JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS Vol. 47, No. 6, Dec. 2012, pp. 1361–1395

doi:10.1017/S0022109012000555

Futures Cross-Hedging with a Stationary Basis

Stefan Ankirchner, Georgi Dimitroff, Gregor Heyne, and Christian Pigorsch*

Abstract

When managing risk, frequently only imperfect hedging instruments are at hand. We show how to optimally *cross-hedge* risk when the spread between the hedging instrument and the risk is *stationary*. For linear risk positions we derive explicit formulas for the hedge error, and for nonlinear positions we show how to obtain numerically efficient estimates. Finally, we demonstrate that even in cases with no clear-cut decision concerning the stationarity of the spread, it is better to allow for *mean reversion of the spread* rather than to neglect it.

I. Introduction

Reducing or even eliminating a particular risk is an important task in risk management. However, often only imperfect hedging instruments are at hand, leading to *basis risk*. This is, for instance, the case if the asset that is hedged does not exactly coincide with the asset underlying the futures contract. A typical example of such a case is an airline company that wants to protect itself against changing kerosene prices. Since there is no liquid kerosene futures market, the airline company may fall back on futures on less refined oil, such as crude oil, for hedging its kerosene risk. This is a reasonable approach if the price evolutions of kerosene and of crude oil are very similar. Graph A of Figure 1 illustrates the close comovement of the 2 price series at the IntercontinentalExchange (ICE).

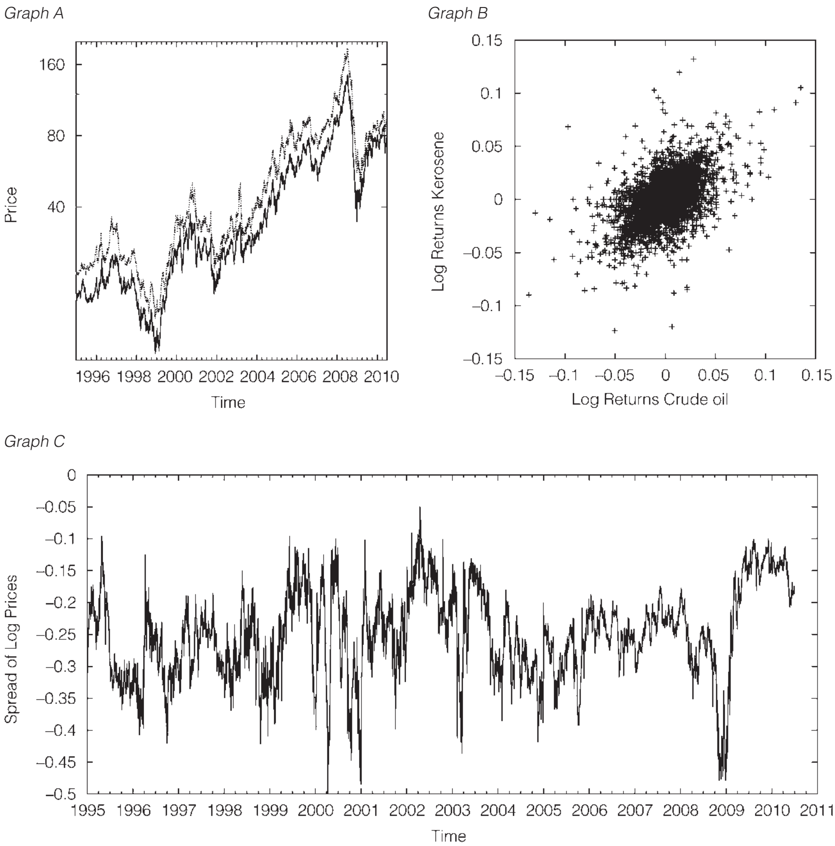
The correlation between the price changes is the crucial determinant of an optimal cross-hedge. A common approach in the literature and in practice is to obtain the optimal hedge ratio by using the most frequent returns or price increments available, irrespective of the time to maturity. This is a valid approach

*Ankirchner, ankirchner@hcm.uni-bonn.de, Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53113 Bonn, Germany; Dimitroff, dimitroff@gmail.com, DZ Bank AG, Platz der Republik, 60265 Frankfurt am Main, Germany; Heyne, heyne@math.hu-berlin.de, Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany; and Pigorsch, christian.pigorsch@uni-bonn.de, Department of Economics, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany. We are grateful to Hendrik Bessembinder (the editor), Mikhail Chernov (associate editor and referee), Michael Monoyios, and seminar participants at Humboldt University Berlin, Technical University Munich, and University of Oxford for helpful comments. Ankirchner and Pigorsch gratefully acknowledge financial support by the German Research Foundation (DFG) through the Hausdorff Center for Mathematics.

FIGURE 1

Crude Oil Spot Price and Jet Kerosene Spot Price

Graph A of Figure 1 depicts the time evolution of the daily price of crude oil in US$/BBL (dashed line) and for jet kerosene in US$/BBL (solid line) from Jan. 2, 1995, until June 30, 2010 (resulting in 4,043 observations). Graph B exhibits the scatter plot of the corresponding daily log returns and shows that there is positive correlation among the 2 series, as mentioned in the text (with a correlation coefficient of 0.52). Graph C depicts the time evolution of the spread of the log prices. Note that crude oil is usually traded in units of barrel (i.e., a volume unit). In contrast, the trading unit of kerosene is the metric ton (i.e., a weight unit). Although a common unit is not necessary for our analysis, we convert the price of kerosene per metric ton to a price per barrel by assuming a density of kerosene of 810 kg/m^3 (at 20 centigrade and 1 atm). This means a metric ton of kerosene has a volume of (1000/810) m^3 or 1234.6 liter. A barrel of crude oil contains 158.987 liter so that the ratio is 0.1288.



if the correlation (between the returns or price increments) and the ratio of the standard deviations are constant with respect to the sampling frequency, such as for correlated (geometric) Brownian motions (BMs). However, in many cases the correlation depends strongly on the selected time interval. For example, in our empirical illustration the correlation of the daily log returns of kerosene and crude oil is only 0.52, which seems unexpectedly low given the strong comovement in the price series. The correlations of the weekly, monthly, and yearly log returns, in contrast, are at 0.72, 0.84, and 0.98, respectively. Thus, the short-term correlation is considerably lower than the long-term correlation, pointing toward a long-term relationship with potential short-term deviations. This property is closely related to the concept of cointegration. It dates back to Engle and Granger (1987) and

Granger (1981) and assumes that a set of time series share a long-term relationship with temporary deviations from this "equilibrium." More precisely, consider 2 integrated time series (of order 1). They are cointegrated if a linear combination of them is stationary. This is supported for our example in Figure 1, which shows on Graph C a clear mean-reverting behavior of the spread between the logarithmic prices of kerosene and crude oil. Note that we do not use an estimated cointegration vector but rather assume that the spread between the log prices is stationary. This is more restrictive, but empirically supported by the *p*-value of the augmented Dickey-Fuller (ADF) test (which is ≤ 0.001), indicating that the null hypothesis of a nonstationary spread is rejected.

From a fundamental point of view the spread of the kerosene and crude oil prices is determined by the marginal costs of producing (the finer) kerosene out of crude oil. Temporary deviations of the spread from the marginal costs may occur due to a kerosene shortage or an oversupply. The speed with which the spread reverts to a mean level essentially reflects how fast the market can compensate the deviations. The common stochastic trend reflects shocks, such as a natural or political crisis in the producing countries, that affect the price of crude oil and, thus, also affect indirectly the price of kerosene due to the intensive and essential use of crude oil in the production process for kerosene. The case of kerosene and crude oil, however, is only one example for a pair of cointegrated processes, and many studies point toward a cointegration relation between asset prices and corresponding hedging instruments (see, e.g., Alexander (1999), Baillie (1989), Brenner and Kroner (1995), Lien and Luo (1993), and Ng and Pirrong (1996) and the references therein).

The long-term relationship between the kerosene price and the crude oil price leads to the observed increasing correlation in our example, so that the optimal hedge ratios are not constant, but depend on time to maturity. Intuitively, for long-term hedges it is likely that the 2 assets are in their equilibrium relationship, whereas in the short term the dynamics are dominated by noisy fluctuations due to shortage or oversupply of kerosene. To account for a time-varying hedge ratio, a possible strategy is to estimate the optimal hedge ratio for different maturity times. However, this strategy is not consistent, as the objective function, for example, the variance of the hedge error, is optimized within a 1-period model (with different times to maturity). But the reestimation implies that this is a multiperiod strategy and therefore results in a suboptimal rule. This is also supported by Lien (1992), who shows that even in a very simple model the multiperiod optimal hedge ratio differs from the one obtained in a 1-period model.

Therefore, applying a 1-period model is imperfect, as it does not account for a dynamic rebalance of the hedge. Howard and D'Antonio (1991), Lien (1992), (2004), and Lien and Luo (1993) were among the first to consider a discrete-time multiperiod planning horizon of the agent with (at least partially) cointegrated time series.[1]

---

[1]Another strand of the literature is based on the recursive estimation of the parameters or states of a "dynamic" model. Among others, Baillie and Myers (1991), Brenner and Kroner (1995), and Cecchetti, Cumby, and Figlewski (1988) consider bivariate generalized autoregressive conditional heteroskedasticity ((G)ARCH) models to estimate the time-varying variances and covariances.

A natural and self-evident generalization is to consider the possibility of adjustments to the hedging position in continuous time. This has the additional advantage that it is often possible to compute the standard deviation of the hedge error for different hedging strategies in *closed form*, which seems to be impossible for discrete-time models. The availability of the standard deviation of the hedge error in turn makes it possible to assess the value of a hedging strategy. Since portfolio risks are usually assessed daily, following a mark-to-market procedure, efficient risk calculation algorithms are needed in order to estimate the risk quickly. Fast hedge error estimation algorithms are also needed for pricing. If a derivative can only be cross-hedged, then a writer will ask for a premium for the hedge error. In order to determine the risk premia, traders need to quickly estimate expected hedge errors. Obviously, the most convenient support is given by closed-form formulas.

The aim of this paper is to set up a model that allows a rigorous study of the effect of a long-term relationship on optimal cross-hedging strategies and at the same time allows an efficient calculation of the basis risk entailed by the optimal cross-hedges. We choose a continuous-time line and reproduce the long-term relationship of the prices by describing the spread as an Ornstein-Uhlenbeck process, a Gaussian mean-reverting process, and by modeling the futures price as a geometric BM (GBM). It is noteworthy that our model differs from the widely studied models where both processes, the risk source and the hedging instrument, are GBMs.[2] In these models, here referred to as 2GBM models, the spread of the log prices is not stationary, since the variance of the spread is proportional to time. We further show that these models underhedge the risk when cointegration is present (see Section V.A). Our model, in contrast, explicitly accounts for a stationary spread. Furthermore, it is easy to estimate, and it is still tractable enough to allow for a quick calculation of the hedge error standard deviation under different trading strategies. In particular, we are able to derive time-consistent strategies, allowing for updating, which minimize the variance of the hedge error.

To this end, we first solve the optimization problem of finding the dynamic strategy that minimizes the variance of the hedge error. In an abstract continuous martingale setting of an incomplete market, variance-optimal hedging strategies have been first described in Föllmer and Sondermann (1986). We make use of their method and transfer it to the specific case of cross-hedging risk with futures contracts within our Markovian model. The optimal hedging strategy can be expressed in terms of the risk's *Greeks* and a hedge ratio decaying with time to maturity. Moreover, for linear risk positions, we are able to derive a closed-form formula for the hedge error standard deviation.

The paper is structured as follows: Section II introduces our model and presents some empirical evidence, while Section III briefly reviews hedging with

Although these models account for time variations in the distribution, the variance of the hedge is still minimized within a 1-period model and with a fixed time to maturity.

[2]Such models are considered, for example, in Ankirchner and Heyne (2012), Duffie and Richardson (1991), and Schweizer (1992), who derive cross-hedging strategies minimizing quadratic objective functions, and in Ankirchner, Imkeller, and Dos Reis (2010), Davis (2006), Monoyios (2004), and Musiela and Zariphopoulou (2004), who provide cross-hedging strategies maximizing the hedger's expected utility.

futures contracts and derives the variance-optimal hedging strategy for our model. Section IV develops the implied hedge errors within our model for linear and nonlinear positions, and Section V compares the hedge errors between different models and (suboptimal) hedging strategies, emphasizing the importance of allowing for a stationary spread. An extension of our model to account for stochastic volatility is given in Section VI. Section VII concludes, while Appendix A contains some empirical results and Appendix B provides the proofs.

II. The Continuous-Time Model with a Stationary Spread

As always in modeling real-world phenomena, there is a trade-off between the accuracy of a model and its tractability. We therefore illustrate the implications of a stationary spread between the futures price and the price of an illiquid asset by assuming a simplified and tractable model, which is presented in Section II.A. This approach allows us not only to derive optimal hedging strategies and their implied hedge errors (see Section III), but also to obtain the transition density in closed form, so that the model can be estimated straightforwardly via the efficient maximum likelihood method. An empirical illustration is provided in Section II.B.

A. Model Specification

Let $I = (I_t)_{t\geq 0}$ denote the price process of an illiquid asset, and suppose that an economic agent aims at hedging a position $h(I_T)$, where $h : \mathbb{R} \to \mathbb{R}$ is a measurable payoff function and $T > 0$ is a fixed-time horizon. Furthermore, we assume that there exists a liquidly traded futures contract with price process $X = (X_t)_{t\geq 0}$, which evolves according to the stochastic differential equation (SDE)

$$\text{(1)} \qquad \mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma_X X_t \,\mathrm{d}W_t^{(X)}, \qquad X_0 = x,$$

with volatility $\sigma_X > 0$ and constant drift rate $\mu \in \mathbb{R}$. The process $W^{(X)} = (W_t^{(X)})_{t\geq 0}$ is a BM on a stochastic basis with probability measure P. We denote the spread of the log prices, in the following simply referred to as the *log spread*, by

$$S_t = \log(X_t) - \log(I_t).$$

Although the nonstationarity of the log spread seems to be a plausible assumption for certain asset classes (e.g., for stock prices), there also exist relevant examples for stationary log spreads, as shown in the Introduction and the mentioned articles. We therefore propose to account for cointegration by first modeling the log spread as a stationary process and then derive the implied dynamics of the illiquid asset.

More precisely, we assume that the log spread follows an (Gaussian) Ornstein-Uhlenbeck process, which is the continuous-time analogue of the stationary discrete-time 1st-order autoregressive process. Under this assumption the log spread solves the mean-reverting SDE

$$\text{(2)} \qquad \mathrm{d}S_t = \kappa(m - S_t)\,\mathrm{d}t + \sigma_S\left(\rho \,\mathrm{d}W_t^{(X)} + \bar{\rho}\,\mathrm{d}W_t^{\perp}\right), \qquad S_0 = s,$$

https://doi.org/10.1017/S0022109012000555 Published online by Cambridge University Press

where $W^{\perp} = \left(W_t^{\perp}\right)_{t\geq 0}$ is a BM independent of $W^{(X)}$, $\kappa \geq 0$ is the mean-reversion speed, and $\rho \in [-1, 1]$ is the correlation. The log spread's volatility σ_S is assumed to be nonnegative. Moreover, we define $\bar{\rho} = \sqrt{1-\rho^2}$ and use, for ease of exposition, the shorthand notation

$$W_t^{(S)} = \rho W_t^{(X)} + \bar{\rho} W_t^{\perp}$$

for the BM driving the log spread. Note that for $\kappa \downarrow 0$ the Ornstein-Uhlenbeck process becomes more and more persistent and in the limit a (scaled and shifted) BM that is correlated with the BM of the futures price process.

The dynamics of X and S determine the dynamics of the illiquid asset price, as $I_t = X_t e^{-S_t}$, $t \geq 0$. A straightforward calculation, appealing to Itō's formula, shows that the dynamics of I satisfy

$$\mathrm{d}\, I_t = I_t \left(\frac{1}{2}\sigma_S^2 - \kappa(m - S_t) + \mu - \rho\sigma_S\sigma_X\right) \mathrm{d}\, t + I_t \sigma_I \,\mathrm{d}\, W_t^{(I)},$$

where $\sigma_I = \sqrt{\sigma_X^2 - 2\rho\sigma_S\sigma_X + \sigma_S^2}$ and $W^{(I)} = (W_t^{(I)})_{t\geq 0}$ is a BM defined by $W_t^{(I)} = ((\sigma_X - \rho\sigma_S)W_t^{(X)} - \bar{\rho}\sigma_S W_t^{\perp})/\sigma_I$, $t \geq 0$.

Note that the correlation ρ_{IX} between the BMs driving I and X is given by

$$\rho_{IX} = \frac{1}{\sigma_I}(\sigma_X - \rho\sigma_S), \tag{3}$$

which is nonnegative if and only if $\sigma_X \geq \rho\sigma_S$. For fixed parameters ρ and σ_X, we can write ρ_{IX} as a function of σ_S:

$$\rho_{IX}(\sigma_S) = \frac{\sigma_X - \rho\sigma_S}{\sqrt{\sigma_X^2 - 2\rho\sigma_S\sigma_X + \sigma_S^2}}. \tag{4}$$

It can be shown that $\rho_{IX}(\sigma_S)$ is strictly decreasing in σ_S, and hence invertible on $\mathbb{R}_+$. For a proof of the following result, see Appendix B.

Lemma 1. Let $\sigma_X > 0$ and $\rho \in (-1, 1)$. Then the mapping $\mathbb{R}_+ \ni \sigma_S \mapsto \rho_{IX}(\sigma_S)$ is strictly decreasing. Moreover, the log spread's volatility σ_S satisfies

$$\sigma_S = \sigma_X \frac{\sqrt{1-\rho_{IX}^2}}{\rho\sqrt{1-\rho_{IX}^2} + \rho_{IX}\sqrt{1-\rho^2}}. \tag{5}$$

Observe that $(\log X_t, S_t, \log I_t)$ is a 3-dimensional Gaussian process. Furthermore, it possesses a closed-form transition density, and hence efficient maximum likelihood estimation becomes feasible. Indeed, a straightforward calculation shows that the triple $(\log X_t, S_t, \log I_t)$ satisfies

$$\left.\begin{pmatrix}\log X_t \\ S_t \\ \log I_t\end{pmatrix}\right| \begin{pmatrix}X_0 \\ S_0 \\ I_0\end{pmatrix} = \begin{pmatrix}x \\ s \\ xe^{-s}\end{pmatrix}$$
$$\sim \mathcal{N}\left(\begin{pmatrix}\log x + \left(\mu - \frac{\sigma_X^2}{2}\right)t \\ se^{-\kappa t} + m\left(1 - e^{-\kappa t}\right) \\ \log x + \left(\mu - \frac{\sigma_X^2}{2}\right)t - se^{-\kappa t} - m\left(1 - e^{-\kappa t}\right)\end{pmatrix}, \Sigma\right),$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} t\sigma_X^2 & \frac{\rho\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) & t\sigma_X^2 - \rho\frac{\sigma_X\sigma_X}{\kappa}\left(1-e^{-\kappa t}\right) \\ \frac{\rho\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) & \frac{\sigma_S^2}{2\kappa}\left(1-e^{-2\kappa t}\right) & \frac{\rho\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) - \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa t}\right) \\ t\sigma_X^2 - \rho\frac{\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) & \begin{array}{c}\frac{\rho\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) \\ -\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa t}\right)\end{array} & \begin{array}{c} t\sigma_X^2 - 2\rho\frac{\sigma_X\sigma_S}{\kappa}\left(1-e^{-\kappa t}\right) \\ +\frac{\sigma_S^2}{2\kappa}\left(1-e^{-2\kappa t}\right)\end{array} \end{pmatrix}$$

and $\mathcal{N}\left(\boldsymbol{m},\boldsymbol{V}\right)$ denotes the normal distribution with mean vector $\boldsymbol{m}$ and covariance matrix $\boldsymbol{V}$.

As we first specify the dynamics of the futures contract as a GBM and the log spread as an Ornstein-Uhlenbeck process, the price of the futures contract leads the risk price. For example, if the futures price is subject to a (demand or supply) shock, the risk price follows and reduces the distance to the futures price. This asymmetric behavior is in line with empirical findings, as there is strong evidence that futures prices lead the spot prices (e.g., see Chan (1992), Kawaller, Koch, and Koch (1987), and Stoll and Whaley (1990)). Note that most studies investigate the relationship between a stock index and the corresponding futures contract. However, their main argument of less frequent trading in the spot market and differences in transaction costs is also valid in our setup. Both lead to asymmetric access to information, which in turn results in asymmetric behavior of the spread. Therefore, for the applications we have in mind, it seems natural to model the futures and the spread first, and then to derive the spot dynamics endogenously. Moreover, it is also possible to specify the relation in the reverse direction, that is, to derive the dynamics of the futures price process based on the dynamics of the risk process and the log spread. One can then proceed in a similar manner.

The model introduced has some similarities with Gaussian commodity spot models (e.g., as in Schwartz (1997) or with the more general model provided in Casassus and Collin-Dufresne (2005)). In these models the triple of futures log price, spot log price, and log spread is a 3-dimensional Gaussian process, too. These spot models, however, have different aims (e.g., they can be used for pricing long-term forward commitments on the *same* commodity). Any forward position can be hedged by using 1 interest rate derivative and 2 short-term futures contracts. This means that the latter models are *complete*, and hence the model dynamics under the *risk-neutral measure* have to be calibrated to current futures and derivative prices.

The main aim of our model instead is to analyze the hedge error entailed when cross-hedging risk exposures with futures written on a correlated, but *different* risk source. Our model includes a nonhedgeable risk factor, the spread, leading to *incompleteness*. We work under the physical measure, since this is the only measure under which the hedge errors characteristics are relevant for risk management. Moreover, in a cross-hedging situation a calibration is not always possible (e.g., if there are no liquid kerosene futures).

B. An Empirical Illustration

Here we illustrate the estimation of the model by reconsidering the example of kerosene and crude oil. We use daily data of the spot kerosene price and the price of different crude oil futures contracts. The maturity of the futures contracts ranges from Jan. 2009 until Oct. 2010, resulting in 21 (overlapping) time series.

In a 1st step we check via the ADF test to see whether the log spread of the kerosene spot price and the corresponding crude oil futures price is stationary. Table A1 in Appendix A reports the results for the different futures contracts. For most of the pairs, we reject the null of a nonstationary log spread at any reasonable level. Of course, as one would expect for a statistical test, the procedure does not suggest the existence of a long-term relationship for every pair even if it is present. For these cases the model uncertainty is obvious, and in Section V.A we examine how the application of an optimal hedging strategy influences the hedging performance if the strategy is derived under our model but is applied to the 2GBM model, and vice versa.

Table A1 in Appendix A also presents the estimation result for our data sets. To concentrate on 1 asset in the remaining part of the paper, we use 1 *representative contract* with moderate, not extreme, parameter values, especially for κ and ρ. We choose the contract with maturity in Aug. 2009. The number of observations at which both assets, the futures contract and the spot kerosene, are traded is 885. Figure 2 shows the time evolution of these 2 price series. Obviously, the price evolutions are very similar, which is also supported by the time-series plot of the log spread of the log prices (depicted in Graph C).

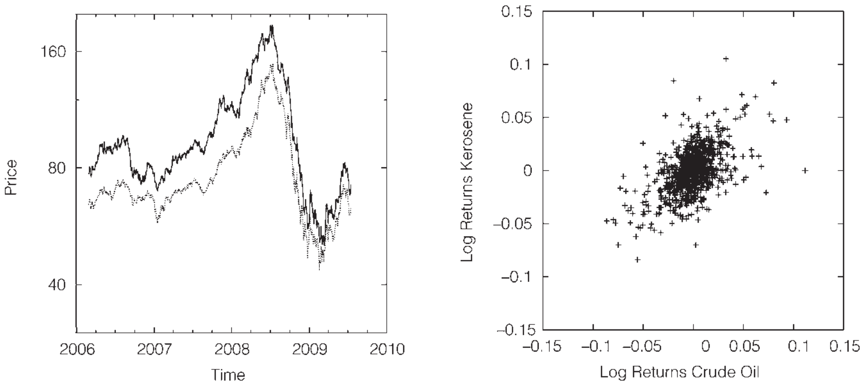
In the next sections we derive the variance-optimal hedge, the corresponding variance of the hedge error, and derive quantitative and qualitative statements in terms of the structural parameters.

FIGURE 2

Crude Oil Futures Price and Jet Kerosene Spot Price

Graph A of Figure 2 depicts the time evolution of the daily price of the crude oil futures with maturity in Aug. 2009 in U.S. dollars per barrel (US$/BBL) (dashed line) and for spot jet kerosene in US$/BBL (solid line) from Feb. 27, 2006, until July 16, 2009 (resulting in 885 observations). The structure of this figure is the same as Figure 1. Graph B exhibits the scatter plot of the daily returns. Graph C depicts the time evolution of the spread of the log prices. Note that the units have been normalized, just as in Figure 1.

Graph A *Graph B*



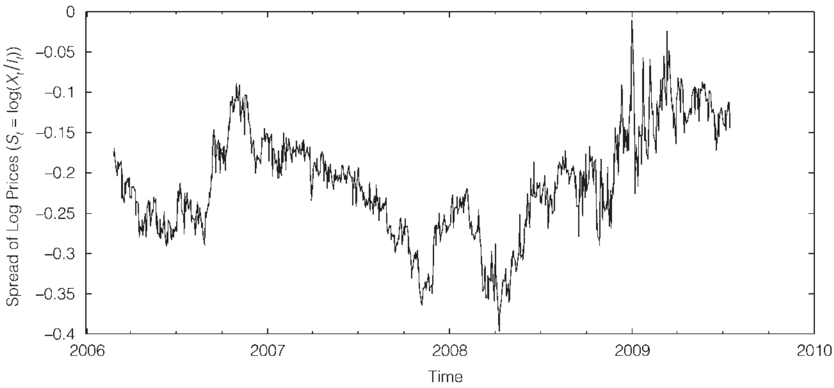
(continued on next page)

https://doi.org/10.1017/S0022109012000555 Published online by Cambridge University Press

FIGURE 2 (continued)

Crude Oil Futures Price and Jet Kerosene Spot Price

Graph C



https://doi.org/10.1017/S0022109012000555 Published online by Cambridge University Press

III. Optimal Variance Hedging with Futures Contracts

Suppose that a hedger sets up a portfolio consisting of futures contracts and cash positions, in order to hedge the risk position $h(I_T)$. In the following, we denote by ξ_t the number of futures contracts held in the portfolio at time t. We assume that any futures position strategy $\xi = (\xi_t)_{t\in[0,T]}$ is nonanticipating (i.e., at any time, it incorporates only information publicly available). In mathematical terms, this means that ξ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, the filtration generated by the BMs $\left(W^{(X)},\ W^{\perp}\right)^{\top}$ and completed with the P-null sets of the basis.

If the futures price changes by ΔX_t from one trading day to the next, the hedger's margin account is adjusted by ΔX_t per futures contract. The cash position in the hedging portfolio is changed accordingly, entailing a portfolio value change due to variation margins of $\Delta V_t^{\mathrm{MAR}} = \xi_t \Delta X_t$.

The *total value* of the hedging portfolio is denoted by $V = (V_t)_{t\in[0,T]}$. Given an interest rate r, the cash position contributes to the portfolio by $rV_t\,\mathrm{d}t$, hence the total value satisfies the continuous-time self-financing condition

(6) $$\mathrm{d}V_t = \xi_t\,\mathrm{d}X_t + rV_t\,\mathrm{d}t.$$

Equation (6) is linear. Therefore, given an initial portfolio value of $V_0 = v$, the portfolio process has the explicit representation

(7) $$V_t = e^{rt}\left(v + \int_0^t e^{-rs}\xi_s\,\mathrm{d}X_s\right),$$

see, for example, Chap. 5.6 in Karatzas and Shreve (1991).

Consider a self-financing hedge portfolio with futures position ξ_t at time t. The *conditional hedge error* of the portfolio at time $t \in [0, T]$ is then given by

$$
\begin{aligned}
C_t(\xi, v) &= \mathbb{E}\left(e^{-r(T-t)} h(I_T) | \mathcal{F}_t\right) - V_t \\
&= e^{rt} \left(\mathbb{E}\left(e^{-rT} h(I_T) | \mathcal{F}_t\right) - v - \int_0^t \xi_s e^{-rs} \, \mathrm{d} X_s \right).
\end{aligned} \tag{8}
$$

$C_T(\xi, v)$ will also be referred to as the *realized hedge error*. Note that if $C_t(\xi, v)$ is negative, the combined value of the risk and the hedge portfolio is expected to end up with a plus.

To determine the variance-optimal strategy within our model, that is, the strategy minimizing the variance of the realized hedge error, we suppose that X is a martingale. This is a plausible assumption, since X is a futures price process. Moreover, the empirical analysis of crude oil futures prices shows that the estimated drift parameter is close to 0 for all contract months and statistically insignificant for most assets (see Table A1 in Appendix A). In addition, as the estimation of the drift is notoriously challenging and very often highly speculative, it can easily distort the main aim of hedging, which is the reduction of risk. We therefore discuss here the martingale case in depth and postpone the discussion of the more general case to Section VI.

Assuming that X is a martingale means that $\mu = 0$ and $\mathrm{d} X_t = \sigma_X X_t \, \mathrm{d} W_t^{(X)}$. Then equation (8) implies that the discounted conditional hedge error is also a martingale. By applying a representation theorem from stochastic analysis, the martingale $e^{-rT} \mathbb{E}\left(h(I_T) | \mathcal{F}_t\right)$ can be written as a stochastic integral process of the form

$$
\begin{aligned}
e^{-rT} \mathbb{E}\left(h(I_T) | \mathcal{F}_t\right) &= e^{-rT} \mathbb{E}\left(h(I_T)\right) + \int_0^t a_s \, \mathrm{d} W_s^{(X)} \\
&\quad + \int_0^t b_s \, \mathrm{d} W_s^{\perp}, \qquad t \in [0, T],
\end{aligned} \tag{9}
$$

where a and b are progressively measurable and square-integrable processes (see, e.g., Chap. 3.4 in Karatzas and Shreve (1991)). The 1st stochastic integral on the right-hand side (RHS) is hedgeable, since it is driven by the same BM as the futures X. More precisely, following the strategy

$$
\xi_t^* = \frac{a_t e^{rt}}{\sigma_X X_t}, \tag{10}
$$

the gain from the futures position up to time t satisfies

$$
\int_0^t \xi_s^* e^{-rs} \, \mathrm{d} X_s = \int_0^t a_s \, \mathrm{d} W_s^{(X)}.
$$

The 2nd integral in equation (9) is orthogonal to $W^{(X)}$, and hence completely nonhedgeable with X. This implies that the strategy ξ^* minimizes the variance of the realized hedge error (see Thm. 1 in Föllmer and Sondermann (1986), where this argument has been employed for the 1st time). Moreover, the conditional hedge error satisfies

$$C_t(\xi^*, v) \quad = \quad e^{rt} \int_0^t b_s \, \mathrm{d}\, W_s^{\perp} + \mathbb{E}\left(e^{-(T-t)r} h(I_T)\right) - e^{rt} v.$$

Profiting from the Markov property of the processes I and S, we may express a and b in terms of sensitivities of the expected risk with respect to the futures and the log spread values. More precisely, let

$$\psi(t, x, s) \quad = \quad e^{-r(T-t)} \mathbb{E}\left(h\left(X_T^{t,x} e^{-S_T^{t,s}}\right)\right),$$

where $X^{t,x}$ and $S^{t,s}$ are the solutions of equation (1) resp. equation (2) on $[t, T]$ with initial conditions $X_t^{t,x} = x$ resp. $S_t^{t,s} = s$. We will refer to ψ as the *value function*. If h is Lipschitz continuous and its weak derivative h' is Lebesgue-almost everywhere differentiable, then ψ is continuously differentiable with respect to x and s, and

(11) $$\psi_x(t, x, s) \quad = \quad \frac{\partial}{\partial x} \psi(t, x, s) \quad = \quad e^{-r(T-t)} \mathbb{E}\left(h'\left(X_T^{t,x} e^{-S_T^{t,s}}\right) X_T^{t,1} e^{-S_T^{t,s}}\right).$$

For details see Lemma 4.8. in Ankirchner and Heyne (2012), where a similar statement is shown. Notice that $\partial S_T^{t,s}/\partial s = e^{-\kappa(T-t)}$, and hence by the same reasoning

(12) $$\begin{aligned} \psi_s(t, x, s) \quad &= \quad \frac{\partial}{\partial s} \psi(t, x, s) \\ &= \quad -e^{-r(T-t)} \mathbb{E}\left(h'\left(X_T^{t,x} e^{-S_T^{t,s}}\right) X_T^{t,x} e^{-S_T^{t,s}} e^{-\kappa(T-t)}\right) \\ &= \quad -e^{-\kappa(T-t)} x \frac{\partial}{\partial x} \psi(t, x, s). \end{aligned}$$

The pair $(X,\ S)^{\top}$ is a 2-dimensional SDE, driven by $\left(W^{(X)},\ W^{\perp}\right)^{\top}$ via the diffusion matrix

$$\tilde{\boldsymbol{\Sigma}}(x, s) \quad = \quad \begin{pmatrix} \sigma_X x & 0 \\ \rho \sigma_S & \bar{\rho} \sigma_S \end{pmatrix}.$$

With Itō's formula we obtain that the processes a and b, appearing in the martingale representation (9), are given by

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} \quad = \quad e^{-rt}\, \tilde{\boldsymbol{\Sigma}}^{\top}(X_t, S_t) \begin{pmatrix} \psi_x(t, X_t, S_t) \\ \psi_s(t, X_t, S_t) \end{pmatrix}.$$

From equations (10) and (12) we can deduce the following result describing the variance-optimal hedge in terms of the futures delta and the minimal hedge error in terms of the log spread delta.

Theorem 1. The variance-optimal futures position in the hedging portfolio is given by

(13) $$\xi_t^* \quad = \quad \left[1 - \frac{\sigma_S}{\sigma_X} \rho e^{-\kappa(T-t)}\right] \psi_x(t, X_t, S_t)$$

and entails a realized hedge error of

(14) $$C_T(\xi^*, v) \quad = \quad \mathbb{E}\left(h(I_T)\right) - e^{rT}\left(v - \bar{\rho} \int_0^T e^{-rt} \sigma_S \psi_s(t, X_t, S_t)\, \mathrm{d}\, W_t^{\perp}\right).$$

Observe that the optimal hedge ξ^* is the delta of the position's expectation, dampened by the *hedge ratio* defined by

$$f(T-t) \quad = \quad 1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}.$$

The factor essentially equals 1 if the product of time to maturity $T-t$ and reversion speed κ is large. In this case, the log spread is expected to return to its mean reversion level before maturity, and hence the position should be fully hedged with the futures. As maturity approaches, the short-term fluctuations have an increasing impact on the terminal hedge performance, making the hedge ratio converge to

(15) $$h \quad = \quad \rho_{IX}\frac{\sigma_I}{\sigma_X}.$$

Indeed, due to equation (3), we have $\lim_{(T-t)\downarrow 0}[1-(\sigma_S\rho e^{-\kappa(T-t)}/\sigma_X)] = 1-(\sigma_S\rho/\sigma_X) = h$. We remark that h, defined in equation (15), is sometimes referred to as the *minimum-variance hedge ratio* (see, e.g., Chap. 3.4 in Hull (2008)).

Observe that if κ is equal to 0, which essentially means that there is no mean reversion, then the log spread is not stationary and its variance increases linearly with time. The hedge ratio is not dampened and it coincides with h. In this case the strategy ξ^* is equal to the optimal strategy in a model where both X and I are modeled as GBMs (see Section V.A for more details).

In formula (13) the optimal hedge is expressed in terms of the delta with respect to the futures price. In order to obtain a representation in terms of the delta with respect to the illiquid asset price I, we first define $\varphi(t,y,s) = e^{-r(T-t)}\mathbb{E}\left(h(I_T^{t,y,s})\right)$, where $I^{t,y,s}$ is the solution of the SDE for the illiquid asset on $[t,T]$, with initial values $I_t^{t,y,s} = y$ and $S_t^{t,s} = s$. Note that $\psi(t,x,s) = \varphi(t,e^{-s}x,s)$, and in particular, $\psi_x(t,x,s) = e^{-s}\varphi_y(t,e^{-s}x,s)$. Thus, the optimal hedge may be rewritten as

(16) $$\xi_t^* \quad = \quad e^{-S_t}\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right]\varphi_y(t,I_t,S_t).$$

If the log spread is positive, then the illiquid asset price is expected to rise relative to the futures price. This explains why in equation (16) the delta is reduced by the factor e^{-S_t}. Conversely, if the log spread is negative, then the illiquid asset is expected to fall relative to the futures price. In this the case the delta is augmented by the factor e^{-S_t}.

Finally, we remark that the hedge ratio remains the same if the hedger uses an option on the futures for hedging the risk exposure $h(I_T)$. We denote the option's delta at time t by $\Delta(t,x)$, given a futures price of $X_t = x$. The dynamics of the option price $P(t,X_t)$ satisfy $\mathrm{d}P(t,X_t) = rP(t,X_t)\,\mathrm{d}t + \Delta(t,X_t)\,\mathrm{d}X_t$, and the value of a self-financing portfolio containing ξ_t options at time t is given by

(17) $$V_t \quad = \quad e^{rt}\left(V_0 + \int_0^t e^{-rs}\xi_s\Delta(s,X_s)\,\mathrm{d}X_s\right).$$

The variance-minimizing *option position* can be shown to be equal to

(18) $$\xi_t^* \quad = \quad \left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right]\frac{\psi_x(t,X_t,S_t)}{\Delta(t,X_t)}.$$

https://doi.org/10.1017/S0022109012000555 Published online by Cambridge University Press

Suppose that a nonlinear position of kerosene is hedged with an option, written on crude oil futures, having a similar payoff profile. Then the ratio of deltas $\psi_x(t, X_t, S_t)/\Delta(t, X_t)$ is usually stable, and hence the hedging portfolio does not need to be rebalanced as strongly as when using futures for hedging.

IV. Standard Deviation of the Hedge Error

Having derived the variance-optimal strategy and the corresponding hedge error (see Theorem 1), we now aim at computing the implied standard deviation of the hedge error. This allows us to quantify the risk associated with the optimal strategy, which is important for risk management and performance evaluation of the hedging strategy. We therefore derive analytic and semianalytic formulas for the standard deviation of the hedge error when minimizing the variance of risk exposures within our model. As in the previous section, we assume that the hedger does not have any directional view concerning the futures. This means that the futures price X is a martingale with dynamics $\mathrm{d}\,X_t = \sigma_X X_t \,\mathrm{d}\,W_t$.

We aim at computing the standard deviation of the hedge error when cross-hedging the position $h(I_T)$ following the strategy ξ^* of equation (13). Note that the standard deviation of $C_T(\xi^*, v)$ coincides with the standard deviation of $e^{rT}\bar{\rho}\int_0^T e^{-rt}\sigma_S\psi_s(t, X_t, S_t)\,\mathrm{d}\,W_t^{\perp}$. The Itō isometry for stochastic integrals implies

$$\text{(19)} \qquad \mathrm{std}(C_T(\xi^*, v)) \quad = \quad e^{rT}\bar{\rho}\sigma_S\sqrt{\int_0^T e^{-2rt}\mathbb{E}\left(\psi_s^2(t, X_t, S_t)\right)\mathrm{d}\,t}.$$

In general, there is no closed-form expression for the formula for the integral in equation (19). For linear positions, however, we may explicitly calculate the variance, since the futures and the spread are lognormally distributed. Besides, for positions corresponding to plain vanilla options, the log spread delta ψ_s has an explicit representation, and thus allows for an efficient Monte Carlo simulation of the error (19). We proceed by discussing both cases separately.

A. Linear Positions

In this subsection we derive an analytic formula for the hedge error variance when cross-hedging a linear position. This is the most relevant case, since most of the risk positions of industrial companies are linear. Think, for instance, of an airline being exposed to a short position of kerosene.

Suppose that the payoff function h is given by $h(y) = cy$, with $c \in \mathbb{R}$. In this case the delta of the value function with respect to the futures price satisfies $\psi_x(t, x, s) = e^{-r(T-t)}\mathbb{E}\left(cX_T^{t,1}e^{-S_T^{t,s}}\right)$ (see equation (11)). Thus, with equation (12), we get

$$\frac{\partial}{\partial s}\psi(t, x, s) \quad = \quad -e^{-\kappa(T-t)}xe^{-r(T-t)}\mathbb{E}\left(cX_T^{t,1}e^{-S_T^{t,s}}\right).$$

In the following we do not only need to compute the expectation of the product $X_T^{t,1}e^{-S_T^{t,s}}$, but also the expectation of the product of higher moments of the log spread and the illiquid asset. We therefore provide the following lemma.

Lemma 2. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$; then

$$\mathbb{E}\left(e^{-aS_t^{0,s}}(X_t^{0,x})^b\right) = A(a,b,x,s,t),$$

where $A(a,b,x,s,t)$ is defined by

$$(20) \qquad A(a,b,x,s,t) = x^b \exp\left[-\frac{1}{2}\sigma_X^2 t\left(b-b^2\right) - ase^{-\kappa t} - a\left(m + b\rho\sigma_X\sigma_S\frac{1}{\kappa}\right)\left(1-e^{-\kappa t}\right) + \frac{1}{2}a^2\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)\right].$$

With this at hand, the standard deviation (19) simplifies to

$$(21) \quad \operatorname{std}(C_T(\xi^*, v)) = |c|\bar{\rho}\sigma_S\sqrt{\int_0^T e^{-2\kappa(T-t)}\mathbb{E}\left(X_t^2 A^2(1,1,1,S_t,T-t)\right)\mathrm{d}t}.$$

From this we are able to derive the following explicit formula for the hedge error variance.

Theorem 2. The variance-optimal cross-hedge of a linear position cI_T entails a hedge error with standard deviation

$$(22) \quad \operatorname{std}(C_T) = \sigma_S\sqrt{1-\rho^2}x\exp\left((m-s)e^{-\kappa T} - m - \rho\sigma_X\sigma_S\frac{1}{\kappa}\left(1-2e^{-\kappa T}\right) + \sigma_S^2\frac{1}{4\kappa}\left(1-2e^{-\kappa T}\right)\right) \times |c|\sqrt{\int_0^T \exp\left(-2\kappa(T-t) + \sigma_X^2 t - 2\rho\sigma_X\sigma_S\frac{1}{\kappa}e^{-\kappa(T-t)} + \sigma_S^2\frac{1}{2\kappa}e^{-2\kappa(T-t)}\right)\mathrm{d}t}.$$

The integral in expression (22) can be computed in a straightforward manner using standard numerical quadratures algorithms.

When analyzing the dependence of the hedge error on the different model parameters, it is convenient to rewrite the hedge error formula (22) as

$$(23) \quad \operatorname{std}(C_T) = |c|\,\sigma_S\sqrt{1-\rho^2}x\exp\left((m-s)e^{-\kappa T} - m\right) \times\left[\int_0^T \exp\left(-2\kappa(T-t)+\sigma_X^2 t\right)\exp\left(-2\rho\sigma_X\frac{\sigma_S}{\kappa}\left(1+e^{-\kappa(T-t)} - 2e^{-\kappa T}\right) + \frac{\sigma_S^2}{2\kappa}\left(1+e^{-2\kappa(T-t)} - 2e^{-\kappa T}\right)\right)\mathrm{d}t\right]^{\frac{1}{2}}.$$

The hedge error (22) can be approximated with simplified formulas. For long maturities, that is, large T, the factor $\exp\left((m-s)e^{-\kappa T} - m\right)$ approximately coincides with e^{-m}. Moreover,

$$\int_0^T e^{-2\kappa(T-t)+\sigma_X^2 t}\,\mathrm{d}t = \frac{e^{\sigma_X^2 T} - e^{-2\kappa T}}{2\kappa + \sigma_X^2} \approx \frac{e^{\sigma_X^2 T}}{2\kappa+\sigma_X^2},$$

and hence for long maturities:

$$\operatorname{std}(C_T) \approx |c|\,xe^{-m}\sigma_S\sqrt{1-\rho^2}e^{\left(-\rho\sigma_S\sigma_X + \frac{\sigma_S^2}{4}\right)/\kappa}\sqrt{\frac{e^{\sigma_X^2 T}}{2\kappa+\sigma_X^2}}.$$

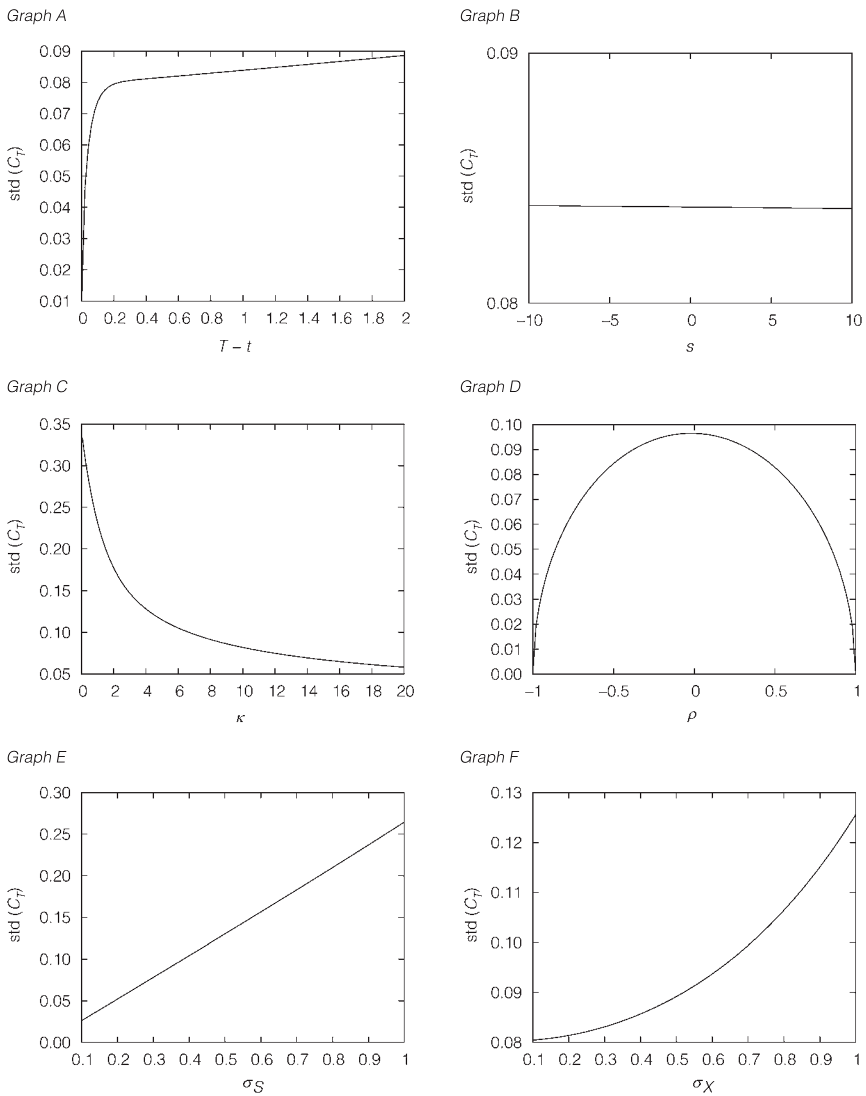
Observe that the hedge error increases exponentially with time to maturity. However, the volatility squared σ_X^2 is usually low (see Table A1 in Appendix A), and thus the hedge error increases approximately linearly, with slope $\sigma_X^2/2$, in the 1st several years (compare with Graph A in Figure 3).

For short maturities, that is, small T, the factor $\exp\left((m-s)e^{-\kappa T}-m\right)$ approximately coincides with e^{-s}. Besides, by linearly approximating exponentials

FIGURE 3

Sensitivity of the Hedge Error

Figure 3 shows the sensitivity of the standard deviation of the hedge error with respect to the time to maturity (in Graph A) and with respect to the parameters of the model (Graphs B–F). In each graph, only the parameter indicated on the abscissa is varied, while the others remain fixed at the estimates from the futures contract with maturity in Aug. 2009.



with the Taylor expansion up to the 1st order, we have

$$\int_0^T e^{-2\kappa(T-t)+\sigma_X^2 t}\,\mathrm{d}t \quad = \quad \frac{e^{-\sigma_X^2 T} - e^{-2\kappa T}}{2\kappa + \sigma_X^2} \quad \approx \quad T.$$

Therefore, we get the approximation for short maturities:

$$\mathrm{std}(C_T) \quad \approx \quad |c|\,\sigma_S\sqrt{1-\rho^2}xe^{-s}\sqrt{T}.$$

Note that the hedge error increases with order $T^{1/2}$ for short maturities (see Graph A in Figure 3).

The kink in Figure 3 is determined by how fast the Ornstein-Uhlenbeck process describing the log spread attains its stationary distribution. The variance of the log spread at time T, as a function of time to maturity $T-t$, is given by $\sigma_S^2(1-e^{-2\kappa(T-t)})/(2\kappa)$. The hedge error is roughly proportional to the variance of the log spread.

The hedge error vanishes as the variance of the stationary log spread distribution, $\sigma_S^2/(2\kappa)$, converges to 0. Moreover, it is straightforward to show

$$\lim_{\kappa\downarrow 0} \mathrm{std}(C_T) \quad = \quad |c|\,xe^{-s}\frac{\sigma_S}{\sigma_X}\sqrt{1-\rho^2}\sqrt{e^{\sigma_X^2 T}-1}. \tag{24}$$

Of course, not only the impact of the time to maturity to the hedge error is of interest, but also the influence of the model's parameters. Figure 3 highlights the sensitivity of the standard deviation of the hedge error toward changes in the parameters. The figure depicts the resulting standard deviation by changing one parameter and keeping the others constant. The fixed parameters are set to the estimates of the futures contract with maturity in Aug. 2009. Graph C shows the decreasing standard deviation in the mean reversion speed κ. This comes as no surprise, as a larger κ results in a faster return to the long-term relationship. The reverse U-shaped behavior with respect to the correlation ρ (in Graph D) highlights the change from the incomplete market setting for $|\rho|<1$ to the complete market setting for $|\rho|=1$. With increasing instantaneous variance of the log spread and the futures price process (σ_S and σ_X), the variance of the hedge error also increases (shown in Graphs E and F).

B. Nonlinear Positions

In practice, the case of nonlinear risk positions is also relevant. For instance, consider the very illiquid German natural gas futures markets. Due to illiquidity, gas traders frequently use futures of neighboring countries for hedging purposes. So, if operators of German gas power plants protect themselves against changing gas prices by buying Dutch gas on the futures market (e.g., natural gas futures of the Dutch market, Title Transfer Facility (TTF)), the basis risk is due to a geographical spread in commodity prices that arises from different trading places for the same underlying. In this case, TTF contracts serve as proxies that are cointegrated with the natural gas prices in the German market area. The profit margin of a gas power plant is essentially determined by the spark spread (i.e., the spread between the electricity price per megawatt-hours (MWh) and the price of the amount of gas the plant needs to produce 1 MWh of electricity). Electricity will only be produced if the profit margin exceeds the operating costs. A gas power plant can thus be seen as a call option, a highly nonlinear position, on the spark spread. We therefore also consider here the hedging of nonlinear risk positions.

When it comes to hedging nonlinear risk positions, the problem is that in general there are no explicit formulas available for the standard deviation of the hedge error. We explain here how a swift simulation analysis can be used to obtain the hedge error characteristics for standard options.

The simulation-based hedge error analysis basically works as follows: First simulate N independent paths of the futures price and the log spread. Then calculate the performance of the hedging portfolio along any simulation path and subtract it from the risk position $h(I_T)$. The collection of hedge errors obtained in this way may be analyzed with respect to the empirical standard deviation, median, and other statistical characteristics.

In order to calculate the portfolio value along any simulation path, one needs to compute the portfolio position and hence the delta at *any* time discretization point. As we show here, for plain vanilla options there are analytic formulas for the deltas, thus allowing one to quickly calculate the portfolio performance for every simulation path. Indeed, the deltas resemble the deltas for plain vanilla options in the Black-Scholes (1973) model. As an example, we provide the relevant formulas for a European call option with strike K (i.e., h is given by $h(y)=(y-K)^+$).

First observe that the value function ψ for call options is given by

$$\psi(t,x,s) \quad = \quad \mathbb{E}\left((I_T^{t,x,s}-K)^+\right).$$

From the proof of Lemma 2 it can be seen that $I_T^{t,x,s}=e^{-S_T^{t,s}}X_T^{t,x}$ can be written as

$$I_T^{t,x,s} \quad = \quad x\exp\left(-\frac{1}{2}\sigma_X^2(T-t)-se^{-\kappa(T-t)}-m\left(1-e^{-\kappa(T-t)}\right)\right)\exp(\sigma N),$$

where N is a standard normal variable and σ^2 is given by

$$\sigma^2 \quad = \quad \sigma_X^2(T-t)-2\rho\sigma_X\sigma_S\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)+\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right).$$

We get

$$(25)\quad \psi(t,x,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(xe^{-\frac{1}{2}\sigma_X^2(T-t)-se^{-\kappa(T-t)}-m\left(1-e^{-\kappa(T-t)}\right)+\sigma y}-K\right)^+e^{-y/2}\,\mathrm{d}y.$$

In analogy to the standard Black-Scholes (1973) case, we define the functions $d_+(t,x,s)$ and $d_-(t,x,s)$ as

$$d_+(t,x,s) \quad = \quad \frac{1}{\sigma}\left[\log\left(\frac{K}{x}\right)+\frac{1}{2}\sigma_X^2(T-t)+se^{-\kappa(T-t)}+m\left(1-e^{-\kappa(T-t)}\right)\right]$$

and $d_-(t,x,s)=d_+(t,x,s)-\sigma$.

Note that the integrand of the integral in equation (25) equals 0 if $y < d_+(t,x,s)$, and hence

$$\begin{aligned}\psi(t,x,s) &= x\exp\left(-\frac{1}{2}\sigma_X^2(T-t) - se^{-\kappa(T-t)} - m\left(1-e^{-\kappa(T-t)}\right) + \frac{1}{2}\sigma^2\right)\\ &\quad \times \Phi(d_-(t,x,s)) - K\Phi(d_+(t,x,s)),\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. The above explicit representation of $\psi(t,x,s)$ yields

$$\begin{aligned}\psi_x(t,x,s) &= \exp\left(-\frac{1}{2}\sigma_X^2(T-t) - se^{-\kappa(T-t)} - m\left(1-e^{-\kappa(T-t)}\right) + \frac{1}{2}\sigma^2\right)\\ &\quad \times \left[\Phi(d_-(t,x,s)) + \varphi(d_-(t,x,s))\partial_x d_-(t,x,s)\right]\\ &\quad - K\varphi(d_+(t,x,s))\partial_x d_+(t,x,s),\end{aligned}$$

where $\partial_x d_+(t,x,s) = \partial_x d_-(t,x,s) = 1/(\sigma x)$.

The analytic and semianalytic formulas for the hedge error allow the efficient computation and the comparison of the hedge error variance for different, potentially usable liquid futures contracts. Up to now, however, we assume that we hedge within the correct model (with a stationary log spread). Although, statistical tests may help to decide whether the log spread is stationary or not, there is always the risk of *hedging within the wrong model*, and a relevant question arises: How sensitive is the hedge error with respect to the model choice? We address this question in the next section.

V. The Performance of Suboptimal Hedging Strategies

So far we have assumed that we know with certainty that the price of the illiquid asset and the price of the liquid futures contract are cointegrated and evolve according to our model. However, it may happen that a statistical test leads to a wrong conclusion or different tests lead to different implications. In other words, we face model uncertainty.

In Section V.A we consider a 2GBM model and derive the hedge error obtained by using the optimal strategy from our model. Furthermore, we analyze the impact of applying the optimal hedging strategy from the 2GBM model to our model. We then proceed by comparing the optimal *dynamic* hedge with its optimal *static* counterpart. In practice, traders often hedge linear positions statically, holding a position in futures that corresponds to the size of the risk. By fully hedging the risk, they intuitively reflect that the hedge ratio essentially equals 1 whenever time to maturity is long. In Section V.B we first derive the optimal *static* hedging strategy and compare it with the hedging strategy ξ^*, which allows for portfolio regrouping.

A. The Costs of Ignoring a Long-Term Relationship or Falsely Assuming a Long-Term Relationship

We next introduce a simple model where both *X and I* are GBMs and hence are not cointegrated, and the log spread does not have a stationary distribution. We will refer to this model as the 2GBM model.

In both models, the futures price is assumed to satisfy the dynamics

$$\mathrm{d}\,X_t \quad = \quad \sigma_X X_t \,\mathrm{d}\,W_t^{(X)},$$

but in contrast to the model with a stationary log spread, discussed in the previous sections, the 2GBM model assumes that the illiquid asset price process is also a GBM with dynamics

$$\mathrm{d}\,I_t \quad = \quad \sigma_I I_t (\rho_{IX}\,\mathrm{d}\,W_t^{(X)} + \bar{\rho}_{IX}\,\mathrm{d}\,W_t^{\perp}).$$

In this model the variance-minimizing hedging strategy for European options with payoff function h is known to have the simple form

$$\zeta_t \quad = \quad \rho_{IX} \frac{\sigma_I I_t}{\sigma_X X_t} \psi_y(t, I_t), \tag{26}$$

where $\psi(t,y) = e^{-r(T-t)}\mathbb{E}\left(h(I_T^{t,y})\right)$. For a derivation of equation (26), we refer to Hulley and McWalter (2008) (see also Ankirchner and Heyne (2012) for a derivation in a slightly more general setting using backward stochastic differential equations (BSDEs)).

The optimal cross-hedge within the 2GBM model (26) is essentially determined by the cross-correlation. If an airline company used a 2GBM model estimated with daily data to hedge kerosene short positions, then it would considerably underhedge its kerosene risk, thus facing unnecessarily high variations in costs. But, by how much does the hedge error increase if we use the wrong model? We next quantify the risk by calculating the hedge error when using the optimal strategy ζ of the 2GBM model while the log prices are cointegrated and evolve according to the dynamics of our model.

We restrict our analysis to linear positions of the form cI_T. As before, we denote the realized hedge error by

$$C_T \quad = \quad cI_T - e^{rT}\left(v + \int_0^T e^{-rs}\zeta_s \,\mathrm{d}\,X_s\right).$$

The following proposition provides the hedge error variance for strategy (26) under our model with cointegration and under the 2GBM model.

Proposition 1. Hedging the linear position cI_T with the strategy ζ entails a hedge error in the cointegration model with variance

$$\begin{aligned} \mathbb{V}\,(C_T) \quad = \quad c^2 \Bigg\{ & A(2,2,X_0,S_0,T) - A^2(1,1,X_0,S_0,T) \\ & - 2\int_0^T \rho_{IX}\sigma_I \left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right) B_t \,\mathrm{d}\,t \\ & + \int_0^T \rho_{IX}^2 \sigma_I^2 A(2,2,X_0,S_0,t)\,\mathrm{d}\,t \Bigg\}, \end{aligned}$$

where B_t is given by

$$B_t = X_0^2 \exp\left(\sigma_X^2 t - \frac{1}{\kappa}\sigma_S\sigma_X\rho\left(3 - 2e^{-\kappa T} - 2e^{-\kappa t} + e^{-\kappa(T-t)}\right) - S_0\left(e^{-\kappa T} + e^{-\kappa t}\right)\right)$$
$$\times \exp\left(\frac{1}{4\kappa}\sigma_S^2\left(2 - e^{-2\kappa T} - e^{-2\kappa t} + 2e^{-\kappa(T-t)} - 2e^{-\kappa(T+t)}\right) - m(2 - e^{-\kappa T} - e^{-\kappa t})\right).$$

Under the correct model, the 2GBM model, the minimal variance of the realized hedge error is given by

$$(27) \qquad \mathbb{V}\left(C_T\right) = c^2 y^2\left(1 - \rho_{IX}^2\right)\left(e^{\sigma_I^2 T} - 1\right).$$

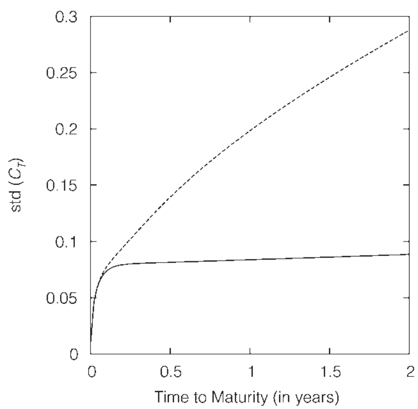
Proposition 1 allows us to analyze the ignorance of a long-term relationship with respect to the variance of the hedge error. Of course the variance of the optimal strategy ξ^* given by equation (13) is less than the standard error using the strategy ζ from the 2GBM model given by equation (26). Graphs A and B of Figure 4 compare the performance of the 2 strategies.[3] When following the strategy ζ, the risk position is underhedged, yielding the hedge error (dashed line) to grow continually with time to maturity. The hedge error does not flatten as strongly as when following strategy ξ^*, whose corresponding hedge error is depicted by the solid line. For very short maturities, the mean reversion has little

FIGURE 4

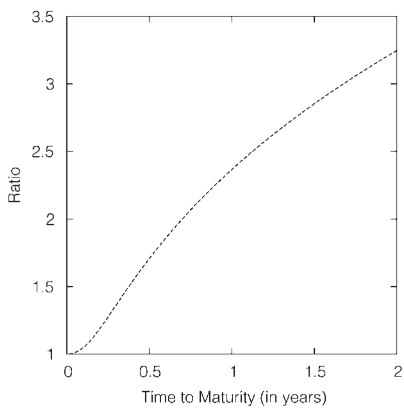
Suboptimal Hedging Strategies

Graph A of Figure 4 shows the standard deviation of the hedge error under cointegration using the optimal strategy (solid line) and the strategy from the 2GBM model (dashed line). Graph C shows the standard deviation under the 2GBM model using the optimal strategy (solid line) and the strategy from the cointegrated model (dashed lines) for $\kappa \in \{2.2450, 0.8365, 0.0927\}$, from top to bottom, respectively. Graphs B and D depict the ratio of the standard deviation of these strategies, respectively.

Graph A



Graph B

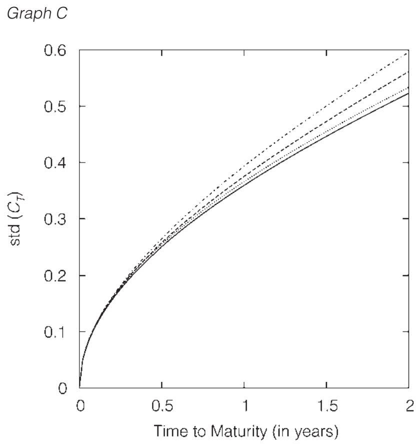


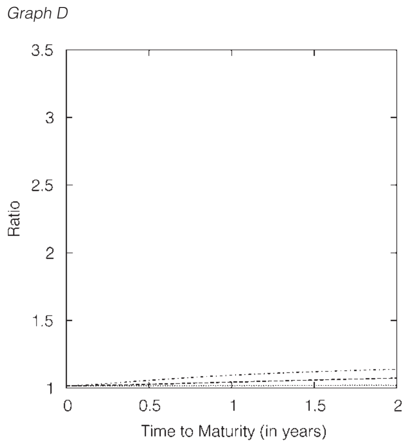
(continued on next page)

[3]Note that we use the estimated parameters obtained for the Aug. 2009 crude oil futures contract (see Table A1 in Appendix A) and that the correlation ρ_{IX} can be expressed in terms of the structural parameters of our model (see equation (4)).

FIGURE 4 (continued)

Suboptimal Hedging Strategies





time to develop, and hence the hedge error entailed by ζ is similar to the hedge error of ξ^*. For long maturities, however, ζ is considerably outperformed by ξ^*, leading, for example, to a more than 3 times higher error standard deviation over a 2-year hedging period.

Since by definition there is no strategy with a smaller variance than the variance-optimal strategy, the proportion of a 3 times larger value is strongly convincing. To fairly compare our model with the 2GBM model, we also consider the inverse case, that is, we study the hedge error of the optimal strategy from the model with the stationary log spread, ξ^*, when there is no cointegration. To this end, we have to derive the resulting hedge error in the 2GBM model, which is given in the next proposition.

Proposition 2. Hedging the linear position cI_T with the strategy ξ^*, see formula (13), entails a hedge error in the 2GBM model with variance

$$(28)\quad \mathbb{V}(C_T) = c^2 \Big\{ B(0,2,T) - B^2(0,1,T) - 2\int_0^T \sigma_I \rho \left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right] \times A(1,1,1,0,T-t)\sigma_X B\Big(1-e^{-\kappa(T-t)}, 1+e^{-\kappa(T-t)}, t\Big)\,\mathrm{d}t + \int_0^T \left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right]^2 A^2(1,1,1,0,T-t) \times \sigma_X^2 B\left(2-2e^{-\kappa(T-t)}, 2e^{-\kappa(T-t)}, t\right)\mathrm{d}t \Big\},$$

where $B(a,b,t)$ is given by

$$(29)\quad B(a,b,t) = \mathbb{E}\left(X_t^a I_t^b\right) = X_0^a I_0^b \exp\left(\frac{1}{2}t\left[\sigma_X^2(a^2-a) + 2ab\sigma_X\sigma_I\rho + \sigma_I^2(b^2-b)\right]\right).$$

Note that now not all parameters are identified. This is in contrast to the previous scenario, where we investigated the impact of ignoring a long-term relationship. As the log spread is not stationary in the 2GBM model, the parameter κ is implicitly set to 0. However, to provide a realistic comparison, we estimate the implied distribution of $\hat{\kappa}$ in the following way: We simulate 10,000 sample paths from the 2GBM model with $T = 885$ observations. Based on these time series, we estimate our model leading to the distribution of $\hat{\kappa}$. For the graphical illustration, we use the corresponding 10%, 50%, and 90% quantiles leading to 0.0927, 0.8365, and 2.2450, respectively. The dashed lines in Graph C of Figure 4 show the hedge error standard deviation for these different mean reversion speeds, when the real prices behave as in the 2GBM model but the risk is hedged according to the cointegration model optimal strategy ξ^*. As a benchmark, the graph also depicts the genuine minimal error standard deviation (solid line) implied by the optimal strategy ζ. The smaller the mean reversion speed, the smaller the hedge error. Moreover, the hedge error converges to the minimal hedge error as the mean reversion speed converges to 0, showing that the cointegration model embeds the 2GBM model.

In many real-world applications, it may not be obvious that there is a long-term relationship between the hedging instrument and the risk to be hedged. Comparing Graphs A and C of Figure 4, we conclude that in ambiguous situations it is nevertheless better to use a model allowing for a stationary spread rather than using a simpler model that does not. The error implied by mistakenly assuming a stationary log spread is significantly smaller than the error made by mistakenly assuming that the hedging instrument is not cointegrated with the risk.

B. The Costs of Using a Static Hedge

In practice, linear positions are often only statically hedged, even though the variance-minimizing hedge is not constant. In the numerical example below, we compare the standard deviations of static and dynamic hedges of a linear position, and address the question by how much the *dynamic* variance-minimizing strategy outperforms the static one.

For that purpose we need to derive the optimal static hedge position $a \in \mathbb{R}$ that minimizes the variance

$$C_T(a) \quad = \quad cI_T - e^{rT}\left(v + \int_0^T e^{-rt} a \,\mathrm{d}X_t\right).$$

With this at hand, we can calculate the minimal error standard deviation that can be achieved by hedging statically with futures.

Proposition 3. The optimal static hedging position $\check{a}$ in futures contracts that minimizes the variance of the hedge error is given by

$$(30) \quad \check{a} \quad = \quad \frac{\mathrm{Cov}\left(cI_T, e^{rT}\int_0^T e^{-rt}\,\mathrm{d}X_t\right)}{\mathbb{V}\left(e^{rT}\int_0^T e^{-rt}\,\mathrm{d}X_t\right)} \quad = \quad ce^{-rT}\frac{\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right)}{\sigma_X^2\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right)}$$

with corresponding variance

$$(31)\quad \mathbb{V}\left(C_T(\check{a})\right) = \mathbb{E}\left(C_T^2(\check{a})\right) = c^2\mathbb{V}\left(I_T\right) - \frac{c^2\left[\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right)\right]^2}{\sigma_X^2\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right)}.$$

The expectation in the denominator is given by

$$\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right) = \begin{cases} \frac{X_0^2}{\sigma_X^2-2r}\left(e^{(\sigma_X^2-2r)T}-1\right), & \text{if } \sigma_X^2\neq 2r, \\ X_0^2T, & \text{if } \sigma_X^2 = 2r. \end{cases}$$

For the expectation in the numerator we have, assuming $\sigma_X^2 > r$,

$$(32)\quad \mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right) = \begin{cases} X_0^2e^{\lambda(T)}\frac{\sigma_X^2}{\sigma_X^2-r}\left(e^{(\sigma_X^2-r)T}-1\right), & \text{if } \rho=0, \\ X_0^2e^{\lambda(T)+\frac{\rho\sigma_X\sigma_S}{\kappa}e^{-\kappa T}}(\Lambda_1(T)-\Lambda_2(T)), & \text{if } \rho\neq 0, \end{cases}$$

with

$$\Lambda_1(T) = \frac{\sigma_X^2 e^{(\sigma_X^2-r)T}\left(|\rho|\sigma_X\sigma_S\right)^{-\frac{\sigma_X^2-r}{\kappa}}}{\kappa^{1-\frac{\sigma_X^2-r}{\kappa}}} \times\left(\gamma\left(\frac{\sigma_X^2-r}{\kappa},\frac{1}{\kappa}|\rho|\sigma_X\sigma_S\right)-\gamma\left(\frac{\sigma_X^2-r}{\kappa},\frac{1}{\kappa}|\rho|\sigma_X\sigma_Se^{-\kappa T}\right)\right),$$

$$\Lambda_2(T) = e^{(\sigma_X^2-r)T}\left(\frac{|\rho|\sigma_X\sigma_S}{\kappa}\right)^{-\frac{\sigma_X^2-r}{\kappa}} \times\left(\gamma\left(\frac{\sigma_X^2-r}{\kappa}+1,\frac{1}{\kappa}|\rho|\sigma_X\sigma_S\right)-\gamma\left(\frac{\sigma_X^2-r}{\kappa}+1,\frac{1}{\kappa}|\rho|\sigma_X\sigma_Se^{-\kappa T}\right)\right),$$

$$\lambda(T) = -S_0e^{-\kappa T}-m(1-e^{-\kappa T})+\frac{\sigma_S^2}{4\kappa}\left(1-e^{-2\kappa T}\right)-\frac{\rho\sigma_S\sigma_X}{\kappa}\left(1-e^{-\kappa T}\right).$$

Here $\gamma(s,x)=\int_0^x y^{s-1}e^{-y}\,\mathrm{d}y$ denotes the incomplete gamma function. Furthermore, we have

$$\mathbb{V}\left(I_T\right) = A(2,2,X_0,S_0,T)-A^2(1,1,X_0,S_0,T),$$

where A is as in equation (20).

Remark 1. Note that the assumption $\sigma_X^2 > r$ in Proposition 3 is only needed in order to get a closed expression with respect to the incomplete gamma function. In case $\sigma_X^2\leq r$, the defining integral of the incomplete gamma function explodes around 0. In this case equation (32) still holds if we replace γ with the *upper incomplete gamma function* $\gamma(s,x)=-\int_x^\infty y^{s-1}e^{-y}\,\mathrm{d}y$. In any case, regardless of $\sigma_X^2>r$, formula (32) holds when we replace $\Lambda_1(T)-\Lambda_2(T)$ with $\int_0^T e^{(\sigma_X^2-r)t}(\sigma_X^2-\rho\sigma_X\sigma_Se^{-\kappa(T-t)})e^{-\rho\sigma_X\sigma_Se^{-\kappa(T-t)}/\kappa}\,\mathrm{d}t$.

Proposition 4. The expressions for the optimal static hedge $\check{a}$ (30) and for the corresponding variance (31) from Proposition 3 hold also in the 2GBM case. For the involved expectations we have

$$\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}\,t\right) = \begin{cases} \frac{X_0^2}{\sigma_X^2-2r}\left(e^{(\sigma_X^2-2r)T}-1\right), & \text{if } \sigma_X^2 \neq 2r, \\ X_0^2 T, & \text{if } \sigma_X^2 = 2r, \end{cases}$$

and

$$\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}\,X_t\right) = \begin{cases} X_0 I_0 \frac{\rho_{IX}\sigma_X\sigma_I}{\rho_{IX}\sigma_X\sigma_I - r}\left(e^{(\rho_{IX}\sigma_X\sigma_I - r)T}-1\right), & \text{if } \rho_{IX}\sigma_X\sigma_I \neq r, \\ X_0 I_0 \rho_{IX}\sigma_X\sigma_I T, & \text{if } \rho_{IX}\sigma_X\sigma_I = r. \end{cases}$$

Furthermore,

$$\mathbb{V}(I_T) = I_0^2\left(e^{\sigma_I^2 T}-1\right).$$

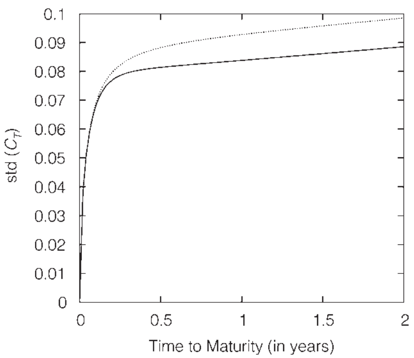
Using the expressions for the standard deviation of the hedge error from Theorem 2 and Proposition 3, we can compare the risks entailed by both strategies. Graph A of Figure 5 depicts the standard deviation of the static and dynamic variance-minimizing hedge against time to maturity. Graph B shows the increase of the standard deviation if one confines with the static hedge. The increase in the variability by more than 10% for positions hedged over a period of 1 year indicates that the hedge should be dynamically adjusted. The figure is again based on the estimated parameter values obtained for the Aug. 2009 crude oil futures contract (see Table A1 in Appendix A).

FIGURE 5

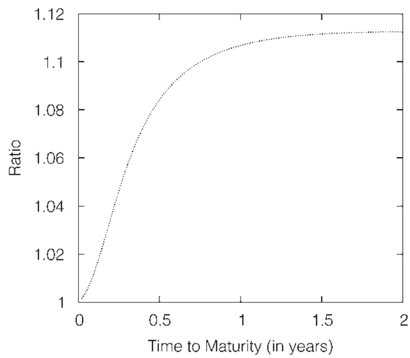
Static versus Dynamic Hedging

Graph A of Figure 5 shows the standard deviation of the static (dashed line) versus the dynamic hedge error (solid line). Graph B depicts the ratio. As we have to assume a constant interest rate for the computation of the variance of the static hedge error, we fix it at $r = 0.02$.

Graph A



Graph B



VI. Including Directional Views and Stochastic Volatility

In this section we extend the model introduced in Section II by allowing for stochastic volatility of the futures and the log spread. We assume that the volatility of both processes are proportional to a Cox-Ingersoll-Ross (1985) process. The futures dynamics thus coincide with the dynamics of the risky asset in the Heston (1993) model.

Let (W^1, W^2, W^3) be a 3-dimensional BM and suppose that the futures price process X and its volatility $\nu = (\nu_t)_{t\geq 0}$ evolve according to the SDE

$$\text{(33)} \qquad \mathrm{d}X_t = \mu(t,\nu_t)X_t\,\mathrm{d}t + \sqrt{\nu_t}X_t\,\mathrm{d}W_t^1,$$

$$\text{(34)} \qquad \mathrm{d}\nu_t = \beta(\vartheta - \nu_t)\,\mathrm{d}t + \sigma_\nu\sqrt{\nu_t}\big(\rho_1\,\mathrm{d}W_t^1 + \bar{\rho}_1\,\mathrm{d}W_t^2\big),$$

where $\rho_1 \in [-1,1]$, $\bar{\rho}_1 = \sqrt{1-\rho_1^2}$, $\beta, \vartheta, \sigma_\nu > 0$, and $\mu : \mathbb{R}_+^2 \to \mathbb{R}$ is measurable. As before, let $S_t = \log(X_t) - \log(I_t)$. Assume that the log spread's volatility is proportional to ν, and that S solves the mean-reverting SDE

$$\text{(35)} \qquad \begin{aligned} \mathrm{d}S_t &= \kappa(m - S_t)\,\mathrm{d}t + \sigma_S\sqrt{\nu_t}\left(\rho\,\mathrm{d}W_t^1 + \bar{\rho}\eta\,\mathrm{d}W_t^2 + \bar{\rho}\bar{\eta}\,\mathrm{d}W_t^3\right),\\ S_0 &= s. \end{aligned}$$

Since we include a directional view in the dynamics of the futures price, we cannot directly invoke the method used in Section III for the derivation of the variance-optimal hedge. When the trading instruments are assumed to be trended, and hence are not martingales, then it is very difficult to determine variance-optimal hedging strategy. There are, however, other quadratic optimality criteria that considerably simplify the calculation of hedging strategies. A very intriguing type of hedging strategy is the so-called *locally risk-minimizing hedging strategies*. These are variance-optimal strategies with respect to a particular martingale measure, usually referred to as the *minimal martingale measure*. For an overview on quadratic hedging approaches, we refer to Schweizer (2008).

In our extended model the minimal martingale measure $\widehat{Q}$ is given by

$$\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}P} = \exp\left(-\int_0^T \omega(t,\nu)\,\mathrm{d}W_t^1 - \frac{1}{2}\int_0^T \omega^2(t,\nu)\,\mathrm{d}t\right),$$

where $\omega(t,\nu_t) = \mu(t,\nu_t)/\sqrt{\nu_t}$ is the market price of risk.[4]

One can proceed as in Section III for the derivation of the local risk-minimizing hedge (i.e., the variance-optimal hedge relative to $\widehat{Q}$). The *value function* of $h(I_T)$ is defined by

$$\text{(36)} \qquad \psi(t,x,v,s) = e^{-r(T-t)}\mathbb{E}^{\widehat{Q}}\left(h\left(X_T^{t,x,v}e^{-S_T^{t,s}}\right)\right).$$

With the same assumptions on h, one can show that the local risk-minimizing hedge is given by

$$\text{(37)} \qquad \widehat{\xi}_t = \psi_x(t,X_t,\nu_t,S_t)\left[1 - \sigma_S\rho e^{-\kappa(T-t)}\right] + \frac{\sigma_\nu\rho_1}{X_t}\psi_v(t,X_t,\nu_t,S_t).$$

[4]A sufficient condition for this to be a proper measure of change is the following growth condition on ω. For $A, B \geq 0$ and $\delta \in [0, ½]$, we assume $|\omega(t,x)| \leq A + Bx^\delta$, $x \geq 0$.

Observe that the local risk-minimizing hedge is now a weighted sum of the delta and vega of the risk position's expectation under the minimal martingale measure $\widehat{Q}$. Clearly, the term involving the vega of the position appears due to the additional nontradable risk induced by the stochastic volatility, which also needs to be cross-hedged.

A similar analysis as in the previous sections (e.g., estimation of model parameters, derivation of hedge errors, and their respective standard deviations) is somewhat more involved. However, one can profit from the affine model structure and express the value function and its gradient in terms of generalized Ricatti equations. Fourier inversion methods then yield semi-explicit formulas for optimal strategies, which are amenable to swift simulation analysis.

VII. Conclusion and Outlook

Hedging is essential for controlling and managing risk, and it is an important area of research. In this paper we show that a long-term relationship between the risk and the hedging instrument has important implications for the optimal hedging strategy and, thus, also for the hedge error. In particular, we propose a model that explicitly takes into account such a long-term relationship. We derive the variance-optimal cross-hedge strategy and provide the variance of the hedge error in terms of the model's parameters. We demonstrate the practical relevance of incorporating the long-term relationship through an empirical example, where we find a long-term relationship between most crude oil futures contracts and the spot kerosene price. Interestingly, the model is also consistent with the commonly observed behavior of commodity traders, who use for cross hedges a hedge ratio of 100% instead of a hedge ratio dampened by the cross-correlation between the risk and the hedging instrument, which is implied by models with only correlated BMs. Furthermore, we show that even for cases where the decision concerning the stationarity of the log spread is not obvious, it is better to allow for a long-term relationship rather than to neglect it.

The model can be extended toward several directions to provide more realistic dynamics for asset prices. The consideration of jumps in the price process seems to be an especially interesting extension. However, several specifications are plausible, and a careful empirical investigation is needed. On an ad hoc basis it is, for example, not clear whether the price processes jump together and how the jump sizes are related. These aspects will have significant impact on the properties of the hedge error, and we plan to investigate these questions in more detail in future research.

Appendix A. Empirical Results

Table A1 shows the number of observations (column 2), the p-value of the augmented Dickey-Fuller (ADF) test (column 3), and the estimation result for different crude oil futures contracts and their corresponding spot kerosene prices.

TABLE A1

Estimates

Column 1 of Table A1 presents the maturity date of the contract, column 2 gives the number of observations, and column 3 reports the *p*-value of the augmented Dickey-Fuller (ADF) test for the null of nonstationarity. The * (**, ***) indicates the rejection of nonstationarity at the 10% (5%, 1%) level. The remaining columns show the parameter estimates, with the corresponding asymptotic standard errors given in parentheses.

Contract	No. of Obs.	ADF	μ	σ_X	σ_S	κ	m	ρ
2010-10	1,147	0.0888*	0.0751 (0.1302)	0.2768 (0.0059)	0.2871 (0.0060)	2.5548 (1.0976)	−0.1661 (0.0534)	0.3401 (0.0260)
2010-09	1,147	0.0750*	0.0742 (0.1324)	0.2795 (0.0059)	0.2877 (0.0060)	2.7131 (1.1078)	−0.1687 (0.0515)	0.3475 (0.0260)
2010-08	1,145	0.0631*	0.0763 (0.1361)	0.2823 (0.0059)	0.2886 (0.0061)	2.8890 (1.1728)	−0.1693 (0.0492)	0.3550 (0.0258)
2010-07	1,123	0.0555*	0.0778 (0.1348)	0.2855 (0.0061)	0.2908 (0.0062)	3.0632 (1.1704)	−0.1731 (0.0473)	0.3646 (0.0259)
2010-06	1,375	0.0310**	0.1703 (0.1209)	0.2758 (0.0053)	0.2993 (0.0058)	2.5899 (1.1150)	−0.1946 (0.0493)	0.3404 (0.0241)
2010-05	1,080	0.0436**	0.1114 (0.1405)	0.2901 (0.0064)	0.2942 (0.0064)	3.5046 (1.2913)	−0.1752 (0.0410)	0.3771 (0.0262)
2010-04	1,058	0.0326**	0.0906 (0.1453)	0.2954 (0.0065)	0.2970 (0.0065)	3.8040 (1.3324)	−0.1808 (0.0390)	0.3867 (0.0262)
2010-03	1,035	0.0250**	0.0737 (0.1494)	0.3002 (0.0066)	0.3006 (0.0067)	4.1174 (1.4241)	−0.1836 (0.0371)	0.3990 (0.0262)
2010-02	1,015	0.0193**	0.0906 (0.1518)	0.3035 (0.0068)	0.3020 (0.0069)	4.5175 (1.6076)	−0.1866 (0.0332)	0.4043 (0.0263)
2010-01	994	0.0126**	0.0789 (0.1551)	0.3098 (0.0070)	0.3045 (0.0069)	4.9939 (1.3783)	−0.1916 (0.0312)	0.4178 (0.0262)
2009-12	1,245	0.0097***	0.1852 (0.1345)	0.2979 (0.0060)	0.3114 (0.0064)	3.8232 (1.2755)	−0.2137 (0.0374)	0.3892 (0.0241)
2009-11	950	0.0046***	0.0860 (0.1564)	0.3197 (0.0073)	0.3108 (0.0072)	6.3841 (2.0112)	−0.1994 (0.0238)	0.4415 (0.0261)
2009-10	928	0.0029***	0.0603 (0.1687)	0.3229 (0.0075)	0.3135 (0.0074)	7.0739 (1.9253)	−0.2032 (0.0231)	0.4532 (0.0261)
2009-09	906	0.0014***	0.0819 (0.1738)	0.3275 (0.0077)	0.3178 (0.0076)	8.0814 (2.1057)	−0.2061 (0.0208)	0.4677 (0.0260)
2009-08	885	$\le$ 0.0010***	0.0430 (0.1807)	0.3321 (0.0079)	0.3223 (0.0078)	9.5437 (2.2822)	−0.2120 (0.0178)	0.4806 (0.0259)
2009-07	862	$\le$ 0.0010***	0.0743 (0.1828)	0.3388 (0.0082)	0.3275 (0.0080)	11.5957 (2.5298)	−0.2166 (0.0153)	0.5011 (0.0256)
2009-06	1,114	$\le$ 0.0010***	0.1416 (0.1498)	0.3191 (0.0068)	0.3298 (0.0072)	6.6833 (1.7074)	−0.2404 (0.0233)	0.4579 (0.0239)
2009-05	819	$\le$ 0.0010***	−0.0114 (0.1953)	0.3516 (0.0086)	0.3407 (0.0086)	15.7614 (2.9919)	−0.2274 (0.0111)	0.5364 (0.0250)
2009-04	797	$\le$ 0.0010***	−0.0655 (0.1973)	0.3513 (0.0088)	0.3429 (0.0087)	15.9689 (3.0523)	−0.2307 (0.0103)	0.5594 (0.0245)
2009-03	775	$\le$ 0.0010***	−0.0672 (0.1953)	0.3436 (0.0086)	0.3351 (0.0086)	15.0723 (2.8192)	−0.2362 (0.0135)	0.5631 (0.0244)
2009-02	755	$\le$ 0.0010***	−0.0694 (0.2005)	0.3469 (0.0089)	0.3257 (0.0085)	12.1794 (2.4786)	−0.2411 (0.0146)	0.5815 (0.0242)
2009-01	733	0.0021***	−0.0802 (0.1966)	0.3306 (0.0086)	0.3053 (0.0081)	10.9528 (2.4631)	−0.2398 (0.0160)	0.5742 (0.0247)

Appendix B. Proofs

Proof of Lemma 1. By distinguishing the cases $\rho \geq 0$ and $\rho < 0$, one can show that $\partial \rho_{IX} / \partial \sigma_S \leq 0$, and that the partial derivative is strictly smaller than 0 if $\sigma_S > 0$. Thus, ρ_{IX} is strictly decreasing in σ_S. From the definition of ρ_{IX} we have $\rho_{IX}^2 = (\sigma_X - \rho\sigma_S)^2 / \left(\sigma_X^2 - 2\rho\sigma_S\sigma_X + \sigma_S^2\right)$, which leads to the quadratic equation in σ_S

$$\text{(B-1)} \qquad \left(\rho_{IX}^2 - \rho^2\right) \sigma_S^2 + 2\rho\sigma_X \left(1 - \rho_{IX}^2\right) \sigma_S - \sigma_X^2 \left(1 - \rho_{IX}^2\right) = 0.$$

If $\rho \neq \rho_{IX}$, then equation (B-1) has 2 solutions. First,

$$\sigma_S = \sigma_X \sqrt{1-\rho_{IX}^2} \left(\frac{-\rho\sqrt{1-\rho_{IX}^2} \pm \rho_{IX}\sqrt{1-\rho^2}}{\rho_{IX}^2 - \rho^2} \right).$$

Since $\rho_{IX}^2 - \rho^2 = (\rho_{IX}\sqrt{1-\rho^2} + \rho\sqrt{1-\rho_{IX}^2})(\rho_{IX}\sqrt{1-\rho^2} - \rho\sqrt{1-\rho_{IX}^2})$, this further yields

$$\sigma_S = \sigma_X \sqrt{1-\rho_{IX}^2} \left(\frac{1}{\rho\sqrt{1-\rho_{IX}^2} \pm \rho_{IX}\sqrt{1-\rho^2}} \right).$$

If $\rho_{IX} > \rho$, then only one of the roots guarantees that $\sigma_S \geq 0$, and we obtain equation (5). If $\rho_{IX} = \rho$, then equation (B-1) has a unique solution, given by $\sigma_S = \sigma_X/(2\rho)$. The inverse function of formula (4) is continuous on $\rho^{-1}(\mathbb{R}_+)$. Therefore, equation (5) must also hold true for $\rho_{IX} < \rho$. □

Proof of Lemma 2. Since $S_t^{0,s} = se^{-\kappa t} + m(1-e^{-\kappa t}) + \int_0^t e^{-\kappa(t-u)}\sigma_S(\rho\, \mathrm{d}W_u + \bar{\rho}\, \mathrm{d}W_u^{\perp})$, we get

$$\begin{aligned} & e^{-aS_t^{0,s}}(X_t^{0,x})^b \\ = \;& x^b \exp\left(-\frac{b}{2}\sigma_X^2 t - ase^{-\kappa t} - am\left(1-e^{-\kappa t}\right) + \int_0^t \left(b\sigma_X - \rho a\sigma_S e^{-\kappa(t-u)}\right) \mathrm{d}W_u\right) \\ & \times \exp\left(-\int_0^t (a\bar{\rho}\sigma_S e^{-\kappa(t-u)})\, \mathrm{d}W_u^{\perp}\right). \end{aligned}$$

We calculate the variances of the independent normal variables given by the integrals in the last 2 factors. We have

$$\begin{aligned} \int_0^t \left(b\sigma_X - \rho a\sigma_S e^{-\kappa(t-u)}\right)^2 \mathrm{d}t \;=\;& b^2\sigma_X^2 t - 2ab\rho\sigma_X\sigma_S\frac{1}{\kappa}\left(1-e^{-\kappa t}\right) \\ & + a^2\rho^2\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right) \end{aligned}$$

and

$$\int_0^t \left(a\bar{\rho}\sigma_S e^{-\kappa(t-u)}\right)^2 \mathrm{d}t = a^2\bar{\rho}^2\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right).$$

We use this to derive

$$\begin{aligned} \mathbb{E}\left(e^{-aS_t^{0,s}}(X_t^{0,x})^b\right) =\;& x^b \exp\left(-\frac{b}{2}\sigma_X^2 t - ase^{-\kappa t} - am\left(1-e^{-\kappa t}\right)\right) \\ & \times \mathbb{E}\left(\exp\left(\int_0^t \left(b\sigma_X - \rho a\sigma_S e^{-\kappa(t-u)}\right) \mathrm{d}W_u\right)\right) \\ & \times \mathbb{E}\left(\exp\left(-\int_0^t \left(a\bar{\rho}\sigma_S e^{-\kappa(t-u)}\right) \mathrm{d}W_u^{\perp}\right)\right) \\ =\;& x^b \exp\left(-\frac{b}{2}\sigma_X^2 t - ase^{-\kappa t} - am\left(1-e^{-\kappa t}\right)\right) \\ & \times \exp\left[\frac{1}{2}\left(b^2\sigma_X^2 t - 2ab\rho\sigma_X\sigma_S\frac{1}{\kappa}\left(1-e^{-\kappa t}\right)\right.\right. \\ & \left.\left. + a^2\rho^2\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)\right)\right] \\ & \times \exp\left[\frac{1}{2}\left(a^2\bar{\rho}^2\sigma_S^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)\right)\right], \end{aligned}$$

from which the result follows. □

Proof of Theorem 2. Recall that from equation (21) we have

$$\text{(B-2)} \qquad \mathbb{V}\left(C_T(\xi^*, v)\right) = c^2 \bar{\rho}^2 \sigma_S^2 \int_0^T e^{-2k(T-t)} \mathbb{E}\left(X_t^2 A^2(1,1,1,S_t,T-t)\right) \mathrm{d}t.$$

By the definition of A, we have

$$\begin{aligned} \text{(B-3)}\ & \mathbb{E}\left(X_t^2 A^2(1,1,1,S_t,T-t)\right) \\ &= \exp\left(-2\left(m + \rho\sigma_X\sigma_S \frac{1}{\kappa}\right)\left(1 - e^{-\kappa(T-t)}\right) + \sigma_S^2 \frac{1}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right) \\ &\quad \times \mathbb{E}\left(X_t^2 \exp\left(-2S_t e^{-\kappa(T-t)}\right)\right), \end{aligned}$$

and again using the definition of A we get

$$\begin{aligned} \mathbb{E}\left(X_t^2 \exp\left(-2S_t e^{-\kappa(T-t)}\right)\right) &= A\left(2e^{-\kappa(T-t)}, 2, x, s, t\right) \\ &= x^2 \exp\left[\sigma_X^2 t - 2se^{-\kappa T} - 2\left(m + 2\rho\sigma_X\sigma_S \frac{1}{\kappa}\right)\right. \\ &\qquad \left. \times \left(e^{-\kappa(T-t)} - e^{-\kappa T}\right)\right] \\ &\quad \times \exp\left[2\sigma_S^2 \frac{1}{2\kappa}\left(e^{-2\kappa(T-t)} - e^{-2\kappa T}\right)\right]. \end{aligned}$$

Combining the last equation with equation (B-3), we further obtain

$$\begin{aligned} \text{(B-4)} \qquad & \mathbb{E}\left(X_t^2 A^2(1,1,1,S_t,T-t)\right) \\ &= x^2 \exp\left(\sigma_X^2 t - 2se^{-\kappa T} - 2\rho\sigma_X\sigma_S \frac{1}{\kappa}\left(1 + e^{-\kappa(T-t)} - 2e^{-\kappa T}\right)\right) \\ &\quad \times \exp\left(-2m\left(1 - e^{-\kappa T}\right) + \sigma_S^2 \frac{1}{2\kappa}\left(1 - 2e^{-\kappa T} + e^{-2\kappa(T-t)}\right)\right). \end{aligned}$$

The previous calculations yield, by combination of equations (B-2) and (B-4), expression (22). □

Proof of Proposition 1. Since I is a GBM in the 2GBM model, the value function ψ associated with the linear position $h(x) = cx$ is given by $\psi(t,y) = e^{-r(T-t)}cy$. Therefore, $\psi_y(t,y) = e^{-r(T-t)}c$, and the realized error variance in Model 1, following strategy $\zeta_t = \rho_{IX}\sigma_I I_t e^{-r(T-t)} c / (\sigma_X X_t)$, satisfies

$$C_T = cI_T - e^{rT}\left(v + \int_0^T e^{-rT} c\rho_{IX}\sigma_I I_t \,\mathrm{d}W_t^{(X)}\right),$$

and consequently we have

$$\mathbb{V}(C_T) = c^2 \mathbb{V}(I_T) - 2c^2 \operatorname{Cov}\left(I_T, \int_0^T \rho_{IX}\sigma_I I_t \,\mathrm{d}W_t^{(X)}\right) + c^2 \mathbb{E}\left(\int_0^T \rho_{IX}^2 \sigma_I^2 I_t^2 \,\mathrm{d}t\right).$$

Note that $\mathbb{V}(I_T) = A(2,2,X_0,S_0,T) - A^2(1,1,X_0,S_0,T)$, and observe further that

$$\mathbb{E}\left(\int_0^T \rho_{IX}^2 \sigma_I^2 I_t^2 \,\mathrm{d}t\right) = \int_0^T \rho_{IX}^2 \sigma_I^2 A(2,2,X_0,S_0,t)\,\mathrm{d}t.$$

It remains to calculate the covariance above. To that effect, we recall the decomposition (B-5) of I_T from the proof of Proposition 3 and borrow the respective notation to write $\int_0^T \rho_{IX}\sigma_I I_t \,\mathrm{d}\, W_t^{(X)} = \int_0^T \rho_{IX}\sigma_I I_t d\tilde{W}_t^{(X)} + \int_0^T \rho_{IX}\sigma_I I_t(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)})\,\mathrm{d}\,t$. Consequently,

$$\begin{aligned}
\mathrm{Cov}\left(I_T, \int_0^T \rho_{IX}\sigma_I I_t \,\mathrm{d}\, W_t^{(X)}\right) &= X_0 e^{\lambda(T)}\mathbb{E}^Q\left(\int_0^T \rho_{IX}\sigma_I I_t\left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right)\mathrm{d}\,t\right)\\
&= X_0 e^{\lambda(T)}\int_0^T \rho_{IX}\sigma_I\left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right)\mathbb{E}^Q\left(I_t\right)\mathrm{d}\,t\\
&= \int_0^T \rho_{IX}\sigma_I\left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right)\mathbb{E}\left(I_T I_t\right)\mathrm{d}\,t.
\end{aligned}$$

In order to calculate $\mathbb{E}\left(I_T I_t\right)$, we proceed as in Lemma 2. We decompose $I_T I_t$ into

$$\begin{aligned}
I_T I_t &= X_0^2\exp\left(\int_0^T\left[\sigma_X(1+1_{u\le t}) - \sigma_S\rho\left(e^{-\kappa(T-u)} + e^{-\kappa(t-u)}1_{u\le t}\right)\right]\mathrm{d}\,W_u^X\right)\\
&\times\exp\left(-\int_0^T \sigma_S\bar{\rho}\left(e^{-\kappa(T-u)} + e^{-\kappa(t-u)}1_{u\le t}\right)\mathrm{d}\,W_u^{\perp}\right)\\
&\times\exp\left(-\frac{1}{2}\int_0^T \sigma_X^2(1+1_{u\le t})\,\mathrm{d}\,u\right)\exp\left(-S_0\left(e^{-\kappa T}+e^{-\kappa t}\right) - m(2-e^{-\kappa T}-e^{-\kappa t})\right).
\end{aligned}$$

The variances of the stochastic integrals are given by

$$\begin{aligned}
&\int_0^T \sigma_X^2(1+2_{u\le t}+1_{u\le t}) - 2\sigma_S\sigma_X\rho\left[e^{-\kappa(T-u)} + 1_{u\le t}\left(2e^{-\kappa(t-u)} + e^{-\kappa(T-u)}\right)\right]\\
&\quad+\sigma_S^2\rho^2\left(e^{-\kappa(T-u)} + e^{-\kappa(t-u)}1_{u\le t}\right)^2\mathrm{d}\,u
\end{aligned}$$

and

$$\int_0^T \sigma_S^2\bar{\rho}^2\left(e^{-\kappa(T-u)} + e^{-\kappa(t-u)}1_{u\le t}\right)^2\mathrm{d}\,u.$$

Hence, taking expectation yields

$$\begin{aligned}
\mathbb{E}\left(I_T I_t\right) &= X_0^2\exp\\
&\times\left(\frac{1}{2}\int_0^T\left[\sigma_X^2 2_{u\le t} - 2\sigma_S\sigma_X\rho\left[e^{-\kappa(T-u)} + 1_{u\le t}\left(2e^{-\kappa(t-u)} + e^{-\kappa(T-u)}\right)\right]\right]\mathrm{d}\,u\right)\\
&\times\exp\left(\frac{1}{2}\int_0^T \sigma_S^2\left(e^{-2\kappa(T-u)} + 2e^{-\kappa(T+t-2u)}1_{u\le t} + e^{-2\kappa(t-u)}1_{u\le t}\right)\mathrm{d}\,u\right)\\
&\times\exp\left(-S_0\left(e^{-\kappa T}+e^{-\kappa t}\right) - m(2-e^{-\kappa T}-e^{-\kappa t})\right).
\end{aligned}$$

The result follows by a simple calculation. □

Proof of Proposition 2. Recall from equation (13) that $\xi_t^* = \left[1 - \sigma_S\rho e^{-\kappa(T-t)}/\sigma_X\right]\psi_x(t, X_t, S_t)$, with $\psi(t,x,s) = e^{-r(T-t)}\mathbb{E}\left(h(X_T^{t,x}e^{-S_T^{t,s}})\right)$.

Hence, for $h(y) = cy$ we get $\psi_x(t,x,s) = ce^{-r(T-t)}\mathbb{E}\left(X_T^{t,1}e^{-S_T^{t,s}}\right) = A(1,1,1,s,T-t)$. Thus, the realized error variance in the 2GBM model, following the strategy above, satisfies

$$C_T = cI_T - e^{rT}\left(v + c\int_0^T e^{-rT}\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right] A(1,1,1,S_t,T-t)\,\mathrm{d}X_t\right).$$

Consequently, setting $v = ce^{-rT}\mathbb{E}(I_T)$, we have

$$\begin{aligned}\mathbb{V}(C_T) &= c^2\mathbb{V}(I_T) - 2c^2\,\mathrm{Cov}\left(I_T, \int_0^T\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right] A(1,1,1,S_t,T-t)\sigma_X X_t\,\mathrm{d}W_t^{(X)}\right)\\ &\quad + c^2\mathbb{E}\left(\int_0^T\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right]^2 A^2(1,1,1,S_t,T-t)\sigma_X^2 X_t^2\,\mathrm{d}t\right).\end{aligned}$$

It is straightforward to see that $B(a,b,t)$ defined as $B(a,b,t) = \mathbb{E}\left(X_t^a I_t^b\right)$, with X and I as in the 2GBM model, fulfills equation (29). Hence, $\mathbb{V}(I_T) = B(0,2,T) - B^2(0,1,T)$. Observe that we may, via Fubini's theorem, write the expectation in the last term in the variance of C_T above as

$$\textstyle\int_0^T\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right]^2 A^2(1,1,1,0,T-t)\sigma_X^2\mathbb{E}\left(\exp\left(-2S_t e^{-\kappa(T-t)}\right)X_t^2\right)\mathrm{d}t.$$

In order to further simplify, recall that $S_t = \log(X_t) - \log(I_t)$. Thus,

$$\begin{aligned}\mathbb{E}\left(\exp\left(-2S_t e^{-\kappa(T-t)}\right)X_t^2\right) &= \mathbb{E}\left(X_t^{2-2e^{-\kappa(T-t)}} I_t^{2e^{-\kappa(T-t)}}\right)\\ &= B\left(2 - 2e^{-\kappa(T-t)}, 2e^{-\kappa(T-t)}, t\right),\end{aligned}$$

which combined with the previous integral yields the last term in expression (28). In order to simplify the remaining term in the variance of C_T, we apply the standard trick of a change of measure, here with $\mathrm{d}Q = (I_T/I_0)\,\mathrm{d}P$, Fubini's Theorem, and reversing the measure change to obtain

$$\textstyle\int_0^T \sigma_I\rho\left[1 - \frac{\sigma_S}{\sigma_X}\rho e^{-\kappa(T-t)}\right] A(1,1,1,0,T-t)\sigma_X\mathbb{E}\left(I_T\exp\left(-S_t e^{-\kappa(T-t)}\right)X_t\right)\mathrm{d}t.$$

Finally, using $S_t = \log(X_t) - \log(I_t)$ and the independent increments of a BM, we get

$$\begin{aligned}\mathbb{E}\left(I_T\exp\left(-S_t e^{-\kappa(T-t)}\right)X_t\right) &= \mathbb{E}\left(I_T I_t^{e^{-\kappa(T-t)}} X_t^{1-e^{-\kappa(T-t)}}\right)\\ &= \mathbb{E}\Bigg(I_t^{1+e^{-\kappa(T-t)}} X_t^{1-e^{-\kappa(T-t)}}\\ &\qquad \times\exp\left(\int_t^T \sigma_I\,\mathrm{d}W_u^I - \frac{1}{2}\int_t^T \sigma_I^2\,\mathrm{d}u\right)\Bigg)\\ &= B\left(1 - e^{-\kappa(T-t)}, 1 + e^{-\kappa(T-t)}, t\right),\end{aligned}$$

which, together with the previous integral, yields the middle term in expression (28) and thus finishes the proof. □

Proof of Proposition 3. The variance of the hedge error does not depend on the initial capital v, and hence we may assume that $e^{rT}v = c\mathbb{E}(I_T)$. Holding the constant position a between 0 and T then entails the hedge error

$$C_T(a) = cI_T - \mathbb{E}(cI_T) - ae^{rT}\int_0^T e^{-rt}\,\mathrm{d}X_t,$$

and since X_t is a martingale, we have $\mathbb{E}(C_T(a)) = 0$. Hence, the variance of $C_T(a)$ is given by

$$\begin{aligned}\mathbb{E}\left(C_T^2(a)\right) &= c^2\mathbb{E}\left((I_T-\mathbb{E}(I_T))^2\right) - 2ace^{rT}\mathbb{E}\left((I_T-\mathbb{E}(I_T))\int_0^T e^{-rt}\,\mathrm{d}X_t\right)\\ &\quad + a^2e^{2rT}\mathbb{E}\left(\left(\int_0^T e^{-rt}\,\mathrm{d}X_t\right)^2\right)\\ &= c^2\mathbb{E}\left((I_T-\mathbb{E}(I_T))^2\right) - 2ace^{rT}\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right)\\ &\quad + a^2e^{2rT}\mathbb{E}\left(\int_0^T e^{-2rt}\sigma_X^2X_t^2\,\mathrm{d}t\right).\end{aligned}$$

The optimal $\check{a}$ that minimizes the variance of the hedge error is given by

$$\check{a} = ce^{-rT}\frac{\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right)}{\sigma_X^2\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right)}.$$

For the variance of the corresponding hedge error, we have

$$\begin{aligned}\mathbb{E}\left(C_T^2(\check{a})\right) &= \mathbb{V}(cI_T) - \frac{\mathrm{Cov}\left(cI_T, e^{rT}\int_0^T e^{-rt}\,\mathrm{d}X_t\right)^2}{\mathbb{V}\left(e^{rT}\int_0^T e^{-rt}\,\mathrm{d}X_t\right)}\\ &= c^2\mathbb{V}(I_T) - \frac{c^2\left[\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right)\right]^2}{\sigma_X^2\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right)}.\end{aligned}$$

The expectation in the denominator is given by

$$\begin{aligned}\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right) &= \int_0^T e^{-2rt}\underbrace{\mathbb{E}\left(X_t^2\right)}_{=X_0^2\exp(\sigma_X^2 t)}\,\mathrm{d}t\\ &= \begin{cases}\frac{X_0^2}{\sigma_X^2-2r}\left(e^{(\sigma_X^2-2r)T}-1\right), & \text{if } \sigma_X^2\neq 2r,\\ X_0^2T, & \text{if } \sigma_X^2 = 2r.\end{cases}\end{aligned}$$

The computations for the expectation in the numerator are somewhat more involved. Using the explicit expressions for S_T and X_T, we may decompose I_T into

(B-5) $$I_T = X_Te^{-S_T} = X_0e^{\lambda(T)}D_T,$$

where D_T is the value at time T of the process D defined by, for all $t\in[0,T]$,

$$\begin{aligned}D_t &= \exp\left(\int_0^t\left(\sigma_X-\rho\sigma_Se^{-\kappa(T-u)}\right)\mathrm{d}W_u^{(X)} - \frac{1}{2}\int_0^t\left(\sigma_X-\rho\sigma_Se^{-\kappa(T-u)}\right)^2\mathrm{d}u\right)\\ &\quad\times\exp\left(-\int_0^t\bar{\rho}\sigma_Se^{-\kappa(T-u)}\,\mathrm{d}W_u^{\perp} - \frac{1}{2}\int_0^t\bar{\rho}^2\sigma_S^2e^{-2\kappa(T-u)}\,\mathrm{d}u\right),\end{aligned}$$

and $\lambda(T)$ is a constant given by

$$\begin{aligned}\lambda(T) &= -S_0e^{-\kappa T} - m\left(1-e^{-\kappa T}\right) - \frac{\sigma_X^2}{2}T + \frac{1}{2}\int_0^T\left(\sigma_X-\rho\sigma_Se^{-\kappa(T-u)}\right)^2\mathrm{d}u\\ &\quad + \frac{1}{2}\int_0^T\bar{\rho}^2\sigma_S^2e^{-2\kappa(T-u)}\,\mathrm{d}u\\ &= -S_0e^{-\kappa T} - m\left(1-e^{-\kappa T}\right) + \frac{\sigma_S^2}{4\kappa}\left(1-e^{-2\kappa T}\right) - \frac{\rho\sigma_S\sigma_X}{\kappa}\left(1-e^{-\kappa T}\right).\end{aligned}$$

Note that D is a strictly positive martingale and satisfies Novikov's condition. Therefore we can define a probability measure Q via

$$\mathrm{d}\,Q \quad = \quad D_T \,\mathrm{d}\,P.$$

Under Q the processes $\tilde{W}^{(X)}$ and $\tilde{W}^{\perp}$, for all $t \in [0, T]$,

$$\begin{aligned} \tilde{W}_t^{(X)} &= W_t^{(X)} - \int_0^t \left(\sigma_X - \rho\sigma_S e^{-\kappa(T-u)}\right) \mathrm{d}\,u, \\ \tilde{W}_t^{\perp} &= W_t^{\perp} + \int_0^t \bar{\rho}\sigma_S e^{-\kappa(T-u)}\, \mathrm{d}\,u \end{aligned}$$

are independent BMs. The dynamics of X, rewritten in terms of $\tilde{W}^{(X)}$, satisfy

$$\mathrm{d}\,X_t \quad = \quad \sigma_X \left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right) X_t \,\mathrm{d}\,t + \sigma_X X_t \,\mathrm{d}\,\tilde{W}_t^{(X)}.$$

Observe that the expectation of X_t with respect to Q is given by

$$\begin{aligned} \mathbb{E}^Q\left(X_t\right) &= X_0 \exp\left(\int_0^t \sigma_X \left(\sigma_X - \rho\sigma_S e^{-\kappa(T-u)}\right) \mathrm{d}\,u\right) \\ &= X_0 \exp\left(\sigma_X^2 t + \frac{\rho\sigma_X\sigma_S}{\kappa} e^{-\kappa T}\right) \exp\left(-\frac{\rho\sigma_X\sigma_S}{\kappa} e^{-\kappa(T-t)}\right). \end{aligned}$$

Now the expectation term in the numerator can be written as

$$\text{(B-6)}\quad \begin{aligned} \mathbb{E}\left(I_T \int_0^T e^{-rt}\,\mathrm{d}\,X_t\right) &= X_0 e^{\lambda(T)} \mathbb{E}^Q\left(\int_0^T e^{-rt}\,\mathrm{d}\,X_t\right) \\ &= X_0 e^{\lambda(T)} \int_0^T e^{-rt}\sigma_X\left(\sigma_X - \rho\sigma_S e^{-\kappa(T-t)}\right) \mathbb{E}^Q\left(X_t\right) \mathrm{d}\,t \\ &= X_0^2 e^{\lambda(T) + \frac{\rho\sigma_X\sigma_S}{\kappa} e^{-\kappa T}} \\ &\quad \times \underbrace{\int_0^T e^{(\sigma_X^2 - r)t}\left(\sigma_X^2 - \rho\sigma_X\sigma_S e^{-\kappa(T-t)}\right) e^{-\frac{\rho\sigma_X\sigma_S}{\kappa} e^{-\kappa(T-t)}}\,\mathrm{d}\,t}_{=A}. \end{aligned}$$

For $\rho = 0$, we are done. For $\rho \neq 0$, we continue by substituting $u = |\rho|\sigma_X\sigma_S e^{-\kappa(T-t)}/\kappa$, which leads to an explicit expression for the above integral A in terms of the incomplete gamma function $\gamma(s, x) = \int_0^x y^{s-1}e^{-y}\,\mathrm{d}\,y$.

$$\begin{aligned} A &= e^{(\sigma_X^2 - r)T}\left(\frac{|\rho|\sigma_X\sigma_S}{\kappa}\right)^{-\frac{\sigma_X^2 - r}{\kappa}} \int\limits_{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S e^{-\kappa T}}^{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S} u^{\frac{\sigma_X^2 - r}{\kappa}}\left(\sigma_X^2 - \kappa u\right)e^{-u}\frac{1}{\kappa u}\,\mathrm{d}\,u \\ &= \frac{\sigma_X^2 e^{(\sigma_X^2 - r)T}\left(|\rho|\sigma_X\sigma_S\right)^{-\frac{\sigma_X^2 - r}{\kappa}}}{\kappa^{1 - \frac{\sigma_X^2 - r}{\kappa}}} \int\limits_{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S e^{-\kappa T}}^{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S} u^{\frac{\sigma_X^2 - r}{\kappa} - 1} e^{-u}\,\mathrm{d}\,u \\ &\quad - e^{(\sigma_X^2 - r)T}\left(\frac{|\rho|\sigma_X\sigma_S}{\kappa}\right)^{-\frac{\sigma_X^2 - r}{\kappa}} \int\limits_{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S e^{-\kappa T}}^{\frac{1}{\kappa}|\rho|\sigma_X\sigma_S} u^{\frac{\sigma_X^2 - r}{\kappa}} e^{-u}\,\mathrm{d}\,u \end{aligned}$$

$$= \underbrace{\frac{\sigma_X^2 e^{(\sigma_X^2-r)T}\left(|\rho|\sigma_X\sigma_S\right)^{-\frac{\sigma_X^2-r}{\kappa}}}{\kappa^{1-\frac{\sigma_X^2-r}{\kappa}}}\left(\gamma\left(\frac{\sigma_X^2-r}{\kappa},\frac{1}{\kappa}|\rho|\sigma_X\sigma_S\right)-\gamma\left(\frac{\sigma_X^2-r}{\kappa},\frac{1}{\kappa}|\rho|\sigma_X\sigma_S e^{-\kappa T}\right)\right)}_{=\Lambda_1(T)}$$

$$\underbrace{-\,e^{(\sigma_X^2-r)T}\left(\frac{|\rho|\sigma_X\sigma_S}{\kappa}\right)^{-\frac{\sigma_X^2-r}{\kappa}}\left(\gamma\left(\frac{\sigma_X^2-r}{\kappa}+1,\frac{|\rho|\sigma_X\sigma_S}{\kappa}\right)-\gamma\left(\frac{\sigma_X^2-r}{\kappa}+1,\frac{|\rho|\sigma_X\sigma_S e^{-\kappa T}}{\kappa}\right)\right)}_{=\Lambda_2(T)}.$$

Plugging this into equation (B-6) gives

$$\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right) = X_0^2 e^{\lambda(T)+\frac{\rho\sigma_X\sigma_S}{\kappa}e^{-\kappa T}}\left(\Lambda_1(T)-\Lambda_2(T)\right).$$

The expression for the $\mathbb{V}\left(I_T\right)$ is straightforward. $\square$

Proof of Proposition 4. The expressions for $\check{a}$ and the corresponding variance $\mathbb{V}\left(C_T(\check{a})\right)$ are model independent and therefore the same as in Proposition 3. In both models X is a GBM, and hence we again have

$$\mathbb{E}\left(\int_0^T e^{-2rt}X_t^2\,\mathrm{d}t\right) = \begin{cases} \frac{X_0^2}{\sigma_X^2-2r}\left(e^{(\sigma_X^2-2r)T}-1\right) & \text{if } \sigma_X^2\neq 2r \\ X_0^2 T & \text{if } \sigma_X^2 = 2r \end{cases}.$$

For the expectation in the numerator we get

$$\mathbb{E}\left(I_T\int_0^T e^{-rt}\,\mathrm{d}X_t\right) = \mathbb{E}\left(\int_0^T e^{-rt}d\langle I,X\rangle_t\right) = \int_0^T e^{-rt}\rho_{IX}\sigma_X\sigma_I\mathbb{E}\left(I_tX_t\right)\mathrm{d}t.$$

Straightforward computations give $\mathbb{E}\left(X_tI_t\right)=X_0I_0e^{\rho_{IX}\sigma_X\sigma_I t}$, which plugged into the above expressions gives the desired result. The expression for the $\mathbb{V}\left(I_T\right)$ is straightforward. $\square$

References

Alexander, C. "Optimal Hedging Using Cointegration." *Philosophical Transactions of the Royal Society A*, 357 (1999), 2039–2058.

Ankirchner, S., and G. Heyne. "Cross Hedging with Stochastic Correlation." *Finance and Stochastics*, 16 (2012), 17–43.

Ankirchner, S.; P. Imkeller; and G. Dos Reis. "Pricing and Hedging of Derivatives Based on Nontradable Underlyings." *Mathematical Finance*, 20 (2010), 289–312.

Baillie, R. T. "Econometric Tests of Rationality and Market Efficiency." *Econometric Reviews*, 8 (1989), 151–186.

Baillie, R. T., and R. J. Myers. "Bivariate GARCH Estimation of the Optimal Commodity Futures Hedge." *Journal of Applied Econometrics*, 6 (1991), 109–124.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy,* 81 (1973), 637–654.

Brenner, R. J., and K. F. Kroner. "Arbitrage, Cointegration, and Testing the Unbiasedness Hypothesis in Financial Markets." *Journal of Financial and Quantitative Analysis*, 30 (1995), 23–42.

Casassus, J., and P. Collin-Dufresne. "Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates." *Journal of Finance*, 60 (2005), 2283–2331.

Cecchetti, S. G.; R. E. Cumby; and S. Figlewski. "Estimation of the Optimal Futures Hedge." *Review of Economics and Statistics*, 70 (1988), 623–630.

Chan, K. "A Further Analysis of the Lead-Lag Relationship between the Cash Market and Stock Index Futures Market." *Review of Financial Studies*, 5 (1992), 123–152.

Cox, J. C.; J. E. Ingersoll; and S. A. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica,* 53 (1985), 385–407.

Davis, M. H. A. "Optimal Hedging with Basis Risk." In *From Stochastic Calculus to Mathematical Finance*, Y. Kabanov, R. Liptser, and J. Stoyanov, eds. Berlin: Springer (2006).

Duffie, D., and H. R. Richardson. "Mean-Variance Hedging in Continuous Time." *Annals of Applied Probability*, 1 (1991), 1–15.

Engle, R. F., and C. W. J. Granger. "Co-Integration and Error Correction: Representation, Estimation, and Testing." *Econometrica*, 55 (1987), 251–276.

Föllmer, H., and D. Sondermann. "Hedging of Non-Redundant Contingent Claims." In *Contributions to Mathematical Economics*, A. Mas-Colell and W. Hildenbrand, eds. Amsterdam: North-Holland (1986).

Granger, C. W. J. "Some Properties of Time Series Data and Their Use in Econometric Model Specification." *Journal of Econometrics*, 16 (1981), 121–130.

Heston, S. L. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies,* 6 (1993), 327–343.

Howard, C. T., and L. J. D'Antonio. "Multiperiod Hedging Using Futures: A Risk Minimization Approach in the Presence of Autocorrelation." *Journal of Futures Markets*, 11 (1991), 697–710.

Hull, J. C. *Options, Futures, and Other Derivatives*, 7th ed. Upper Saddle River, NJ: Prentice Hall (2008).

Hulley, H., and T. A. McWalter. "Quadratic Hedging of Basis Risk." Research Paper Series 225, Quantitative Finance Research Centre, University of Technology, Sydney (2008).

Karatzas, I., and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, 2nd ed. Berlin: Springer (1991).

Kawaller, I. G.; P. D. Koch; and T. W. Koch. "The Temporal Price Relationship between S&P 500 Futures and the S&P 500 Index." *Journal of Finance*, 42 (1987), 1309–1329.

Lien, D. "Optimal Hedging and Spreading in Cointegrated Markets." *Economics Letters*, 40 (1992), 91–95.

Lien, D. "Cointegration and the Optimal Hedge Ratio: The General Case." *Quarterly Review of Economics and Finance*, 44 (2004), 654–658.

Lien, D., and X. Luo. "Estimating Multiperiod Hedge Ratios in Cointegrated Markets." *Journal of Futures Markets*, 13 (1993), 909–920.

Monoyios, M. "Performance of Utility-Based Strategies for Hedging Basis Risk." *Quantitative Finance*, 4 (2004), 245–255.

Musiela, M., and T. Zariphopoulou. "An Example of Indifference Prices under Exponential Preferences." *Finance and Stochastics*, 8 (2004), 229–239.

Ng, V. K., and S. C. Pirrong. "Price Dynamics in Refined Petroleum Spot and Futures Markets." *Journal of Empirical Finance*, 2 (1996), 359–388.

Schwartz, E. S. "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging." *Journal of Finance*, 52 (1997), 923–973.

Schweizer, M. "Mean-Variance Hedging for General Claims." *Annals of Applied Probability*, 2 (1992), 171–179.

Schweizer, M. "Local Risk-Minimization for Multidimensional Assets and Payment Streams." In *Advances in Mathematics of Finance*, Ł. Stettner, ed. Warsaw: Banach Center Publications (2008).

Stoll, H. R., and R. E. Whaley. "The Dynamics of Stock Index and Stock Index Futures Returns." *Journal of Financial and Quantitative Analysis*, 25 (1990), 441–468.