive regimes. For this reason, we observe only a single price change of the best bid price of $F$ during the inactive regime. Moreover, the drift of the ask queue of $G$ is still negative while the drift of the bid queue of $G$ is zero, so that the bid price of $G$ moves upward during the inactive regime. These model assumptions might replicate a market in which the actual national bid price in $G$ is higher than in $F$ and the high number of trading volume in $F$ during the active regime is only caused due to the possibility to sell goods to $G$.


Figure 2.14: Simulation of the bid queues, the ask queue sizes, and the prices of $F$ (orange) and $G$ (turquoise) as well as the capacity process for $n=10,000$ (imbalanced setting, change point in the model parameters). The white areas represent the active regimes whereas the gray ones represent the inactive regimes.

### 2.6.2 Impact of the market coupling on the evolution of bid prices

We study the impact of coupling two markets on the evolution of bid prices in our cross-border market model. For this reason, we simulate the active and inactive dynamics from the same underlying order flow process and compare their empirical mean number of price changes and bid price ranges. We simulate $n=10,000$ time steps and replicate the simulations $m=1,000$ times. The order types are assumed to be independent and have the same probabilities, i.e., $\mathbb{P}\left[\left(\phi_{1}^{(n)}, \psi_{1}^{(n)}\right)=(i, I)\right]=0.25$ for all $(i, I) \in\{b, a\} \times\{F, G\}$.

We analyze four scenarios: in a) we study a completely balanced cross-border market model in which the frequency of all market and limit orders is of comparable size, i.e., $\mu^{i, I}=0$ for all $(i, I) \in\{b, a\} \times\{F, G\}$. Second, we analyze in b) a cross-border market model in which the frequency of market orders leading to possible imports and those leading to possible exports is balanced, but the frequency of market orders at the bid side of the shared order book is higher than on the ask side, i.e., $\mu^{b, F}=\mu^{b, G}=-2.5$ and $\mu^{a, F}=\mu^{a, G}=0$. In scenario c) we analyze a balanced cross-border market in which the frequency of market orders with origin $F$ is higher than the frequency of market orders with origin $G$, i.e., $\mu^{b, F}=\mu^{a, F}=-2.5$ and $\mu^{b, G}=\mu^{a, G}=0$. Finally, we study in d) an imbalanced cross-border market in which the frequency of market orders leading to possible exports from $F$ is higher than the frequency of those orders leading to possible imports to $F$, i.e., $\mu^{b, F}=\mu^{a, G}=-2.5$ and $\mu^{a, F}=\mu^{b, G}=0$.
In the following, let us denote by $N^{\text {shared }}, N^{F}$, and $N^{G}$ the empirical mean number of price changes and by $\mathcal{R}^{\text {shared }}, \mathcal{R}^{F}$, and $\mathcal{R}^{G}$ the empirical mean bid price ranges in the shared limit order book obtained from simulating the dynamics of $\widetilde{S}^{(n)}$, and in the national limit order book of $F$ respectively $G$ obtained from simulating the dynamics of $\widetilde{S}^{(n)}$.

| scenario | $N^{\text {shared }}$ | $N^{F}$ | $N^{G}$ |
| :---: | :---: | :---: | :---: |
| a) $\mu^{b, F}=\mu^{a, F}=\mu^{b, G}=\mu^{a, G}=0$ | 6.88 | 11.91 | 11.86 |
| b) $\mu^{b, F}=\mu^{b, G}=-2.5, \mu^{a, F}=\mu^{a, G}=0$ | 27.91 | 34.99 | 35.36 |
| c) $\mu^{b, F}=\mu^{a, F}=-2.5, \mu^{b, G}=\mu^{a, G}=0$ | 23.39 | 50.34 | 11.72 |
| d) $\mu^{b, F}=\mu^{a, G}=-2.5, \mu^{a, F}=\mu^{b, G}=0$ | 23.6 | 35.19 | 35.31 |

Figure 2.15: The mean number of price changes in different simulations with $n=10,000$ time steps and $m=1,000$ iterations.

We note that $\mathcal{R}^{\text {shared }}, \mathcal{R}^{F}$, and $\mathcal{R}^{G}$ are stated relative to the tick size, i.e., they describe the mean number of ticks between the maximum and minimum bid prices.

| scenario | $\mathcal{R}^{\text {shared }}$ | $\mathcal{R}^{F}$ | $\mathcal{R}^{G}$ |
| :---: | :---: | :---: | :---: |
| a) $\mu^{b, F}=\mu^{a, F}=\mu^{b, G}=\mu^{a, G}=0$ | 3.10 | 4.48 | 4.44 |
| b) $\mu^{b, F}=\mu^{b, G}=-2.5, \mu^{a, F}=\mu^{a, G}=0$ | 18.36 | 15.03 | 15.13 |
| c) $\mu^{b, F}=\mu^{a, F}=-2.5, \mu^{b, G}=\mu^{a, G}=0$ | 6.85 | 10.25 | 4.43 |
| d) $\mu^{b, F}=\mu^{a, G}=-2.5, \mu^{a, F}=\mu^{b, G}=0$ | 6.71 | 14.93 | 15.48 |

Figure 2.16: The mean bid price ranges relative to the tick size in different simulations with $n=10,000$ time steps and $m=1,000$ iterations.

Recall that a price change in the shared order book always yields price changes in both national limit order books. First, we observe that the coupling of the national markets reduces the number of price changes in the shared limit order book (cf. Table 2.15). Except from scenario b) the coupling of the national markets also reduces the mean price ranges (cf. Table 2.16). This can be explained as follows: on the one hand, the coupling of two limit order books always leads to an increase of the order volume at the best bid and ask queues. On the other hand, however, also the drift, the volatility, and the correlation parameters of the cumulative order flow process differ from the parameters of order flow processes corresponding to the national order books. The difference of the model parameters could amplify or cancel out the effects caused due to the increase in trading volume. For this reason, the bid price range in scenario b) is higher in the shared order book than in the national ones. More precisely, even though the order volume has increased due to the coupling, the drifts and volatilities of the cumulative order flow process have doubled. In particular, the drift of the bid side of the shared order book of size -5 compared to the drift of the bid side of the national order books of size -2.5 leads to a more extreme price evolution in the shared order book. In contrast, in the scenarios a) and d), the drifts of the cumulative order flow process equals the drifts of the national order flow processes. Hence, the coupling of the two limit order books leads to a decrease of the number of price changes and of the bid price ranges since the standing volumes at the best bid and ask queues have increased. Finally, in scenario c) in which we have a high imbalance of market and limit orders in favor of market orders in only one of the two countries, we observe a kind of balancing effect, i.e., the number of price changes as well as the bid price range in the shared order book are smaller than in the national order book with the high imbalance but higher than in the balanced one.

We finish this section by studying the effect of coupling two markets using a theoretical result presented in Cont and de Larrard [19]. It describes the conditional distribution of the duration between price changes in a single country limit order book model and is based on the results in [90] on the first exit time of a planar Brownian motion with drift. This theoretical result can also be used to formulate the conditional distribution of the duration between price changes in the shared limit order book: let $\mathcal{Q}^{b}$ and $\mathcal{Q}^{a}$ be the cumulative bid and ask components of the limit of the queue size process of the shared order book. To analyze the market coupling effect on bid price evolution, we must
relate the parameters of the cumulative queues to the model parameters introduced in Assumption 2.2. To this end, we note (cf. Corollary 2.7.1) that

$$
\begin{aligned}
\mu_{b} & =\mu^{b, F}+\mu^{b, G} \\
\mu_{a} & =\mu^{a, F}+\mu^{a, G} \\
\sigma_{b}^{2} & =\left(\sigma^{b, F}\right)^{2}+2 \sigma^{(b, F),(b, G)}+\left(\sigma^{b, G}\right)^{2}, \\
\sigma_{a}^{2} & =\left(\sigma^{a, F}\right)^{2}+2 \sigma^{(a, F),(a, G)}+\left(\sigma^{a, G}\right)^{2} \\
\rho \sigma_{a} \sigma_{b} & =\sigma^{(b, F),(a, F)}+\sigma^{(b, F),(a, G)}+\sigma^{(b, G),(a, F)}+\sigma^{(b, G),(a, G)} .
\end{aligned}
$$

Let the random time $\tau$ denote the duration until the next price change in $\mathcal{Q}=$ $\left(\mathcal{Q}^{b}, \mathcal{Q}^{a}\right)$. Then, the conditional distribution of the duration between price changes is given by

$$
\begin{align*}
\mathbb{P}[\tau & >t \mid \mathcal{Q}(0)=(x, y)] \\
& =\frac{2}{\alpha t} \exp \left(a_{1} x+a_{2} y+a_{t} t-\frac{U}{2 t}\right) \sum_{j=1}^{\infty} \sin \left(\frac{j \pi \theta_{0}}{\alpha}\right) \int_{0}^{\alpha} \sin \left(\frac{j \pi \theta}{\alpha}\right) g_{j}(\theta) d \theta \tag{2.6.1}
\end{align*}
$$

where

$$
\begin{gathered}
U:=\left(1-\rho^{2}\right)^{-1}\left(\frac{x^{2}}{\sigma_{b}^{2}}+\frac{y^{2}}{\sigma_{a}^{2}}-\frac{2 \rho x y}{\sigma_{b} \sigma_{a}}\right), \\
\alpha:=\left\{\begin{array}{ll}
\pi+\arctan \left(-\frac{\sqrt{1-\rho^{2}}}{\rho}\right) & \rho>0 \\
\frac{\pi}{2} & \rho=0 \\
\arctan \left(-\frac{\sqrt{1-\rho^{2}}}{\rho}\right) & \rho<0
\end{array}, \quad \theta_{0}:= \begin{cases}\pi+\arctan \left(\frac{y \sigma_{b} \sqrt{1-\rho^{2}}}{x \sigma_{a}-\rho y \sigma_{b}}\right) & x \sigma_{a}<\rho y \sigma_{b} \\
\frac{\pi}{2} & x \sigma_{a}=\rho y \sigma_{b}, \\
\arctan \left(\frac{y \sigma_{b} \sqrt{1-\rho^{2}}}{x \sigma_{a}-\rho y \sigma_{b}}\right) & x \sigma_{a}>\rho y \sigma_{b}\end{cases} \right. \\
g_{j}(\theta):=\int_{0}^{\infty} r \exp \left(-\frac{r^{2}}{2 t}\right) \exp \left(d_{1} r \sin (\theta-\alpha)-d_{2} r \cos (\theta-\alpha)\right) I_{j \pi / \alpha}\left(\frac{r \sqrt{U}}{t}\right) d r,
\end{gathered}
$$

where $I_{j}$ denotes the $j$-th modified Bessel function of first kind, and

$$
\begin{array}{rlr}
a_{1}:=\frac{\rho \mu_{a} \sigma_{b}-\mu_{b} \sigma_{a}}{\left(1-\rho^{2}\right) \sigma_{b}^{2} \sigma_{a}}, & a_{2}:=\frac{\rho \mu_{b} \sigma_{a}-\mu_{a} \sigma_{b}}{\left(1-\rho^{2}\right) \sigma_{a}^{2} \sigma_{b}} \\
d_{1}:=a_{1} \sigma_{b}+\rho a_{2} \sigma_{a}, & d_{2}:=a_{2} \sigma_{a} \sqrt{1-\rho^{2}}
\end{array}
$$

and

$$
a_{t}=\frac{a_{1}^{2} \sigma_{b}^{2}}{2}+\rho a_{1} a_{2} \sigma_{b} \sigma_{a}+\frac{a_{2}^{2} \sigma_{a}^{2}}{2}+a_{1} \mu_{b}+a_{2} \mu_{a}
$$

At first glance, the survival probability introduced in 2.6.1 gives only little insights
into its dependence on the model parameters and on the market coupling effect. Therefore, let us numerically study its behavior for different model parameters. In the following, let $t=1$ and $x=y=1$.

To analyze the effect of the mean and variance, we assume for simplicity independence between incoming order events. Then, thanks to Assumption 2.2, we have $\sigma_{b}^{2}=1-\sigma_{a}^{2}$. We observe that the survival probability increases with the mean, while the choice of different variances affects the skewness and kurtosis of the curve (cf. Figure 2.17 left). This behavior is not surprising as the bid mean describes the relation between incoming market and limit orders affecting the bid side. A high value indicates that limit orders are more likely than market orders yielding an increase of the bid queue and hence stabilizes the bid price. For fixed mean $\mu_{b}=\mu_{a}=0$, the variance has only little influence on the size of the survival probability and the symmetry of the curve follows from the relation $\sigma_{b}^{2}=1-\sigma_{a}^{2}$ (cf. Figure 2.17 right). In contrast, for a negative bid mean $\mu_{b}$ (negative ask mean $\mu_{a}$ ) the probability decreases (increases) with the bid variance $\sigma_{b}^{2}$, since an increasing bid variance increases activity on the bid side while simultaneously decreasing activity on the ask side.


Figure 2.17: Influence of model parameters on the survival probability. Left: Influence of the mean $\mu_{b}$ for different values of $\sigma_{b}^{2}$. Right: Influence of the variance $\sigma_{b}^{2}$ for different values of $\mu_{b}$ and $\mu_{a}$.

Next, we study the influence of the correlations between incoming order sizes with origin $F$ and $G$ on the survival probability. If we have a non-zero correlation parameter $\rho^{(b, F),(b, G)}$ or $\rho^{(a, F),(a, G)}$, note that the relation $\sigma_{b}^{2}=1-\sigma_{a}^{2}$ is violated as the correlation directly effects the size of the bid/ask variance $\sigma_{b}^{2}$ or $\sigma_{a}^{2}$. In Figure 2.18 (left), we study the case in which $\rho^{(b, F),(b, G)}$ is not necessarily zero, while all other correlation parameters equal zero. Then, a negative (positive) correlation decreases (increases) the bid variance and hence decreases (increases) the activity on the bid side while the activity on the ask side stays unchanged. For this reason, the survival probability decreases with $\rho^{(b, F),(b, G)}$. In Figure 2.18 (right), we study the influence of the correlation parameter $\rho^{(b, F),(a, G)}$ while the other correlation parameters are set to zero. Since the size of $\rho^{(b, F),(a, G)}$ only effects the size of the correlation parameter $\rho$ in the equation for the survival probability,
the relation $\sigma_{b}^{2}=1-\sigma_{a}^{2}$ is again satisfied. Moreover, we observe for different choices of means and variances that the survival probability increases with $\rho^{(b, F),(a, G)}$. Hence, a positive correlation between bid orders with origin $F$ and ask orders with origin $G$ stabilizes the bid price.


Figure 2.18: Influence of model parameters on the survival probability. Left: Influence of $\rho^{(b, F),(b, G)}$ for different values of $\mu_{b}$ and $\mu_{a}$, where $\left(\sigma^{i, I}\right)^{2}=0.25$ for all $(i, I) \in\{b, a\} \times\{F, G\}$. Right: Influence of $\rho^{(b, F),(a, G)}$ for different values of $\mu_{b}, \mu_{a}$, and $\sigma_{b}^{2}$.

### 2.7 Technical details

In the following, we denote by $\Lambda_{T}$ the class of strictly increasing, continuous mappings from $[0, T]$ onto itself. Moreover, let $d_{J_{1}}$ be the distance that induces the Skorokhod $\left(J_{1^{-}}\right)$topology on the space $D\left([0, T], \mathbb{R}^{k}\right)$, for $k \geqslant 1$ (cf. e.g. Billingsley 8 for a detailed definition).

### 2.7.1 A functional central limit theorem for the net order flow process

In this subsection, we state the proof of the functional central limit theorem for the net order flow process $X^{(n)}$, cf. Proposition 2.3.1.

Proof of Proposition 2.3.1. In order to ease notation, we assume that for $(i, I),(j, J) \in$ $\{b, a\} \times\{F, G\}$ we have $\sigma^{(i, I),(j, J),(n)}=0$ whenever $I \neq J$, i.e., we assume independence of the order sizes between different countries. However, our strategy (based on the Cramer-Wold device) can be easily extended to the more general case, where we allow dependencies between all order sizes.

Let $X^{(n)}=\left(X^{b, F,(n)}, X^{a, F,(n)}, X^{b, G,(n)}, X^{a, G,(n)}\right)$ denote the piecewise constant interpolation of the net order flow process (cf. its detailed definition in (2.3.1). Using the Cramer-Wold device, it is sufficient to prove that for $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^{4}$ and a standard

Brownian motion $B$,

$$
\begin{align*}
\left(\alpha X^{b, F,(n)}(t)+\beta X^{a, F,(n)}(t)\right. & \left.+\gamma X^{b, G,(n)}(t)+\delta X^{a, G,(n)}(t)\right)_{t \geqslant 0}  \tag{2.7.1}\\
& \Rightarrow\left(\mu_{*} t+\sigma_{*} B(t)\right)_{t \geqslant 0}
\end{align*}
$$

in the Skorokhod topology on the space $D([0, T], \mathbb{R})$, where the drift and diffusion components of the limit process are given by
$\mu_{*}:=\alpha \mu^{b, F}+\beta \mu^{a, F}+\gamma \mu^{b, G}+\delta \mu^{a, G}$,
$\sigma_{*}^{2}:=\alpha^{2}\left(\sigma^{b, F}\right)^{2}+\beta^{2}\left(\sigma^{a, F}\right)^{2}+\gamma^{2}\left(\sigma^{b, G}\right)^{2}+\delta^{2}\left(\sigma^{a, G}\right)^{2}+2 \alpha \beta \sigma^{(b, F),(a, F)}+2 \gamma \delta \sigma^{(b, G),(a, G)}$.
For all $n \in \mathbb{N}$, we define

$$
W_{k}^{(n)}:=\left(\alpha V_{k}^{b, F,(n)}+\beta V_{k}^{a, F,(n)}+\gamma V_{k}^{b, G,(n)}+\delta V_{k}^{a, G,(n)}\right)
$$

and

$$
\mathcal{W}_{k}^{(n)}:=\sum_{j=1}^{k} W_{j}^{(n)}, \quad \mathcal{W}^{(n)}(t):=\sum_{k=1}^{T_{n}} \mathcal{W}_{k}^{(n)} \mathbb{1}_{\left[t_{k}^{(n)}, t_{k+1}^{(n)}\right)}(t)
$$

Then, for all $t \in[0, T]$, the left hand side of 2.7 .1 equals

$$
\alpha X^{b, F,(n)}(t)+\beta X^{a, F,(n)}(t)+\gamma X^{b, G,(n)}(t)+\delta X^{a, G,(n)}(t)=\mathcal{W}^{(n)}(t)
$$

For all $n \in \mathbb{N},\left(W_{k}^{(n)}, k \geqslant 1\right)$ forms a sequence of stationary random variables thanks to Assumption 2.2. Next, by Assumption 2.2 and Assumption 2.3, observe that

$$
\begin{aligned}
\sum_{k=1}^{\left\lfloor t / \Delta t^{(n)}\right\rfloor} \mathbb{E}\left[W_{k}^{(n)}\right] & =\left\lfloor t / \Delta t^{(n)}\right\rfloor\left(\Delta v^{(n)}\right)^{2}\left(\alpha \mu^{b, F,(n)}+\beta \mu^{a, F,(n)}+\gamma \mu^{b, G,(n)}+\delta \mu^{a, G,(n)}\right) \\
& \rightarrow t\left(\alpha \mu^{b, F}+\beta \mu^{a, F}+\gamma \mu^{b, G}+\delta \mu^{a, G}\right)=t \mu_{*}
\end{aligned}
$$

Now, we want to apply Theorem 19.1 in Billingsley [8, Section 19, p.197] and therefore, we need to show that

$$
\begin{equation*}
\frac{1}{\left(\Delta v^{(n)}\right)^{2}}\left(\operatorname{Var}\left[W_{1}^{(n)}\right]+2 \sum_{k=2}^{T_{n}} \operatorname{Cov}\left[W_{1}^{(n)}, W_{k}^{(n)}\right]\right) \rightarrow \sigma_{*}^{2} \tag{2.7.2}
\end{equation*}
$$

Applying the definition of the $W_{k}^{(n)}$ 's, for $k \geqslant 1$, we have

$$
\begin{aligned}
\operatorname{Cov}\left[W_{1}^{(n)}, W_{k}^{(n)}\right]= & \alpha^{2} \operatorname{Cov}\left[V_{1}^{b, F,(n)}, V_{k}^{b, F,(n)}\right]+\beta^{2} \operatorname{Cov}\left[V_{1}^{a, F,(n)}, V_{k}^{a, F,(n)}\right] \\
& +\gamma^{2} \operatorname{Cov}\left[V_{1}^{b, G,(n)}, V_{k}^{b, G,(n)}\right]+\delta^{2} \operatorname{Cov}\left[V_{1}^{a, G,(n)}, V_{k}^{a, G,(n)}\right] \\
& +\alpha \beta \operatorname{Cov}\left[V_{1}^{b, F,(n)}, V_{k}^{a, F,(n)}\right]+\alpha \beta \operatorname{Cov}\left[V_{1}^{a, F,(n)}, V_{k}^{b, F,(n)}\right] \\
& +\gamma \delta \operatorname{Cov}\left[V_{1}^{b, G,(n)}, V_{k}^{a, G,(n)}\right]+\gamma \delta \operatorname{Cov}\left[V_{1}^{a, G,(n)}, V_{k}^{b, G,(n)}\right]
\end{aligned}
$$

where we used the independence of the order sizes between different countries. Applying Assumption 2.2 i), we conclude

$$
\begin{aligned}
& \operatorname{Var}\left[W_{1}^{(n)}\right]+2 \sum_{k=2}^{T_{n}} \operatorname{Cov}\left[W_{1}^{(n)}, W_{k}^{(n)}\right] \\
& =\left(\Delta v^{(n)}\right)^{2}\left(\alpha^{2}\left(\sigma^{b, F,(n)}\right)^{2}+\beta^{2}\left(\sigma^{a, F,(n)}\right)^{2}+\gamma^{2}\left(\sigma^{b, G,(n)}\right)^{2}+\delta^{2}\left(\sigma^{a, G,(n)}\right)^{2}\right) \\
& \quad+\left(\Delta v^{(n)}\right)^{2}\left(\alpha \beta \sigma^{(b, F),(a, F),(n)}+\gamma \delta \sigma^{(b, G),(a, G),(n)}\right)
\end{aligned}
$$

An application of Assumption 2.2 ii) finally yields

$$
\frac{1}{\left(\Delta v^{(n)}\right)^{2}}\left(\operatorname{Var}\left[W_{1}^{(n)}\right]+2 \sum_{k=2}^{T_{n}} \operatorname{Cov}\left[W_{1}^{(n)}, W_{k}^{(n)}\right]\right) \rightarrow \sigma_{*}^{2}
$$

Now, by Assumption 2.3 we have $\lim _{n \rightarrow \infty} \Delta t^{(n)} /\left(\Delta v^{(n)}\right)^{2}=1$ and therefore, applying Theorem 19.1 in Billingsley 8 , the sequence of processes $\left(\mathcal{W}^{(n)}\right)_{n \geqslant 1}$ converges weakly in the Skorokhod topology to a Brownian motion with drift $\mu_{*}$ and volatility $\sigma_{*}$. In particular, we conclude 2.7.1 finishing the proof.

As a direct consequence, we obtain a functional limit theorem for the cumulative net order flow process $h\left(X^{(n)}\right)$.

Corollary 2.7.1. Under the assumptions of Proposition 2.3.1, the cumulative net order flow process $h\left(X^{(n)}\right)$ converges weakly in the Skorokhod topology to a planar Brownian motion, i.e.,

$$
h\left(X^{(n)}\right) \Rightarrow h(X):=\left(t \hat{\mu}+\hat{\Sigma}^{1 / 2} \hat{B}(t)\right)_{t \geqslant 0}
$$

where $\hat{B}$ is a standard planar Brownian motion and

$$
\hat{\mu}:=\binom{\hat{\mu}^{b}}{\hat{\mu}^{a}}:=\binom{\mu^{(b, F)}+\mu^{(b, G)}}{\mu^{(a, F)}+\mu^{(a, G)}}, \quad \hat{\Sigma}:=\left(\begin{array}{cc}
\hat{\sigma}_{1,1}^{2} & \hat{\sigma}_{1,2} \\
\hat{\sigma}_{1,2} & \hat{\sigma}_{2,2}^{2}
\end{array}\right)
$$

and

$$
\begin{aligned}
& \hat{\sigma}_{1,1}^{2}:=\left(\sigma^{(b, F)}\right)^{2}+2 \sigma^{(b, F),(b, G)}+\left(\sigma^{(b, G)}\right)^{2}, \\
& \hat{\sigma}_{2,2}^{2}:=\left(\sigma^{(a, F)}\right)^{2}+2 \sigma^{(a, F),(a, G)}+\left(\sigma^{(a, G)}\right)^{2}, \\
& \hat{\sigma}_{1,2}:=\sigma^{(b, F),(a, F)}+\sigma^{(b, F),(a, G)}+\sigma^{(b, G),(a, F)}+\sigma^{(b, G),(a, G)} .
\end{aligned}
$$

Proof. Note that the function $h: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ is continuous at $\omega \in$ $C\left([0, T], \mathbb{R}^{4}\right)$ with respect to the Skorokhod topology. Since $X$ is a four-dimensional linear Brownian motion (thanks to Proposition 2.3.1), we conclude that $X$ lies, with probability one, in the continuity set of $h$. Hence, a combination of Proposition 2.3.1 and the continuous mapping theorem yields

$$
h\left(X^{(n)}\right) \Rightarrow h(X)
$$

in Skorokhod the topology on $D\left([0, T], \mathbb{R}^{2}\right)$. Again, by Proposition 2.3.1, we conclude that $h(X)$ is a planar Brownian motion with the stated drift $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$. The positive-definiteness of $\hat{\Sigma}$ can be directly deduced from that of $\Sigma$.

### 2.7.2 Continuity properties of important functions

In this subsection, we describe the continuity sets of the first hitting time map, the function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$, the function $\widetilde{\Psi}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times$ $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$, and some other related functions.

### 2.7.2.1 Continuity of the first hitting time map

In order to prove our convergence results, we frequently apply the continuity of the first hitting time and last value map. Therefore, let us introduce for any $z \in \mathbb{R}$, the maps

$$
\begin{array}{ll}
\tau_{z}: D([0, T], \mathbb{R}) \rightarrow[0, T], & \tau_{z}(\omega):=\inf \{t \geqslant 0: \omega(t) \leqslant z\} \wedge T \\
l_{z}: D([0, T], \mathbb{R}) \rightarrow \mathbb{R}, & l_{z}(\omega):=\omega\left(\tau_{z}(\omega)-\right) \tag{2.7.3}
\end{array}
$$

In contrast to the definition of the first hitting time and last value map in Whitt [86], we allow equality in the definition of the first hitting time map. For this reason, we cannot directly apply the corresponding continuity result in Whitt [86, Theorem 13.6.4] for the first hitting time map with respect to the Skorokhod topology. Let us introduce for some $z \in \mathbb{R}$ the function space $C_{z}^{\prime}([0, T], \mathbb{R})$ in which the continuity of these maps can still be established. We define

$$
\begin{align*}
& C_{z}^{\prime}([0, T], \mathbb{R}) \\
& \quad:=\left\{\omega \in C([0, T], \mathbb{R}): \quad z \notin \omega\left(\left(\tau_{z}^{\prime}(\omega)-\varepsilon, \tau_{z}^{\prime}(\omega)\right)\right) \quad \text { for all } \varepsilon>0\right\} \tag{2.7.4}
\end{align*}
$$

where

$$
\tau_{z}^{\prime}: D([0, T], \mathbb{R}) \rightarrow[0, T], \quad \tau_{z}^{\prime}(\omega):=\inf \{t \geqslant 0: \omega(t)<z\} \wedge T
$$

In words, the functions space $C_{z}^{\prime}([0, T], \mathbb{R})$ contains all continuous functions $\omega$ that are not equal to $z$ throughout the interval $\left(\tau_{z}^{\prime}(\omega)-\varepsilon, \tau_{z}^{\prime}(\omega)\right)$ for any $\varepsilon>0$. Later, we work with functions in its $k$-dimensional version $C_{z}^{\prime}\left([0, T], \mathbb{R}^{k}\right):=\left\{\omega \in C\left([0, T], \mathbb{R}^{k}\right)\right.$ : $\left.\pi_{j}^{(k)} \omega \in C_{z}^{\prime}([0, T], \mathbb{R}), j=1, \cdots, k\right\}$, for $k \geqslant 1$. Moreover, we denote by $C_{z}^{\prime}\left[[0, T], \mathbb{R}^{k} \backslash\right.$ $\{(0, \cdots, 0)\})$ the space containing functions $\omega \in C_{z}^{\prime}\left([0, T], \mathbb{R}^{k}\right)$ avoiding the origin, i.e., $(0, \cdots, 0) \notin \omega([0, T])$.
Lemma 2.7.2 (Continuity of the first hitting time and last value map). Let $z \in$ $\mathbb{R}, \omega \in C_{z}^{\prime}([0, T], \mathbb{R})$, and $\left(\omega_{n}\right)_{n \geqslant 1}$ be a sequence taking values in $D([0, T], \mathbb{R})$ with $d_{J_{1}}\left(\omega, \omega_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. Then, for the maps $\tau_{z}$ and $l_{z}$ in 2.7.3), we have

$$
\left(\tau_{z}\left(\omega_{n}\right), l_{z}\left(\omega_{n}\right)\right) \rightarrow\left(\tau_{z}(\omega), l_{z}(\omega)\right) \quad \text { as } n \rightarrow \infty
$$

Proof. For each $x \in D([0, T], \mathbb{R})$, let $\tau_{z}^{\prime}(x):=\inf \{t \geqslant 0: x(t)<z\} \wedge T$ and $l_{z}^{\prime}(x):=$ $x\left(\tau_{z}^{\prime}(x)-\right)$. Since $\omega \in C_{z}^{\prime}([0, T], \mathbb{R})$ and convergence in the Skorokhod $J_{1}$-topology implies convergence in the Skorokhod $M_{2}$-topology, we can apply Theorem 13.6.4 in Whitt 86 and conclude that

$$
\tau_{z}^{\prime}\left(\omega_{n}\right) \rightarrow \tau_{z}^{\prime}(\omega), \quad l_{z}^{\prime}\left(\omega_{n}\right) \rightarrow l_{z}^{\prime}(\omega)
$$

as $n \rightarrow \infty$. Moreover, since $\omega \in C_{z}^{\prime}([0, T], \mathbb{R})$, we have $\tau_{z}^{\prime}(\omega)=\tau_{z}(\omega)$ and $l_{z}^{\prime}(\omega)=l_{z}(\omega)$ for $\tau_{z}$ and $l_{z}$ defined in 2.7.3). Hence, $\left(\tau_{z}\left(\omega_{n}\right), l_{z}\left(\omega_{n}\right)\right) \rightarrow\left(\tau_{z}(\omega), l_{z}(\omega)\right)$ as $n \rightarrow \infty$.

Next, let us introduce maps $\tau_{1}, \tau_{2}, \tau: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ by setting

$$
\tau_{1}(\omega):=\inf \left\{t \geqslant 0: \pi_{1}^{(2)} \omega(t) \leqslant 0\right\} \wedge T, \quad \tau_{2}(\omega):=\inf \left\{t \geqslant 0: \pi_{2}^{(2)} \omega(t) \leqslant 0\right\} \wedge T
$$

and $\tau(\omega):=\tau_{1}(\omega) \wedge \tau_{2}(\omega)$. The function $\tau$ defines the first hitting time of $\omega \in$ $D\left([0, T], \mathbb{R}^{2}\right)$ of the axes $\{(0, y): y>0\} \cup\{(x, 0): x \geqslant 0\}$.
Lemma 2.7.3. For $i=1,2$, the map

$$
\begin{aligned}
H_{i}:(D([0, T], & \left.\left.\mathbb{R}^{2}\right), d_{J_{1}}\right) \rightarrow \mathbb{R}, \\
\omega & \mapsto \mathbb{1}_{\left\{\tau(\omega)=\tau_{i}(\omega)\right\}}
\end{aligned}
$$

is continuous on the set $\left\{\omega \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right), \tau(\omega)<T\right\}$.
Proof. We prove the stated result for the function $H_{1}$. When $\tau(\omega)<T$, the fact that $H_{1}(\omega)=1$ indicates that $\omega$ first hits the $x$-axis. Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\omega_{0} \notin B_{2}(0,1 / n):=\left\{x \in \mathbb{R}^{2}:\|x\|<\right.$ $1 / n\}$. Let $\varepsilon>0$. Moreover, let $\omega^{\prime} \in D\left([0, T], \mathbb{R}^{2}\right)$ with $d_{J_{1}}\left(\omega_{0}, \omega^{\prime}\right)<\varepsilon$. In particular, there exists $\lambda \in \Lambda_{T}$ such that $\left\|\omega_{0} \circ \lambda-\omega^{\prime}\right\|_{\infty}<\varepsilon$ and $\|\lambda-\mathrm{id}\|_{\infty}<\varepsilon$. Furthermore, assume that $\varepsilon+\eta_{\omega_{0}}(\varepsilon)+\eta_{\omega_{0} \circ \lambda}(\varepsilon)<1 / n$, where $\eta_{\omega}$ defines the modulus of continuity of $\omega$, cf. equation (7.1) in Billingsley [8]. Now, we want to show that for all such $\varepsilon$, $\omega^{\prime} \in D\left([0, T], \mathbb{R}^{2}\right)$, and $\lambda \in \Lambda_{T}$, we have

$$
\mathbb{1}_{\left\{\tau\left(\omega_{0}\right)=\tau_{1}\left(\omega_{0}\right)\right\}}=\mathbb{1}_{\left\{\tau\left(\omega^{\prime}\right)=\tau_{1}\left(\omega^{\prime}\right)\right\}} .
$$

Without loss of generality, by the continuity of the first hitting time map $\tau$ with respect to the Skorokhod topology (cf. Lemma 2.7.2), we might assume that also $\left|\tau\left(\omega_{0}\right)-\tau\left(\omega^{\prime}\right)\right|<\varepsilon$. Next, we have that

$$
\begin{aligned}
& \left\|\omega_{0}\left(\tau\left(\omega_{0}\right)\right)-\omega^{\prime}\left(\tau\left(\omega^{\prime}\right)\right)\right\| \\
& \leqslant\left\|\omega_{0} \circ \lambda-\omega^{\prime}\right\|_{\infty}+\left\|\omega_{0} \circ \lambda\left(\tau\left(\omega_{0}\right)\right)-\omega_{0}\left(\tau\left(\omega_{0}\right)\right)\right\|+\left\|\omega_{0} \circ \lambda\left(\tau\left(\omega^{\prime}\right)\right)-\omega_{0} \circ \lambda\left(\tau\left(\omega_{0}\right)\right)\right\| \\
& \leqslant \varepsilon+\eta_{\omega_{0}}(\varepsilon)+\eta_{\omega_{0} \circ \lambda}(\varepsilon) .
\end{aligned}
$$

Since $\varepsilon+\eta_{\omega_{0}}(\varepsilon)+\eta_{\omega_{0} \circ \lambda}(\varepsilon)<1 / n$ and $\omega_{0} \in C^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{0,0\}\right)$, we finally conclude for all $n \geqslant N$ that $\mathbb{1}_{\left\{\tau\left(\omega_{0}\right)=\tau_{1}\left(\omega_{0}\right)\right\}}=\mathbb{1}_{\left\{\tau\left(\omega^{\prime}\right)=\tau_{1}\left(\omega^{\prime}\right)\right\}}$.

### 2.7.2.2 Continuity of the function $g$

Let us first analyze for $k \in \mathbb{N}$ the functions $\hat{\tau}_{k}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ given in Definition 2.3.2 for $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$, we can write $\hat{\tau}_{1}(\omega)=\hat{\tau}_{1,1}(\omega) \wedge \hat{\tau}_{1,2}(\omega)$, where

$$
\begin{equation*}
\hat{\tau}_{1, i}(\omega):=\inf \left\{t \geqslant 0: \pi_{i}^{(2)} \omega(t) \leqslant 0\right\} \wedge T \quad \text { for } i=1,2 \tag{2.7.5}
\end{equation*}
$$

For $k \geqslant 2$ with $\widetilde{g}_{k-1}(\omega)\left(\hat{\tau}_{k-1}(\omega)\right) \notin \mathbb{R}_{-}^{2}$, we have $\hat{\tau}_{k}(\omega):=\hat{\tau}_{k, 1}(\omega) \mathbb{1}\left(\hat{\tau}_{k-1}(\omega)=\right.$ $\left.\hat{\tau}_{k-1,2}(\omega)\right)+\hat{\tau}_{k, 2}(\omega) \mathbb{1}\left(\hat{\tau}_{k-1}(\omega)=\hat{\tau}_{k-1,1}(\omega)\right)$ and

$$
\begin{equation*}
\hat{\tau}_{k, i}(\omega):=\inf \left\{t \geqslant \hat{\tau}_{k-1}(\omega): \pi_{i}^{(2)} \widetilde{g}_{k}(\omega)(t) \leqslant 0\right\} \wedge T \quad \text { for } i=1,2 . \tag{2.7.6}
\end{equation*}
$$

We recall the definition of $\tau: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ introduced in Lemma 2.3.4 i.e.,

$$
\tau(\omega):=\inf \{t \geqslant 0: g(\omega)(t)=(0,0)\} \wedge T=\inf \left\{t \geqslant 0: h_{1}(\omega)(t) \leqslant 0\right\} \wedge T
$$

and denote $\|\omega\|_{[a, b]}:=\left\|\left.\omega\right|_{[a, b]}\right\|_{\infty}$ for any $[a, b] \subset[0, T]$ and $\omega \in D\left([0, T], \mathbb{R}^{k}\right), k \geqslant 1$.
Lemma 2.7.4 (Continuity of the $\widetilde{g}_{k}$ 's, $g_{k}$ 's and $\hat{\tau}_{k}$ 's). Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ satisfy condition (I) with $h_{1}\left(\omega_{0}\right) \in C_{0}^{\prime}([0, T], \mathbb{R})$. Then, for $k \in \mathbb{N}$ the functions $\widetilde{g}_{k}, g_{k}$ : $D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ and $\hat{\tau}_{k}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ are continuous at $\omega_{0}$.

Proof. Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ and $h_{1}\left(\omega_{0}\right) \in C_{0}^{\prime}([0, T], \mathbb{R})$. Denote $K:=$ $K_{1} \vee K_{2}$, where $K_{1}:=\left\|h_{1}\left(\omega_{0}\right)\right\|_{\infty}<\infty$ and $K_{2}:=\left\|\omega_{0}\right\|_{\infty}<\infty$. Let $\varepsilon>0$. Moreover, let $\omega^{\prime} \in D\left([0, T], \mathbb{R}^{2}\right)$ and $\lambda \in \Lambda_{T}$ be such that $\left\|\omega_{0}-\omega^{\prime} \circ \lambda\right\|_{\infty}<\delta$ and $\| \lambda$-id $\|_{\infty}<\delta$ for some $\delta>0$. Since $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, the function $h_{1}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D([0, T], \mathbb{R})$ is continuous at $\omega_{0}$ with respect to the Skorokhod topology. Hence, there exists $\delta_{1}>0$ such that for all $\delta \leqslant \delta_{1}$, we have

$$
\left\|h_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant 2 K \quad \text { and } \quad\left\|\omega^{\prime} \circ \lambda\right\|_{\infty} \leqslant 2 K
$$

Studying Definition 2.3.2, we conclude for $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$ and all $k \in \mathbb{N}$ that $\left\|g_{k}(\omega)\right\|_{\left[0, \hat{\tau}_{k}(\omega)\right)}=\left\|\widetilde{g}_{k}(\omega)\right\|_{\left[0, \hat{\epsilon}_{k}(\omega)\right)} \leqslant\left\|h_{1}(\omega)\right\|_{\infty}$. Let us perform an induction over $k \geqslant 1$.

Induction start: Let $k=1$. By Definition 2.3.2, we have $g_{1}(\omega)=\widetilde{g}_{1}(\omega)=\omega$ for $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$. Hence, for all $\delta \leqslant \varepsilon$, we conclude that $\left\|g_{1}\left(\omega_{0}\right)-g_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}=$
$\left\|\widetilde{g}_{1}\left(\omega_{0}\right)-\widetilde{g}_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}=\left\|\omega_{0}-\omega^{\prime} \circ \lambda\right\|_{\infty}<\varepsilon$. By construction, $\hat{\tau}_{1}(\omega)$ defines the first hitting time map of $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$ of the axes. Since $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, we conclude that $\hat{\tau}_{1}$ is continuous at $\omega_{0}$ (cf. Lemma $\sqrt{2.7 .2}$ ) and for $\delta$ small enough, applying Lemma 2.7.3, we further have

$$
\mathbb{1}_{\left\{\hat{\tau}_{1}\left(\omega_{0}\right)=\hat{\tau}_{1, i}\left(\omega_{0}\right)\right\}}=\mathbb{1}_{\left\{\hat{\tau}_{1}\left(\omega^{\prime} \circ \lambda\right)=\hat{\tau}_{1, i}\left(\omega^{\prime} \circ \lambda\right)\right\}} \quad \text { for } i=1,2,
$$

where the functions $\left(\hat{\tau}_{k, 1}\right)_{k \geqslant 1}$ and $\left(\hat{\tau}_{k, 2}\right)_{k \geqslant 1}$ are introduced in (2.7.5) and 2.7.6).
Induction hypothesis: For $k=2, \cdots, j-1$ with $\hat{\tau}_{k}\left(\omega_{0}\right)<T$ the functions $\widetilde{g}_{k}, g_{k}$ : $D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ and $\hat{\tau}_{k}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow[0, T]$ are continuous at $\omega_{0}$, and there exists a $\delta^{\prime}>0$ such that for all $\delta \leqslant \delta^{\prime}$, we have

$$
\mathbb{1}_{\left\{\hat{\tau}_{k}\left(\omega_{0}\right)=\hat{\tau}_{k, i}\left(\omega_{0}\right)\right\}}=\mathbb{1}_{\left\{\hat{\tau}_{k}\left(\omega^{\prime} \circ \lambda\right)=\hat{\tau}_{k, i}\left(\omega^{\prime} \circ \lambda\right)\right\}} \quad \text { for } i=1,2 .
$$

Induction step: Let $k=j$. By the induction hypothesis, the functions $g_{j-1}$ and $\hat{\tau}_{j-1}$ are continuous at $\omega_{0}$. Hence, there exists a $\delta_{2}>0$ such that for all $\delta \leqslant \delta_{2}$, we have $\left\|g_{j-1}\left(\omega_{0}\right)-g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \frac{\varepsilon}{6}$. By the definition of $\widetilde{g}_{j}$ and the Lipschitz property of the reflection map at zero $t \mapsto \ell_{t}(x):=\sup _{s \leqslant t}(-x(s))^{+}$for $x \in D([0, T], \mathbb{R})$ (cf. e.g. Lemma 13.5.1 in Whitt [86]), we conclude

$$
\begin{aligned}
& \left\|\pi_{1}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right)-\pi_{1}^{(2)} \widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \\
& \leqslant\left\|\pi_{1}^{(2)} g_{j-1}\left(\omega_{0}\right)-\pi_{1}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|\ell .\left(\pi_{1}^{(2)} g_{j-1}\left(\omega_{0}\right)\right)-\ell .\left(\pi_{1}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right)\right\|_{\infty} \\
& \quad \quad+\left\|\ell .\left(\pi_{2}^{(2)} g_{j-1}\left(\omega_{0}\right)\right)-\ell .\left(\pi_{2}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right)\right\|_{\infty} \\
& \leqslant 2\left\|\pi_{1}^{(2)} g_{j-1}\left(\omega_{0}\right)-\pi_{1}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|\pi_{2}^{(2)} g_{j-1}\left(\omega_{0}\right)-\pi_{2}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} .
\end{aligned}
$$

Deriving a similar bound for $\left\|\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right)-\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}$, we conclude that

$$
\begin{aligned}
& \left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \\
& \leqslant\left\|\pi_{1}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right)-\pi_{1}^{(2)} \widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right)-\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \\
& \leqslant 3\left(\left\|\pi_{1}^{(2)} g_{j-1}\left(\omega_{0}\right)-\pi_{1}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|\pi_{2}^{(2)} g_{j-1}\left(\omega_{0}\right)-\pi_{2}^{(2)} g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}\right) \\
& \leqslant 6\left\|g_{j-1}\left(\omega_{0}\right)-g_{j-1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \varepsilon .
\end{aligned}
$$

This proves the continuity of $\tilde{g}_{j}$ at $\omega_{0}$. In the following, let $\delta_{3}>0$ be such that for all $\delta \leqslant \delta_{3}$, we have $\left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \frac{\varepsilon}{8}$. Next, without loss of generality, assume that $\hat{\tau}_{j-1}\left(\omega_{0}\right)=\hat{\tau}_{j-1,1}\left(\omega_{0}\right)$. Now, by the induction hypothesis, for all $\delta \leqslant \delta^{\prime}$, we also have $\hat{\tau}_{j-1}\left(\omega^{\prime} \circ \lambda\right)=\hat{\tau}_{j-1,1}\left(\omega^{\prime} \circ \lambda\right)$. By Definition 2.3.2, we conclude for $\omega \in\left\{\omega_{0}, \omega^{\prime} \circ \lambda\right\}$ that

$$
g_{j}(\omega)(t)=\widetilde{g}_{j}(\omega)\left(t \wedge \hat{\tau}_{j}(\omega)-\right)+\omega(t)-\omega\left(t \wedge \hat{\tau}_{j}(\omega)-\right) \quad \text { for } t \in[0, T],
$$

where

$$
\tilde{g}_{j}(\omega)(t)=g_{j-1}(\omega)(t)+\ell_{t}^{(2)}\left(g_{j-1}(\omega)\right) R \quad \text { for } t \in[0, T]
$$

and $\hat{\tau}_{j}(\omega)=\hat{\tau}_{j, 2}(\omega)$. Hence, for all $\delta \leqslant \delta^{\prime}$, we have

$$
\mathbb{1}_{\left\{\hat{\tau}_{j}\left(\omega_{0}\right)=\hat{\tau}_{j, i}\left(\omega_{0}\right)\right\}}=\mathbb{1}_{\left\{\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)=\hat{\tau}_{j, i}\left(\omega^{\prime} \circ \lambda\right)\right\}} \quad \text { for } i=1,2
$$

If $\hat{\tau}_{j}\left(\omega_{0}\right)<T$, then $\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right) \in C_{0}^{\prime}\left(\left[\hat{\tau}_{j-1}\left(\omega_{0}\right), \hat{\tau}_{j}\left(\omega_{0}\right)\right], \mathbb{R}\right)$ and $\pi_{2}^{(2)} \widetilde{g}_{j}\left(\omega_{0}\right)>0$ on $\left[\hat{\tau}_{j-1}\left(\omega_{0}\right), \hat{\tau}_{j}\left(\omega_{0}\right)\right)$. Since $\hat{\tau}_{j}\left(\omega_{0}\right)=\hat{\tau}_{j, 2}\left(\omega_{0}\right)$ and applying Lemma 2.7.2, we conclude that $\hat{\tau}_{j}$ is continuous at $\omega_{0}$. In contrast, if $\hat{\tau}_{j}\left(\omega_{0}\right)=\tau\left(\omega_{0}\right)=T$ and since $h_{1}\left(\omega_{0}\right) \in$ $C_{0}^{\prime}([0, T], \mathbb{R})$, we also conclude the continuity of $\hat{\tau}_{j}$ at $\omega_{0}$ by the continuity of the first hitting time map $\tau$ at $\omega_{0}$ (cf. Lemma 2.7.2). Hence, there exists a $\delta_{4}>0$ such that for all $\delta \leqslant \delta_{4} \wedge \delta^{\prime}$, we also have $\left|\hat{\tau}_{j}\left(\omega_{0}\right)-\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)\right| \leqslant \frac{\varepsilon}{30 K}$. Analyzing the cases $\hat{\tau}_{j}\left(\omega_{0}\right) \leqslant \hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)$ and $\hat{\tau}_{j}\left(\omega_{0}\right)>\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)$ separately and applying that $\left\|\widetilde{g}_{j}(\omega)\right\|_{\left[0, \hat{\tau}_{j}(\omega)\right)} \leqslant\left\|h_{1}(\omega)\right\|_{\infty}$ for $\omega \in D([0, T], \mathbb{R})$, we get

$$
\begin{aligned}
& \left\|\widetilde{g}_{j}\left(\omega_{0}\right)\left(\hat{\tau}_{j}\left(\omega_{0}\right)-\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\left(\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)-\right)\right\| \\
& \quad \leqslant\left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+2 K\left|\hat{\tau}_{j}\left(\omega_{0}\right)-\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)\right| .
\end{aligned}
$$

Finally, for all $\delta \leqslant \delta_{1} \wedge \delta_{3} \wedge \delta_{4} \wedge \delta^{\prime}$, we observe that

$$
\begin{aligned}
& \left\|g_{j}\left(\omega_{0}\right)-g_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \\
& \leqslant\left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[0, \hat{\tau}_{j}\left(\omega_{0}\right) \wedge \hat{\wedge}_{j}\left(\omega^{\prime} \circ \lambda\right)\right)} \\
& +\left\|\widetilde{g}_{j}\left(\omega_{0}\right)\left(\hat{\tau}_{j}\left(\omega_{0}\right)-\right)+\omega_{0}-\omega_{0}\left(\hat{\tau}_{j}\left(\omega_{0}\right)-\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[\hat{\tau}_{j}\left(\omega_{0}\right) \wedge \hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right), \hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)\right)} \\
& +\left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\left(\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)-\right)-\omega^{\prime} \circ \lambda+\omega^{\prime} \circ \lambda\left(\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)-\right)\right\|_{\left[\hat{\tau}_{j}\left(\omega_{0}\right) \wedge \hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right), \hat{\tau}_{j}\left(\omega_{0}\right)\right)} \\
& +\| \widetilde{g}_{j}\left(\omega_{0}\right)\left(\hat{\tau}_{j}\left(\omega_{0}\right)-\right)+\omega_{0}-\omega_{0}\left(\hat{\tau}_{j}\left(\omega_{0}\right)-\right) \\
& \quad-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\left(\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)-\right)-\omega^{\prime} \circ \lambda+\omega^{\prime} \circ \lambda\left(\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)-\right) \|_{\left[\hat{\tau}_{j}\left(\omega_{0}\right) \vee \hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right), T\right]} \\
& \leqslant 2\left\|\omega_{0}-\omega^{\prime} \circ \lambda\right\|_{\infty}+2\left\|\widetilde{g}_{j}\left(\omega_{0}\right)-\widetilde{g}_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+15 K\left|\hat{\tau}_{j}\left(\omega_{0}\right)-\hat{\tau}_{j}\left(\omega^{\prime} \circ \lambda\right)\right| \\
& \leqslant 2 \delta+\frac{3 \varepsilon}{4} .
\end{aligned}
$$

Now, if $\delta \leqslant \delta_{1} \wedge \delta_{3} \wedge \delta_{4} \wedge \delta^{\prime} \wedge \varepsilon / 8$, we conclude that $\left\|g_{j}\left(\omega_{0}\right)-g_{j}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \varepsilon$. This proves the induction and finishes the proof.

Proof of Lemma 2.3.5. Let $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$ and $h_{1}\left(\omega_{0}\right) \in C_{0}^{\prime}([0, T], \mathbb{R})$. Let us first assume, that there exists a finite $\kappa:=\kappa\left(\omega_{0}\right) \in \mathbb{N}$ such that $\hat{\tau}_{\infty}\left(\omega_{0}\right)=\hat{\tau}_{\kappa}\left(\omega_{0}\right)=T$ and $g\left(\omega_{0}\right)(T) \in(0, \infty)^{2}$. By Definition 2.3.2, we have that $g\left(\omega_{0}\right)(t)=\widetilde{g}_{\kappa}\left(\omega_{0}\right)(t)$ for all $t \in[0, T]$ and $\kappa$ is continuous at $\omega_{0}$. Then, the continuity of $g$ at $\omega_{0}$ follows from Lemma 2.7.4,
In contrast, assume that $\hat{\tau}_{k}\left(\omega_{0}\right)<\hat{\tau}_{\infty}\left(\omega_{0}\right)<T$ for all $k \in \mathbb{N}$ and that $\hat{\tau}_{\infty}\left(\omega_{0}\right)=$ $\tau\left(\omega_{0}\right)$. Moreover, let $\varepsilon>0$ and $K:=K_{1} \vee K_{2}$, where $K_{1}:=\left\|h_{1}\left(\omega_{0}\right)\right\|_{\infty}<\infty$ and $K_{2}:=\left\|\omega_{0}\right\|_{\infty}<\infty$. Now, we can choose $\kappa:=\kappa\left(\omega_{0}\right)$ such that $\left|\hat{\tau}_{\kappa}\left(\omega_{0}\right)-\hat{\tau}_{\infty}\left(\omega_{0}\right)\right|<\frac{\varepsilon}{12 K}$. Moreover, let $\omega^{\prime} \in D\left([0, T], \mathbb{R}^{2}\right)$ and $\lambda \in \Lambda_{T}$ be such that

$$
\left\|\omega_{0}-\omega^{\prime} \circ \lambda\right\|_{\infty}<\delta \quad \text { and } \quad\|\lambda-\mathrm{id}\|_{\infty}<\delta,
$$

for $\delta>0$ small enough. Since $\omega_{0} \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, the map $h_{1}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow$ $D([0, T], \mathbb{R})$ is continuous at $\omega_{0}$. Hence, there exists a $\delta_{1}>0$ such that for all $\delta \leqslant \delta_{1}$, we have

$$
\left\|h_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant 2 K \quad \text { and } \quad\left\|\omega^{\prime} \circ \lambda\right\|_{\infty} \leqslant 2 K .
$$

Moreover, by Definition 2.3.2, observe that for all $\omega \in D\left([0, T], \mathbb{R}^{2}\right)$ and $k \in \mathbb{N}$, we have $\|g(\omega)\|_{\infty} \leqslant\left\|h_{1}(\omega)\right\|_{\infty}$ and $\left\|g_{k}(\omega)\right\|_{\left[0, \hat{\tau}_{k}(\omega)\right)} \leqslant\left\|h_{1}(\omega)\right\|_{\infty}$. Next, since $h_{1}\left(\omega_{0}\right) \in$ $C_{0}^{\prime}([0, T], \mathbb{R}), \hat{\tau}_{\infty} \equiv \tau$ is continuous at $\omega_{0}$ (cf. Lemma 2.7.2). Hence, there exists a $\delta_{2}>0$ such that for all $\delta \leqslant \delta_{2}$, we have $\left|\hat{\tau}_{\infty}\left(\omega_{0}\right)-\hat{\tau}_{\infty}\left(\omega^{\prime} \circ \lambda\right)\right|<\frac{\varepsilon}{12 K}$. Then, for all $\delta \leqslant \delta_{1} \wedge \delta_{2}$, we conclude

$$
\begin{aligned}
& \left\|g\left(\omega_{0}\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[\hat{\tau}_{\kappa}\left(\omega_{0}\right), T\right]} \\
& \quad \leqslant\left\|g\left(\omega_{0}\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[\hat{\tau}_{\kappa}\left(\omega_{0}\right), \hat{\tau}_{\infty}\left(\omega_{0}\right)\right)}+\left\|g\left(\omega_{0}\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[\hat{\tau}_{\infty}\left(\omega_{0}\right), \hat{\tau}_{\infty}\left(\omega_{0}\right) \vee \hat{\tau}_{\infty}\left(\omega^{\prime} \circ \lambda\right)\right]} \\
& \quad \leqslant\left(\left\|h_{1}\left(\omega_{0}\right)\right\|_{\infty}+\left\|h_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}\right)\left(\left|\hat{\tau}_{\kappa}\left(\omega_{0}\right)-\hat{\tau}_{\infty}\left(\omega_{0}\right)\right|+\left|\hat{\tau}_{\infty}\left(\omega_{0}\right)-\hat{\tau}_{\infty}\left(\omega^{\prime} \circ \lambda\right)\right|\right) \\
& \quad \leqslant 3 K\left(\frac{\varepsilon}{12 K}+\frac{\varepsilon}{12 K}\right)=\frac{\varepsilon}{2} .
\end{aligned}
$$

Next, applying Lemma 2.7.4, there exists a $\delta_{3}>0$ such that for all $\delta \leqslant \delta_{3}$, we have $\left\|g_{\kappa}\left(\omega_{0}\right)-g_{\kappa}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \frac{\varepsilon}{4}$ and $\left|\hat{\tau}_{\kappa}\left(\omega_{0}\right)-\hat{\tau}_{\kappa}\left(\omega^{\prime} \circ \lambda\right)\right| \leqslant \frac{\varepsilon}{32 K}$. Then, for all $\delta \leqslant \delta_{1} \wedge \delta_{3}$, we conclude

$$
\begin{aligned}
& \left\|g\left(\omega_{0}\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[0, \hat{\tau}_{\kappa}\left(\omega_{0}\right)\right)} \\
& \quad \leqslant\left\|g_{\kappa}\left(\omega_{0}\right)-g_{\kappa}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|g_{\kappa}\left(\omega^{\prime} \circ \lambda\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\left[\hat{\kappa}_{\kappa}\left(\omega_{0}\right) \wedge \hat{\tau}_{\kappa}\left(\omega^{\prime} \circ \lambda\right), \hat{\tau}_{\kappa}\left(\omega_{0}\right)\right)} \\
& \quad \leqslant \frac{\varepsilon}{4}+2\left(\left\|h_{1}\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty}+\left\|\omega^{\prime} \circ \lambda\right\|_{\infty}\right)\left|\hat{\tau}_{\kappa}\left(\omega_{0}\right)-\hat{\tau}_{\kappa}\left(\omega^{\prime} \circ \lambda\right)\right| \\
& \quad \leqslant \frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2} .
\end{aligned}
$$

Finally, choosing $\delta \leqslant \delta_{1} \wedge \delta_{2} \wedge \delta_{3}$, we conclude that $\left\|g\left(\omega_{0}\right)-g\left(\omega^{\prime} \circ \lambda\right)\right\|_{\infty} \leqslant \varepsilon$.
We finish this section by showing that a planar Brownian motion $\mathbb{P}$-almost surely satisfy the conditions of Lemma 2.3.5. In the following, let $W=\left(W_{1}, W_{2}\right)$ be a planar Brownian motion starting in $x \in \mathbb{R}_{+}^{2} \backslash\{(0,0)\}$ with mean $\mu \in \mathbb{R}^{2}$ and covariance matrix $\Sigma \in \mathbb{R}^{2 \times 2}$, whose inverse $\Sigma^{-1} \in \mathbb{R}^{2 \times 2}$ exists. In particular, its components $W_{1}$ and $W_{2}$ are not degenerate and not perfectly correlated.

Lemma 2.7.5. Let $W$ be a planar Brownian motion as defined above. Then, the paths of $W$ satisfy condition (I) $\mathbb{P}$-almost surely.

Proof. Let $W=\left(W_{1}, W_{2}\right)$ be a planar Brownian motion starting in $x=\left(x_{1}, x_{2}\right) \in$ $\mathbb{R}_{+}^{2} \backslash\{(0,0)\}$ with drift $\mu \in \mathbb{R}^{2}$ and covariance matrix

$$
\Sigma:=\left(\begin{array}{cc}
\sigma_{1}^{2} & \rho \sigma_{1} \sigma_{2} \\
\rho \sigma_{1} \sigma_{2} & \sigma_{2}^{2}
\end{array}\right)
$$

where $|\rho|<1$ and $\sigma_{1}, \sigma_{2}>0$. Let us denote $\hat{\tau}_{k}:=\hat{\tau}_{k}(W), \hat{\tau}_{\infty}:=\hat{\tau}_{\infty}(W)$, and
$\tau:=\tau(W):=\inf \left\{t \geqslant 0: h_{1}(W)(t) \leqslant 0\right\} \wedge T$. Thanks to Lemma 2.7.6, we have $\hat{\tau}_{\infty}=\tau$.
First, we will show on the event $\left\{\hat{\tau}_{\infty}<T\right\}$ that $\mathbb{P}$-almost surely $\hat{\tau}_{k}<\hat{\tau}_{\infty}$ for all $k \in \mathbb{N}$. Since $W$ has, with probability one, sample paths in $C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, we conclude that $\hat{\tau}_{1}<\tau \mathbb{P}$-almost surely. Thanks to Theorem 2.3 .26 , the process $g(W)$ is a Markov process. Hence, we can restrict our analysis to the interval $\left[\hat{\tau}_{1}, \hat{\tau}_{2}\right]$ and show that $\mathbb{P}$-almost surely $\hat{\tau}_{2}<\tau$. Let us concentrate on the event $\left\{\hat{\tau}_{1}=\hat{\tau}_{1,1}\right\}$, i.e., $W$ first hits the $x$-axis and note that on the event $\left\{\hat{\tau}_{1}=\hat{\tau}_{1,2}\right\}$ we can argue completely analogously. On this event, observe that $\left(\pi_{1}^{(2)} g(W), W_{1}, \frac{1}{2} L .\left(\pi_{1}^{(2)} g(W)\right)\right)$ solves the Skorokhod problem on $\left[0, \hat{\tau}_{2}\right.$ ].

Now, let $\left(Y_{1}, B_{1}\right)$ be a weak solution, unique in law, of the Tanaka SDE (with volatility $\sigma_{1}$ not necessarily equal to one) given by

$$
d Y_{1}(t)=\operatorname{sign}\left(Y_{1}(s)\right) d B_{1}(s), \quad Y_{1}(0)=B_{1}(0)=x_{1}
$$

We have $Y_{1} \simeq B_{1}$ and by Tanaka's formula (cf. e.g. [89, Proposition 2.11]), we obtain

$$
\begin{aligned}
\left|Y_{1}(t)\right| & =x_{1}+\int_{0}^{t} \operatorname{sign}\left(Y_{1}(s)\right) d Y_{1}(s)+L_{t}\left(Y_{1}\right) \\
& =x_{1}+\int_{0}^{t}\left(\operatorname{sign}\left(Y_{1}(s)\right)\right)^{2} d B_{1}(s)+L_{t}\left(Y_{1}\right) \\
& =B_{1}(t)+L_{t}\left(Y_{1}\right)
\end{aligned}
$$

In particular, $\left(\left|Y_{1}\right|, B_{1}, L .\left(Y_{1}\right)\right)$ solves the one-dimensional Skorokhod problem. Next, let us consider another Brownian motion $B_{1}^{\perp}$, possibly on a larger probability space, such that $B_{1}^{\perp}$ is independent of $\left(Y_{1}, B_{1}\right)$, starts in $\frac{\sigma_{1}}{\sigma_{2} \sqrt{1-\rho^{2}}} x_{2}-\frac{\rho}{\sqrt{1-\rho^{2}}} x_{1}$, and has volatility $\sigma_{1}$. Let us introduce $B_{2}:=\frac{\sigma_{2}}{\sigma_{1}}\left(\rho B_{1}+\sqrt{1-\rho^{2}} B_{1}^{\perp}\right)$. By construction, we have $B:=\left(B_{1}, B_{2}\right) \simeq(W(t)-\mu t)_{t \geqslant 0}$. Next, define $Y_{2}:=B_{2}-L .\left(Y_{1}\right)$. Then,

$$
\begin{equation*}
\mathbb{P}\left[\exists t \in[0, T]:\left(\left|Y_{1}\right|, Y_{2}\right)(t)=(0,0)\right]=\mathbb{P}\left[\exists t \in[0, T]:\left(Y_{1}, B_{1}+B_{2}\right)(t)=(0,0)\right] \tag{2.7.7}
\end{equation*}
$$

Next observe that, even though $\left(Y_{1}, B_{1}+B_{2}\right)$ is not a planar Brownian motion, it obeys the scaling and Markov property. Moreover, its marginal law at any time $t$ has a Lebesgue density in $\mathbb{R}^{2}$, which is strictly positive almost everywhere. Therefore, the proof of Lévy's theorem on the area of a planar Brownian motion (cf. Theorem 2.24 in 67]) can be adapted to the process $\left(Y_{1}, B_{1}+B_{2}\right)$ and one obtains $\mathcal{L}\left(\left(Y_{1}, B_{1}+B_{2}\right)[0, T]\right)=0$. Fubini's theorem then gives $\mathbb{P}\left[z \in\left(Y_{1}, B_{1}+B_{2}\right)[0, T]\right]=0$ for any $z \neq\left(x_{1}, x_{1}+x_{2}\right)$. Especially, the probability in 2.7 .7 must be zero. Now, applying the Cameron-MartinGirsanov theorem, there exists an equivalent measure $\mathbb{Q}$ of $\mathbb{P}$ such that $B$ behaves like a planar Brownian motion starting in $x \in \mathbb{R}_{+}^{2} \backslash\{(0,0)\}$ with covariance $\Sigma$ and drift $\mu$ under the measure $\mathbb{Q}$. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, $\left(\left|Y_{1}\right|, Y_{2}\right)$ does not hit the origin $\mathbb{Q}$-almost surely. By the uniqueness of the solution of the Skorokhod problem, we conclude that $g(W)$ on $\left[0, \hat{\tau}_{2}\right]$ has the same distribution as $\left(\left|Y_{1}\right|, Y_{2}\right)$, under $\mathbb{Q}$, until $Y_{2}$ hits zero the first time. Hence, $g(W)\left(\hat{\tau}_{2}\right) \neq(0,0)$ with probability one implying that
$\hat{\tau}_{2}<\tau \mathbb{P}$-almost surely. Applying the Markov property of $g(W)$, we further conclude that $\hat{\tau}_{k}<\tau \mathbb{P}$-almost surely for all $k \in \mathbb{N}$.

In contrast, let us consider the event $\left\{\hat{\tau}_{\infty}=T\right\}$. Since $\hat{\tau}_{\infty}=\tau$ is the first time that $h_{1}(W)$ hits zero, $h_{1}(W)$ is a one-dimensional Brownian motion, and $\mathbb{P}\left[h_{1}(W)(T)=\right.$ $0]=0$, we conclude that $h_{1}(W)>0$ and $g(W) \in \mathbb{R}_{+}^{2} \backslash\{(0,0)\}$ on $[0, T]$ with probability one. By the path property of a planar Brownian motion, there exists a finite $\kappa \in \mathbb{N}$ such that $\hat{\tau}_{\kappa}=T$ and $g(W)=\widetilde{g}_{\kappa+1}(W)$. Finally, studying the components of $g(W)$ on $\left[\hat{\tau}_{\kappa-1}, T\right]$, we conclude that $g(W)(T) \in(0, \infty)^{2}$ with probability one implying that $g(W)=\widetilde{g}_{\kappa+1}(W)=g_{\kappa}(W)$ on $[0, T]$.

Lemma 2.7.6. Let $W$ be a planar Brownian motion as defined above. On the event $\left\{\hat{\tau}_{\infty}(W)<T\right\}$, we have $\lim _{t \rightarrow \hat{\tau}_{\infty}(W)} g(W)(t)=(0,0) \mathbb{P}$-almost surely.

Proof. Let us denote $\hat{\tau}_{k}:=\hat{\tau}_{k}(W)$ and $\hat{\tau}_{\infty}:=\hat{\tau}_{\infty}(W)$. Now, we can rewrite $\hat{\tau}_{1}=$ $\hat{\tau}_{1,1}(W) \wedge \hat{\tau}_{1,2}(W)$ and $\hat{\tau}_{k}=\hat{\tau}_{k, 1}(W) \mathbb{1}\left(\hat{\tau}_{k-1}=\hat{\tau}_{k-1,2}(W)\right)+\hat{\tau}_{k, 2}(W) \mathbb{1}\left(\hat{\tau}_{k-1}=\hat{\tau}_{k-1,1}(W)\right)$ for $k \geqslant 2$, where $\left(\hat{\tau}_{k, 1}\right)_{k \geqslant 1}$ and $\left(\hat{\tau}_{k, 2}\right)_{k \geqslant 2}$ are introduced in 2.7.5 and 2.7.6. In the following, let us concentrate on the event $\left\{\hat{\tau}_{1}=\hat{\tau}_{1,1}\right\}$ and note that on the event $\left\{\hat{\tau}_{1}=\right.$ $\left.\hat{\tau}_{1,2}\right\}$, we can argue completely analogously. First, for $k \in \mathbb{N}$, we have $\hat{\tau}_{2 k-1}=\hat{\tau}_{2 k-1,1}(W)$ and $\hat{\tau}_{2 k}=\hat{\tau}_{2 k, 2}(W)$ implying that $\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 k-1}(W)\right)=0$ and $\pi_{2}^{(2)} g(W)\left(\hat{\tau}_{2 k}(W)\right)=0$ $\mathbb{P}$-almost surely. Next, for $k \in \mathbb{N}$, observe that

$$
\begin{aligned}
\sup _{t \geqslant \hat{\tau}_{k}} & \pi_{1}^{(2)} g(W)(t) \leqslant \sup _{j \geqslant\lfloor k / 2\rfloor} \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+2}\right)} \pi_{1}^{(2)} g(W)(t) \\
& \leqslant \sup _{j \geqslant\lfloor k / 2\rfloor} \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+2}\right)}\left\{\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)+\left|\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)-\pi_{1}^{(2)} g(W)(t)\right|\right\} \\
& =\sup _{j \geqslant\lfloor k / 2\rfloor} \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+2}\right)}\left|\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)-\pi_{1}^{(2)} g(W)(t)\right|
\end{aligned}
$$

Using the representation of $g(W)$ as a semimartingale reflecting Brownian motion $(g(W), W, l)$ (cf. Proposition 2.3.21) and the Hölder continuity of the Brownian motion, we have on $\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)$

$$
\begin{aligned}
& \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left|\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)-\pi_{1}^{(2)} g(W)(t)\right| \\
& =\sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)-l_{2}\left(\hat{\tau}_{2 j+1}\right)+l_{2}(t)\right| \\
& \leqslant \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)\right|+\sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left(-\pi_{2}^{(2)} g(W)\left(\hat{\tau}_{2 j}\right)-W_{2}(t)+W_{2}\left(\hat{\tau}_{2 j}\right)\right)^{+} \\
& \leqslant \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)\right|+\sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+1}\right)}\left|W_{2}(t)-W_{2}\left(\hat{\tau}_{2 j}\right)\right| \\
& \leqslant\left(C_{1, \alpha}+C_{2, \alpha}\right)\left|\hat{\tau}_{2 j+1}-\hat{\tau}_{2 j}\right|^{\alpha} \rightarrow 0
\end{aligned}
$$

$\mathbb{P}$-almost surely as $j \rightarrow \infty$, for $0<\alpha<\frac{1}{2}$ and positive, integrable random variables
$C_{1, \alpha}$ and $C_{2, \alpha}$. Similarly, we get on $\left[\hat{\tau}_{2 j+1}, \hat{\tau}_{2 j+2}\right)$

$$
\begin{aligned}
& \sup _{t \in\left[\hat{\tau}_{2 j+1}, \hat{\tau}_{2 j+2}\right)}\left|\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)-\pi_{1}^{(2)} g(W)(t)\right| \\
& \leqslant \sup _{t \in\left[\hat{\tau}_{2 j+1}, \hat{\tau}_{2 j+2}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)\right|+\left|l_{1}\left(\hat{\tau}_{2 j+2}\right)-l_{1}\left(\hat{\tau}_{2 j+1}\right)\right| \\
& \leqslant \sup _{t \in\left[\hat{\tau}_{2 j+1}, \hat{\tau}_{2 j+2}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)\right|+\sup _{t \in\left[\hat{\tau}_{2 j+1}, \hat{\tau}_{2 j+2}\right)}\left|W_{1}\left(\hat{\tau}_{2 j+1}\right)-W_{1}(t)\right| \\
& \leqslant 2 C_{1, \alpha}\left|\hat{\tau}_{2 j+2}-\hat{\tau}_{2 j+1}\right|^{\alpha} \rightarrow 0
\end{aligned}
$$

$\mathbb{P}$-almost surely as $j \rightarrow \infty$. Together, we conclude $\mathbb{P}$-almost surely that

$$
\sup _{t \geqslant \hat{\tau}_{k}} \pi_{1}^{(2)} g(W)(t) \leqslant \sup _{j \geqslant\lfloor k / 2\rfloor} \sup _{t \in\left[\hat{\tau}_{2 j}, \hat{\tau}_{2 j+2}\right)}\left|\pi_{1}^{(2)} g(W)\left(\hat{\tau}_{2 j+1}\right)-\pi_{1}^{(2)} g(W)(t)\right| \rightarrow 0
$$

as $k \rightarrow \infty$. Similarly, we can show that

$$
\sup _{t \geqslant \hat{\tau}_{k}} \pi_{2}^{(2)} g(W)(t) \leqslant \sup _{j \geqslant\lfloor k / 2\rfloor t \in\left[\hat{\tau}_{2 j-1}, \hat{\tau}_{2 j+1}\right)} \sup _{2}\left|\pi_{2}^{(2)} g(W)\left(\hat{\tau}_{2 j}\right)-\pi_{2}^{(2)} g(W)(t)\right| \rightarrow 0
$$

$\mathbb{P}$-almost surely as $k \rightarrow \infty$. Hence, it holds that $\lim _{t \rightarrow \hat{\tau}_{\infty}} g(W)(t)=(0,0)$.
Corollary 2.7.7. Let $W$ be a planar Brownian motion as defined above. Then, the function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is $\mathbb{P}$-almost surely continuous at $W$.

Proof. Let us verify the conditions of Lemma 2.3.5. since the components of $W$ are not degenerate and not perfectly correlated, the paths of $W$ are almost surely contained in $C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$. Moreover, $h_{1}(W)$ defines a one-dimensional Brownian motion. Hence, its paths are almost surely contained in $C_{0}^{\prime}([0, T], \mathbb{R})$. Now, applying Lemma 2.7.5 and Lemma 2.7.6. $W$ satisfies condition (I) with probability one and $\hat{\tau}_{\infty}(W)=$ $\tau(W)$ with probability one. Hence, the assumptions in Lemma 2.3.5 are satisfied with probability one, yielding that the function $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is $\mathbb{P}$-almost surely continuous at $W$.

### 2.7.2.3 Continuity of the function $\widetilde{\Psi}^{Q}$

The goal of this subsection is to characterize the continuity set of the map

$$
\widetilde{\Psi}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)
$$

introduced in Definition 2.3.7. To study the continuity of the map $\widetilde{\Psi}^{Q}$, we endow $D\left([0, T], \mathbb{R}^{4}\right)$ with the Skorokhod topology. The set $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ is endowed with the topology induced by cylindrical semi-norms defined as follows: for $R:=\left(R_{k}\right)_{k \geqslant 1}, R^{\prime}:=\left(R_{k}^{\prime}\right)_{k \geqslant 1} \in$ $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ and $\varepsilon>0$, we define

$$
d_{\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}}\left(R, R^{\prime}\right)<\varepsilon \quad: \Leftrightarrow \quad \forall j \geqslant 1, \sup \left\{\left\|R_{1}-R_{1}^{\prime}\right\|, \cdots,\left\|R_{j}-R_{j}^{\prime}\right\|\right\}<\varepsilon
$$

The space $D\left([0, \infty), \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ is then endowed with the corresponding product topology, i.e.,

$$
d\left((\omega, R, \widetilde{R}),\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}\right)\right):=d_{J_{1}}\left(\omega, \omega^{\prime}\right)+d_{\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}}\left(R, R^{\prime}\right)+d_{\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}}\left(\widetilde{R}, \widetilde{R}^{\prime}\right)
$$

In the following, we inductively analyze the continuity sets of the functions $\widetilde{\Psi}_{k}^{Q}$ : $D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right), k \geqslant 1$, introduced in Definition 2.3.7.
Lemma 2.7.8 (Continuity of $\left.\widetilde{\Psi}_{1}^{Q}\right)$. Let $\widetilde{\tau}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow[0, T]$ be the first hitting time map introduced in (2.3.6). Let $\left(\omega, R_{1}, \widetilde{R}_{1}\right) \in C\left([0, T], \mathbb{R}^{4}\right) \times \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4}$ satisfy the following three conditions:
i) $G$ is continuous at $\omega, R_{1}+\omega(\cdot+\widetilde{\tau}(\omega))-\omega(\widetilde{\tau}(\omega))$, and $\widetilde{R}_{1}+\omega(\cdot+\widetilde{\tau}(\omega))-\omega(\widetilde{\tau}(\omega))$,
ii) $h(\omega) \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, and
iii) $\widetilde{\tau}(\omega) \in(0, T)$.

Then, the map $\widetilde{\Psi}_{1}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is continuous at $\left(\omega, R_{1}, \widetilde{R}_{1}\right)$ with respect to the following distance on $D\left([0, T], \mathbb{R}^{4}\right) \times \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4}$ :

$$
d\left(\left(\omega, R_{1}, \widetilde{R}_{1}\right),\left(\omega^{\prime}, R_{1}^{\prime}, \widetilde{R}_{1}^{\prime}\right)\right):=d_{J_{1}}\left(\omega, \omega^{\prime}\right)+\left\|R_{1}-R_{1}^{\prime}\right\|+\left\|\widetilde{R}_{1}-\widetilde{R}_{1}^{\prime}\right\| .
$$

Proof. Let $\left(\omega_{0}, R_{1}, \widetilde{R}_{1}\right)$ satisfy the conditions stated in Lemma 2.7.8. Since $\widetilde{\tau}\left(\omega_{0}\right) \in$ $(0, T)$ and $h\left(\omega_{0}\right) \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$, there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N, 1 / n<\widetilde{\tau}\left(\omega_{0}\right)<T-1 / n$ and $h\left(\omega_{0}\right) \notin B_{2}(0,1 / n)$ on $[0, T]$. Moreover, let $0<\varepsilon \leqslant \frac{1}{2 n}$ and $\left(\omega^{\prime}, R_{1}^{\prime}, \widetilde{R}_{1}^{\prime}\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4}$ be such that

$$
d\left(\left(\omega_{0}, R_{1}, \widetilde{R}_{1}\right),\left(\omega^{\prime}, R_{1}^{\prime}, \widetilde{R}_{1}^{\prime}\right)\right)<\varepsilon
$$

Since $d_{J_{1}}\left(\omega_{0}, \omega^{\prime}\right)<\varepsilon$, there exists $\lambda \in \Lambda_{T}$ such that $\left\|\omega_{0} \circ \lambda-\omega^{\prime}\right\|_{\infty}<\varepsilon$ and $\|\lambda-\mathrm{id}\|_{\infty}<\varepsilon$. By condition i), without loss of generality, we can assume for the same $\lambda \in \Lambda_{T}$ that $\left\|G\left(\omega_{0} \circ \lambda\right)-G\left(\omega^{\prime}\right)\right\|_{\infty}<\varepsilon$. Now, the function $\omega \mapsto \widetilde{\tau}(G(\omega))$ can be identified with the first hitting time map of $h(\omega) \in D\left([0, T], \mathbb{R}^{2}\right)$ of the axes $\{(0, y): y>0\} \cup\{(x, 0): x \geqslant 0\}$, i.e., we have $\widetilde{\tau}(G(\omega))=\widetilde{\tau}(\omega)$. By assumption, $h\left(\omega_{0}\right) \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$. Hence, by the continuity of the first hitting time map (cf. Lemma 2.7.2), one can also assume, without loss of generality, that $\left|\widetilde{\tau}\left(\omega_{0} \circ \lambda\right)-\widetilde{\tau}\left(\omega^{\prime}\right)\right|<\varepsilon$. Moreover, since $h\left(\omega_{0}\right)$ does not intersect with $B_{2}(0,1 / n)$ and $0<\varepsilon \leqslant \frac{1}{2 n}$, we can argue as in the proof of Lemma 2.7.3 and conclude

$$
\mathbb{1}_{\left\{\widetilde{\tau}\left(\omega_{0} \circ \lambda\right)=\widetilde{\tau}_{i}\left(\omega_{0} \circ \lambda\right)\right\}}=\mathbb{1}_{\left\{\widetilde{\tau}\left(\omega^{\prime}\right)=\widetilde{\tau}_{i}\left(\omega^{\prime}\right)\right\}}, \quad \text { for } i=b, a .
$$

Now, we define

$$
\begin{equation*}
\lambda^{\varepsilon}(t):=\lambda(t)+\widetilde{\tau}\left(\omega^{\prime}\right)-\widetilde{\tau}\left(\omega_{0} \circ \lambda\right) \tag{2.7.8}
\end{equation*}
$$

on $[1 / n, T-1 / n]$ and extend its definition to $[0, T]$ by linear interpolation in the gaps with $\lambda^{\varepsilon}(0)=0$ and $\lambda^{\varepsilon}(T)=T$. Since $\|\lambda-\mathrm{id}\|_{\infty}<\varepsilon,\left|\widetilde{\tau}\left(\omega_{0} \circ \lambda\right)-\widetilde{\tau}\left(\omega^{\prime}\right)\right|<\varepsilon$, and $0<\varepsilon \leqslant \frac{1}{2 n}$, the map $t \mapsto \lambda^{\varepsilon}(t)$ is well-defined and an element of $\Lambda_{T}$. Then,

$$
\begin{aligned}
\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{\infty} & =\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{[0,1 / n)}+\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{[1 / n, T-1 / n]}+\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{(T-1 / n, T]} \\
& \leqslant 3\|\lambda-\mathrm{id}\|_{\infty}+3\left|\widetilde{\tau}\left(\omega^{\prime}\right)-\widetilde{\tau}\left(\omega_{0} \circ \lambda\right)\right|<6 \varepsilon
\end{aligned}
$$

since $\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{[1 / n, T-1 / n]} \leqslant\|\lambda-\mathrm{id}\|_{\infty}+\left|\widetilde{\tau}\left(\omega^{\prime}\right)-\widetilde{\tau}\left(\omega_{0} \circ \lambda\right)\right|$,

$$
\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{[0,1 / n)} \leqslant\left|\lambda^{\varepsilon}\left(n^{-1}\right)-n^{-1}\right|, \quad\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{(T-1 / n, T]} \leqslant\left|\lambda^{\varepsilon}\left(T-n^{-1}\right)-\left(T-n^{-1}\right)\right|
$$

On the other hand,

$$
\begin{aligned}
\left\|\omega_{0} \circ \lambda^{\varepsilon}-\omega^{\prime}\right\|_{\infty} & =\left\|\omega_{0} \circ \lambda^{\varepsilon}-\omega_{0} \circ \lambda+\omega_{0} \circ \lambda-\omega^{\prime}\right\|_{\infty} \\
& \leqslant\left\|\omega_{0} \circ \lambda^{\varepsilon}-\omega_{0}\right\|_{\infty}+\left\|\omega_{0}-\omega_{0} \circ \lambda\right\|_{\infty}+\left\|\omega_{0} \circ \lambda-\omega^{\prime}\right\|_{\infty} \\
& <\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon
\end{aligned}
$$

Again, by Lemma 2.3.5, we assume without loss of generality for the same $\lambda \in \Lambda_{T}$ that

$$
\begin{equation*}
\left\|G\left(\omega_{0} \circ \lambda^{\varepsilon}\right)-G\left(\omega^{\prime}\right)\right\|_{\infty}<\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon \tag{2.7.9}
\end{equation*}
$$

Next, observe that

$$
\begin{aligned}
\| R_{1}^{\prime} & +\omega^{\prime}\left(\cdot+\widetilde{\tau}\left(\omega^{\prime}\right)\right)-\omega^{\prime}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)-R_{1}-\omega_{0} \circ \lambda^{\varepsilon}\left(\cdot+\widetilde{\tau}\left(\omega^{\prime}\right)\right)+\omega_{0} \circ \lambda^{\varepsilon}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right) \|_{\infty} \\
& <\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon+\left\|\omega_{0} \circ \lambda^{\varepsilon}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)-\omega^{\prime}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right\|<2\left(\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon\right)
\end{aligned}
$$

By the continuity of $G$ at $R_{1}+\omega_{0}\left(\cdot+\widetilde{\tau}\left(\omega_{0}\right)\right)-\omega_{0}\left(\widetilde{\tau}\left(\omega_{0}\right)\right)$ and the continuity of $t \mapsto$ $G\left(\omega_{0}\right)(t)$ since $\omega_{0} \in C\left([0, T], \mathbb{R}^{4}\right)$, without loss of generality, we might assume that

$$
\begin{align*}
\| G\left(R_{1}^{\prime}+\omega^{\prime}\left(\cdot+\widetilde{\tau}\left(\omega^{\prime}\right)\right)-\omega^{\prime}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right) & -G\left(R_{1}+\omega_{0} \circ \lambda^{\varepsilon}\left(\cdot+\widetilde{\tau}\left(\omega^{\prime}\right)\right)-\omega_{0} \circ \lambda^{\varepsilon}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right) \|_{\infty} \\
& <2\left(\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon\right) \tag{2.7.10}
\end{align*}
$$

By definition of $\lambda^{\varepsilon}$ we have $\widetilde{\tau}\left(\omega_{0} \circ \lambda^{\varepsilon}\right)=\widetilde{\tau}\left(\omega^{\prime}\right)$ and hence also $\mathbb{1}_{\left[0, \widetilde{\tau}\left(\omega_{0} \circ \lambda^{\varepsilon}\right)\right)}=\mathbb{1}_{\left[0, \widetilde{\tau}\left(\omega^{\prime}\right)\right)}$ and $\mathbb{1}_{\left[\tau\left(\omega_{0} \circ \lambda^{\varepsilon}\right), T\right]}=\mathbb{1}_{\left[\widetilde{\tau}\left(\omega^{\prime}\right), T\right]}$. Therefore,

$$
\begin{aligned}
& \widetilde{\Psi}_{1}^{Q}\left(\omega_{0}, R_{1}, \widetilde{R}_{1}\right) \circ \lambda^{\varepsilon}-\widetilde{\Psi}_{1}^{Q}\left(\omega^{\prime}, R_{1}^{\prime}, \widetilde{R}_{1}^{\prime}\right) \\
= & \left(G\left(\omega_{0} \circ \lambda^{\varepsilon}\right)-G\left(\omega^{\prime}\right)\right) \mathbb{1}_{\left[0, \widetilde{\tau}\left(\omega^{\prime}\right)\right)}+\mathbb{1}_{\left[\widetilde{\tau}\left(\omega^{\prime}\right), T\right]}\{ \\
& +\mathbb{1}_{\left\{\widetilde{\tau}\left(\omega^{\prime}\right)=\widetilde{\tau}_{a}\left(\omega^{\prime}\right)\right\}}\left(G\left(R_{1}^{\prime}+\omega^{\prime}-\omega^{\prime}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right)-G\left(R_{1}+\omega_{0} \circ \lambda^{\varepsilon}-\omega_{0} \circ \lambda^{\varepsilon}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right)\right) \\
& \left.+\mathbb{1}_{\left\{\widetilde{\tau}\left(\omega^{\prime}\right)=\widetilde{\tau}_{b}\left(\omega^{\prime}\right)\right\}}\left(G\left(\widetilde{R}_{1}^{\prime}+\omega^{\prime}-\omega^{\prime}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right)-G\left(\widetilde{R}_{1}+\omega_{0} \circ \lambda^{\varepsilon}-\omega_{0} \circ \lambda^{\varepsilon}\left(\widetilde{\tau}\left(\omega^{\prime}\right)\right)\right)\right)\right\}
\end{aligned}
$$

Finally, a combination of 2.7.9 and 2.7.10 yields

$$
\left\|\widetilde{\Psi}_{1}^{Q}\left(\omega_{0}, R_{1}, \widetilde{R}_{1}\right) \circ \lambda^{\varepsilon}-\widetilde{\Psi}_{1}^{Q}\left(\omega^{\prime}, R_{1}^{\prime}, \widetilde{R}_{1}^{\prime}\right)\right\|_{\infty}<3\left(\eta_{\omega_{0}}(6 \varepsilon)+\eta_{\omega_{0}}(\varepsilon)+\varepsilon\right)
$$

which proves that $\left(\omega_{0}, R_{1}, \widetilde{R}_{1}\right)$ is indeed a continuity point of $\widetilde{\Psi}_{1}^{Q}$.
To ease notation, let us introduce the shift operators $\widetilde{\varphi}_{k}^{(1)}, \widetilde{\varphi}_{k}^{(2)}, k \geqslant 1$, given by

$$
\begin{align*}
& \widetilde{\varphi}_{k}^{(1)}(\omega, R, \widetilde{R}):=R_{k}+\omega\left(\cdot+\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right)-\omega\left(\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right),  \tag{2.7.11}\\
& \widetilde{\varphi}_{k}^{(2)}(\omega, R, \widetilde{R}):=\widetilde{R}_{k}+\omega\left(\cdot+\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right)-\omega\left(\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\right)\right),
\end{align*}
$$

where $\widetilde{\Psi}_{k-1}^{Q}:=\widetilde{\Psi}_{k-1}^{Q}(\omega, R, \widetilde{R})$ for $k \geqslant 1$. The next lemma characterizes the continuity sets of the functions $\widetilde{\Psi}_{k}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right), k \geqslant 2$, being introduced in Definition 2.3.7.

Lemma 2.7.9 (Continuity of $\left.\widetilde{\Psi}_{k}^{Q}\right)$. Let $k \geqslant 1$ and assume that $(\omega, R, \widetilde{R}) \in C\left([0, T], \mathbb{R}^{4}\right) \times$ $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfies the following:
i) The function $G$ is continuous at $\omega, \widetilde{\varphi}_{j}^{(1)}(\omega, R, \widetilde{R})$, and $\widetilde{\varphi}_{j}^{(2)}(\omega, R, \widetilde{R})$ for all $j \leqslant k$,
ii) $h(\omega) \in C_{0}^{\prime}\left([0, T], \mathbb{R}^{2} \backslash\{(0,0)\}\right)$,
iii) $\widetilde{\tau}\left(\widetilde{\Psi}_{j-1}^{Q}(\omega, R, \widetilde{R})\right) \in(0, T)$ for all $j \leqslant k$, and
iv) $(0,0) \notin\left(h \circ \widetilde{\Psi}_{j-1}^{Q}\right)(\omega, R, \widetilde{R})\left(\left[0, \widetilde{\tau}\left(\widetilde{\Psi}_{j-1}^{Q}(\omega, R, \widetilde{R})\right)\right]\right)$ for all $j \leqslant k$.

Then, the function $\widetilde{\Psi}_{k}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is continuous at $(\omega, R, \widetilde{R})$.

Proof. Let $\left(\omega_{0}, R, \widetilde{R}\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ fulfill the assumptions in Lemma 2.7.9. We will show the stated result by induction: Let $k=1$. Applying Lemma 2.7.8, we conclude that $\widetilde{\Psi}_{1}^{Q}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$. Next, let us assume for all $k=1, \cdots, l-1$ that $\widetilde{\Psi}_{k}^{Q}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$. Now, let us consider $k=l$. Then, the assumptions of Lemma 2.7 .9 are also satisfied for $k=l-1$ and hence, by the induction hypothesis, we conclude that $\widetilde{\Psi}_{l-1}^{Q}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$. Next, let $\varepsilon>0$ be small and let $\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ be such that

$$
d_{J_{1}}\left(\omega_{0}, \omega^{\prime}\right)+\sup _{i=1, \cdots, l}\left\|R_{i}-R_{i}^{\prime}\right\|+\sup _{i=1, \cdots, l}\left\|\widetilde{R}_{i}-\widetilde{R}_{i}^{\prime}\right\|<\varepsilon .
$$

Now, replicating the proof of Lemma 2.7.8. since $\widetilde{\Psi}_{l-1}^{Q}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$, we can construct $\lambda^{\varepsilon} \in \Lambda_{T}$ appropriately such that

$$
\left\|\omega_{0} \circ \lambda^{\varepsilon}-\omega^{\prime}\right\|_{\infty}<\delta(\varepsilon), \quad\left\|\lambda^{\varepsilon}-\mathrm{id}\right\|_{\infty}<\delta(\varepsilon)
$$

and $\widetilde{\tau}\left(\widetilde{\Psi}_{l-1}^{Q}\left(\omega_{0}, R, \widetilde{R}\right) \circ \lambda^{\varepsilon}\right)=\widetilde{\tau}\left(\widetilde{\Psi}_{l-1}^{Q}\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}\right)\right)$, where $\delta(\varepsilon)>0$ and $\delta(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$. In particular, for such $\lambda^{\varepsilon}$, we can show that

$$
\left\|\widetilde{\Psi}_{l}^{Q}\left(\omega_{0}, R, \widetilde{R}\right) \circ \lambda^{\varepsilon}-\widetilde{\Psi}_{l}^{Q}\left(\omega^{\prime}, R, \widetilde{R}^{\prime}\right)\right\|_{\infty}<\delta(\varepsilon)
$$

implying that $\widetilde{\Psi}_{l}^{Q}$ is also continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$.
Now, for all $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ such that there exists a finite number $N_{T}:=\inf \left\{k \geqslant 0: \widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R})\right)=T\right\}$, the function $\widetilde{\Psi}^{Q}$ in Definition 2.3.7 is given by $\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})=\widetilde{\Psi}_{N_{T}}^{Q}(\omega, R, \widetilde{R})$. To describe the continuity set of $\widetilde{\Psi}^{Q}$, we further need to analyze the continuity set of the map $N_{T}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow$ $\mathbb{N}_{0} \cup\{+\infty\}$, where $N_{T}(\omega, R, \widetilde{R})=+\infty$ if the above infimum does not exist.

Obviously, the function $N_{T}$ depends on the upper boundary $T$ of the domain of the function $\omega \in D\left([0, T], \mathbb{R}^{4}\right)$. When proving our main result (cf. Theorem 2.5.1), we will also need some flexibility in this upper boundary. Setting $\widetilde{\tau}_{\sigma}(\omega):=\widetilde{\tau}(\omega) \wedge \sigma$ for some $\sigma \in(0, T]$, we introduce the function $N: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times(0, T] \rightarrow$ $\mathbb{N}_{0} \cup\{+\infty\}$ by

$$
\begin{equation*}
N(\omega, R, \widetilde{R}, \sigma):=\inf \left\{k \geqslant 0: \widetilde{\tau}_{\sigma}\left(\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R})\right)=\sigma\right\} \tag{2.7.12}
\end{equation*}
$$

and $N(\omega, R, \widetilde{R}, \sigma)=+\infty$ if infimum does not exist. Note that $N(\omega, R, \widetilde{R}, T)=$ $N_{T}(\omega, R, \widetilde{R})$. In the following lemma, we characterize the continuity set of the function $N: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times(0, T] \rightarrow \mathbb{N}_{0} \cup\{+\infty\}$.
Lemma 2.7.10 (Continuity of $N$ ). Let $\left(\omega_{0}, R, \widetilde{R}, \sigma\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times$ $(0, T]$. Moreover, assume that the following three conditions hold true:
i) $N_{0}:=N\left(\omega_{0}, R, \widetilde{R}, \sigma\right)<\infty$.
ii) If $N_{0}=0$ then the assumptions of Lemma 2.3.6 hold, otherwise the assumptions of Lemma 2.7.9 hold for $k=N_{0}$.
iii) $\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)(\sigma) \in(0, \infty)^{2}$.

Then, the function $N: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times(0, T] \rightarrow \mathbb{N}_{0} \cup\{+\infty\}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}, \sigma\right)$ with respect to the distance

$$
d^{\prime}\left((\omega, R, \widetilde{R}, \sigma),\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}, \sigma^{\prime}\right)\right):=d\left((\omega, R, \widetilde{R}),\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}\right)\right)+\left|\sigma-\sigma^{\prime}\right| .
$$

Proof. Let $\left(\omega_{0}, R, \widetilde{R}, \sigma\right)$ satisfy the assumptions of Lemma 2.7.10 and denote $N_{0}:=$ $N\left(\omega_{0}, R, \widetilde{R}, \sigma\right)$. In the following, let $\widetilde{\tau}_{0}^{0}:=0$ and $\widetilde{\tau}_{k}^{0}:=\widetilde{\tau}_{\sigma}\left(\widetilde{\Psi}_{k-1}^{(2)}\left(\omega_{0}, R, \widetilde{R}\right)\right)$ for $k \geqslant 1$.

Thanks to condition i), we conclude that $N_{0}<\infty$. Next, for some $\delta>0$, let $\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}, \sigma^{\prime}\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times(0, T]$ and $\lambda \in \Lambda_{T}$ be such that $d^{\prime}\left(\left(\omega_{0}, R, \widetilde{R}, \sigma\right),\left(\omega^{\prime}, R^{\prime}, \widetilde{R}^{\prime}, \sigma^{\prime}\right)\right)<\delta,\left\|\omega_{0}-\omega^{\prime} \circ \lambda\right\|_{\infty}<\delta$, and $\|\lambda-\mathrm{id}\|_{\infty}<\delta$. In the following, denote $N^{\prime}:=N\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right), \widetilde{\tau}_{0}^{\prime}:=0$, and $\widetilde{\tau}_{k}^{\prime}:=\widetilde{\tau}_{\sigma^{\prime}}\left(\widetilde{\Psi}_{k-1}^{Q}\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\right)$ for $k \geqslant 1$.

By condition ii), we conclude that $t \mapsto \widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)(t)$ is continuous on $\left[\widetilde{\tau}_{N_{0}}^{0}, T\right]$. Together with condition iii), the definition of $N_{0}$, and since $\left|\sigma-\sigma^{\prime}\right|<\delta$, there exists an $\varepsilon>0$ and a $\delta_{1}>0$ such that for all $\delta \leqslant \delta_{1}$,
a) $\sigma-\widetilde{\tau}_{N_{0}}^{0}>3 \varepsilon$,
b) $\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right) \in(\varepsilon, \infty)^{2}$ on $\left[\widetilde{\tau}_{N_{0}}^{0}, \sigma \vee \sigma^{\prime}\right]$.

By condition ii), we conclude that $\widetilde{\Psi}_{N_{0}}^{Q}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$. Moreover, since also $h$ is continuous at $\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)$ restricted on $\left[\widetilde{\tau}_{N_{0}}^{0}, T\right]$, there exists a $\delta_{2}>0$ such that for all $\delta \leqslant \delta_{2}$, we have

$$
\begin{equation*}
\left\|\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)-\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\right\|_{\left[\widetilde{\tau}_{N_{0}}^{0}, T\right]}<\varepsilon \tag{2.7.13}
\end{equation*}
$$

By the continuity of the first hitting time map, there exists a $\delta_{3}>0$ such that for all $\delta \leqslant \delta_{3}$, we have $\left|\widetilde{\tau}_{N_{0}}^{0}-\widetilde{\tau}_{N_{0}}^{\prime}\right|<\varepsilon$. Together with condition a) and $\left|\sigma-\sigma^{\prime}\right|<\delta$, we conclude that $\sigma^{\prime}-\widetilde{\tau}_{N_{0}}^{\prime}>\varepsilon$ for all $\delta \leqslant \delta_{1} \wedge \delta_{2} \wedge \delta_{3} \wedge \varepsilon$. Hence, we must have $N^{\prime} \geqslant N_{0}$ for all such $\delta$.

Assume that $N^{\prime}>N_{0}$. Then, $\widetilde{\tau}_{N_{0}+1}^{\prime}<\sigma^{\prime}$. Again, by the continuity of the first hitting time map, there exists a $\delta_{4}>0$ such that for all $\delta \leqslant \delta_{4}$, we have $\mid \widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)\right)-$ $\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\right) \mid<\varepsilon$. Applying condition b), we obtain

$$
\begin{aligned}
& \left|\sigma-\widetilde{\tau}_{N_{0}+1}^{\prime}\right| \\
& \leqslant\left|\sigma-\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)\right) \wedge \sigma^{\prime}\right|+\left|\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)\right) \wedge \sigma^{\prime}-\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\right) \wedge \sigma^{\prime}\right| \\
& \leqslant\left|\sigma-\sigma^{\prime}\right|+\left|\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)\right)-\widetilde{\tau}\left(\widetilde{\Psi}_{N_{0}}^{Q}\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\right)\right| \\
& \leqslant \delta+\varepsilon \leqslant 2 \varepsilon
\end{aligned}
$$

Applying condition a), we conclude that $\widetilde{\tau}_{N_{0}+1}^{\prime} \in\left[\widetilde{\tau}_{N_{0}}^{0}, \sigma^{\prime}\right)$ for all $\delta \leqslant \delta_{1} \wedge \delta_{2} \wedge \delta_{3} \wedge \delta_{4} \wedge \varepsilon$. By construction, there exists an $i \in\{1,2\}$ such that $\left(\pi_{i} \circ h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega^{\prime} \circ \lambda, R, \widetilde{R}\right)\left(\widetilde{\tau}_{N_{0}+1}^{\prime}\right) \leqslant 0$. Hence, for this $i \in\{1,2\}$, we finally conclude by applying equation (2.7.13) and condition b)

$$
\begin{aligned}
\varepsilon & >\left\|\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)-\left(h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega^{\prime} \circ \lambda, R, \widetilde{R}\right)\right\|\left[\widetilde{\tau}_{N_{0}}^{0}, T\right] \\
& \geqslant\left|\left(\pi_{i} \circ h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)\left(\widetilde{\tau}_{N_{0}+1}^{\prime}\right)-\left(\pi_{i} \circ h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega^{\prime} \circ \lambda, R^{\prime}, \widetilde{R}^{\prime}\right)\left(\widetilde{\tau}_{N_{0}+1}^{\prime}\right)\right| \\
& \geqslant\left|\left(\pi_{i} \circ h \circ \widetilde{\Psi}_{N_{0}}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)\left(\widetilde{\tau}_{N_{0}+1}^{\prime}\right)\right|>\varepsilon .
\end{aligned}
$$

Hence, we end up in a contradiction yielding that $N^{\prime}=N_{0}$ for all $\delta \leqslant \delta_{1} \wedge \delta_{2} \wedge \delta_{3} \wedge$ $\delta_{4} \wedge \varepsilon$.

With all these preparations done, we are finally ready to give the proof of Theorem 2.3 .9 i.e., we are able to characterize the continuity set of $\widetilde{\Psi}^{Q}$.

Proof of Theorem 2.3.9. Let $\left(\omega_{0}, R, \widetilde{R}\right)$ satisfy conditions i)-iv) in Theorem 2.3.9. Let us denote by $\widetilde{\tau}_{k}=\widetilde{\tau}\left(\widetilde{\Psi}_{k-1}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)\right), k \geqslant 1$, and $\widetilde{\tau}_{0}:=0$. Then, for all $k \geqslant 0$, observe that

$$
\left(h \circ \widetilde{\Psi}_{k}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)\left(\cdot+\widetilde{\tau}_{k}\right)=h\left(\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega_{0}\left(\cdot+\widetilde{\tau}_{k}\right)-\omega_{0}\left(\widetilde{\tau}_{k}\right)\right)
$$

In the following, we write $\widetilde{Y}_{k}\left(\omega_{0}, R, \widetilde{R}\right):=\left(h \circ \widetilde{\Psi}_{k}^{Q}\right)\left(\omega_{0}, R, \widetilde{R}\right)\left(\cdot+\widetilde{\tau}_{k}\right)$. Then, we can identify the jumps times of $\widetilde{\Psi}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)$ with the first hitting times of $\widetilde{Y}_{k}\left(\omega_{0}, R, \widetilde{R}\right), k \geqslant 0$, of the axes $\{(0, y): y>0\} \cup\{(x, 0): x \geqslant 0\}$ (cf. Lemma 2.3.4. Observe that $\left(\omega_{0}, R, \widetilde{R}\right)$ satisfies the conditions of Lemma 2.7 .9 for any $k \geqslant 0$ such that $\widetilde{\tau}_{k}<T$. Moreover, condition ii) yields that $\widetilde{\Psi}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)$ has only $N_{T}$ reinitializations in $[0, T]$, is not reinitialized at time $T$ and hence, implies that the function $N_{T}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow \mathbb{N}_{0} \cup\{+\infty\}$ is continuous at $\left(\omega_{0}, R, \widetilde{R}\right)$. Hence, $\widetilde{\Psi}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)=\widetilde{\Psi}_{N_{T}}^{Q}\left(\omega_{0}, R, \widetilde{R}\right)$. Finally, Lemma 2.7.9 and Lemma 2.7.10 yields that $\left(\omega_{0}, R, \widetilde{R}\right)$ lies in the continuity set of $\widetilde{\Psi}^{Q}$.

### 2.7.2.4 Continuity of $N_{b}$ and $N_{a}$

The goal of this subsection is to characterize the continuity sets of the maps

$$
N_{b}, N_{a}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{N}_{0} \cup\{+\infty\}\right)
$$

introduced in Definition 2.3.17. First, let us introduce the functions $N_{b, 1}, N_{a, 1}$ : $D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow\{0,1\}$ by

$$
\begin{align*}
& N_{b, 1}(\omega, R, \widetilde{R})=N_{b, 1}(\omega):=\mathbb{1}_{[\widetilde{\tau}(\omega), T]} \mathbb{1}_{\left\{\widetilde{\tau}_{b}(\omega)=\widetilde{\tau}(\omega)\right\}}  \tag{2.7.14}\\
& N_{a, 1}(\omega, R, \widetilde{R})=N_{a, 1}(\omega):=\mathbb{1}_{[\widetilde{\tau}(\omega), T]} \mathbb{1}_{\left\{\widetilde{\tau}_{a}(\omega)=\widetilde{\tau}(\omega)\right\}}
\end{align*}
$$

where the first hitting time maps $\widetilde{\tau}_{b}, \widetilde{\tau}_{a}$, and $\widetilde{\tau}$ are introduced in 2.3.6 and 2.3.7).
Corollary 2.7.11 (Continuity of $N_{b, 1}$ and $\left.N_{a, 1}\right) . \operatorname{Let}\left(\omega, R_{1}, \widetilde{R}_{1}\right) \in D\left([0, T], \mathbb{R}^{4}\right) \times$ $\left(\mathbb{R}_{+}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)$ satisfy the conditions of Lemma 2.7.8. Then, the maps $N_{b, 1}, N_{a, 1}$ are continuous at $\left(\omega, R_{1}, \widetilde{R}_{1}\right)$.

Proof. The result can be shown by replicating the proof of Lemma 2.7.8.
Now, for $k \geqslant 1$ and the short-hand notation $\widetilde{\Psi}_{k}^{Q}:=\widetilde{\Psi}_{k}^{Q}(\omega, R, \widetilde{R})$, let us introduce

$$
\begin{align*}
& N_{b, k+1}(\omega, R, \widetilde{R}):=N_{b, k}+\mathbb{1}_{\left[\widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}\right), T\right]} \mathbb{1}_{\left\{\widetilde{\tau}_{b}\left(\widetilde{\Psi}_{k}^{Q}\right)=\widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}\right)\right\}}  \tag{2.7.15}\\
& N_{a, k+1}(\omega, R, \widetilde{R}):=N_{a, k}+\mathbb{1}_{\left[\widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}\right), T\right]} \mathbb{1}_{\left\{\widetilde{\tau}_{a}\left(\widetilde{\Psi}_{k}^{Q}\right)=\widetilde{\tau}\left(\widetilde{\Psi}_{k}^{Q}\right)\right\}}
\end{align*}
$$

Corollary 2.7.12 (Continuity of $N_{b, k}$ and $\left.N_{a, k}\right)$. For $k \geqslant 1$, let $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right)$ $\times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfy the conditions of Lemma 2.7.9. Then, the maps $N_{b, k}$ and $N_{a, k}$ are continuous at $(\omega, R, \widetilde{R})$.

Proof. Again, the result can be shown by replicating the proof of Lemma 2.7.9.

A combination of the above corollaries with Lemma 2.7.10 yields a characterization of the continuity sets of the maps $N_{b}$ and $N_{a}$ introduced in Definition 2.3.17

Corollary 2.7.13 (Continuity of $N_{b}$ and $N_{a}$ ). Let $(\omega, R, \widetilde{R}) \in D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times$ $\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}}$ satisfy the conditions in Theorem 2.3.9. Then, the maps $N_{b}$ and $N_{a}$ introduced in Definition 2.3.17 are continuous at $(\omega, R, R)$.

Proof. Again, the result can be shown by replicating the proof of Theorem 2.3.9 but applying Corollary 2.7 .12 instead of Lemma 2.7.9.

### 2.7.3 Proofs of the auxiliary results

Proof of Lemma 2.3.4. For $k \in \mathbb{N}$, let $\hat{\tau}_{k}:=\hat{\tau}_{k}(\omega), \hat{\tau}_{\infty}:=\hat{\tau}_{\infty}(\omega)$, and $\tau:=\tau(\omega)$. Note that the function $g$ is inductively defined through the sequences of functions $\left(g_{k}\right)_{k \geqslant 1}$ and $\left(\widetilde{g}_{k}\right)_{k \geqslant 1}$ (cf. Definition 2.3.2). First, we assume that $\hat{\tau}_{k}<\hat{\tau}_{\infty}$ for all $k \in \mathbb{N}$. Then, we perform an induction over the sequence $\left(g_{k}\right)_{k \geqslant 1}$ and show that for all $g_{k}, k \in \mathbb{N}$, we have

$$
\left(h_{1} \circ g_{k}\right)(\omega)(t)=h_{1}(\omega)(t) \quad \text { for } t \in[0, T] .
$$

For $k=1$ we have $g_{1}(\omega)=\omega$ and hence $\left(h_{1} \circ g_{1}\right)(\omega)(t)=h_{1}(\omega)(t)$ for $t \in[0, T]$. Now, for $k=1, \cdots, l-1$, we assume that $\left(h_{1} \circ g_{k}\right)(\omega)(t)=h_{1}(\omega)(t)$ for $t \in[0, T]$. Next, let $k=l$. Then, by definition of $g_{l}$, we have

$$
\begin{aligned}
g_{l}(\omega)(t)= & \widetilde{g}_{l}(\omega)\left(t \wedge \hat{\tau}_{l}-\right)+\omega(t)-\omega\left(t \wedge \hat{\tau}_{l}-\right) \\
= & g_{l-1}(\omega)\left(t \wedge \hat{\tau}_{l}-\right)+\omega(t)-\omega\left(t \wedge \hat{\tau}_{l}-\right) \\
& +\sup _{s \leqslant t \wedge \hat{\tau}_{l}-}\left(-\pi_{1}^{(2)} g_{l-1}(\omega)(s)\right)^{+}(1,-1)+\sup _{s \leqslant t \wedge \hat{\tau}_{l}-}\left(-\pi_{2}^{(2)} g_{l-1}(\omega)(s)\right)^{+}(-1,1) .
\end{aligned}
$$

We note, that $h_{1}(-x, x) \equiv h_{1}(x,-x) \equiv 0$ for all $x \in \mathbb{R}$. Hence, using the induction hypothesis, we conclude for all $t \in[0, T]$

$$
\begin{aligned}
\left(h_{1} \circ g_{l}\right)(\omega)(t) & =\left(h_{1} \circ g_{l-1}\right)(\omega)\left(t \wedge \hat{\tau}_{l}-\right)+h_{1}(\omega)(t)-h_{1}(\omega)\left(t \wedge \hat{\tau}_{l}-\right) \\
& =h_{1}(\omega)\left(t \wedge \hat{\tau}_{l}-\right)+h_{1}(\omega)(t)-h_{1}(\omega)\left(t \wedge \hat{\tau}_{l}-\right) \\
& =h_{1}(\omega)(t) .
\end{aligned}
$$

This finishes our induction. By definition of $g$ we conclude

$$
\left(h_{1} \circ g\right)(\omega)(t)=h_{1}(\omega)(t) \quad \text { for } t \in\left[0, \hat{\tau}_{\infty}\right) .
$$

By assumption, we have that $\hat{\tau}_{\infty}=\tau$. It is left to consider the case, when $t=\tau$. Then, we have $g(\omega)(\tau)=(0,0)$. By construction, we must further have $h_{1}(\omega)(\tau) \leqslant 0$. Hence, $h_{1}(\omega)(\tau)+\sup _{s \leqslant \tau}\left(-h_{1}(\omega)(s)\right)^{+}=0=\left(h_{1} \circ g\right)(\omega)(\tau)$ and since $\sup _{s \leqslant t}\left(-h_{1}(\omega)(s)\right)^{+}=0$ for all $t \in[0, \tau)$, we finally conclude that

$$
\left(h_{1} \circ g\right)(\omega)(t)=h_{1}(\omega)(t)+\sup _{s \leqslant t}\left(-h_{1}(\omega)(s)\right)^{+} \quad \text { for } t \in[0, \tau] .
$$

In contrast, let us assume that there exists a finite $\kappa \in \mathbb{N}$ such that $\hat{\tau}_{\kappa}=\hat{\tau}_{\infty}=\tau$. If $\tau<T$, we have $g(\omega)=g_{\kappa}(\omega) \mathbb{1}_{[0, \tau)}$ on $[0, T]$ and we can argue as above to conclude that also $\left(h_{1} \circ g\right)(\omega)(t)=h_{1}(\omega)(t)+\sup _{s \leqslant t}\left(-h_{1}(\omega)(s)\right)^{+}$for $t \in[0, \tau]$. If $\tau=T$, then $g(\omega)=$ $\widetilde{g}_{\kappa+1}(\omega)$ which equals either $\tilde{g}_{\kappa+1}(\omega)=g_{\kappa}(\omega) \mathbb{1}_{[0, T)}$ or $\widetilde{g}_{\kappa+1}(\omega)=g_{\kappa}(\omega)+\ell^{(2)}\left(g_{\kappa}(\omega)\right) R$ depending on whether $\widetilde{g}_{\kappa}(\omega)(T) \in \mathbb{R}_{-}^{2}$ or not. Thus, we can again argue as above to conclude the stated result.

Proof of Lemma 2.3.31. Let $\left(\widetilde{\tau}_{k}^{*}\right)_{k \geqslant 0}$ be the sequence of stopping times introduced in (2.3.16) at which we observe a price change in $\widetilde{S}$ (except from $\widetilde{\tau}_{0}^{*}:=0$ ). Moreover, recall from the proof of Theorem 2.3.19 that for each $k \in \mathbb{N}_{0}$,

$$
\widetilde{B}_{k}:=\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)+X\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-X\left(\widetilde{\tau}_{k}^{*}\right)
$$

defines a four-dimensional linear Brownian motion starting in $\widetilde{Q}\left(\widetilde{\tau}_{k}^{*}\right)$ with drift $\mu$ and covariance matrix $\Sigma$. Note that by construction the increments of $B_{k}$ are independent of $\sigma\left(\widetilde{S}\left(t \wedge \widetilde{\tau}_{k}^{*}\right): t \in[0, T]\right)$, the sigma algebra generated by $\left\{\widetilde{S}\left(t \wedge \widetilde{\tau}_{k}^{*}\right): t \in[0, T]\right\}$. According to the proof of Theorem 2.3 .19 , we have $\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, b}^{*} \wedge \widetilde{\tau}_{k+1, a}^{*}$, where for $i=b, a$,

$$
\widetilde{\tau}_{k+1, i}^{*}:=\inf \left\{t>\widetilde{\tau}_{k}^{*}: \widetilde{B}_{k}^{i, F}+\widetilde{B}_{k}^{i, G}=0\right\} \wedge T, \quad k \in \mathbb{N}_{0} .
$$

Now let us introduce on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right]$ the processes

$$
\begin{aligned}
& C_{k}^{+}:=\frac{1}{2} L \cdot\left(\pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}_{k}\right)\right)+\frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}_{k}\right)\right), \\
& C_{k}^{-}:=\frac{1}{2} L \cdot\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}_{k}\right)\right)+\frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}_{k}\right)\right),
\end{aligned}
$$

where $g\left(\pi_{b} \widetilde{B}_{k}\right)$ and $g\left(\pi_{a} \widetilde{B}_{k}\right)$ behave as two non-identical, non-trivial semimartingale reflecting Brownian motions on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right]$ (cf. equation 2.3.22 ). Thanks to Assumption 2.2 the components of $\widetilde{B}_{k}$ are not perfectly correlated. Therefore, for $i=b, a$, all $x \in \mathbb{R} \backslash\{0\}$, and all $\pi_{i} \widetilde{B}_{k}$-measurable times $\eta$, we have

$$
\mathbb{P}\left[\left(C_{k}^{+}-C_{k}^{-}\right)(\eta)=x \mid \sigma\left(\widetilde{S}\left(t \wedge \widetilde{\tau}_{k}^{*}\right), \pi_{i} \widetilde{B}_{k}(t): t \in[0, T]\right)\right]=0
$$

By Theorem 2.3.28, we have $\widetilde{C}\left(\cdot+\widetilde{\tau}_{k}^{*}\right)-\widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right)=C_{k}^{+}-C_{k}^{-}$on $\left[0, \widetilde{\tau}_{k+1}^{*}-\widetilde{\tau}_{k}^{*}\right]$. Hence, for $i=b, a$,

$$
\begin{aligned}
\mathbb{P} & {\left[\left\{\sigma=\widetilde{\tau}_{k+1, i}^{*}\right\} \cap\left\{\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, i}^{*}\right\} \mid \sigma\left(\pi_{i} \widetilde{B}_{k}, \widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right)\right)\right] } \\
& =\mathbb{P}\left[\left\{\widetilde{C}\left(\widetilde{\tau}_{k+1, i}^{*}\right) \in\left\{-\kappa_{-}, \kappa_{+}\right\}\right\} \cap\left\{\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, i}^{*}\right\} \mid \sigma\left(\pi_{i} \widetilde{B}_{k}, \widetilde{C}\left(\widetilde{\tau}_{k}^{*}\right)\right)\right]=0,
\end{aligned}
$$

since $\widetilde{\tau}_{k+1, i}^{*}$ is $\pi_{i} \widetilde{B}_{k}$-measurable and $\kappa_{-}, \kappa_{+}$are fixed. In particular, for $i=b, a$, we also
have $\mathbb{P}\left[\left\{\sigma=\widetilde{\tau}_{k+1, i}^{*}\right\} \cap\left\{\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, i}^{*}\right\}\right]=0$ and hence for any $k \in \mathbb{N}_{0}$,

$$
\begin{aligned}
& \mathbb{P}\left[\sigma=\widetilde{\tau}_{k+1}^{*}\right] \\
& =\mathbb{P}\left[\left\{\sigma=\widetilde{\tau}_{k+1, b}^{*}\right\} \cap\left\{\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, b}^{*}\right\}\right]+\mathbb{P}\left[\left\{\sigma=\widetilde{\tau}_{k+1, a}^{*}\right\} \cap\left\{\widetilde{\tau}_{k+1}^{*}=\widetilde{\tau}_{k+1, a}^{*}\right\}\right]=0 .
\end{aligned}
$$

Proof of Lemma 2.3.32. We will show the result for $I=J=F$. Let $\widetilde{B}_{k}, k \geqslant 0$, be the sequence of four-dimensional linear Brownian motions introduced in Theorem 2.3.25, whose increments are independent over $k$, and denote $\widetilde{B}:=\widetilde{B}_{0}$. Note that $\widetilde{B}$ starts in $\widetilde{Q}_{0}=x=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in(0, \infty)^{4}$, has mean $\mu \in \mathbb{R}^{4}$ and covariance matrix $\Sigma \in \mathbb{R}^{4 \times 4}$ (cf. Proposition 2.3.1). Since $\widetilde{Q}$ is a Markov process and $\left(\widetilde{Q}^{b, F}, \widetilde{Q}^{a, F}\right)=$ $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ on $\left[0, \widetilde{\tau}_{1}^{*}\right)$, it is enough to prove that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[0, \widetilde{\tau}_{1}^{*}\right]$ with probability one, where $\widetilde{\tau}_{1}^{*}$ denotes the time of the first price change. Moreover, we introduce $\hat{\tau}_{1}^{*}:=\hat{\tau}^{*}\left(\pi_{b} \widetilde{B}\right) \wedge \hat{\tau}^{*}\left(\pi_{a} \widetilde{B}\right), \hat{\tau}_{2}^{*}:=$ $\hat{\tau}^{*}\left(\pi_{b} \widetilde{B}\right) \mathbb{1}\left(\hat{\tau}_{1}^{*}=\hat{\tau}^{*}\left(\pi_{a} \widetilde{B}\right)\right)+\hat{\tau}^{*}\left(\pi_{a} \widetilde{B}\right) \mathbb{1}\left(\hat{\tau}_{1}^{*}=\hat{\tau}^{*}\left(\pi_{b} \widetilde{B}\right)\right)$, and $\hat{\tau}^{*}(\omega):=\inf \{t \geqslant 0:$ $\left.\pi_{1}^{(2)} g(\omega)(t)=0\right\} \wedge T$.

Step 1: Show that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[0, \hat{\tau}_{1}^{*}\right] \mathbb{P}$-almost surely. On $\left[0, \hat{\tau}_{1}^{*}\right]$ we have

$$
\widetilde{Q}=\left(\begin{array}{c}
\widetilde{B}^{b, F}-\frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right) \\
\widetilde{B}^{b, G}+\frac{1}{2} L .\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right) \\
\widetilde{B}^{a, F}-\frac{1}{2} L .\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right) \\
\widetilde{B}^{a, G}+\frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)
\end{array}\right)
$$

and observe that

$$
\left(\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right),\left(\widetilde{B}^{b, G}, \widetilde{B}^{a, G}\right),\left(\frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right), \frac{1}{2} L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)\right)\right)
$$

solves the two-dimensional Skorokhod problem with reflection matrix being the identity on $\left[0, \hat{\tau}_{1}^{*}\right]$. Let $\hat{Y}_{1}:=\left(Y_{1}^{b, G}, Y_{1}^{a, G}\right)$ be a planar Brownian motion starting in $\left(x_{2}, x_{4}\right)$ with zero mean and covariance matrix

$$
\hat{\Sigma}_{1}:=\left(\begin{array}{cc}
\left(\sigma^{b, G}\right)^{2} & \sigma^{(b, G),(a, G)} \\
\sigma^{(b, G),(a, G)} & \left(\sigma^{a, G}\right)^{2}
\end{array}\right)
$$

and define $\hat{W}_{1}:=\left(W_{1}^{b, G}, W_{1}^{a, G}\right)$ by $\hat{W}_{1}(t)=\left(x_{2}, x_{4}\right)+\int_{0}^{t} \operatorname{sign}_{2}\left(\hat{Y}_{1}(s)\right) d \hat{Y}_{1}(s)$, where

$$
\operatorname{sign}_{2}\left(\hat{Y}_{1}(t)\right):=\left(\begin{array}{cc}
\operatorname{sign}\left(Y_{1}^{b, G}(t)\right) & 0 \\
0 & \operatorname{sign}\left(Y_{1}^{a, G}(t)\right)
\end{array}\right) .
$$

In particular, $d \hat{W}_{1}(t)=\operatorname{sign}_{2}\left(\hat{Y}_{1}(t)\right) d \hat{Y}_{1}(t)$. Hence, $\left(\hat{Y}_{1}, \hat{W}_{1}\right)$ is a weak solution of the

## SDE

$$
d \hat{Y}_{1}(t)=\operatorname{sign}_{2}\left(\hat{Y}_{1}(t)\right) d \hat{W}_{1}(t)
$$

With Levy's characterization of a Brownian motion, we conclude that ( $\hat{Y}_{1}, \hat{W}_{1}$ ) is unique in law. Note that this SDE is a two-dimensional version of the Tanaka SDE with covariance $\hat{\Sigma}_{1}$ being not necessarily the identity matrix. Next, let us apply Tanaka's formula (cf. e.g. [89, Proposition 2.11]) and observe that

$$
\begin{aligned}
\left|Y_{1}^{b, G}(t)\right| & =x_{2}+\int_{0}^{t} \operatorname{sign}\left(Y_{1}^{b, G}(s)\right) d Y_{1}^{b, G}(s)+L_{t}\left(Y_{1}^{b, G}\right) \\
& =x_{2}+\int_{0}^{t}\left(\operatorname{sign}\left(Y_{1}^{b, G}(s)\right)\right)^{2} d W_{1}^{b, G}(s)+L_{t}\left(Y_{1}^{b, G}\right) \\
& =W_{1}^{b, G}(t)+L_{t}\left(Y_{1}^{b, G}\right)
\end{aligned}
$$

Moreover, we can show that $\left|Y_{1}^{a, G}(t)\right|=W^{a, G}(t)+L_{t}\left(Y_{1}^{a, G}\right)$. Then,

$$
\left(\left(\left|Y_{1}^{b, G}\right|,\left|Y_{1}^{a, G}\right|\right),\left(W_{1}^{b, G}, W_{1}^{a, G}\right),\left(L \cdot\left(Y_{1}^{b, G}\right), L \cdot\left(Y_{1}^{a, G}\right)\right)\right)
$$

solves the two-dimensional Skorokhod problem with reflection matrix being the identity. Similarly as in the proof of Lemma 2.7.5, let us take another planar Brownian motion $\widetilde{W}_{1}:=\left(W_{1}^{b, F}, W_{1}^{a, F}\right)$, possibly on a larger probability space, starting in $\left(x_{1}, x_{3}\right)$ and being correlated with $\hat{W}_{1}$ such that $W_{1}:=\left(W_{1}^{b, F}, W_{1}^{b, G}, W_{1}^{a, F}, W_{1}^{a, G}\right) \simeq(\widetilde{B}(t)-\mu t)_{t \geqslant 0}$. Let $Y_{1}^{i, F}:=W_{1}^{i, F}-L .\left(Y_{1}^{i, G}\right)=W_{1}^{i, F}+W_{1}^{i, G}-\left|Y_{1}^{i, G}\right|$ for $i=b, a$. Then,

$$
\begin{aligned}
\mathbb{P}[\exists t \in & {\left.[0, T]:\left(Y_{1}^{b, F}(t), Y_{1}^{a, F}(t)\right)=(0,0)\right] } \\
= & \mathbb{P}\left[\exists t \in[0, T]:\left(W_{1}^{b, F}+W_{1}^{b, G}-\left|Y_{1}^{b, G}\right|, W_{1}^{a, F}+W_{1}^{a, G}-\left|Y_{1}^{a, G}\right|\right)(t)=(0,0)\right] \\
\leqslant \mathbb{P} & {\left[\exists t:\left(W_{1}^{b, F}+W_{1}^{b, G}-Y_{1}^{b, G}, W_{1}^{a, F}+W_{1}^{a, G}-Y_{1}^{a, G}\right)(t)=(0,0)\right] } \\
& +\mathbb{P}\left[\exists t:\left(W_{1}^{b, F}+W_{1}^{b, G}+Y_{1}^{b, G}, W_{1}^{a, F}+W^{a, G}-Y_{1}^{a, G}\right)(t)=(0,0)\right] \\
& +\mathbb{P}\left[\exists t:\left(W_{1}^{b, F}+W^{b, G}-Y_{1}^{b, G}, W_{1}^{a, F}+W_{1}^{a, G}+Y_{1}^{a, G}\right)(t)=(0,0)\right] \\
& +\mathbb{P}\left[\exists t:\left(W^{b, F}+W^{b, G}+Y_{1}^{b, G}, W_{1}^{a, F}+W_{1}^{a, G}+Y_{1}^{a, G}\right)(t)=(0,0)\right] .
\end{aligned}
$$

As in the proof of Lemma 2.7.5 one can argue that each probability on the RHS equals zero. Now, applying the Cameron-Martin-Girsanov theorem, there exists an equivalent measure $\mathbb{Q}$ of $\mathbb{P}$ such that $W_{1}$ behaves like a four-dimensional linear Brownian motion starting in $x \in(0, \infty)^{4}$ with covariance $\Sigma$ and drift $\mu$ under $\mathbb{Q}$. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, $\left(Y_{1}^{b, F}, Y_{1}^{a, F}\right)$ does not hit the origin $\mathbb{Q}$-almost surely. By the uniqueness of the solution of the Skorokhod problem, we conclude that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ on $\left[0, \hat{\tau}_{1}^{*}\right]$ has the same distribution as $\left(Y_{1}^{b, F}, Y_{1}^{a, F}\right)$, under $\mathbb{Q}$, until one of its components hits zero the first time. Hence, $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on [ $\left.0, \hat{\tau}_{1}^{*}\right]$ with probability one.

Step 2: On the event $\left\{\hat{\tau}_{1}^{*}=\hat{\tau}^{*}\left(\pi_{b} \widetilde{B}\right)\right\}$, prove that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right)\right] \mathbb{P}$-almost surely. Let us concentrate on the event
$\left\{\hat{\tau}_{1}^{*}=\hat{\tau}^{*}\left(\pi_{b} \widetilde{B}\right)\right\}$ and introduce $\bar{\tau}:=\inf \left\{t \geqslant \hat{\tau}_{1}^{*}: \pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right)(t)=0\right\} \wedge T$ the first time after $\hat{\tau}_{1}^{*}$ that $\pi_{2}^{(2)} g\left(\pi_{b} \widetilde{B}\right)$ hits zero. On $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \bar{\tau}_{2}\right]$, we have

$$
\widetilde{Q}=G(\widetilde{B})\left(\hat{\tau}_{1}^{*}\right)+\left(\begin{array}{c}
\widetilde{B}^{b, F}-\widetilde{B}^{b, F}\left(\hat{\tau}_{1}^{*}\right)+\frac{1}{2} L .\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right) \\
\widetilde{B}^{b, G}-\widetilde{B}^{b, G}\left(\hat{\tau}_{1}^{*}\right)-\frac{1}{2} L .\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right) \\
\widetilde{B}^{a, F}-\widetilde{B}^{a, F}\left(\hat{\tau}_{1}^{*}\right)-\frac{1}{2}\left(L .\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)-L_{\hat{\tau}_{1}^{*}}\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)\right) \\
\widetilde{B}^{a, G}-\widetilde{B}^{a, G}\left(\hat{\tau}_{1}^{*}\right)+\frac{1}{2}\left(L .\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)-L_{\hat{\tau}_{1}^{*}}\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)\right)
\end{array}\right)
$$

and observe that

$$
\begin{gathered}
\left(\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right),\left(\widetilde{B}^{b, F}-\widetilde{B}^{b, F}\left(\hat{\tau}_{1}^{*}\right), \pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\left(\hat{\tau}_{1}^{*}\right)+\widetilde{B}^{a, G}-\widetilde{B}^{a, G}\left(\hat{\tau}_{1}^{*}\right)\right),\right. \\
\left.\left(\frac{1}{2} L \cdot\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right)\right), \frac{1}{2}\left\{L \cdot\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)-L_{\hat{\tau}_{1}^{*}}\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)\right\}\right)\right)
\end{gathered}
$$

solves the two-dimensional Skorokhod problem with reflection matrix being the identity on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \bar{\tau}_{2}\right]$. In the following, let $G(\widetilde{B})\left(\hat{\tau}_{1}^{*}\right)=y:=\left(0, y_{2}, y_{3}, y_{4}\right)$. Then analogously as above, for a planar Brownian motion $\hat{Y}_{2}:=\left(Y_{2}^{b, F}, Y_{2}^{a, G}\right)$ starting in $\left(0, y_{4}\right)$ with mean zero and covariance matrix

$$
\hat{\Sigma}_{2}:=\left(\begin{array}{cc}
\left(\sigma^{b, F}\right)^{2} & \sigma^{(b, F),(a, G)} \\
\sigma^{(b, F),(a, G)} & \left(\sigma^{a, G}\right)^{2}
\end{array}\right)
$$

we define $\hat{W}_{2}:=\left(W_{2}^{b, F}, W_{2}^{a, G}\right)$ by $\hat{W}_{2}=\left(0, y_{4}\right)+\int_{0}^{t} \operatorname{sign}_{2}\left(\hat{Y}_{2}(s)\right) d \hat{Y}_{2}(s)$. Then, $\left(\hat{Y}_{2}, \hat{W}_{2}\right)$ is again a weak solution, unique in law, of the two-dimensional Tanaka SDE with covariance matrix $\hat{\Sigma}_{2}$. Applying Tanaka's formula, it follows that $\left|Y_{2}^{b, F}(t)\right|=W_{2}^{b, F}(t)+L_{t}\left(Y_{2}^{b, F}\right)$ and $\left|Y_{2}^{a, G}(t)\right|=W_{2}^{a, G}(t)+L_{t}\left(Y_{2}^{a, G}\right)$, and

$$
\left(\left(\left|Y_{2}^{b, F}\right|,\left|Y_{2}^{a, G}\right|\right),\left(W_{2}^{b, F}, W_{2}^{a, G}\right),\left(L .\left(Y_{2}^{b, F}\right), L .\left(Y_{2}^{a, G}\right)\right)\right)
$$

solves the two-dimensional Skorokhod problem with reflection matrix being the identity. Next, let us take another planar Brownian motion $\widetilde{W}_{2}:=\left(W_{2}^{b, G}, W_{2}^{a, F}\right)$, possibly on a larger probability space, starting in $\left(y_{2}, y_{3}\right)$ and being correlated with $\hat{W}_{2}$ such that $W_{2}:=\left(W_{2}^{b, F}, W_{2}^{b, G}, W_{2}^{a, F}, W_{2}^{a, G}\right) \simeq((y-x)+\widetilde{B}(t)-\mu t)_{t \geqslant 0}$. Moreover, let $Y_{2}^{b, G}:=$ $W_{2}^{b, G}-L .\left(Y_{2}^{b, F}\right)=W_{2}^{b, F}+W_{2}^{b, G}-\left|Y_{2}^{b, F}\right|$ and $Y_{2}^{a, F}:=W_{2}^{a, F}-L .\left(Y_{2}^{a, G}\right)=W_{2}^{a, F}+$ $W_{2}^{a, G}-\left|Y_{2}^{a, F}\right|$. Then, similarly as above, we conclude that

$$
\mathbb{P}\left[\exists t \in[0, T]:\left(\left|Y_{2}^{b, F}(t)\right|, Y_{2}^{a, F}(t)\right)=(0,0)\right]=0 .
$$

Again, applying the Cameron-Martin-Girsanov theorem, we can change to the equivalent probability measure $\mathbb{Q}$ of $\mathbb{P}$, under which $W_{2}$ behaves like a four-dimensional linear Brownian motion with drift $\mu$ and $\left(\left|Y_{2}^{b, F}\right|, Y_{2}^{a, F}\right)$ does not hit the origin $\mathbb{Q}$-almost surely. By the uniqueness of the solution of the Skorokhod problem, we conclude that
$\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \bar{\tau}_{2}\right]$ has the same distribution as $\left(\left|Y_{2}^{b, F}\right|, Y_{2}^{a, F}\right)$, under $\mathbb{Q}$, until $Y_{2}^{b, G}$ or $Y_{2}^{a, F}$ hits zero the first time. Hence, $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \bar{\tau}_{2}\right]$.

Applying the Markov property of $G(\widetilde{B})$, we conclude from the above together with step 1 that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \hat{\tau}_{k}\left(\pi_{b} \widetilde{B}\right)\right]$ with probability one for all $k \in \mathbb{N}$. Since $\hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right)$ equals the first hitting time of $h_{1}\left(\pi_{b} \widetilde{B}\right)$ at zero (cf. Lemma 2.7.6, we can study the process $\left(h_{1}\left(\pi_{b} \widetilde{B}\right), \widetilde{B}^{a, F}-\frac{1}{2} L .\left(\pi_{2}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)\right.$ ) in a similar manner to conclude that also $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[\hat{\tau}_{1}^{*}, \hat{\tau}_{2}^{*} \wedge \hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right)\right]$ with probability one.

Step 3: Show that $\hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right) \neq \hat{\tau}_{\infty}\left(\pi_{a} \widetilde{B}\right)$ with probability one. Applying Lemma 2.7.6 we conclude that $\hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right)$ and $\hat{\tau}_{\infty}\left(\pi_{a} \widetilde{B}\right)$ agree with the first hitting times of the onedimensional Brownian motions $h_{1}\left(\pi_{b} \widetilde{B}\right)$ and $h_{1}\left(\pi_{a} \widetilde{B}\right)$ at zero. Since $\left(h_{1}\left(\pi_{b} \widetilde{B}\right), h_{1}\left(\pi_{a} \widetilde{B}\right)\right.$ ) is again a planar Brownian motion whose components are not perfectly correlated, we conclude that $\left(h_{1}\left(\pi_{b} \widetilde{B}\right), h_{1}\left(\pi_{a} \widetilde{B}\right)\right.$ ) does not hit the origin with probability one. Hence, $\hat{\tau}_{\infty}\left(\pi_{b} \widetilde{B}\right) \neq \hat{\tau}_{\infty}\left(\pi_{a} \widetilde{B}\right) \mathbb{P}$-almost surely.

Step 4: End of the proof. Combining the results of step 2 and step 3 together with the Markov property of $\widetilde{Q}$, we conclude that $\left(\pi_{1}^{(2)} g\left(\pi_{b} \widetilde{B}\right), \pi_{1}^{(2)} g\left(\pi_{a} \widetilde{B}\right)\right)$ does not hit the origin on $\left[0, \widetilde{\tau}_{1}^{*}\right] \mathbb{P}$-almost surely. Since the process $\widetilde{Q}$ is reinitialized at times of price changes at a new value in $(0, \infty)^{4}$ and by applying again the Markov property of $\widetilde{Q}$, we conclude that ( $\widetilde{Q}^{b, F}, \widetilde{Q}^{a, F}$ ) does not hit the origin with probability one on the whole interval $[0, T]$. This finishes the proof.

Proof of Theorem 2.3.26. To identify the infinitesimal generator of the process $\widetilde{Q}$, we note that $h \in C^{0}\left(\mathbb{R}_{+}^{4}, \mathbb{R}\right)$ is in the domain of the infinitesimal generator if for all $x \in \mathbb{R}_{+}^{4}$,

$$
\lim _{t \rightarrow 0} \frac{\mathbb{E}_{x}[h(\widetilde{Q}(t))]-h(x)}{t}<\infty,
$$

where $\mathbb{E}_{x}[h(\widetilde{Q}(t))]:=\mathbb{E}[h(\widetilde{Q}(t)) \mid \widetilde{Q}(0)=x]$. For $x \in \widetilde{\mathbb{R}}_{+}^{4}$ and $h \in C^{2}\left(\widetilde{\mathbb{R}_{+}^{4}}, \mathbb{R}\right)$, we can apply the Itô-Tanaka formula and obtain

$$
\begin{align*}
\mathbb{E}_{x}[h(\widetilde{Q}(t))]-h(x) & -\mathbb{E}_{x}\left[\int_{0}^{t} \mathcal{A} h(\widetilde{Q}(s)) d s\right] \\
& =\frac{1}{2} \sum_{j=1}^{4} \mathbb{E}_{x}\left[\int_{0}^{t} \widetilde{\partial_{x_{j}} h}(\widetilde{Q}(s)) d L_{s}\left(\pi_{j} \widetilde{Q}\right)\right], \tag{2.7.16}
\end{align*}
$$

where $\mathcal{A} h$ is given in 2.3.24) and

$$
\begin{array}{ll}
\widetilde{\partial_{x_{1}} h}:=\frac{\partial h}{\partial x_{1}}-\frac{\partial h}{\partial x_{3}}, & \widetilde{\partial_{x_{3}} h}:=-\frac{\partial h}{\partial x_{1}}+\frac{\partial h}{\partial x_{3}}, \\
\widetilde{\partial_{x_{2}} h}:=\frac{\partial h}{\partial x_{2}}-\frac{\partial h}{\partial x_{4}}, & \widetilde{\partial_{x_{4}} h}:=-\frac{\partial h}{\partial x_{2}}+\frac{\partial h}{\partial x_{4}} .
\end{array}
$$

Note, that $t \mapsto L_{t}\left(\pi_{j} \widetilde{Q}\right), j=1,2,3,4$, increases only on the set $\left\{t \geqslant 0: \pi_{j} \widetilde{Q}(t)=0\right\}$.

Hence, if $h$ satisfies, for all $x_{1}, x_{2}, x_{3}, x_{4}>0$ and all $y=\left(y_{1}, y_{2}, y_{3}, y_{4}\right) \in \widetilde{\mathbb{R}}_{+}^{4}$,

$$
\begin{array}{ll}
\widetilde{\partial_{x_{1}} h}\left(0, y_{2}, x_{3}, y_{4}\right)=0, & \widetilde{\partial_{x_{2}} h}\left(y_{1}, 0, y_{3}, x_{4}\right)=0, \\
\widetilde{\partial_{x_{3}} h} h\left(x_{1}, y_{2}, 0, y_{4}\right)=0, & \widetilde{\partial_{x_{4}} h}\left(y_{1}, x_{2}, y_{3}, 0\right)=0, \tag{2.7.17}
\end{array}
$$

then the right hand side in 2.7.16) equals zero and therefore, $\mathcal{A} h$ is indeed the infinitesimal generator of $\widetilde{Q}$ on $\widetilde{\mathbb{R}}_{+}^{4}$ since $h(\widetilde{Q}(t))-h(\widetilde{Q}(0))-\int_{0}^{t} \mathcal{A} h(\widetilde{Q}(s)) d s$ is a martingale. This yields equation (2.3.24) and the first boundary condition. Let $f^{+}, f^{-}$be the distributions introduced in Assumption 2.4. Then, for all $x=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \widetilde{\mathbb{R}}_{+}^{4}$, we have

$$
\begin{aligned}
\mathbb{E} & {\left[h(\widetilde{Q}(t)) \mid \widetilde{Q}(0)=\left(x_{1}, 0, x_{3}, 0\right)\right] } \\
& =\int_{\mathbb{R}_{+}^{4}} \mathbb{E}\left[h(\widetilde{Q}(t)) \mid \widetilde{Q}(0+)=\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right] f^{+}(d u) \\
& =\int_{\mathbb{R}_{+}^{4}}\left(\operatorname{tAh}\left(\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right)+h\left(\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right)\right) f^{+}(d u)+o(t) .
\end{aligned}
$$

As $t \rightarrow 0$, this leads to

$$
\begin{aligned}
& \frac{\mathbb{E}\left[h(\widetilde{Q}(t)) \mid \widetilde{Q}(0)=\left(x_{1}, 0, x_{3}, 0\right)\right]-h\left(x_{1}, 0, x_{3}, 0\right)}{t} \\
& \begin{aligned}
&=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right) f^{+}(d u) \\
& \quad+\frac{1}{t} \int_{\mathbb{R}_{+}^{4}}\left[h\left(\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right)-h\left(x_{1}, 0, x_{3}, 0\right)\right] f^{+}(d u)+o(1) .
\end{aligned}
\end{aligned}
$$

Similarly, we obtain as $t \rightarrow \infty$ that

$$
\begin{aligned}
& \frac{\mathbb{E}\left[h(\widetilde{Q}(t)) \mid \widetilde{Q}(0)=\left(0, x_{2}, 0, x_{4}\right)\right]-h\left(0, x_{2}, 0, x_{4}\right)}{t} \\
& \begin{aligned}
&=\int_{\mathbb{R}_{+}^{4}} \mathcal{A} h\left(\Phi\left(\left(0, x_{2}, 0, x_{4}\right), u\right)\right) f^{-}(d u) \\
& \quad+\frac{1}{t} \int_{\mathbb{R}_{+}^{4}}\left[h\left(\Phi\left(\left(0, x_{2}, 0, x_{4}\right),(u)\right)\right)-h\left(0, x_{2}, 0, x_{4}\right)\right] f^{-}(d u)+o(1)
\end{aligned}
\end{aligned}
$$

Thus the limit $t \rightarrow 0$ is only well defined if $h$ further verifies for all $x=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in$ $\widetilde{\mathbb{R}}_{+}^{4}$,

$$
\begin{align*}
& h\left(x_{1}, 0, x_{3}, 0\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\Phi\left(\left(x_{1}, 0, x_{3}, 0\right), u\right)\right) f^{+}(d u), \\
& h\left(0, x_{2}, 0, x_{4}\right)=\int_{\mathbb{R}_{+}^{4}} h\left(\Phi\left(\left(0, x_{2}, 0, x_{4}\right), u\right)\right) f^{-}(d u) . \tag{2.7.18}
\end{align*}
$$

This yields equations 2.3.25 and the second boundary condition. Note that the boundary conditions in 2.7 .17 and 2.7 .18 are together a Wentzell boundary condition (cf. Taira $[76]$ ) which corresponds to a reflection and a jump to the interior, respectively, whenever the process reaches the boundary. Hence, the domain $\operatorname{dom}(\mathcal{A})$ of $\mathcal{A}$ is given by the set

$$
\operatorname{dom}(\mathcal{A}):=\left\{h \in C^{2}\left(\widetilde{\mathbb{R}}_{+}^{4}, \mathbb{R}\right) \cap C^{0}\left(\mathbb{R}_{+}^{4}, \mathbb{R}\right): \quad h \text { verifies 2.7.17 and 2.7.18 }\right\}
$$

It follows that $\mathcal{A}$ is an elliptic operator defined by the Laplacian on $(0, \infty)^{4}$ with Wentzell boundary condition. Thanks to Assumption 2.4, our boundary condition is transversal. Hence, Theorem 1 in 76 implies the existence of an $\mathbb{R}_{+}^{4}$-valued Feller process, unique in law, whose infinitesimal generator is $(\mathcal{A}, \operatorname{dom}(\mathcal{A}))$. Therefore, the limit process $\widetilde{Q}$ is an $\mathbb{R}_{+}^{4}$-valued Markov process associated with this semigroup.

## 3 Parametric change point detection with random occurrence of the change point


#### Abstract

We are concerned with the problem of detecting a single change point in the model parameters of time series data generated from an exponential family. In contrast to the existing literature, we allow that the true location of the change point is itself random, possibly depending on the data. Under the alternative, we study the case when the size of the change $\Delta^{2}$ in the parameter converges to zero while the sample size goes to infinity. Moreover, we concentrate on change points in the "middle of the data", i.e., we assume that the change point fraction $\lambda_{n}^{*}$ (the location of the change point relative to the sample size) satisfies $\left|\lambda_{n}^{*}-\lambda^{*}\right|=o_{\mathbb{P}}\left(\left(n \Delta^{2}\right)^{-1}\right)$ and $\lambda^{*}$ is a random variable which takes its values almost surely in a closed subset of $(0,1)$. We show that the known statistical results from the literature also transfer to this setting. We substantiate our theoretical results with a simulation study.


### 3.1 Introduction

Detecting structural changes in the parameters of time series data is of great interest from both econometric and statistical perspectives. Traditionally, one is faced with the question of whether the underlying time series data contain one or more change points. While some literature (cf. e.g. [3, 7, 23]) has also investigated the detection of multiple change points, in this chapter we focus only on so-called "at most one change point" (AMOC) models. Assuming the location of the change point is known, one can interpret the question of deciding whether or not the data contain a change point as a two-sample test problem. However, the location of a change point is typically unknown. Many works on change point models already provide statistical tests to answer this question for unknown change points. While many authors build their tests assuming that the change point occurs only in a single model parameter (typically in its mean, cf. e.g. [51] or in its variance, cf. e.g. [3. 74]), Horváth [44], Gombay and Horváth [34-36], and Csörgő and Horváth [23] provide likelihood ratio-based tests that check for a simultaneous change in the parameter of quite general parametric distributions including exponential families, see Csörgő and Horváth [23] for an overview. Under a long-span asymptotic scheme, in which the time span of data is assumed to go to infinity, the existing literature provides theory for the estimation of a fractional change point (the location of the change point relative to the sample size), including the consistency, the rate of convergence, and the limiting distribution (cf. e.g. [3, 23, 51]). Although in

### 3.1. INTRODUCTION

most of the literature, time series data are considered, some authors study the detection of a change in the drift and/or the volatility process of continuous-time diffusions or more general Itô-semimartingales assuming that a continuous record or a discrete-time record with mesh size converging to zero over a finite time span is available (cf. e.g. $[7,47,51])$. If the size of the model parameters relates appropriately with the samples size, one might approximate the time series data by a continuous-time model and hence might be able to connect the findings for time series data with the theory developed for continuous-time processes. For example, if the time series data are normally distributed and the mean is of order $n^{-1 / 2}$ while the volatility is of order 1 ( $n$ denotes the sample size), for large $n$, the $n^{-1 / 2}$-scaled partial sum can be approximated by a diffusion process. Because of the different scaling in $n$ in the mean and volatility, a change in the mean is typically much harder to detect than in the volatility. Even more, when studying the detection of changes in the mean, the existing literature reveals that the consistency of an estimator for the location of the change point can only be obtained if the size of the change is of larger order than $n^{-1 / 2}$. But then, the approximation of the time series data by a continuous-time model fails since the size of the change in the mean explodes as $n \rightarrow \infty$. For this reason, [51] studied the asymptotic properties of the change point estimator in the crucial case when the shift in the mean is of order $n^{-1 / 2}$. While in this work, we mainly focus on parametric models, the recent literature also provides tools for the detection of structural changes in non-parametric models such as in the volatility process of an Itô-semimartingale (cf. e.g. [7]), or the mean or location parameter of time series data (cf. e.g. [23]).

Calibration of mathematical models is one of the main concerns from a practitioner's point of view. It is well known that change points are present in high-frequency financial data. In the referenced literature the location of the change point is unknown but deterministic. However, if the change point is caused by endogenous effects, the dependence on the underlying data must be considered. The integrated European intraday electricity market "Single Intraday Coupling" (SIDC) is a real-world example in which change points might be endogenously caused. In this market, multiple national limit order books are coupled, i.e., summarized in a single shared order book such that market orders are allowed to be matched with standing volumes of the domestic and foreign limit order books. However, the coupling of multiple markets is only maintained as long as transmission capacities are available. In contrast, if the transmission capacities are fully occupied, market orders can only be matched with standing volumes of the same origin. The switch between these two regimes typically leads to structural changes in the trading behavior. In Chapter 2, we construct cross-border market dynamics including prices, standing volumes at the best bid and ask prices, and capacities from the underlying net order flow process and the total available transmission capacities. The time of a regime switch is then modeled by a stopping time depending on the net order flow and on the total available capacities. While the order flow is publicly available, the transmission capacities are harder to obtain and therefore often unknown. Hence, in order to calibrate the model to high-frequency data, the time of a regime switch, that depends on the observed data, must be estimated.

Models that have been studied in the literature, in which the location of a change
point is itself random are, for example, so-called Markov switching models. In these models, the type of the regime depends on an unobserved Markov process which is independent of the data (cf. e.g. 17,26$]$ ). Despite this, to the best of our knowledge, the existing literature on randomly occurring change points is rather limited. In our work, we extend the statistical results in Csörgő and Horváth [23 to randomly occurring change points, possibly depending on the data. Throughout, we assume that the data points are independent and only study the case when the size of the change $\Delta^{2}$ in the parameter converges to zero while the sample size $n$ goes to infinity. From a statistical point of view, this case describes the crucial setting as it answers the question which minimum size of a change in the parameter is detectable. If the null hypothesis $H_{0}$ is "no change point" and the alternative $H_{1}$ is "there is one change point", then we discuss the problem of distinguishing between $H_{0}$ and $H_{1}$. Moreover, under the alternative $H_{1}$, if $k_{n}^{*} \in\{1, \cdots, n-1\}$ denotes the true but random location of the change point, we concentrate on change points in the "middle of the data", i.e., we assume that the change point fraction $\lambda_{n}^{*}:=k_{n}^{*} / n$ satisfies $\left|\lambda_{n}^{*}-\lambda^{*}\right|=o_{\mathbb{P}}\left(\left(n \Delta^{2}\right)^{-1}\right)$, where $\lambda^{*}$ is a random variable taking values in a closed subset of $(0,1)$ with probability one.

Our model should be understood as a first step toward the extension of the very general change point theory in $\sqrt[23]{ }$ to randomly occurring change points. We show that the statistical properties of the test statistic as well as of the estimator for the location of the change point transfer from the deterministic setting considered in 23 to randomly occurring change points. While this might be clear under the null hypothesis $H_{0}$, this is not obvious under the alternative. In particular, our work shows that the theory in [23] can also be applied in the model framework introduced in Chapter 2 in which the location of a regime switch depends on the underlying net order flow process.

To extend the results in [23], the main difficulty is to show that the limit result for the test statistic under the alternative holds true uniformly for all possible values of the location of the change point. Therefore, we introduce an alternative test statistic depending on two time parameters (i.e. on the true and estimated location of the change point) and show that if this test statistic is scaled appropriately, it converges weakly in the Skorokhod topology to a Gaussian process with two time parameters. The hard part of the proof is to correctly identify the finite-dimensional distributions of the limit process. This can be nicely simplified by an application of the fourth moment theorem (cf. Theorem 1 in 68]) since we concentrate on normally distributed data when studying the asymptotics under the alternative. After establishing the limit theorem of the test statistic under the alternative, it is indeed straight-forward to prove the known results in 23 also for randomly occurring change points. We provide empirical support for our theoretical results through a detailed simulation study. Moreover, in this simulation study we also discuss two important generalizations of our model: weakly dependent observations and non-parametric change point detection in the volatility process of an Itô-semimartingale. It turns out, at least empirically, that change point detection works for these cases as well, even if the location of the change point depends on the data.

Structure of the chapter: In Section 2, we introduce the model framework and the test statistic based on the so-called maximally selected log-likelihood ratio. Since under
the null hypothesis, no change point occurs, in Section 3, we repeat the results for the asymptotics of the considered test statistic under the null hypothesis in [23]. Under the alternative, our more general setting of a change point with random location becomes important. Therefore, in Section 4, we present a new test statistic depending on the location of a change point and derive its limit distribution relative to the location of a change point (cf. Theorem 3.4.4). To simplify the proof, we assume in Section 4 that the observations are normally distributed. Moreover, we introduce an estimator for the fractional change point and establish its consistency, the convergence rate, and the limit distribution. The latter is also stated in a distribution-free version, which allows to build confidence intervals for the true location of the change point based on the data. We finish this chapter by a detailed simulation study in Section 5 .

Notation: In the following, for each $x \in \mathbb{R}^{d}, d \geqslant 1$, let us denote by $\|x\|:=\left(x^{T} x\right)^{1 / 2}$ the euclidean norm in $\mathbb{R}^{d}$. For a family of random variables $\left(X_{n}\right)_{n \in \mathbb{N}}$ and a positive deterministic sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$, we write $X_{n}=o_{\mathbb{P}}\left(a_{n}\right)$ if $X_{n} / a_{n}$ converges to zero in probability as $n \rightarrow \infty$. Moreover, we denote $X_{n}=\mathcal{O}_{\mathbb{P}}\left(a_{n}\right)$ if for any $\varepsilon>0$, there exist $\eta>0$ and $N \in \mathbb{N}$ such that such that $\mathbb{P}\left[\left|X_{n} / a_{n}\right|>\eta\right]<\varepsilon$ for all $n>N$. Similarly, for another deterministic sequence $\left(b_{n}\right)_{n \in \mathbb{N}}$, we write $b_{n}=o\left(a_{n}\right)$ if $b_{n} / a_{n}$ converges to zero as $n \rightarrow \infty$ and $b_{n}=\mathcal{O}\left(a_{n}\right)$ if there exist $\eta>0$ and $N \in \mathbb{N}$ such that $b_{n} / a_{n} \leqslant \eta$ for all $n>N$. Furthermore, we write $\mathbb{P}[A, B]:=\mathbb{P}[A \cap B]$ for $A, B \in \mathcal{F}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

### 3.2 Setup

Throughout, we assume that all random variables are defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X_{1}, \cdots, X_{n}$ be independent observations in $\mathbb{R}^{m}$ which have densities $f_{X_{j}}, j=1, \cdots, n$, with respect to some $\sigma$-finite measure $\nu$ being element of the exponential family, i.e.,

$$
\begin{equation*}
f_{X_{j}}(x)=f\left(x ; \theta_{j}\right)=\exp \left(\theta_{j}^{T} T(x)+S(x)-A\left(\theta_{j}\right)\right) \mathbb{1}_{\{x \in C\}}, \tag{3.2.1}
\end{equation*}
$$

where $x=\left(x_{1}, \cdots, x_{m}\right)^{T}, \theta_{j}=\left(\theta_{j, 1}, \cdots, \theta_{j, d}\right)^{T} \in \Theta \subset \mathbb{R}^{d}, S, T_{1}, \cdots, T_{d}:(\Omega, \mathcal{F}) \rightarrow$ $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable functions with $T=\left(T_{1}, \cdots, T_{d}\right)^{T}, A: \mathbb{R}^{d} \rightarrow \mathbb{R}$, and $C \subset \mathbb{R}^{m}$. Note that the representation of the density in (3.2.1) is often referred to as the natural parametrization of an exponential family.

In our work, we want to test the null hypothesis "no change point"

$$
H_{0}: \quad \theta_{1}=\cdots=\theta_{n}
$$

against the alternative "there exists one change point"

$$
\begin{aligned}
H_{1}: & \text { There exists an } k_{n}^{*} \in\{1, \cdots, n-1\} \text { such that } \\
& \theta_{1}=\cdots=\theta_{k_{n}^{*}} \neq \theta_{k_{n}^{*}+1}=\cdots=\theta_{n} .
\end{aligned}
$$

This is a so-called "at most one change point" (AMOC) model. Such a model is frequently studied in the literature (cf. e.g. [3, 23, 34, 36,44$]$ ) provided that the true location of a change point $k_{n}^{*}$ is unknown but deterministic. Following Csörgő and Horváth [23], a natural approach to build an appropriate test statistic is based on the likelihood ratio, i.e., if the change point occurs at $k=k_{n}^{*}$ known, then we should reject $H_{0}$ for small values of $\Lambda_{k}$, where

$$
\begin{equation*}
\Lambda_{k}:=\frac{\sup _{\theta_{0} \in \Theta} \prod_{1 \leqslant i \leqslant n} f\left(X_{i} ; \theta_{0}\right)}{\sup _{\theta_{0}^{(1)}, \theta_{0}^{(2)} \in \Theta} \prod_{1 \leqslant i \leqslant k} f\left(X_{i} ; \theta_{0}^{(1)}\right) \prod_{k<i \leqslant n} f\left(X_{i} ; \theta_{0}^{(2)}\right)} \in(0,1] . \tag{3.2.2}
\end{equation*}
$$

Remark 3.2.1. In the definition of the likelihood ratio in (3.2.2), we follow the notation in [23]. Note however, that in several other literature, the likelihood ratio has been introduced by $\Lambda_{k}^{-1}$. Then, of course, $\Lambda_{k}^{-1} \in[1, \infty)$ and the null hypothesis $H_{0}$ should be rejected for large values of $\Lambda_{k}^{-1}$.

In order to guarantee the existence of the maximum likelihood estimators and later to study their asymptotics, we need some additional regularity assumptions.
Assumption 3.1. There exists an open set $\Theta_{0} \subset \Theta \subset \mathbb{R}^{d}$ such that for all $\theta=$ $\left(\theta_{1}, \cdots, \theta_{d}\right)^{T} \in \Theta_{0}$, we have
i) $A(\theta)$ has continuous derivatives up to the third order and $A^{\prime \prime}(\theta):=\left\{\frac{\partial^{2}}{\partial \theta_{i} \partial \theta_{j}} A(\theta)\right.$, $1 \leqslant i, j \leqslant d\}$ is a positive definite matrix.
ii) $\operatorname{inv} A^{\prime}(\theta)$, the unique inverse of $\vartheta \mapsto A^{\prime}(\vartheta):=\left(\frac{\partial}{\partial \vartheta_{1}} A(\vartheta), \cdots, \frac{\partial}{\partial \vartheta_{d}} A(\vartheta)\right)^{T}$ at $\theta$, exists.
Under Assumption 3.1 ii) we can find unique maximum likelihood estimators (MLEs) for the parameters before and after the change provided that their true values are contained in $\Theta_{0}$. Elementary calculations reveal that for each $k=1, \cdots, n-1$, the MLEs for the parameters before and after a change point $k$ are given by $\operatorname{inv} A^{\prime}\left(B_{n}(k)\right)$ and $\operatorname{inv} A^{\prime}\left(B_{n}^{*}(k)\right)$, respectively, where

$$
B_{n}(k):=\frac{1}{k} \sum_{1 \leqslant i \leqslant k} T\left(X_{i}\right), \quad \text { and } \quad B_{n}^{*}(k):=\frac{1}{n-k} \sum_{k<i \leqslant n} T\left(X_{i}\right) .
$$

Plugging these estimators into (3.2.2), we can rewrite the log-likelihood ratio as

$$
\begin{equation*}
S_{n}(k):=-\log \Lambda_{k}=k H\left(B_{n}(k)\right)+(n-k) H\left(B_{n}^{*}(k)\right)-n H\left(B_{n}(n)\right), \tag{3.2.3}
\end{equation*}
$$

where

$$
\begin{equation*}
H(x):=\left(\operatorname{inv} A^{\prime}(x)\right)^{T} x-A\left(\operatorname{inv} A^{\prime}(x)\right) . \tag{3.2.4}
\end{equation*}
$$

Remark 3.2.2. Note that under Assumption 3.1 for all $x \in \Theta_{0}$, the derivatives of $H$ up to the third order exist, are continuous in $x$, and

$$
H^{\prime}(x)=\operatorname{inv} A^{\prime}(x), \quad H^{\prime \prime}(x)=\left(A^{\prime \prime}\left(H^{\prime}(x)\right)\right)^{-1}
$$

Since the true location of a change point $k_{n}^{*}$ is unknown, it is natural to use the maximally selected log-likelihood ratio and reject $H_{0}$, if

$$
\mathcal{S}_{n}:=\max _{1 \leqslant k \leqslant n}\left\{2 S_{n}(k)\right\}
$$

is large.
In our work, under $H_{1}$ "there exists one change point", we will assume that the true location of the change point $k_{n}^{*}: \Omega \rightarrow\{1, \cdots, n-1\}$ is a random variable. This new framework is of particular interest if the location of the change point equals a stopping time, often depending on the data $X_{1}, \cdots, X_{n}$ itself as we already discussed in Section 3.1. In the following, we will study the convergence rate and asymptotic distribution of the test statistic $\mathcal{S}_{n}$ under the null and under the alternative hypothesis. Moreover, under the alternative, we introduce an estimator $\hat{k}_{n}$ of $k_{n}^{*}$ based on an estimator of the fractional change point $\hat{\lambda}_{n}:=\hat{k}_{n} / n$ of $\lambda_{n}^{*}:=k_{n}^{*} / n$, for which we establish the consistency, the rate of convergence, and the limit distribution.

### 3.3 Asymptotics under the null

Under the null hypothesis $H_{0}$, since there is no change point in the data, we may consult the result in [35, Theorem 1.1] which gives a limit theorem for the distribution of $\mathcal{S}_{n}:=\max _{1 \leqslant k \leqslant n}\left\{2 S_{n}(k)\right\}$ under $H_{0}$. Note that this result is a corollary of the more general result in [34] as we restrict our considerations to densities of exponential form (cf. the assumption in (3.2.1)). Let $a(x):=(2 \log (x))^{1 / 2}$,

$$
b_{d}(x):=2 \log (x)+\frac{d}{2} \log \log (x)-\log (\Gamma(d / 2))
$$

and $\Gamma(t):=\int_{0}^{\infty} y^{t-1} e^{-y} d y$ be the Gamma function.
Theorem 3.3.1 (Asymptotics under the null hypothesis). Let $H_{0}$ and Assumption 3.1 be satisfied. Moreover, let $\theta_{0} \in \Theta$ be the true value of the parameter in (3.2.1) which is contained in $\Theta_{0}$. Then, for all $t \in \mathbb{R}$,

$$
\lim _{n \rightarrow \infty} \mathbb{P}\left[a(\log n) \mathcal{S}_{n}^{1 / 2} \leqslant t+b_{d}(\log n)\right]=\exp \left(-2 e^{-t}\right)
$$

We omit the proof. The statement can be found in [35, Theorem 1.1], whereas the proof in a more general setting is stated in 34 .

The above theorem states that under the null hypothesis $H_{0}$ when no change point occurs the test statistic $\mathcal{S}_{n}^{1 / 2}$ follows a Gumbel distribution asymptotically. This is not surprising since $\left(S_{n}^{1 / 2}(k), 1 \leqslant k \leqslant n\right)$ converges weakly to a sequence of normally distributed random variables under the null hypothesis and a Gumbel distribution describes the maximum (or minimum) of normally distributed data.

With the help of Theorem 3.3.1, we are able to derive rejection regions of the test statistic $\mathcal{S}_{n}:=\max _{1 \leqslant k \leqslant n}\left\{2 S_{n}(k)\right\}$ under the null hypothesis. For example, let $m=1$,
$d=2, n=10,000$, and consider different significance levels $\alpha=0.1,0.05,0.01$, where $1-\alpha=\exp \left(-2 e^{-t}\right)$ for appropriate $t \in \mathbb{R}$. Then, we should reject the null hypothesis if $\mathcal{S}_{n}^{1 / 2}$ is larger than the corresponding critical value $\kappa_{\alpha}$ satisfying $\mathbb{P}\left[\mathcal{S}_{n}^{1 / 2}>\kappa_{\alpha}\right]=$ $1-\exp \left(-2 e^{-t}\right)=\alpha$ and Theorem 3.3.1. In the table below, we have presented the critical values $\kappa_{\alpha}$ for different values of $\alpha$.

| $\alpha$ | $\kappa_{\alpha}$ |
| :---: | :---: |
| 0.1 | 3.8827 |
| 0.05 | 4.2242 |
| 0.01 | 4.9977 |

Figure 3.1: Depiction of the critical values $\kappa_{\alpha}$ for different values of $\alpha \in$ $\{0.1,0.05,0.01\}$

The rate of convergence to the Gumbel distribution in Theorem 3.3.1 is usually believed to be very slow. Consequently, a very large sample is necessary to test $H_{0}$ versus $H_{1}$ with the help of Theorem 3.3.1. In a simulation study, the authors in 23 showed that for a moderate sample size the critical values derived from Theorem 3.3.1 tend to be much larger than the true ones and therefore, the distribution in Theorem 3.3 .1 yields conservative rejection regions. For this reason, the authors in [23] state a second limit theorem for the distribution of the test statistic $\mathcal{S}_{n}$ under the null hypothesis. In this second limit result, it is shown that the distribution of the test statistic can be approximated by that of the supremum of the continuous-time process $\left(B^{(d)}(t) /(t(1-t))\right)_{t \in(0,1)}$ taken over a slightly shorter time interval, where $B^{(d)}(t):=$ $\sum_{1 \leqslant i \leqslant d} B_{i}^{2}(t)$ and $B_{1}, \cdots, B_{d}$ are independent Brownian bridges (c.f. Theorem 1.3.2 in [23]). Moreover, they showed that the critical values obtained from the distribution in Theorem 1.3.2 in [23] are often preferable to the ones obtained from the distribution in Theorem 3.3.1.

### 3.4 Asymptotics under the alternative

Assume that $H_{1}$ "there exists one change point" holds true and let us denote by $\theta_{0}^{(1)}, \theta_{0}^{(2)} \in \Theta \subset \mathbb{R}^{d}$ the true values of the parameters before and after the change point of $X_{1}, \cdots, X_{n}$. Let $\left(X_{1, i}, i \geqslant 1\right)$ and $\left(X_{2, i}, i \geqslant 1\right)$ be two independent sequences of iid random variables, where $X_{1,1} \sim f\left(x ; \theta_{0}^{(1)}\right)$ and $X_{2,1} \sim f\left(x ; \theta_{0}^{(2)}\right)$ satisfy

$$
X_{i}=\left\{\begin{array}{ll}
X_{1, i} & \text { for } i=1, \cdots, k_{n}^{*} \\
X_{2, i} & \text { for } i=k_{n}^{*}+1, \cdots, n
\end{array} .\right.
$$

Since the densities of the $X_{i}$ 's are elements of an exponential family, we have

$$
\begin{array}{ll}
\frac{\partial A\left(\theta_{0}^{(1)}\right)}{\partial \theta_{j}}=\mathbb{E}\left[T_{j}\left(X_{1,1}\right)\right], & \frac{\partial^{2} A\left(\theta_{0}^{(1)}\right)}{\partial \theta_{i} \partial \theta_{j}}=\operatorname{Cov}\left[T_{i}\left(X_{1,1}\right), T_{j}\left(X_{1,1}\right)\right]  \tag{3.4.1}\\
\frac{\partial A\left(\theta_{0}^{(2)}\right)}{\partial \theta_{j}}=\mathbb{E}\left[T_{j}\left(X_{2,1}\right)\right], & \frac{\partial^{2} A\left(\theta_{0}^{(2)}\right)}{\partial \theta_{i} \partial \theta_{j}}=\operatorname{Cov}\left[T_{i}\left(X_{2,1}\right), T_{j}\left(X_{2,1}\right)\right]
\end{array}
$$

where $\theta=\left(\theta_{1}, \cdots, \theta_{d}\right)^{T} \in \Theta \subset \mathbb{R}^{d}$. In the following, we introduce

$$
\begin{equation*}
\tau_{1}:=A^{\prime}\left(\theta_{0}^{(1)}\right), \quad \tau_{2}:=A^{\prime}\left(\theta_{0}^{(2)}\right), \quad \Sigma_{1}:=A^{\prime \prime}\left(\theta_{0}^{(1)}\right), \quad \text { and } \quad \Sigma_{2}:=A^{\prime \prime}\left(\theta_{0}^{(2)}\right) \tag{3.4.2}
\end{equation*}
$$

In order to study the asymptotics of the statistic $\mathcal{S}_{n}$ for a possibly random occurrence of the change point $k_{n}^{*}$, we will study its asymptotics for all possible true values of $k_{n}^{*}$. Therefore, under $H_{1}$, we can rewrite the test statistic $S_{n}$ in 3.2.3) as a discrete-time process of two time parameters $k, k^{*} \in\{1, \cdots, n-1\}$, i.e.,

$$
\begin{equation*}
S_{n}\left(k, k^{*}\right)=k H\left(B_{n}\left(k, k^{*}\right)\right)+(n-k) H\left(B_{n}^{*}\left(k, k^{*}\right)\right)-n H\left(B_{n}\left(n, k^{*}\right)\right) \tag{3.4.3}
\end{equation*}
$$

where $H$ is given as in 3.2.4,

$$
B_{n}\left(k, k^{*}\right)= \begin{cases}\frac{1}{k} \sum_{i=1}^{k} T\left(X_{1, i}\right) & \text { if } k \leqslant k^{*} \\ \frac{1}{k}\left(\sum_{1=1}^{k^{*}} T\left(X_{1, i}\right)+\sum_{i=k^{*}+1}^{k} T\left(X_{2, i}\right)\right) & \text { if } k>k^{*}\end{cases}
$$

and

$$
B_{n}^{*}\left(k, k^{*}\right)= \begin{cases}\frac{1}{n-k}\left(\sum_{i=k+1}^{k^{*}} T\left(X_{1, i}\right)+\sum_{i=k^{*}+1}^{n} T\left(X_{2, i}\right)\right) & \text { if } k \leqslant k^{*} \\ \frac{1}{n-k} \sum_{i=k+1}^{n} T\left(X_{2, i}\right) & \text { if } k>k^{*}\end{cases}
$$

In the following, we will study the limit distribution of $S_{n}$. It turns out that its distribution depends on the limit distribution of $k_{n}^{*} / n$ and on the size of the change in the parameter

$$
\begin{equation*}
\Delta^{2}:=\left\|\theta_{0}^{(1)}-\theta_{0}^{(2)}\right\|^{2} \tag{3.4.4}
\end{equation*}
$$

Assumption 3.2. Let the true location of the change point $k_{n}^{*}$ be a random variable taking values in $\{1, \cdots, n-1\}$. Let $\gamma \in(0,1 / 2)$ be a constant and $\lambda^{*}$ be a random variable taking values in $[\gamma, 1-\gamma]$ with probability one such that $\lambda_{n}^{*}:=k_{n}^{*} / n$ satisfies $\left|\lambda_{n}^{*}-\lambda^{*}\right|=o_{\mathbb{P}}\left(\left(n \Delta^{2}\right)^{-1}\right)$. Moreover, we assume that $\theta_{0}^{(1)}:=\theta_{0}^{(1)}(n) \rightarrow \theta_{A}$ and $\theta_{0}^{(2)}:=$ $\theta_{0}^{(2)}(n) \rightarrow \theta_{A}$ as $n \rightarrow \infty$ for some $\theta_{A}$ in the interior of $\Theta \subset \mathbb{R}^{d}$ with

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{n \Delta^{2}}{\log \log (n)}=\infty \tag{3.4.5}
\end{equation*}
$$

where the size of the change $\Delta^{2}$ is given in (3.4.4).

Assuming that $k_{n}^{*} / n \rightarrow \lambda^{*}$ in probability as $n \rightarrow \infty$ for some random variable $\lambda^{*}$ taking values almost surely in $[\gamma, 1-\gamma]$ and $\gamma \in(0,1 / 2)$ ensures that the change point occurs in "the middle of the data". Moreover, we concentrate on the critical case in which the size of the change $\Delta^{2}$ converges to zero as $n \rightarrow \infty$. We will show that as long as $\Delta^{2}$ satisfies the condition in 3.4 .5 , we are still able to detect the change point in the data.

Remark 3.4.1. Csörgő and Horváth [23] studied this problem provided that $k_{n}^{*}$ is deterministic and $k_{n}^{*} / n \rightarrow \lambda \in(0,1)$ as $n \rightarrow \infty$. Moreover, they studied slight modifications of Assumption 3.2, e.g.,
i) the occurrence of an early change point, i.e., $k_{n}^{*} / n \rightarrow 0$ as $n \rightarrow \infty$, and
ii) the size of the change is large compared to the sample size in the sense that $\Delta^{2}$ is independent of $n$.

Combining our subsequent analysis with the arguments in [23], we expect to be able to derive similar results in these settings.

Remark 3.4.2. If Assumption 3.2 is satisfied, the true values of the parameters $\theta_{0}^{(1)}$ and $\theta_{0}^{(2)}$, and hence also the true values of the transformed parameters $\tau_{1}, \tau_{2}, \Sigma_{1}$, $\Sigma_{2}$, and $\Delta^{2}$ introduced in (3.4.2) and (3.4.4), respectively, depend on $n$. However, for reasons of notation, this dependence will often be omitted.

Since $\left(X_{1, i}, i \geqslant 1\right)$ and $\left(X_{2, i}, i \geqslant 1\right)$ are independent sequences containing independent and identically distributed random variables, we can state our first limit theorem. It is a direct consequence of Donsker's theorem in higher dimensions.

Lemma 3.4.3. Let Assumptions 3.1 and 3.2 be satisfied and $\theta_{A} \in \Theta_{0}$. Moreover, we define $W_{l}^{(n)}:=\left(W_{l}^{(n)}(t)\right)_{t \in[0,1]}$, where $W_{l}^{(n)}(t):=\sum_{k=1}^{n} W_{l, k}^{(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}, W_{l, k}^{(n)}:=$ $n^{-1 / 2} \sum_{i=1}^{k}\left(T\left(X_{l, i}\right)-\tau_{l}\right)$, and $l=1,2$. Then,

$$
\left(W_{1}^{(n)}, W_{2}^{(n)}\right) \Rightarrow \Sigma_{A}^{1 / 2}\left(W_{1}, W_{2}\right)
$$

in the Skorokhod topology on $D\left([0,1], \mathbb{R}^{2 d}\right)$, where $\Sigma_{A}:=A^{\prime \prime}\left(\theta_{A}\right)$ and $W_{1}, W_{2}$ are two independent d-dimensional standard Brownian motions.

Proof. This is a direct application of Donsker's theorem in higher dimensions to the iid sequences $\left(T\left(X_{1, i}\right)-\tau_{1}, i \geqslant 1\right)$ and $\left(T\left(X_{2, i}\right)-\tau_{2}, i \geqslant 1\right)$.

Next, for all $k, k^{*} \in\{1, \cdots, n-1\}$, let us introduce

$$
\mu_{n}\left(k, k^{*}\right):=\left\{\begin{align*}
k H\left(\tau_{1}\right)+(n-k) H\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right) &  \tag{3.4.6}\\
-n H\left(\frac{k^{*}}{n} \tau_{1}+\frac{n-k^{*}}{n} \tau_{2}\right) & \text { if } k \leqslant k^{*} \\
k H\left(\frac{k^{*}}{k} \tau_{1}+\frac{k-k^{*}}{k} \tau_{2}\right)+(n-k) H\left(\tau_{2}\right) & \\
-n H\left(\frac{k^{*}}{n} \tau_{1}+\frac{n-k^{*}}{n} \tau_{2}\right) & \text { if } k>k^{*}
\end{align*}\right.
$$

Note that $\mu_{n}\left(k, k^{*}\right)$ is asymptotically the expected value of the statistic $S_{n}\left(k, k^{*}\right)$. Then, for all $k, k^{*} \in\{1, \cdots, n-1\}$, applying Taylor's formula of the first order, we can write

$$
\begin{equation*}
S_{n}\left(k, k^{*}\right)-\mu_{n}\left(k, k^{*}\right)=Z_{n}\left(k, k^{*}\right)+R_{n}\left(k, k^{*}\right), \tag{3.4.7}
\end{equation*}
$$

where for $h_{n}(x):=H^{\prime}\left(x \tau_{1}(n)+(1-x) \tau_{2}(n)\right)^{T}, k \leqslant k^{*}$,

$$
\begin{align*}
Z_{n}\left(k, k^{*}\right):= & h_{n}(1) \sum_{i=1}^{k}\left(T\left(X_{1, i}\right)-\tau_{1}\right) \\
& +h_{n}\left(\frac{k^{*}-k}{n-k}\right)\left(\sum_{i=k+1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k^{*}+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right)\right)  \tag{3.4.8}\\
& -h_{n}\left(\frac{k^{*}}{n}\right)\left(\sum_{i=1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k^{*}+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right)\right)
\end{align*}
$$

for $k>k^{*}$,

$$
\begin{align*}
Z_{n}\left(k, k^{*}\right):= & h_{n}\left(\frac{k^{*}}{k}\right)\left(\sum_{i=1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k^{*}+1}^{k}\left(T\left(X_{2, i}\right)-\tau_{2}\right)\right) \\
& +h_{n}(0) \sum_{i=k+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right)  \tag{3.4.9}\\
& -h_{n}\left(\frac{k^{*}}{n}\right)\left(\sum_{i=1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k^{*}+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right)\right)
\end{align*}
$$

and $R_{n}\left(k, k^{*}\right)$ is the corresponding remainder of Lagrange form such that the equation in (3.4.7 holds true. With a little abuse of notation, we define by $Z_{n}:=\left(Z_{n}(t, \lambda)\right)_{t, \lambda \in[0,1]}$ the piecewise constant interpolation of $\left(Z_{n}\left(k, k^{*}\right) ; k, k^{*} \in\{1, \cdots, n-1\}\right)$, where

$$
Z_{n}(t, \lambda):=\sum_{k=1}^{n-1} \sum_{k^{*}=1}^{n-1} Z_{n}\left(k, k^{*}\right) \mathbb{1}_{\{n t \in[k, k+1)\}} \mathbb{1}_{\left\{n \lambda \in\left[k^{*}, k^{*}+1\right)\right\}}
$$

Similarly, we define $\mu_{n}:=\left(\mu_{n}(t, \lambda)\right)_{t, \lambda \in[0,1]}$ and $R_{n}:=\left(R_{n}(t, \lambda)\right)_{t, \lambda \in[0,1]}$.
In the following, we assume for simplicity that $f(x ; \theta)$ is the density of an $m$ dimensional normal distribution in its natural parametrization. Note, moreover, that under this assumption, also Assumption 3.1 holds true.

Assumption 3.3. Let $f(x ; \theta)$ be the density of an m-dimensional normal distribution with parameter $\theta \in \Theta \subset \mathbb{R}^{d}$ given in its natural parametrization. In more detail, for mean $\mu \in \mathbb{R}^{m}$ and covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$ symmetric and positive definite, let $\theta:=\theta(\mu, \Sigma)=\left(\Sigma^{-1} \mu,-\frac{1}{2} \Sigma^{-1}\right)=:\left(\theta_{1}, \theta_{2}\right) \in \mathbb{R}^{m} \times \mathbb{R}^{m \times m}$. Then, the density in (3.2.1) is given by

$$
f(x ; \theta)=\exp \left(\theta_{1}^{T} x+x^{T} \theta_{2} x+\frac{1}{4} \theta_{1}^{T} \theta_{2}^{-1} \theta_{1}-\frac{1}{2} \log \left(\operatorname{det}\left(-\pi \theta_{2}^{-1}\right)\right)\right)
$$

and $T(x)=\left(x, x x^{T}\right), A(\theta)=-\frac{1}{4} \theta_{1}^{T} \theta_{2}^{-1} \theta_{1}+\frac{1}{2} \log \left(\operatorname{det}\left(-\pi \theta_{2}^{-1}\right)\right)$ for $x \in \mathbb{R}^{m}, \theta=$ $\left(\theta_{1}, \theta_{2}\right) \in \mathbb{R}^{d}=\mathbb{R}^{m+m^{2}}$, and $H(y)=-\frac{1}{2} \operatorname{det}\left(2 \pi\left(y_{2}-y_{1} y_{1}^{T}\right)\right)$, for $y=\left(y_{1}, y_{2}\right) \in \mathbb{R}^{m} \times$ $\mathbb{R}^{m \times m}$.

Assuming that $f(x, \theta)$ is the density of an $m$-dimensional normal distribution will remarkably simplify the proof of the following limit theorem as the identification of the finite-dimensional distributions can be derived by the fourth moment theorem (cf. Nualart and Peccati [68, Theorem 1]).

Theorem 3.4.4 (A limit theorem for $Z_{n}$ under the alternative). Let Assumptions 3.2 and 3.3 be satisfied. Moreover, let us denote $\delta^{2}:=\delta^{2}(n):=\left\|\tau_{1}(n)-\tau_{2}(n)\right\|^{2} \rightarrow 0$ as $n \rightarrow \infty$ and $\tau_{A}:=\lim _{n \rightarrow \infty} \tau_{1}(n)$. Then,

$$
\begin{equation*}
\left(n \delta^{2}\right)^{-1 / 2} Z_{n} \Rightarrow Z^{*} \tag{3.4.10}
\end{equation*}
$$

in the Skorokhod topology on $D\left([0,1]^{2}, \mathbb{R}\right)$, where $Z^{*}$ is a Gaussian process with mean zero and covariance function

$$
c\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)=\sigma_{A}^{2} \begin{cases}(1-\lambda)\left(1-\lambda^{\prime}\right) \min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\}, & \text { if } t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime} \\ (1-\lambda) \lambda^{\prime} \min \left\{\frac{t\left(1-t^{\prime}\right)}{(1-t) t^{\prime}}, 1\right\}, & \text { if } t \leqslant \lambda, t^{\prime}>\lambda^{\prime} \\ \lambda\left(1-\lambda^{\prime}\right) \min \left\{\frac{(1-t) t^{\prime}}{t\left(1-t^{\prime}\right)}, 1\right\}, & \text { if } t>\lambda, t^{\prime} \leqslant \lambda^{\prime} \\ \lambda \lambda^{\prime} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t>\lambda, t^{\prime}>\lambda^{\prime}\end{cases}
$$

for $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in[0,1]^{2} \times[0,1]^{2} \backslash\{((1,1),(1,1))\}$ and $c((1,1),(1,1))=0$, and $\sigma_{A}^{2}$ given by

$$
\sigma_{A}^{2}:=\lim _{n \rightarrow \infty} \sigma_{A}^{2}(n):=\lim _{n \rightarrow \infty} \frac{\left(\tau_{1}(n)-\tau_{2}(n)\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}(n)-\tau_{2}(n)\right)}{\left\|\tau_{1}(n)-\tau_{2}(n)\right\|^{2}}
$$

provided that the limit on the right hand side exists. Otherwise, we multiply the left hand side of (3.4.10) with $\left(\sigma_{A}^{2}(n)\right)^{-1 / 2}$.

In the following, without loss of generality and without further comment, we will assume that the limit $\sigma_{A}^{2}$ introduced in the above theorem exists.

Corollary 3.4.5. Let the assumptions of Theorem 3.4.4 be satisfied. Then,

$$
\left(\frac{Z_{n}(t, t)}{\sqrt{n \delta^{2}}}\right)_{t \in[0,1]} \Rightarrow \sigma_{A} B
$$

in the Skorokhod topology on the space $D([0,1], \mathbb{R})$, where $B$ is a one-dimensional Brownian bridge, i.e., $B$ is a Gaussian process with mean zero and covariance function $c\left(t, t^{\prime}\right)=\min \left\{t, t^{\prime}\right\}-t t^{\prime}$.


Figure 3.2: Depiction of the variance structure of the limit process $Z^{*}$.
Proof of Theorem 3.4.4. For $l=1,2$, recall that $W_{l}^{(n)}(t):=\sum_{k=1}^{n} W_{l, k}^{(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}$, where $W_{l, k}^{(n)}:=n^{-1 / 2} \sum_{i=1}^{k}\left(T\left(X_{l, i}\right)-\tau_{l}\right)$ and let $\widetilde{Z}_{n}(t, \lambda):=\left(n \delta^{2}\right)^{-1 / 2} Z_{n}(t, \lambda)$ for $t, \lambda \in[0,1]$. In the following, we assume the existence of the limit $\sigma_{A}^{2}$ given in Theorem 3.4.4 Otherwise, we simply study the process $\bar{Z}_{n}(t, \lambda)=\left(\sigma_{A}^{2}(n)\right)^{-1 / 2} \widetilde{Z}_{n}$.

We will first establish tightness of the sequence $\left(\widetilde{Z}_{n}\right)_{n \in \mathbb{N}}$ and then prove that its finite-dimensional distributions converge to those of $Z^{*}$.

Tightness: For $t, \lambda \in[0,1]$, let us introduce the short-hand notations

$$
\alpha_{n}(t, \lambda):=\frac{\lfloor n \lambda\rfloor-\lfloor n t\rfloor}{n-\lfloor n t\rfloor} \mathbb{1}_{\{t<\lambda\}}, \quad \beta_{n}(\lambda):=\frac{\lfloor n \lambda\rfloor}{n}, \quad \text { and } \quad \gamma_{n}(t, \lambda):=\frac{\lfloor n \lambda\rfloor}{\lfloor n t\rfloor} \mathbb{1}_{\{t>\lambda\}} .
$$

As $n \rightarrow \infty$, we have for each $t, \lambda \in[0,1]$, that $\alpha_{n}(t, \lambda) \rightarrow(\lambda-t) /(1-t) \mathbb{1}_{\{t<\lambda\}}$, $\beta_{n}(\lambda) \rightarrow \lambda$, and $\gamma_{n}(t, \lambda) \rightarrow \lambda / t \mathbb{1}_{\{t>\lambda\}}$. Now, for $t \leqslant \lambda$ and $\alpha_{n}:=\alpha_{n}(t, \lambda), \beta_{n}:=\beta_{n}(\lambda)$, we can write

$$
\begin{gathered}
Z_{n}(t, \lambda)=\left(h_{n}(1)-h_{n}\left(\beta_{n}\right)\right) n^{1 / 2} W_{1}^{(n)}(\lambda)+\left(h_{n}\left(\alpha_{n}\right)-h_{n}(1)\right) n^{1 / 2}\left(W_{1}^{(n)}(\lambda)-W_{1}^{(n)}(t)\right) \\
+\left(h_{n}\left(\alpha_{n}\right)-h_{n}\left(\beta_{n}\right)\right) n^{1 / 2}\left(W_{2}^{(n)}(1)-W_{2}^{(n)}(\lambda)\right)
\end{gathered}
$$

and for $t>\lambda$ and $\gamma_{n}:=\gamma_{n}(t, \lambda)$,

$$
\begin{aligned}
Z_{n}(t, \lambda)= & \left(h_{n}\left(\gamma_{n}\right)-h_{n}\left(\beta_{n}\right)\right) n^{1 / 2} W_{1}^{(n)}(\lambda)+\left(h_{n}(0)-h_{n}\left(\beta_{n}\right)\right) n^{1 / 2}\left(W_{2}^{(n)}(1)-W_{2}^{(n)}(\lambda)\right) \\
& +\left(h_{n}\left(\gamma_{n}\right)-h_{n}(0)\right) n^{1 / 2}\left(W_{2}^{(n)}(t)-W_{2}^{(n)}(\lambda)\right) .
\end{aligned}
$$

Since $f(x, \theta)$ is the density of an $m$-dimensional normal distribution, we conclude that $H$ has continuous derivatives up to the second order and by Assumption $3.2,\left(\tau_{1}, \tau_{2}\right) \rightarrow$ $\left(\tau_{A}, \tau_{A}\right)$ as $n \rightarrow \infty$, where $\tau_{A}:=A^{\prime}\left(\theta_{A}\right)$. Hence, for all sequences $\left(x_{n}\right)_{n \in \mathbb{N}},\left(y_{n}\right)_{n \in \mathbb{N}} \subset$ $[0,1]$ with $x_{n} \rightarrow x \in[0,1], y_{n} \rightarrow y \in[0,1]$ as $n \rightarrow \infty$, and $x_{n} \leqslant y_{n}$ for all $n \in \mathbb{N}$, we conclude by an application of the definition of the derivative in higher dimensions and Remark 3.2.2 that

$$
\begin{align*}
\frac{h_{n}\left(y_{n}\right)-h_{n}\left(x_{n}\right)}{\delta} & =\frac{\left(H^{\prime}\left(y_{n} \tau_{1}+\left(1-y_{n}\right) \tau_{2}\right)-H^{\prime}\left(x_{n} \tau_{1}+\left(1-x_{n}\right) \tau_{2}\right)\right)^{T}}{\left\|\tau_{1}-\tau_{2}\right\|} \\
& =\left(y_{n}-x_{n}\right) \frac{\left(\tau_{1}-\tau_{2}\right)^{T}}{\left\|\tau_{1}-\tau_{2}\right\|} H^{\prime \prime}\left(\tau_{2}\right)+o(1)  \tag{3.4.11}\\
& \rightarrow(y-x) \widetilde{\Sigma}^{1 / 2}
\end{align*}
$$

as $n \rightarrow \infty$, where $\widetilde{\Sigma}^{1 / 2}:=\lim _{n \rightarrow \infty}\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right) / \delta$. In the following, we denote $G_{n}(x, y):=\left(h_{n}(x)-h_{n}(y)\right) / \delta$. Then, the process $\tilde{Z}_{n}$ can be written as follows: for $t \leqslant \lambda$,

$$
\begin{aligned}
& \widetilde{Z}_{n}(t, \lambda)=G_{n}\left(1, \beta_{n}\right) W_{1}^{(n)}(\lambda)-G_{n}\left(1, \alpha_{n}\right) W_{1}^{(n)}(\lambda)+G_{n}\left(1, \alpha_{n}\right) W_{1}^{(n)}(t) \\
&-G_{n}\left(\beta_{n}, \alpha_{n}\right) W_{2}^{(n)}(1)+G_{n}\left(\beta_{n}, \alpha_{n}\right) W_{2}^{(n)}(\lambda),
\end{aligned}
$$

and for $t>\lambda$,

$$
\begin{aligned}
\widetilde{Z}_{n}(t, \lambda)= & G_{n}\left(\gamma_{n}, \beta_{n}\right) W_{1}^{(n)}(\lambda)-G_{n}\left(\beta_{n}, 0\right) W_{2}^{(n)}(1)+G_{n}\left(\beta_{n}, 0\right) W_{2}^{(n)}(\lambda) \\
& +G_{n}\left(\gamma_{n}, 0\right) W_{2}^{(n)}(t)-G_{n}\left(\gamma_{n}, 0\right) W_{2}^{(n)}(\lambda) .
\end{aligned}
$$

Hence, the process $\widetilde{Z}_{n}$ can be represented as a sum of five scalar products, multiplying the $d$-dimensional partial sums $W_{l}^{(n)}(t), W_{l}^{(n)}(\lambda)$, or $W_{l}^{(n)}(1), l=1,2$, with the non-random $d$-dimensional vector $\left(G_{n} \circ\left(x_{n}, y_{n}\right)\right)(t, \lambda)$, where $x_{n}(t, \lambda), y_{n}(t, \lambda) \in\left\{0, \alpha_{n}(t, \lambda), \beta_{n}(\lambda)\right.$, $\left.\gamma_{n}(t, \lambda), 1\right\}$. Thus, we can identify each summand of $\widetilde{Z}_{n}$ as a discrete-time process in two time parameters which converges weakly in the Skorokhod topology to a onedimensional Gaussian process thanks to Lemma 3.4.3 and (3.4.11). In particular, the limit process of each summand of $\widetilde{Z}_{n}$ is a continuous process in both time parameters. By Theorem 12.6.1 in Whitt [86], we conclude their joint convergence implying that the sequence $\left(\widetilde{Z}_{n}\right)_{n \in \mathbb{N}}$ is tight.

Convergence of the finite-dimensional distributions: It is left prove to the convergence of the finite-dimensional distributions of $\widetilde{Z}_{n}$, i.e., for all $k \geqslant 1$ and $\left(t_{1}, \lambda_{1}\right), \cdots,\left(t_{k}, \lambda_{k}\right) \in$ $[0,1]^{2}$, we want to show that

$$
\left(\widetilde{Z}_{n}\left(t_{1}, \lambda_{1}\right), \cdots, \widetilde{Z}_{n}\left(t_{k}, \lambda_{k}\right)\right) \Rightarrow\left(Z^{*}\left(t_{1}, \lambda_{1}\right), \cdots, Z^{*}\left(t_{k}, \lambda_{k}\right)\right)
$$

as $n \rightarrow \infty$, where $Z^{*}$ is a Gaussian process with mean zero and covariance function $c:[0,1]^{2} \times[0,1]^{2} \rightarrow \mathbb{R}$ defined in Theorem 3.4.4. For sake of notation, we only analyze the case $k=2$, i.e., for $(t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right) \in[0,1]^{2}$, we will prove the joint convergence

$$
\begin{equation*}
\left(\widetilde{Z}_{n}(t, \lambda), \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right) \Rightarrow\left(Z^{*}(t, \lambda), Z^{*}\left(t^{\prime}, \lambda^{\prime}\right)\right) \tag{3.4.12}
\end{equation*}
$$

as $n \rightarrow \infty$ and note that for $k>2$, we can argue completely analogously. Applying the Cramér-Wold device, (3.4.12) is equivalent to

$$
\begin{equation*}
\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right):=x \widetilde{Z}_{n}(t, \lambda)+y \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right) \Rightarrow x Z^{*}(t, \lambda)+y Z^{*}\left(t^{\prime}, \lambda^{\prime}\right) \tag{3.4.13}
\end{equation*}
$$

where $x, y \in \mathbb{R}$ are arbitrary. Since the $X_{i}$ 's are assumed to be normally distributed, the first $m$ components of $W_{l}^{(n)}$ belong to the Wiener chaos of order one and the last $m^{2}$ components of $W_{l}^{(n)}$ belong to the Wiener chaos of order two, for $l=1,2$ and all $n \in \mathbb{N}$. Hence, $\hat{Z}_{n}$ belongs to the Wiener chaos of order two for all $n \in \mathbb{N}$. In order to prove (3.4.13), we will apply the fourth moment theorem by Nualart and Peccati 68]. According to the fourth moment theorem, for all $(t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right) \in[0,1]^{2}$, the convergence in (3.4.13) is satisfied if the following two conditions hold true: as $n \rightarrow \infty$, we have
i) $\operatorname{Var}\left[\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right] \rightarrow \hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$ and
ii) $\mathbb{E}\left[\hat{Z}_{n}^{4}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right] \rightarrow 3\left(\hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right)^{2}$,
where

$$
\hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)=x^{2} c((t, \lambda),(t, \lambda))+2 x y c\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)+y^{2} c\left(\left(t^{\prime}, \lambda^{\prime}\right),\left(t^{\prime}, \lambda^{\prime}\right)\right)
$$

and the covariance function $c:[0,1]^{2} \times[0,1]^{2} \rightarrow \mathbb{R}$ is defined in Theorem 3.4.4. Since $\mathbb{E}\left[T\left(X_{l, 1}\right)\right]=\tau_{l}$ for $l=1,2$, we conclude that $\mathbb{E}\left[\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right]=0$. In order to analyze the second and fourth moment of $\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$, we need to differentiate between the following four cases:

1) $t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime}$,
2) $t \leqslant \lambda, t^{\prime}>\lambda^{\prime}$,
3) $t>\lambda, t^{\prime} \leqslant \lambda^{\prime}, \quad$ and 4$) ~ t>\lambda, t^{\prime}>\lambda^{\prime}$.

First, let us consider case 1$)$, let $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in[0,1]^{2} \times[0,1]^{2} \backslash\{((1,1),(1,1))\}$ with $t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime}$, and denote $\alpha_{n}:=\alpha_{n}(t, \lambda), \alpha_{n}^{\prime}:=\alpha_{n}\left(t^{\prime}, \lambda^{\prime}\right), \beta_{n}:=\beta_{n}(\lambda)$, and $\beta_{n}^{\prime}:=\beta_{n}\left(\lambda^{\prime}\right)$. Then, for $\sigma_{A}^{2}$ defined in Theorem 3.4.4, applying Lemma 3.4.3 and
(3.4.11), we conclude

$$
\begin{aligned}
& \mathbb{E}\left[\widetilde{Z}_{n}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right]=G_{n}\left(1, \beta_{n}\right) \mathbb{E}\left[W_{1}^{(n)}(\lambda)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(1, \beta_{n}^{\prime}\right)\right)^{T} \\
&-G_{n}\left(1, \beta_{n}\right) \mathbb{E}\left[W_{1}^{(n)}(\lambda)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)-W_{1}^{(n)}\left(t^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(1, \alpha_{n}^{\prime}\right)\right)^{T} \\
&-G_{n}\left(1, \beta_{n}\right) \mathbb{E}\left[W_{1}^{(n)}(\lambda)\left(W_{2}^{(n)}(1)-W_{2}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(\beta_{n}^{\prime}, \alpha_{n}^{\prime}\right)\right)^{T} \\
&-G_{n}\left(1, \alpha_{n}\right) \mathbb{E}\left[\left(W_{1}^{(n)}(\lambda)-W_{1}^{(n)}(t)\right)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(1, \beta_{n}^{\prime}\right)\right)^{T} \\
& \quad+G_{n}\left(1, \alpha_{n}\right) \mathbb{E}\left[\left(W_{1}^{(n)}(\lambda)-W_{1}^{(n)}(t)\right)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)-W_{1}^{(n)}\left(t^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(1, \alpha_{n}^{\prime}\right)\right)^{T} \\
&+G_{n}\left(1, \alpha_{n}\right) \mathbb{E}\left[\left(W_{1}^{(n)}(\lambda)-W_{1}^{(n)}(t)\right)\left(W_{2}^{(n)}(1)-W_{2}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G_{n}\left(\beta_{n}^{\prime}, \alpha_{n}^{\prime}\right)\right)^{T} \\
&-G_{n}\left(\beta_{n}, \alpha_{n}\right) \mathbb{E}\left[\left(W_{2}^{(n)}(1)-W_{2}^{(n)}(\lambda)\right)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G\left(1, \beta_{n}^{\prime}\right)\right)^{T} \\
& \quad+G_{n}\left(\beta_{n}, \alpha_{n}\right) \mathbb{E}\left[\left(W_{2}^{(n)}(1)-W_{2}^{(n)}(\lambda)\right)\left(W_{1}^{(n)}\left(\lambda^{\prime}\right)-W_{1}^{(n)}\left(t^{\prime}\right)\right)^{T}\right]\left(G\left(1, \alpha_{n}^{\prime}\right)\right)^{T} \\
& \quad+G_{n}\left(\beta_{n}, \alpha_{n}\right) \mathbb{E}\left[\left(W_{2}^{(n)}(1)-W_{2}^{(n)}(\lambda)\right)\left(W_{2}^{(n)}(1)-W_{2}^{(n)}\left(\lambda^{\prime}\right)\right)^{T}\right]\left(G\left(\beta_{n}^{\prime}, \alpha_{n}^{\prime}\right)\right)^{T} \\
& \rightarrow \sigma_{A}^{2}(1-\lambda)\left(1-\lambda^{\prime}\right)\left(\min \left\{\lambda, \lambda^{\prime}\right\}-\frac{1}{1-t^{\prime}} \max \left\{\min \left\{\lambda, \lambda^{\prime}\right\}-t^{\prime}, 0\right\}-\frac{t^{\prime}}{1-t^{\prime}} \max \left\{\lambda-\lambda^{\prime}, 0\right\}\right. \\
& \quad \quad \frac{1}{1-t} \max \left\{\min \left\{\lambda, \lambda^{\prime}\right\}-t, 0\right\}+\frac{1}{(1-t)\left(1-t^{\prime}\right)} \max \left\{\min \left\{\lambda, \lambda^{\prime}\right\}-\max \left\{t, t^{\prime}\right\}, 0\right\} \\
& \quad+\frac{t^{\prime}}{(1-t)\left(1-t^{\prime}\right)} \max \left\{\lambda-\max \left\{t, \lambda^{\prime}\right\}, 0\right\}-\frac{t}{1-t} \max \left\{\lambda^{\prime}-\lambda, 0\right\} \\
&\left.\quad+\frac{t}{(1-t)\left(1-t^{\prime}\right)} \max \left\{\lambda^{\prime}-\max \left\{\lambda, t^{\prime}\right\}, 0\right\}+\frac{t t^{\prime}}{(1-t)\left(1-t^{\prime}\right)}\left(1-\max \left\{\lambda, \lambda^{\prime}\right\}\right)\right) \\
&=\sigma_{A}^{2}(1-\lambda)\left(1-\lambda^{\prime}\right) \min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\} .
\end{aligned}
$$

Moreover, by definition of $c:[0,1]^{2} \times[0,1]^{2} \rightarrow \mathbb{R}$ at $((1,1),(1,1))$, we have $\mathbb{E}\left[\widetilde{Z}_{n}^{2}(1,1)\right] \rightarrow$ $0=c((1,1),(1,1))$ as $n \rightarrow \infty$. Analogously, applying Donsker's theorem in higher dimensions for the partial sums $W_{l}^{(n)}, l=1,2$, (cf. Lemma 3.4.3) and (3.4.11), we can derive the limit of the covariance of $\widetilde{Z}_{n}$ in the remaining three cases, where cases 2 ) and 3) are symmetric. Finally, we conclude that for $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in[0,1]^{2} \times[0,1]^{2} \backslash$
$\{((1,1),(1,1))\}$,

$$
\mathbb{E}\left[\widetilde{Z}_{n}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right] \rightarrow \sigma_{A}^{2} \begin{cases}(1-\lambda)\left(1-\lambda^{\prime}\right) \min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\}, & \text { if } t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime} \\ (1-\lambda) \lambda^{\prime} \min \left\{\frac{t\left(1-t^{\prime}\right)}{(1-t) t^{\prime}}, 1\right\}, & \text { if } t \leqslant \lambda, t^{\prime}>\lambda^{\prime} \\ \lambda\left(1-\lambda^{\prime}\right) \min \left\{\frac{(1-t) t^{\prime}}{t\left(1-t^{\prime}\right)}, 1\right\}, & \text { if } t>\lambda, t^{\prime} \leqslant \lambda^{\prime} \\ \lambda \lambda^{\prime} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t>\lambda, t^{\prime}>\lambda^{\prime}\end{cases}
$$

In particular, by definition of $\hat{Z}_{n}$, we conclude for all $(t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right) \in[0,1]^{2}$ that $\operatorname{Var}\left[\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right] \rightarrow \hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$ as desired.

Next, we analyze $\underset{\sim}{\mathbb{E}}\left[\hat{Z}_{n}^{4}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right]$. For this reason, we will calculate the mixed fourth moments $\mathbb{E}\left[\widetilde{Z}_{n}^{3}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right]$ and $\mathbb{E}\left[\widetilde{Z}_{n}^{2}(t, \lambda) \widetilde{Z}_{n}^{2}\left(t^{\prime}, \lambda^{\prime}\right)\right]$. Recall that $\widetilde{Z}_{n}$ can be represented by finitely many scalar products of the partial sums $W_{l}^{(n)}, l=1,2$, and the non-random function $G_{n}$ evaluated at $\left(x_{n}(t, \lambda), y_{n}(t, \lambda)\right)$, where $x_{n}(t, \lambda), y_{n}(t, \lambda) \in$ $\left\{0, \alpha_{n}(t, \lambda), \beta_{n}(\lambda), \gamma_{n}(t, \lambda), 1\right\}$. Because of Lemma 3.4.3. $\left(W_{l}^{(n)}\right)_{n \in \mathbb{N}}, l=1,2$, converges weakly in the Skorokhod topology to a $d$-dimensional Brownian motion. Thus, we can apply the same arguments as for the computation of the mixed fourth moments of partial sums of iid standard normal distributed random variables.
Lemma 3.4.6. Let $\left(\xi_{i}, i \geqslant 1\right)$ be a sequence of iid one-dimensional standard normal random variables and $W^{(n)}(t):=\sum_{k=1}^{n} W_{k}^{(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}$, where $W_{k}^{(n)}:=n^{-1 / 2} \sum_{i=1}^{k} \xi_{i}$. Moreover, let $\left\{\alpha_{i}\right\}_{i=1,2,3,4},\left\{\beta_{i}\right\}_{i=1,2,3,4} \subset[0,1]$ with $\alpha_{i} \leqslant \beta_{i}$ for all $i=1,2,3,4$, and denote by $\check{\alpha}:=\max \left\{\alpha_{i}: i=1,2,3,4\right\}, \check{\alpha}_{i j}:=\max \left\{\alpha_{i}, \alpha_{j}\right\}, \hat{\beta}:=\min \left\{\beta_{i}: i=1,2,3,4\right\}$, and $\hat{\beta}_{i j}:=\min \left\{\beta_{i}, \beta_{j}\right\}$. Then, as $n \rightarrow \infty$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\prod_{i=1}^{4}\left(W^{(n)}\left(\beta_{i}\right)-W^{(n)}\left(\alpha_{i}\right)\right)\right]=\max \left\{\hat{\beta}_{12}-\check{\alpha}_{12}, 0\right\} \max \left\{\hat{\beta}_{34}-\check{\alpha}_{34}, 0\right\} \\
& \quad+\max \left\{\hat{\beta}_{13}-\check{\alpha}_{13}, 0\right\} \max \left\{\hat{\beta}_{24}-\check{\alpha}_{24}, 0\right\}+\max \left\{\hat{\beta}_{14}-\check{\alpha}_{14}, 0\right\} \max \left\{\hat{\beta}_{23}-\check{\alpha}_{23}, 0\right\}
\end{aligned}
$$

Proof. Since $\left(\xi_{i}, i \geqslant 1\right)$ is a sequence of iid standard normal random variables and by the definition of $W^{(n)}$, we have as $n \rightarrow \infty$,

$$
\begin{aligned}
& \mathbb{E}\left[\prod_{i=1}^{4}\left(W^{(n)}\left(\beta_{i}\right)-W^{(n)}\left(\alpha_{i}\right)\right)\right]=\frac{1}{n^{2}} \sum_{i=\lfloor n \check{\alpha}\rfloor+1}^{\lfloor n \hat{\beta}\rfloor} \mathbb{E}\left[\xi_{i}^{4}\right]+\frac{1}{n^{2}} \sum_{i=\left\lfloor n \check{\alpha}_{12}\right\rfloor+1}^{\left\lfloor n \hat{\beta}_{12}\right\rfloor} \sum_{\substack{j=\left\lfloor n \check{\alpha}_{34}\right\rfloor+1 \\
j \neq i}}^{\left\lfloor n \hat{\beta}_{34}\right\rfloor} \mathbb{E}\left[\xi_{i}^{2} \xi_{j}^{2}\right] \\
& +\frac{1}{n^{2}} \sum_{i=\left\lfloor n \check{\alpha}_{13}\right\rfloor+1}^{\left\lfloor n \hat{\beta}_{13}\right\rfloor} \sum_{\substack{j=\left\lfloor n \check{\alpha}_{24}\right\rfloor+1 \\
j \neq i}}^{\left\lfloor n \hat{\beta}_{24}\right\rfloor} \mathbb{E}\left[\xi_{i}^{2} \xi_{j}^{2}\right]+\frac{1}{n^{2}} \sum_{i=\left\lfloor n \check{\alpha}_{14}\right\rfloor+1}^{\left\lfloor n \hat{\beta}_{14}\right\rfloor} \sum_{\substack{\left\lfloor n \check{\alpha}_{23}\right\rfloor+1 \\
j \neq i}}^{\left\lfloor n \hat{\beta}_{23}\right\rfloor} \mathbb{E}\left[\xi_{i}^{2} \xi_{j}^{2}\right] \\
& \rightarrow \max \left\{\hat{\beta}_{12}-\check{\alpha}_{12}, 0\right\} \max \left\{\hat{\beta}_{34}-\check{\alpha}_{34}, 0\right\}+\max \left\{\hat{\beta}_{13}-\check{\alpha}_{13}, 0\right\} \max \left\{\hat{\beta}_{24}-\check{\alpha}_{24}, 0\right\} \\
& +\max \left\{\hat{\beta}_{14}-\check{\alpha}_{14}, 0\right\} \max \left\{\hat{\beta}_{23}-\check{\alpha}_{23}, 0\right\} .
\end{aligned}
$$

Now, applying Lemma 3.4.6 and the convergence of $G_{n}$ in 3.4.11, elementary calculations yield for $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in[0,1]^{2} \times[0,1]^{2} \backslash\left\{\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right): t=\lambda=\right.$ 1 or $\left.t^{\prime}=\lambda^{\prime}=1\right\}$,

$$
\begin{aligned}
& \mathbb{E}\left[\widetilde{Z}_{n}^{3}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right] \\
& \rightarrow 3 \sigma_{A}^{4} \begin{cases}(1-\lambda)^{3}\left(1-\lambda^{\prime}\right) \frac{t}{1-t} \min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\}, & \text { if } t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime} \\
(1-\lambda)^{3} \lambda^{\prime}\left(\frac{t}{1-t}\right)^{2} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t \leqslant \lambda, t^{\prime}>\lambda^{\prime} \\
\lambda^{3}\left(1-\lambda^{\prime}\right) \frac{1-t}{t} \frac{t^{\prime}}{1-t^{\prime}} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t>\lambda, t^{\prime} \leqslant \lambda^{\prime} \\
\lambda^{3} \lambda^{\prime} \frac{1-t}{t} \min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}, & \text { if } t>\lambda, t^{\prime}>\lambda^{\prime}\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}\left[\widetilde{Z}_{n}^{2}(t, \lambda) \widetilde{Z}_{n}^{2}\left(t^{\prime}, \lambda^{\prime}\right)\right] \\
& \rightarrow \sigma_{A}^{4} \begin{cases}(1-\lambda)^{2}\left(1-\lambda^{\prime}\right)^{2}\left\{\frac{t t^{\prime}}{(1-t)\left(1-t^{\prime}\right)}+2\left(\min \left\{\frac{t}{1-t}, \frac{t^{\prime}}{1-t^{\prime}}\right\}\right)^{2}\right\}, & \text { if } t \leqslant \lambda, t^{\prime} \leqslant \lambda^{\prime} \\
(1-\lambda)^{2}\left(\lambda^{\prime}\right)^{2}\left\{\frac{t\left(1-t^{\prime}\right)}{(1-t) t^{\prime}}+2\left(\frac{t}{1-t}\right)^{2}\left(\min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}\right)^{2}\right\}, & \text { if } t \leqslant \lambda, t^{\prime}>\lambda^{\prime} \\
\lambda^{2}\left(1-\lambda^{\prime}\right)^{2}\left\{\frac{(1-t) t^{\prime}}{t\left(1-t^{\prime}\right)}+2\left(\frac{t^{\prime}}{1-t^{\prime}}\right)^{2}\left(\min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}\right)^{2}\right\}, & \text { if } t>\lambda, t^{\prime} \leqslant \lambda^{\prime} \\
\lambda^{2}\left(\lambda^{\prime}\right)^{2}\left\{\frac{(1-t)\left(1-t^{\prime}\right)}{t t^{\prime}}+2\left(\min \left\{\frac{1-t}{t}, \frac{1-t^{\prime}}{t^{\prime}}\right\}\right)^{2}\right\}, & \text { if } t>\lambda, t^{\prime}>\lambda^{\prime}\end{cases}
\end{aligned}
$$

Note that for all $\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \in\left\{\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right): t=\lambda=1\right.$ or $\left.t^{\prime}=\lambda^{\prime}=1\right\}$ and $n \in \mathbb{N}$, we have

$$
\mathbb{E}\left[\widetilde{Z}_{n}^{3}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right]=\mathbb{E}\left[\widetilde{Z}_{n}^{2}(t, \lambda) \widetilde{Z}_{n}^{2}\left(t^{\prime}, \lambda^{\prime}\right)\right]=0
$$

Hence, the fourth moment of $\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$ satisfies for all $(t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right) \in[0,1]^{2}$,

$$
\begin{array}{rl}
\mathbb{E}\left[\hat{Z}_{n}^{4}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right]=x^{4} & \mathbb{E}\left[\widetilde{Z}_{n}^{4}(t, \lambda)\right]+4 x^{3} y \mathbb{E}\left[\widetilde{Z}_{n}^{3}(t, \lambda) \widetilde{Z}_{n}\left(t^{\prime}, \lambda^{\prime}\right)\right] \\
& +6 x^{2} y^{2} \mathbb{E}\left[\widetilde{Z}_{n}^{2}(t, \lambda) \widetilde{Z}_{n}^{2}\left(t^{\prime}, \lambda^{\prime}\right)\right] \\
& +4 x y^{3} \mathbb{E}\left[\widetilde{Z}_{n}(t, \lambda) \widetilde{Z}_{n}^{3}\left(t^{\prime}, \lambda^{\prime}\right)\right]+y^{4} \mathbb{E}\left[\widetilde{Z}_{n}^{4}\left(t^{\prime}, \lambda^{\prime}\right)\right] \\
\rightarrow 3\left(\hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)\right)^{2}
\end{array}
$$

as desired. An application of the fourth moment theorem [68, Theorem 1] yields that

$$
\hat{Z}_{n}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right) \Rightarrow x Z^{*}(t, \lambda)+y Z^{*}\left(t^{\prime}, \lambda^{\prime}\right)=: \hat{Z}^{*}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)
$$

where $\hat{Z}^{*}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$ is a normally distributed random variable with mean zero and variance $\hat{c}\left((t, \lambda),\left(t^{\prime}, \lambda^{\prime}\right)\right)$. Together with the tightness of the sequence $\left(\widetilde{Z}_{n}\right)_{n \in \mathbb{N}}$, the statement of Theorem 3.4.4 follows.

Example 3.1 (Change in the mean with known covariance). We have $d=m$ and the density function in (3.2.1 equals

$$
f(x ; \theta)=\exp \left(\theta^{T} \Sigma^{-1} x-\frac{1}{2} \theta^{T} \Sigma^{-1} \theta-\frac{1}{2} x^{T} \Sigma^{-1} x-\log \left((2 \pi)^{m / 2} \operatorname{det}(\Sigma)\right)\right)
$$

where $\Sigma \in \mathbb{R}^{m \times m}$ is symmetric, positive definite. Hence, $T(x)=\Sigma^{-1} x, H(x)=\frac{1}{2} x^{T} \Sigma x$, and $A(\theta)=\frac{1}{2} \theta^{T} \Sigma^{-1} \theta$. Elementary calculations yield that $H^{\prime \prime}(x)=\Sigma, A^{\prime \prime}(\theta)=\Sigma^{-1}$.
Example 3.2 (Simultaneous change in the mean and covariance). We have $d=m+m^{2}$ and for $\mu \in \mathbb{R}^{m}$ and $\Sigma \in \mathbb{R}^{m \times m}$ being symmetric and positive definite, the density function equals

$$
f(x ; \mu, \Sigma)=\exp \left(\mu^{T} \Sigma^{-1} x-\frac{1}{2} \mu^{T} \Sigma^{-1} \mu-\frac{1}{2} x^{T} \Sigma^{-1} x-\log \left((2 \pi)^{m / 2} \operatorname{det}(\Sigma)\right)\right)
$$

With a little abuse of notation, we identify $\mathbb{R}^{m \times m} \equiv \mathbb{R}^{m^{2}}$ and $\Sigma \equiv \operatorname{vec}(\Sigma) \in \mathbb{R}^{m^{2}}$. Then, setting $\theta(\mu, \Sigma)=\left(\theta_{1}, \theta_{2}\right)^{T}=\left(\Sigma^{-1} \mu,-\frac{1}{2} \Sigma^{-1}\right) \in \mathbb{R}^{m+m^{2}}$ the density can be rewritten in form of its natural parametrization, i.e.,

$$
f(x ; \theta)=\exp \left(\theta_{1}^{T} x+x^{T} \theta_{2} x+\frac{1}{4} \theta_{1}^{T} \theta_{2}^{-1} \theta_{1}-\frac{1}{2} \log \left(\operatorname{det}\left(-\pi \theta_{2}^{-1}\right)\right)\right)
$$

and therefore $T(x)=\left(x, x x^{T}\right) \in \mathbb{R}^{m+m^{2}}, H(x)=-\frac{1}{2} \log \left(\operatorname{det}\left(2 \pi\left(x_{2}-x_{1} x_{1}^{T}\right)\right)\right)$, for $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{m} \times \mathbb{R}^{m^{2}}$, and $A(\theta)=-\frac{1}{4} \theta_{1}^{T} \theta_{2}^{-1} \theta_{1}+\frac{1}{2} \log \left(\operatorname{det}\left(-\pi \theta_{2}^{-1}\right)\right)$. Elementary calculations yield that

$$
A^{\prime}(\theta):=\left(A_{\theta_{1}}, A_{\theta_{2}}\right)^{T}:=\left(-\frac{1}{2} \theta_{1}^{T} \theta_{2}^{-1}, \frac{1}{4}\left(\theta_{1}^{T} \theta_{2}^{-1} \otimes \theta_{1}^{T} \theta_{2}^{-1}\right)-\frac{1}{2} v e c\left(\theta_{2}^{-1}\right)^{T}\right)^{T}
$$

and

$$
A^{\prime \prime}(\theta):=\left(\begin{array}{ll}
\frac{\partial A_{\theta_{1}}}{\partial \theta_{1}} & \frac{\partial A_{\theta_{1}}}{\partial \theta_{2}} \\
\frac{\partial A_{\theta_{2}}}{\partial \theta_{1}} & \frac{\partial A_{\theta_{2}}}{\partial \theta_{2}},
\end{array}\right)
$$

where $\frac{\partial A_{\theta_{1}}}{\partial \theta_{1}}=-\frac{1}{2} \theta_{2}^{-1} \in \mathbb{R}^{m \times m}, \frac{\partial A_{\theta_{1}}}{\partial \theta_{2}}=\left(\frac{\partial A_{\theta_{2}}}{\partial \theta_{1}}\right)^{T}=\left(\theta_{1}^{T} \theta_{2}^{-1}\right)^{T} \otimes \operatorname{vec}\left(\theta_{2}^{-1}\right)^{T} \in \mathbb{R}^{m \times m^{2}}$, and $\frac{\partial A_{\theta_{2}}}{\partial \theta_{2}}=\frac{1}{2}\left(\theta_{1}^{T} \theta_{2}^{-1}\right) \otimes\left(\theta_{1}^{T} \theta_{2}^{-1}\right) \otimes \operatorname{vec}\left(\theta_{2}^{-1}\right) \in \mathbb{R}^{m^{2} \times m^{2}}$.

Once the null hypothesis $H_{0}$ "no change point" is rejected, one is interested in locating the change point $k_{n}^{*}$ or the change point fraction $\lambda_{n}^{*}:=k_{n}^{*} / n$. For this, we suggest the estimator

$$
\begin{equation*}
\hat{\lambda}_{n}:=\frac{1}{n} \underset{1 \leqslant k \leqslant n-1}{\arg \max }\left\{2 S_{n}(k)\right\} . \tag{3.4.14}
\end{equation*}
$$

In the following, we denote by $\hat{k}_{n}:=n \hat{\lambda}_{n}$ the estimator of the location of the change point $k_{n}^{*}$. Note that, as always, the estimator in (3.4.14) is random, as it depends on the data $X_{1}, \cdots, X_{n}$ and therefore, also on the true location of the change point $k_{n}^{*}$. In the following, we will study the properties of this estimator.

Theorem 3.4.7 (Consistency of $\hat{\lambda}_{n}$ ). Let the assumptions of Theorem 3.4.4 hold. Then,

$$
\delta^{2}\left|\hat{k}_{n}-k_{n}^{*}\right|=\mathcal{O}_{\mathbb{P}}(1) .
$$

In particular, $\hat{\lambda}_{n}$ is a consistent estimator of $\lambda_{n}^{*}$ with $\left|\hat{\lambda}_{n}-\lambda_{n}^{*}\right|=\mathcal{O}_{\mathbb{P}}\left(\left(\delta^{2} n\right)^{-1}\right)$.
Remark 3.4.8 (Convergence rates and minimum detectable size in slightly different models). Csörgő and Horváth [23] also studied the consistency of the estimator in (3.4.14) provided that the true location of the change point $k_{n}^{*}$ is deterministic under slightly different assumptions:
i) If the size of the change is independent of $n$, they still obtain a convergence rate for the estimator $\hat{\lambda}_{n}$ of order $n^{-1}$.
ii) If the change point fraction satisfies $k_{n}^{*} / n \rightarrow 0$ as $n \rightarrow \infty$, i.e., the data contain an early change point, they obtain the same convergence rate as in Theorem 3.4.7, but the detectable size of the change has to be generally of larger order satisfying

$$
\frac{k_{n}^{*} \Delta^{2}}{\log \log (n)} \rightarrow \infty
$$

Although, we do not study these cases in our work, we expect to obtain similar results assuming that $k_{n}^{*}$ is random itself.

Proof of Theorem [3.4.7. First, we consider the case $1 \leqslant k \leqslant k^{*}$ where $k^{*} \in[n \gamma, n(1-\gamma)]$ for some $\gamma \in(0,1 / 2)$. Then, observe that

$$
\begin{aligned}
& \mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right) \\
& =\left(k-k^{*}\right) H\left(\tau_{1}\right)+(n-k) H\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)-\left(n-k^{*}\right) H\left(\tau_{2}\right) \\
& =\left(k^{*}-k\right)\left(H\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)-H\left(\tau_{1}\right)\right) \\
& \quad \quad+\left(n-k^{*}\right)\left(H\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)-H\left(\tau_{2}\right)\right) .
\end{aligned}
$$

Now, two applications of Taylor's formula of the second order yield that

$$
\begin{aligned}
& \mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)=\frac{1}{2} \frac{\left(k^{*}-k\right)\left(n-k^{*}\right)}{n-k}\left(2\left(\tau_{2}-\tau_{1}\right)^{T}\left(H^{\prime}\left(\tau_{1}\right)-H^{\prime}\left(\tau_{2}\right)\right)\right. \\
& \left.+\frac{n-k^{*}}{n-k}\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(\tau_{1}-\tau_{2}\right)+\frac{k^{*}-k}{n-k}\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{2}\right)\left(\tau_{1}-\tau_{2}\right)\right) \\
& +o\left(\frac{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}}{n-k}\right),
\end{aligned}
$$

where we obtain the last summand by bounding the Lagrange remainder term. By the
mean value theorem, there exists $\xi$ in the interval connecting $\tau_{1}$ and $\tau_{2}$ such that

$$
\begin{align*}
& \mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)=\frac{1}{2} \frac{\left(k^{*}-k\right)\left(n-k^{*}\right)}{n-k}\left(-2\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}(\xi)\left(\tau_{1}-\tau_{2}\right)\right. \\
& \left.+\frac{n-k^{*}}{n-k}\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(\tau_{1}-\tau_{2}\right)+\frac{k^{*}-k}{n-k}\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{2}\right)\left(\tau_{1}-\tau_{2}\right)\right)  \tag{3.4.15}\\
& +o\left(\frac{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}}{n-k}\right)
\end{align*}
$$

Now, since $\left(\tau_{1}, \tau_{2}\right) \rightarrow\left(\tau_{A}, \tau_{A}\right)$ as $n \rightarrow \infty$ thanks to Assumption 3.2, we conclude for $n$ large enough, that the above difference is negative, increasing in $k$ for fixed $k^{*}$, and we can find a constant $C>0$ such that

$$
\begin{equation*}
\mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right) \leqslant-C \frac{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}}{n-k} . \tag{3.4.16}
\end{equation*}
$$

In the following, we denote $\hat{\mu}_{n}\left(k, k^{*}\right):=\mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)$. Next, we apply Taylor's formula of the second order to obtain

$$
\begin{aligned}
& S_{n}\left(k, k^{*}\right)-S_{n}\left(k^{*}, k^{*}\right)-\hat{\mu}_{n}\left(k, k^{*}\right) \\
& =Z_{n}\left(k, k^{*}\right)-Z_{n}\left(k^{*}, k^{*}\right)+\frac{k}{2}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right) \\
& \quad-\frac{k^{*}}{2}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right) \\
& -\frac{n-k^{*}}{2}\left(B_{n}^{*}\left(k^{*}, k^{*}\right)-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{2}\right)\left(B_{n}^{*}\left(k^{*}, k^{*}\right)-\tau_{2}\right) \\
& +\frac{n-k}{2}\left(B_{n}^{*}\left(k, k^{*}\right)-\frac{k^{*}-k}{n-k} \tau_{1}-\frac{n-k^{*}}{n-k} \tau_{2}\right)^{T} H^{\prime \prime}\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right) \\
& \quad \times\left(B_{n}^{*}\left(k, k^{*}\right)-\frac{k^{*}-k}{n-k} \tau_{1}-\frac{n-k^{*}}{n-k} \tau_{2}\right)+R_{1, n}\left(k, k^{*}\right),
\end{aligned}
$$

where $R_{1, n}\left(k, k^{*}\right)$ is the remainder of Lagrange form satisfying

$$
\begin{aligned}
& \left|R_{1, n}\left(k, k^{*}\right)\right| \leqslant C\left(k\left\|B_{n}\left(k, k^{*}\right)-\tau_{1}\right\|^{3}+k^{*}\left\|B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right\|^{3}\right. \\
& \left.\quad+\left(n-k^{*}\right)\left\|B_{n}^{*}\left(k^{*}, k^{*}\right)-\tau_{2}\right\|^{3}+(n-k)\left\|B_{n}^{*}\left(k, k^{*}\right)-\frac{k^{*}-k}{n-k} \tau_{1}-\frac{n-k^{*}}{n-k} \tau_{2}\right\|^{3}\right)
\end{aligned}
$$

for some $C>0$. Applying the law of the iterated logarithm, we conclude

$$
\max _{1 \leqslant k \leqslant k^{*}} k^{3 / 2}\left\|B_{n}\left(k, k^{*}\right)-\tau_{1}\right\|^{3}=\mathcal{O}_{\mathbb{P}}\left(\left(\log \log \left(k^{*}\right)\right)^{3 / 2}\right),
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$ and applying Donsker's theorem (cf. Lemma 3.4.3),
we get

$$
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)}\left(k^{*}\right)^{3 / 2}\left\|B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right\|^{3}=\mathcal{O}_{\mathbb{P}}(1) .
$$

Bounding the remaining terms in a similar way and applying the upper bound for the difference $\hat{\mu}_{n}\left(k, k^{*}\right)$ in (3.4.16), we conclude for every $\varepsilon>0$ and $\eta>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$
\begin{equation*}
\mathbb{P}\left[\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{\left|R_{1, n}\left(k, k^{*}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|}>\eta\right]<\varepsilon . \tag{3.4.17}
\end{equation*}
$$

Next, let us analyze the difference $Z_{n}\left(k, k^{*}\right)-Z_{n}\left(k^{*}, k^{*}\right)$ which equals for all $1 \leqslant k \leqslant k^{*}$ and $k^{*} \in\{1, \cdots, n-1\}$

$$
\begin{aligned}
Z_{n}\left(k, k^{*}\right)-Z_{n}\left(k^{*}, k^{*}\right)= & \left(h_{n}\left(\frac{k^{*}-k}{n-k}\right)-h_{n}(1)\right) \sum_{i=k+1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right) \\
& +\left(h_{n}\left(\frac{k^{*}-k}{n-k}\right)-h_{n}(0)\right) \sum_{i=k^{*}+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right) \\
= & Z_{n}^{(1)}\left(k, k^{*}\right)+Z_{n}^{(2)}\left(k, k^{*}\right)
\end{aligned}
$$

For the first summand $Z_{n}^{(1)}\left(k, k^{*}\right)$, studying the calculations in (3.4.11) and applying the law of the iterated logarithm, we obtain uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$,

$$
\begin{aligned}
\max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} & \frac{n-k}{\left(n-k^{*}\right) \delta} \frac{1}{\sqrt{\left(k^{*}-k\right) \log \log \left(k^{*}-k\right)}}\left|Z_{n}^{(1)}\left(k, k^{*}\right)\right| \\
& =\max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{1}{\sqrt{\left(k^{*}-k\right) \log \log \left(k^{*}-k\right)}}\left|\widetilde{\Sigma}^{1 / 2} \sum_{i=k+1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right|+o(1) \\
& =\mathcal{O}_{\mathbb{P}}(1) .
\end{aligned}
$$

For the second summand $Z_{n}^{(2)}\left(k, k^{*}\right)$, applying Donsker's theorem, we obtain

$$
\begin{aligned}
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} & \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{n-k}{\left(k^{*}-k\right) \delta} \frac{1}{\sqrt{n-k^{*}}}\left|Z_{n}^{(2)}\left(k, k^{*}\right)\right| \\
& =\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \sqrt{\frac{n}{n-k^{*}}} \frac{1}{\sqrt{n}}\left|\widetilde{\Sigma}^{1 / 2} \sum_{i=k^{*}+1}^{n}\left(T\left(X_{2, i}\right)-\tau_{2}\right)\right|+o(1) \\
& =\mathcal{O}_{\mathbb{P}}(1) .
\end{aligned}
$$

Applying (3.4.5) and (3.4.16), we conclude for every $\varepsilon>0$ and $\eta>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$
\begin{equation*}
\mathbb{P}\left[\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{\left|Z_{n}^{(1)}\left(k, k^{*}\right)+Z_{n}^{(2)}\left(k, k^{*}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|}>\eta\right]<\varepsilon . \tag{3.4.18}
\end{equation*}
$$

Next, applying again the law of the iterated logarithm, we have

$$
\max _{1 \leqslant k \leqslant k^{*}} \frac{k}{2}\left|\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)\right|=\mathcal{O}_{\mathbb{P}}\left(\log \log \left(k^{*}\right)\right)
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Then, our assumption in (3.4.5) implies

$$
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant \alpha n} \frac{(n-k) \log \log (n)}{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

for all $0<\alpha<\gamma$. Hence, applying (3.4.16), we conclude

$$
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant \alpha n} \frac{k\left|\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|}=\mathcal{O}_{\mathbb{P}}(1)
$$

In contrast, applying Donsker's theorem, the assumption in (3.4.5), and equation (3.4.16), it follows that

$$
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant \alpha n} \frac{k^{*}\left|\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|}=\mathcal{O}_{\mathbb{P}}(1)
$$

Next, we can rewrite

$$
\begin{aligned}
& \frac{k}{2}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right) \\
& \quad-\frac{k^{*}}{2}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right) \\
& =\frac{1}{2}\left\{\sqrt{k}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T}-\sqrt{k^{*}}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T}\right\} H^{\prime \prime}\left(\tau_{1}\right) \sqrt{k}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right) \\
& \quad+\frac{1}{2} \sqrt{k^{*}}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left\{\sqrt{k}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)-\sqrt{k^{*}}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)\right\}
\end{aligned}
$$

Then, for all $0<\alpha<\gamma$, applying (3.4.16), the law of the iterated logarithm, and

Donsker's theorem, we obtain

$$
\begin{aligned}
& \max _{\alpha n \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{\left|\left(\sqrt{k}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T}-\sqrt{k^{*}}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T}\right) H^{\prime \prime}\left(\tau_{1}\right) \sqrt{k}\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|} \\
& =\mathcal{O}_{\mathbb{P}}(1) \max _{\alpha n \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{(n-k) \sqrt{\log \log (k)}}{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}}\left(\frac{1}{\sqrt{k}}\left\|\sum_{i=k+1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\|\right. \\
& \left.\quad+\frac{k^{*}-k}{\sqrt{k\left(k^{*}+k\right)}} \frac{1}{\sqrt{k}}\left\|\sum_{i=1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\|\right) \\
& =\mathcal{O}_{\mathbb{P}}(1) \max _{\alpha n \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{(n-k) \sqrt{\log \log (k)}}{\left(k^{*}-k\right)\left(n-k^{*}\right) \delta^{2}}\left(\sqrt{\frac{\left(k^{*}-k\right) \log \log \left(k^{*}-k\right)}{k}}+\frac{k^{*}-k}{\sqrt{k k^{*}}}\right) \\
& =\mathcal{O}_{\mathbb{P}}(1) \sqrt{\frac{\log \log (\kappa)}{\kappa}}
\end{aligned}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$ and note that the $\mathcal{O}_{\mathbb{P}}(1)$-term is independent of $\kappa$. Combining the above probability bounds, we conclude for every $\varepsilon>0$ and $\eta>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$
\begin{align*}
\mathbb{P} & \max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \mid k\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k, k^{*}\right)-\tau_{1}\right)  \tag{3.4.19}\\
& -k^{*}\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left(B_{n}\left(k^{*}, k^{*}\right)-\tau_{1}\right)\left|/\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|>\eta\right]<\varepsilon .
\end{align*}
$$

With similar arguments, we can also show for every $\varepsilon>0$ and $\eta>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant \mathbb{N}$,

$$
\begin{align*}
& \mathbb{P}\left[\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma) 1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \max ^{(n-k)}\left(B_{n}^{*}\left(k, k^{*}\right)-\frac{k^{*}-k}{n-k} \tau_{1}-\frac{n-k^{*}}{n-k} \tau_{2}\right)^{T}\right. \\
& \quad \times H^{\prime \prime}\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)\left(B_{n}^{*}\left(k, k^{*}\right)-\frac{k^{*}-k}{n-k} \tau_{1}-\frac{n-k^{*}}{n-k} \tau_{2}\right) \\
&-\left(n-k^{*}\right)\left(B_{n}^{*}\left(k^{*}, k^{*}\right)-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{2}\right)\left(B_{n}^{*}\left(k^{*}, k^{*}\right)-\tau_{2}\right)\left|/\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|>\eta\right]<\varepsilon . \tag{3.4.20}
\end{align*}
$$

Finally, combining the results in (3.4.17), (3.4.18), (3.4.19), and (3.4.20), we conclude for every $\varepsilon>0$ and $\eta>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$
\begin{equation*}
\mathbb{P}\left[\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{\left|S_{n}\left(k, k^{*}\right)-S_{n}\left(k^{*}, k^{*}\right)-\hat{\mu}_{n}\left(k, k^{*}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|}>\eta\right]<\varepsilon . \tag{3.4.21}
\end{equation*}
$$

Now, by Assumption $3.2, \lambda^{*} \in[\gamma, 1-\gamma]$ for some $\gamma \in(0,1 / 2)$ with probability one. Hence, for each $\varepsilon>0$, there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$
\mathbb{P}\left[\lambda_{n}^{*} \notin[\gamma / 2,1-\gamma / 2]\right]<\varepsilon
$$

Next, let us define the set $A_{n}:=\left\{\hat{k}_{n} \leqslant k_{n}^{*}\right\} \cap\left\{\lambda_{n}^{*} \in[\gamma / 2,1-\gamma / 2]\right\}$. Then, the result in (3.4.21) implies for every $\varepsilon>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$
\begin{aligned}
& \mathbb{P}\left[k_{n}^{*}-\hat{k}_{n}>\frac{\kappa}{\delta^{2}}, A_{n}\right] \leqslant \mathbb{P}\left[\max _{1 \leqslant k \leqslant k_{n}^{*}-\kappa / \delta^{2}} S_{n}\left(k, k_{n}^{*}\right) \geqslant S_{n}\left(k_{n}^{*}, k_{n}^{*}\right), A_{n}\right] \\
& \quad \leqslant \mathbb{P}\left[\max _{n \gamma / 2 \leqslant k^{*} \leqslant n(1-\gamma / 2)} \max _{1 \leqslant k \leqslant k^{*}-\kappa / \delta^{2}} \frac{\left|S_{n}\left(k, k^{*}\right)-S_{n}\left(k^{*}, k^{*}\right)-\hat{\mu}_{n}\left(k, k^{*}\right)\right|}{\left|\hat{\mu}_{n}\left(k, k^{*}\right)\right|} \geqslant \eta, A_{n}\right]<\varepsilon,
\end{aligned}
$$

where $\eta>0$. Hence, for every $\varepsilon>0$, there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$
\begin{aligned}
\mathbb{P}\left[\left\{k_{n}^{*}-\hat{k}_{n}>\frac{\kappa}{\delta^{2}}\right\},\left\{\hat{k}_{n} \leqslant k_{n}^{*}\right\}\right] & \leqslant \mathbb{P}\left[\left\{k_{n}^{*}-\hat{k}_{n}>\frac{\kappa}{\delta^{2}}\right\}, A_{n}\right]+\mathbb{P}\left[\lambda_{n}^{*} \notin[\gamma / 2,1-\gamma / 2]\right] \\
& <2 \varepsilon
\end{aligned}
$$

With similar arguments, we are able to show for every $\varepsilon>0$, that there exist $\kappa>0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$
\mathbb{P}\left[\left\{\hat{k}_{n}-k_{n}^{*}>\frac{\kappa}{\delta^{2}}\right\},\left\{\hat{k}_{n}>k_{n}^{*}\right\}\right]<2 \varepsilon
$$

Hence, $\delta^{2}\left|k_{n}^{*}-\hat{k}_{n}\right|=\mathcal{O}_{\mathbb{P}}(1)$. Finally, since

$$
\delta^{2}\left|k_{n}^{*}-\hat{k}_{n}\right|=n \delta^{2}\left|\lambda_{n}^{*}-\hat{\lambda}_{n}\right|
$$

we conclude that $\hat{\lambda}_{n}$ is a consistent estimator of $\lambda_{n}^{*}$ and $\left|\lambda_{n}^{*}-\hat{\lambda}_{n}\right|=\mathcal{O}_{\mathbb{P}}\left(\left(n \delta^{2}\right)^{-1}\right)$.
In order to construct confidence intervals for $k_{n}^{*}$, we need to establish the limit distribution of the deviation $\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$ provided that $H_{1}$ holds true. The next theorem gives us a first limit result for this deviation. However, it is only of theoretical interest, since $\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)$, the size of a change, is unknown.

Theorem 3.4.9 (Limit distribution of $\hat{k}_{n}$ under the alternative). Let the assumptions of Theorem 3.4 .4 be satisfied. Moreover, let us introduce the process

$$
W^{*}(t):= \begin{cases}\sigma_{A} W_{1}(-t)-\frac{1}{2} \sigma_{A}^{2}|t|, & t<0, \\ 0, & t=0 \\ \sigma_{A} W_{2}(t)-\frac{1}{2} \sigma_{A}^{2}|t|, & t>0\end{cases}
$$

where $W_{1}$ and $W_{2}$ are two independent Brownian motions. Then,

$$
\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right) \Rightarrow \underset{u \in(-\infty, \infty)}{\arg \max } W^{*}(u)
$$

Note that $\arg \max _{u \in(-\infty, \infty)} W^{*}(u)$ is the canonical limit distribution from the literature (cf. e.g. 23,51) for the deviation $\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$ provided that the size of the change vanishes as $n \rightarrow \infty$. Moreover, the law of the iterated logarithm for the Brownian motion implies the almost sure finiteness of arg $\max _{u \in(-\infty, \infty)} W^{*}(u)$.
We state the proof of this theorem at the end of this section. With a slight rescaling of the left hand side in Theorem 3.4.9, we can establish a distribution-free limit process.

Corollary 3.4.10. Let the assumptions of Theorem 3.4 .4 be satisfied. Then,

$$
\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)\left(\hat{k}_{n}-k_{n}^{*}\right) \Rightarrow \underset{u \in(-\infty, \infty)}{\arg \max } \hat{W}(u),
$$

where the limit process $\hat{W}$ is defined by

$$
\hat{W}(t):= \begin{cases}W_{1}(-t)-\frac{1}{2}|t|, & t<0, \\ 0, & t=0, \\ W_{2}(t)-\frac{1}{2}|t|, & t>0,\end{cases}
$$

for two independent Brownian motions $W_{1}$ and $W_{2}$.
Proof. By the scaling property of the Brownian motion, we conclude that

$$
\underset{u \in(-\infty, \infty)}{\arg \max } W^{*}(u) \quad \text { and } \quad \frac{1}{\sigma_{A}^{2}} \underset{u \in(-\infty, \infty)}{\arg \max } \hat{W}(s)
$$

have the same distribution, where

$$
\sigma_{A}^{2}:=\lim _{n \rightarrow \infty} \frac{\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)}{\delta^{2}} .
$$

This finishes the proof.
Even if the right hand side in Corollary 3.4.10 is distribution-free, $\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\right.$ $\tau_{2}$ ), the size of a change, occurring on the left hand side, is still unknown. Hence, in order to be able to construct confidence intervals for $k_{n}^{*}$, we need to estimate
$\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)$, the size of a change. For that, we use the estimator

$$
\begin{equation*}
\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right) \tag{3.4.22}
\end{equation*}
$$

For this reason, we will show in the next lemma that this is indeed a consistent estimator for the size of a change.

Lemma 3.4.11. Under the assumptions of Theorem 3.4.4, we have

$$
\frac{\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)}{\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)} \rightarrow 1 \quad \text { in probability }
$$

Proof. Since $\mathbb{E}\left[T\left(X_{1,1}\right)\right]=A^{\prime}\left(\theta_{0}^{(1)}\right)=\tau_{1}$ and $\mathbb{E}\left[T\left(X_{2,1}\right)\right]=A^{\prime}\left(\theta_{0}^{(2)}\right)=\tau_{2}$, we conclude by Assumptions 3.2 and 3.3 , and an application of the law of large numbers that

$$
B_{n}(n)=\frac{1}{n} \sum_{i=1}^{n} T\left(X_{i}\right) \rightarrow A^{\prime}\left(\theta_{A}\right)=: \tau_{A} \quad \text { in probability. }
$$

Since $H$ has continuous derivatives up to the third order, we also have

$$
\begin{equation*}
H^{\prime \prime}\left(B_{n}(n)\right) \rightarrow H^{\prime \prime}\left(\tau_{A}\right) \quad \text { in probability } \tag{3.4.23}
\end{equation*}
$$

In the following, let us concentrate on the event $A_{n}:=\left\{\lambda_{n}^{*} \in[\gamma / 2,1-\gamma / 2]\right\}$. First, note that

$$
\begin{aligned}
& \left\|B_{n}\left(\hat{k}_{n}\right)-\tau_{1}\right\|=\left\|\frac{1}{\hat{k}_{n}}\left(\sum_{i=1}^{\hat{k}_{n}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right)\right\| \mathbb{1}_{\left\{\hat{k}_{n} \leqslant k_{n}^{*}\right\}} \\
& \quad+\left\|\frac{1}{\hat{k}_{n}}\left(\sum_{i=1}^{k_{n}^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k_{n}^{*}+1}^{\hat{k}_{n}}\left(T\left(X_{2, i}\right)-\tau_{2}\right)-\left(\hat{k}_{n}-k_{n}^{*}\right)\left(\tau_{1}-\tau_{2}\right)\right)\right\| \mathbb{1}_{\left\{\hat{k}_{n}>k_{n}^{*}\right\}}
\end{aligned}
$$

On the event $A_{n}$, we obtain by an application of the law of the iterated logarithm that

$$
\begin{align*}
& \left\|\frac{1}{k_{n}^{*}} \sum_{i=1}^{k_{n}^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\| \leqslant \max _{n \gamma / 2 \leqslant k^{*} \leqslant n(1-\gamma / 2)}\left\|\frac{1}{k^{*}} \sum_{i=1}^{k^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\| \\
& \quad=\mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log \log (n)}{n}}\right) \tag{3.4.24}
\end{align*}
$$

Since $\hat{\lambda}_{n}$ is a consistent estimator of $\lambda_{n}^{*}$, we further conclude

$$
\left\|\frac{1}{\hat{k}_{n}} \sum_{i=1}^{\hat{k}_{n}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\|=\mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log \log (n)}{n}}\right)
$$

and

$$
\begin{aligned}
& \left\|\frac{1}{\hat{k}_{n}}\left(\sum_{i=1}^{k_{n}^{*}}\left(T\left(X_{1, i}\right)-\tau_{1}\right)+\sum_{i=k_{n}^{*}+1}^{\hat{k}_{n}}\left(T\left(X_{2, i}\right)-\tau_{2}\right)-\left(\hat{k}_{n}-k_{n}^{*}\right)\left(\tau_{1}-\tau_{2}\right)\right)\right\| \\
& \quad=\mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log \log (n)}{n}}\right) .
\end{aligned}
$$

Hence, on the event $A_{n}$, we obtain that

$$
\begin{equation*}
\left\|B_{n}\left(\hat{k}_{n}\right)-\tau_{1}\right\|=\mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log \log (n)}{n}}\right) . \tag{3.4.25}
\end{equation*}
$$

Similarly, another application of the law of the iterated logarithm shows that, restricted on the event $A_{n}$,

$$
\begin{equation*}
\left|B_{n}^{*}\left(\hat{k}_{n}\right)-\tau_{2}\right|=\mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log \log (n)}{n}}\right) . \tag{3.4.26}
\end{equation*}
$$

Moreover, applying our assumption in (3.4.5), we conclude that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\log \log (n)}{n\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)}=0 \tag{3.4.27}
\end{equation*}
$$

Hence, on the event $A_{n}$, combining the results in (3.4.23), (3.4.25), (3.4.26), and (3.4.27), it follows that

$$
\frac{\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)}{\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)} \rightarrow 1 \quad \text { in probability. }
$$

Finally, by Assumption [3.2, it holds that $\lambda^{*} \in[\gamma, 1-\gamma]$ for some $\gamma \in(0,1 / 2)$ with probability one. Hence, for every $\varepsilon>0$, there exists an $N_{1} \in \mathbb{N}$ such that for all $n \geqslant N_{1}$,

$$
\mathbb{P}\left[\lambda_{n}^{*} \notin[\gamma / 2,1-\gamma / 2]\right]<\varepsilon .
$$

Then, for every $\varepsilon>0$ and $\eta>0$ there exists an $N_{2}>0$ such that for all $n \geqslant N_{1} \vee N_{2}$, we have

$$
\begin{aligned}
& \mathbb{P}\left[\left|\frac{\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)}{\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)}-1\right|>\eta\right] \\
& \leqslant \mathbb{P}\left[\left|\frac{\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)}{\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right)}-1\right|>\eta, A_{n}\right] \\
& \quad+\mathbb{P}\left[\lambda_{n}^{*} \notin[\gamma / 2,1-\gamma / 2]\right]<2 \varepsilon .
\end{aligned}
$$

This finishes the proof.

Remark 3.4.12. The probability bound of order $\sqrt{\log \log (n) / n}$ in (3.4.24) can be improved to the order $1 / \sqrt{n}$ : since $\left(T\left(X_{1, i}\right)-\tau_{1}\right)_{i \geqslant 1}$ is a sequence of iid, centered random variables, the discrete-time process $S_{k}:=\sum_{i=1}^{k}\left(T\left(X_{1, i}\right)-\tau_{1}\right), k=1, \cdots, n$, is a martingale and by Jensen's inequality, $\left(\left\|S_{k}\right\|\right)_{k \geqslant 1}$ is a submartingale. Applying Doob's martingale inequality for $p=2$, we get the following probability bound:

$$
\mathbb{P}\left[\max _{1 \leqslant k \leqslant n}\left\|\frac{1}{\sqrt{n}} \sum_{i=1}^{k}\left(T\left(X_{1, i}\right)-\tau_{1}\right)\right\| \geqslant \lambda\right] \leqslant \frac{C}{\lambda^{2}},
$$

for each $\lambda>0$, a constant $C$ depending on $\Sigma_{A}$, and $n$ large enough. In particular, this implies that the term on the left hand side in (3.4.24) is indeed of order $\mathcal{O}_{\mathbb{P}}\left(n^{-1 / 2}\right)$. Arguing in this way, the assumption in (3.4.5) on the convergence speed of the size of the change in the parameters can be relaxed to $n \Delta^{2} \rightarrow \infty$ as $n \rightarrow \infty$.

With all these preparations done, we are finally ready to state a distribution-free limit theorem for the deviation $\left(\hat{k}_{n}-k_{n}^{*}\right)$ under the alternative. This result can be used to build confidence intervals for $k_{n}^{*}$.

Corollary 3.4.13. Let the assumptions of Theorem 3.4 .4 be satisfied. Then,

$$
\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)^{T} H^{\prime \prime}\left(B_{n}(n)\right)\left(B_{n}\left(\hat{k}_{n}\right)-B_{n}^{*}\left(\hat{k}_{n}\right)\right)\left(\hat{k}_{n}-k_{n}^{*}\right) \Rightarrow \underset{u \in(-\infty, \infty)}{\arg \max } \hat{W}(u) .
$$

Proof. This follows directly from Corollary 3.4.10, Lemma 3.4.11, and an application of Slutzky's Lemma.

Finally, we finish this section by stating the proof of Theorem 3.4.9.
Proof of Theorem 3.4.9. We show that for some arbitrary $\kappa>0$, it holds

$$
S_{n}\left(\lambda+\cdot /\left(n \delta^{2}\right), \lambda\right)-S_{n}(\lambda, \lambda) \Rightarrow W^{*}
$$

in the Skorokhod topology on the space $D([-\kappa, \kappa], \mathbb{R})$ uniformly over $\lambda \in[\gamma, 1-\gamma]$. First, let us consider $k \in\left[k^{*}-\kappa / \delta^{2}, k^{*}\right]$, where $k^{*} / n \in[\gamma, 1-\gamma]$. A Taylor expansion of the second order yields for all $1 \leqslant k \leqslant k^{*}$,

$$
\begin{aligned}
S_{n}\left(k, k^{*}\right) & -S_{n}\left(k^{*}, k^{*}\right)-\left(\mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)\right) \\
& =V_{1, n}\left(k, k^{*}\right)+V_{2, n}\left(k, k^{*}\right)+V_{3, n}\left(k, k^{*}\right)+V_{4, n}\left(k, k^{*}\right)+R_{1, n}\left(k, k^{*}\right),
\end{aligned}
$$

where

$$
\begin{gathered}
V_{1, n}\left(k, k^{*}\right):=H^{\prime}\left(\tau_{1}\right)^{T} n^{1 / 2}\left(W_{1, k}^{(n)}-W_{1, k^{*}}^{(n)}\right), \\
V_{2, n}\left(k, k^{*}\right):=\frac{n}{2 k}\left(W_{1, k}^{(n)}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right) W_{1, k}^{(n)}-\frac{n}{2 k^{*}}\left(W_{1, k^{*}}^{(n)}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right) W_{1, k^{*}}^{(n)},
\end{gathered}
$$

$$
\begin{aligned}
V_{3, n}\left(k, k^{*}\right):= & H^{\prime}\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)^{T} n^{1 / 2}\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}+W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right) \\
& -H^{\prime}\left(\tau_{2}\right)^{T} n^{1 / 2}\left(W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
V_{4, n}\left(k, k^{*}\right):= & \frac{n}{2(n-k)}\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}+W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right)^{T} \\
& \quad \times H^{\prime \prime}\left(\frac{k^{*}-k}{n-k} \tau_{1}+\frac{n-k^{*}}{n-k} \tau_{2}\right)\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}+W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right) \\
& -\frac{n}{2\left(n-k^{*}\right)}\left(W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right)^{T} H^{\prime \prime}\left(\tau_{2}\right)\left(W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right) .
\end{aligned}
$$

Moreover, studying the treatment of the Lagrange remainder term in the proof of Theorem 3.4.7. we conclude that $R_{1, n}\left(k, k^{*}\right)$ is of order $o_{\mathbb{P}}(1)$ uniformly over $k^{*}-\kappa / \delta^{2} \leqslant$ $k \leqslant k^{*}$ and $n \gamma \leqslant k^{*} \leqslant n(1-\gamma)$ and hence vanishes in probability as $n \rightarrow \infty$. Applying Donsker's theorem, we obtain for all $\kappa>0$,

$$
\begin{equation*}
\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}}\left\|n^{1 / 2} \delta\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}\right)\right\|=\max _{u \in[-\kappa, 0]}\left\|\delta \sum_{j=0}^{\left\lfloor-u / \delta^{2}\right\rfloor}\left(T\left(X_{1, k^{*}-j}\right)-\tau_{1}\right)\right\|=\mathcal{O}_{\mathbb{P}}(1) \tag{3.4.28}
\end{equation*}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Moreover, for $k \in\left[k^{*}-\kappa / \delta^{2}, k^{*}\right]$, we have

$$
\begin{equation*}
\frac{n\left(k^{*}-k\right)}{k k^{*}}=\mathcal{O}\left(\left(n \delta^{2}\right)^{-1}\right) \tag{3.4.29}
\end{equation*}
$$

Hence, applying the Assumptions 3.1 and 3.2, the equations (3.4.28) and (3.4.29), and Donsker's theorem (cf. Lemma 3.4.3), we obtain

$$
\begin{aligned}
\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} & \left|V_{2, n}\left(k, k^{*}\right)\right| \leqslant \max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} \frac{1}{2} \frac{n\left(k^{*}-k\right)}{k k^{*}}\left|\left(W_{1, k}^{(n)}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right) W_{1, k}^{(n)}\right| \\
& +\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} \frac{1}{2} \frac{n}{k^{*}}\left|\left\{W_{1, k}^{(n)}-W_{1, k^{*}}^{(n)}\right\}^{T} H^{\prime \prime}\left(\tau_{1}\right) W_{1, k}^{(n)}\right| \\
& +\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} \frac{1}{2} \frac{n}{k^{*}}\left|\left(W_{1, k^{*}}^{(n)}\right)^{T} H^{\prime \prime}\left(\tau_{1}\right)\left\{W_{1, k}^{(n)}-W_{1, k^{*}}^{(n)}\right\}\right| \\
& =o_{\mathbb{P}}(1)
\end{aligned}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Bounding $V_{4, n}$ in a similar way, we obtain for arbitrary $\kappa>0$ and $n$ large enough

$$
\begin{equation*}
\max _{n \gamma \leqslant k^{*} \leqslant n(1-\gamma)} \max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}}\left|V_{2, n}\left(k, k^{*}\right)+V_{4, n}\left(k, k^{*}\right)\right|=o_{\mathbb{P}}(1) . \tag{3.4.30}
\end{equation*}
$$

Next, applying Donsker's theorem (cf. Lemma 3.4.3), we obtain again for $\kappa>0$ and $n$
large enough

$$
\begin{aligned}
& \max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}}\left|V_{1, n}\left(k, k^{*}\right)+V_{3, n}\left(k, k^{*}\right)-\left(H^{\prime}\left(\tau_{2}\right)-H^{\prime}\left(\tau_{1}\right)\right)^{T} n^{1 / 2}\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}\right)\right| \\
&=\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} \left\lvert\, n^{1 / 2}\left(H ^ { \prime } \left(\frac{k^{*}-k}{n-k} \tau_{1}\right.\right.\right.\left.\left.+\frac{n-k^{*}}{n-k} \tau_{2}\right)-H^{\prime}\left(\tau_{2}\right)\right)^{T} \\
& \times\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}+W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right) \mid \\
&=\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} n^{1 / 2} \delta \frac{k^{*}-k}{n-k}\left|\widetilde{\Sigma}^{1 / 2}\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}+W_{2, n}^{(n)}-W_{2, k^{*}}^{(n)}\right)\right|+o(1) \\
&=o_{\mathbb{P}}(1)
\end{aligned}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Combining the above bounds, we conclude that

$$
\begin{align*}
\max _{k^{*}-\kappa / \delta^{2} \leqslant k \leqslant k^{*}} \mid S_{n}\left(k, k^{*}\right) & -S_{n}\left(k^{*}, k^{*}\right)-\left(\mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)\right)  \tag{3.4.31}\\
& -n^{1 / 2}\left(H^{\prime}\left(\tau_{2}\right)-H^{\prime}\left(\tau_{1}\right)\right)^{T}\left(W_{1, k^{*}}^{(n)}-W_{1, k}^{(n)}\right) \mid=o_{\mathbb{P}}(1)
\end{align*}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Similarly, we can show that

$$
\begin{align*}
\max _{k^{*} \leqslant k \leqslant k^{*}+\kappa / \delta^{2}} \mid S_{n}\left(k, k^{*}\right) & -S_{n}\left(k^{*}, k^{*}\right)-\left(\mu_{n}\left(k, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)\right)  \tag{3.4.32}\\
& -n^{1 / 2}\left(H^{\prime}\left(\tau_{1}\right)-H^{\prime}\left(\tau_{2}\right)\right)^{T}\left(W_{2, k}^{(n)}-W_{2, k^{*}}^{(n)}\right)=o_{\mathbb{P}}(1)
\end{align*}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Furthermore, studying equation 3.4.15), for arbitrary $\kappa>0$, we have

$$
\begin{align*}
& \max _{-\kappa \leqslant u \leqslant \kappa} \mid \mu_{n}\left(k^{*}+u / \delta^{2}, k^{*}\right)-\mu_{n}\left(k^{*}, k^{*}\right)  \tag{3.4.33}\\
& \left.-\frac{1}{2}|u|\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right) / \delta^{2} \right\rvert\,=o(1)
\end{align*}
$$

uniformly over $k^{*} \in[n \gamma, n(1-\gamma)]$. Combining equations (3.4.30), (3.4.31), (3.4.32), and (3.4.33) with equation (3.4.28) and Donsker's theorem (cf. Lemma 3.4.3), we finally obtain for arbitrary $\kappa>0$ that

$$
S_{n}\left(\lambda+\cdot /\left(\delta^{2} n\right), \lambda\right)-S_{n}(\lambda, \lambda) \Rightarrow W^{*}
$$

in the Skorokhod topology on the space $D([-\kappa, \kappa], \mathbb{R})$ uniformly over $\lambda \in[\gamma, 1-\gamma]$. Since the random variable $\lambda^{*} \in[\gamma, 1-\gamma]$ with probability one (cf. Assumption 3.2), we
then conclude that

$$
\begin{equation*}
S_{n}\left(\lambda^{*}+\cdot /\left(\delta^{2} n\right), \lambda^{*}\right)-S_{n}\left(\lambda^{*}, \lambda^{*}\right) \Rightarrow W^{*} \tag{3.4.34}
\end{equation*}
$$

in the Skorokhod topology on the space $D([-\kappa, \kappa], \mathbb{R})$ and all $\kappa>0$. In the following, we denote $\widetilde{S}_{n}(\cdot, \lambda):=S_{n}\left(\lambda+\cdot /\left(\delta^{2} n\right), \lambda\right)-S_{n}(\lambda, \lambda)$ and $\widetilde{\mu}_{n}(\cdot, \lambda):=\mu_{n}\left(\lambda+\cdot /\left(\delta^{2} n\right), \lambda\right)-$ $\mu_{n}(\lambda, \lambda)$ for $\lambda \in[0,1]$. Next, we want to show that $\widetilde{S}_{n}\left(\cdot, \lambda_{n}^{*}\right) \Rightarrow W^{*}$ in the Skorokhod topology on the space $D([-\kappa, \kappa], \mathbb{R})$ for arbitrary $\kappa>0$. Without loss of generality, for all $n$ large enough, let $\lambda_{n}^{*} \in[\gamma, 1-\gamma]$ with probability one. Otherwise, we can argue analogously as in the proof of Theorem 3.4.7. Moreover, let us concentrate on the event $\left\{k_{n}^{*} \leqslant k^{*}\right\}$ and note that on the event $\left\{k_{n}^{*}>k^{*}\right\}$, we can argue completely analogously. Then, studying equations (3.4.30), (3.4.31), (3.4.32), (3.4.33), and the calculations in (3.4.11), for arbitrary $\kappa>0$, we conclude that

$$
\begin{aligned}
& \max _{u \in[-\kappa, 0]}\left|\widetilde{S}_{n}\left(u, \lambda_{n}^{*}\right)-\widetilde{S}_{n}\left(u, \lambda^{*}\right)\right| \\
& \leqslant o_{\mathbb{P}}(1)+\max _{u \in[-\kappa, 0]}\left|\widetilde{\mu}_{n}\left(u, \lambda_{n}^{*}\right)-\widetilde{\mu}_{n}\left(u, \lambda^{*}\right)\right| \\
& +\max _{u \in[-\kappa, 0]}\left|\left(H^{\prime}\left(\tau_{2}\right)-H^{\prime}\left(\tau_{1}\right)\right)^{T}\left\{\sum_{j=0}^{\left\lfloor-u / \delta^{2}\right\rfloor}\left(T\left(X_{1, k_{n}^{*}-j}\right)-\tau_{1}\right)-\sum_{j=0}^{\left\lfloor-u / \delta^{2}\right\rfloor}\left(T\left(X_{1, k^{*}-j}\right)-\tau_{1}\right)\right\}\right| \\
& \leqslant \max _{\lambda \in[\gamma, 1-\gamma] u \in[-\kappa, 0]} \max 2\left|\mu_{n}\left(\lambda+u /\left(\delta^{2} n\right), \lambda\right)-\mu_{n}(\lambda, \lambda)-\frac{1}{2}\right| u\left|\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right) / \delta^{2}\right| \\
& \quad+\max _{u \in[-\kappa, 0]} 2\left|\widetilde{\Sigma}^{1 / 2} \delta \sum_{j=k_{n}^{*}-\left\lfloor u / \delta^{2}\right\rfloor}^{k^{*}-\left\lfloor u / \delta^{2}\right\rfloor}\left(T\left(X_{1, j}\right)-\tau_{1}\right)\right|+o_{\mathbb{P}}(1) .
\end{aligned}
$$

We observe, that the first summand is of order $o(1)$ thanks to equation (3.4.33). Moreover, since $\left|k_{n}^{*}-k^{*}\right|=o_{\mathbb{P}}\left(\delta^{-2}\right)$ (cf. Assumption 3.2), the second summand is of order $o_{\mathbb{P}}(1)$ yielding that $\max _{u \in[-\kappa, 0]}\left|\widetilde{S}_{n}\left(u, \lambda_{n}^{*}\right)-\widetilde{S}_{n}\left(u, \lambda^{*}\right)\right|=o_{\mathbb{P}}(1)$ for arbitrary $\kappa>0$. Similarly, we can show that $\max _{u \in[0, k]}\left|\widetilde{S}_{n}\left(u, \lambda_{n}^{*}\right)-\widetilde{S}_{n}\left(u, \lambda^{*}\right)\right|=o_{\mathbb{P}}(1)$. An application of Slutzky's lemma together with (3.4.34) implies

$$
\widetilde{S}_{n}\left(\cdot, \lambda_{n}^{*}\right)=\widetilde{S}_{n}\left(\cdot, \lambda^{*}\right)+\left\{\widetilde{S}_{n}\left(\cdot, \lambda_{n}^{*}\right)-\widetilde{S}_{n}\left(\cdot, \lambda^{*}\right)\right\} \Rightarrow W^{*}
$$

in the Skorokhod topology on the space $D([-\kappa, \kappa], \mathbb{R})$ for arbitrary $\kappa>0$. Finally, since $\delta^{2}\left|\hat{k}_{n}-k_{n}^{*}\right|=\mathcal{O}_{\mathbb{P}}(1)$ (cf. Theorem 3.4.7), we conclude by an application of the continuous mapping theorem together with the continuity of the arg max function (cf. e.g. [56]) that

$$
\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right) \Rightarrow \underset{u \in(-\infty, \infty)}{\arg \max } W^{*}(u)
$$

### 3.5 Simulation study

In this section, we discuss the derived asymptotic properties for the test statistic $\mathcal{S}_{n}:=\max _{1 \leqslant k \leqslant n}\left\{2 S_{n}(k)\right\}$ and the estimator $\hat{k}_{n}$ of $k_{n}^{*}$ through several simulation studies.

In the following, we simulate time series data as follows: first, let us consider data sets $\mathcal{Y}^{i,(n)}:=\left\{Y_{k}^{i,(n)}: k=1, \cdots, n\right\}, i=1,2$, that are simulated from two independent normal distributions. For $\mu_{1}, \mu_{2} \in \mathbb{R}$ and $\sigma_{1}, \sigma_{2}>0$, let $N_{1}$ and $N_{2}$ be two independent normal distributions such that $N_{i} \sim \mathcal{N}\left(n^{-1 / 2} \mu_{i}, \sigma_{i}^{2}\right)$ and $Y_{k}^{i,(n)} \sim N_{i}$ for all $k=1, \cdots, n$ and $i=1,2$. Moreover, we simulate the location of the change point by a stopping time that depends on the data $\mathcal{Y}^{1,(n)}$. Therefore, let us introduce the discrete-time process $X^{1,(n)}(t):=\sum_{k=1}^{n} X_{k}^{1,(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}$ with $X_{k}^{1,(n)}:=n^{-1 / 2} \sum_{j=1}^{k} Y_{j}^{1,(n)}$.

Then, the change point fraction $\lambda_{n}^{*}:=k_{n}^{*} / n$ is generated, for $\kappa \in \mathbb{R}$ fixed, by

$$
\begin{equation*}
\lambda_{n}^{*}=\inf \left\{t \geqslant \gamma: X^{1,(n)}(t)<\kappa\right\} \wedge(1-\gamma) \tag{3.5.1}
\end{equation*}
$$

Remark 3.5.1. We present the empirical results where the location of a change point is generated from the stopping time above. Moreover, we have also run simulations when $\lambda_{n}^{*}$ is sampled from a uniform distribution on $[\gamma, 1-\gamma]$ or a truncated normal distribution with mean $\frac{1}{2}$ and volatility $\frac{1}{6}-\frac{\gamma}{3}$. In both cases, the empirical observations are not significantly different from those we discuss below.

Next, we analyze the data set $\mathcal{Y}^{(n)}:=\left\{Y_{k}^{(n)}: k=1, \cdots, n\right\}$ which may contain a change point and which is given by

$$
Y_{k}^{(n)}=Y_{k}^{1,(n)} \mathbb{1}_{\left\{k \leqslant k_{n}^{*}\right\}}+Y_{k}^{2,(n)} \mathbb{1}_{\left\{k>k_{n}^{*}\right\}}
$$

for $k=1, \cdots, n$. Then, for $n$ large enough, the discrete-time process $X^{(n)}(t)=$ $\sum_{k=1}^{n} X_{k}^{(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}$ with $X_{k}^{(n)}:=n^{-1 / 2} \sum_{j=1}^{k} Y_{j}^{(n)}$ can be approximated by the continuous-time diffusion process

$$
\begin{equation*}
X(t)=\int_{0}^{t}\left(\mu_{1} \mathbb{1}_{\left\{t \leqslant \lambda^{*}\right\}}+\mu_{2} \mathbb{1}_{\left\{t>\lambda^{*}\right\}}\right) d t+\int_{0}^{t}\left(\sigma_{1} \mathbb{1}_{\left\{t \leqslant \lambda^{*}\right\}}+\sigma_{2} \mathbb{1}_{\left\{t>\lambda^{*}\right\}}\right) d W(t), \tag{3.5.2}
\end{equation*}
$$

where $W$ is a standard Brownian motion and $\lambda^{*} \in[\gamma, 1-\gamma]$ is the limit of the true change point fraction $\lambda_{n}^{*}$ (cf. Assumption 3.2). In this setting, the data points $\mathcal{Y}^{(n)}$ might be interpreted as the scaled increments of $X$ recorded at discrete, equidistant time steps $t_{k}^{(n)}:=k / n, k=1, \cdots, n$.

In the following, we choose $\gamma=0.1, \kappa=-1$ and simulate $n=10,000$ time steps. All depicted empirical distributions are generated from $m=10,000$ Monte Carlo runs.

Parametric change point detection in the mean and volatility: Let us choose $\mu_{1}=\mu_{2}=-2, \sigma_{1}=1$, and $\sigma_{2}=1.1$, i.e., we consider a jump in the volatility of $N_{1}$ versus $N_{2}$ of size 0.1. Then we have $n \Delta^{2}=n\left(\sigma_{1}-\sigma_{2}\right)^{2}=100$. So, we might be in the studied setting of Theorem 3.4.7 and hope to detect the quite small jump in the
volatility. In Figure 3.3, we depict one realization of $X^{(n)}$ under the alternative and the empirical values of the test statistic $\mathcal{S}_{n}^{1 / 2}$ under the null and under the alternative hypothesis. While the jump in volatility is not visible to the naked eye, the empirical values of the test statistic show that our test can very well separate the null hypothesis "no change point" from the alternative hypothesis "there exists one change point".

Change point model with parameters $\sigma_{1}=1$ and $\sigma_{2}=1.1$


Figure 3.3: Left: One path of $X^{(n)}$ under the alternative with a change point after $n=2620$ time steps. Right: Empirical values of $\mathcal{S}_{n}^{1 / 2}$ under the null (orange) and the alternative (turquoise).

In a second simulation, we choose $\mu_{1}=-2, \mu_{2}=-12, \sigma_{1}=1$, and $\sigma_{2}=1$, i.e., we consider a jump in the mean of $N_{1}$ versus $N_{2}$ of size $n^{-1 / 2} \cdot(-10)=-0.1$. Again, we have $n \Delta^{2}=100$, so that we might hope to detect the jump in the mean. In Figure 3.4 we depict one realization of $X^{(n)}$ under the alternative and the empirical values of the test statistic $\mathcal{S}_{n}^{1 / 2}$ under the null and under the alternative hypothesis. Even though we can see the jump in the expected value after $n \approx 3000$ time steps in the realization of $X^{(n)}$ very clearly, the empirical distributions of the test statistic $\mathcal{S}_{n}^{1 / 2}$ suggest that the null hypothesis "no change point" is harder to distinguish from the alternative hypothesis "there exists one change point" compared to our first simulation in Figure 3.3. Moreover, if we interpret the observations $\mathcal{Y}^{(n)}$ as the discretely observed scaled increments of a diffusion process $X$, we see from a comparison of these two simulation studies that the detection of a jump in its drift component of $X$ is harder than in its volatility component. In more detail, in order to guarantee that the condition in 3.4.5) holds true and hence we are able to distinguish between the null and alternative, a change in the drift component has to converge to infinity, while the change in the volatility component might even go to zero as $n \rightarrow \infty$. Note that these observations are also consistent with the theoretical results in [3,51.

For both simulations, we also calculate the empirical distribution of $\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$. In both cases, we obtain that the empirical distribution replicates the theoretical result from Theorem 3.4.9. In Figure 3.5 we depict the empirical distribution of $\arg \max _{u \in(-\infty, \infty)} \hat{W}(u)$ versus the empirical distribution of $\sigma_{A}^{2} \delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$ for a change in the mean.

Change point model with parameters $\mu_{1}=-2$ and $\mu_{2}=-12$



Figure 3.4: Left: One path of $X^{(n)}$ under the alternative with a change point at $n=3024$ time steps. Right: Empirical values of $\mathcal{S}_{n}^{1 / 2}$ under the null (orange) and under the alternative (turquoise).


Figure 3.5: Left: Empirical distributions of $\arg \max _{u \in(-\infty, \infty)} \hat{W}(u)$. Right: Empirical distribution of the deviation $\sigma_{A}^{2} \delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$.

Parametric change point detection for weakly dependent observations: Although we developed our theory for independent observations, weak dependencies between subsequent observations do not ruin our empirical results. To see that, for $a \in(-1,1)$, let

$$
\widetilde{Y}_{1}^{(n)}:=Y_{1}^{(n)}, \quad \widetilde{Y}_{k}^{(n)}:=a Y_{k-1}^{(n)}+\sqrt{1-a^{2}} Y_{k}^{(n)} \quad \text { for } k \geqslant 2,
$$

and $\widetilde{X}^{(n)}(t):=\sum_{k=1}^{n} \widetilde{X}_{k}^{(n)} \mathbb{1}_{\{n t \in[k, k+1)\}}$ with $\widetilde{X}_{k}^{(n)}:=n^{-1 / 2} \sum_{j=1}^{k} \widetilde{Y}_{j}^{(n)}$. In the following simulation, we choose again $\mu_{1}=\mu_{2}=-2, \sigma_{1}=1$, and $\sigma_{2}=1.1$. Moreover, we choose $a=1 / 2$. In Figure 3.6, we depict one realization of $\widetilde{X}^{(n)}$ and the empirical values of the test statistic $\mathcal{S}_{n}^{1 / 2}$ under the null and alternative hypothesis. We observe that even for weakly dependent observations, the test statistic $\mathcal{S}_{n}^{1 / 2}$ is still able to distinguish between the null and alternative hypothesis. Moreover, also the empirical distribution of $\delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$ replicates the theoretical result from Theorem 3.4.9 (cf. Figure 3.7).

Change point model for weakly dependent observations



Figure 3.6: Left: One path of $\tilde{X}^{(n)}$ under the alternative with a change point at $n=2522$ time steps. Right: Empirical values of $\mathcal{S}_{n}^{1 / 2}$ under the null (orange) and under the alternative (turquoise).


Figure 3.7: Left: Empirical distribution of $\arg \max _{u \in(-\infty, \infty)} \hat{W}(u)$. Right: Empirical distribution of the deviation $\sigma_{A}^{2} \delta^{2}\left(\hat{k}_{n}-k_{n}^{*}\right)$.

Our empirical results therefore suggest that at least for weakly dependent observations, we are probably able to establish the stated asymptotic properties for the test statistic $\mathcal{S}_{n}$ and the estimators $\hat{k}_{n}$ and $\hat{\lambda}_{n}$.

Non-parametric change point detection in the volatility process: For many practical applications, an approximation as in 3.5 .2 does not describe the underlying structure of the observations well. In contrast, one might be interested in whether or not there is a jump in the volatility process $\sigma: \Omega \times[0, T] \rightarrow \mathbb{R}_{+}$of an Itô-semimartingale. The authors in [7] developed a statistical change point theory to detect, among others, a "local jump" in the volatility process such that $\left|\sigma^{2}(\lambda)-\lim _{s \uparrow \lambda} \sigma^{2}(s)\right|>0$ for some $\lambda \in(0,1)$. Let us consider a volatility process of the form

$$
\sigma(t)=\left(\int_{0}^{t} c \cdot \rho d W(s)+\int_{0}^{t} \sqrt{1-\rho^{2}} \cdot c d W^{\perp}(s)+1\right) \cdot v(t)
$$

which fluctuates around a deterministic seasonality function

$$
v(t)=1-0.2 \sin \left(\frac{3}{4} \pi t\right), \quad t \in[0,1]
$$

with $c=0.1$ and $\rho=0.5$, where $W^{\perp}$ is a standard Brownian motion independent of $W$. Note that the authors of [7] studied the same volatility process but for a deterministic location of the change point. Again, we simulate $\lambda_{n}^{*}$ by (3.5.1) and add one jump of size 0.3 at time $\lambda_{n}^{*}$ to $\sigma$. Since the volatility process is time-dependent, we apply the test statistic $V_{n, u_{n}}^{*}$ introduced in 7] instead of $\mathcal{S}_{n}$. Let $\Delta X_{k}^{(n)}:=X_{k}^{(n)}-X_{k-1}^{(n)}, k \geqslant 1$, be the increments of an Itô-semimartingale $X$ with the volatility process above and constant drift equal to -2 , recorded at discrete time steps $t_{k}^{(n)}, k \geqslant 1$. Then, a reasonable test statistic is

$$
V_{n, u_{n}}^{*}:=\max _{i=k_{n}, \cdots, n-k_{n}}\left|\frac{\frac{n}{k_{n}} \sum_{j=i-k_{n}+1}^{i}\left(\Delta X_{j}^{(n)}\right)^{2} \mathbb{1}_{\left\{\left|\Delta X_{j}^{(n)}\right| \leqslant u_{n}\right\}}}{\frac{n}{k_{n}} \sum_{j=i+1}^{i+k_{n}}\left(\Delta X_{j}^{(n)}\right)^{2} \mathbb{1}_{\left\{\left|\Delta X_{j}^{(n)}\right| \leqslant u_{n}\right\}}}-1\right|
$$

where $k_{n} \rightarrow \infty$. The core idea of the test statistic is to utilize a local two-sample $t$-test over $k_{n}$ asymptotically small blocks and take all overlapping blocks of $k_{n}$ increments into account. Moreover, we truncate the increments of $X$ by $u_{n}$ to exclude large squared increments which are ascribed to jumps. In [7] the authors suggest to take $u_{n}=\sqrt{2 \log (n)} n^{-1 / 2}$ and $k_{n}=C(\log (n))^{1 / 2} n^{1 / 2}$, for some $C>1$.

In Figure 3.8, we depict one realization of $\sigma$ under the null and under the alternative hypothesis. Moreover, in Figure 3.9 , we depict one realization of $X$ under the alternative and the empirical values of the test statistic

$$
\mathcal{V}_{n}:=\sqrt{\frac{\log \left(m_{n}\right) k_{n}}{2}} V_{n, u_{n}}^{*}-2 \log \left(m_{n}\right)-\frac{1}{2} \log \log \left(m_{n}\right)-\log (3)
$$

under the null and under the alternative hypothesis. Here, $m_{n}:=\left\lfloor n / k_{n}\right\rfloor$. Accoring to Proposition 3.5 in $[7]$, the test statistic $\mathcal{V}_{n}$ converges in distribution under the null hypothesis to a Gumbel distribution. We observe that the test statistic in [7] can fairly good distinguish between the null and alternative even if the location of the change point has been sampled according to 3.5.1.


Figure 3.8: One realization of the volatility process $\sigma$ under the null (left) and under the alternative (right).

Non-parametric change point model with a jump in the volatility process of size 0.3



Figure 3.9: Left: One path of $X$ under the alternative with a change point at $n=1830$ time steps. Right: Empirical values of $\mathcal{V}_{n}$ under the null (orange) and under the alternative (turquoise).

Conclusion: The starting point of our work was to generalize the theory in Csörgő and Horváth [23 to randomly occurring changes in the model parameters, where, in particular, the location of the change point is allowed to depend on the data itself. In our simulation study, we generated the location of the change point from the stopping time in (3.5.1). This stopping time is a rather simple way to choose the location of the change point depending on the data. From a financial point of view it is still quite interesting: the process X in 3.5 .2 might be an approximation for log prices of a financial asset containing a change point. Then, the stopping time in (3.5.1) causes the change in the model parameters of the log price if the price drops below some critical value $\kappa \in \mathbb{R}$. It also shows that our theory is flexible enough to be applied to even more complex dependence relationships between the location of the change point and the observed data.
Finally, our simulations for the case of weakly dependent observations as well as the non-parametric case suggest that change point theory in these settings still works even if the location of the change point depends on the data.

## Notation

## General

| $\\|\cdot\\|,\|\cdot\|$ | euclidean norm in $\mathbb{R}^{\text {d }} /$ absolute value |
| :---: | :---: |
| $\\|\cdot\\| \infty,\\|\cdot\\|_{[a, b]}$ | sup norm / sup norm on interval [a,b] for some $a<b$ |
| $\\|\cdot\\|_{L^{2}}$ | $L^{2}(\overline{\mathbb{R}})$ norm |
| $\vee, \wedge$ | $x \vee y:=\max \{x, y\}$ and $x \wedge y:=\min \{x, y\}$ for $x, y \in \mathbb{R}$ |
| $\mathbb{P}[A, B]$ | $:=\mathbb{P}[A \cap B]$ for $A, B \in \mathcal{F}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ |
| $\mathbb{R}_{+}, \mathbb{R}_{-}$ | positive/negative real half-line |
| $C\left([0, T], \mathbb{R}^{k}\right)$ | function space of continuous functions $f:[0, T] \rightarrow \mathbb{R}^{k}$ |
| $\underline{D}\left([0, T], \mathbb{R}^{k}\right)$ | function space of càdiàg functions $f:[0, T] \rightarrow \mathbb{R}^{k}$ |

## Chapter 1

| General |  |
| :---: | :---: |
| $\lambda$ | Lebesgue measure |
| $\varepsilon_{x}$ | Dirac measure at $x \in \mathbb{R}$ |
| $\delta X_{k}$ | $:=X_{k}-X_{k-1}, k \in \mathbb{N}$, is the $k$-th increment of $X:=\left\{X_{k}: k \in \mathbb{N}_{0}\right\}$ |
| $\Delta X(t)$ | $:=X(t)-X(t-)$ is the jump of $X:=(X)(t)) t \in[0, T]$ at $t>0$ |
| $f^{+}$ | positive part of a real-valued function $f: f^{+}(x):=\max \{f(x), 0\}$ |
| $f$ | negative part of a real-valued function $f: f-(x):=-\min \{f(x), 0\}$ |
| $I$ | index denoting the bid/ask side of the LOB if $I=b$ (resp. $I=a$ ) |
| State space |  |
| $\left(E,\\|\cdot\\|_{E}\right)$ | Hilbert space and state space of LOB-sequence, where $E:=\mathbb{R} \times$ $L^{2}(\mathbb{R}) \times \mathbb{R} \times L^{2}(\mathbb{R}) \times[0, T]$ and $\\|(b, v, a, w, t)\\|_{E}^{2}:=\|b\|^{2}+\\|v\\|_{L^{2}}^{2}+$ $\|a\|^{2}+\\|w\\|_{L^{2}}^{2}+\|t\|^{2}$ |
| Scaling parameters |  |
| $\Delta x^{(n)}$ | tick size |
| $\Delta v^{(n)}$ | average size of a passive limit order placement |
| $\Delta t^{(n)}$ | time scaling parameter |
| $\delta_{n}$ | null sequence separating small/large price changes (cf. Ass. 1.4 ) |
| $\eta_{n}$ | null sequence guaranteeing that the diffusion coefficients do not vanish in the $n$-th model (cf. Ass 1.4) |
| Microscopic model dynamics |  |
| $S^{(n)}$ | LOB-dynamics with values in $E$, where $S^{(n)}=\left(B^{(n)}, v_{b}^{(n)}, A^{(n)}\right.$, $v_{a}^{(n)}, \tau^{(n)}$ ) (cf. Eq. 1.2.8p) |

$\left.\begin{array}{ll}B^{(n)}, A^{(n)} \\ v_{b}^{(n)}, v_{a}^{(n)}\end{array} \quad \begin{array}{l}\text { bid/ask price dynamics with values in } \mathbb{R} \text { (cf. Eq. (1.2.5) } \\ \text { dynamics of the buy/sell side volume density function relative to } \\ \left.\text { the best bid/ask price with values in } L^{2}(\mathbb{R}) \text { (cf. Eq. .1.2.7) }\right)\end{array}\right\}$

| $\theta_{I}^{(n)}, \theta_{I}$ | $\theta_{I}^{(n)}, \theta_{I}: E \times[-M, M] \rightarrow \mathbb{R}$ are the coefficient functions corresponding to the large price dynamics for $I=b, a$ |
| :---: | :---: |
| $f_{I}^{(n)}, f_{I}$ | $f_{I}^{(n)}, f_{I}: E \rightarrow L^{2}(\mathbb{R})$ are the coefficient functions corresponding to the volume dynamics for $I=b, a$ |
| Integrands |  |
| $K_{I}^{(n)}, K_{I}$ | $K_{I}^{(n)}, K_{I}: E \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}_{+}$are kernels representing the conditional distributions of the large price changes for $I=b, a$ (cf. Eq. 1.2.9) and Ass. 1.6) |
| $\ddot{Q}_{I}$ | finite measure on $\mathcal{B}(\mathbb{R})$ with compact support in $[-M, M]$ and $Q_{I}(\{0\})=0$ for $I=b, a$ (cf. Ass. 1.6) |
| $Z_{b}, Z_{a}$ | independent standard Brownian motions |
| $\mu_{b}^{Q}, \mu_{a}^{Q}$ | independent homogeneous Poisson random measures with intensities $\lambda \times Q_{b}$ and $\lambda \times Q_{a}$, independent of $Z_{b}, Z_{a}$ |
| $\delta Z_{I, k}^{(n)}$ | (nearly) normalized small price increments for $I=b, a$ (cf. step 2 in Section 1.5) |
| $\ddot{Z}_{I}^{(n)}$ | piecewise constant interpolation of $\left(Z_{I, k}^{(n)}\right)_{k \geqslant 1}$, where $Z_{I, k}^{(n)}:=$ $\sum_{j=1}^{k} \delta Z_{I, j}^{(n)}$ for $I=b, a$ (cf. step 2 in Section 1.5) |
| $\mu^{\eta},(\dot{n})$ | joint jump measure of the large price dynamics $B^{\eta}, \ell,(n)$ and $A^{\eta, \ell,(n)}$ with compensator $\nu^{\eta,(n)}$ (cf. step 2 in Section 1.5) |
| $\mu_{I}^{\eta,(n)}$ | individual jump measure of the large price dynamics with compensator $\nu_{I}^{\eta,(n)}$ for $I=b, a$ (cf. step 2 in Section 1.5 ) |
| $\cdots{ }^{(j n}$ | joint jump measure with respect to some transformation of the large price jumps with compensator $\nu^{J^{(n)}}$ (cf. step 2 in Section 1.5) |
| $\mu_{I}^{j(n)}$ | individual jump measure with respect to some transformation of the large price jumps with compensator $\nu_{I}^{J^{(n)}}$ for $I=b, a$ (cf. step 2 in Section 1.5 |
| Technicalities |  |
| $I^{(n)}(x)$ | $I^{(n)}(x):=\left[j \Delta x^{(n)},(j+1) \Delta x^{(n)}\right)$ for $j \Delta x^{(n)} \leqslant x<(j+1) \Delta x^{(n)}$ |
| $Z_{M}^{(n)}$ | $Z_{M}^{(n)}:=\left\{j \in \mathbb{Z}:-M \leqslant j \Delta x^{(n)} \leqslant M\right\}$ |
| $\psi$ | $\psi: E \rightarrow E$ is a random shift operator shifting the volume density functions by their corresponding current best bid/ask prices (cf. step 4 in Section 1.5 |

## Chapter 2

|  | General |
| :---: | :---: |
| $d_{J_{1}}$ | distance inducing the Skorokhod topology on $D\left([0, T], \mathbb{R}^{k}\right), k \geqslant 1$ |
| $X \simeq Y$ | the finite dimensional distribution of processes $X, Y$ coincide |
| $F, G$ | two countries |
| $(i, I)$ | indices denoting the order types, where $i=b, a$ and $I=F, G$ |


| $\mathcal{I}^{I m}, \mathcal{I}^{\text {Ex }}$ | $\mathcal{I}^{I m}:=\{(a, F),(b, G)\}$ and $\mathcal{I}^{E x}:=\{(b, F),(a, G)\}$ |
| :---: | :---: |
| $\dddot{\ell}_{t}(x)$ | $:=\sup _{s \leq t}(-x(s))^{+}$, reflection at zero of $x \in D([0, T], \mathbb{R})$ |
| $\left(L_{t}(X)\right)_{t \geqslant 0}$ | local time of a continuous semimartingale $X$ at zero |
| $C_{0}^{\prime}\left([0, T], \mathbb{R}^{k}\right)$ | function space containing all continuous function $\omega$ avoiding the origin and whose components cross the axes each time they touch them (cf. Eq. (2.7.4)) |
| Projection maps |  |
| $\pi_{j}^{(k)}, \pi_{j}$ | $\begin{aligned} & \pi_{j}^{(k)} \omega:=\omega_{j} \in D([0, T], \mathbb{R}) \text { denotes the } j \text {-th projection map of } \\ & \omega=\left(\omega_{1}, \cdots, \omega_{k}\right) \in D\left([0, T], \mathbb{R}^{k}\right), 1 \leqslant j \leqslant k, k \geqslant 1 \text {, and } \pi_{j}:=\pi_{j}^{(4)} \end{aligned}$ |
| $\pi_{i, j}^{(k)}, \pi_{i, j}$ | $\begin{aligned} & \pi_{i, j}^{(k)} \omega:=\left(\omega_{i}, \omega_{j}\right) \in D\left([0, T], \mathbb{R}^{2}\right) \text { denotes the }(i, j) \text {-th projection of } \\ & \omega=\left(\omega_{1}, \cdots, \omega_{k}\right) \in D\left([0, T], \mathbb{R}^{k}\right) 1 \leqslant i, j \leqslant k, \text { and } \pi_{i, j}:=\pi_{i, j}^{(4)} \end{aligned}$ |
| $\pi_{F}, \pi_{G}$ | $\pi_{F}:=\pi_{1,2}$ and $\pi_{G}:=\pi_{3,4}$ |
| $\pi_{b}, \pi_{a}$ | $\pi_{b}:=\pi_{1,3}$ and $\pi_{a}:=\pi_{2,4}$ |
| $\grave{h}$ | $\overparen{h}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}^{2}\right), h: \omega \mapsto\left(\pi_{1} \omega+\pi_{3} \omega, \pi_{2} \omega+\pi_{4} \omega\right)$ |
| $h_{1}$ | $h_{1}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D([0, T], \mathbb{R}), h_{1}: \omega \mapsto \pi_{1}^{(2)} \omega+\pi_{2}^{(2)} \omega$ |
| State space |  |
| E | state space of the LOB-dynamics $E:=\mathbb{R}^{2} \times \mathbb{R}_{+}^{4} \times \mathbb{R}$ |
| Model parameters |  |
| $\delta$ | tick size |
| $\Delta t^{(n)}$ | time between two consecutive order arrivals |
| $\Delta v^{(n)}$ | average size of a limit order placement |
| $\kappa_{+}, \kappa_{-}$ | total transmission capacities in direction $G$ to $F$ and vice versa |
| $\mu^{i, 1,} ;(n)$ | scaled mean of order sizes with limit $\mu^{i, 1}$ (cf. Ass. 2.2 ) |
| $\sigma^{i, 1 ;}(\underline{n})$ | scaled variance of order sizes with limit $\sigma^{i, I}$ (cf. Ass 2.2) |
| $\dddot{\sigma}^{(i, i) j,(\bar{j}, \bar{j}) ;(\bar{n})}$ | scaled covariance of the order sizes with limit $\sigma^{(i, I),(j, J)}$, where $(i, I) \neq(j, J)$ (cf. Ass. 2.2 ) |
| $f_{n}^{+}, f_{n}^{-}$ | distributions to determine the size of order queues after price increases/decreases with limits $f^{+}$and $f^{-}$(cf. Ass. 2.4) |
| Microscopic model dynamics |  |
| $X^{(n)}$ | net order flow process (cf. Eq. (2.3.1)) |
| $S^{(n)}$ | market dynamics with values in $E$, where $S^{(n)}=\left(B^{(n)}, Q^{(n)}, C^{(n)}\right)$ (cf. Section 2.2 and Def. 2.2.7) |
| $\ddot{B}^{(n)}$ | bid price dynamics of countries $F$ and $G$ |
| $Q^{(n)}$ | dynamics of limit orders at the best bid/ask queues of $F$ and $G$ |
| $C^{(n)}$ | dynamics of the net number of cross-border trades between $F, G$ |
| $M^{(n)}$ | dynamics of the number of cross-border trades (cf. Eq. (2.2.7) |
| $\widetilde{S}^{(n)}$ | active market dynamics with $\widetilde{S}^{(n)}=\left(\widetilde{B}^{(n)}, \widetilde{Q}^{(n)}, \widetilde{C}^{(n)}\right)$ |
| $\widetilde{S}^{(n)}$ | inactive market dynamics with $\widetilde{S}^{(n)}=\left(\widetilde{\widetilde{B}}^{(n)}, \widetilde{\widetilde{Q}}^{(n)}, \widetilde{C}^{(n)}\right)$ |
| $\tilde{\widetilde{Z}}_{k}^{I \prime(n)}$ | dynamics to determine the starting values of $S^{(n)}$ of an inactive regime (cf. Eq. 2.2 .12 ) |


| $l^{(n)}$ | number of price changes in $S^{(n)}$ |
| :---: | :---: |
| Continuous-time limit approximations |  |
| $X$ | limit of $X^{(n)}$ being a four-dimensional linear Brownian motion |
| $S$ | limit dynamics of $S$, similarly $\dddot{\widetilde{S}}, \tilde{\widetilde{S}}, B, Q, C, \widetilde{B}, \widetilde{Q}, M, \tilde{\widetilde{C}}, \tilde{B}$, and $\widetilde{Q}$ are the limit dynamics of their corr. microscopic dynamics |
| $l$ | number of price changes in $S$ |
| Sequences of random variables |  |
| $\left(\phi_{k}^{(n)}\right)_{k \geqslant 1}$ | $\phi_{k}^{(n)}=b$ (resp. $\left.\phi_{k}^{(n)}=a\right)$ if the $k$-th incoming order event affects the bid side (resp. the ask side) |
| $\left(\psi_{k}^{(n)}\right)_{k \geqslant 1}$ | $\psi_{k}^{(n)}=F\left(\right.$ resp. $\left.\psi_{k}^{(n)}=G\right)$ if the $k$-th incoming order has origin $F$ (resp. origin $G$ ) |
| $\left(V_{k}^{(n)}\right)_{k \geqslant 1}$ | sizes of incoming order events |
| $\left(V_{k}^{i, I \prime,},(\underline{n})\right)_{k \geqslant 1}$ | sizes of incoming order events of type $(i, I) \in\{b, a\} \times\{F, G\}$ |
| $\left.\left(\epsilon_{k}^{+}\right)-(\bar{n})\right)_{k \geqslant 1}$ | iid random variables with $\epsilon_{1}^{+\prime-(n)} \sim f_{n}^{+} /$(cf. Ass. 2.4 ) |
| $\left.\left(R_{k}^{+}\right)-(n)\right)_{k \geqslant 1}$ | order sizes after price increases/decreases in $S^{(n)}$ (cf. Ass. 2.4) |
| $\left.\left(\widetilde{R}_{k}^{+}\right)-,(n)\right)_{k \geqslant 1}$ | order sizes after price increases/decreases in $\widetilde{\widetilde{S}}^{(n)}$ |
| $\left.\left(\widetilde{\widetilde{R}}_{k}^{++}\right)-(n)\right)_{k \geqslant 1}$ | order sizes after price increases/decreases in $\widetilde{\widetilde{S}}^{(n)}$ |
| $\left(U_{k}^{(n)}\right)_{k \geqslant 1}$ | iid Bernoulli random variables with $U_{k}^{(n)} \in\{-1,1\}$ |
| Stopping times |  |
| $\left(\tau_{k}^{(n)}, \tau_{k}\right)_{k \geqslant 1}$ | random times of price changes in $S^{(n)}$ and $S$ |
| $\left(\widetilde{\tau}_{k}^{(n)}, \widetilde{\tau}_{k}^{*}\right)_{k \geqslant 1}$ | random times of price changes in $\widetilde{S}^{(n)}$ and $\widetilde{S}$ |
| $\left(\tilde{\widetilde{\tau}}_{k}(\underline{n}), \tilde{\widetilde{\tau}}_{k}^{*}\right)_{k \geqslant 1}$ | random times of price changes in $\widetilde{\widetilde{S}}^{(n)}$ and $\widetilde{\widetilde{S}}$ |
| $\left(\rho_{k}^{(n)}, \rho_{k}\right)_{k \geqslant 0}$ | starts of active regimes in $S^{(n)}$ and $S$ (cf. Def. 2.2.7, Section 2.5) |
| $\left(\sigma_{k}^{(n)}, \sigma_{k}\right)_{k \geqslant 1}$ | starts of inactive regimes in $S^{(n)}$ and $S$ (cf. Def. 2.2.7. Section 2.5) |
| First hitting time maps |  |
| $\left(\hat{\tau}_{k}\right)_{k \geqslant 1}$ | first hitting time maps (cf. Def. 2.3.2) |
| $\left(\ddot{\widetilde{\tau}}_{k}\right)_{k \geqslant 1}$ | first hitting time maps (cf. Def. 2.3 .7 ) |
| $\ddot{\tau}, \widetilde{\tau}_{b / a}$ | first hitting time maps (cf. Eq. 2.3.7) and (2.3.6) |
| Important order book functions |  |
| $\Phi^{(n)}, \Phi$ | $\Phi^{(n)}: \mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4} \rightarrow\left(\Delta v^{(n)} \mathbb{N}\right)^{4}$ and $\Phi \in C^{2}\left(\mathbb{R}_{+}^{4} \times \mathbb{R}_{+}^{4},(0, \infty)^{4}\right)$ are used to describe the sequences of queue sizes after price changes in the $n$-th model and the scaling limit (cf. Ass. 2.4) |
| $g$ | $g: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is used to describe the evolution of the one-sided queue size dynamics in the active regimes between price changes (cf. Def. 2.3.2) |
| $\overline{\bar{g}}$ | $\bar{g}: D\left([0, T], \mathbb{R}^{2}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{2}\right)$ is used to describe the evolution of the one-sided number of cross-border trades in the active regimes between price changes (cf. Def. 2.3.11) |


| $G$ | $G: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is used to describe the evolution of the queue size dynamics in the active regimes between price changes (cf. Eq. 2.3.4) |
| :---: | :---: |
| $\stackrel{\rightharpoonup}{G}$ | $\bar{G}: D\left([0, T], \mathbb{R}^{4}\right) \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is used to describe the evolution of the number of cross-border trades in the active regimes between price changes (cf. Eq. 2.3 .9 ) |
| $\widetilde{\Psi}^{Q},\left(\widetilde{\Psi}_{k}^{Q}\right)_{k \geqslant 0}$ | $\widetilde{\Psi}^{Q}, \widetilde{\Psi}_{k}^{Q}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right), k \geqslant 0$, is used to describe the evolution of the queue size dynamics in the active regimes over time (cf. Def. 2.3.7) |
| $\left(\widetilde{\widetilde{\varphi}}_{k}\right)_{k \geqslant 1}$ | $\widetilde{\varphi}_{k}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{4}\right), k \geqslant 1$, are random shifts defined by $\widetilde{\varphi}_{k}(\omega, R, \widetilde{R}):=\widetilde{\Psi}^{Q}(\omega, R, \widetilde{R})\left(\widetilde{\tau}_{k}\right)+\omega(\cdot+$ $\left.\widetilde{\tau}_{k}\right)-\omega\left(\widetilde{\tau}_{k}\right)$ (cf. Def. 2.3.7) |
| $\Psi^{M}$ | $\widetilde{\Psi}^{M}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}_{+}^{4}\right)$ is used to describe the evolution of the number of cross-border trades in the active regimes over time (cf. Def. 2.3 .14 ) |
| $N_{a / b}$ | $N_{a / b}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{N}_{0}\right)$ is used to describe the number of price increases/decreases in the active regimes over time (cf. Def. 2.3.17) |
| $N_{T}$ | $N_{T}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{N}_{0}\right)$ is used to describe the number of price changes in the active regimes over time (cf. Def. 2.7 .12 ) |
| $\widetilde{\Psi}^{B}$ | $\widetilde{\Psi}^{B}: D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D\left([0, T], \mathbb{R}^{2}\right)$ is used to describe the evolution of the bid price dynamics in the active regimes over time (cf. Def. 2.3.17) |
| $\Psi$ | $\tilde{\Psi}: E \times D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D([0, T], E)$ is used to describe the evolution of the active dynamics $\widetilde{S}^{(n)}$ over time |
| $\widetilde{\widetilde{T}}$ | $\tilde{\Psi}: E \times D\left([0, T], \mathbb{R}^{4}\right) \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \times\left(\mathbb{R}_{+}^{4}\right)^{\mathbb{N}} \rightarrow D([0, T], E)$ is used to describe the evolution of the inactive dynamics $\tilde{S}^{(n)}$ over time |

## Chapter 3

| Landau symbols |  |  |  |  |
| :---: | :--- | :---: | :---: | :---: |
| $o_{\mathbb{P}}\left(a_{n}\right), \mathcal{O}_{\mathbb{P}}\left(a_{n}\right)$ | for a family of random variables $\left(X_{n}\right)_{n \in \mathbb{N}}$ and a positive deter- <br> ministic sequence $\left(a_{n}\right)_{n \in \mathbb{N}}, X_{n}=o_{\mathbb{P}}\left(a_{n}\right)$ if $X_{n} / a_{n}$ converges in <br> probability to zero as $n \rightarrow \infty$. We write $X_{n}=\mathcal{O}_{\mathbb{P}}\left(a_{n}\right)$ if the set <br> $\left(X_{n} / a_{n}\right)_{n \in \mathbb{N}}$ is stochastically bounded. |  |  |  |
| $o\left(a_{n}\right), \mathcal{O}\left(a_{n}\right)$ | for two deterministic sequences $\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}},\left(a_{n}\right)_{n \in \mathbb{N}}$ positive, <br> we write $b_{n}=o\left(a_{n}\right)$ if $b_{n} / a_{n}$ converges to zero as $n \rightarrow \infty$ and <br> $b_{n}=\mathcal{O}\left(a_{n}\right)$ if there exist $M>0$ and $N \in \mathbb{N}$ such that $b_{n} / a_{n} \leqslant M$ <br> for all $n>N$. |  |  |  |

## The model

| $\left(X_{i}\right)_{i=1, \cdots, n}$ | independent observations in $\mathbb{R}^{m}$ |
| :---: | :---: |
| $f(x, \theta)$ | $:=\exp \left(\theta^{T} T(x)+S(x)-A(\theta)\right) \mathbb{1}_{\{x \in C\}}$ being the distribution of the data belonging to an exponential family, where $x \in \mathbb{R}^{m}$ and $\theta \in \mathbb{R}^{d}$ |
| inv $A^{\prime}(\theta)$ | unique inverse of $\vartheta \mapsto A^{\prime}(\vartheta)$ at $\theta$ (cf. Ass. 3.1 ii)) |
| H | $:=\left(\operatorname{inv} A^{\prime}(x)\right)^{T} x-A\left(\operatorname{inv} A^{\prime}(x)\right)$ |
| ${ }_{k}^{*}$ | true location of the change point under the alternative $H_{1}$ "there exists one change point" |
| $\lambda_{n}^{*}$ | $:=k_{n}^{*} / n$, true location of a change point relative to the sample size |
| $\lambda^{*}$ | limit of $\lambda_{n}^{*}$ with almost surely values in $[\gamma, 1-\gamma] \subset(0,1)$ |
| $\theta_{0}^{(1)}, \theta_{0}^{(2)}$ | true parameters before and after the change (under $H_{1}$ ) |
| $\theta_{A}$ | limit of $\theta_{0}^{(1)}$ and $\theta_{0}^{(2)}$ (cf. Ass. 3.2) |
| $\Delta^{2}$ | $:=\left\\|\theta_{0}^{(1)}-\theta_{0}^{(2)}\right\\|^{2}$, the (squared) size of the change |
| $\left(X_{i, j}\right)_{j \geqslant 1}$ | sequences of iid random variables with $X_{i, 1} \sim f\left(\cdot ; \theta_{0}^{(i)}\right)$ for $i=1,2$ and $\left(X_{1, j}\right)_{j \geqslant 1}$ and $\left(X_{2, j}\right)_{j \geqslant 1}$ are independent |
| $\tau_{1}, \tau_{2}$ | $\tau_{1}:=A^{\prime}\left(\theta_{0}^{(1)}\right)$ and $\tau_{2}:=A^{\prime}\left(\theta_{0}^{(2)}\right)$ |
| $\Sigma_{1}, \Sigma_{2}$ | $\Sigma_{1}:=A^{\prime \prime}\left(\theta_{0}^{(1)}\right)$ and $\Sigma_{2}:=A^{\prime \prime}\left(\theta_{0}^{(2)}\right)$ |
| $\tau_{A}$ | $:=A^{\prime}\left(\theta_{A}\right)$, limit of $\tau_{1}, \tau_{2}$ |
| $\delta^{2}$ | $\because=\left\\|\tau_{1}-\tau_{2}\right\\|^{2}$ |
| $\sigma_{A}^{2}$ | limit of $\left(\tau_{1}-\tau_{2}\right)^{T} H^{\prime \prime}\left(\tau_{A}\right)\left(\tau_{1}-\tau_{2}\right) / \delta^{2}$ (cf. Theorem 3.4.4, some transformation of the (squared) size of the change |
|  | Test statistics |
| $S_{n}(k)$ | $=:-\log \Lambda_{k}$ for the likelihood ratio $\Lambda_{k}$ (cf. Eq. (3.2.2), (3.2.3)) |
| $\mathcal{S}_{n}$ | $=: \max _{1 \leqslant k \leqslant n}\left\{2 S_{n}(k)\right\}$, maximally selected log-likelihood ratio |
| $B_{n}(k)$ | $=: k^{-1} \sum_{1 \leqslant i \leqslant k} T\left(X_{i}\right)$ |
| $B_{n}^{*}(k)$ | $=:(n-k)^{-1} \sum_{k \leqslant i \leqslant n} T\left(X_{i}\right)$ |
| $S_{n}\left(k, k^{*}\right)$ | similar as $S_{n}(k)$ but also tracking the dependence on the true location $k^{*}$ of the change point (cf. Eq. (3.4.3)) |
| $B_{n}\left(k, k^{*}\right)$ | similar as $B_{n}(k)$ |
| $B_{n}^{*}\left(k, k^{*}\right)$ | similar as $B_{n}^{*}(k)$ |
| $\mu_{n}\left(k, k^{*}\right)$ | mean of $S_{n}\left(k, k^{*}\right)$ (cf. Eq. (3.4.6) |
| $\dddot{Z}_{n}\left(\underline{k}, k^{*}\right)$ | transformed test statistic (cf. Eq. 3.4.8, 3.4.8) |
|  | Estimators |
| $\hat{\lambda}_{n}$ | $:=n^{-1} \arg \max _{1 \leqslant k \leqslant n-1}\left\{2 S_{n}(k)\right\}$, estimator of $\lambda_{n}^{*}$ |
| $\hat{k}_{n}$ | $:=n \hat{\lambda}_{n}$, estimator of $k_{n}^{*}$ |

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## Selbständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig verfasst und noch nicht für andere Prüfungen eingereicht habe. Sämtliche Quellen einschließlich Internetquellen, die unverändert oder abgewandelt wiedergegeben werden, insbesondere Quellen für Texte, Grafiken, Tabellen und Bilder, sind als solche kenntlich gemacht. Mir ist bekannt, dass bei Verstößen gegen diese Grundsätze ein Verfahren wegen Täuschungsversuchs bzw. Täuschung eingeleitet wird.

Berlin, den
Cassandra Milbradt


[^0]:    ${ }^{1}$ also Skorokhod $J_{1}$-topology to differentiate between the Skorokhod $J_{2^{-}}, M_{1^{-}}$, and $M_{2}$-topologies
    ${ }^{2}$ In Billingsley [8, Section 6], the concept of weak ${ }^{\circ}$ convergence of probability measures has been introduced, an analogue of the classical concept of weak convergence of probability measures for non-separable topological spaces.

[^1]:    ${ }^{1}$ Indeed, for this reason in 42 the absolute volume function is taken as part of the state variable. However, as argued above from a modeling point of view taking the relative volume function is more sensible.
    ${ }^{2}$ The data is publicly available at https://www.epexspot.com/

[^2]:    ${ }^{3}$ Note that we choose to model the standing volumes at the bid side through the positive instead of the negative half-line so that we can treat both volume density functions in the same manner.

[^3]:    ${ }^{4}$ The bid price changes must be subtracted from the state variable since the standing volumes of the bid side are modeled through the positive half-line of its volume density function.

[^4]:    ${ }^{5}$ In fact, strictly decreasing would work as well. For the ease of exposition we consider the increasing case.
    ${ }^{6}$ For notational simplicity, we write $\theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right)\right), j \in \mathbb{Z}$, and always think of the well-defined sets $\theta_{I}\left(s,\left[x_{j}^{(n)}, x_{j+1}^{(n)}\right) \cap[-M, M]\right), j \in \mathbb{Z}$, for all $s \in E$.

[^5]:    ${ }^{1}$ The data set is publicly available at EPEX SPOT. We consider the German and Austrian market area and hour products for different delivery times.
    ${ }^{2}$ The data set is publicly available at the at OMIE and focuses on France and Spain.

