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Homotopy Algebras in Cosmology and Quantum Mechanics

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von
Allison F. Pinto

Präsidentin der Humboldt-Universität zu Berlin
Prof. Dr. Julia von Blumenthal
Dekanin der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Caren Tischendorf

Gutachter:
1. Prof. Dr. Robert Brandenberger
2. Prof. Dr. Olaf Hohm
3. Dr. Emanuel Malek

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Declaration of Authorship

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Zusammenfassung

In dieser Arbeit werden die Grundlagen von zwei häufig auftretenden Merkmalen unserer Naturgesetze untersucht: Eichsymmetrien und Quantisierung. Durch die Betrachtung dieser Merkmale im mathematischen Rahmen von Homotopie-Algebren wollen wir neue Methoden zur Berechnung physikalischer Observablen beschreiben, insbesondere in der Kosmologie und der Quantenmechanik.

Zunächst befassen wir uns mit dem Problem der Eichredundanzen, die es schwer machen zu erkennen, welche Größen eine physikalische Bedeutung haben. Im Jahr 1980 erreichte Bardeen dieses Ziel in der kosmologischen Störungstheorie zu erster Ordnung. Die Frage, ob dieses Verfahren auf die perturbative Expansion von Eichtheorien aller Ordnungen ausgedehnt werden kann, ist seitdem jedoch offen geblieben. Wir zeigen, dass die Umformulierung von Eichtheorien in eichinvariante Felder als ein Transfer von homotopie-algebraischer Strukturen verstanden werden kann. Unter Verwendung dieses mathematischen Rahmens erweitern wir dann die Gültigkeit der Bardeen-Variablen auf perturbative Eichtheorien zu allen Ordnungen.

Nach der Einführung eines systematisches Verfahrens für die eichinvariante Störungstheorie betrachten wir die Berechnung von Observablen in der Doppelfeldtheorie um zeitabhängige Hintergründe. Indem wir die Doppelfeldtheorie um zeitabhängige Hintergründe quadratischer und kubischer Ordnung erweitern und die quadratische Wirkung in den eichinvarianten Variablen ausdrücken, schaffen wir eine Grundlage für zukünftige Berechnungen, insbesondere zur Untersuchung des Einflusses massiver Stringmoden in kosmologischen Hintergründen.

Zum Schluss betrachten wir einen anderen Ansatz zur Berechnung von Erwartungswerten in der Quantenmechanik. Obwohl die Pfadintegralformulierung der Quantenmechanik für den Fortschritt der Quantentheorie von entscheidender Bedeutung war, fehlt ihr immer noch eine strenge mathematische Definition. Die Reduktion eines unendlich-dimensionalen Raums von klassisch erlaubten Trajektorien auf einen Erwartungswert, der lediglich eine Funktion der Anfangs- und Endrandbedingungen ist, hat jedoch eine homotopiealgebraische Interpretation. Mit Hilfe des Batalin-Vilkovisky-Formalismus, der eng mit Homotopie-Lie-Algebren verwandt ist, entwickeln wir einen homologischen Ansatz zur Berechnung von Quantenerwartungswerten. Als Beispiel betrachten wir den harmonischen Oszillator und zeigen, dass unsere Methode auch im Kontext der Quantenfeldtheorie in gekrümmter Raumzeit verwendet werden kann, indem wir den Unruh-Effekt berechnen.

Abstract

This thesis examines the underpinnings of two frequently manifest features of our laws of nature: gauge symmetries and quantization. By studying these features through the mathematical framework of homotopy algebras, we aim to describe new methods towards the computation of physical observables, in particular for cosmology and quantum mechanics.

First, we deal with the problem of gauge redundancies, which make it difficult to discern which quantities have physical meaning. In 1980, Bardeen introduced a procedure to achieve this goal in first order cosmological perturbation theory. However, the question whether this procedure can be extended to the perturbative expansion of gauge theories to all orders has remained open since then. We show that, in general, the reformulation of gauge theories in gauge invariant fields has the interpretation of transferring homotopy algebraic structure. Utilising this mathematical framework, we then generalize Bardeen's procedure to perturbative expansions of gauge theories to all orders in perturbations.

After establishing a systematic procedure for gauge invariant perturbation theory, we set up the stage for computing observables in double field theory around time-dependent backgrounds. Double field theory not only has T-duality as a manifest symmetry, which is expected to be important in string cosmology proposals, but is also (in its weakly constrained form) a description of massive string modes, and hence is a suitable arena to investigate the imprint of massive string modes in cosmological backgrounds. By expanding double field theory around time-dependent backgrounds to quadratic and cubic order and expressing the quadratic action in terms of gauge invariant variables, we provide a basis for future computations.

Finally, we describe a different approach for computing expectation values in quantum mechanics. Though having been essential for the progress of quantum theory, the path integral formulation of quantum mechanics still lacks a rigorous mathematical definition. However, the act of reducing an infinite-dimensional space of classically allowed trajectories into an expectation value which is merely a function of the initial and final boundary conditions does have a homotopy algebraic interpretation. Through the Batalin-Vilkovisky formalism, which is closely related to homotopy Lie algebras, we build a homological approach for computing quantum expectation values. We demonstrate our method for the harmonic oscillator and we show that our method can also be used in the context of quantum field theory in curved spacetime by rederiving the Unruh effect.

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Chapter 1

Introduction

1.1 Motivation and Overview

In the development of modern physics, symmetries have become guiding principles in the way we characterize the laws of nature. A key discovery in the late 1800s, the speed of light was found to be the same regardless of the reference frame. Shortly after, physicists realized that Maxwell's equations of electromagnetism are invariant under what came to be known as Lorentz transformations. Between 1905 and 1907, special relativity was born with two postulates: the speed of light is the same in all inertial reference frames, which are frames that move at constant velocity relative to each other, and the laws of physics must be the same in all inertial reference frames.

In 1918, Noether paved the way for physicists to look at symmetries as not physical consequences but rather as features from which physics can be derived. Noether's first theorem states that for every continuous global symmetry of an action of a theory there is a conserved current. She gave us the intuition that symmetries give rise to physical laws. Since then, many symmetries and conservation laws were uncovered.

Moreover, it was realized that a kind of symmetry, called gauge symmetry, can simplify the formulations of our fundamental theories, such as general relativity and electromagnetism. Gauge symmetries are symmetries of the way we characterize theories: observables are measured relative to a reference value called a gauge, and our laws of physics must stay the same regardless of the choice of gauge. Unlike global symmetries, which imply conserved quantities, gauge symmetries are local symmetries that are redundancies of our formulations. Despite their redundancies, they have helped us attain the most concise descriptions of the four fundamental forces: general relativity, whose gauge group is the group of spacetime diffeomorphisms, and the Standard Model, which contains

$U(1) \times SU(2)$ for the electroweak interaction and $SU(3)$ for the strong interaction. With gauge symmetries, new particles were predicted and experimentally measured. Most recently, the Higgs boson, hypothesized through symmetry arguments, was verified almost 50 years after its prediction. In addition, the observation of gravitational waves and their two polarizations predicted through gauge symmetry have attested to the applicability to general relativity. In our quest as theoretical physicists to find a unified description of nature and answer unsolved problems, we have constructed theories with more complicated symmetries, such as supersymmetry, supergravity theories and string theories. Though we have yet to experimentally verify our new theories beyond the Standard Model plus general relativity, it is likely that symmetries will continue to play an important role along the way.

Although gauge symmetries have been helpful for finding new physics, the redundancies that accompany them have been obstacles in computing observables necessary to test our theoretical models of the universe. Symmetries and redundancies are two sides of the same coin. On one side, a system has a symmetry when it remains unchanged after the application of a particular set of transformations. On the other side, this is a redundancy in the formulation; there are different ways of describing the same physical system. Different solutions to an equation of motion are equivalent and related through a symmetry transformation. In electrodynamics for instance, the magnetic field is unchanged with the addition of a curl-free vector field to the electromagnetic potential. Such redundancies have caused problems, especially in the quantum realm where many of our research questions in theoretical physics lie.

Redundancies make it difficult to quantize a theory. It is easy to see the problem via the path integral formulation of quantum mechanics, from which quantum field theory and hence the Standard Model are built upon. In the path integral formulation of quantum mechanics, a particle can take any path on its way from point A to point B, as long as its trajectory preserves the endpoints. Each path is not equally likely– the contribution of a path depends on the action evaluated on that path– but there is an infinite sea of classically allowed trajectories. In order to find the probability amplitude of a particle travelling from point A to point B, one must take the weighted average of its possible paths. Although the path integral lacks a mathematically rigorous definition, it has the power to successfully compute observables. Quantum field theory uses the same technique, with the integral taken over all possible field configurations that preserve the boundary conditions. For a theory with gauge redundancy, one must sum over all gauge-inequivalent field configurations. For this reason, theoretical physicists must often face the challenge of wandering through the forest of gauge redundancies and follow only the invariant paths. The primary objective of this thesis is to shed light inside this beautifully symmetric but mysteriously redundant forest and unearth its reliable routes.

Taking a bird's eye perspective, the problem of untangling redundant formulations can be seen as classifying redundant objects into equivalence classes. Once one determines the appropriate equivalence classes– the ones which are physically relevant – one only needs to find a representative of the class. By choosing a representative, one fixes a gauge. Ideally, this choice leads to a convenient computation of observables, but it may not be obvious how to make this choice.

No matter what method may be used to overcome redundancies and solve for observables, one must be able to produce the physically relevant data of the theory. This can be described in a rather general way. Considering a free theory (without interactions), an observable belongs to the kernel of the linear map,

$$\partial_0 : \ f \to \mathcal{E} \tag{1.1}$$

where f is the space of fields and $\mathcal{E}$ is the space of equations of motion. For instance, in electromagnetism, ∂_0 acts on the gauge field A_μ

$$\partial_0(A)_\mu = \Box A_\mu - \partial_\mu(\partial^\nu A_\nu)\,. \tag{1.2}$$

For on-shell fields, the RHS of (1.2) must be zero. The off-shell fields which do not satisfy the equation of motion play a role in the quantum theory. The gauge redundancy of fields is encoded by the linear map,

$$\partial_1 : \ g \to f \tag{1.3}$$

where g is the space of gauge parameters. This means that given a gauge parameter, one can map to a field by its gauge transformation in terms of the gauge parameter. Taking electromagnetism again as an example, ∂_1 would act on a gauge parameter Λ as

$$\partial_1(\Lambda)_\mu = \partial_\mu\Lambda\,. \tag{1.4}$$

Because of gauge invariance of the equation of motion, ∂_0 acting on a field which comes from a gauge transformation must be zero:

$$\partial_0 \circ \partial_1 = 0\,. \tag{1.5}$$

Consequently, given an on-shell field A which satisfies the equation of motion $\partial_0 A = 0$, when shifted by a gauge transformation, e.g. $A + \partial_1 \Lambda$, it still satisfies the equation of motion. Thus, by taking out the field configurations that are pure gauge transformations, one obtains the physically relevant observables which belong to the quotient space,

$$H := \text{Ker}\partial_0 / \text{Im}\partial_1\,. \tag{1.6}$$

One approach to deal with redundancies besides fixing a gauge is to reformulate the theory in terms of gauge invariant variables. This method was developed by Bardeen in the 1980s in the context of cosmological perturbation theory [1]. In this method, one examines the gauge transformations of the fields in the theory, and builds gauge invariant combinations. We can view this as some sort of projection from the space of fields ϕ to a space of gauge invariant fields $\bar{\phi}$:

$$p : \phi \to \bar{\phi}\,. \tag{1.7}$$

Once the theory is rewritten in terms of the gauge invariant variables, the gauge redundancy is eliminated. One can then proceed to compute observables without worrying about any redundancies. At the level of the free theory, without interactions, expressing the theory in terms of gauge invariant variables sounds promising. However, Bardeen's procedure does not have a prescription for taking into account higher order interactions in perturbation theory.

Since perturbation theory is ubiquitous in theoretical physics, one cannot help but wonder if there is any way to improve Bardeen's procedure to be able to apply it at all orders in perturbations. The complexity arises because one needs to be able to take products of fields, or products of fields and gauge parameters– for instance, the gauge transformations of the fields would take on a non-linear piece. This requires the definition of an algebraic structure on the free theory. Consequently, the reformulation of the interacting theory in terms of gauge invariant variables should also have an algebraic structure. We know that any reformulation should reproduce the physically relevant data of the theory. If we can find a modification of $p : \phi \to \bar{\phi}$, such that the gauge invariant perturbation theory gives rise to the same physical observables, then we will have a successful gauge invariant reformulation of an interacting theory. To study algebraic structures of perturbation theories which are equivalent in a way that maintains the physical data belongs to the realm of homotopy algebras.

Roughly speaking, a homotopy algebra is a more generalized notion of an algebra, in that its multiplication rules only hold up to some error terms. It is easier to see this with an example. For instance, an associative algebra $(A, \cdot)$ is an algebra whose multiplication is associative in that

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\,, \quad a, b, c \in A\,. \tag{1.8}$$

A homotopy associative algebra is an algebra whose associativity does not hold in general, but rather up to an error term:

$$a \cdot (b \cdot c) - (a \cdot b) \cdot c = \mathcal{A}(a, b, c)\,, \tag{1.9}$$

where $\mathcal{A}$ (called the associator) can be thought of as a total derivative. Because of this failure to uphold the usual multiplication rules, one does not define isomorphisms of homotopy algebras, but rather quasi-isomorphisms (which we will define later in section 1.3). As Vallette put it neatly: *"Algebra is the study of algebraic structures with respect to isomorphisms....Homotopical algebra is the study of algebraic structures with respect to quasi-isomorphisms..."*[2]. The homotopy algebras that we will be considering in this thesis are homotopy Lie algebras called L_∞ algebras. In contrast to a Lie algebra, whose product obeys the Jacobi identity, for an L_∞ algebra the Jacobi identity is only satisfied "up to homotopy". [1]

The study of homotopy algebraic structures of open and closed string field theory began in the 90s. It was shown that open string field theory has a homotopy associative (A_∞) algebra, while closed string field theory has a homotopy Lie (L_∞) algebra [3–5]. In 2016, Sen found a prescription for a consistent truncation of closed string field theory for massless modes [6]. It turned out that the algebraic structure of this effective field theory is an L_∞ algebra. For further reviews see [7–9]. L_∞ algebras have since made their presence known in more conventional gauge theories such as Yang-Mills theory and Einstein gravity [10, 11]. In addition, it is closely related to the Batalin-Vilkovisky (BV) algebra, the structure

[1]The term homotopy comes from topology. Two functions are homotopic if they can be connected by a continuous path of continuous maps. Topological structures are homotopy equivalent when there exists a continuous map f which has an inverse g up to homotopy, i.e. $f \circ g$ and $g \circ f$ are homotopic to the identity map. Our usage of the term relates to the structures these notions induce on certain algebraic invariants associated to topological spaces, namely homology.

behind the BV formalism which deals with the gauge redundancies of quantum field theories [12, 13].

By considering the homotopy algebraic structures of our gauge theories, we can view their reformulations in terms of gauge invariant variables as a so-called homotopy transfer, the transfer of homotopy algebraic structure from the original gauge redundant theory to the gauge invariant theory. Through this interpretation, we gain a systematic understanding of how theories can be expressed in terms of gauge invariant variables. As we will illustrate in this thesis, homotopy transfer is at the core of reducing extraneous information in a theory to a smaller space of physically meaningful data.

After reformulating a theory in terms of gauge invariant variables, one can wonder whether homotopy algebras can also be applied to the computation of observables, especially at the quantum level. Indeed, one can think of computing quantum expectation values as a problem which also involves the reduction of superfluous data– like the infinitely many allowed trajectories of a particle to travel from one point to another – to observables. Using homotopy algebras as our guide, we introduce a new partial reformulation of quantum mechanics.

Outline of the Thesis

In this thesis, we will explore the application of homotopy algebras in two major areas of theoretical physics: cosmology and quantum mechanics. In section 1.2 we will dive deeper into how homotopy algebras can encode gauge theories. In section 1.3 we will review mathematical concepts needed to understand the homotopy algebraic machinery that will be applied to physics later. In chapter 2 we will extend Bardeen's procedure to higher orders in perturbation theory by applying what is called the homological perturbation lemma. We will demonstrate how the new procedure works for Yang-Mills theory and gravity on flat and cosmological backgrounds. We reproduce an important result in cosmological perturbation theory which is the Mukhanov-Sasaki action for the gauge invariant scalar perturbation known as the Mukhanov variable.

In chapter 3, our procedure will be applied in the direction of new physics, namely for double field theory on cosmological backgrounds. Double field theory is a string-inspired theory which is manifestly invariant under T-duality, a duality of closed string theory [14–17]. It is a point-particle theory, not a theory of a strings, formulated on a doubled spacetime. Upon eliminating half of the spacetime coordinates it coincides with the low-energy sector of closed string theory: the graviton, antisymmetric B-field, and the dilaton. The expansion of double field theory on cosmological backgrounds to quadratic and cubic order and the identification of gauge invariant variables are the first steps towards the goal of learning whether strings could have an imprint in cosmological observations.

In chapter 4, we will advance to the quantum arena. Although there has been a lot of progress made for addressing the quantization of gauge quantum field theories through the Batalin-Vilkovisky (BV) formalism, we will have a close look at the intermediate step: quantum mechanics. In particular we will study the quantization of a theory with no gauge symmetries at all, namely the harmonic oscillator in one dimension. It turns out that the same mechanism that takes a gauge redundant theory and reformulates it in terms of gauge invariant variables can be used to sum over all the paths that a particle can take between an initial and a final position and compute a physical observable which is the expectation value. This leads to a new reformulation of quantum mechanics based on the BV formalism.

Chapters 2 to 4 are heavily based on the content of the author's publications. The results presented in this thesis have been published in the following papers:

[18] Christoph Chiaffrino, Olaf Hohm, and Allison F. Pinto. Gauge Invariant Perturbation Theory via Homotopy Transfer. JHEP, 05:236, 2021. doi: 10.1007/JHEP05(2021)236.

[19] Christoph Chiaffrino, Olaf Hohm, and Allison F. Pinto. Homological Quantum Mechanics. 12 2021. arXiv: 2112.11495.

[20] Olaf Hohm and Allison F. Pinto. Cosmological Perturbations in Double Field Theory. 7 2022. arXiv: 2207.14788.

It is important to acknowledge that the novel mathematical approaches in [18] and [19] were mostly developed by the coauthor, Christoph Chiaffrino. The author has contributed to the more applied aspects of these works, most notably, the applications to cosmological perturbation theory and the computation of the Unruh effect using the new methods. The results from [20] that are described in chapter 3 of this thesis were predominantly obtained by the author.

1.2 Physics in Terms of Homotopy Algebras

The main ingredients of a gauge theory are its gauge fields, equations of motion, and its gauge symmetries. For linear theories, these objects are typically elements of vector spaces. For example, a scalar field is an element of $C^\infty(M)$, the vector space of all smooth functions on a manifold M, and the electromagnetic vector potential is an element of the vector space of differential one-forms. Given a gauge theory, one often separates its free part, whose equations of motion are linear differential equations, from its interacting part, whose equations of motion are non-linear. In the L_∞ algebra formulation of a field theory, one organizes the free theory into a sequence of vector spaces related via linear maps,

$$\text{gauge parameters} \longrightarrow \text{fields} \longrightarrow \text{field equations} \longrightarrow \text{Noether identities} \tag{1.10}$$

and the interactions of the full theory are given by multi-linear maps on these spaces. Let us elaborate on how these structures encode field theories. For supplementary material, see the reviews [21–23].

The sequence in (1.10) is an example of a chain complex. A chain complex $(V_\bullet, \partial)$ is a sequence of vector spaces V_i,

$$\cdots \xrightarrow{\partial} V_2 \xrightarrow{\partial} V_1 \xrightarrow{\partial} V_0 \xrightarrow{\partial} V_{-1} \xrightarrow{\partial} \cdots \tag{1.11}$$

where the differential ∂_i is a map which takes an element in V_i and maps it to an element in V_{i-1} and squares to zero:

$$\partial_{i-1}\partial_i = 0\,. \tag{1.12}$$

The subscript i is a label called the degree. The chain complex is an example of a graded vector space, which we will define in the next section.

How does this chain complex actually encode a free theory? Let us first assign the convention that the vector space V_1 is the space of gauge parameters, V_0 is the space of fields, V_{-1} is the space of field equations, and V_{-2} is the space of Noether identities. The differential acting on a gauge parameter is defined to be the infinitesimal gauge transformation of a field with respect to that parameter to lowest order in perturbations. Given a gauge parameter $\lambda \in V_1$, the differential ∂_1 acts as

$$\partial_1\lambda = \delta_\lambda\phi \tag{1.13}$$

where the gauge transformation $\delta_\lambda \phi$ is a linear function of λ. The differential acting on a field is defined to be its equation of motion:

$$\partial_0 \phi = E(\phi)\,. \tag{1.14}$$

The condition for the differential squaring to zero requires:

$$\partial_0 \partial_1 \lambda = E(\delta_\lambda \phi) \overset{!}{=} 0\,. \tag{1.15}$$

This makes sense because it means that the equations of motion are gauge invariant. For a free theory, the Noether identity is a linear function of the equations of motion and is identically zero when the equation of motion is written explicitly in terms of the fields, as it should be:

$$\partial_{-1}\partial_0 \phi = \partial_{-1} E(\phi) = f(E(\phi)) = 0\,. \tag{1.16}$$

One can obtain the action by defining a non-degenerate pairing $\langle \cdot\,,\cdot \rangle$. The action for the free theory is given by

$$S = \frac{1}{2}\langle \phi\,, \partial\phi \rangle\,. \tag{1.17}$$

Varying the action with respect to the fields,

$$\delta S = \langle \delta\phi\,, E \rangle\,. \tag{1.18}$$

$\delta S = 0$ gives us the field equations $E = 0$. In this way, the chain complex describes the free theory, and with a suitable inner product, one can construct the free action.

To see how this works with an explicit example, let us check these relations for a simple free theory with gauge symmetry– Maxwell's theory. Let us use the same convention for labelling the vector spaces as we did previously, and define the chain complex starting with V_1 as the space of gauge parameters:

$$\begin{array}{ccccccc} V_1 & \xrightarrow{\partial_1} & V_0 & \xrightarrow{\partial_0} & V_{-1} & \xrightarrow{\partial_{-1}} & V_{-2} \\ \{\lambda\} & & \{A^\mu\} & & \{E_\mu\} & & \{f\} \end{array} \tag{1.19}$$

The differentials act as

$$(\partial_1 \lambda)_\mu = \partial_\mu \lambda\,, \tag{1.20}$$
$$(\partial_0 A)_\mu = \Box A_\mu - \partial_\mu(\partial^\nu A_\nu)\,, \tag{1.21}$$
$$\partial_{-1}(E) = \partial_\mu E^\mu\,. \tag{1.22}$$

We can check that the differential squares to zero:

$$\partial_0(\partial_1\lambda)_\mu = \partial_0(\partial_\mu\lambda) = \Box\partial_\mu\lambda - \partial_\mu(\partial^\nu\partial_\nu\lambda) = 0\,, \tag{1.23}$$

$$\partial_{-1}(\partial_0 A) = \partial_{-1}\big(\Box A^\mu - \partial^\mu(\partial^\nu A_\nu)\big) = \partial_\mu\Box A^\mu - \partial_\mu\partial^\mu\partial^\nu A_\nu = 0\,. \tag{1.24}$$

We define the inner product $\langle f\,,g\rangle$ to be

$$\langle f\,,g\rangle = \int f\cdot g\; d^4x\,. \tag{1.25}$$

The action is then

$$S = \frac{1}{2}\int A_\mu\big(\Box A^\mu - \partial^\mu(\partial^\nu A_\nu)\big)d^4x\,, \tag{1.26}$$

which, by integrating by parts, yields the familiar form of Maxwell's action

$$S = -\frac{1}{4}\int d^4x\; F^{\mu\nu}F_{\mu\nu} \tag{1.27}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

We have now organized a free theory into a chain complex. To define the full theory with interactions, one incorporates multi-linear maps. For example, if we equip Maxwell's theory with the appropriate products, we obtain Yang-Mills theory which has 3-gluon and 4-gluon vertices. For instance, the gauge transformation for the field gains a non-linear term:

$$\delta_\lambda A_\mu = \partial_\mu\lambda + [A_\mu, \lambda]\,, \tag{1.28}$$

where $[\cdot,\cdot]$ is the Lie bracket. So the 2-bracket acting on one field and one gauge parameter is

$$b_2^\mu(A,\lambda) = [A^\mu, \lambda]\,. \tag{1.29}$$

By inspecting the equations of motion, one can read off the 2-bracket acting on fields

$$b_2^\mu(A,B) = \partial_\nu[A^\nu, B^\mu] + [\partial^\mu A^\nu - \partial^\nu A^\mu, B_\nu] + (A \leftrightarrow B) \tag{1.30}$$

and the 3-bracket

$$b_3^\mu(A,B,C) = [A_\nu,[B^\nu,C^\mu]] + \text{permutations}\,. \tag{1.31}$$

One can show that the algebra of Yang-Mills theory is a so-called L_∞ algebra. An L_∞ algebra is a generalization of a Lie algebra, where the Jacobi identity is satisfied up to an error term. The error term involves a higher product:

$$\begin{aligned} &b_2(b_2(v,w),z) + (-1)^{|z|}b_2(b_2(z,v),w) + (-1)^{|w|+|z|}b_2(b_2(w,z),v) \\ &= -\,b_1(b_3(v,w,z)) - b_3(b_1(v),w,z) - (-1)^{|v|}b_3(v,b_1(w),z) \\ &\quad - (-1)^{|v|+|w|}b_3(v,w,b_1(z))\,, \end{aligned} \tag{1.32}$$

where $|v|$ is the degree of v. The factors of -1 are present because of the grading of the algebra, which will be clarified in the next section. These definitions are in the b-picture of the L_∞ algebra, meaning that the products b_i have degree -1 [11]. Since the terms on the RHS of (1.32) look like a total derivative, one says that the Jacobi identity is satisfied "up to homotopy". For an L_∞ algebra, this relation extends to higher products, meaning there is a Jacobi identity for b_3 which is satisfied up to terms involving a higher map b_4, which satisfies another identity involving a b_5, and so on. An L_∞ algebra is a chain complex equipped with multilinear maps which satisfy a generalized Jacobi identity. Since there are infinitely many products with infinitely many entries, one may wonder whether one can define an L_∞ algebra in closed form. In fact, this can be done and the cleanest way to define an L_∞ algebra is through its coalgebra. For this reason we start the next section by reviewing coalgebras, whose defining operations are coproducts, which take in one input and yield two or more outputs, and are thus in a sense dual to algebras. Although the coalgebra picture may be structurally simpler to work with, the algebra picture is still important and perhaps more intuitive.

An interacting theory is encoded through equipping an L_∞ algebra on the chain complex which describes the free theory. The action for fields $\phi \in X_0$ in the full interacting theory is given by

$$S = \frac{1}{2}\langle\phi,\partial\phi\rangle + \frac{1}{3!}\langle\phi,b_2(\phi,\phi)\rangle + \frac{1}{4!}\langle\phi,b_3(\phi,\phi,\phi)\rangle + \cdots \frac{1}{(n+1)!}\langle\phi,b_n(\phi,\cdots,\phi)\rangle \tag{1.33}$$

and the equations of motion are given by

$$E = \partial\phi + \frac{1}{2}b_2(\phi,\phi) + \frac{1}{3!}b_3(\phi,\phi,\phi) + \cdots = 0\,. \tag{1.34}$$

Using this L_∞ algebra framework, let us now sketch how reformulations are described. Given a reformulation of a physical theory, the physically relevant data must be preserved. Just as we described the physical observables as the quotient space (1.6) of the kernel of the equations of motion modulo gauge transformations, it is easy to see that the physical observables of the chain complex $(V_\bullet, \partial)$ are in its homology

$$H_0(V_\bullet) = \text{Ker}\partial_0/\text{Im}\partial_1\,, \tag{1.35}$$

where ∂_0 maps fields in V_0 to equations of motion in V_{-1} and ∂_1 maps gauge parameters in V_1 to fields. As hinted earlier, the reformulation of a theory in terms of its gauge invariant variables has a mathematical meaning, and in the language of L_∞ algebras, it constitutes an operation called homotopy transfer. One starts with a chain complex with an L_∞ algebraic structure, and we ask whether this algebraic structure can be transferred such that the homologies are equivalent. The conditions that are needed for the transfer are given by the homotopy transfer theorem, which will be explained in the next section. Once we verify that the algebraic structure can be transferred, we can determine the non-linear corrections to the gauge invariant variables through a recipe given by the homological perturbation lemma. This will be discussed in more detail in section 1.3.2.

In order to build a quantum formulation, one can make use of the Batalin-Vilkovisky (BV) formalism, whose underlying structure is the BV algebra which is closely related to L_∞ algebras. In fact, the BV algebra is often used to derive a quantum L_∞ algebra [24]. In our work, we apply the BV formalism to compute quantum expectation values. The BV formalism was originally invented to make sense of path integrals over gauge redundant field configurations. It is a generalization of BRST quantization and allows for the quantization of theories that could not otherwise be quantized via BRST, such as theories with open gauge algebras where the algebra of the gauge transformations is closed only when the equations of motion are satisfied. Unlike with BRST quantization, the BV formalism has an underlying algebraic structure on which one can apply homotopy algebraic techniques. In section 1.3.3 we will define BV algebras and briefly discuss how they are related to L_∞ algebras.

1.3 Mathematical Preliminaries

1.3.1 Coalgebras

Coalgebras are dual to algebras in the sense that its defining operation is a coproduct, which takes in one input and gives out two or more outputs. Although this operation being the opposite of a product might seem unusual, the notion of a coalgebra comes naturally from an algebra. An algebra (V, b) over a field $\mathbb{K}$ is a vector space V with a bilinear product $b : V \otimes V \to V$. Consider the dual vector space V^*. One can ask, what structure does the product induce on the dual space? The product induces the map:

$$\Delta : V^* \to V^* \otimes V^* \,, \tag{1.36}$$

where Δ is defined by

$$\Delta(\lambda)(v, w) = \lambda(b(v, w)) \,, \qquad \text{for all } \; v, w \in V \text{ and } \lambda \in V^* \,. \tag{1.37}$$

We consider associative algebras with a unit. From these properties, one can induce conditions on the coproduct. From associativity one can verify that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \,. \tag{1.38}$$

Since the algebra has a unit, one can show

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta \,, \tag{1.39}$$

given the map $\epsilon : V^* \to \mathbb{K}$, which is also called a counit. We now give the formal definition of a coalgebra. A coalgebra (W, Δ) over a field $\mathbb{K}$ is a vector space W with a coproduct Δ which satisfies (1.38) and (1.39).

As an aside, a notable appearance of the coproduct in physics is in quantum mechanics, namely in the addition of angular momenta. Consider two particles, each one being an irreducible representation of the algebra $su(2)$. The total system of these two particles is a tensor product representation. The tensor product representation decomposes into a sum of irreducible representations and the total angular momentum operator acts as a coproduct.

Just as we can have homomorphisms of algebras, we can have cohomomorphisms of coalgebras. Given two coalgebras, $S_1 = (W_1, \Delta_1)$ and $S_2 = (W_2, \Delta_2)$, a coalgebra morphism $f : S_1 \to S_2$ preserves the above properties of the coproduct if:

$$(f \otimes f) \circ \Delta_1 = \Delta_2 \circ f \tag{1.40}$$

and

$$\epsilon_2 \circ f = \epsilon_1 , \tag{1.41}$$

where ϵ_1 and ϵ_2 are the counits of coalgebras S_1 and S_2, respectively. Analogous to a derivation on algebras satisfying the Leibniz rule, a coderivation D is a linear map $D : W \to W$ which satisfies the co-Leibniz rule,

$$\Delta \circ D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D) \circ \Delta . \tag{1.42}$$

If a coalgebra is equipped with a coderivation which squares to zero, it is called a differential coalgebra.

Let us now consider coalgebras on a graded vector space. A graded vector space V is a vector space that can be decomposed into a sum of vector subspaces V_n,

$$V = \bigoplus_n V_n , \tag{1.43}$$

and n is called the degree. The notion of a graded vector space is rather general. One can take any vector space and assign labels to its subspaces. For example, $\mathbf{R}^2$ can be a graded vector space– one could label the elements on the x-axis by degree "blue" and the elements on the y-axis by degree "orange". Another example of a graded vector space is the tensor algebra $T(V)$, which is the sum of all nth tensor powers of V

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n} . \tag{1.44}$$

This has a natural grading, where the degree-n subspace is the nth tensor power of V,

$$T(V) = \bigoplus_{n \in \mathbb{N}} T^n V , \qquad \text{where} \quad T^n V = V^{\otimes n} . \tag{1.45}$$

The product is the map $b : T^n V \otimes T^m V \to T^{n+m} V$. Hence the tensor algebra is a graded algebra.

Given a tensor algebra, one can define two distinct coalgebra structures. The first one is the cofree coalgebra which is dual to the algebra $T(V^*)$ in the sense that was described at the beginning of this section. It is defined by the coproduct

$$\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} (v_1 \cdots v_i) \otimes (v_{i+1} \cdots v_n) \,, \tag{1.46}$$

where $v_i \in V$ and the tensor product in $T(V)$ has been omitted. The brackets $(v_1 \cdots v_0)$ and $(v_{n+1} \cdots v_n)$ are set to the 1. For example,

$$\begin{aligned} \Delta(v) &= 1 \otimes v + v \otimes 1 \,, \\ \Delta(vw) &= 1 \otimes (vw) + v \otimes w + (vw) \otimes 1 \end{aligned} \tag{1.47}$$

where $v, w \in V$. This coproduct gives all the possible splittings of the object $v_1 \cdots v_n$ into two parts and preserves the order of the elements. It also preserves the grading, meaning that the sum of the degrees of the objects on the LHS is the sum of the degrees of the objects on the RHS.

The second coalgebra is defined by the coproduct

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in S(i,n-i)} (v_{\sigma(1)} \cdots v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \cdots v_{\sigma(n)}) \,, \tag{1.48}$$

where $S(p,q)$ is a permutation of $p+q$ elements called an unshuffle. A (p,q) unshuffle is a permutation that preserves the order of the first p elements and the order of the second q elements. An example of an unshuffle is the bridge shuffle or a riffle of cards, where one takes a deck of cards, splits it into two decks, and shuffles the two decks together but maintains the relative order in each deck. The above coproduct (1.48) is defined in two steps. The first is its definition acting on a degree 1 object,

$$\Delta(v) = v \otimes 1 + 1 \otimes v \,, \quad \text{where} \quad v \in V \subset T(V) \,. \tag{1.49}$$

The second step is to demand that the coproduct is a homomorphism of algebras $\Delta : T(V) \to T(V) \otimes T(V)$ (the tensor product of two algebras over fields is also an algebra),

$$\Delta(v_1 v_2) = \Delta(v_1) \otimes \Delta(v_2) \,, \qquad v_1, v_2 \in T(V) \,. \tag{1.50}$$

With these two definitions, one can extend (1.50) on elements of degree n. The property of the coproduct being an algebra homomorphism defines what is called a bialgebra.[2]

In the above examples, we have seen that from a vector space, one can build a tensor algebra. When the vector space itself has no explicit grading, one can make the tensor algebra a symmetric algebra, by imposing invariance under permutations. However one could also consider the vector space with a grading, from which a graded symmetric algebra can be defined.

A graded symmetric algebra $S^c(V_\bullet)$ is defined as

$$S^c(V_\bullet) = \bigoplus_{n\geq 1} V_\bullet^{\wedge n} \tag{1.51}$$

where $\wedge$ is the graded symmetric product and $V_\bullet$ is a graded vector space. The graded symmetric product respects the grading by following the Koszul sign rule,

$$v \wedge w = (-1)^{|v||w|} w \wedge v\,, \tag{1.52}$$

where v and w are homogeneous elements in $V_\bullet$, and $|v|$ and $|w|$ are their respective degrees. For the graded symmetric product of inhomogeneous elements, one extends this rule linearly.

One can obtain a graded symmetric coalgebra defined by the coproduct,

$$\Delta : S^c(V_\bullet) \to S^c(V_\bullet) \otimes S^c(V_\bullet)\,, \tag{1.53}$$

$$\Delta(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{n-1} \sum_{\sigma\in S(i,n-i)} e(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \wedge \cdots \wedge v_{\sigma(n)})\,, \tag{1.54}$$

where $e(\sigma)$ is the appropriate sign following the rule in (1.52), i.e.

$$v_1 \cdots v_n = e(\sigma) v_{\sigma(1)} \cdots v_{\sigma(n)}\,, \tag{1.55}$$

and $S(i, n-i)$ are $(i, n-i)$-unshuffles. The $i = 0$ term corresponds to 1. In particular, on a degree 1 element $v \in V_\bullet$, the coproduct acts as

$$\Delta(v) = v \otimes 1 + (-1)^{|v|} 1 \otimes v = v - v = 0\,. \tag{1.56}$$

[2]In fact, bialgebras appear elsewhere in physics, e.g. Hopf algebras which can be applied in renormalization techniques in quantum field theory[25, 26].

For our purposes, let $S^c(V_\bullet)$ be equipped with a coderivation of degree -1 which squares to zero,

$$D : S^c(V_\bullet) \to S^c(V_\bullet)\,, \qquad D^2 = 0\,. \tag{1.57}$$

If one restricts the coderivation to degree n and projects to $V_\bullet$, one obtains the products

$$b_n : V_\bullet^{\wedge n} \to V_\bullet\,, \tag{1.58}$$

of degree -1. These products can be extended to the graded symmetric coalgebra by defining

$$D_k(v_1 \wedge \cdots \wedge v_n) := \sum_{\sigma \in S(k,n-k)} e(\sigma) b_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \wedge \cdots \wedge v_{\sigma(n)}\,, \quad \text{for } n < k\,. \tag{1.59}$$

One can show that D_k is a coderivation on $S^c(V_\bullet)$. Now we are ready to define an L_∞ algebra.

An L_∞ algebra is a graded vector space $V_\bullet$ with the products b_n taken from the coderivation acting on the differential graded symmetric coalgebra given by $\big(S^c(V_\bullet), \Delta, D\big)$. By the nilpotency of the coderivation D, we can derive relations among the products b_n. The first three are:

$$\begin{aligned}
0 &= b_1^2\,, && (1.60)\\
0 &= b_1(b_2(vw)) + b_2(b_1(v)w) + (-1)^{|v|} b_2(v b_1(w))\,, && (1.61)\\
0 &= b_2(b_2(v,w),z) + (-1)^{|z|} b_2(b_2(z,v),w) + (-1)^{|w|+|z|} b_2(b_2(w,z),v)\\
&\quad + b_1(b_3(v,w,z)) + b_3(b_1(v),w,z) + (-1)^{|v|} b_3(v,b_1(w),z)\\
&\quad + (-1)^{|v|+|w|} b_3(v,w,b_1(z))\,, && (1.62)\\
&\ \ \vdots
\end{aligned}$$

These relations continue to infinity and they are known as the L_∞ relations.[3] The first relation (1.60) states that b_1 is nilpotent; this is the differential on $V_\bullet$. Thus, we see that $V_\bullet$ together with $b_1 : V_\bullet \to V_\bullet$ defines a chain complex. This was our starting point for defining an L_∞ algebra in the previous section. The second relation (1.61) states that b_1 acts as a derivation of b_2, and the third relation (1.62) is the generalized Jacobi identity of b_2, as already introduced in (1.32). The condition that the coderivation on the differential graded symmetric coalgebra

[3] In the literature one often finds these relations with different sign conventions in what is called the l-picture, where the products l_n have degree $n-2$, see for instance [22].

is nilpotent implies the generalized Jacobi identities satisfied by the multi-linear maps of the L_∞ algebra.

We began this subsection with an algebra on a vector space and finding a coalgebra structure on the dual space. Then we discussed what would happen if we had an algebraic structure on a graded vector space. Finally, we considered the graded symmetric algebra and constructed a differential graded symmetric coalgebra. Given a differential graded symmetric coalgebra $\big(S^c(V_\bullet), \Delta, D\big)$, we found an L_∞ algebraic structure on the graded vector space $V_\bullet$.

1.3.2 Homotopy Transfer

Given an algebra (V, b) and a vector space W, can one transfer the algebraic structure from V to W such that the two algebraic structures are equivalent, i.e. isomorphic? The answer is yes, when there exists an isomorphism between V and W. Let $p : V \to W$ be an isomorphism and $i : W \to V$ its inverse, then the transferred product c on W is

$$c(w_1, w_2) = p\big(b(i(w_1), i(w_2))\big) \,. \tag{1.63}$$

For algebras on chain complexes, the story is not so simple. A morphism $f : (V_\bullet, \partial) \to (W_\bullet, \bar{\partial})$ between two chain complexes, also known as a chain map, is a family of homomorphisms $f_i : V_i \to W_i$ that satisfy

$$f_{n-1}\partial_n = \bar{\partial}_n f_n \,. \tag{1.64}$$

This means that the following diagram must commute:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & V_0 & \xrightarrow{\partial_0} & V_{-1} & \longrightarrow & \cdots \\ & & \big\downarrow f_0 & & \big\downarrow f_{-1} & & \\ \cdots & \longrightarrow & W_0 & \xrightarrow{\bar{\partial}_0} & W_{-1} & \longrightarrow & \cdots \end{array} \tag{1.65}$$

In general one does not have an isomorphism between chain complexes. However, it is sufficient to consider isomorphisms between the homologies of the chain complexes, because this is where the physical data live. Given a chain complex $(V_\bullet, \partial)$ the nth homology group is

$$H_n(V_\bullet) = \frac{\text{Ker }\partial_n}{\text{Im }\partial_{n+1}}\,. \tag{1.66}$$

One can show that a chain map induces a homomorphism on homology groups: $H_n(f) : H_n(V_\bullet) \to H_n(W_\bullet)$. If $H_n(f)$ is an isomorphism for all n, then the chain map f is called a quasi-isomorphism. Quasi-isomorphisms of chain complexes induce isomorphisms on homologies.

If there exist quasi-isomorphisms between the chain complexes, the algebraic structure can always be transferred. The homotopy transfer theorem states that if chain maps $p : V_\bullet \to W_\bullet$ and $i : W_\bullet \to V_\bullet$ satisfy the relation:

$$i \circ p = \text{id}_V - \partial \circ h - h \circ \partial\,, \tag{1.67}$$

where h is a degree $+1$ map $h_i : V_i \to V_{i+1}$, then p and i are quasi-isomorphisms and therefore the algebraic structure on $V_\bullet$ can be transferred to $W_\bullet$. $W_\bullet$ is called a homotopy retract of $V_\bullet$. The new algebraic structure on $W_\bullet$ is equivalent to that on $V_\bullet$ up to homotopy, meaning that if $V_\bullet$ is equipped with a Lie algebra, then the transferred structure on $W_\bullet$ is an L_∞ algebra. This is exactly what we need since the L_∞ structure is what we want to start with and what we want to transfer.

The chain complexes together with the quasi-isomorphisms and homotopy map h,

$$(V_\bullet, \partial), h \underset{i}{\overset{p}{\rightleftarrows}} (W_\bullet, \bar{\partial})\,, \tag{1.68}$$

make up what is called a homotopy equivalence data. If in addition $p \circ i = \text{id}_W$, $W_\bullet$ is called a deformation retract of $V_\bullet$. This condition is important when we want to invoke the homological perturbation lemma, which will help us find how these new transferred products act on $W_\bullet$.

Let us now figure out the expressions for the transferred products. For a 2-product, it is easy to see that we obtain the same expression in (1.63). To find the transferred 3-product, we have to consider all the possible ways of taking a product in $V_\bullet$,

$$\begin{aligned} c_3(w_1, w_2, w_3) \;=\; & p\big(\, b_3(i(w_1)\,, i(w_2)\,, i(w_3))\,\big) \\ & + b_2\big(h(b_2(i(w_1)\,, i(w_2)))\,, i(w_3)\big) \\ & + \cdots\,. \end{aligned} \tag{1.69}$$

In order to find general expressions for transferring multi-linear products, there is an efficient recipe that comes from the homological perturbation lemma. If the multi linear products are small perturbations of the differential on a chain complex, the homological perturbation lemma gives the necessary conditions for transferring the algebraic structure of a perturbed chain complex.

Let us first state the homological perturbation lemma. Consider the perturbed data with perturbation δ,

$$(V_\bullet, \partial + \delta),\, h' \underset{i'}{\overset{p'}{\rightleftarrows}} (W_\bullet, \bar{\partial})\,, \tag{1.70}$$

where the perturbed differential squares to zero

$$(\partial + \delta)^2 = 0\,. \tag{1.71}$$

The homological perturbation lemma states that the perturbed data is a homotopy equivalence data, if i', p', and h' are written as [27]:

$$\begin{aligned} i' &= i - h(\mathrm{id} + \delta h)^{-1}\delta i\,, \quad & p' &= p - p(\mathrm{id} + \delta h)^{-1}\delta h\,, \\ h' &= -h(\mathrm{id} + \delta h)^{-1} h\,, & \bar{\partial} &= p\partial i + p(\mathrm{id} + \delta h)^{-1}\delta i\,. \end{aligned} \tag{1.72}$$

By using the identities $(\mathrm{id} + \delta h)^{-1} = \mathrm{id} - (\mathrm{id} + \delta h)^{-1}\delta h$ and $(\mathrm{id} + h\delta)^{-1} = \mathrm{id} - h(\mathrm{id} + \delta h)^{-1}\delta$[4], the expressions for i' and p' can be brought to simpler forms:

$$i' = (\mathrm{id} + h\delta)^{-1} i\,, \quad p' = p(\mathrm{id} + \delta h)^{-1}\,. \tag{1.73}$$

Because we are interested in the homotopy transfer of L_∞ algebras, we want δ to consist of the products $b_n : V_\bullet^{\wedge n} \to V_\bullet$ except for the linear piece. However, in order to define the perturbation as $\delta = \sum_{n\geq 2} b_n$, the homological perturbation lemma needs to be applied to coalgebras and coalgebra morphisms. In other

[4]To prove the identity, $(\mathrm{id} + h\delta)^{-1} = \mathrm{id} - h(\mathrm{id} + \delta h)^{-1}\delta$, let $a = (\mathrm{id} + \delta h)^{-1}$. Applying the identity $(\mathrm{id} + \delta h)a = \mathrm{id}$ on the left of δ yields

$$(\mathrm{id} + \delta h)a\delta = a\delta + \delta h a\delta = \delta\,.$$

Using this relation, it follows that

$$(\mathrm{id} + h\delta)(\mathrm{id} - ha\delta) = \mathrm{id} + h\delta - ha\delta - h\delta h a\delta = \mathrm{id} + h(\delta - a\delta - \delta h a\delta) = \mathrm{id}\,.$$

words, we need to consider the situation

$$(S^c(V_\bullet), \partial + \delta), h' \underset{i'}{\overset{p'}{\rightleftarrows}} (S^c(W_\bullet), \bar{\partial}) \tag{1.74}$$

where ∂ and δ are extended to act on $S^c(V_\bullet)$ as in (1.59). Then since $\partial + \delta$ make up a coderivation (1.57) which is nilpotent, the square-zero condition (1.71) is satisfied. For ensuring the homotopy equivalence of coalgebra structures equipped with coderivations, one cannot use the homological perturbation lemma alone, since it only maintains that there exist quasi-isomorphisms between chain complexes. However if one imposes extra conditions given by

$$h \circ i = 0\,, \quad p \circ h = 0\,, \quad h^2 = 0\,, \tag{1.75}$$

then $W_\bullet$ is called a strong deformation retract of $V_\bullet$ and the transferred structure has the desired coalgebra structure [28]. Then the modified projection and inclusion maps p' and i' in (1.73) are cohomomorphisms. By lifting the maps (1.73) to the graded symmetric coalgebras, one can derive the transferred products on $W_\bullet$. This concludes our review of L_∞ algebras.

1.3.3 BV Algebra

A BV algebra is a type of Poisson algebra, the algebraic structure that appears in classical mechanics. In Hamiltonian mechanics, the possible configurations of a system can be modelled as a (symplectic) manifold, and the space of real smooth functions over this manifold is a Poisson algebra. In the BV formalism, in addition to the classical field content, one introduces opposite parity variables. Hence the Poisson algebra acquires a grading and is promoted to a graded Poisson algebra. Furthermore, the BV formalism gives rise to quantization by equipping the graded Poisson algebra with an operator called the BV Laplacian. The resulting algebraic structure is a BV algebra.

Let us first define a Poisson algebra. A Poisson algebra is an associative commutative algebra $(A, \cdot)$ with a Lie bracket $\{-,-\}$ that satisfies the Leibniz rule, meaning that it acts as a derivation on the product $\cdot$,

$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}\,, \quad \text{for any} \quad a, b, c \in A\,. \tag{1.76}$$

The product $\{-,-\}$ is called a Poisson bracket. This is the product that appears in Hamiltonian mechanics– the familiar Poisson bracket which acts on two functions $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ on the phase space given by canonical coordinates (q_i, p_i):

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) . \tag{1.77}$$

Following our previously discussed logic, let us add a grading to the Poisson algebra $(A, \cdot, \{-, -\})$. Now the product $\cdot$ is graded commutative which means:

$$a \cdot b = (-1)^{|a||b|} b \cdot a\,, \quad \text{for} \quad a, b \in A\,, \tag{1.78}$$

where $|a|$ denotes the degree of the element a. The Poisson bracket is assigned a degree n meaning that

$$|\{a, b\}| = |a| + |b| + n\,. \tag{1.79}$$

The grading changes the properties of the Poisson bracket, for example its antisymmetry,

$$\{a, b\} = -(-1)^{(|a|+n)(|b|+n)} \{b, a\}\,, \tag{1.80}$$

and the Leibniz rule in (1.76) modified as

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(|a|+n)|b|} b \cdot \{a, c\}\,. \tag{1.81}$$

The bracket also obeys a graded Jacobi identity

$$\{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+|c|)} \{\{b, c\}, a\} + (-1)^{(|c|+n)(|a|+|b|)} \{\{c, a\}, b\} = 0\,. \tag{1.82}$$

There are various choices for the degree of the Poisson bracket. To define a BV algebra, we will set $n = 1$.

A BV algebra is a graded Poisson algebra with Poisson bracket of degree $+1$ equipped with a nilpotent operator Δ of degree $+1$ which acts as a derivation for $\{-, -\}$,

$$\Delta\{a, b\} = \{\Delta a, b\} + (-1)^{a+1} \{a, \Delta b\}\,, \tag{1.83}$$

and satisfies

$$\Delta(a \cdot b) = \Delta a \cdot b + (-1)^{|a|} a \cdot \Delta b + (-1)^{|a|} \{a, b\}\,. \tag{1.84}$$

Another way of stating the second condition (1.84) is that the Poisson bracket is the failure of Δ being a derivation for the product $\cdot$. In the context of BV algebras, the bracket $\{-,-\}$ is often referred to as the anti-bracket, and the operator Δ is called the BV Laplacian.

In the BV formalism, the graded vector space on which we define a BV algebra is also a chain complex. The differential on the chain complex is given by

$$\delta = \{S, -\} - i\hbar\Delta\,, \tag{1.85}$$

where S is the action of the theory and has degree 0. The nilpotency of δ gives a condition on S, which is the so-called Maurer-Cartan equation:

$$\frac{1}{2}\{S,S\} - i\hbar\Delta S = 0\,. \tag{1.86}$$

This is how the structure of a gauge theory is described: a BV algebra and an action S which satisfies the Maurer-Cartan equation.

In the previous section, we mentioned that the BV algebra is used to derive a quantum L_∞ algebra. Let us now briefly shed some light on this. An L_∞ algebra $(V_\bullet, b_n)$ is defined on a chain complex made up of fields, their equations of motion, gauge parameters, etc.. The BV complex is actually the dual space: it contains the functionals of fields and degree -1 objects (belonging to the space of equations of motion) which are called anti-fields in the BV formalism. The classical part of the BV differential, $Q \equiv \{S, -\}$, is the dual operator to the coderivation D on the coalgebra from which one can define an L_∞ algebra. One can see this by expanding Q in a basis (x^i, z^i) where x are commuting variables and z are anti-commuting variables:

$$Q = \sum Q^i \frac{\partial}{\partial z^i} = \sum f^i_{j_1\ldots j_n} z^{j_1}\cdots z^{j_n}\frac{\partial}{\partial z^i}\,, \tag{1.87}$$

where $f^i_{j_1\ldots j_n}$ are coefficients. These coefficients are related to the L_∞ brackets in the basis e_i of $V_\bullet$,

$$b_n(e_{i_1},\ldots,e_{i_n}) = f^j_{i_1\ldots i_n} e_j\,. \tag{1.88}$$

By adding a quantum part to Q and defining the full BV differential $\delta = Q - i\hbar\Delta$ and applying homotopy algebraic techniques, the BV formalism allows for a procedure to find a quantum version of an L_∞ algebra [3, 29].

Chapter 2

Gauge Invariant Perturbation Theory via Homotopy Transfer

In [1] Bardeen introduced a procedure to construct gauge invariant variables in linear cosmological perturbation theory, which considers linear perturbations to Einstein gravity around an expanding background spacetime. It is especially useful to construct gauge invariant variables in this context, due to the complication that both the perturbed spacetime and the time-dependent background spacetime are affected by coordinate transformations. Non-linear cosmological perturbation theory remains a challenge, see for instance [30, 31], and even second-order computations have been quite involved [32–40].

In this chapter, we will extend Bardeen's procedure in order to express perturbation theory in terms of gauge invariant variables to all orders in perturbations. To this end we will interpret this construction through the L_∞ algebraic framework and we will see that this constitutes building a homotopy equivalence data. By applying the perturbation lemma, we will show how to derive gauge invariant variables to all orders in perturbations. Yang-Mills theory and gravity on flat and cosmological backgrounds will be worked out as examples. The results in this chapter are published in [18].

Before going into detail, let us summarize our general approach. Starting with a free theory with a gauge field A, one can find the gauge invariant variables $\bar{A}$, for instance with Bardeen's procedure to be reviewed shortly in section 2.1. Let this be encoded by a map $p_0 : \phi \to \bar{\phi}$ between the space of gauge fields ϕ and the space of gauge invariant fields $\bar{\phi}$. The crucial (yet perhaps unspectacular) point to note is that any configuration of the gauge field will differ from its gauge invariant configuration by a gauge transformation:

$$A = p_0(A) + \delta A \,, \tag{2.1}$$

where $p_0(A) = \bar{A}$. Then once we insert the expression for A into the action, the pure gauge terms with δA will drop out, because the free action is invariant under these (linearized) gauge transformations. The resulting action will be in terms of the gauge invariant $\bar{A}$.

In the homotopy algebraic framework, linear gauge transformations are encoded by the map $\partial : X_{-1} \to X_0$, where X_{-1} is the space of gauge transformations and X_0 is the space of gauge fields. The gauge transformation on the RHS of (2.1) can be interpreted as the map ∂ acting on some function of A:

$$A = p_0(A) + \partial \cdot s(A)\,. \tag{2.2}$$

For degree reasons, one can infer that $s(A)$ belongs to the space of gauge parameters X_{-1}. We will see that $s(A)$ is given by the homotopy map $s : X_0 \to X_{-1}$ (called h in section 1.3 but in this chapter, h will denote metric perturbations) and the equation (2.2) is the condition necessary for the homotopy equivalence between the chain complex of the free theory to the gauge invariant complex. To extend this to all orders in perturbations, we use the same logic to write the gauge field in terms of its gauge invariant part plus a gauge transformation, but instead of considering only infinitesimal linearized gauge transformations, we take finite gauge transformations,

$$A = e^{\Delta} p(A)\,, \tag{2.3}$$

where Δ is the operation that encodes the infinitesimal non-linear gauge transformations and $p(A)$ is the gauge invariant variable to all orders. We will show how p is determined by perturbing the free part p_0 by the interactions of the theory using the homological perturbation lemma and why (2.3) is the correct object that we can insert into the action to obtain the action in terms of gauge invariant variables to all orders.

2.1 Gauge Invariant Variables

Let us review Bardeen's procedure for constructing gauge invariant variables, based on the original article [1] and the reviews [41, 42]. The first step to building gauge invariant variables is by splitting the perturbations into various scalar, vector, and tensor modes, i.e. performing what is known as a scalar-vector-tensor (SVT) decomposition. Then one inspects the gauge transformations for these modes and combines them to obtain gauge invariant variables.

We demonstrate this procedure with a simple example in Maxwell's theory of electromagnetism. Consider the action,

$$S[A] = -\frac{1}{4}\int d^4x \; F^{\mu\nu}F_{\mu\nu} \,, \tag{2.4}$$

where the field strength of the electromagnetic field A_μ is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action is invariant under the gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda \,. \tag{2.5}$$

The gauge field can be split into its temporal and spatial components: $A_\mu = (A_0\,, A_i)$. Then the 3-vector A_i can be split even further into a divergenceless vector and a gradient of a scalar

$$A_i = \hat{A}_i + \partial_i \Phi \,. \tag{2.6}$$

where $\partial^i \hat{A}_i = 0$. This decomposition in (2.6) is the familiar Helmholtz decomposition of smooth bounded vectors in 3 dimensions. For this decomposition to be well-defined, one needs to be able to express the components Φ and $\hat{A}_i$ in terms of the original field A_i. By taking the divergence of (2.6), we obtain

$$\partial^i A_i = \Delta \Phi \,, \tag{2.7}$$

where $\Delta \equiv \partial^i \partial_i$. Assuming the Laplacian Δ is invertible, we find that

$$\Phi = \Delta^{-1}(\partial^i A_i) \tag{2.8}$$

which with (2.6) implies

$$\hat{A}_i = A_i - \partial_i \Delta^{-1}(\partial^j A_j) \,. \tag{2.9}$$

The assumption of the invertibility of the Laplacian follows from the assumption that all components decay rapidly at infinity. Concretely, to check that one can invert the Laplacian, we need all harmonic functions to be zero, i.e.

$$\Delta f = 0 \,, \quad \rightarrow \quad f = 0 \,. \tag{2.10}$$

Since the harmonic functions are bounded, one can show that they must be constant, and therefore we can set their values to zero. Thus by this assumption, the

Laplacian is invertible.

Let us check that this decomposition preserves the number of degrees of freedom, i.e. independent components. Since the divergenceless vector $\hat{A}_i$ satisfies one constraint, it has 2 degrees of freedom. The scalar modes A_0 and Φ each have one degree of freedom. Together, A_0, Φ, and $\hat{A}_i$ encode in total 4 degrees of freedom, which is exactly what we started with in A_μ.

Let us now take a look at the gauge transformations of these components under (2.5). For A_0 we simply take the zeroth component of (2.5), $\delta A_0 = \dot{\Lambda}$ (where now the dot represents the time derivative). For the vector component,

$$\delta A_i = \delta \hat{A}_i + \partial_i(\delta\Phi) = \partial_i \Lambda\,. \tag{2.11}$$

By taking the divergence of both sides and inverting the Laplacian, we find $\delta\Phi = \Lambda$. It follows that the divergenceless components $\hat{A}_i$ are gauge invariant. Collecting all the gauge transformations:

$$\delta A_0 = \dot{\Lambda}\,, \qquad \delta \hat{A}_i = 0\,, \qquad \delta\Phi = \Lambda\,. \tag{2.12}$$

It is easy to see that one can build an additional gauge invariant combination:

$$\hat{\Phi} = A_0 - \dot{\Phi}\,. \tag{2.13}$$

Finally, we are ready to rewrite the action in terms of gauge invariant variables. After expanding the action in terms of the components, $(A_0, \Phi, \hat{A}_i)$, reorganizing the action, and integrating by parts, we obtain the manifestly gauge invariant action,

$$S = \frac{1}{2}\int d^4x \left(\hat{A}^i \Box \hat{A}_i - \hat{\Phi}\Delta\hat{\Phi}\right). \tag{2.14}$$

By varying with respect to the new fields, the equations of motion are

$$\Box \hat{A}_i = 0\,, \qquad \Delta\hat{\Phi} = 0\,. \tag{2.15}$$

Here we see one of the advantages of this decomposition: it is easy to see which modes of the gauge field are propagating degrees of freedom. We realize that $\hat{A}_i$ encodes the 2 propagating degrees of freedom of the photon. With the invertibility of the Laplacian, the equation of motion for $\hat{\Phi}$ is $\hat{\Phi} = 0$, signifying that the scalar mode $\hat{\Phi}$ does not propagate.

Another advantage is that the gauge invariant scalar and vector modes do not couple and thus their equations of motion are simpler to solve. Since a scalar can only couple to a vector through its divergence, and in this procedure the divergence of the vector mode is absorbed into the gauge invariant scalar perturbation, the scalar-vector coupling does not appear in the manifestly gauge invariant action. Ensuring the absence of these couplings can also be seen as a guiding principle for performing the decomposition of the gauge field.

2.1.1 Linearized Gravity on Flat Space

We now perform the same procedure for linearized gravity on flat space. The Einstein-Hilbert action is

$$S = \int d^4x \sqrt{-g}\, R \tag{2.16}$$

where the g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci curvature scalar. The metric is expanded around the Minkowski metric $\eta_{\mu\nu}$ as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, . \tag{2.17}$$

The action is invariant under the gauge transformations

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \, . \tag{2.18}$$

From this we can find the gauge transformation of the fluctuation $h_{\mu\nu}$. Let ξ^μ be a first-order gauge parameter so the transformation generated by ξ^μ does not act on the background. Hence the fluctuation of the metric transforms as:

$$\delta g_{\mu\nu} = \delta(\eta_{\mu\nu} + h_{\mu\nu}) = \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \, . \tag{2.19}$$

We first split the metric fluctuation into the following independent components $h_{\mu\nu} = (h_{00}\, , h_{0i}\, , h_{ij})$. With the foresight that we want scalar, vector, and tensor modes to eventually decouple in the action, we perform the decomposition:

$$\begin{aligned} h_{00} &= -2\phi \, , \\ h_{0i} &= B_i + \partial_i B \, , \\ h_{ij} &= \widehat{h}_{ij} + 2C\delta_{ij} + \partial_i E_j + \partial_j E_i + 2\left(\partial_i \partial_j E - \frac{1}{3}\delta_{ij}\Delta E\right) , \end{aligned} \tag{2.20}$$

where B_i and E_i are divergenceless,

$$\partial^i B_i = \partial^i E_i = 0\,, \tag{2.21}$$

and $\widehat{h}_{ij}$ is transverse and traceless,

$$\partial^i \widehat{h}_{ij} = 0\,, \qquad \delta^{ij}\widehat{h}_{ij} = 0\,. \tag{2.22}$$

To find the gauge transformations of the components, we also need to decompose the gauge parameters into scalar and divergenceless vector parts:

$$\xi_\mu = (\xi_0, \xi_i)\,, \qquad \xi_i = \zeta_i + \partial_i \chi\,, \tag{2.23}$$

where $\partial^i \zeta_i = 0$. With the decomposition of the gauge parameters, the gauge transformations in (2.19) can be written as:

$$\delta h_{00} = 2\dot{\xi}_0\,, \qquad \delta h_{0i} = \dot{\zeta}_i + \partial_i \dot{\chi}\,, \qquad \delta h_{ij} = \partial_i \zeta_j + \partial_j \zeta_i + 2\partial_i \partial_j \chi\,. \tag{2.24}$$

One can then deduce the gauge transformations of the components (2.20):

$$\begin{aligned} \delta\phi &= -\dot{\xi}_0\,, \quad \delta B_i = \dot{\zeta}_i\,, \qquad \delta B = \dot{\chi} + \xi_0\,, \\ \delta\widehat{h}_{ij} &= 0\,, \qquad \delta C = \frac{1}{3}\Delta\chi\,, \quad \delta E_i = \zeta_i\,, \qquad \delta E = \chi\,. \end{aligned} \tag{2.25}$$

In addition to the tensor modes $\widehat{h}_{ij}$, there are three more gauge invariant combinations:

$$\Sigma_i \equiv \dot{E}_i - B_i\,, \qquad \Psi \equiv -C + \frac{1}{3}\Delta E\,, \qquad \Phi \equiv \phi + \dot{B} - \ddot{E}\,. \tag{2.26}$$

We count in total 6 gauge invariant degrees of freedom: 2 tensor modes, 2 vector modes, and 2 scalar modes. $\widehat{h}_{ij}$ is symmetric so it starts with 6 independent components, but it is subject to 3 transverse plus 1 traceless constraints in (2.22), ending up with 2 independent components. Σ_i starts with 3 components and is subject to 1 constraint, thus having 2 independent components. However, from general relativity we expect only the 2 tensor modes to be physical– these are the two polarizations of the gravitational wave.

Let us now check this and write the action to quadratic order in terms of these gauge invariant variables. To quadratic order in fluctuations on a flat background, the action can be written as

$$S_{FP} = -\frac{1}{2}\int d^4x \; h^{\mu\nu}G_{\mu\nu} \tag{2.27}$$

where $G_{\mu\nu}$ is the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu}\,, \tag{2.28}$$

where $R_{\mu\nu}$ is the Ricci tensor to linear order,

$$R_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\mu h_{\mu\rho} + \partial_\mu\partial_\nu h)\,, \tag{2.29}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, and R is the Ricci scalar to linear order

$$R = \eta^{\mu\nu}R_{\mu\nu} = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h\,. \tag{2.30}$$

Replacing the metric fluctuation in (2.27) with its components (2.20) and reorganizing the action into the gauge invariant variables $(\widehat{h}_{ij},\ \Sigma_i,\ \Psi,\ \Phi)$, we obtain

$$S_{FP} = \int d^4x \left(\frac{1}{4}\widehat{h}^{ij}\Box\widehat{h}_{ij} - \frac{1}{2}\Sigma^i\Delta\Sigma_i + (4\Phi - 2\Psi)\Delta\Psi + 6\Psi\ddot{\Psi}\right). \tag{2.31}$$

Varying the action with respect to the scalars Φ and Ψ, we obtain the equations

$$\Delta\Psi = 0\,, \qquad 4\Delta(\Phi - \Psi) + 12\ddot{\Psi} = 0\,. \tag{2.32}$$

By the invertibility of the Laplacian, we infer that $\Psi = 0$. It follows that the second equation of (2.32) becomes $\Delta\Phi = 0$, which then leads us to $\Phi = 0$. Similarly, Σ_i can also be integrated out. We are left with one equation of motion:

$$\Box\widehat{h}_{ij} = 0\,. \tag{2.33}$$

This indeed describes the dynamics of the two propagating degrees of freedom of the transverse-traceless gravitational wave.

It turns out that we can write the quadratic action in terms of the gauge invariant variables (2.31) in an even simpler fashion, by implementing the field redefinition

$$\Phi \quad \to \quad \Phi = \Phi' + \frac{1}{2}\Psi - \frac{3}{2}\Delta^{-1}\ddot{\Psi}\,. \tag{2.34}$$

With this redefinition, one easily arrives at

$$S_{FP} = \int d^4x \left(\frac{1}{4}\hat{h}^{ij}\Box\hat{h}_{ij} - \frac{1}{2}\Sigma^i\Delta\Sigma_i + 4\Phi'\Delta\Psi \right). \tag{2.35}$$

The last term can be made diagonal by using a second redefinition:

$$\Phi_\pm = 2\Psi \pm \Phi' . \tag{2.36}$$

Finally we obtain the diagonal form of the gauge invariant action:

$$S_{FP} = \int d^4x \left(\frac{1}{4}\hat{h}^{ij}\Box\hat{h}_{ij} - \frac{1}{2}\Sigma^i\Delta\Sigma_i + \frac{1}{2}\Phi_+\Delta\Phi_+ - \frac{1}{2}\Phi_-\Delta\Phi_- \right). \tag{2.37}$$

In this form, one can immediately see that Φ_+ and Φ_- (and hence Φ and Ψ) do not propagate and that they can be integrated out.

This procedure can also be applied in the presence of matter perturbations. For linearized gravity on flat space, the matter couplings are introduced via the energy-momentum tensor $T^{\mu\nu}$:

$$S_{\text{matter}} = \int d^4x \left(h^{\mu\nu}G_{\mu\nu} + \frac{1}{2}h_{\mu\nu}T^{\mu\nu} \right). \tag{2.38}$$

If we perform an SVT decomposition of the energy-momentum tensor, we can expect to separate the matter couplings into purely scalar, vector, and tensor parts. Let us decompose $T^{\mu\nu}$ as:

$$\begin{aligned} T^{00} &= \rho\,, \\ T^{0i} &= q^i + \partial^i q\,, \\ T^{ij} &= \Pi^{ij} + \partial^i\Pi^j + \partial^j\Pi^i + \partial^i\partial^j\Pi - \frac{1}{3}\delta^{ij}\Delta\Pi + p\delta^{ij}\,, \end{aligned} \tag{2.39}$$

where $\partial_i q^i = 0$ and $\partial_i\Pi^i = \delta_{ij}\Pi^{ij} = \partial_i\Pi^{ij} = 0$. In addition, the energy-momentum tensor must satisfy the conservation equation $\partial_\mu T^{\mu\nu} = 0$, which expressed in terms of the above components is:

$$\begin{aligned} \dot{\rho} + \Delta q &= 0\,, \\ \dot{q}^i + \Delta\Pi^i &= 0\,, \\ p + \dot{q} + \frac{2}{3}\Delta\Pi &= 0\,. \end{aligned} \tag{2.40}$$

By inserting (2.39) into the full action with matter couplings, together with the result in (2.31) we obtain

$$S_{\text{matter}} = \int d^4x \left(\frac{1}{4}\hat{h}^{ij}\Box\hat{h}_{ij} - \frac{1}{2}\Sigma^i\Delta\Sigma_i + (4\Phi - 2\Psi)\Delta\Psi + 6\Psi\ddot{\Psi} + \frac{1}{2}\hat{h}_{ij}\Pi^{ij} - \Sigma_i q^i - \Phi\rho - 3\Psi p \right). \tag{2.41}$$

After performing the field redefinitions (2.34) and (2.36), integrating by parts, and applying the conservation equations (2.40), the full action becomes

$$S_{\text{matter}} = \int d^4x \left(\frac{1}{4}\widehat{h}^{ij}\Box\widehat{h}_{ij} - \frac{1}{2}\Sigma^i\Delta\Sigma_i + \frac{1}{2}\Phi_+\Delta\Phi_+ - \frac{1}{2}\Phi_-\Delta\Phi_- + \frac{1}{2}\widehat{h}_{ij}\Pi^{ij} - \Sigma_i q^i - \frac{1}{2}(\Phi_+ - \Phi_-)\rho - \frac{1}{8}(\Phi_+ + \Phi_-)(\rho + 3p - 2\Delta\Pi) \right). \tag{2.42}$$

Since Φ_+, Φ_-, and Σ_i can be integrated out, what we are left with is the equation of motion for $\widehat{h}_{ij}$:

$$\Box\widehat{h}_{ij} = -\Pi_{ij}\,. \tag{2.43}$$

2.1.2 Gravity on FLRW Backgrounds

Let us continue to determine the gauge invariant variables for gravity on Friedmann-Lemâitre-Robertson-Walker (FLRW) backgrounds. The FLRW metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{2.44}$$

where $a(t)$ is the scale factor and γ_{ij} is the spatial metric with constant curvature. It is often convenient to define the conformal time η by

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}\,. \tag{2.45}$$

Then the FLRW metric (2.44) can be rewritten as

$$\mathrm{d}s^2 = a(t)^2(\mathrm{d}\eta^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j) \tag{2.46}$$

We consider quadratic perturbations in an FLRW universe with minimally-coupled scalar matter. The theory is described by the action,

$$S = \int d^4x \sqrt{-g} \left\{ R - \frac{1}{2} g^{\mu\nu} \partial_\mu \mathcal{X} \partial_\nu \mathcal{X} - V(\mathcal{X}) \right\} \tag{2.47}$$

where $\mathcal{X}$ denotes the scalar matter field and $V(\mathcal{X})$ is its potential. The expansion of the fields around the purely time-dependent background is as follows:

$$\begin{aligned} g_{\mu\nu}(\eta, x) &= a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}(\eta, x)), \\ \mathcal{X}(\eta, x) &= \mathcal{X}^{(0)}(\eta) + \varphi(\eta, x), \end{aligned} \tag{2.48}$$

where η is conformal time, $a(\eta)$ is the scale factor, $\mathcal{X}^{(0)}$ describes the background matter, and φ is the matter fluctuation. The background dynamics are governed by the Friedmann equations and the equation of motion of $\mathcal{X}^{(0)}$:

$$H^2 = \frac{1}{6} a^2 \rho, \tag{2.49}$$

$$\dot{H} + H^2 = \frac{1}{12} a^2 (\rho - 3p), \tag{2.50}$$

$$\ddot{\mathcal{X}}^{(0)} + 2H\dot{\mathcal{X}}^{(0)} + a^2 V'\left(\mathcal{X}^{(0)}\right) = 0, \tag{2.51}$$

where the dot now denotes the derivative with respect to conformal time, $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter, and the prime indicates a derivative with respect to $\mathcal{X}^{(0)}$. ρ and p are the background density and pressure, respectively:

$$\rho = \frac{1}{2} a^{-2} \dot{\mathcal{X}}^{(0)2} + V\left(\mathcal{X}^{(0)}\right), \tag{2.52}$$

$$p = \frac{1}{2} a^{-2} \dot{\mathcal{X}}^{(0)2} - V\left(\mathcal{X}^{(0)}\right), \tag{2.53}$$

which satisfy the conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.54}$$

The gauge transformations (2.18) to linear order around the FLRW background read:

$$\delta h_{\mu\nu} = a^{-2} \xi^\rho \partial_\rho (a^2 \eta_{\mu\nu}) + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta\varphi = \xi^\rho \partial_\rho \mathcal{X}^{(0)}. \tag{2.55}$$

We now begin the SVT decomposition of the metric and the gauge parameters as done in the previous subsection. First, we make the split of the fluctuations $(h_{\mu\nu}, \varphi) = (h_{00}, h_{0i}, h_{ij}, \varphi)$ whose components transform under (2.55) as:

$$\begin{aligned} \delta h_{00} &= 2H\xi_0 + 2\dot{\xi}_0\,, \\ \delta h_{0i} &= \dot{\xi}_i + \partial_i \xi_0\,, \\ \delta h_{ij} &= -2H\xi_0\delta_{ij} + 2\partial_{(i}\xi_{j)}\,, \\ \delta\varphi &= -\dot{\mathcal{X}}^{(0)}\xi_0\,. \end{aligned} \tag{2.56}$$

Using the same decomposition of the gauge parameter in (2.23), it follows that the gauge transformations of the components in (2.56) can be re-expressed as:

$$\begin{aligned} \delta h_{00} &= 2H\xi_0 + 2\dot{\xi}_0\,, \\ \delta h_{0i} &= \dot{\zeta}_i + \partial_i(\dot{\chi} + \xi_0)\,, \\ \delta h_{ij} &= -2H\xi_0\delta_{ij} + 2\partial_{(i}\zeta_{j)} + 2\partial_i\partial_j\chi\,, \\ \delta\varphi &= -\dot{\mathcal{X}}^{(0)}\xi_0\,. \end{aligned} \tag{2.57}$$

The metric fluctuation is decomposed as done previously in the flat space case in (2.20). With (2.20) and (2.57), the gauge transformations of each irreducible component of the metric fluctuation read:

$$\begin{aligned} \delta\phi &= -H\xi_0 - \dot{\xi}_0\,, \quad \delta B_i = \dot{\zeta}_i\,, \qquad\qquad \delta B = \dot{\chi} + \xi_0\,, \\ \delta\hat{h}_{ij} &= 0\,, \qquad\qquad \delta C = H\xi_0 + \frac{1}{3}\Delta\chi\,, \quad \delta E_i = \zeta_i\,, \quad \delta E = \chi\,. \end{aligned} \tag{2.58}$$

By looking at which transformations cancel each other out, we can find invariant combinations:

$$(\hat{h}_{ij}\,,\ \Sigma_i\,,\ \Psi\,,\ \Phi\,,\ \Theta) \tag{2.59}$$

where

$$\begin{aligned} \Sigma_i &= \dot{E}_i - B_i\,, \\ \Psi &= -C + \frac{1}{3}\Delta E - H(B - \dot{E})\,, \quad \Phi = \phi + H(B - \dot{E}) + \dot{B} - \ddot{E}\,, \\ \Theta &= \varphi + \dot{\mathcal{X}}^{(0)}(B - \dot{E})\,. \end{aligned} \tag{2.60}$$

Since the fluctuations are on an FLRW background instead of a flat one, note that there are additional terms with the Hubble parameter entering in Ψ and Φ, as well as an additional invariant scalar from the scalar matter fluctuation.

Following the procedure, the next step is to rewrite the action in terms of the gauge invariant variables. Unfortunately, expanding the Einstein-Hilbert action to quadratic order around a time-dependent background is already computationally cumbersome, and to reorganize the terms into the gauge invariant terms is quite tedious. For now, we proceed with these computations. However, in the next section, we describe a new method which will drastically simplify the reorganization of the action in terms of the gauge invariant variables.

In order to compute the quadratic action around FLRW we make use of the vielbein formalism. The vielbein satisfies

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu} \, , \tag{2.61}$$

and its inverse $e_a{}^\mu$ is defined by $e_a{}^\mu e_\mu{}^b = \delta_a{}^b$ and $e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu$. The Einstein-Hilbert action can be expressed in terms of the vielbein as:

$$\int d^4x \, \sqrt{-g} \, R \; = \; \int d^4x \, e\Big(-\frac{1}{4}\Omega^{abc}\Omega_{abc} + \frac{1}{2}\Omega^{abc}\Omega_{bca} + \Omega_a\Omega^a \Big) \, , \tag{2.62}$$

where e is the determinant of the vielbein and

$$\Omega_{abc} \; \equiv \; e_a{}^\mu e_b{}^\nu (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \, , \qquad \Omega_a \; \equiv \; \Omega_{ab}{}^b \tag{2.63}$$

are the anholonomy coefficients. The vielbein is expanded as

$$e_\mu{}^a(\eta, \mathbf{x}) \; = \; \bar{e}_\mu{}^a(\eta) + a(\eta) h_\mu{}^a(\eta, \mathbf{x}) \, , \tag{2.64}$$

where

$$\bar{e}_\mu{}^a(\eta) \; = \; a(\eta) \begin{pmatrix} 1 & 0 \\ 0 & \delta_i{}^\alpha \end{pmatrix} \tag{2.65}$$

is the background FLRW frame which satisfies $\bar{e}_\mu{}^a \bar{e}_\nu{}^b \eta_{ab} = \bar{g}_{\mu\nu}$, and

$$h_\mu{}^a \; = \; \begin{pmatrix} h_0{}^{\bar{0}} & h_0{}^\alpha \\ h_i{}^{\bar{0}} & h_i{}^\alpha \end{pmatrix} \; = \; \begin{pmatrix} \phi & \mathcal{B}^\alpha \\ 0 & h_i{}^\alpha \end{pmatrix} \tag{2.66}$$

is the (rescaled) fluctuation. Here we performed a $3+1$ split of indices:

$$\mu \; = \; (0, i) \, , \quad a \; = \; (\bar{0}, \alpha) \, , \tag{2.67}$$

and picked a gauge for the local Lorentz transformations with $h_i{}^{\bar{0}} = 0$. The computation of the quadratic action requires up to second order in fluctuations of the inverse vielbein:

$$e_a{}^\mu \;=\; \bar{e}_a{}^\mu - a(\eta)\bar{e}_a{}^\nu h_\nu{}^b \bar{e}_b{}^\mu + a^2(\eta)\bar{e}_a{}^\nu h_\nu{}^b \bar{e}_b{}^\rho h_\rho{}^c \bar{e}_c{}^\mu \,. \tag{2.68}$$

Writing this in components we can summarize the vielbein and its inverse as

$$\begin{aligned} e_\mu{}^a \;&=\; \begin{pmatrix} e_0{}^{\bar{0}} & e_0{}^\alpha \\ e_i{}^{\bar{0}} & e_i{}^\alpha \end{pmatrix} \;=\; a(\eta)\begin{pmatrix} 1+\phi & \mathcal{B}^\alpha \\ 0 & \delta_i{}^\alpha + h_i{}^\alpha \end{pmatrix} \,, \\ e_a{}^\mu \;&=\; \begin{pmatrix} e_{\bar{0}}{}^0 & e_{\bar{0}}{}^i \\ e_\alpha{}^0 & e_\alpha{}^i \end{pmatrix} \;=\; a^{-1}(\eta)\begin{pmatrix} 1-\phi+\phi^2 & -\mathcal{B}^i + \phi\mathcal{B}^i + \mathcal{B}^j h_j{}^\alpha \delta_\alpha{}^i \\ 0 & \delta_\alpha{}^i - h_\alpha{}^i + h_\alpha{}^j h_j{}^\beta \delta_\beta{}^i \end{pmatrix} \,. \end{aligned} \tag{2.69}$$

This vielbein leads to the same parameterizations of the first order fluctuations of the metric that is standard in cosmology, namely in (2.20), with h^{0i} identified with $\mathcal{B}^\alpha = -h^{0i}\delta_i{}^\alpha$ via the background vielbein, and with

$$h_{ij} \;=\; 2\, h_{(i}{}^\alpha \delta_{j)\alpha} \,. \tag{2.70}$$

To second order, we collect the fluctuations of the metric and its inverse:

$$\begin{aligned} g_{00} \;&=\; a^2(-1 - 2\phi - \phi^2 + \mathcal{B}^\alpha \mathcal{B}_\alpha) \,, \\ g_{0i} \;&=\; a^2(\mathcal{B}_i + \mathcal{B}_\alpha h_i{}^\alpha) \,, \\ g_{ij} \;&=\; a^2(\delta_{ij} + h_{ij} + h_i{}^\alpha h_{j\alpha}) \,, \\ g^{00} \;&=\; a^{-2}(-1 + 2\phi - 3\phi^2) \,, \\ g^{0i} \;&=\; a^{-2}(\mathcal{B}^i - 2\phi\mathcal{B}^i - \mathcal{B}^j h_j{}^\alpha \delta_\alpha{}^i) \,, \\ g^{ij} \;&=\; a^{-2}(\delta^{ij} - 2h_\alpha{}^{(i}\delta^{|\alpha|j)} + 2h_\alpha{}^k h_k{}^\gamma \delta_\gamma{}^{(i}\delta^{|\alpha|j)}) \,. \end{aligned} \tag{2.71}$$

The SVT decomposition in (2.20) translates to that of $h_i{}^\alpha$ as

$$h_i{}^\alpha \;=\; \widehat{h}_i{}^\alpha + \partial_i E^\alpha + \partial_i \partial^\alpha E + \delta_i{}^\alpha \Big(C - \frac{1}{3}\Delta E\Big) \,, \tag{2.72}$$

with $\widehat{h}_i{}^\alpha$ satisfying the constraints:

$$\delta_\alpha{}^i \widehat{h}_i{}^\alpha \;=\; 0 \,, \quad \partial^i \widehat{h}_i{}^\alpha \;=\; 0 \,. \tag{2.73}$$

Similarly, the vector $\mathcal{B}^\alpha$ is decomposed as:

$$\mathcal{B}^\alpha \;=\; B^\alpha + \partial^\alpha B \,, \quad \partial_\alpha B^\alpha \;=\; 0 \,. \tag{2.74}$$

Inserting the decompositions into (2.62), we compute the full action with matter coupling (2.47) to quadratic order. The linear terms of the action drop out, assuming the background field equations are satisfied. The quadratic action is:

$$
\begin{aligned}
S = \int d^4x\, a^2 \Big\{ & \frac{1}{4}\dot{h}^{ij}\dot{h}_{ij} + \frac{1}{4}h_{ij}\Delta h^{ij} + \frac{1}{2}\partial_j h^{ij}\partial^k h_{ik} + \frac{1}{2}(h_i{}^i - h_{00})\partial^j\partial^k h_{jk} \\
& + \frac{1}{2}h_{0i}\Delta h^{0i} - \frac{1}{2}(\partial_i h^{0i})^2 + \partial^i h^{0j}\dot{h}_{ij} + \partial^i h_{0i}\dot{h}_j{}^j \\
& - \frac{1}{4}(\dot{h}_i{}^i)^2 - \frac{1}{4}h_i{}^i\Delta h_j{}^j + \frac{1}{2}h_i{}^i\Delta h_{00} \\
& - \frac{1}{2}(\dot{H} + 2H^2)h_{00}^2 - Hh_{00}\dot{h}_i{}^i + 2Hh_{00}\partial^i h_{0i} \\
& + \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\varphi\Delta\varphi - \frac{1}{2}a^2 V''(\mathcal{X}^{(0)})\varphi^2 \\
& - \frac{1}{2}\dot{\mathcal{X}}^{(0)}\varphi\big(\dot{h}_{00} + \dot{h}_i{}^i + 2\partial_i h^{0i}\big) + a^2 V'(\mathcal{X}^{(0)})\varphi h_{00} \Big\} \,.
\end{aligned}
\tag{2.75}
$$

By inserting the Bardeen variables in (2.60), after a tedious computation the quadratic action can be organized into its gauge invariant form:

$$
\begin{aligned}
S \;=\; \int d^4x\, a^2 \Big\{ & \frac{1}{4}\dot{\widehat{h}}{}^{ij}\dot{\widehat{h}}_{ij} + \frac{1}{4}\widehat{h}^{ij}\Delta\widehat{h}_{ij} - \frac{1}{2}\Sigma_i\Delta\Sigma^i + 4\Psi\Delta\Phi - 2\Psi\Delta\Psi + \frac{1}{2}\dot{\mathcal{X}}^{(0)2}\Phi^2 \\
& - 6(\dot{\Psi} + H\Phi)^2 + \frac{1}{2}\dot{\Theta}^2 + \frac{1}{2}\Theta\Delta\Theta - \frac{1}{2}a^2 V''(\mathcal{X}^{(0)})\Theta^2 \\
& + \dot{\mathcal{X}}^{(0)}\Theta(\dot{\Phi} + 3\dot{\Psi}) - 2a^2 V'(\mathcal{X}^{(0)})\Theta\Phi \Big\} \,.
\end{aligned}
\tag{2.76}
$$

In section 2.5 we will return to cosmological perturbation theory and show how our method systematizes this computation.

2.2 Homotopy Transfer to Gauge Invariant Variables

As explained in section 1.2, the data of a free theory can be encoded in a chain complex, i.e.

$$
X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \xrightarrow{\partial} X_{-2}\,, \tag{2.77}
$$

where X_1 is the space of gauge parameters, X_0 is the space of fields, X_{-1} is the space of field equations, and X_{-2} is the space of Noether identities. Once the free theory is in terms of the gauge invariant fields, it has no gauge symmetry, and this property should be reflected in the chain complex. The chain complex of the gauge invariant free theory would take the form:

$$0 \longrightarrow \bar{X}_0 \xrightarrow{\bar{\partial}} \bar{X}_{-1} \longrightarrow 0 \tag{2.78}$$

where X_0 is the space of gauge invariant fields and X_{-1} is the space of their corresponding equations of motion. If no gauge symmetries are present, then there are no Noether identities either, leaving zeroes at the ends of the complex.

Let us now relate the chain complexes of the free theory data $X_\bullet$ and its gauge invariant data $\bar{X}_\bullet$. We define a projection $p_0 : X_\bullet \to \bar{X}_\bullet$ from the chain complex of gauge fields to the chain complex of gauge invariant variables. Here the subscript 0 is assigned because we are dealing with objects of the free theory instead of the full interacting theory. For example, in Maxwell's theory, in degree 0 the projection acts as

$$\bar{A}_\mu = p_0(A_\mu) = (\bar{A}_0, \bar{A}_i) = (\hat{\Phi}, \hat{A}_i)\,, \tag{2.79}$$

where $\hat{\Phi}$ and $\hat{A}_i$ are defined in (2.13) and (2.9) respectively. In degree -1, the projector is defined by

$$\bar{E}^\mu = p_0(E^\mu) = E^\mu - (0, \partial^i \Delta^{-1}(\partial_\mu E^\mu)) \tag{2.80}$$

so that the space $\bar{X}_{-1}$ consists of 4-vectors with zero divergence, i.e. $\partial_\mu \bar{E}^\mu \equiv 0$, which ensures that the projection to the space of Noether identities in degree -2 is trivial. The inclusion map $i_0 : \bar{X}_\bullet \to X_\bullet$ takes a gauge invariant object and treats it as an element of the original space (as $\bar{X}_\bullet$ is a subspace of $X_\bullet$). For example, $i_0(\bar{A}_\mu) = \bar{A}_\mu$ and $i_0(\bar{E}^\mu) = \bar{E}^\mu$. By definition, the projection and inclusion maps satisfy $p_0 i_0 = \mathrm{id}_{\bar{X}}$ and are chain maps, meaning they satisfy (1.64). The chain complexes and the morphisms can be represented by the following diagram.

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{\partial} & X_{-1} & \xrightarrow{\partial} & X_{-2} \\ {\scriptstyle i_0}\uparrow\downarrow{\scriptstyle p_0} & & {\scriptstyle i_0}\uparrow\downarrow{\scriptstyle p_0} & & {\scriptstyle i_0}\uparrow\downarrow{\scriptstyle p_0} & & {\scriptstyle i_0}\uparrow\downarrow{\scriptstyle p_0} \\ 0 & \longrightarrow & \bar{X}_0 & \xrightarrow{\bar{\partial}} & \bar{X}_{-1} & \longrightarrow & 0 \end{array} \tag{2.81}$$

It is easy to derive the new differential $\bar{\partial}$ on the gauge invariant chain complex, by using the chain map property (1.64) and that $p_0 i_0 = \mathrm{id}_{\bar{X}}$:

$$\bar{\partial} = \bar{\partial} p_0 i_0 = p_0 \partial i_0\,. \tag{2.82}$$

We proceed to determine the interacting theory in terms of gauge invariant variables. We recall the homotopy transfer theorem which states that if we have quasi-isomorphisms between $X_\bullet$ and $\bar{X}_\bullet$, then the algebraic structure on $X_\bullet$ can be transferred to $\bar{X}_\bullet$. For i_0, p_0 to be quasi-isomorphisms, there needs to exist a homotopy map s_0 of degree $+1$ such that

$$i_0 \circ p_0 = \mathrm{id}_X - \partial \circ s_0 - s_0 \circ \partial \,. \tag{2.83}$$

Let us consider this relation for the Maxwell example:

$$i_0 p_0(A_\mu) = \bar{A}_\mu = A_\mu - \partial_\mu(s_0(A)) \tag{2.84}$$

where we have assumed that s_0 acting on objects of degree -1 yields zero. By using (2.79), (2.6) and (2.13) to rewrite $\bar{A}_\mu$ as

$$\bar{A}_\mu = (\hat{\Phi}, \hat{A}_i) = (A_0 - \dot{\Phi}, A_i - \partial_i \Phi) = A_\mu - \partial_\mu \Phi \tag{2.85}$$

and by recalling the definition for Φ in (2.8), we can deduce $s_0(A)$:

$$s_0(A) = \Phi = \Delta^{-1}(\partial_i A^i)\,. \tag{2.86}$$

2.3 Applying the Homological Perturbation Lemma

Now that we have established a homotopy equivalence data of the free theory $(X_\bullet, \partial)$ and the gauge invariant complex $(\bar{X}_\bullet, \bar{\partial})$, we can utilize the homological perturbation lemma to determine the gauge invariant variables to higher orders in perturbations. We recall the discussion around (1.74). Let the products $b_k : X_\bullet^{\wedge n} \to X_\bullet$ which encode the non-linear interactions be the small perturbation $\delta = \sum_{k\geq 2} b_k$ to the differential in the chain complex $(X_\bullet, \partial)$. In order to apply the homological perturbation lemma, we lift the maps b_k and ∂ to act on the graded symmetric coalgebra $S^c(X_\bullet)$. Assuming that we have a strong deformation retract given by the conditions (1.75) (in addition to $p_0 i_0 = \mathrm{id}$), the perturbed projection and inclusion maps are given by

$$i = (1 + s_0\delta)^{-1} i_0\,, \quad p = p_0(1 + \delta s_0)^{-1}\,, \tag{2.87}$$

where i and p are coalgebra morphisms. To realize the expressions for i and p, we first lift the chain maps p_0 and i_0 to coalgebra morphisms, which is straightforward since they are linear.

$$p_0(x_1 \wedge \cdots \wedge x_n) = p_0(x_1) \wedge \cdots \wedge p_0(x_n) \tag{2.88}$$

where p_0 on the LHS is a cohomomorphism $S^c(X_\bullet) \to S^c(\bar{X}_\bullet)$ and p_0 on the RHS is the chain map $X_\bullet \to \bar{X}_\bullet$. Similarly, the inclusion map is lifted to a cohomomorphism $S^c(\bar{X}_\bullet) \to S^c(X_\bullet)$:

$$i_0(x_1 \wedge \cdots \wedge x_n) = i_0(x_1) \wedge \cdots \wedge i_0(x_n) \tag{2.89}$$

The lift of the homotopy map s_0 requires a few steps. First we define its action on tensor powers $h_n : X_\bullet^{\otimes n} \to X_\bullet^{\wedge n}$

$$\begin{aligned} h_n(x_1 \otimes \cdots \otimes x_n) =& s_0(x_1) \wedge x_2 \wedge \cdots \wedge x_n + (-)^{x_1} i_0 \circ p_0(x_1) \wedge s_0(x_2) \wedge x_3 \wedge \cdots \wedge x_n \\ &+ \ldots + (-)^{x_1+\ldots+x_{n-1}} i_0 \circ p_0(x_1) \wedge \cdots \wedge i_0 \circ p_0(x_{n-1}) \wedge s_0(x_n)\,. \end{aligned} \tag{2.90}$$

Then we define a map $q_n : X_\bullet^{\wedge n} \to X_\bullet^{\otimes n}$ which accounts for the graded symmetrization:

$$q_n(x_1 \wedge \cdots \wedge x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \tag{2.91}$$

By defining the composition,

$$h_n \circ q_n : X_\bullet^{\wedge n} \to X_\bullet^{\wedge n} \tag{2.92}$$

we have a well-defined homotopy map $s_0 : S^c(X_\bullet) \to S^c(X_\bullet)$:

$$s_0 := \sum_{n \geq 1} h_n \circ q_n\,. \tag{2.93}$$

The expressions for the perturbed maps p and i are expanded order by order in k. The inverses in (2.87) are defined as a geometric series, $(1+x)^{-1} = \sum_{n\geq 0}(-x)^n$.

Let us work out what p would be for Yang-Mills to second-order in fields, meaning that we would have to implement the b_2 map in (1.30).

$$\begin{aligned} &-p_0 b_2 s_0(A\wedge B)\\ &= -p_0 b_2 \frac{1}{2}(s_0(A)\wedge B + A\wedge s_0(B) + s_0(A)\wedge i_0 p_0(B) + i_0 p_0(A)\wedge s_0(B))\\ &= -\frac{1}{2}p_0([(1+i_0p_0)(A_\mu), s_0(B)] - [s_0(A),(1+i_0p_0)(B)])\,, \end{aligned} \tag{2.94}$$

where $[\cdot,\cdot]$ is the Lie bracket of the Yang-Mills gauge algebra. We know that the gauge invariant part is the projection of the gauge field:

$$\hat{A} = p(A)\,. \tag{2.95}$$

For the above example, to quadratic order in fields,

$$\begin{aligned} \hat{A} = p(A) &= p_0(A_\mu) - \frac{1}{4}p_0([(1+i_0p_0)(A_\mu), s_0(B)] - [s_0(A),(1+i_0p_0)(B)])\\ &= p_0(A_\mu) + \frac{1}{2}p_0[s_0(A),(1+i_0p_0)(A_\mu)]\,. \end{aligned} \tag{2.96}$$

Let us check that this is indeed gauge invariant. By taking the linear variation of the quadratic term,

$$\begin{aligned} &\frac{1}{2}p_0[s_0(\partial_\mu\lambda),(1+i_0p_0)(A_\mu)] + \frac{1}{2}p_0[s_0(A),(1+i_0p_0)(\partial_\mu\lambda)]\\ &= \frac{1}{2}p_0([\lambda,(1+i_0p_0)(A_\mu)] + [s_0(A),\partial_\mu\lambda])\\ &= \frac{1}{2}p_0([\lambda,(1+i_0p_0)(A_\mu)] - [(1-i_0p_0)(A_\mu),\lambda])\\ &= -p_0([A_\mu,\lambda]) \end{aligned} \tag{2.97}$$

The variation of the linear term gives:

$$p_0(\partial_\mu\lambda + [A_\mu,\lambda]) = p_0([A_\mu,\lambda])\,, \tag{2.98}$$

which cancels with the variation of the quadratic term.

Now that we have obtained the gauge invariant variable $\hat{A} = p(A)$ to all orders in perturbations, we want to find a suitable replacement of the gauge field in terms of the gauge invariant variable, $A(\hat{A})$, to obtain a manifestly gauge invariant action $S[\hat{A}]$. Even though our end result would only depend on the gauge invariant fields, the inclusion map $i : \bar{X}_0 \to X_0$, which views the gauge invariant field as a member of the larger gauge variant space, does not provide enough information. This is because the action is originally defined on gauge redundant fields and these have more degrees of freedom than their gauge invariant projections. For example, the gauge invariant variable in Maxwell theory $\hat{A}_i$ has only 2 degrees of freedom and the other 2 degrees of freedom in A_μ turn out to be pure gauge (see the discussion at the beginning of section 2.1). In other words, the space of gauge fields X_0 is of the same size as the space of gauge invariant fields $\bar{X}_0$ plus the space of pure gauge fields, which we now denote as Y_0. By finding a sort of extension to the inclusion map which accounts for the pure gauge fields

$$F : \bar{X}_0 \oplus Y_0 \to X_0, \tag{2.99}$$

one can reinsert the forgotten gauge degrees of freedom together with the invariant ones into the action.

We expect the map to be given by the finite gauge transformation

$$A_\mu = e^{\Delta_\phi} \hat{A}_\mu \,, \tag{2.100}$$

where Δ_ϕ is the operator defining the infinitesimal (non-linear) gauge transformations. Upon inserting this expression into the action, by gauge invariance, the pure gauge terms will drop out and we will be left with the action in terms of the gauge invariant variable. We will now illustrate how the above is the appropriate map and how this fits into our L_∞ framework.

Let us first consider what properties this map F should satisfy in general. Part of it is given by the inclusion $i : \bar{X}_0 \to X_0$ and the other part should produce a pure gauge piece:

$$\begin{aligned} F : \;& \bar{X}_0 \oplus Y_0 \to X_0, \\ & (\hat{A}, \phi) \mapsto i(\hat{A}) + j(\hat{A}, \phi) \,. \end{aligned} \tag{2.101}$$

For Maxwell theory as an example, the gauge field can be written (as in (2.85))

$$A_\mu = \bar{A}_\mu + \partial_\mu \phi \,, \tag{2.102}$$

so the map F is specified by $i(\bar{A}_\mu) = \bar{A}_\mu$ and $j(\bar{A}_\mu, \phi) = \partial_\mu \phi$. The inverse of F is

$$\begin{aligned} F^{-1} : X_0 &\to \bar{X}_0 \oplus Y_0 \,, \\ A &\mapsto (p(A), q(A)) \,, \end{aligned} \tag{2.103}$$

where p is the projection and q encodes the rest of the gauge degrees of freedom. In the Maxwell example, this just represents the rewriting

$$A_\mu = p_0(A_\mu) + \partial_\mu q_0(A) \,. \tag{2.104}$$

and since according to (2.85) and (2.86) the projection acts as

$$p_0(A_\mu) = \bar{A}_\mu = A_\mu - \partial_\mu(\Delta^{-1}(\partial_i A^i)) \,, \tag{2.105}$$

by inserting this into (2.104) we can deduce $q_0(A)$:

$$q_0(A) = \Delta^{-1}(\partial_i A^i) \,. \tag{2.106}$$

$q_0(A)$ coincides with the homotopy $s_0(A)$ but is a degree zero map. What we want to find is a map $F = i + j$ whose inverse is given by $F^{-1} = (p, q)$ where i and p are the full inclusion and projection maps given by the perturbation lemma (2.87). At the linear level, we see that it is relatively straightforward to find the maps j and q. To account for the full non-linear theory, our ansatz is $\mathcal{F} : \bar{X}_0 \oplus Y_0 \to X_0$

$$\mathcal{F}(\hat{A}_\mu, \phi) = e^{\Delta_\phi} \hat{A}_\mu \,. \tag{2.107}$$

We will show that this is the correct expression by using the example of Yang-Mills theory.

Before we proceed, let us comment about the invertibility of these maps. At the linear level, we know that F is invertible because the spaces $\bar{X}_0 \oplus Y_0$ and X_0 are isomorphic. At the non-linear level, we are dealing with morphisms of L_∞ algebras, so the proof of invertibility is more involved, however it can be shown that an L_∞ algebra morphism is invertible if its linear piece is [43].

For Yang-Mills, the infinitesimal gauge transformation is given by

$$\Delta_\phi(A_\mu) = \partial_\mu \phi + [A_\mu, \phi] \,, \tag{2.108}$$

which upon inserting into (2.107) to quadratic order yields

$$A_\mu = \mathcal{F}(\hat{A}_\mu, \phi) = \widehat{A}_\mu + \partial_\mu \phi + \left[\hat{A}_\mu, \phi\right] + \frac{1}{2}\left[\partial_\mu \phi, \phi\right] + \cdots . \tag{2.109}$$

The inverse map is

$$\mathcal{F}^{-1}(A) = (p(A), q(A)) , \tag{2.110}$$

where $p(A_\mu) = \hat{A}_\mu$ and $q(A)$ can be computed order by order (as is typically done for inverting L_∞ morphisms). For instance, by applying q_0 in (2.106) to both sides of (2.109), the first term on the RHS vanishes because of the divergenceless vector constraint $\partial_\mu \hat{A}^\mu = 0$, and the second term is $q_0(\partial_\mu \phi) = \phi$. Then by bringing A_μ to the right and ϕ to the left hand side, (2.109) becomes

$$\phi = q_0(A_\mu) - q_0\left[\hat{A}_\mu, \phi\right] - \frac{1}{2} q_0\left[\partial_\mu \phi, \phi\right] . \tag{2.111}$$

We now view ϕ as a function of A, and identify $\phi(A)$ as $q(A)$. Then by writing $\hat{A}_\mu$ and ϕ to linear order in A_μ, by recalling homotopy relation in (2.84) and that to leading order $q_0(A) = s_0(A)$,

$$\begin{aligned} q(A) &= q_0(A_\mu) - q_0\left[i_0 p_0(A_\mu), s_0(A)\right] - \frac{1}{2} q_0\left[(1 - i_0 p_0)(A_\mu), s_0(A)\right] \\ &= q_0(A_\mu) - \frac{1}{2} q_0\left[(1 + i_0 p_0)(A_\mu), s_0(A)\right] . \end{aligned} \tag{2.112}$$

Thus we have computed our ansatz for q to first non-trivial order.

Let us check whether the maps that we have defined are indeed inverses of each other, with i and p given by the perturbation lemma (2.87), namely that

$$\mathcal{F}^{-1}(\mathcal{F}(\hat{A}_\mu, \phi)) = \left(p(\mathcal{F}(\hat{A}_\mu, \phi)), q(\mathcal{F}(\hat{A}_\mu, \phi))\right) = (\hat{A}_\mu, \phi) . \tag{2.113}$$

Since we have already computed q such that $q(\mathcal{F}(\hat{A}_\mu, \phi)) = \phi$, we must only check the identity with p. To lowest order, we apply p_0 to $\mathcal{F}(\hat{A}_\mu, \phi)$ in (2.109) and rearrange to have $\hat{A}_\mu$ on the LHS:

$$\hat{A}_\mu = p_0(A_\mu) - p_0\left[\hat{A}_\mu, \phi\right] - \frac{1}{2} p_0\left[\partial_\mu \phi, \phi\right] . \tag{2.114}$$

Expressing $\hat{A}_\mu$ and ϕ in terms of A_μ to linear order gives

$$\begin{aligned}\hat{A}_\mu &= p_0(A_\mu) - p_0\left[i_0 p_0(A_\mu), s_0(A)\right] - \frac{1}{2}p_0\left[(1 - i_0 p_0)(A_\mu), s_0(A)\right] \\ &= p_0(A_\mu) - \frac{1}{2}p_0\left[(1 + i_0 p_0)(A_\mu), s_0(A)\right], \end{aligned} \tag{2.115}$$

which is exactly the correct expression of $\hat{A}_\mu$ to quadratic order in (2.96). Thus our ansatz in (2.107) is compatible with the definition of the perturbed projections and inclusion maps. We have shown that this works for quadratic order in Yang-Mills, but one can in principle repeat these steps for higher orders.

In addition to being able to express the action in terms of gauge invariant variables to all orders, one can also ask what the effective equations of motion are. Recall the general formula for the perturbed coderivation given by (1.72).

$$\bar{\partial} = p_0 \partial i_0 + p_0(\text{id} + \delta s_0)^{-1}\delta i_0. \tag{2.116}$$

In order to simplify this expression, we make use of the geometric series

$$(\text{id} + \delta s_0)^{-1} = \sum_{n=0}^{\infty}(-\delta s_0)^n\,. \tag{2.117}$$

In the case of Yang-Mills, the gauge invariant complex only consists of elements in degree 0 and -1 since there are no gauge symmetries left, and this implies that $\bar{\partial}$ can only be nonzero when acting on degree zero elements. Since δ is a degree -1 map and s_0 acting on degree -1 elements is zero, the second term on the RHS of (2.116) reduces to

$$p_0(\text{id} + \delta s_0)^{-1}\delta i_0 = p_0 \sum_{n=0}^{\infty}(-\delta s_0)^n \delta i_0 = p_0 \delta i_0\,. \tag{2.118}$$

Finally, the perturbed coderivation is obtained:

$$\bar{\partial} = p_0 \partial i_0 + p_0 \delta i_0 = p_0(\partial + \delta) i_0\,. \tag{2.119}$$

This states that the equation of motion of the gauge invariant variable $\hat{A}$ is computed by inserting $\hat{A}$ into the equation of motion of the gauge field A. The resulting equation of motion is simplified compared to the original one, since the gauge invariant variable has fewer degrees of freedom and obeys some constraints, e.g. the divergenceless constraint $\partial_i \hat{A}^i = 0$ in Yang-Mills.

This concludes the description of our procedure for finding the action in terms of gauge invariant variables. Although we have developed the procedure for gauge invariant perturbation theory to all orders, it can also be applied at the level of the free theory. In the next sections, we will demonstrate this for linearized gravity and cosmological perturbation theory.

2.4 Linearized Gravity

We now take linearized gravity and its gauge invariant variables which we described in section 2.1.1 and provide its homotopy algebraic interpretation. First, we organize the theory in the chain complex

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{\partial} & X_{-1} & \xrightarrow{\partial} & X_{-2} \\ \{\xi_\mu\} & & \{h_{\mu\nu}\} & & \{E_{\mu\nu}\} & & \{F_\mu\} \end{array} \tag{2.120}$$

where the differentials act as

$$\begin{aligned} \partial(\xi)_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \, , \\ \partial(h)_{\mu\nu} &= G_{\mu\nu}(h) \, , \\ \partial(E)_\mu &= \partial^\nu E_{\nu\mu} \, . \end{aligned} \tag{2.121}$$

In (2.26) we have identified the gauge invariant variables, Σ_i, Ψ, Φ, along with $\widehat{h}_{ij}$.

We would like to define the gauge invariant chain complex $\bar{X}_\bullet$. Let us define the projection $p : X_0 \to \bar{X}_0$ as

$$p(h_{\mu\nu}) = \bar{h}_{\mu\nu} \, , \tag{2.122}$$

where

$$\bar{h}_{ij} = \widehat{h}_{ij} - 2\Psi\delta_{ij} \, , \qquad \bar{h}_{0i} = -\Sigma_i \, , \qquad \bar{h}_{00} = -2\Phi \, . \tag{2.123}$$

Although it might not be obvious that this is a good projection, we can check that it is, by seeing whether we can write the original metric field as the gauge invariant field plus a pure gauge term:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \psi_\nu + \partial_\nu \psi_\mu \, . \tag{2.124}$$

We find that with (2.122) and (2.26), the pure gauge field is given by

$$\psi_\mu = (\psi_0, \psi_i) = (B - \dot{E}, E_i + \partial_i E)\,. \tag{2.125}$$

The inclusion $i : \bar{X}_0 \to X_0$ is defined in the usual way:

$$i(\bar{h}_{\mu\nu}) = \bar{h}_{\mu\nu}\,. \tag{2.126}$$

Let us proceed to find the homotopy retract from $X_\bullet$ to $\bar{X}_\bullet$. We will find the homotopy map s can be found by evaluating the homotopy retract condition in (2.83) on elements of each degree. On degree zero elements which are the fields:

$$(\mathrm{id} - ip)(h_{\mu\nu}) = h_{\mu\nu} - \bar{h}_{\mu\nu} = \partial_\mu\psi_\nu + \partial_\nu\psi_\mu = \partial(\psi)_{\mu\nu}\,, \tag{2.127}$$

where we have treated ψ_μ as an element of X_1. By comparing this with (2.83) and assuming that s acting on elements in X_{-1} is zero, we find that the homotopy map acts as

$$s(h) = \psi\,. \tag{2.128}$$

For elements of degree 1, namely the gauge parameters, we compute

$$(\mathrm{id} - ip)\xi_\mu = \xi_\mu = s(\partial\xi)_\mu \tag{2.129}$$

where $p(\xi) = 0$ because we project down to gauge invariant variables, and $s(\xi) = 0$ since there are no elements of degree 2. Therefore we have

$$s(\partial\xi)_\mu = \xi_\mu \tag{2.130}$$

For the homotopy map on degree -2 elements, we must first find the projection from X_{-1} to $\bar{X}_{-1}$. The elements of $\bar{X}_{-1}$ must be divergence-free Lorentz tensors

$$\partial^\mu \bar{E}_{\mu\nu} = 0\,, \tag{2.131}$$

since the space of Noether identities X_{-2} must project to zero. The projection which satisfies this condition is

$$\bar{E}_{\mu\nu} \equiv p(E)_{\mu\nu} = \begin{pmatrix} E_{00} & E_{0j} - \partial_j\Delta^{-1}(\partial^\nu E_{\nu 0}) \\ E_{i0} - \partial_i\Delta^{-1}(\partial^\nu E_{\nu 0}) & E_{ij} - 2\partial_{(i}\Delta^{-1}\partial^\nu E_{j)\nu} + \partial_i\partial_j\Delta^{-2}(\partial^\mu\partial^\nu E_{\mu\nu}) \end{pmatrix}. \tag{2.132}$$

With this we can compute the homotopy relation on $E \in X_{-1}$,

$$\begin{aligned}(\mathrm{id}-ip)(E)_{\mu\nu} &= \begin{pmatrix} 0 & -\partial_i \Delta^{-1}(\partial^\nu E_{\nu 0}) \\ \partial_i \Delta^{-1}(\partial^\nu E_{\nu 0}) & 2\partial_{(i}\Delta^{-1}(\partial^\nu E_{j)\nu}) - \partial_i\partial_j \Delta^{-2}(\partial^\mu \partial^\nu E_{\mu\nu}) \end{pmatrix} \\ &= s(\partial E)_{\mu\nu}\,, \end{aligned} \tag{2.133}$$

where we have assumed $s(E) = 0$. Since $\partial(E)_\mu = \partial^\nu E_{\nu\mu}$, we can define the homotopy map on degree -2 objects as

$$s(F)_{\mu\nu} = \begin{pmatrix} s(F)_{00} & s(F)_{0j} \\ s(F)_{i0} & s(F)_{ij} \end{pmatrix} = \begin{pmatrix} 0 & \partial_j \Delta^{-1} F_0 \\ \partial_i \Delta^{-1} F_0 & 2\partial_{(i}\Delta^{-1} F_{j)} - \partial_i \partial_j \Delta^{-2}(\partial_\mu F^\mu) \end{pmatrix}. \tag{2.134}$$

We check that this is the correct expression by evaluating the homotopy relation on $F \in X_{-2}$

$$(\mathrm{id}-ip)(F)_\mu = \partial(s(F))_\mu + s(\partial F)_\mu = F_\mu\,. \tag{2.135}$$

Since $s(\partial F) = 0$ and by using (2.134),

$$\begin{aligned}\partial(s(F))_\mu = \partial^\nu s(F)_{\nu\mu} &= \begin{pmatrix} \partial^0 s(F)_{00} + \partial^i s(F)_{i0} \\ \partial^0 s(F)_{0i} + \partial^j s(F)_{ji} \end{pmatrix} \\ &= \begin{pmatrix} F_0 \\ \partial_i \Delta^{-1}(\partial^0 F_0) + \partial_i \Delta^{-1}(\partial^j F_j) + F_i - \partial_i \Delta^{-1}(\partial_\mu F^\mu) \end{pmatrix} \\ &= \begin{pmatrix} F_0 \\ F_i \end{pmatrix}, \end{aligned} \tag{2.136}$$

as it should be. We have now finished specifying all the homotopy maps which satisfy the homotopy relations and have defined the homotopy retract from the free theory to the theory in terms of gauge invariant variables. In order to compute gauge invariant variables to higher orders, one would require the higher products $\delta = \sum_{k\geq 2} b_k$ from the higher order perturbation theory and apply the homological perturbation lemma to these maps.

2.5 Cosmological Perturbation Theory

In this section we give the homotopy retract of cosmological perturbation theory to quadratic order around an FLRW background and extend the discussion in section 2.4. Recalling the initial description of the theory in section 2.1.2, our space of fields X_0 must now include the matter field φ in addition to the metric field $h_{\mu\nu}$. The differential acts on the gauge parameters as

$$\begin{aligned} \partial(\xi)_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2H\xi_0 \eta_{\mu\nu} \,, \\ \partial(\xi)_\bullet &= -\dot{\mathcal{X}}^{(0)} \xi_0 \,, \end{aligned} \tag{2.137}$$

where the $\bullet$ indicates the component in the direction of the scalar field φ. We define the projection from X_0 to $\bar{X}_0$ as:

$$p(h)_{\mu\nu} = \bar{h}_{\mu\nu} \,, \qquad p\varphi = \Theta \,, \tag{2.138}$$

where

$$\bar{h}_{ij} = \hat{h}_{ij} - 2\Psi \delta_{ij} \,, \quad \bar{h}_{0i} = -\Sigma_i \,, \quad \bar{h}_{00} = -2\Phi \,. \tag{2.139}$$

and $(\Sigma_i, \Psi, \Phi, \Theta)$ are defined in (2.60). We check that the original fields can be written in terms of the projected fields plus pure gauge terms:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \psi_\nu + \partial_\nu \psi_\mu - 2H\psi_0 \eta_{\mu\nu} \,, \tag{2.140}$$

$$\varphi = \bar{\varphi} - \dot{\mathcal{X}}^{(0)} \psi_0 \,, \tag{2.141}$$

where

$$\psi_\mu = (\psi_0, \psi_i) = (B - \dot{E}, E_i + \partial_i E) \,. \tag{2.142}$$

In order to find the homotopy map $s : X_0 \to X_1$, we compute

$$(ip - \mathrm{id})(h_{\mu\nu}) = \bar{h}_{\mu\nu} - h_{\mu\nu} = -\partial_\mu \psi_\nu - \partial_\nu \psi_\mu + 2H\psi_0 \eta_{\mu\nu} = -\partial(\psi)_{\mu\nu} \,, \tag{2.143}$$

$$(ip - \mathrm{id})(\varphi) = \bar{\varphi} - \varphi = \dot{\mathcal{X}}^{(0)} \psi_0 = -\partial(\psi)_\bullet \,, \tag{2.144}$$

from which we can infer:

$$\begin{aligned} s(h)_\mu &= \psi_\mu \in X_1 \,, \\ s(\varphi) &= \psi_\bullet \in X_1 \,, \end{aligned} \tag{2.145}$$

and s on degree -1 elements is zero.

One can repeat the computation for the projection and homotopy maps for the spaces just like for the flat space case. However, let us turn to discussing the computation of organizing the action (2.75) into gauge invariant form. The computation is dramatically shortened by substituting the fields with their projections plus pure gauge terms, (2.140) and (2.141), into the action. All the pure gauge terms drop out by gauge invariance. The remaining terms containing the gauge invariant variables are also simplified due to the constraints on the projected fields, given by

$$\partial^i \bar{h}_{ij} - \tfrac{1}{3}\partial_j (\bar{h}^i{}_i) = 0\,, \qquad \partial^i \bar{h}_{i0} = 0\,, \tag{2.146}$$

which can be deduced from the constraints on the SVT components of $h_{\mu\nu}$ in (2.21) and (2.22). We recover the manifestly gauge invariant action in (2.76).

To provide a consistency check with the literature, in order to compare with the results of [41], we set to zero all modes except the scalar modes and deconstruct the scalar Bardeen variables in (2.76), i.e. by expressing them in terms of the scalars (C, E, B, ϕ, and φ) as in (2.60). We indeed reproduce the quadratic action (10.68) obtained in [41].

As a final check we use our action (2.76) to re-derive the Mukhanov-Sasaki action for the scalar modes. It turns out that the dynamics of the three scalar modes Ψ, Φ, Θ can be reduced to the dynamics of only one mode known as the Mukhanov variable. To start our derivation, we again set all the modes in (2.76) to zero except for the scalar modes, and introduce the following combination of gauge invariant variables,

$$W \equiv \Theta + f\Psi\,, \quad \text{where} \quad f \equiv \frac{\dot{\mathcal{X}}^{(0)}}{H}\,. \tag{2.147}$$

Substituting Θ in terms of W and Ψ, the action (2.76) takes the form:

$$\begin{aligned} S = \int d^4x\, a^2 \bigg\{ & \frac{1}{2}\dot{W}^2 + \frac{1}{2}W\Delta W - \frac{1}{2}a^2 V'' W^2 \\ & + 4\Psi\Delta\Phi - 2\Psi\Delta\Psi - f\Psi\Delta W + \frac{1}{2}f^2\Psi\Delta\Psi \\ & + \frac{1}{2}f^2\Big(-\dot{H} + 2H^2 + \frac{2\ddot{H}}{H} - \frac{2\dot{H}^2}{H^2} + \frac{6\ddot{\mathcal{X}}}{f}\Big)\Psi^2 \\ & - \dot{W}(\dot{f}\Psi + f\dot{\Psi}) + a^2 V'' f\Psi W + 3\dot{\mathcal{X}}W\dot{\Psi} + \Big(\frac{1}{2}f^2 - 6\Big)(\dot{\Psi} + H\Phi)^2 \\ & - f^2(\dot{H} + 2H^2)\Psi\Phi + \dot{\mathcal{X}}W\dot{\Phi} - 2a^2V'W\Phi \bigg\}\,, \end{aligned} \tag{2.148}$$

where the superscript on the background quantity $\mathcal{X}^{(0)}$ has been omitted for readability. We first absorb the dependence of $\dot{\Phi}$ and $\dot{\Psi}$ by performing the field redefinition,

$$\Phi \to \Phi = \Upsilon - \frac{1}{H}\dot{\Psi} + \frac{f}{4}W - \frac{f^2}{4}\Psi\,. \tag{2.149}$$

The action then reduces to a function of Ψ, Υ, and W:

$$S = \int d^4x\, a^2 \left\{\frac{1}{2}\dot{W}^2 + \frac{1}{2}W\Delta W + \frac{1}{2}\Big(-a^2V'' - 2\dot{H} + 4H^2 + \frac{2\dot{H}^2}{H^2} - \frac{2\ddot{H}}{H}\Big)W^2 \right.$$
$$\left. + 4\Psi\Delta\Upsilon + \Big(\frac{1}{2}f^2 - 6\Big)H^2\Upsilon^2 + \Big(\ddot{\mathcal{X}} - \frac{\dot{H}}{H}\dot{\mathcal{X}}\Big)\Upsilon W - \dot{\mathcal{X}}\Upsilon\dot{W}\right\}. \tag{2.150}$$

With the invertibility of the Laplacian, we make use of another field redefinition

$$\Psi \to \Psi = \Gamma - \frac{1}{4}\Delta^{-1}\left[\Big(\frac{1}{2}f^2 - 6\Big)H^2\Upsilon + \Big(\ddot{\mathcal{X}} - \frac{\dot{H}}{H}\dot{\mathcal{X}}\Big)W - \dot{\mathcal{X}}\dot{W}\right], \tag{2.151}$$

which simplifies the second line of (2.150) into a form in which Γ and Υ are clearly auxiliary:

$$S = \int d^4x\, a^2 \left\{\frac{1}{2}\dot{W}^2 + \frac{1}{2}W\Delta W + \frac{1}{2}\Big(-a^2V'' - 2\dot{H} + 4H^2 + \frac{2\dot{H}^2}{H^2} - \frac{2\ddot{H}}{H}\Big)W^2 \right.$$
$$\left. + 4\Gamma\Delta\Upsilon\right\}. \tag{2.152}$$

Let us now introduce the Mukhanov variable $v \equiv aW$ and diagonalize the last term in (2.152) by defining

$$\Phi_\pm \equiv a(\Gamma \pm \Upsilon)\,, \tag{2.153}$$

to obtain

$$S = \int d^4x\, \left\{\frac{1}{2}v\Delta v + \frac{1}{2}\dot{v}^2 + \frac{1}{2}\frac{\ddot{z}}{z}v^2 + \Phi_+\Delta\Phi_+ - \Phi_-\Delta\Phi_-\right\}, \quad \text{where} \quad z \equiv af\,. \tag{2.154}$$

Finally, we can eliminate the auxiliary fields Φ_+ and Φ_- with their equations of motion and the invertibility of the Laplacian to obtain the Mukhanov-Sasaki action,

$$S = \int d^4x\, \left\{\frac{1}{2}v\Delta v + \frac{1}{2}\dot{v}^2 + \frac{1}{2}\frac{\ddot{z}}{z}v^2\right\}. \tag{2.155}$$

This concludes the re-derivation of the Mukhanov-Sasaki action for the gauge invariant scalar mode.

Chapter 3

Cosmological Perturbations in Double Field Theory

In this chapter, we apply our homotopy algebra methods to cosmological perturbations in double field theory. Before going into detail, let us introduce double field theory and our motivations for studying it especially for cosmological applications. The defining symmetry of double field theory is T-duality, which is a duality exhibited by closed string theory. Let us briefly address the difference between dualities and symmetries. Both share the notion of equivalence under transformations. However, a symmetry transformation leaves a physical system invariant, whereas a duality transformation relates two different descriptions of a system or two different theories.

T-duality relates the mass spectra of the closed string on toroidal backgrounds of radius R and $1/R$. We recall that a closed string on a toroidal background is not only described by its momentum but also its winding number, the number of times it can wind around periodic dimensions. Upon quantization, momentum takes discrete integer values, which can be easily seen by applying the translation operator in the periodic directions. Let us consider the simple case of a closed string in a spacetime with one compactified dimension. Its mass spectrum is

$$\begin{aligned} M^2 &= p^2 + w^2 + \frac{2}{\alpha'}(N_L + N_R - 2) \\ &= \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N_L + N_R - 2)\,, \qquad n, m, N_L, N_R \in \mathbb{Z}\,, \end{aligned} \tag{3.1}$$

where α' is the inverse string tension, n/R is the momentum quantum number, mR/α' is the winding quantum number, N_L and N_R are the number of left- and right-moving oscillatory modes which obey the so-called level-matching constraint

$$N_R - N_L = nm\,. \tag{3.2}$$

The mass spectrum (3.1) is invariant under the exchange:

$$n \leftrightarrow m\,, \qquad R \leftrightarrow \frac{\alpha'}{R}\,. \tag{3.3}$$

This implies that closed string theory on large tori shares the same physics as that on small tori as long as the momentum and winding modes are switched. This equivalence is called T-duality. The above transformations belong to the group $\mathbb{Z}_2$. One can extend this example to compactification on a d-torus, where the corresponding T-duality transformations belong to the group $O(d,d;\mathbb{Z})$. Because this invariance appears only upon compactification and is not manifest as a symmetry in the action, one often refers to T-duality as a hidden symmetry of string theory.

The way T-duality relates physics on large (compact) spaces of radius R and small spaces of radius α'/R is partly what makes string theory an attractive perspective to answer questions in cosmology [44–48]. For example, string gas cosmology, a model based on a gas of strings, offers an alternative scenario to inflation and proposes a resolution to the Big Bang singularity [44, 46, 47, 49, 50] (see [51–56] for more recent material). Making measurable predictions of cosmological observables by applying string theory alone has not yet been successful, in particular, string theory does not provide an equivalent of the Friedmann equations. However, it may be advantageous to use a field theoretical description of the background spacetime (instead of the worldsheet). Double field theory is the corresponding field theory which is manifestly invariant under T-duality and thus it is desirable to study its cosmological perturbations.

In the next section we give a short introduction of the general formulation of double field theory. More extensive reviews of double field theory can be found in [57–59]. After this introduction, we will expand the theory to quadratic and cubic order around time-dependent backgrounds, and reformulate the quadratic theory in terms of gauge invariant variables. This will provide a basis for the future computation of observables in cosmological double field theory. The results in section 3.2 have been published in [18] and the results in sections 3.3 and 3.4 have been published in [20]. Many passages have been adapted from [18] and [20].

3.1 Overview of DFT

Double field theory is defined on a doubled spacetime[1], given by coordinates $\mathbf{X}^M \equiv (\tilde{x}_i, x^i)$, where $i = 1, ..., d$. x^i are coordinates which are conjugate to momentum, and $\tilde{x}_i$ are conjugate to winding. The derivatives are denoted as $\partial_M = (\tilde{\partial}^i, \partial_i)$. Indices are raised and lowered by the $O(d,d)$ metric,

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} . \tag{3.4}$$

The $O(d,d)$ metric is invariant under $O(d,d)$ transformations

$$h^t \eta h = \eta \,, \qquad h \in O(d,d) \,. \tag{3.5}$$

The fields in double field theory are the generalized metric $\mathcal{H}_{MN}$ and the dilaton φ, where the index $M = 0, 1, ..., 2d$. These are $O(d,d)$ covariant objects, in the sense that under an $O(d,d)$ transformation h,

$$f'(\mathbf{X}') = f(\mathbf{X}) \,, \qquad \mathbf{X}' = h\mathbf{X} \,, \tag{3.6}$$

enabling double field theory to have a manifest global $O(d,d)$ symmetry. The generalized metric satisfies the constraint

$$\mathcal{H}_{MN}\mathcal{H}^{NQ} = \delta_M{}^Q \,, \tag{3.7}$$

where $\mathcal{H}^{MN}$ is obtained by raising indices with η. Solving its constraint, the generalized metric can be parameterized in the following way,

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} . \tag{3.8}$$

where g_{ij} is the metric field and b_{ij} is the antisymmetric B-field.

The diffeomorphisms on the doubled spacetime are called generalized diffeomorphisms. These are the gauge transformations of double field theory. Under generalized coordinate transformations, $\mathbf{X}^M \to \mathbf{X}'^M = \mathbf{X}^M - \xi^M$, where ξ^M is an

[1]For a more precise definition of doubled geometry, see [60, 61].

infinitesimal parameter, fields transform according to the generalized Lie derivatives $\mathcal{L}_\xi$. On scalars these act as

$$\mathcal{L}_\xi f \equiv \xi^M \partial_M f \,. \tag{3.9}$$

On $O(d,d)$ vectors,

$$\mathcal{L}_\xi V^M \equiv \xi^N \partial_N V^M + K^M{}_N(\xi) V^N \,, \tag{3.10}$$

where

$$K^M{}_N(\xi) = \partial^M \xi_N - \partial_N \xi^M \,. \tag{3.11}$$

For a general $O(d,d)$ tensor $T^M{}_N$,

$$\mathcal{L}_\xi T^M{}_N \equiv \xi^L \partial_L T^M{}_N + K^M{}_L(\xi) T^L{}_N + K_N{}^L(\xi) T^M{}_L \,. \tag{3.12}$$

It is worth noting that, upon the strong constraint which will be introduced next, the generalized Lie derivatives form an algebra

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]_c} \,, \tag{3.13}$$

where $[\cdot,\cdot]_c$ is the C-bracket,

$$[\xi_1,\xi_2]_c^M = \xi_1^N \partial_N \xi_2^M - \frac{1}{2}\xi_{1N}\partial^M \xi_2^N - (1 \leftrightarrow 2) \,. \tag{3.14}$$

This does not satisfy the Jacobi identity [62]. In fact, the gauge algebra of double field theory is an L_∞ algebra [11].

Consistency of double field theory, including the gauge invariance of the action, requires the constraint,

$$\eta_{MN}\partial^M \partial^N f = \partial^M \partial_M f = 2\partial_i \tilde{\partial}^i f = 0 \,, \tag{3.15}$$

for all objects in the theory. It is motivated by the level-matching constraint in string theory (3.2). An immediate solution to the constraint is to demand that all functions only depend on half of the coordinates, i.e. either x^i or $\tilde{x}_i$. This version of the constraint is called the strong constraint, and hence (3.15) by itself is referred to as the weak constraint. Since the constraint (3.15) is $O(d,d)$

invariant, all solutions can be related to this choice via an $O(d,d)$ transformation. Upon eliminating half of the coordinates, one recovers the low-energy effective target space action of closed string theory; in this case, double field theory can be thought of as a manifestly T-duality-invariant reformulation of the theory of massless string modes. When one considers a weakly-constrained double field theory, the functions can depend on more than half of the coordinates, and thus one is not restricted to the massless string modes and instead can also describe massive modes. However, this becomes an obstacle when constructing a full weakly-constrained double field theory beyond quadratic order. For instance the cubic theory requires a modification of the point-wise product such that products of functions also obey the weak constraint. This is addressed in our work in [20], but will not be discussed in this thesis.

The action of double field theory reads

$$\begin{aligned} S_{DFT} = \int \mathrm{d}^{2d}\mathbf{X}\, e^{-2\varphi}\Big(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} \\ + 4\mathcal{H}^{MN}\partial_M\varphi\partial_N\varphi - 2\partial_M\mathcal{H}^{MN}\partial_N\varphi\Big)\,. \end{aligned} \tag{3.16}$$

It is invariant under the gauge transformations,

$$\begin{aligned} \delta_\xi\mathcal{H}_{MN} &= \mathcal{L}_\xi\mathcal{H}_{MN} = \xi^L\partial_L\mathcal{H}_{MN} + K_M{}^L(\xi)\mathcal{H}_{LN} + K_N{}^L(\xi)\mathcal{H}_{ML}\,, \\ \delta_\xi\varphi &= \mathcal{L}_\xi\varphi \equiv \xi^M\partial_M\varphi - \frac{1}{2}\partial_M\xi^M\,. \end{aligned} \tag{3.17}$$

The gauge transformation of the dilaton is defined such that the quantity $e^{-2\varphi}$ transforms as a scalar density. For the full double field theory, gauge invariance of the action requires the strong constraint. However, for the quadratic and cubic theories, as will be discussed later on in the chapter, only the weak version of the constraint is needed for consistency.

The canonical formulation of the double field theory action was derived in [63], via splitting the doubled coordinates into temporal and spatial components. The dependence on the resulting "dual time" coordinates is eliminated so that all fields depend on $(t, \mathbf{X}^M)$ and the action is formulated on a $(1+2d)$-dimensional space. This results in a theory with fields:

$$\mathcal{H}_{MN}\,, \quad \Phi\,, \quad n\,, \quad \mathcal{N}^M\,, \tag{3.18}$$

where $\mathcal{H}_{MN}$ is the spatial generalized metric, Φ is the duality invariant dilaton, n

denotes the lapse function ensuring time reparametrization invariance, and $\mathcal{N}^M$ denotes the doubled shift vector. The action reads

$$S = \int \mathrm{d}t \int \mathrm{d}^{2d}\mathbf{X}\, n\, e^{-2\Phi} \Big(-4(D_t\Phi)^2 - \frac{1}{8} D_t\mathcal{H}_{MN} D_t\mathcal{H}^{MN} + \mathcal{R}(\Phi, \mathcal{H}_{MN}) \Big) . \tag{3.19}$$

The covariant derivatives are defined as

$$D_t \equiv \frac{1}{n}\left(\partial_t - \mathcal{L}_{\mathcal{N}}\right) , \tag{3.20}$$

where $\mathcal{L}_{\mathcal{N}}$ is the generalized Lie derivative with respect to the generalized shift vector $\mathcal{N}^M$:

$$\begin{aligned} \mathcal{L}_{\mathcal{N}}\Phi &= \mathcal{N}^M \partial_M \Phi - \frac{1}{2}\partial_M \mathcal{N}^M , \\ \mathcal{L}_{\mathcal{N}}\mathcal{H}_{MN} &= \mathcal{N}^K \partial_K \mathcal{H}_{MN} + K_M{}^K(\mathcal{N})\mathcal{H}_{KN} + K_N{}^K(\mathcal{N})\mathcal{H}_{KN} . \end{aligned} \tag{3.21}$$

The last term in the action (3.19) uses the generalized curvature scalar

$$\begin{aligned} \mathcal{R}(\Phi, \mathcal{H}_{MN}) \equiv\; & 4\mathcal{H}^{MN}\partial_M\partial_N\Phi - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M\Phi\partial_N\Phi + 4\partial_M\mathcal{H}^{MN}\partial_N\Phi \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} . \end{aligned} \tag{3.22}$$

For now we assume all fields to be subject to the strong version of the constraint meaning that also terms of the form $\partial^M f\, \partial_M g$ are set to zero.

In the following we will use a frame formalism for the spatial generalized metric. We introduce a generalized frame or vielbein $E_A{}^M$, with inverse $E_M{}^A$, satisfying $E_A{}^M E_M{}^B = \delta_A{}^B$, from which the generalized metric can be constructed via

$$\mathcal{H}_{MN} = E_M{}^A E_N{}^B S_{AB} , \tag{3.23}$$

where S_{AB} denotes a positive-definite (tangent space) metric to be given momentarily. The flat indices split as $A = (a, \bar{a})$. The frame field is subject to the constraint that the "flattened" version of the $O(d,d)$ metric (3.4) is block-diagonal:

$$\mathcal{G}_{AB} \equiv E_A{}^M E_B{}^N \eta_{MN} = \begin{pmatrix} \mathcal{G}_{ab} & 0 \\ 0 & \mathcal{G}_{\bar{a}\bar{b}} \end{pmatrix} , \tag{3.24}$$

with no further constraints on the (generally spacetime dependent) metrics $\mathcal{G}_{ab}$ and $\mathcal{G}_{\bar{a}\bar{b}}$. Thus, the local frame transformations comprise the group $GL(d, \mathbb{R}) \times$

$GL(d,\mathbb{R})$. The metric $\mathcal{G}_{AB}$, which is used to raise and lower flat indices, has signature (d,d), and so we can assume without loss of generality that $\mathcal{G}_{ab}$ is negative-definite and that $\mathcal{G}_{\bar{a}\bar{b}}$ is positive-definite. Then, the metric S_{AB} is defined as

$$S_{AB} = \begin{pmatrix} -\mathcal{G}_{ab} & 0 \\ 0 & \mathcal{G}_{\bar{a}\bar{b}} \end{pmatrix} . \tag{3.25}$$

Finally, we give the symmetries of the action, which is invariant under $O(d,d)$ transformations and generalized diffeomorphisms. The transformation rules can be inferred from [64]. The gauge parameters are ξ^0, ξ_0, ξ^M, and $\Lambda_A{}^B$, all depending on coordinates $(t,\mathbf{X}^M)$, and act infinitesimally via

$$\begin{aligned} \delta E_A{}^M &= \mathcal{L}_\xi E_A{}^M + n\xi^0 D_t E_A{}^M + \Lambda_A{}^B E_B{}^M , \\ \delta n &= n\xi^0 D_t n + n^2 D_t \xi^0 + \xi^M \partial_M n , \\ \delta \mathcal{N}^M &= \partial^M \xi_0 + \partial_t \xi^M - n^2 \mathcal{H}^{MN} \partial_N \xi^0 + \mathcal{L}_\xi \mathcal{N}^M , \\ \delta \Phi &= n\xi^0 D_t \Phi + \mathcal{L}_\xi \Phi . \end{aligned} \tag{3.26}$$

We are now ready to expand the theory around flat backgrounds as well as time-dependent ones.

3.2 DFT on Flat Space

3.2.1 SVT Decomposition

Let us start with the expansion of double field theory around flat space. The fields can be split into their constant background and fluctuation parts:

$$E_A{}^M(t,\mathbf{X}) = \bar{E}_A{}^M - h_A{}^B(t,\mathbf{X})\bar{E}_B{}^M(t) , \tag{3.27}$$

$$\Phi(t,\mathbf{X}) = \bar{\Phi} + \varphi(t,\mathbf{X}) , \tag{3.28}$$

$$n(t,\mathbf{X}) = \bar{n}(1+\phi(t,\mathbf{X})) , \tag{3.29}$$

$$\mathcal{N}^M(t,\mathbf{X}) = \mathcal{A}^M(t,\mathbf{X}) . \tag{3.30}$$

Here we have taken the background value of the shift vector to be zero. The background tangent space metric $\bar{\mathcal{G}}_{AB}$ constructed as in (3.24) will be used to raise and lower flat indices, while the background $\bar{E}_A{}^M$ and its inverse will be used to flatten and unflatten indices. In the following we will often omit the bar on the background quantities. A gauge (using Λ from the frame transformations) has been chosen such that $h_{ab} = h_{\bar{a}\bar{b}} = 0$ and $h_A{}^B$ is solely described by the components

$$h_{a\bar{b}} = -h_{\bar{b}a} \,. \tag{3.31}$$

This equality follows from the fact that (3.24) is block-diagonal. Let us define the derivatives,

$$D_A \equiv E_A{}^M \partial_M \equiv (D_a, D_{\bar{a}}) \,. \tag{3.32}$$

The weak constraint (3.15) is re-expressed as

$$\Delta f \equiv -2D^a D_a f = 2D^{\bar{a}} D_{\bar{a}} f \,. \tag{3.33}$$

This shows that despite the doubling of spatial coordinates, there is a unique Laplacian Δ.

The double field theory action (3.19) expanded to quadratic order around flat space yields

$$\begin{aligned} S_0 = \int \mathrm{d}t \int \mathrm{d}^{2d}\mathbf{X}\, n e^{-2\Phi} \Big(& -4(D_t\varphi)^2 - D_t h^{a\bar{b}} D_t h_{a\bar{b}} \\ & + 8D^a\phi D_a\varphi - 8D^a\varphi D_a\varphi - 8D_a\varphi D_{\bar{b}} h^{a\bar{b}} + 4D_a\phi D_{\bar{b}} h^{a\bar{b}} \\ & - 2D^a h^{b\bar{c}} D_a h_{b\bar{c}} + 2D^c h^{a\bar{b}} D_a h_{c\bar{b}} - 2D^{\bar{c}} h^{a\bar{b}} D_{\bar{b}} h_{a\bar{c}} \Big) \,, \end{aligned} \tag{3.34}$$

where the covariant time derivatives (3.20) act as

$$\begin{aligned} D_t\varphi &= \partial_t\varphi + \frac{1}{2} D_a \mathcal{A}^a + \frac{1}{2} D_{\bar{a}} \mathcal{A}^{\bar{a}} \,, \\ D_t h_{a\bar{b}} &= \partial_t h_{a\bar{b}} - D_a \mathcal{A}_{\bar{b}} + D_{\bar{b}} \mathcal{A}_a \,. \end{aligned} \tag{3.35}$$

As unbarred and barred indices are consistently contracted the above action has a manifest $SO(d)_L \times SO(d)_R$ invariance in addition to the $O(d,d)$ duality. The action is invariant under the gauge symmetries with parameters $(\xi_a, \xi_{\bar{a}}, \xi^0, \xi_0)$, given by

$$\begin{aligned} \delta h_{a\bar{b}} &= D_a \xi_{\bar{b}} - D_{\bar{b}} \xi_a \,, \\ \delta\phi &= \dot{\xi}^0 \,, \\ \delta\varphi &= -\frac{1}{2} D_a \xi^a - \frac{1}{2} D_{\bar{a}} \xi^{\bar{a}} \,, \\ \delta\mathcal{A}_a &= \dot{\xi}_a + D_a(\xi_0 + \xi^0) \,, \\ \delta\mathcal{A}_{\bar{a}} &= \dot{\xi}_{\bar{a}} + D_{\bar{a}}(\xi_0 - \xi^0) \,. \end{aligned} \tag{3.36}$$

There is a gauge symmetry for gauge symmetries since gauge parameters of the form

$$\xi_a = D_a\chi\,, \quad \xi_{\bar{a}} = D_{\bar{a}}\chi\,, \quad \tilde{\xi}_0 = -\dot{\chi}\,, \tag{3.37}$$

where χ is an arbitrary scalar, do not generate a transformation of fields. Note that the derivatives (3.35) are invariant under gauge transformations w.r.t. $(\xi_a, \xi_{\bar{a}})$. They are also invariant under ξ_0 transformations thanks to the weak constraint (3.33). This leaves ξ^0 transformations as the only symmetry linking the terms in the first line of (3.34) to the rest of the action. The total invariance, subject to the weak constraint (3.33), is easy to verify. It should be emphasized that at the level of the free theory one does not have to worry about how to implement the weak constraint (3.33) on products of fields, since under an integral, e.g. $\int D^a f D_a g = -\int f D^a D_a g$ through partial integration, and thus it is sufficient that all fields and gauge parameters satisfy the constraint [15].

In the following we will follow our procedure for rewriting a theory in terms of gauge invariant variables for linearized double field theory. We start with a scalar-vector-tensor (SVT) decomposition:

$$\begin{aligned} h_{a\bar{b}} &= \widehat{h}_{a\bar{b}} + D_a B_{\bar{b}} - D_{\bar{b}} B_a + D_a D_{\bar{b}} E\,, \\ \mathcal{A}_a &= A_a + D_a A\,, \\ \mathcal{A}_{\bar{a}} &= A_{\bar{a}} + D_{\bar{a}}\bar{A}\,, \end{aligned} \tag{3.38}$$

where

$$D^a \widehat{h}_{a\bar{b}} = D^{\bar{b}} \widehat{h}_{a\bar{b}} = 0\,, \qquad D^a B_a = D^{\bar{a}} B_{\bar{a}} = 0\,, \qquad D^a A_a = D^{\bar{a}} A_{\bar{a}} = 0\,. \tag{3.39}$$

Since there is no $SO(d)_L \times SO(d)_R$ invariant way to take a trace of $h_{a\bar{b}}$, we do not have an additional scalar component as we do in (2.20) in standard gravity. The gauge parameters are decomposed as:

$$\xi_a = \zeta_a + D_a\lambda\,, \qquad \xi_{\bar{a}} = \zeta_{\bar{a}} + D_{\bar{a}}\bar{\lambda}\,, \tag{3.40}$$

where

$$D^a\zeta_a = D^{\bar{a}}\zeta_{\bar{a}} = 0\,. \tag{3.41}$$

We can now collect the gauge transformations of the various components:

$$\begin{aligned} &\delta\widehat{h}_{a\bar{b}} = 0\,, \quad \delta B_a = \zeta_a\,, \quad \delta B_{\bar{a}} = \zeta_{\bar{a}}\,, \quad \delta E = \bar{\lambda} - \lambda\,, \\ &\delta A_a = \dot{\zeta}_a\,, \quad \delta A_{\bar{a}} = \dot{\zeta}_{\bar{a}}\,, \quad \delta A = \dot{\lambda} + \xi_0 + \xi^0\,, \quad \delta\bar{A} = \dot{\bar{\lambda}} + \xi_0 - \xi^0\,, \\ &\delta\varphi = \frac{1}{4}\Delta(\lambda - \bar{\lambda})\,, \quad \delta\phi = \dot{\xi}^0\,. \end{aligned} \tag{3.42}$$

In addition to the gauge invariant tensor mode $\widehat{h}_{a\bar{b}}$, we can find two vector modes and two scalar modes which are gauge invariant:

$$\begin{aligned} \widehat{A}_a &= A_a - \dot{B}_a\,, \\ \widehat{A}_{\bar{a}} &= A_{\bar{a}} - \dot{B}_{\bar{a}}\,, \\ \Psi &= \phi - \frac{1}{2}(\dot{A} - \dot{\bar{A}} + \ddot{E})\,, \\ \Phi &= \varphi + \frac{1}{4}\Delta E\,. \end{aligned} \tag{3.43}$$

We notice that compared to standard gravity on flat space, there is one more gauge invariant vector mode, coming from the vector component of the B-field. However, we do not obtain an additional scalar mode corresponding to the dilaton. Let us check that the number of gauge invariant variables agrees with what we expect given the number of gauge redundancies. Choosing $d = 3$ for definiteness, there are 17 off-shell field components in total:

$$h_{a\bar{b}} \;:\; 9 \quad \mathcal{A}_a \;:\; 3 \quad \mathcal{A}_{\bar{a}} \;:\; 3 \quad \phi \;:\; 1 \quad \varphi \;:\; 1\,. \tag{3.44}$$

The gauge parameters $\zeta_a, \zeta_{\bar{a}}, \xi_0, \xi^0$ generate a total of $3 + 3 + 1 + 1 = 8$ gauge redundancies, and subtracting the redundancy in the gauge symmetries (3.37), we have seven gauge redundancies. Therefore we must have ten gauge invariant components, which matches with the above gauge invariant variables:

$$10 \;=\; 4(\widehat{h}_{a\bar{b}}) \;+\; 2(\widehat{A}_a) \;+\; 2(\widehat{A}_{\bar{a}}) \;+\; 1(\Psi) \;+\; 1(\Phi)\,. \tag{3.45}$$

3.2.2 Chain Complex of Double Field Theory

Let us define the chain complex of the free theory to set up the homotopy retract to gauge invariant variables. The chain complex reads:

$$\begin{array}{ccccccccccc} X_2 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{\partial} & X_{-1} & \xrightarrow{\partial} & X_{-2} & \xrightarrow{\partial} & X_{-3} \\ \{\chi\} & & \{\xi\} & & \{\Psi\} & & \{\mathcal{E}\} & & \{\mathcal{G}\} & & \{\rho\} \end{array} \tag{3.46}$$

where X_2 is the space of trivial gauge parameters, which here are scalars χ, X_1 is the space of gauge parameters ξ, and X_0 is the space of fields Ψ. The gauge parameters and fields are organized in terms of components:

$$\xi = \begin{pmatrix} \xi^a \\ \xi^{\bar{a}} \\ \xi_0 \\ \xi^0 \end{pmatrix} \in X_1 \,, \qquad \Psi = \begin{pmatrix} h_{a\bar{b}} \\ \phi \\ \varphi \\ \mathcal{A}_a \\ \mathcal{A}_{\bar{a}} \end{pmatrix} \in X_0 \,. \tag{3.47}$$

Next, X_{-1} is the space of field equations $\mathcal{E}$, which have the same index structure as fields, and X_{-2} is the space of Noether identities, which have the same index structure as the gauge parameters. Finally, X_{-3} is the space of Noether identities for Noether identities, which have the same index structure as the trivial gauge parameters and are thus scalars.

We now list the differential maps ∂. First, looking at the trivial gauge transformations (3.37), we define $\partial_2 : X_2 \to X_1$,

$$\partial(\chi) = \begin{pmatrix} D_a\chi \\ D_{\bar{a}}\chi \\ -\dot{\chi} \\ 0 \end{pmatrix} \in X_1 \,. \tag{3.48}$$

Second, from the gauge transformations (3.36) we define $\partial_1 : X_1 \to X_0$:

$$\partial(\xi) = \begin{pmatrix} D_a\xi_{\bar{b}} - D_{\bar{b}}\xi_a \\ \dot{\xi}^0 \\ -\frac{1}{2}D_a\xi^a - \frac{1}{2}D_{\bar{a}}\xi^{\bar{a}} \\ \dot{\xi}_a + D_a(\xi_0 + \xi^0) \\ \dot{\xi}_{\bar{a}} + D_{\bar{a}}(\xi_0 - \xi^0) \end{pmatrix} \in X_0 \,. \tag{3.49}$$

Next, we obtain the differential $\partial_0 : X_0 \to X_{-1}$ which encode the field equations

$$\begin{pmatrix} \mathcal{E}_{a\bar{b}} \\ \mathcal{E}_{\phi} \\ \mathcal{E}_{\varphi} \\ \mathcal{E}_{a} \\ \mathcal{E}_{\bar{a}} \end{pmatrix} = \partial \begin{pmatrix} h_{a\bar{b}} \\ \phi \\ \varphi \\ \mathcal{A}_{a} \\ \mathcal{A}_{\bar{a}} \end{pmatrix} \in X_{-1}\,, \tag{3.50}$$

which we have derived by varying the quadratic action (3.34),

$$\begin{aligned} \mathcal{E}_{a\bar{b}} &= \partial_t(D_t h_{a\bar{b}}) - 2D_a D_{\bar{b}}\phi + \mathcal{R}_{a\bar{b}}\,, \\ \mathcal{E}_{\phi} &= \mathcal{R}\,, \\ \mathcal{E}_{\varphi} &= \partial_t(D_t\varphi) + \frac{1}{2}\Delta\phi + \mathcal{R}\,, \\ \mathcal{E}_{a} &= D_a(D_t\varphi) + \frac{1}{2}D^{\bar{b}}(D_t h_{a\bar{b}})\,, \\ \mathcal{E}_{\bar{a}} &= D_{\bar{a}}(D_t\varphi) - \frac{1}{2}D^{b}(D_t h_{b\bar{a}})\,, \end{aligned} \tag{3.51}$$

with the spatial curvatures,

$$\begin{aligned} \mathcal{R} &= -\Delta\varphi + D_a D_{\bar{b}} h^{a\bar{b}}\,, \\ \mathcal{R}_{a\bar{b}} &= 2\left(-\frac{1}{2}\Delta h_{a\bar{b}} - D_a D^{c} h_{c\bar{b}} + D_{\bar{b}} D^{\bar{c}} h_{a\bar{c}} + 2D_a D_{\bar{b}}\varphi\right)\,. \end{aligned} \tag{3.52}$$

It is easy to check the gauge invariance of the equations of motion (3.51), i.e. $\partial_0 \circ \partial_1 = 0$, with the gauge variations

$$\delta(D_t h_{a\bar{b}}) = 2D_a D_{\bar{b}}\xi^0\,, \qquad \delta(D_t\varphi) = -\frac{1}{2}\Delta\xi^0\,. \tag{3.53}$$

We next observe that the curvatures (3.52) obey the identities

$$\begin{aligned} D^{a}\mathcal{R}_{a\bar{b}} - 2D_{\bar{b}}\mathcal{R} &= 0\,, \\ D^{\bar{b}}\mathcal{R}_{a\bar{b}} + 2D_{a}\mathcal{R} &= 0\,, \end{aligned} \tag{3.54}$$

which can be used to derive the Noehter identities for $\mathcal{E}$, defining the differential $\partial_{-1} : X_{-1} \to X_{-2}$:

$$\partial\mathcal{E} \equiv \begin{pmatrix} (\partial\mathcal{E})^a \\ (\partial\mathcal{E})^{\bar{a}} \\ (\partial\mathcal{E})_0 \\ (\partial\mathcal{E})^0 \end{pmatrix} \equiv \partial \begin{pmatrix} \mathcal{E}_{a\bar{b}} \\ \mathcal{E}_\phi \\ \mathcal{E}_\varphi \\ \mathcal{E}_a \\ \mathcal{E}_{\bar{a}} \end{pmatrix} = \begin{pmatrix} D_{\bar{b}}\mathcal{E}^{a\bar{b}} + 2D^a\mathcal{E}_\varphi - 2\partial_t\mathcal{E}^a \\ D_b\mathcal{E}^{b\bar{a}} - 2D^{\bar{a}}\mathcal{E}_\varphi + 2\partial_t\mathcal{E}^{\bar{a}} \\ D_a\mathcal{E}^a + D_{\bar{a}}\mathcal{E}^{\bar{a}} \\ D_a\mathcal{E}^a - D_{\bar{a}}\mathcal{E}^{\bar{a}} - \partial_t\mathcal{E}_\phi \end{pmatrix} . \tag{3.55}$$

The combinations in (3.55) yield zero upon inserting $\mathcal{E}$ given by (3.51), verifying $\partial_{-1} \circ \partial_0 = 0$. The last differential $\partial_{-2} : X_{-1} \to X_{-3}$ is defined by

$$\partial \begin{pmatrix} \mathcal{G}^a \\ \mathcal{G}^{\bar{a}} \\ \mathcal{G}_0 \\ \mathcal{G}^0 \end{pmatrix} \equiv \dot{\mathcal{G}}_0 + \frac{1}{2}D_a\mathcal{G}^a - \frac{1}{2}D_{\bar{a}}\mathcal{G}^{\bar{a}} . \tag{3.56}$$

The expression on the RHS vanishes identically upon rewriting $\mathcal{G}$ as the expression on the RHS of (3.55), i.e. $\partial_{-2} \circ \partial_{-1} = 0$. This completes the construction of the chain complex and verification that the differential is indeed nilpotent.

3.2.3 Homotopy Interpretation

We are ready to construct a homotopy retract from the chain complex to the chain complex of gauge invariant variables. In contrast to conventional gravity, we have gauge symmetries for gauge symmetries, and although in principle one might think that these extra spaces are projected to something non-zero but trivial, in the following we will show that the homotopy relations pan out with

$$p_2 = p_1 = 0 , \qquad \bar{X}_2 = \{0\} = \bar{X}_1 . \tag{3.57}$$

First we treat the homotopy relation on degree 2. The gauge transformations of the gauge parameters (3.48) decompose with (3.40) as

$$\begin{aligned} \delta_\chi\zeta^a &= 0 , & \delta_\chi\zeta^{\bar{a}} &= 0 , \\ \delta_\chi\lambda &= \chi , & \delta_\chi\bar{\lambda} &= \chi , \\ \delta_\chi\xi_0 &= -\dot{\chi} , & \delta_\chi\xi^0 &= 0 . \end{aligned} \tag{3.58}$$

We can combine the components into the gauge invariant gauge parameters,

$$\begin{aligned}
\bar{\xi}^a &\equiv \zeta^a + \frac{1}{2}D^a(\lambda - \bar{\lambda}) \,, \\
\bar{\xi}^{\bar{a}} &\equiv \zeta^{\bar{a}} - \frac{1}{2}D^{\bar{a}}(\lambda - \bar{\lambda}) \,, \\
\bar{\xi}_0 &\equiv \xi_0 + \frac{1}{2}(\dot{\lambda} + \dot{\bar{\lambda}}) \,, \\
\bar{\xi}^0 &\equiv \xi^0 \,.
\end{aligned} \tag{3.59}$$

With this combination, we can express the gauge parameters into an invariant piece plus pure gauge piece,

$$\xi = \begin{pmatrix} \xi^a \\ \xi^{\bar{a}} \\ \xi_0 \\ \xi^0 \end{pmatrix} = \begin{pmatrix} \bar{\xi}^a + D^a f \\ \bar{\xi}^{\bar{a}} + D^{\bar{a}} f \\ \bar{\xi}_0 - \dot{f} \\ \bar{\xi}^0 \end{pmatrix} \,, \tag{3.60}$$

where

$$f \equiv \frac{1}{2}(\lambda + \bar{\lambda}) \,. \tag{3.61}$$

Written more concisely using the differential,

$$\xi = \bar{\xi} + \partial(f) \,. \tag{3.62}$$

If we define the homotopy map s acting on gauge parameters as

$$s(\xi) = f \,, \tag{3.63}$$

and by defining $s(\chi) = 0$, the homotopy relation on X_2 is satisfied:

$$(\mathrm{id} - ip)(\chi) = \chi = s(\partial\chi) \,. \tag{3.64}$$

Before defining the homotopy map $s_0 : X_0 \to X_1$ from fields to gauge parameters, for consistency of notation let us denote the gauge invariant quantities with a bar and rename (3.43),

$$\bar{h}_{a\bar{b}} = \widehat{h}_{a\bar{b}} \,, \quad \bar{\varphi} = \Phi \,, \quad \bar{\phi} = \Psi \,, \quad \bar{\mathcal{A}}_a = \widehat{A}_a \,, \quad \bar{\mathcal{A}}_{\bar{a}} = \widehat{A}_{\bar{a}} \,. \tag{3.65}$$

We then define the projection $p_0 : X_0 \to \bar{X}_0$ these quantities,

$$p_0(\Psi) = \bar{\Psi}\,, \tag{3.66}$$

using the notation (3.47). It follows by a simple computation that the original fields are can be written in terms of their projections as

$$\begin{aligned} h_{a\bar{b}} &= \bar{h}_{a\bar{b}} + D_a F_{\bar{b}} - D_{\bar{b}} F_a\,, \\ \phi &= \bar{\phi} + \dot{F}^0\,, \\ \varphi &= \bar{\varphi} - \frac{1}{2} D_a F^a - \frac{1}{2} D_{\bar{a}} F^{\bar{a}}\,, \\ \mathcal{A}_a &= \bar{\mathcal{A}}_a + \dot{F}_a + D_a(\widetilde{F}_0 + F^0)\,, \\ \mathcal{A}_a &= \bar{\mathcal{A}}_{\bar{a}} + \dot{F}_{\bar{a}} + D_{\bar{a}}(\widetilde{F}_0 - F^0)\,, \end{aligned} \tag{3.67}$$

where the pure gauge F terms read

$$\begin{aligned} F_a &= B_a - \frac{1}{2} D_a E\,, \\ F_{\bar{a}} &= B_{\bar{a}} + \frac{1}{2} D_{\bar{a}} E\,, \\ F^0 &= \frac{1}{2}(\dot{E} + A - \bar{A})\,, \\ \widetilde{F}_0 &= \frac{1}{2}(A + \bar{A})\,. \end{aligned} \tag{3.68}$$

We can deduce the homotopy map s_0 from fields to gauge parameters

$$s_0(\Psi) = \begin{pmatrix} F^a \\ F^{\bar{a}} \\ \widetilde{F}_0 \\ F^0 \end{pmatrix}\,. \tag{3.69}$$

Acting s_0 on a field that is pure gauge yields the invariant part (3.59) of the gauge parameter:

$$s(\partial\xi) = \bar{\bar{\xi}}\,. \tag{3.70}$$

Then with (3.57), (3.62), and (3.63), we see that the homotopy relation on $\xi \in X_1$ is satisfied

$$(\mathrm{id} - ip)(\xi) = \xi = \partial(s(\xi)) + s(\partial\xi) = \partial f + \bar{\bar{\xi}}\,. \tag{3.71}$$

Lastly, the homotopy relation on $\Psi \in X_0$ reads

$$(\mathrm{id} - ip)(\Psi) = \Psi - \bar{\Psi} = \partial(F) = \partial(s(\Psi))\,, \tag{3.72}$$

where we used (3.67) and (3.69). It is satisfied if we take the homotopy map to act trivially on the space of field equations,

$$s_{-1} = 0\,. \tag{3.73}$$

At this point, we have established a homotopy retract from the sub-complex $X_2 \to X_1 \to X_0$ to the gauge invariant complex.

Let us now extend the homotopy retract to the entire chain complex, which will need two more new non-trivial homotopy maps. As is the case for Yang-Mills theory and gravity, the spaces of Noether identities and the Noether identities of Noether identities are projected to zero:

$$\bar{X}_{-2} = \bar{X}_{-3} = \{0\}\,, \qquad p_{-2} = p_{-3} = 0\,. \tag{3.74}$$

In degree -1 the projector $p_{-1} : X_{-1} \to \bar{X}_{-1}$ is non-trivial and projects onto the space of tensors that satisfy the Noether identities identically. These are defined by $\bar{\mathcal{E}} = p(\mathcal{E})$, where

$$\begin{aligned}
\bar{\mathcal{E}}_{a\bar{b}} &= \mathcal{E}_{a\bar{b}} + 2D_a\Delta^{-1}(\partial\mathcal{E})_{\bar{b}} - 2D_{\bar{b}}\Delta^{-1}(\partial\mathcal{E})_a \\
&\qquad + 4D_aD_{\bar{b}}\Delta^{-2}\left((\partial\dot{\mathcal{E}})^0 - \frac{1}{2}D^c(\partial\mathcal{E})_c - \frac{1}{2}D^{\bar{c}}(\partial\mathcal{E})_{\bar{c}}\right)\,, \\
\bar{\mathcal{E}}_\phi &= \mathcal{E}_\phi\,, \\
\bar{\mathcal{E}}_\varphi &= \mathcal{E}_\varphi\,, \\
\bar{\mathcal{E}}_a &= \mathcal{E}_a + D_a\Delta^{-1}\left((\partial\mathcal{E})_0 + (\partial\mathcal{E})^0\right)\,, \\
\bar{\mathcal{E}}_{\bar{a}} &= \mathcal{E}_{\bar{a}} - D_{\bar{a}}\Delta^{-1}\left((\partial\mathcal{E})_0 - (\partial\mathcal{E})^0\right)\,,
\end{aligned} \tag{3.75}$$

where the components of $(\partial\mathcal{E})$ are defined in (3.55). Consequently, one can confirm that the relations in (3.55) hold identically:

$$\begin{aligned}
0 &\equiv D_{\bar{b}}\bar{\mathcal{E}}^{a\bar{b}} + 2D^a\bar{\mathcal{E}}_\varphi - 2\partial_t\bar{\mathcal{E}}^a\,, \\
0 &\equiv D_b\bar{\mathcal{E}}^{b\bar{a}} - 2D^{\bar{a}}\bar{\mathcal{E}}_\varphi + 2\partial_t\bar{\mathcal{E}}^{\bar{a}}\,, \\
0 &\equiv D_a\bar{\mathcal{E}}^a + D_{\bar{a}}\bar{\mathcal{E}}^{\bar{a}}\,, \\
0 &\equiv D_a\bar{\mathcal{E}}^a - D_{\bar{a}}\bar{\mathcal{E}}^{\bar{a}} - \partial_t\bar{\mathcal{E}}_\phi\,.
\end{aligned} \tag{3.76}$$

Now that the projections have been defined, we will now find the remaining homotopy maps.

Assuming the homotopy from the space of field equations to the space of fields to be trivial, $s(\mathcal{E}) = 0$. In order to find the homotopy map on the space of Noether identities X_{-1} let us evaluate the homotopy relation on $\mathcal{E} \in X_{-1}$:

$$(\mathrm{id} - ip)(\mathcal{E}) = \mathcal{E} - \bar{\mathcal{E}} = s(\partial \mathcal{E}) \ . \tag{3.77}$$

By inspecting the expression of the projection (3.75) in terms of $\partial\mathcal{E}$, we can immediately write the homotopy map:

$$\begin{aligned}
s(\mathcal{G})_{a\bar{b}} &= -2D_a\Delta^{-1}\mathcal{G}_{\bar{b}} + 2D_{\bar{b}}\Delta^{-1}\mathcal{G}_a - 4D_aD_{\bar{b}}\Delta^{-2}\left(\dot{\mathcal{G}}^0 - \frac{1}{2}D^c\mathcal{G}_c - \frac{1}{2}D^{\bar{c}}\mathcal{G}_{\bar{c}}\right) , \\
s(\mathcal{G})_\phi &= 0 \, , \\
s(\mathcal{G})_\varphi &= 0 \, , \\
s(\mathcal{G})_a &= -D_a\Delta^{-1}\left(\mathcal{G}_0 + \mathcal{G}^0\right) , \\
s(\mathcal{G})_{\bar{a}} &= D_{\bar{a}}\Delta^{-1}\left(\mathcal{G}_0 - \mathcal{G}^0\right) .
\end{aligned} \tag{3.78}$$

Next we evaluate the homotopy relation on $\mathcal{G} \in X_{-2}$ and recalling (3.74),

$$(\mathrm{id} - ip)(\mathcal{G}) = \mathcal{G} = \partial(s(\mathcal{G})) + s(\partial\mathcal{G}) \ . \tag{3.79}$$

The first term on the RHS can be computed with (3.55) and (3.78):

$$\partial(s(\mathcal{G})) = \mathcal{G} + 2\begin{pmatrix} D_a\Delta^{-1}(\dot{\mathcal{G}}_0 + \frac{1}{2}D^c\mathcal{G}_c - \frac{1}{2}D^{\bar{c}}\mathcal{G}_{\bar{c}}) \\ D_{\bar{a}}\Delta^{-1}(\dot{\mathcal{G}}_0 + \frac{1}{2}D^c\mathcal{G}_c - \frac{1}{2}D^{\bar{c}}\mathcal{G}_{\bar{c}}) \\ 0 \\ 0 \end{pmatrix} . \tag{3.80}$$

Looking back at (3.56) we infer that the failure of $\partial(s(\mathcal{G}))$ to give back $\mathcal{G}$ involves $\partial\mathcal{G}$. This implies that (3.79) is satisfied if we define a non-trivial homotopy map $s : X_{-3} \to X_{-2}$ as follows

$$s(\rho) = -2\begin{pmatrix} D_a\Delta^{-1}\rho \\ D_{\bar{a}}\Delta^{-1}\rho \\ 0 \\ 0 \end{pmatrix} . \tag{3.81}$$

Lastly we verify that the homotopy relation holds on $\rho \in X_{-3}$:

$$(\mathrm{id} - ip)(\rho) = \rho = \partial(s(\rho)) \, , \tag{3.82}$$

using $\partial\rho = 0$, (3.56) and (3.81). We have finally completed the proof of the homotopy retract from the entire chain complex (3.46) of linearized double field theory on flat space can be to the complex $\bar{X}_0 \to \bar{X}_{-1}$ of gauge invariant fields and their field equations, where all redundancies have been eliminated.

3.2.4 Gauge Invariant Action

By replacing the fields with their projections, we can express the quadratic double field theory Lagrangian (3.34) in terms of the gauge invariant variables.

$$\mathcal{L} = -4\dot{\Phi}^2 - \widehat{h}^{a\bar{b}} \,\square\, \widehat{h}_{a\bar{b}} + \frac{1}{2}\widehat{A}^a \Delta \widehat{A}_a - \frac{1}{2}\widehat{A}^{\bar{a}} \Delta \widehat{A}_{\bar{a}} - 4\Phi\Delta\Phi + 4\Psi\Delta\Phi \, , \tag{3.83}$$

where $\square = -\frac{\partial^2}{\partial t^2} + \Delta$. Following a similar diagonalization procedure as we have done in conventional gravity, we perform the field redefinition

$$\Psi \to \Psi' = \Psi - \Phi + \Delta^{-1}\ddot{\Phi} \, , \tag{3.84}$$

which removes the $\dot{\Phi}^2$ term, resulting in the Lagrangian,

$$\mathcal{L} = -\widehat{h}^{a\bar{b}} \,\square\, \widehat{h}_{a\bar{b}} + \frac{1}{2}\widehat{A}^a \Delta \widehat{A}_a - \frac{1}{2}\widehat{A}^{\bar{a}} \Delta \widehat{A}_{\bar{a}} + 4\Psi' \Delta\Phi \, . \tag{3.85}$$

In this form, it is clear that only $\widehat{h}^{a\bar{b}}$ propagates, while the other modes could be integrated out to eliminate them, i.e., in the vacuum case the Lagrangian reduces to

$$\mathcal{L} = -\widehat{h}^{a\bar{b}} \,\square\, \widehat{h}_{a\bar{b}} \, . \tag{3.86}$$

We infer that the four propagating degrees in $\widehat{h}^{a\bar{b}}$ are the spin-2 tensor modes with two degrees of freedom, the scalar mode given by the B-field and the scalar mode given by the dilaton.

3.3 DFT on Time-Dependent Backgrounds

Now let us expand around a background that is purely time-dependent as follows:

$$
\begin{aligned}
E_A{}^M(t,\mathbf{X}) &= \bar{E}_A{}^M(t) - h_A{}^B(t,\mathbf{X})\bar{E}_B{}^M(t)\,, &(3.87)\\
\Phi(t,\mathbf{X}) &= \bar{\Phi}(t) + \varphi(t,\mathbf{X})\,, &(3.88)\\
n(t,\mathbf{X}) &= \bar{n}(t)(1+\phi(t,\mathbf{X}))\,, &(3.89)\\
\mathcal{N}^M(t,\mathbf{X}) &= \bar{n}(t)\mathcal{A}^M(t,\mathbf{X})\,. &(3.90)
\end{aligned}
$$

Again we will omit the bar on the background quantities for readability. We assume that a gauge has been chosen for the (background) frame transformations for which $\bar{\mathcal{G}}_{AB}$ is constant and does not depend on time. The corresponding background version of the metric (3.25) is then also constant, which allows us to raise and lower flat indices under time derivatives. As done previously for the flat space case, we fix a gauge so that the independent fluctuation is given by $h_{a\bar{b}} = -h_{\bar{b}a}$.

3.3.1 Background Equations

Inserting the above expansion into the action, one obtains to leading order an action for the purely time-dependent background fields encoding their dynamics:

$$
S_0 = \int dt \int d^{2d}\mathbf{X}\, n^{-1}\, e^{-2\Phi}\Big(-4\dot{\Phi}^2 - \frac{1}{8}\mathrm{tr}(\dot{\mathcal{S}}^2)\Big)\,, \tag{3.91}
$$

where we employ matrix notation, with $\mathcal{S}^M{}_N$ the background generalized metric with one index raised, and the dot denotes the time derivative. The field equations read

$$
\begin{aligned}
\ddot{\mathcal{S}} + \mathcal{S}\dot{\mathcal{S}}^2 - 2\Big(\dot{\Phi} + \frac{1}{2}\partial_t \log n\Big)\dot{\mathcal{S}} &= 0\,,\\
-4\ddot{\Phi} + 4\dot{\Phi}^2 + 4\dot{\Phi}\partial_t \log n - \frac{1}{8}\mathrm{tr}(\dot{\mathcal{S}}^2) &= 0\,,\\
4\dot{\Phi}^2 + \frac{1}{8}\mathrm{tr}(\dot{\mathcal{S}}^2) &= 0\,.
\end{aligned}
\tag{3.92}
$$

It will be convenient to express these equations explicitly in terms of the background frame field. We define

$$L_A{}^B \;\equiv\; \frac{1}{n}\partial_t E_A{}^M E_M{}^B\,, \tag{3.93}$$

in terms of which the equations of motion are

$$\begin{aligned}
\frac{1}{n}\dot{L}_a{}^{\bar{b}} - L_a{}^c L_c{}^{\bar{b}} + L_a{}^{\bar{c}} L_{\bar{c}}{}^{\bar{b}} - \frac{2}{n}\dot{\Phi} L_a{}^{\bar{b}} \;&=\; 0\,, &(3.94)\\
-4\ddot{\Phi} + 4\dot{\Phi}^2 + 4\dot{\Phi}\partial_t \log n + n^2\, L_a{}^{\bar{b}} L_{\bar{b}}{}^a \;&=\; 0\,, &(3.95)\\
4\dot{\Phi}^2 - n^2\, L_a{}^{\bar{b}} L_{\bar{b}}{}^a \;&=\; 0\,. &(3.96)
\end{aligned}$$

It is instructive to rewrite these equations in terms of derivatives that are covariant under background frame transformations with parameter $\bar{\Lambda}_A{}^B$ and time reparametrization with parameter $\bar{\xi}^0$. These transformations act on $L_A{}^B$, the lapse function n and generic vectors $\mathcal{V}_a$ and $\mathcal{V}_{\bar{a}}$ as

$$\begin{aligned}
\bar{\delta} L_A{}^B &= \bar{\xi}^0 \partial_t L_A{}^B + \frac{1}{n}\partial_t \bar{\Lambda}_A{}^B + \bar{\Lambda}_A{}^C L_C{}^B - \bar{\Lambda}_C{}^B L_A{}^C\,,\\
\bar{\delta} n &= \partial_t(\bar{\xi}^0 n)\,,\\
\bar{\delta}\mathcal{V}_a &= \bar{\xi}^0 \partial_t \mathcal{V}_a + \bar{\Lambda}_a{}^b \mathcal{V}_b\,,\\
\bar{\delta}\mathcal{V}_{\bar{a}} &= \bar{\xi}^0 \partial_t \mathcal{V}_{\bar{a}} + \bar{\Lambda}_{\bar{a}}{}^{\bar{b}} \mathcal{V}_{\bar{b}}\,.
\end{aligned} \tag{3.97}$$

All other fields transform as scalars under $\bar{\xi}^0$ and as tensors under $\bar{\Lambda}_A{}^B$. The first line of (3.97) implies that $L_a{}^b$ and $L_{\bar{a}}{}^{\bar{b}}$ transform as connections under background frame transformations, while $L_a{}^{\bar{b}}$ transforms as a tensor. We then have the covariant derivatives

$$\begin{aligned}
\nabla_t \mathcal{V}_a \;&=\; \frac{1}{n}\partial_t \mathcal{V}_a - L_a{}^b \mathcal{V}_b\,,\\
\nabla_t \mathcal{V}_{\bar{a}} \;&=\; \frac{1}{n}\partial_t \mathcal{V}_{\bar{a}} - L_{\bar{a}}{}^{\bar{b}} \mathcal{V}_{\bar{b}}\,,
\end{aligned} \tag{3.98}$$

which transform covariantly in that

$$\begin{aligned}
\bar{\delta}\left(\nabla_t \mathcal{V}_a\right) \;&=\; \bar{\xi}^0 \partial_t \big(\nabla_t \mathcal{V}_a\big) + \bar{\Lambda}_a{}^b \nabla_t \mathcal{V}_b\,,\\
\bar{\delta}\left(\nabla_t \mathcal{V}_{\bar{a}}\right) \;&=\; \bar{\xi}^0 \partial_t \big(\nabla_t \mathcal{V}_{\bar{a}}\big) + \bar{\Lambda}_{\bar{a}}{}^{\bar{b}} \nabla_t \mathcal{V}_{\bar{b}}\,.
\end{aligned} \tag{3.99}$$

(3.94) can now be written more compactly as

$$\nabla_t L_a{}^{\bar{b}} - \frac{2}{n}\dot{\Phi} L_a{}^{\bar{b}} \; = \; 0\,, \tag{3.100}$$

or, equivalently,

$$\nabla_t \left(e^{-2\Phi} L_{a\bar{b}} \right) \; = \; 0\,. \tag{3.101}$$

In addition, upon adding (3.95) and (3.96) one obtains the useful equation

$$\ddot{\Phi} - 2\dot{\Phi}^2 - \dot{\Phi}\,\partial_t \log n \; = \; 0\,, \tag{3.102}$$

or, equivalently,

$$\partial_t \left(e^{-2\Phi} n^{-1} \dot{\Phi} \right) \; = \; 0\,. \tag{3.103}$$

3.3.2 Quadratic Fluctuations

Next, we assume that the background equations are satisfied, so that the terms linear in fluctuations drop out. The action for the quadratic fluctuations is given by

$$S^{(2)} = \int dt \int d^{2d}\mathbb{X}\; \mathcal{L}\,, \tag{3.104}$$

where

$$\begin{aligned}\mathcal{L} = n\, e^{-2\Phi} \Big\{ & -4\Gamma^2 + 2\phi \left(4\frac{\dot{\Phi}}{n}\Gamma - L^{a\bar{b}}\omega_{a\bar{b}} \right) - \omega_{a\bar{b}}\omega^{a\bar{b}} + \frac{1}{2}(\omega_{ab}\omega^{ab} + \omega_{\bar{a}\bar{b}}\omega^{\bar{a}\bar{b}}) \\ & - \frac{1}{2}(K_{ab}K^{ab} + K_{\bar{a}\bar{b}}K^{\bar{a}\bar{b}}) + 4L^{a\bar{b}}(\varphi K_{a\bar{b}} + \Gamma h_{a\bar{b}}) + \mathcal{V}^{(2)}(\phi, \varphi, h) \Big\}\,,\end{aligned} \tag{3.105}$$

and we defined

$$\begin{aligned} \Gamma &\equiv \nabla_t \varphi + \frac{1}{2}\partial_M \mathcal{A}^M\,, \\ \omega_{a\bar{b}} &\equiv \nabla_t h_{a\bar{b}} - K_{a\bar{b}}\,, \\ \omega^{ab} &\equiv 2h^{[a}{}_{\bar{c}} L^{b]\bar{c}} + K^{ab}\,, \\ \omega^{\bar{a}\bar{b}} &\equiv 2h^{c[\bar{a}} L_c{}^{\bar{b}]} + K^{\bar{a}\bar{b}}\,, \end{aligned} \tag{3.106}$$

with

$$K_{AB} \equiv K_{AB}(\mathcal{A}) = D_A \mathcal{A}_B - D_B \mathcal{A}_A\,. \tag{3.107}$$

K_{AB} satisfies the identities:

$$D_A K_{BC} + D_B K_{CA} + D_C K_{AB} = 0\,, \tag{3.108}$$

$$D^B K_{AB} - D_A D^B \mathcal{A}_B = 0\,, \tag{3.109}$$

the latter being a consequence of the strong constraint (3.15), which in terms of this flattened derivative takes the form

$$D^a D_a + D^{\bar{a}} D_{\bar{a}} \;=\; 0\,. \tag{3.110}$$

Also note that the flattened spatial derivatives and the covariant (time) derivative satisfy the commutation relations:

$$\big[D_a, \nabla_t\big]\mathcal{V}_B = -L_a{}^{\bar{c}} D_{\bar{c}} \mathcal{V}_B\,, \tag{3.111}$$

$$\big[D_{\bar{a}}, \nabla_t\big]\mathcal{V}_B = -L_{\bar{a}}{}^{c} D_{c} \mathcal{V}_B\,. \tag{3.112}$$

In the action we collected the terms with only spatial derivatives into $\mathcal{V}^{(2)}$, defined as

$$\begin{aligned}\mathcal{V}^{(2)}(\phi,\varphi,h_{a\bar{b}}) \;=\;& 8D^a\phi D_a\varphi - 8D^a\varphi D_a\varphi - 8D_a\varphi D_{\bar{b}} h^{a\bar{b}} + 4D_a\phi D_{\bar{b}} h^{a\bar{b}} - 2D^a h^{b\bar{c}} D_a h_{b\bar{c}}\\ &+ 2D^c h^{a\bar{b}} D_a h_{c\bar{b}} - 2D^{\bar{c}} h^{a\bar{b}} D_{\bar{b}} h_{a\bar{c}}\,.\end{aligned} \tag{3.113}$$

This can be rewritten in terms of

$$\varphi_\pm \;\equiv\; \varphi \pm \frac{1}{2}\phi\,, \tag{3.114}$$

which yields

$$\begin{aligned}\mathcal{V}^{(2)}(\phi,\varphi_-,h_{a\bar{b}}) \;=\;& 2D^a\phi D_a\phi - 8D^a\varphi_- D_a\varphi_- - 8D_a\varphi_- D_{\bar{b}} h^{a\bar{b}} - 2D^a h^{b\bar{c}} D_a h_{b\bar{c}}\\ &+ 2D^c h^{a\bar{b}} D_a h_{c\bar{b}} - 2D^{\bar{c}} h^{a\bar{b}} D_{\bar{b}} h_{a\bar{c}}\,.\end{aligned} \tag{3.115}$$

We next turn to the gauge invariance of the quadratic action, which for time-dependent backgrounds is quite subtle even for the free theory. Under linearized gauge transformations one finds

$$\begin{aligned}\delta^{(0)}h_{a\bar{b}} &= D_a\xi_{\bar{b}} - D_{\bar{b}}\xi_a - \xi^0 L_{a\bar{b}}\,,\\ \delta^{(0)}\phi &= \frac{1}{n}\partial_t\xi^0\,,\\ \delta^{(0)}\varphi &= \xi^0\frac{\dot{\Phi}}{n} - \frac{1}{2}D_a\xi^a - \frac{1}{2}D_{\bar{a}}\xi^{\bar{a}}\,,\\ \delta^{(0)}\mathcal{A}^a &= \nabla_t\xi^a + L_{\bar{b}}{}^a\xi^{\bar{b}} + D^a(\xi_0+\xi^0)\,,\\ \delta^{(0)}\mathcal{A}^{\bar{a}} &= \nabla_t\xi^{\bar{a}} + L_b{}^{\bar{a}}\xi^b + D^{\bar{a}}(\xi_0-\xi^0)\,,\end{aligned} \tag{3.116}$$

where we performed the following rescaling of gauge parameters,

$$\xi^0 \;\to\; \frac{1}{n}\xi^0\,, \qquad \xi_0 \;\to\; n\xi_0\,. \tag{3.117}$$

It is also convenient to note that with respect to spatial generalized diffeomorphisms with parameters $\xi^M = \bar{E}_A{}^M\xi^A$ the gauge transformations for φ and $\mathcal{A}^M = \bar{E}_A{}^M\mathcal{A}^A$ simplify as follows:

$$\begin{aligned}\delta^{(0)}\varphi &= -\frac{1}{2}\partial_M\xi^M\,,\\ \delta^{(0)}\mathcal{A}^M &= \frac{1}{n}\partial_t\xi^M\,.\end{aligned} \tag{3.118}$$

The gauge transformations in (3.116) act trivially for the special case:

$$\xi^a \;=\; D^a\chi\,, \quad \xi^{\bar{a}} \;=\; D^{\bar{a}}\chi\,, \quad \xi_0 = -\nabla_t\chi\,, \tag{3.119}$$

where χ is an arbitrary function. For $\delta^{(0)}h_{a\bar{b}}$ this can be seen by inspection, for $\delta^{(0)}\varphi$ by the constraint (3.15), and for $\delta^{(0)}\mathcal{A}^a$ and $\delta^{(0)}\mathcal{A}^{\bar{a}}$ by using the commutators (3.111) and (3.112).

3.3.3 Canonical Formulation of the Quadratic Theory

In order to elucidate the gauge structure of the quadratic double field theory on a time-dependent backgrounds we find it convenient to introduce a canonical formulation. We begin by computing the canonical momenta of the fields φ and

$h_{a\bar{b}}$:

$$p_\varphi \equiv \frac{\delta \mathcal{L}}{\delta \partial_t \varphi} = e^{-2\Phi}\big(-8\Gamma + 8n^{-1}\dot{\Phi}\phi + 4L^{a\bar{b}}h_{a\bar{b}}\big)\,,$$
$$p^{a\bar{b}} \equiv \frac{\delta \mathcal{L}}{\delta \partial_t h_{a\bar{b}}} = e^{-2\Phi}\big(-2\phi L^{a\bar{b}} - 2\omega^{a\bar{b}}\big)\,. \tag{3.120}$$

In the following we rescale the dot by a factor of n^{-1}: $\dot{f} \equiv n^{-1}\partial_t f$. It is also convenient to rescale the canonical momenta by multiplying by the background quantity $e^{2\Phi}$:

$$P_\varphi \equiv e^{2\Phi}p_\varphi \;=\; -8\Gamma + 8\dot{\Phi}\phi + 4L^{a\bar{b}}h_{a\bar{b}}\,,$$
$$P_{a\bar{b}} \equiv e^{2\Phi}p^{a\bar{b}} = -2\phi L_{a\bar{b}} - 2\omega_{a\bar{b}}\,. \tag{3.121}$$

Under the gauge transformations in (3.116), the canonical momenta transform as:

$$\delta P_\varphi = 4\Delta\xi^0 + 4L^{a\bar{b}}(D_a\xi_{\bar{b}} - D_{\bar{b}}\xi_a)\,,$$
$$\delta P_{a\bar{b}} = 4\dot{\Phi}\xi^0 L_{a\bar{b}} - 4D_aD_{\bar{b}}\xi^0 - 2L_a{}^{\bar{c}}(D_{\bar{c}}\xi_{\bar{b}} - D_{\bar{b}}\xi_{\bar{c}}) + 2L_{\bar{b}}{}^{c}(D_c\xi_a - D_a\xi_c)\,. \tag{3.122}$$

The Hamiltonian density is the Legendre transform of $\mathcal{L}$, defined as follows:

$$\mathcal{H} = n\,e^{-2\Phi}(P_\varphi\dot{\varphi} + P^{a\bar{b}}\dot{h}_{a\bar{b}}) - \mathcal{L}\,, \tag{3.123}$$

which yields explicitly for the above fields

$$\begin{aligned}\mathcal{H} = n\,e^{-2\Phi}\Big\{ &-\frac{1}{4}P_{a\bar{b}}P^{a\bar{b}} - \frac{1}{16}P_\varphi^2 - \phi L_{a\bar{b}}P^{a\bar{b}} + P^{a\bar{b}}(K_{a\bar{b}} + L_a{}^{c}h_{c\bar{b}} + L_{\bar{b}}{}^{\bar{c}}h_{a\bar{c}})\\ &-\frac{1}{2}\partial_M\mathcal{A}^M P_\varphi + \dot{\Phi}\phi P_\varphi + \frac{1}{2}(L^{a\bar{b}}h_{a\bar{b}})P_\varphi\\ &-4\dot{\Phi}(L^{a\bar{b}}h_{a\bar{b}})\phi + L_b{}^{\bar{d}}L_{\bar{c}}{}^{b}h_a{}^{\bar{c}}h^a{}_{\bar{d}} + L_c{}^{\bar{b}}L_{\bar{b}}{}^{d}h^c{}_{\bar{a}}h_d{}^{\bar{a}} + 2L_a{}^{\bar{b}}L_c{}^{\bar{d}}h^a{}_{\bar{d}}h^c{}_{\bar{b}}\\ &-2L_c{}^{\bar{b}}h^{c\bar{a}}K_{\bar{a}\bar{b}} - 2L^{b\bar{c}}h^a{}_{\bar{c}}K_{ab}\\ &-4L^{a\bar{b}}K_{a\bar{b}}\varphi - (L^{a\bar{b}}h_{a\bar{b}})^2 - \mathcal{V}^{(2)}\Big\}\,.\end{aligned} \tag{3.124}$$

In terms of this Hamiltonian density the complete quadratic action can be written with the original fields and the canonical momenta in the following first-order form:

$$S = \int dt \int d^{2d}\mathbf{X}\Big[n\,e^{-2\Phi}(P^{a\bar{b}}\dot{h}_{a\bar{b}} + P_\varphi\dot{\varphi}) - \mathcal{H}\Big]\,. \tag{3.125}$$

Naturally, upon integrating out the canonical momenta one recovers the original second-order action. Specifically, the equations of motion $E = 0$ following from this action are encoded in the following components:

$$E_{P_\varphi} = \frac{1}{8}P_\varphi + \dot{\varphi} + \frac{1}{2}\partial_M \mathcal{A}^M - \dot{\Phi}\phi - \frac{1}{2}L^{a\bar{b}}h_{a\bar{b}}\,, \tag{3.126}$$

$$E_{Pa\bar{b}} = \frac{1}{2}P_{a\bar{b}} + \nabla_t h_{a\bar{b}} - K_{a\bar{b}} + \phi L_{a\bar{b}}\,, \tag{3.127}$$

$$E_\varphi = -\nabla_t P_\varphi + 2\dot{\Phi}P_\varphi + 4L^{a\bar{b}}K_{a\bar{b}} + 4\Delta\phi - 8\Delta\varphi + 8D^a D^{\bar{b}}h_{a\bar{b}}\,, \tag{3.128}$$

$$\begin{aligned} E_{a\bar{b}} = {} & -\nabla_t P_{a\bar{b}} + 2\dot{\Phi}P_{a\bar{b}} - \frac{1}{2}L_{a\bar{b}}P_\varphi + 2(L^{c\bar{d}}h_{c\bar{d}})L_{a\bar{b}} + 4\dot{\Phi}\phi L_{a\bar{b}} \\ & - 2L_{\bar{d}}{}^{c}L_{c\bar{b}}h_a{}^{\bar{d}} - 2L_a{}^{\bar{d}}L_{\bar{d}}{}^{c}h_{c\bar{b}} + 4L_a{}^{\bar{d}}L_{\bar{b}}{}^{c}h_{c\bar{d}} + 2L_a{}^{\bar{d}}K_{\bar{b}\bar{d}} + 2L^c{}_{\bar{b}}K_{ac} \\ & + 8D_a D_{\bar{b}}\varphi - 4D_a D_{\bar{b}}\phi - 2\Delta h_{a\bar{b}} - 4D_a D^c h_{c\bar{b}} + 4D_{\bar{b}}D^{\bar{c}}h_{a\bar{c}}\,, \end{aligned} \tag{3.129}$$

$$E_a = -D^{\bar{b}}P_{a\bar{b}} - \frac{1}{2}D_a P_\varphi + 4L_{a\bar{b}}D^{\bar{b}}\varphi + 2L_b{}^{\bar{c}}D^b h_{a\bar{c}} - 2L_a{}^{\bar{c}}D_b h^b{}_{\bar{c}}\,, \tag{3.130}$$

$$E_{\bar{a}} = D^b P_{b\bar{a}} - \frac{1}{2}D_{\bar{a}}P_\varphi - 4L_{b\bar{a}}D^b\varphi - 2L_{c\bar{a}}D_{\bar{b}}h^{c\bar{b}} + 2L_c{}^{\bar{b}}D_{\bar{b}}h^c{}_{\bar{a}}\,, \tag{3.131}$$

$$E_\phi = L_{a\bar{b}}P^{a\bar{b}} - \dot{\Phi}P_\varphi + 4\dot{\Phi}L^{a\bar{b}}h_{a\bar{b}} + 4\Delta\varphi - 4D^a D^{\bar{b}}h_{a\bar{b}}\,. \tag{3.132}$$

If one solves $E_{P_\varphi} = 0$ and $E_{Pa\bar{b}} = 0$ for the canonical momenta and substitutes the expressions into the action, one recovers the original Lagrangian (3.105). The tensors defining the equations of motion above satisfy Bianchi identities $G = 0$ with the following components:

$$G^0 \equiv D_a E^a + D_{\bar{a}}E^{\bar{a}}\,, \tag{3.133}$$

$$\begin{aligned} G_0 \equiv {} & D^{\bar{a}}E_{\bar{a}} - D^a E_a - \dot{E}_\phi + 2\dot{\Phi}E_\phi + \dot{\Phi}E_\varphi - L^{a\bar{b}}E_{a\bar{b}} + 4\dot{\Phi}L^{a\bar{b}}E_{Pa\bar{b}} \\ & + 4\Delta E_{P\varphi} - 4D^a D^{\bar{b}}E_{Pa\bar{b}}\,, \end{aligned} \tag{3.134}$$

$$\begin{aligned} G_a \equiv {} & D^{\bar{b}}E_{a\bar{b}} - \nabla_t E_a + 2\dot{\Phi}E_a + L_a{}^{\bar{b}}E_{\bar{b}} + \frac{1}{2}D_a E_\varphi + 2L_c{}^{\bar{b}}D^c E_{Pa\bar{b}} \\ & - 2L_a{}^{\bar{b}}D^c E_{Pc\bar{b}} + 4L_a{}^{\bar{b}}D_{\bar{b}}E_{P\varphi}\,, \end{aligned} \tag{3.135}$$

$$\begin{aligned} G_{\bar{a}} \equiv {} & -D^b E_{b\bar{a}} - \nabla_t E_{\bar{a}} + 2\dot{\Phi}E_{\bar{a}} + L_{\bar{a}}{}^{b}E_b + \frac{1}{2}D_{\bar{a}}E_\varphi - 2L_{\bar{b}}{}^{c}D^{\bar{b}}E_{Pc\bar{a}} \\ & + 2L_{\bar{a}}{}^{c}D^{\bar{b}}E_{Pc\bar{b}} + 4L_{\bar{a}}{}^{b}D_b E_{P\varphi}. \end{aligned} \tag{3.136}$$

For the explicit expressions for the E tensors in (3.126)–(3.132) the G tensors vanish identically. This fact expresses the Bianchi identities. Furthermore, the Bianchi identities provide a straightforward proof of gauge invariance of the action, since one may verify with (3.116) and (3.122) that the gauge variation of the action can be written as the sum over the gauge parameter times the corresponding Bianchi identity:

$$\delta^{(0)}S = \int dt \int d^{2d}\mathbf{X}\, n\, e^{-2\Phi}\big(\xi_0 G^0 + \xi^0 G_0 + \xi^a G_a + \xi^{\bar{a}} G_{\bar{a}}\big) = 0\,. \tag{3.137}$$

3.3.4 Cubic Theory

We now turn to the cubic terms in the action obtained by expanding about a generic time-dependent background to third order in fluctuations. The corresponding Lagrangian reads

$$\begin{aligned}\mathcal{L}^{(3)} = n\, e^{-2\Phi}\Big\{ & 2\omega^{a\bar{b}}\omega_a{}^c h_{c\bar{b}} + 2\omega^{a\bar{b}}\omega_{\bar{b}}{}^{\bar{c}} h_{a\bar{c}} - 2L^{a\bar{b}}\omega^{c\bar{d}} h_{a\bar{d}} h_{c\bar{b}} - 2L^{a\bar{b}}K^{c\bar{d}} h_{c\bar{b}} h_{a\bar{d}} \\ & + 4L_{a\bar{b}}\omega^{a\bar{b}}\varphi^2 + 2\omega^{a\bar{b}}\mathcal{A}^M\partial_M h_{a\bar{b}} + 8\Big(\Gamma - \frac{\dot{\Phi}}{n}\phi\Big)\mathcal{A}^M\partial_M\varphi \\ & + 4L^{a\bar{b}}\varphi_+\mathcal{A}^M\partial_M h_{a\bar{b}} - 2\varphi_+\mathcal{W}(\phi,\mathcal{A},\varphi,h) + \mathcal{V}^{(3)}(\phi,\varphi_-,h)\Big\}\,,\end{aligned} \tag{3.138}$$

where we have grouped objects familiar from the quadratic action into

$$\begin{aligned}\mathcal{W}(\phi,\mathcal{A},\varphi,h) \;=\; & -4\Gamma^2 - \omega_{a\bar{b}}\omega^{a\bar{b}} + \frac{1}{2}\big(\omega_{ab}\omega^{ab} + \omega_{\bar{a}\bar{b}}\omega^{\bar{a}\bar{b}}\big) - \frac{1}{2}\big(K_{ab}K^{ab} + K_{\bar{a}\bar{b}}K^{\bar{a}\bar{b}}\big) \\ & + 2\phi\Big(4\frac{\dot{\Phi}}{n}\Gamma - L_{a\bar{b}}\omega^{a\bar{b}}\Big)\,.\end{aligned} \tag{3.139}$$

Moreover, we defined

$$\mathcal{V}^{(3)}(\phi,\varphi_-,h) = \mathcal{T} + \mathcal{U}\,, \tag{3.140}$$

where

$$\begin{aligned}\mathcal{T} = \; & 4h^{a\bar{b}}\big(D_a h^{c\bar{d}} D_{\bar{b}} h_{c\bar{d}} - D_a h^{c\bar{d}} D_{\bar{d}} h_{c\bar{b}} - D_{\bar{b}} h^{c\bar{d}} D_c h_{a\bar{d}}\big) \\ & + 4\varphi_-\big(D^a h^{c\bar{d}} D_a h_{c\bar{d}} + D^a h_{a\bar{b}} D_c h^{c\bar{b}} - D^{\bar{b}} h_{a\bar{b}} D_{\bar{c}} h^{a\bar{c}} + 2h_{a\bar{b}} D^a D_c h^{c\bar{b}} - 2h_{a\bar{b}} D^{\bar{b}} D_{\bar{c}} h^{a\bar{c}}\big) \\ & - 16 h_{a\bar{b}}\varphi_- D^a D^{\bar{b}}\varphi_- - 8\varphi_-^2 D^a D_a\varphi_-\,, \\ \mathcal{U} = \; & 4\varphi_- D^a\phi D_a\phi + 8\varphi_-\phi D^a D_a\phi - 4\phi D^a\phi D_a\phi + 4h^{a\bar{b}}\phi D_a D_{\bar{b}}\phi\,.\end{aligned} \tag{3.141}$$

We note that $\mathcal{T}$ matches the form of the cubic expansion of the double field theory action around a constant background without space-time split in [65], with φ_-

playing the role of the dilaton.

Finally, expanding the full gauge transformations (3.26) to first order in fields (i.e. including all terms quadratic in fields and gauge parameters) one finds:

$$\begin{aligned}
\delta^{(1)} h_{a\bar{b}} &= \xi^0 \omega_{a\bar{b}} + \xi^N \partial_N h_{a\bar{b}} + (D_{\bar{b}}\xi^{\bar{c}} - D^{\bar{c}}\xi_{\bar{b}}) h_{a\bar{c}} + (D_a \xi^c - D^c \xi_a) h_{c\bar{b}}\,, \\
\delta^{(1)} \phi &= \frac{1}{n}\partial_t(\xi^0 \phi) - \mathcal{A}^M \partial_M \xi^0 + \xi^M \partial_M \phi\,, \\
\delta^{(1)} \varphi &= \xi^0 \Gamma + \xi^N \partial_N \varphi\,, \\
\delta^{(1)} \mathcal{A}^a &= -2h^{a\bar{b}} D_{\bar{b}} \xi^0 + 2\phi D^a \xi^0 + \xi^B D_B \mathcal{A}^a + (D^a \xi_B - D_B \xi^a)\mathcal{A}^B\,, \\
\delta^{(1)} \mathcal{A}^{\bar{a}} &= -2h^{b\bar{a}} D_b \xi^0 - 2\phi D^{\bar{a}} \xi^0 + \xi^B D_B \mathcal{A}^{\bar{a}} + (D^{\bar{a}} \xi_B - D_B \xi^{\bar{a}})\mathcal{A}^B\,.
\end{aligned} \tag{3.142}$$

Gauge invariance of the action to cubic order in fluctuations requires

$$\delta^{(1)} S^{(2)} + \delta^{(0)} S^{(3)} = 0\,, \tag{3.143}$$

as the reader may verify with the above formulas by a straightforward but tedious computation.

3.4 Gauge Invariant Cosmological Perturbations

In this section we decompose the fundamental fields of double field theory into irreducible components by performing an SVT decomposition. This is done for a special class of FLRW backgrounds which we will describe in the first subsection. After constructing the gauge invariant variables, we will express the quadratic double field theory action in terms of them. This analysis extends what was already done for fluctuations with respect to a flat background in [18].

3.4.1 FLRW backgrounds

A general time-dependent metric $G_{ij}(t)$ and Kalb-Ramond B-field can be packaged into

$$\mathcal{E}_{ij}(t) = G_{ij}(t) + B_{ij}(t)\,. \tag{3.144}$$

In order to write a general frame $E_A{}^M(t)$ *without gauge fixing* one needs to introduce on top of this two independent d-dimensional frame fields $e_a{}^i(t)$ and $\bar{e}_{\bar{a}}{}^i(t)$, for then we can write

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{E}_{ji} e_a{}^j & -e_a{}^i \\ \mathcal{E}_{ij} \bar{e}_{\bar{a}}{}^j & \bar{e}_{\bar{a}}{}^i \end{pmatrix} , \tag{3.145}$$

or, in matrix notation with $E = (E_A{}^M)$, $e = (e_a{}^i)$ and $\bar{e} = (\bar{e}_{\bar{a}}{}^i)$,

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} e\mathcal{E} & -e \\ \bar{e}\mathcal{E}^t & \bar{e} \end{pmatrix} . \tag{3.146}$$

This background satisfies the constraint (3.24) in that

$$\mathcal{G}_{AB} = \begin{pmatrix} -e_a{}^i e_b{}^j G_{ij} & 0 \\ 0 & \bar{e}_{\bar{a}}{}^i \bar{e}_{\bar{b}}{}^j G_{ij} \end{pmatrix} . \tag{3.147}$$

This parametrization of the frame preserves the full $O(d,d)$ and $GL(d) \times GL(d)$ covariance, since we have $3d^2$ degrees of freedom (e, $\bar{e}$ and $\mathcal{E}$), as it should be for a frame with $(2d)^2 = 4d^2$ components satisfying the d^2 constraints (3.24) (i.e., the $GL(d) \times GL(d)$ covariant constraints that the off-diagonal blocks of (3.147) vanish).

We now turn to the class of time-dependent backgrounds with the largest degree of symmetry: the FLRW spaces with vanishing spatial curvature. These are characterized by a single time-dependent function, the scale factor $a(t)$. Specifically, since the B-field vanishes, the metric in (3.144) reduces to $\mathcal{E}_{ij}(t) = a^2(t)\delta_{ij}$. It will furthermore be convenient to introduce two constant but otherwise arbitrary bases or frames $e_a{}^i$ and $\bar{e}_{\bar{a}}{}^i$ for the doubled spatial geometry. Since these bases are unconstrained, the 'tangent space' metrics defined in terms of the flat spatial metric by

$$g_{ab} = e_a{}^i e_b{}^j \delta_{ij} , \qquad g_{\bar{a}\bar{b}} = \bar{e}_{\bar{a}}{}^i \bar{e}_{\bar{b}}{}^j \delta_{ij} , \tag{3.148}$$

are then independent constant metrics (of Euclidean signature). As usual, we use δ_{ij} and δ^{ij} to lower and raise indices $i, j, \ldots$, while g_{ab} and g^{ab}, respectively $g_{\bar{a}\bar{b}}$ and $g^{\bar{a}\bar{b}}$, are used to lower and raise indices $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$. Using the frames $e_a{}^i$ and $\bar{e}_{\bar{a}}{}^i$, together with their inverses denoted by $e_i{}^a$ and $\bar{e}_i{}^{\bar{a}}$, to convert indices $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$ to $i, j, \ldots$, one may verify that the different operations of raising and lowering indices are mutually compatible. For instance, in

$$e^{ia} = \delta^{ij} e_j{}^a = g^{ab} e_b{}^i , \tag{3.149}$$

the second equation follows by contraction with $e_a{}^k$ and using the inverse of (3.148). Notably, there are tensors

$$g_a{}^{\bar{b}} := e_a{}^i \, \bar{e}_i{}^{\bar{b}} \,, \qquad g_{\bar{a}}{}^b := \bar{e}_{\bar{a}}{}^i \, e_i{}^b \tag{3.150}$$

that connect unbarred and barred indices and that satisfy

$$g_a{}^{\bar{b}} \, g_{\bar{b}}{}^c = \delta_a{}^c \,, \quad \text{etc.} \tag{3.151}$$

With these objects we can write the full background frame as

$$E_A{}^M(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} a(t)\delta_{ij}e_a{}^j & -a^{-1}(t)e_a{}^i \\ a(t)\delta_{ij}\bar{e}_{\bar{a}}{}^j & a^{-1}(t)\bar{e}_{\bar{a}}{}^i \end{pmatrix} . \tag{3.152}$$

The corresponding tangent space metric then satisfies the constraint (3.24) with the constant metric on the doubled tangent space

$$\mathcal{G}_{AB} = \begin{pmatrix} -g_{ab} & 0 \\ 0 & g_{\bar{a}\bar{b}} \end{pmatrix} . \tag{3.153}$$

Note the relative sign in the upper-left block, which will lead to a change of convention in raising and lowering indices later in 3.4.2. The differential operators D_A are given by

$$\begin{aligned} D_a &= -\tfrac{1}{\sqrt{2}} \left(a^{-1}(t)\partial_a - a(t)\tilde{\partial}_a \right) \,, \\ D_{\bar{a}} &= \tfrac{1}{\sqrt{2}} \left(a^{-1}(t)\partial_{\bar{a}} \, + \, a(t)\tilde{\partial}_{\bar{a}} \right) \,, \end{aligned} \tag{3.154}$$

using the notation $\partial_a = e_a{}^i\partial_i$, $\tilde{\partial}_a = e_a{}^i\tilde{\partial}_i$, and similarly for barred indices. Note that despite the notation there is only one kind of momentum and one kind of winding derivative: with (3.150) we have $\partial_{\bar{a}} = g_{\bar{a}}{}^b\partial_b$ and $\tilde{\partial}_{\bar{a}} = g_{\bar{a}}{}^b\tilde{\partial}_b$.

We can next give the explicit form of the tensor $L_A{}^B$ in (3.93). Using (3.152) one finds for its components

$$L_a{}^{\bar{b}} = Hg_a{}^{\bar{b}} \,, \qquad L_{\bar{a}}{}^b = Hg_{\bar{a}}{}^b \,, \qquad L_a{}^b = L_{\bar{a}}{}^{\bar{b}} = 0 \,, \tag{3.155}$$

with Hubble parameter

$$H \equiv \frac{\dot{a}}{a} \,. \tag{3.156}$$

With this choice of backgrounds, the background equations of motion in (3.94)-(3.96) become:

$$(\dot{H} - 2\dot{\Phi} H) g_a{}^{\bar{b}} = 0\,, \tag{3.157}$$

$$-4\ddot{\Phi} + 4\dot{\Phi}^2 + dH^2 = 0\,, \tag{3.158}$$

$$4\dot{\Phi}^2 - dH^2 = 0\,. \tag{3.159}$$

As done before in sec. 2, we can add (3.158) and (3.159) to obtain the equation

$$-\ddot{\Phi} + 2\dot{\Phi}^2 = 0\,. \tag{3.160}$$

The equations of motion imply that the following quantity is conserved,

$$\beta \equiv \frac{\dot{\Phi}}{H}\,, \tag{3.161}$$

since by taking its time derivative we have

$$\dot{\beta} = \frac{(\ddot{\Phi} H - \dot{\Phi} \dot{H})}{H^2} = \frac{2\dot{\Phi}^2 H - \dot{\Phi}(2\dot{\Phi} H)}{H^2} = 0\,. \tag{3.162}$$

We note that since we are now considering a rather special class of backgrounds, under a general $O(d,d)$ or $GL(d) \times GL(d)$ transformation the background frame will of course not stay in the same class. However, under constant or time independent $GL(d) \times GL(d)$ transformations the above backgrounds transform into themselves, just with the frames $e_a{}^i$ and $\bar{e}_{\bar{a}}{}^i$ rotated. The genuine duality transformation left in $O(d,d)$ is given by

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \qquad \mathcal{E}'(t) = \mathcal{E}^{-1}(t) \qquad \Leftrightarrow \qquad a'(t) = \frac{1}{a(t)}\,. \tag{3.163}$$

This is the expected T-duality or scale-factor duality property of string cosmology. A closer analysis in our work in [20] shows that the invariance group of generic FRW backgrounds is given by the diagonal subgroup $\text{diag}(O(d) \times O(d))$.

3.4.2 Scalar-Vector-Tensor Decomposition

We now take the backgrounds to be of the form (3.152) with scale factor $a(t)$. According to (3.155) we then have $L_a{}^b = L_{\bar{a}}{}^{\bar{b}} = 0$, so that the covariant derivatives defined in (3.98) simplify:

$$\nabla_t \mathcal{V}_a = \dot{\mathcal{V}}_a\,, \qquad \nabla_t \mathcal{V}_{\bar{a}} = \dot{\mathcal{V}}_{\bar{a}}\,. \tag{3.164}$$

As for double field theory on flat space we also have the spatial Laplacians that are related by the level-matching constraint as follows:

$$\Delta \equiv 2g^{ab}D_aD_b = 2g^{\bar{a}\bar{b}}D_{\bar{a}}D_{\bar{b}}\,. \tag{3.165}$$

In contrast to flat space there is a new differential operator based on the tensor (3.150):

$$\Diamond \equiv 2g_a{}^{\bar{b}}D^aD_{\bar{b}} = 2g_{\bar{b}}{}^{a}D_aD^{\bar{b}}\,, \tag{3.166}$$

which satisfies the commutation relations:

$$\begin{aligned} [\nabla_t, \Delta] &= 2H\Diamond\,, \\ [\nabla_t, \Diamond] &= 2H\Delta\,. \end{aligned} \tag{3.167}$$

Explicitly, these two Laplace-type operators are given by

$$\begin{aligned} \Delta &= a^{-2}(t)\partial^2 + a^2(t)\tilde{\partial}^2\,, \\ \Diamond &= -a^{-2}(t)\partial^2 + a^2(t)\tilde{\partial}^2\,, \end{aligned} \tag{3.168}$$

where

$$\partial^2 \equiv \partial^i\partial_i \equiv \delta^{ij}\partial_i\partial_j\,, \qquad \tilde{\partial}^2 \equiv \tilde{\partial}^i\tilde{\partial}_i \equiv \delta_{ij}\tilde{\partial}^i\tilde{\partial}^j\,, \tag{3.169}$$

are the independent spatial (Euclidean) Laplacians of the doubled space.

At this stage a comment is in order regarding our conventions for raising and lowering indices. In the general frame formulation of double field theory the tangent space metric $\mathcal{G}_{AB}$ of signature (d, d) is used to raise and lower flat indices. For the FLRW backgrounds to be used here this metric takes the form (3.153) in terms of the positive-definite (Euclidean) metrics g_{ab} and $g_{\bar{a}\bar{b}}$, respectively, which will be used from now on to raise and lower indices (as alluded to after (3.148)). Since $\mathcal{G}_{ab} = -g_{ab}$ this amounts to a change in convention. To be definite, let us take the elementary fields of the theory to be given by

$$\phi\,, \quad \varphi\,, \quad h_{a\bar{b}}\,, \quad \mathcal{A}_a\,, \quad \mathcal{A}_{\bar{a}}\,, \tag{3.170}$$

i.e., these fields are considered to be metric-independent, as are the differential operators ∂_a, $\tilde{\partial}_a$, etc., with lower indices. Any expression involving these objects with an upper index is then interpreted to mean that the index is raised with g^{ab} or $g^{\bar{a}\bar{b}}$, respectively. This change of convention leads to sign changes but has desirable internal consistency properties. For instance, the Laplacians in (3.169) are then also writable as $\partial^2 = \partial^a \partial_a = \partial^{\bar{a}} \partial_{\bar{a}}$ and $\tilde{\partial}^2 = \tilde{\partial}^a \tilde{\partial}_a = \tilde{\partial}^{\bar{a}} \tilde{\partial}_{\bar{a}}$, as follows quickly by recalling $\partial_a = e_a{}^i \partial_i$, $\tilde{\partial}_a = e_a{}^i \tilde{\partial}_i$, and similarly for barred indices. As a consequence, in all formulas to follow we may freely raise and lower indices without having to worry about sign factors.

After this digression, we turn to the problem of decomposing the complete list of fields (3.170) into 'irreducible' components by performing a scalar-vector-tensor (SVT) decomposition. For the $\mathcal{A}$ fields we write

$$\begin{aligned} \mathcal{A}_a &= A_a + \partial_a A + \tilde{\partial}_a \tilde{A} \, , \\ \mathcal{A}_{\bar{a}} &= A_{\bar{a}} + \partial_{\bar{a}} \bar{A} + \tilde{\partial}_{\bar{a}} \tilde{\bar{A}} \, . \end{aligned} \tag{3.171}$$

Here we see the first instance of an important novelty of a genuinely doubled field theory on cosmological backgrounds: the SVT decomposition of a vector yields, compared to standard gravity, an additional scalar mode, corresponding to the possibility of subtracting the divergence with respect to winding derivatives, in addition to ordinary derivatives. Correspondingly, the remaining vector mode is now divergence-free (transverse) with respect to both derivatives:

$$\partial^a A_a = \tilde{\partial}^a A_a = \partial^{\bar{a}} A_{\bar{a}} = \tilde{\partial}^{\bar{a}} A_{\bar{a}} = 0 \, . \tag{3.172}$$

Thus, each of the transverse vectors has $d-2$ degrees of freedom. The logic here is that one imposes as many constraints as possible on the remaining vector or tensor mode. This ultimately guarantees the complete decoupling among tensor, vector and scalar modes. Turning then to the tensor field $h_{a\bar{b}}$ we postulate the decomposition

$$h_{a\bar{b}} = \widehat{h}_{a\bar{b}} + g_{a\bar{b}} E + \partial_a B_{\bar{b}} - \partial_{\bar{b}} B_a + \tilde{\partial}_a \tilde{B}_{\bar{b}} - \tilde{\partial}_{\bar{b}} \tilde{B}_a + \partial_a \partial_{\bar{b}} C + \tilde{\partial}_a \tilde{\partial}_{\bar{b}} \tilde{C} + \partial_a \tilde{\partial}_{\bar{b}} D + \tilde{\partial}_a \partial_{\bar{b}} \tilde{D} \, , \tag{3.173}$$

with now five independent scalar modes and four independent vector modes, which are subject to the constraints analogous to (3.172), i.e., every divergence vanishes:

$$\partial^a B_a = \tilde{\partial}^a B_a = \partial^{\bar{a}} B_{\bar{a}} = \tilde{\partial}^{\bar{a}} B_{\bar{a}} = 0 \, , \qquad \partial^a \tilde{B}_a = \tilde{\partial}^a \tilde{B}_a = \partial^{\bar{a}} \tilde{B}_{\bar{a}} = \tilde{\partial}^{\bar{a}} \tilde{B}_{\bar{a}} = 0 \, . \tag{3.174}$$

Similarly, the irreducible tensor mode obeys

$$\begin{aligned} &Q_a \equiv \partial^{\bar b}\widehat{h}_{a\bar b} = 0\,, \qquad Q_{\bar b} \equiv \partial^{a}\widehat{h}_{a\bar b} = 0\,, \\ &\tilde Q_a \equiv \tilde\partial^{\bar b}\widehat{h}_{a\bar b} = 0\,, \qquad \tilde Q_{\bar b} \equiv \tilde\partial^{a}\widehat{h}_{a\bar b} = 0\,, \qquad Q \equiv g_{\bar b}{}^{a}\widehat{h}_{a}{}^{\bar b} = 0\,. \end{aligned} \tag{3.175}$$

These constraints are not independent but rather subject to

$$\begin{aligned} \partial^a Q_a = \partial^{\bar b} Q_{\bar b}\,, \qquad \tilde\partial^a Q_a = \partial^{\bar b}\tilde Q_{\bar b}\,, \\ \partial^a \tilde Q_a = \tilde\partial^{\bar b} Q_{\bar b}\,, \qquad \tilde\partial^a \tilde Q_a = \tilde\partial^{\bar b}\tilde Q_{\bar b}\,. \end{aligned} \tag{3.176}$$

Let us verify that $h_{a\bar b}$ so written encodes the right number of (off-shell) degrees of freedom. As for A, the vector modes B_a, $B_{\bar a}$, $\tilde B_a$, $\tilde B_{\bar a}$ together encode $4(d-2)$ degrees of freedom. The number of components of $\widehat{h}_{a\bar b}$ is d^2 minus the number of constraints. Since the constraints (3.175) in turn are subject to (3.176) we have $4d+1-4 = 4d-3$ independent constraints, so that $\widehat{h}_{a\bar b}$ carries d^2-4d+3 degrees of freedom. In total, together with the five scalar modes $E, C, \tilde C, D, \tilde D$ and the $4(d-2)$ vector modes, the irreducible components carry $4(d-2)+d^2-4d+3+5 = d^2$ degrees of freedom, as it should be.

At this stage it is appropriate to briefly discuss the counting of degrees of freedom done above and to relate it to the familiar counting in, say, four spacetime dimensions (for which $d=3$). Consider, for instance, the Fourier expansion of a vector mode in (3.171):

$$A_a(x,\tilde x) = \sum_{k,\tilde k} A_a(k,\tilde k)e^{i(k\cdot x+\tilde k\cdot\tilde x)}\,. \tag{3.177}$$

As before we assume that the Fourier modes $A_a(k,\tilde k)$ are only non-zero provided $k\cdot\tilde k = 0$, so that the weak constraint is obeyed. The two constraints in (3.172) yield

$$k^a A_a(k,\tilde k) = \tilde k^a A_a(k,\tilde k) = 0\,, \tag{3.178}$$

which implies that among the three components of A_a (for $d=3$) generically only one survives. For instance, for $k=(0,0,1)$ and $\tilde k=(0,1,0)$, for which $k\cdot\tilde k=0$, the constraints imply $A_2 = A_3 = 0$, so that only A_1 remains as a physical degree of freedom. So what happened to the familiar two polarizations of a spin-1 vector mode in four dimensions? These spin-1 modes are still present for the special case that the individual modes carry only momentum or only winding, say in the form

$$A_a(x,\tilde{x}) = \sum_k A_a(k)e^{ik\cdot x} + \sum_{\tilde{k}} \tilde{A}_a(\tilde{k})e^{i\tilde{k}\cdot\tilde{x}}\,, \tag{3.179}$$

for which the level-matching constraint is trivially satisfied. Then for each of the two terms one of the constraints (3.172) trivializes, so that the corresponding vector mode carries the expected two polarizations. Therefore, if we consider vector modes that live only in x-space, or vector modes that live only in $\tilde{x}$-space, they do carry the familiar two polarizations (effectively eliminating the additional scalar modes). It is only for modes that both carry genuine momentum and winding that the degrees of freedom organize differently. Similar remarks apply to the tensor modes. Indeed, naively $\widehat{h}_{a\bar{b}}$ carries zero degrees of freedom in four dimensions, but if we consider tensor modes that live only in x-space, or only in $\tilde{x}$-space, some of the constraints trivialize, so that they do carry the two polarizations of a spin-2 mode.

Returning to our discussion of the SVT decomposition, let us verify that these decompositions exist by proving that the SVT components satisfying the appropriate constraints can always be defined from the given fields (3.170). More precisely, this is the case if ∂^2 and $\tilde{\partial}^2$ are invertible operators, as we will assume in the following. For instance, the scalar modes of $h_{a\bar{b}}$ can be expressed in terms of the original fields as:

$$\begin{aligned}
C &= \frac{d-1}{d-2}\partial^{-4}(\partial^a\partial^{\bar{b}}h_{a\bar{b}}) - \frac{1}{d-2}\partial^{-2}\big(g_{\bar{b}}{}^a h_a{}^{\bar{b}} - \tilde{\partial}^{-2}(\tilde{\partial}^a\tilde{\partial}^{\bar{b}}h_{a\bar{b}})\big)\,,\\
\tilde{C} &= \frac{d-1}{d-2}\tilde{\partial}^{-4}(\tilde{\partial}^a\tilde{\partial}^{\bar{b}}h_{a\bar{b}}) - \frac{1}{d-2}\tilde{\partial}^{-2}\big(g_{\bar{b}}{}^a h_a{}^{\bar{b}} - \partial^{-2}(\partial^a\partial^{\bar{b}}h_{a\bar{b}})\big)\,,\\
D &= \partial^{-2}\tilde{\partial}^{-2}(\partial^a\tilde{\partial}^{\bar{b}}h_{a\bar{b}})\,, \qquad \tilde{D} = \tilde{\partial}^{-2}\partial^{-2}(\tilde{\partial}^a\partial^{\bar{b}}h_{a\bar{b}})\,,\\
E &= \frac{1}{d-2}\big(g_{\bar{b}}{}^a h_a{}^{\bar{b}} - \partial^{-2}\partial^a\partial^{\bar{b}}h_{a\bar{b}} - \tilde{\partial}^{-2}\tilde{\partial}^a\tilde{\partial}^{\bar{b}}h_{a\bar{b}}\big)\,.
\end{aligned} \tag{3.180}$$

The vector modes of $h_{a\bar{b}}$ in turn can be defined as

$$\begin{aligned}
B_a &= -\partial^{-2}\big(\partial^{\bar{b}}h_{a\bar{b}} - g_{a\bar{b}}\partial^{\bar{b}}E\big) + \partial_a C + \tilde{\partial}_a\tilde{D}\,,\\
B_{\bar{b}} &= \partial^{-2}\big(\partial^a h_{a\bar{b}} - g_{a\bar{b}}\partial^a E\big) - \partial_{\bar{b}}C - \tilde{\partial}_{\bar{b}}D\,,\\
\tilde{B}_a &= -\tilde{\partial}^{-2}\big(\tilde{\partial}^{\bar{b}}h_{a\bar{b}} - g_{a\bar{b}}\tilde{\partial}^{\bar{b}}E\big) + \tilde{\partial}_a\tilde{C} + \partial_a\tilde{D}\,,\\
\tilde{B}_{\bar{b}} &= \tilde{\partial}^{-2}\big(\tilde{\partial}^a h_{a\bar{b}} - g_{a\bar{b}}\tilde{\partial}^a E\big) - \tilde{\partial}_{\bar{b}}\tilde{C} - \partial_{\bar{b}}D\,,
\end{aligned} \tag{3.181}$$

where one should view the scalar modes in here as defined in terms of $h_{a\bar{b}}$ via

(3.180). Finally, $\widehat{h}_{a\bar{b}}$ can then be defined by inserting these scalar and vector modes (3.180) and (3.181) into (3.173) and solving for $\widehat{h}_{a\bar{b}}$. Similarly, the scalar and vector modes of the vector fields $\mathcal{A}_a$ and $\mathcal{A}_{\bar{a}}$ can be defined in terms of these fields as

$$\begin{aligned}
&A = \partial^{-2}(\partial^a \mathcal{A}_a)\,, \qquad \tilde{A} = \tilde{\partial}^2(\tilde{\partial}^a \mathcal{A}_a)\,, \qquad \bar{A} = \partial^{-2}(\partial^{\bar{a}} \mathcal{A}_{\bar{a}})\,, \qquad \bar{\tilde{A}} = \tilde{\partial}^{-2}(\tilde{\partial}^{\bar{a}} \mathcal{A}_{\bar{a}})\,,\\
&A_a = \mathcal{A}_a - \partial_a\big(\partial^{-2}(\partial^b \mathcal{A}_b)\big) - \tilde{\partial}_a\big(\partial^{-2}(\partial^{\bar{b}} \mathcal{A}_{\bar{b}})\big)\,,\\
&A_{\bar{a}} = \mathcal{A}_{\bar{a}} - \partial_{\bar{a}}\big(\tilde{\partial}^{-2}(\tilde{\partial}^{\bar{b}} \mathcal{A}_{\bar{b}})\big) - \tilde{\partial}_{\bar{a}}\big(\tilde{\partial}^{-2}(\tilde{\partial}^{\bar{b}} \mathcal{A}_{\bar{b}})\big)\,.
\end{aligned} \tag{3.182}$$

We will next determine the gauge transformations of the SVT components. To this end we decompose the gauge parameters ξ_a and $\xi_{\bar{a}}$ into scalar and divergenceless vector components:

$$\xi_a = \zeta_a + \partial_a\lambda + \tilde{\partial}_a\chi\,, \qquad \xi_{\bar{a}} = \zeta_{\bar{a}} + \partial_{\bar{a}}\bar{\lambda} + \tilde{\partial}_{\bar{a}}\bar{\chi}\,, \tag{3.183}$$

where $\partial^a\zeta_a = \tilde{\partial}^a\zeta_a = \partial^{\bar{a}}\zeta_{\bar{a}} = \tilde{\partial}^{\bar{a}}\zeta_{\bar{a}} = 0$. With this decomposition, the gauge transformations (3.116) become:

$$\begin{aligned}
\delta h_{a\bar{b}} =& -\xi^0 L_{a\bar{b}} + \frac{1}{\sqrt{2}}\big(a\tilde{\partial}_a\zeta_{\bar{b}} - a^{-1}\partial_a\zeta_{\bar{b}} - a\tilde{\partial}_{\bar{b}}\zeta_a - a^{-1}\partial_{\bar{b}}\zeta_a\big)\\
&+ \frac{1}{\sqrt{2}}\big(\tilde{\partial}_a\partial_{\bar{b}}(a\bar{\lambda} - a^{-1}\chi) + a\tilde{\partial}_a\tilde{\partial}_{\bar{b}}(\bar{\chi} - \chi) - a^{-1}\partial_a\partial_{\bar{b}}(\lambda + \bar{\lambda})\\
&- \partial_a\tilde{\partial}_{\bar{b}}(a^{-1}\bar{\chi} + a\lambda)\big)\,,\\
\delta\phi =& \dot{\xi}^0\,,\\
\delta\varphi =& \dot{\Phi}\xi^0 + \frac{1}{2\sqrt{2}}\big(-a^{-1}\partial^2(\lambda + \bar{\lambda}) + a\tilde{\partial}^2(\chi - \bar{\chi})\big)\,,\\
\delta\mathcal{A}_a =& \dot{\zeta}_a - L_a{}^{\bar{b}}\zeta_{\bar{b}} + \partial_a\left(\dot{\lambda} - H\bar{\lambda} - \frac{a^{-1}}{\sqrt{2}}(\xi_0 + \xi^0)\right) + \tilde{\partial}_a\left(\dot{\chi} - H\bar{\chi} + \frac{a}{\sqrt{2}}(\xi_0 + \xi^0)\right),\\
\delta\mathcal{A}_{\bar{a}} =& \dot{\zeta}_{\bar{a}} - L_{\bar{a}}{}^{b}\zeta_b + \partial_{\bar{a}}\left(\dot{\bar{\lambda}} - H\lambda + \frac{a^{-1}}{\sqrt{2}}(\xi_0 - \xi^0)\right) + \tilde{\partial}_{\bar{a}}\left(\dot{\bar{\chi}} - H\chi + \frac{a}{\sqrt{2}}(\xi_0 - \xi^0)\right).
\end{aligned} \tag{3.184}$$

Comparing this with (3.171), (3.173) we can then read off the gauge transformation of the SVT components of the fields:

$$
\begin{aligned}
&\delta\widehat{h}_{a\bar{b}} = 0\,, \qquad \delta E = -H\xi^0\,, \\
&\delta B_a = \frac{a^{-1}}{\sqrt{2}}\zeta_a\,, \qquad \delta B_{\bar{a}} = -\frac{a^{-1}}{\sqrt{2}}\zeta_{\bar{a}}\,, \qquad \delta\tilde{B}_a = \frac{a}{\sqrt{2}}\zeta_a\,, \qquad \delta\tilde{B}_{\bar{a}} = \frac{a}{\sqrt{2}}\zeta_{\bar{a}}\,, \\
&\delta C = -\frac{a^{-1}}{\sqrt{2}}(\lambda + \bar{\lambda})\,, \qquad \delta\tilde{C} = \frac{a}{\sqrt{2}}(\bar{\chi} - \chi)\,, \\
&\delta D = \frac{1}{\sqrt{2}}(-a^{-1}\bar{\chi} - a\lambda)\,, \qquad \delta\tilde{D} = \frac{1}{\sqrt{2}}(a\bar{\lambda} - a^{-1}\chi)\,, \\
&\delta A_a = \dot{\zeta}_a - L_a{}^{\bar{b}}\zeta_{\bar{b}}\,, \qquad \delta A_{\bar{a}} = \dot{\zeta}_{\bar{a}} - L_{\bar{a}}{}^{b}\zeta_b\,, \\
&\delta A = \dot{\lambda} - H\bar{\lambda} - \frac{a^{-1}}{\sqrt{2}}(\xi_0 + \xi^0)\,, \qquad \delta\tilde{A} = \dot{\chi} - H\bar{\chi} + \frac{a}{\sqrt{2}}(\xi_0 + \xi^0)\,, \\
&\delta\bar{A} = \dot{\bar{\lambda}} - H\lambda + \frac{a^{-1}}{\sqrt{2}}(\xi_0 - \xi^0)\,, \qquad \delta\bar{\tilde{A}} = \dot{\bar{\chi}} - H\chi + \frac{a}{\sqrt{2}}(\xi_0 - \xi^0)\,.
\end{aligned}
\tag{3.185}
$$

Note that the tensor mode $\widehat{h}_{a\bar{b}}$ is gauge invariant.

3.4.3 Gauge Fixing

In order to verify that the number of gauge independent fields is as expected (i.e. equal to the number of off-shell degrees of freedom minus the number of gauge redundancies) we can impose simple gauge fixing conditions, as we do in the following.

From the second line in (3.185) we see that ζ_a and $\zeta_{\bar{a}}$ can be used to gauge fix two vectors to zero, e.g.,

$$\tilde{B}_a = \tilde{B}_{\bar{a}} = 0\,. \tag{3.186}$$

This fixes the gauge invariance under ζ_a, $\zeta_{\bar{a}}$ completely and does not require any compensating gauge transformations. Next, we observe with the last two lines in (3.185) that ξ^0 and ξ_0 can be used to gauge away two scalars, say:

$$\tilde{A} = \bar{\tilde{A}} = 0\,. \tag{3.187}$$

This again fixes the gauge invariance under ξ^0, ξ_0 completely, but now we require compensating gauge transformations that determine these parameters in terms of the remaining scalar gauge parameters:

$$\frac{a}{\sqrt{2}}(\xi_0+\xi^0) = -\dot{\chi} + H\bar{\chi}\,, \qquad \frac{a}{\sqrt{2}}(\xi_0-\xi^0) = -\dot{\bar{\chi}} + H\chi\,. \tag{3.188}$$

We are now left with four scalar gauge parameters (λ, χ, $\bar{\lambda}$, $\bar{\chi}$), so naively we would expect that we can gauge away four more scalar field modes, but it turns out that, due to the gauge for gauge symmetry, only three more scalar modes can be gauged away. Let us pick the gauge that, say,

$$\tilde{C} = D = \tilde{D} = 0\,. \tag{3.189}$$

The first condition fixes, say, the parameter $\bar{\chi} = \chi$. The second gauge condition can then be achieved by means of λ, which in turn fixes the compensating gauge transformation to be $\lambda = -a^{-2}\bar{\chi} = -a^{-2}\chi$. Finally, the third gauge condition can be achieved by means of $\bar{\lambda}$, which in turn fixes the compensating gauge transformation to be $\bar{\lambda} = a^{-2}\chi$. In total we have reduced the gauge redundancy to one, with independent parameter χ and compensating transformations in terms of χ:

$$\bar{\chi} = \chi\,, \qquad \lambda = -a^{-2}\chi\,, \qquad \bar{\lambda} = a^{-2}\chi\,. \tag{3.190}$$

However, from the third line of (3.185) we then infer that C is gauge invariant and thus cannot be set to zero. Similarly, using (3.190) in (3.188) yields

$$\xi_0 = -\sqrt{2}a^{-1}(\dot{\chi} - H\chi)\,, \qquad \xi^0 = 0\,, \tag{3.191}$$

and with this it follows that the other remaining scalar modes (E, A and $\bar{A}$) are all gauge invariant and can thus not be gauged away. Thus, the apparent remaining parameter χ in fact does not act at all on the remaining fields. This is just a consequence of there being a scalar gauge-for-gauge symmetry. Thus, we have fixed the gauge redundancy completely.

Summarizing, the gauge independent fields can be chosen to be, for instance:

- tensor modes: $\widehat{h}_{a\bar{b}}$ $\quad [d^2-4d+3]\,,$
- vector modes: $A_a\,, A_{\bar{a}}\,, B_a\,, B_{\bar{a}}$ $\quad [4(d-2)]\,,$
- scalar modes: $E\,, C\,, A\,, \bar{A}\,, \phi,\, \varphi$ $\quad [6]\,,$

where we also included the two scalar modes ϕ, φ that were present from the beginning. We displayed in parenthesis the number of degrees of freedom, which adds to d^2+1. This is equal to the total number of off-shell degrees of freedom of the fields (3.170), given by d^2+2d+2, minus the number of gauge redundancies for parameters ξ_a, $\xi_{\bar{a}}$, ξ_0, ξ^0, which is given by $2d+1$ (taking into account the one gauge-for-gauge redundancy).

3.4.4 Gauge Invariant Variables

We now aim to rewrite the field variables and the theory in terms of gauge invariant combinations of the SVT components. This is essentially equivalent to fixing a gauge in the sense that the resulting gauge invariant variables are subject to constraints that are formally identical to gauge fixing conditions. A gauge invariant formulation may always be reconstructed from a gauge fixed one. We have already noted that $\widehat{h}_{a\bar{b}}$ gauge invariant. In addition, one can build the following gauge invariant variables:

$$\begin{aligned}
\widehat{\phi} =&\phi + \frac{1}{2\sqrt{2}}\nabla_t\big(a(A+\bar{A}+\sqrt{2}a\dot{C}) - a^{-1}(\tilde{A}-\tilde{\bar{A}}+\sqrt{2}a^{-1}\dot{\tilde{C}})\big)\,,\\
\widehat{A}_a =&A_a - \frac{1}{\sqrt{2}}\nabla_t(aB_a + a^{-1}\tilde{B}_a) - \frac{1}{\sqrt{2}}L_a{}^{\bar{b}}(aB_{\bar{b}} - a^{-1}\tilde{B}_{\bar{b}})\,,\\
\widehat{A}_{\bar{a}} =&A_{\bar{a}} + \frac{1}{\sqrt{2}}\nabla_t(aB_{\bar{a}} - a^{-1}\tilde{B}_{\bar{a}}) + \frac{1}{\sqrt{2}}L_{\bar{a}}{}^{b}(aB_b + a^{-1}\tilde{B}_b)\,,\\
\widehat{A} =&a(A+\bar{A}+\sqrt{2}a\dot{C}) + a^{-1}(\tilde{A}-\tilde{\bar{A}}+\sqrt{2}a^{-1}\dot{\tilde{C}})\,,\\
\widehat{\tilde{A}} =&a(A-\bar{A}-\sqrt{2}a\dot{C}-2\sqrt{2}aHC) + a^{-1}(\tilde{\bar{A}}+\tilde{A}+\sqrt{2}a^{-1}\dot{\tilde{C}}-2\sqrt{2}a^{-1}H\tilde{C})\\
&+2\sqrt{2}\dot{D}\,,\\
\widehat{\bar{A}} =&a(\bar{A}-A-\sqrt{2}a\dot{C}-2\sqrt{2}aHC) - a^{-1}(\tilde{\bar{A}}+\tilde{A}-\sqrt{2}a^{-1}\dot{\tilde{C}}+2\sqrt{2}a^{-1}H\tilde{C})\\
&-2\sqrt{2}\dot{\tilde{D}}\,,\\
\widehat{C} =&D-\tilde{D}-a^2C+a^{-2}\tilde{C}\,,\\
\widehat{B}_a =&\frac{1}{2}(aB_a - a^{-1}\tilde{B}_a)\,,\\
\widehat{B}_{\bar{a}} =&\frac{1}{2}(aB_{\bar{a}} + a^{-1}\tilde{B}_{\bar{a}})\,,\\
\widehat{\varphi} =&\varphi + \frac{1}{2\sqrt{2}}\dot{\Phi}\big(a(A+\bar{A}+\sqrt{2}a\dot{C}) - a^{-1}(\tilde{A}-\tilde{\bar{A}}+\sqrt{2}a^{-1}\dot{\tilde{C}})\big)\\
&+\frac{1}{2}(-\partial^2C+\tilde{\partial}^2\tilde{C})\,,\\
\widehat{E} =&E - \frac{1}{2\sqrt{2}}H\big(a(A+\bar{A}+\sqrt{2}a\dot{C}) - a^{-1}(\tilde{A}-\tilde{\bar{A}}+\sqrt{2}a^{-1}\dot{\tilde{C}})\big)\,,
\end{aligned} \tag{3.192}$$

as follows with simple computations using (3.185). These gauge invariant variables are of course not unique, as any linear combination of gauge invariant fields is also gauge invariant. Moreover, there is one relation among them: the scalars $\widehat{\tilde{A}}$ and $\widehat{\bar{A}}$ satisfy

$$\widehat{\tilde{A}} + \widehat{\bar{A}} = 2\sqrt{2}\dot{\widehat{C}}\,. \tag{3.193}$$

We next aim to rewrite the quadratic double field theory action directly in terms of gauge invariant variables that are linear combinations of (3.192). Given that the latter take a rather complicated form, a priori this seems to be a difficult task that, however, is simplified by the following trick: We use that the original fields can be expressed in terms of specific combinations of the gauge invariant fields plus terms that take the form of an infinitesimal gauge transformation (3.116). Specifically, we claim that

$$\begin{aligned}
h_{a\bar{b}} &= \bar{h}_{a\bar{b}} + D_a F_{\bar{b}} - D_{\bar{b}} F_a - F^0 L_{a\bar{b}}\,, \\
\mathcal{A}_a &= \bar{\mathcal{A}}_a + \dot{F}_a - L_a{}^{\bar{b}} F_{\bar{b}} + D_a\big(\tilde{F}_0 + F^0\big) \\
\mathcal{A}_{\bar{a}} &= \bar{\mathcal{A}}_{\bar{a}} + \dot{F}_{\bar{a}} - L_{\bar{a}}{}^{b} F_b + D_{\bar{a}}\big(\tilde{F}_0 - F^0\big)\,, \\
\varphi &= \bar{\varphi} + \dot{\Phi} F^0 - \tfrac{1}{2} D_a F^a - \tfrac{1}{2} D_{\bar{a}} F^{\bar{a}}\,, \\
\phi &= \bar{\phi} - \dot{F}^0\,,
\end{aligned} \tag{3.194}$$

where the gauge invariant fields, denoted by a bar, are given by

$$\begin{aligned}
\bar{h}_{a\bar{b}} = \widehat{h}_{a\bar{b}} &+ (a^{-1}\partial_a + a\tilde{\partial}_a)\widehat{B}_{\bar{b}} - (a^{-1}\partial_{\bar{b}} - a\tilde{\partial}_{\bar{b}})\widehat{B}_a + \widehat{E} g_{a\bar{b}} \\
&+ \frac{1}{2}(a^{-2}\partial_a\partial_{\bar{b}} + 2\partial_a\tilde{\partial}_{\bar{b}} - 2\tilde{\partial}_a\partial_{\bar{b}} - a^2\tilde{\partial}_a\tilde{\partial}_{\bar{b}})\widehat{C}\,,
\end{aligned} \tag{3.195}$$

$$\bar{\mathcal{A}}_a = \widehat{\mathcal{A}}_a + \frac{a^{-1}}{4}\partial_a\big(\widehat{A} - \widehat{\tilde{A}} + 2\sqrt{2}H\widehat{C}\big) + \frac{a}{4}\tilde{\partial}_a\big(\widehat{A} + \widehat{\tilde{A}} + 2\sqrt{2}H\widehat{C}\big)\,, \tag{3.196}$$

$$\bar{\mathcal{A}}_{\bar{a}} = \widehat{\mathcal{A}}_{\bar{a}} + \frac{a^{-1}}{4}\partial_{\bar{a}}\big(\widehat{A} - \widehat{\tilde{A}} + 2\sqrt{2}H\widehat{C}\big) - \frac{a}{4}\tilde{\partial}_{\bar{a}}\big(\widehat{A} + \widehat{\tilde{A}} + 2\sqrt{2}H\widehat{C}\big)\,, \tag{3.197}$$

$$\bar{\varphi} = \widehat{\varphi} + \frac{1}{4}\Delta\widehat{C}\,, \tag{3.198}$$

$$\bar{\phi} = \widehat{\phi}\,, \tag{3.199}$$

while the 'effective gauge parameters', denoted by F, are given by

$$\begin{aligned}
F^0 =& -\frac{1}{2\sqrt{2}}\left(a(A+\bar{A}+\sqrt{2}a\dot{C}) - a^{-1}(\tilde{A}-\bar{\tilde{A}}+\sqrt{2}a^{-1}\dot{\tilde{C}})\right),\\
F_0 =& \frac{1}{2\sqrt{2}}\left(a^{-1}(\tilde{A}+\bar{\tilde{A}}) - a(A-\bar{A})\right),\\
F_a =& \frac{1}{\sqrt{2}}(aB_a + a^{-1}\tilde{B}_a) + \frac{\sqrt{2}a^{-1}}{4}\partial_a(-3a^2C - 2\tilde{D} + a^{-2}\tilde{C})\\
&+ \frac{\sqrt{2}a}{4}\tilde{\partial}_a(-3a^{-2}\tilde{C} - 2D + a^2C),\\
F_{\bar{a}} =& -\frac{1}{\sqrt{2}}(aB_{\bar{a}} - a^{-1}\tilde{B}_{\bar{a}}) + \frac{\sqrt{2}a^{-1}}{4}\partial_{\bar{a}}(-3a^2C + 2D + a^{-2}\tilde{C})\\
&+ \frac{\sqrt{2}a}{4}\tilde{\partial}_{\bar{a}}(3a^{-2}\tilde{C} - 2\tilde{D} - a^2C).
\end{aligned} \tag{3.200}$$

These quantities transform under gauge transformations as

$$\begin{aligned}
&\delta F^0 = \xi^0,\\
&\delta F_0 = \xi_0 - \dot{\eta},\\
&\delta F_a = \zeta_a + \partial_a(\lambda - \tfrac{1}{\sqrt{2}}a^{-1}\eta) + \tilde{\partial}_a(\chi + \tfrac{1}{\sqrt{2}}a\eta) = \xi_a - \tfrac{1}{\sqrt{2}}a^{-1}\partial_a\eta + \tfrac{1}{\sqrt{2}}a\tilde{\partial}_a\eta,\\
&\delta F_{\bar{a}} = \zeta_{\bar{a}} + \partial_{\bar{a}}(\bar{\lambda} + \tfrac{1}{\sqrt{2}}a^{-1}\eta) + \tilde{\partial}_{\bar{a}}(\bar{\chi} + \tfrac{1}{\sqrt{2}}a\eta) = \xi_{\bar{a}} + \frac{1}{\sqrt{2}}a^{-1}\partial_{\bar{a}}\eta + \frac{1}{\sqrt{2}}a\tilde{\partial}_{\bar{a}}\eta,
\end{aligned} \tag{3.201}$$

where the gauge-for-gauge parameter η is given by

$$\eta = \frac{\sqrt{2}}{4}\left(a\lambda - a\bar{\lambda} - a^{-1}\chi - a^{-1}\bar{\chi}\right). \tag{3.202}$$

Using the expressions (3.200) and those for the gauge invariant variables (3.192) one may verify that (3.194) are just identities. These identities are very useful, however, since they decompose the fields into their gauge invariant parts plus terms of the 'pure gauge form' (3.116).

We can now replace each appearance of a field in the quadratic action by its right-hand side in (3.194). Gauge invariance then implies that the pure gauge terms drop out, so that in effect we may simply replace each field by its gauge invariant version, i.e., we just put a bar on each field. Afterwards we use the expressions in terms of the SVT components, which in turn allows us to express each divergence, trace, etc., of a tensor or vector in terms of appropriate scalar modes, thereby achieving complete decoupling.

In order to perform this computation it is convenient to first display the quadratic Lagrangian (3.105) in terms of the derivatives ∂ and $\tilde{\partial}$:

$$
\begin{aligned}
\mathcal{L} = ne^{-2\Phi}\Big\{ & -4\dot{\varphi}^2 + 8\dot{\Phi}\phi\dot{\varphi} + H(2\phi+4\varphi)g_{\bar{b}}{}^{a}\dot{h}_a{}^{\bar{b}} \\
& + \dot{h}_{a\bar{b}}\dot{h}^{a\bar{b}} + 2H^2 h^{a\bar{b}}h_{a\bar{b}} - 2H^2 g_a{}^{\bar{c}}g_{\bar{d}}{}^{b}h^{a\bar{d}}h_{b\bar{c}} \\
& + \frac{1}{\sqrt{2}}(a\tilde{\partial}^{\bar{a}}\mathcal{A}_{\bar{a}} + a^{-1}\partial^{\bar{a}}\mathcal{A}_{\bar{a}} - a\tilde{\partial}^{a}\mathcal{A}_{a} + a^{-1}\partial^{a}\mathcal{A}_{a})(-4\dot{\varphi} + 4\dot{\Phi}\phi - 2Hg_{\bar{b}}{}^{a}h_a{}^{\bar{b}}) \\
& + \frac{1}{\sqrt{2}}H(4\varphi+2\phi)(-a\tilde{\partial}^{\bar{a}}\mathcal{A}_{\bar{a}} + a^{-1}\partial^{\bar{a}}\mathcal{A}_{\bar{a}} + a\tilde{\partial}^{a}\mathcal{A}_{a} + a^{-1}\partial^{a}\mathcal{A}_{a}) \\
& - \frac{1}{2}\mathcal{A}_{\bar{a}}\Delta\mathcal{A}^{\bar{a}} - \frac{1}{2}\mathcal{A}^{a}\Delta\mathcal{A}_{a} - \frac{1}{2}(a\tilde{\partial}^{a}\mathcal{A}_{a} - a^{-1}\partial^{a}\mathcal{A}_{a})^2 - \frac{1}{2}(a\tilde{\partial}^{\bar{a}}\mathcal{A}_{\bar{a}} + a^{-1}\partial^{\bar{a}}\mathcal{A}_{\bar{a}})^2 \\
& + \sqrt{2}\mathcal{A}^{\bar{b}}(a\tilde{\partial}^{a} - a^{-1}\partial^{a})\dot{h}_{a\bar{b}} - \sqrt{2}\mathcal{A}^{a}(a\tilde{\partial}^{\bar{b}} + a^{-1}\partial^{\bar{b}})\dot{h}_{a\bar{b}} \\
& - \sqrt{2}Hg_b{}^{\bar{d}}\mathcal{A}^{b}(a\tilde{\partial}^{a}h_{a\bar{d}} - a^{-1}\partial^{a}h_{a\bar{d}}) + \sqrt{2}Hg_{\bar{b}}{}^{c}\mathcal{A}^{\bar{b}}(a\tilde{\partial}^{\bar{a}}h_{c\bar{a}} + a^{-1}\partial^{\bar{a}}h_{c\bar{a}}) \\
& - \sqrt{2}H\mathcal{A}^{\bar{b}}(a\tilde{\partial}^{a}h_{a\bar{b}} + a^{-1}\partial^{a}h_{a\bar{b}}) + \sqrt{2}H\mathcal{A}^{a}(a\tilde{\partial}^{\bar{b}}h_{a\bar{b}} - a^{-1}\partial^{\bar{b}}h_{a\bar{b}}) \\
& + 4\phi\Delta\varphi - 4\varphi\Delta\varphi + (4\varphi - 2\phi)(a^{-2}\partial^{a}\partial^{\bar{b}} - \tilde{\partial}^{a}\partial^{\bar{b}} + \partial^{a}\tilde{\partial}^{\bar{b}} - a^2\tilde{\partial}^{a}\tilde{\partial}^{\bar{b}})h_{a\bar{b}} \\
& + h^{a\bar{b}}\Delta h_{a\bar{b}} + (a^{-1}\partial^{a} - a\tilde{\partial}^{a})h_a{}^{\bar{b}}(a^{-1}\partial^{c} - a\tilde{\partial}^{c})h_{c\bar{b}} \\
& + (a^{-1}\partial^{\bar{b}} + a\tilde{\partial}^{\bar{b}})h_{a\bar{b}}(a^{-1}\partial_{\bar{c}} + a\tilde{\partial}_{\bar{c}})h^{a\bar{c}}\Big\}\,.
\end{aligned}
\tag{3.203}
$$

Following the above procedure one obtains the decoupled quadratic action in terms of gauge invariant variables:

$$
S = \int dt \int d^{2d}\mathbf{X}\, n\, e^{-2\Phi}(\mathcal{L}_T + \mathcal{L}_V + \mathcal{L}_S)\,, \tag{3.204}
$$

where $\mathcal{L}_T$, $\mathcal{L}_V$ and $\mathcal{L}_S$ denote the Lagrangians for the tensor, vector and scalar modes, respectively. As emphasized before, the action should completely decouple among these modes, as indeed it does. Specifically, the action for the tensor modes is given by

$$
\mathcal{L}_T = \dot{\widehat{h}}_{a\bar{b}}\dot{\widehat{h}}^{a\bar{b}} + 2H^2\widehat{h}_{a\bar{b}}\widehat{h}^{a\bar{b}} - 2H^2 g_a{}^{\bar{c}}g_{\bar{d}}{}^{b}\widehat{h}^{a\bar{d}}\widehat{h}_{b\bar{c}} + \widehat{h}_{a\bar{b}}\Delta\widehat{h}^{a\bar{b}}\,, \tag{3.205}
$$

while the action for the vector modes reads

$$\begin{aligned}\mathcal{L}_V = &-\dot{\widehat{B}}_{\bar{a}}\Delta\dot{\widehat{B}}^{\bar{a}} - \dot{\widehat{B}}^a\Delta\dot{\widehat{B}}_a - H^2\widehat{B}_a\Delta\widehat{B}^a - H^2\widehat{B}_{\bar{a}}\Delta\widehat{B}^{\bar{a}} \\ &+ 4H^2 g_a{}^{\bar{b}}\widehat{B}^a\Diamond\widehat{B}_{\bar{b}} - \widehat{B}_a(\Delta^2-\Diamond^2)\widehat{B}^a - \widehat{B}_{\bar{a}}(\Delta^2-\Diamond^2)\widehat{B}^{\bar{a}} \\ &-\frac{1}{2}\widehat{A}_{\bar{a}}\Delta\widehat{A}^{\bar{a}} - \frac{1}{2}\widehat{A}_a\Delta\widehat{A}^a - \sqrt{2}\widehat{A}^a\Diamond(\dot{\widehat{B}}_a + Hg_a{}^{\bar{b}}\widehat{B}_{\bar{b}}) + \sqrt{2}\widehat{A}^{\bar{a}}\Diamond(\dot{\widehat{B}}_{\bar{a}} + Hg_{\bar{a}}{}^b\widehat{B}_b)\,.\end{aligned} \tag{3.206}$$

Note that while this action carries two more vector modes than in standard gravity, the A modes are actually auxiliary as they may be eliminated by their own equations of motions. Indeed, varying with respect to $\widehat{A}_a$ and $\widehat{A}_{\bar{a}}$ and assuming Δ is invertible, we obtain

$$\begin{aligned}\widehat{A}_a &= -\sqrt{2}\Delta^{-1}\Diamond(\dot{\widehat{B}}_a + Hg_a{}^{\bar{b}}\widehat{B}_{\bar{b}})\,, \\ \widehat{A}_{\bar{a}} &= \sqrt{2}\Delta^{-1}\Diamond(\dot{\widehat{B}}_{\bar{a}} + Hg_{\bar{a}}{}^b\widehat{B}_b)\,,\end{aligned} \tag{3.207}$$

which may be reinserted into the action to obtain:

$$\begin{aligned}\mathcal{L}_V(B) = &-\dot{\widehat{B}}_{\bar{a}}\Delta\dot{\widehat{B}}^{\bar{a}} - \dot{\widehat{B}}^a\Delta\dot{\widehat{B}}_a - H^2\widehat{B}_a\Delta\widehat{B}^a - H^2\widehat{B}_{\bar{a}}\Delta\widehat{B}^{\bar{a}} \\ &+ 4H^2 g_{\bar{b}}{}^a\widehat{B}^{\bar{b}}\Diamond\widehat{B}_a - \widehat{B}_{\bar{a}}(\Delta^2-\Diamond^2)\widehat{B}^{\bar{a}} - \widehat{B}_a(\Delta^2-\Diamond^2)\widehat{B}^a \\ &+ (\dot{\widehat{B}}_{\bar{a}} + Hg_{\bar{a}}{}^b\widehat{B}_b)\Delta^{-1}\Diamond^2(\dot{\widehat{B}}^{\bar{a}} + Hg^{\bar{a}c}\widehat{B}_c) + (\dot{\widehat{B}}_a + Hg_a{}^{\bar{b}}\widehat{B}_{\bar{b}})\Delta^{-1}\Diamond^2(\dot{\widehat{B}}^a + Hg^{a\bar{c}}\widehat{B}_{\bar{c}})\,.\end{aligned} \tag{3.208}$$

The action for the scalar modes is

$$\begin{aligned}\mathcal{L}_S = &-4\dot{\widehat{\varphi}}^2 + 8\dot{\Phi}\phi\widehat{\varphi} + d\dot{\widehat{E}}^2 + 2dH(\widehat{\phi}+2\widehat{\varphi})\dot{\widehat{E}} + \frac{1}{4}\dot{\widehat{C}}(\Delta^2-\Diamond^2)\dot{\widehat{C}} + \frac{1}{2}H^2\widehat{C}(\Delta^2-\Diamond^2)\widehat{C} \\ &+\frac{\sqrt{2}}{2}\widehat{A}\Diamond(2\dot{\widehat{\varphi}} - 2\dot{\Phi}\widehat{\phi} + dH\widehat{E}) + \frac{\sqrt{2}}{2}\widehat{A}\Delta\big(H(\widehat{\phi}+2\widehat{\varphi}) + \dot{\widehat{E}}\big) \\ &-\frac{\sqrt{2}}{4}H\widehat{A}(\Delta^2-\Diamond^2)\widehat{C} + \frac{1}{16}\widehat{A}(\Delta^2-\Diamond^2)\widehat{A} - \frac{\sqrt{2}}{8}\widehat{\widehat{A}}(\Delta^2-\Diamond^2)\dot{\widehat{C}} + \frac{1}{16}\widehat{\widehat{A}}(\Delta^2-\Diamond^2)\widehat{\widehat{A}} \\ &+4\widehat{\phi}\Delta\widehat{\varphi} - 4\widehat{\varphi}\Delta\widehat{\varphi} + (d-2)\widehat{E}\Delta\widehat{E} + 2(\widehat{\phi}-2\widehat{\varphi})\Diamond\widehat{E} - \frac{1}{2}(\widehat{\phi}-2\widehat{\varphi})(\Delta^2-\Diamond^2)\widehat{C}\,.\end{aligned} \tag{3.209}$$

Here, $\widehat{\widehat{A}}$, and $\widehat{A}$ are auxiliary fields. Varying the action with respect to these fields yields

$$(\Delta^2-\Diamond^2)(-\sqrt{2}\dot{\widehat{C}} + \widehat{\widehat{A}}) = 0\,, \tag{3.210}$$

$$\frac{1}{8}(\Delta^2-\lozenge^2)(\widehat{A}-2\sqrt{2}H\widehat{C})+\frac{\sqrt{2}}{2}\lozenge(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})+\frac{\sqrt{2}}{2}\Delta\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)=0\,. \tag{3.211}$$

Since $\Delta^2-\lozenge^2=4\partial^2\tilde{\partial}^2$ and ∂^2 and $\tilde{\partial}^2$ are invertible, we can solve for $\widehat{\bar{A}}$ and $\widehat{A}$,

$$\widehat{\bar{A}}=\sqrt{2}\dot{\widehat{C}}\,, \tag{3.212}$$

$$\widehat{A}=2\sqrt{2}H\widehat{C}-\sqrt{2}\partial^{-2}\tilde{\partial}^{-2}\big(\lozenge(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})+\Delta\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)\,, \tag{3.213}$$

and re-insert these into the action, which yields

$$\begin{aligned}\mathcal{L}_S=\;&(d-1)\dot{\widehat{E}}^2+2(d-1)H(\widehat{\phi}+2\widehat{\varphi})\dot{\widehat{E}}+\frac{1}{8}\dot{\widehat{C}}(\Delta^2-\lozenge^2)\dot{\widehat{C}}\\&+2\widehat{C}\lozenge(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})+2\widehat{C}\Delta\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)\\&+(d-1)H^2\widehat{\phi}^2-4H^2\widehat{\phi}\widehat{\varphi}-4H^2\widehat{\varphi}^2+d^2H^2\widehat{E}^2+4dH\dot{\widehat{\varphi}}\widehat{E}-4dH\dot{\Phi}\widehat{E}\widehat{\phi}\\&+4\widehat{\phi}\Delta\widehat{\varphi}-4\widehat{\varphi}\Delta\widehat{\varphi}+(d-2)\widehat{E}\Delta\widehat{E}+2(\widehat{\phi}-2\widehat{\varphi})\lozenge\widehat{E}-\frac{1}{2}(\widehat{\phi}-2\widehat{\varphi})(\Delta^2-\lozenge^2)\widehat{C}\\&-\frac{1}{2}(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})\partial^{-2}\tilde{\partial}^{-2}(a^4\tilde{\partial}^4+a^{-4}\partial^4)(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})\\&-\frac{1}{2}\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)\partial^{-2}\tilde{\partial}^{-2}(a^4\tilde{\partial}^4+a^{-4}\partial^4)\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)\\&-(2\dot{\widehat{\varphi}}-2\dot{\Phi}\widehat{\phi}+dH\widehat{E})\partial^{-2}\tilde{\partial}^{-2}(a^4\tilde{\partial}^4-a^{-4}\partial^4)\big(H(\widehat{\phi}+2\widehat{\varphi})+\dot{\widehat{E}}\big)\,.\end{aligned} \tag{3.214}$$

We close this section by pointing out that the definition of gauge invariant variables can be extended to cubic order, using our procedure of gauge invariant variables described in Chapter 2. As before, the cubic action in terms of these variables takes the same form as the gauge fixed action, but with the fields being the fully gauge invariant ones. We will leave the details to future work.

Chapter 4

Homological Quantum Mechanics

Let us now turn to the application of homotopy algebras to quantum mechanics. In this chapter, our goal is to compute physically relevant observables in quantum mechanics, namely, quantum expectation values. Traditionally these are computed via the canonical formulation or the path integral. In the canonical formulation, one defines the state of a quantum system in terms of vectors in a Hilbert space. Observables, such as position, momentum or energy, are no longer numbers as in classical mechanics but are rather self-adjoint operators on the Hilbert space. A measurement of an observable is an expectation value, which is computed by taking an initial state, acting on it with the operator corresponding to this observable, and projecting the result onto a final state. In the path integral formulation, one computes expectation values by summing over all the paths a system can take when evolving from its initial to final state. The paths are weighted by $\exp(iS/\hbar)$ where S is the action of the system. Since the path integral works with the Lagrangian instead of the Hamiltonian, it is advantageous to canonical quantization by preserving symmetries of the system, and thus has been essential in the development of quantum field theory. However, summing over infinitely many paths lacks in mathematical rigour. Our partial reformulation presented in [19] circumvents this issue by computing expectation values algebraically instead of through analysis. It has yet to be a full reformulation, since we only demonstrate our method for computing expectation values for the harmonic oscillator and its perturbations. As a first step towards an extension to quantum field theory, we apply our method to rederive the Unruh effect in the context of quantum field theory in curved spacetime. Whether our method can be applied broadly in quantum field theory, especially for gauge theories, still remains an open question.

Given the tools and systematics that we have learned about so far, to accomplish our goal of computing quantum expectation values, we might think of computing the cohomology of the algebraic structure which encodes the theory, because that is where we expect to find our physical observables. To start with, we observe that the cohomology of the L_∞ algebra of the theory does not provide sufficient information, since this corresponds to the classical configurations of the system that satisfy the equations of motion. In order to do quantum mechanics, additional structures seem to be required – for us this is the BV algebra. As mentioned in 1.3.3, the BV algebra is defined on the space of functionals $\mathcal{F}(V)$ which is dual to the vector space V underlying the theory. Although at first glance it might seem unintuitive to work with functionals in the dual space, it is not surprising when one recalls that the path integral does the same – it is indeed a functional integral.

Nevertheless, the resulting expectation value is a function, namely of the initial and final boundary conditions which live in $\mathbb{R}^{2n}$ for a quantum mechanical system with n degrees of freedom. Instead of computing the functional integral, in our approach, we reach the resulting function by defining a projection from the space of functionals to the space of functions of boundary conditions,

$$P: \ \mathcal{F}(V) \to \mathcal{F}(\mathbb{R}^2)\,. \tag{4.1}$$

Our claim is that given a functional F, its quantum expectation value is given by its projection $P(F)$ whose inclusion to the space of functionals is a member of the δ-cohomology class of F, i.e.

$$I(P(F)) = F - \delta G\,, \tag{4.2}$$

where $G \in \mathcal{F}(V)$. This turns out to be exactly the relation given by the homotopy retract from the BV algebra to the space of functions of the boundary conditions. After reviewing the standard formulation to establish notation, we will show how to compute $P(F)$ for the harmonic oscillator and compare our results with the standard path integral.

The notion that the cohomology of the BV differential is related to quantum expectation values had already been developed for the finite-dimensional model[1] in section 2.3 of [66] and in [67]. The quantum mechanical system described by

[1] In the finite-dimensional model, the action is a function instead of a functional and so the path integral is over $\mathbb{R}$ instead of an infinite-dimensional space of functions.

the action

$$S(x) = \frac{1}{2}ax^2\,, \tag{4.3}$$

is considered as an example. It can be shown that the cohomology of the BV differential computes the expectation value of polynomials x^n as defined as

$$\langle x^n \rangle = \frac{\int \mathrm{d}x\, x^n e^{-ax^2/2\hbar}}{\int \mathrm{d}x e^{-ax^2/2\hbar}}\Bigg|_{a\to -ia} \tag{4.4}$$

where the substitution $a \to -ia$ is performed after integration. One starts by defining the BV complex on the graded vector space given by the space of functions of the form

$$F(x, x^*) = f(x) + x^* g(x)\,, \tag{4.5}$$

where x^* is an anti-commuting number and f and g are polynomials. The space in degree zero is given by the functions that have $g = 0$ and the space in degree -1 is given by the functions that have $g \neq 0$. The BV Laplacian is

$$\Delta = -\frac{\partial^2}{\partial x \partial x^*} \tag{4.6}$$

and the bracket is given by

$$\{a, b\} = \frac{\partial_r a}{\partial x^*}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial_l b}{\partial x^*}\,. \tag{4.7}$$

The BV-differential corresponding to this action is

$$\delta = -ax\frac{\partial}{\partial x^*} + i\hbar\frac{\partial^2}{\partial x \partial x^*}\,. \tag{4.8}$$

Let us compute the cohomology. First we find the kernel of δ by applying δ on $F(x, x^*)$ in (4.5):

$$\delta F(x, x^*) = -axg(x) + i\hbar g'(x)\,. \tag{4.9}$$

Setting this to zero, we find that $g(x) = e^{iax^2/2\hbar}$ is a solution but is not a polynomial, so $g(x)$ must be zero. Thus the kernel of δ is given by the functions $F(x, x^*)$ where $g(x) = 0$:

$$\ker \delta = \{F(x, x^*) \equiv f(x)\}\,. \tag{4.10}$$

Next we find the image of δ. We take a look at how δ acts on functions where $f(x) = 0$ and $g(x) = x^n$, i.e. functions $G(x, x^*) = x^* x^n$:

$$\delta G(x, x^*) = -ax^{n+1} + i\hbar n x^{n-1} \,. \tag{4.11}$$

In cohomology, we therefore have

$$ax^{n+1} \sim i\hbar n x^{n-1} \,. \tag{4.12}$$

Inserting $n = 0$ into (4.12), we see that $x \sim 0$. If $n = 1$, then $x^2 \sim \frac{i\hbar}{a}$. For $n = 2$, we deduce that $x^3 \sim \frac{3i\hbar}{a} x \sim 0$. And for $n = 3$, we get $x^4 \sim 3(\frac{i\hbar}{a})x^2 \sim 3(\frac{i\hbar}{a})^2$. Continuing the pattern, we have

$$x^n \sim \begin{cases} 0 & \text{for } n \text{ odd} \\ \left(\frac{i\hbar}{a}\right)^{\frac{n}{2}} (n-1)(n-3) \cdot \dots \cdot 1 & \text{for } n \text{ even} \end{cases} \tag{4.13}$$

These are the cohomology classes for x^n for all n. Each polynomial x^n can be represented by one number, and this is the expectation value $\langle x^n \rangle$ (see for instance p. 14 in Zee's textbook [68]). In our formulation, we will extend this to infinite-dimensional models for computing expectation values for the harmonic oscillator with respect to position and momentum eigenstates, as well as coherent states.

Path integral

In order to compare our formulation with the path integral formulation, we would like to briefly review the path integral and establish some general notation.

Consider a non-relativistic particle moving in one dimension with the Hamiltonian

$$H[x, p] = \frac{p^2}{2m} + V(x) \,, \tag{4.14}$$

where $x = x(t)$ and $p = p(t)$ are its position and momentum respectively parametrized by time t and where $V(x)$ is an arbitrary potential. In the path integral formulation of quantum mechanics, the amplitude of measuring a particle at a final position $x(t_f) = x_f$ given its initial position $x(t_i) = x_i$ is

$$\langle x_f | x_i \rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x \, \exp(iS[x]/\hbar) \,. \tag{4.15}$$

where S is the classical action functional $S[x] = \int \mathrm{d}t L[x, \dot{x}] = \int \mathrm{d}t\,(p\dot{x} - H[x, p])$. The functional integral $\int \mathcal{D}x$ sums over all paths $x(t)$ which begin at $x(t_i) = x_i$ and end at $x(t_f) = x_f$.

Suppose we have a function $F(x)$ which defines an operator $F(\hat{x})$ in the Schrödinger picture in the sense that

$$\hat{F}(\hat{x}) = \int dx\,|x\rangle\,\langle x|\,F(x)\,. \tag{4.16}$$

where $|x\rangle$ are the eigenstates of $\hat{x}$, i.e. $\hat{x}\,|x\rangle = x\,|x\rangle$. One can show that the expectation value of $\hat{F}(\hat{x})$ is given by

$$\langle x_f|\hat{F}(\hat{x})|x_i\rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x\,F[x]\exp(iS[x]/\hbar)\,. \tag{4.17}$$

Wick contractions

In our work we will reproduce Wick contractions using a homological approach. In the standard formulation, when computing vacuum expectation values of products of field operators, one often uses Wick's theorem. Just as position and momentum operators of the quantum harmonic oscillator can be rewritten in terms of creation and annihilation operators, field operators Φ are split into its creation part Φ^+ and annihilation part Φ^-:

$$\Phi = \Phi^+ + \Phi^-\,, \tag{4.18}$$

where Φ^- annihilates the vacuum, $\Phi^-\,|\Omega\rangle = 0$. Wick's theorem dictates how to organize time-ordered products of operators. Time-ordering of a product places an operator to the right of another if it is defined at an earlier time, so that the time at which an operator is defined in the product increases from right to left. For a product of two operators, time-ordering yields

$$\begin{aligned} T(\Phi(x)\Phi(y)) =& \Phi^+(x)\Phi^+(y) + \Phi^-(x)\Phi^-(y) \\ &+ \Theta(t_y - t_x)(\Phi^+(y)\Phi^-(x) + \Phi^-(y)\Phi^+(x)) \\ &+ \Theta(t_x - t_y)(\Phi^+(x)\Phi^-(y) + \Phi^-(x)\Phi^+(y))\,, \end{aligned} \tag{4.19}$$

where Θ is the step function. Wick's theorem says that the time-ordered product is the normal-ordered product plus Green's functions. A product of operators Φ

is normal-ordered when all the Φ^+ operators are on the left of Φ^-. For example,

$$N(\Phi(x)\Phi(y)) = \Phi^+(x)\Phi^+(y) + \Phi^+(x)\Phi^-(y) + \Phi^+(y)\Phi^-(x) + \Phi^-(x)\Phi^-(y)\,. \tag{4.20}$$

One can show that the time-ordered product can be given as

$$T(\Phi(x)\Phi(y)) = N(\Phi(x)\Phi(y)) - iG_F(x-y)\,, \tag{4.21}$$

where G_F is the Feynman propagator. When evaluating the expectation value with respect to the vacuum, the normal-ordered product vanishes since the annihilation operators are on the right. Wick's theorem shows how to extend this result to compute products of an arbitrary number of field operators.

$$\langle\Omega|T(\Phi(x_1)\cdots\Phi(x_n)|\Omega\rangle = \sum_{\text{pairs}} G_F(x_{i_1}-x_{i_2})G_F(x_{i_2}-x_{i_3})\cdots G_F(x_{i_{n-1}}-x_{i_n})\,. \tag{4.22}$$

4.1 Homological approach

Let us now describe our general approach. We would like to do quantum mechanics for a theory with an action of the form,

$$S[\phi] = \int_{t_i}^{t_f} dt\, \mathcal{L}(\phi(t)\,, \dot{\phi}(t), t)\,, \quad \text{where} \quad \phi(t) \in C^\infty([t_i, t_f])\,. \tag{4.23}$$

The free theory can be organized into a chain complex (V, ∂):

$$0 \longrightarrow V^0 \xrightarrow{\ \partial\ } V^1 \longrightarrow 0\,, \tag{4.24}$$

where V^0 and V^1 are vector spaces $C^\infty([t_i, t_f])$ of the smooth functions on the interval $[t_i, t_f]$. The differential ∂ acting on a field $\phi \in V^0$ yields its equation of motion, i.e.

$$\partial\phi = \mathrm{EL}(\phi(t)) \in V^1\,, \tag{4.25}$$

where $\mathrm{EL}(\phi(t)) = 0$ are the Euler-Lagrange equations. The fields in V^0 are assigned even degree and the elements in V^1, called anti-fields, have odd degree.

Let us define functionals on V,

$$F[\phi, \phi^*] = \int dt_1 \cdots dt_k ds_1 \cdots ds_l f(t_1, ..., t_k, s_1, ...s_l) \phi(t_1) \cdots \phi(t_k) \phi^*(s_1) \cdots \phi^*(s_l) , \tag{4.26}$$

where $\phi(t) \in V^0$ and $\phi^*(s) \in V^1$ and the coefficient functions $f(t_1, ..., t_k, s_1, ...s_l)$ are totally symmetric in t_i and totally antisymmetric in s_i. The vector space $\mathcal{F}(V)$ of these functionals admits a grading,

$$\mathcal{F}(V) = \cdots \oplus F(V)^{-2} \oplus F(V)^{-1} \oplus F(V)^0 , \tag{4.27}$$

where the degree of a functional is given by minus the number of ϕ^* it contains, $|F| = -l$. The BV complex is defined by equipping $\mathcal{F}(V)$ with the BV differential $\delta = Q - i\hbar\Delta$, where

$$Q = -\int_{t_i}^{t_f} dt\, \mathrm{EL}(\phi(t)) \frac{\delta}{\delta\phi^*(t)} , \quad \text{and} \quad \Delta = -\int_{t_i}^{t_f} dt\, \frac{\delta^2}{\delta\phi(t)\phi^*(t)} , \tag{4.28}$$

and δ is a degree $+1$ map since Q and Δ both decrease the number of ϕ^* by 1. One can check that $\delta^2 = 0$ and that in addition, $Q^2 = 0$. One may notice that the definition of δ is different from what we expect from the definition in (1.85). For Δ to satisfy (1.83) and (1.84), the Poisson bracket is defined as

$$\{F, G\} = \frac{\delta F}{\delta\phi^*} \frac{\delta G}{\delta\phi} - \frac{\delta F}{\delta\phi} \frac{\delta G}{\delta\phi^*} , \tag{4.29}$$

With (1.85) and the above definition, we expect Q to be of the form

$$\{S, -\} = -\frac{\delta S}{\delta\phi} \frac{\delta}{\delta\phi^*} . \tag{4.30}$$

However, $\frac{\delta S}{\delta\phi}$ and the Euler-Lagrange equations are only equal up to boundary terms. For traditional applications of the BV formalism, the specific form in (4.30) is important, and in fact there is an extension of the BV formalism, called the BV-BFV formalism [69, 70] which deals with boundary terms.[2] For our purposes, however, what we define in (4.28) is sufficient.

Given the BV complex $(\mathcal{F}(V), \delta)$ of the theory, we would like to compute

[2]In particular the symplectic form which induces the bracket $\{-, -\}$ is no longer invariant under Q due to boundary terms, but this is irrelevant in our formulation since we do not consider the symplectic form.

expectation values of functionals with respect to initial and final boundary conditions corresponding to position or momentum eigenstates in the canonical formulation, for example,

$$\langle T(\hat{F}(\hat{x}))\rangle = \frac{\langle y; t_f | T(\hat{F}(\hat{x})) | x; t_i\rangle}{\langle y; t_f | x; t_i\rangle}\,, \tag{4.31}$$

where $\hat{x}\,|x;t_i\rangle = |x;t_i\rangle$. Let us stress that the words "operator" and "state" are only used in the context of comparing our method to the canonical formulation, and that these concepts are non-existent in our formulation. As sketched at the beginning of this chapter, the expectation value is given by the projection to the space of functions of boundary conditions. To define this projection requires some steps. Our procedure starts by establishing a homotopy retract from the chain complex of the free theory (V, ∂) to the space of the boundary conditions, also called the phase space $\mathbb{R}^2$. Then, we lift this homotopy retract to a homotopy retract on the dual complexes. By applying the perturbation lemma, we can incorporate the interacting theory structure and the quantum piece of the BV differential to finally obtain the cohomology of δ. In the following we elaborate on each of these steps.

We begin by considering only the free theory without interactions. We build a chain complex of the on-shell functions, namely, solutions to the free equations of motion parametrized by boundary conditions. As an example, let $\phi_{x_i x_f}$ be a unique solution with the boundary conditions,

$$(\phi_{x_i x_f}(t_i), \phi_{x_i x_f}(t_f)) = (x_i, x_f)\,, \qquad (x_i, x_f) \in \mathbb{R}^2\,. \tag{4.32}$$

In other words, for every pair of boundary conditions, we can map to a solution of the equations of motion, hence defining the inclusion map,

$$i : \mathbb{R}^2 \to V\,, \qquad (x_i, x_f) \mapsto (\phi, \phi^*) = (\phi_{x_i x_f}, 0)\,. \tag{4.33}$$

For a field in V, we can project to its boundary values,

$$p : V \to \mathbb{R}^2\,, \qquad (\phi, \phi^*) \mapsto (\phi(t_i), \phi(t_f))\,. \tag{4.34}$$

Elements in V^1 map to 0 because on-shell fields trivially satisfy the equations of motion, so there are no equations of motion left in the phase space. We can also consider more general boundary conditions, namely the projections

$$\begin{aligned} p: \ V &\to \mathbb{R}^2\,, \\ \phi &\mapsto (a_i\phi(t_i) + b_i\dot{\phi}(t_i)\,, a_f\phi(t_f) + b_f\dot{\phi}(t_f))\,. \end{aligned} \tag{4.35}$$

Summarizing the above data into a diagram, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^0 & \xrightarrow{\partial} & V^1 & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow 0 & & \\ 0 & \longrightarrow & \mathbb{R}^2 & \xrightarrow{0} & 0 & \longrightarrow & 0 \end{array} \tag{4.36}$$

The two chain complexes are homotopy equivalent if there exists a homotopy retract between them. This means that their cohomologies are the same and in this case it is obvious since the phase space is the cohomology of ∂. The homotopy retract requires the existence of a homotopy map, h, of degree -1, which satisfies

$$i \circ p = \text{id} - h \circ \partial - \partial \circ h\,. \tag{4.37}$$

Acting on elements in V^0, the third term on the RHS of the equation vanishes since there is no V^{-1}. Thus for objects in V^0, we have the relation

$$i \circ p = \text{id} - h \circ \partial\,. \tag{4.38}$$

Acting (4.37) on elements in V^1, we will have 0 on the LHS and the second term on the RHS vanishes since there is no V^2. Thus, V^1 elements, we have the relation

$$\text{id} = \partial \circ h\,. \tag{4.39}$$

Equation (4.39) might look familiar. Given a differential operator $\text{L}(t)$, the Green's function $K(t,s)$ is a solution to the equation $\text{L}(t)K(t,s) = \delta(t-s)$, where $\delta(t-s)$ is the Dirac delta function. In our case, the Green's function for the differential operator ∂ is the integral kernel of the homotopy map h. In other words, we define the map $h : V^1 \to V^0$ for any function $f \in V^1$,

$$h(f)(t) = \int_{t_i}^{t_f} \text{d}s K(t,s) f(s)\,, \tag{4.40}$$

where $K(t,s)$ is defined such that

$$\phi_f(t) = \int_{t_i}^{t_f} \mathrm{d}sK(t,s)f(s) \tag{4.41}$$

is a solution of

$$\partial\phi = f\,. \tag{4.42}$$

This observation will become useful later on when we compare our formalism to the path integral.

Now that we have established the homotopy retract from (V, ∂) to the phase space $(\mathbb{R}^2, 0)$, we will lift the maps to act on the space of functionals $\mathcal{F}(V)$. First we lift the inclusion and projection maps. These are defined by the pullbacks:

$$i^* : \mathcal{F}(V) \to \mathcal{F}(\mathbb{R}^2)\,, \qquad i^*(F) := F \circ i\,. \tag{4.43}$$

$$p^* : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(V)\,, \qquad p^*(f) := f \circ p\,. \tag{4.44}$$

For readability we rename the maps:

$$I \equiv p^*\,, \qquad P \equiv i^*\,. \tag{4.45}$$

The differential on the space of functionals $\mathcal{F}(V)$ is the free, classical piece of the BV differential, which we call Q_0. The homotopy map on this space satisfies

$$Q_0 H(t)[\phi] = t[\phi] - IPt[\phi] \tag{4.46}$$

and

$$HQ_0(t)[\phi^*] = t[\phi^*]\,, \tag{4.47}$$

where $t[\phi]$ is the evaluation functional that takes in a function ϕ and returns $\phi(t)$, i.e. $t[\phi] = \phi(t)$. Consequently, to define the homotopy map on any functional in $\mathcal{F}(V)$, the homotopy map is defined to act on products of $t[\phi]$ and $t[\phi^*]$ as

$$H(FG) = \frac{1}{2}(H(F)G + (-)^F p^* i^*(F) H(G) + (-)^{FG}\{H(G)F + (-)^G p^* i^*(G) H(F)\}. \tag{4.48}$$

This definition of the homotopy map satisfies

$$Q_0 H + HQ_0 = \mathrm{id} - IP \tag{4.49}$$

on products FG, as long as it holds on F and G individually, meaning that the action of H on any functional can be successively reduced to its action on $t[\phi]$ and $t[\phi^*]$ in (4.46) and (4.47). In summary, a homotopy retract from (V, ∂) to $(\mathbb{R}^2, 0)$ gives rise to a homotopy retract from $(\mathcal{F}(V), Q_0)$ to $(\mathcal{F}(\mathbb{R}^2), 0)$.

What we are really after is a homotopy retract from $(\mathcal{F}(V), \delta)$ to $(\mathcal{F}(\mathbb{R}^2), 0)$. We can apply the homological perturbation lemma, with the interaction term $Q_I = \{S_I, -\}$ and the quantum piece $-i\hbar\Delta$ as a perturbation:

$$\delta = Q_0 + \eta\,, \qquad \eta = Q_I - i\hbar\Delta\,. \tag{4.50}$$

The homological perturbation lemma gives us the homotopy retract $(\mathcal{F}(V), \delta, H') \to (\mathcal{F}(\mathbb{R}^2), 0)$ with the new projection and inclusion maps.

$$P' = P \sum_{n \geq 0} (-\eta H)^n\,, \qquad I' = I\,. \tag{4.51}$$

The inclusion map remains unperturbed because of degree reasons, and in the dual picture it implies that p is unperturbed. This means p is untouched by interactions.

With these new projection and inclusion maps, given a functional $F \in \mathcal{F}(V)$, let us define

$$f := P'(F), \qquad F' := I'P'(F)\,. \tag{4.52}$$

The homotopy retract tells us that

$$F - F' = (1 - I'P')(F) = \delta(H'(F))\,. \tag{4.53}$$

This means that F and F' are in the same cohomology class with respect to δ. For a functional $F \in \mathcal{F}(V)$, we can find a member of its δ-cohomology class that can be written as $F' = I'(f) = p^*(f)$ for a function f on phase space $\mathbb{R}^2$. Our claim is that the expectation value of F in (4.31) is equal to the function $f = P'(F)$:

$$\langle T(\hat{F}(\hat{x}))\rangle = \frac{\langle y; t_f | T(\hat{F}(\hat{x})) | x; t_i\rangle}{\langle y; t_f | x; t_i\rangle} = f(x, y)\,. \tag{4.54}$$

We can apply this method for more general boundary conditions and this will be discussed in 4.2.4.

4.1.1 Uniqueness of the Expectation Value

Let us prove that the expectation value that we have computed in our approach is unique. Concretely, we would like to prove that there is a unique function $f \in \mathcal{F}(\mathbb{R}^2)$ such that there is a functional F' in the same cohomology class of F such that $F' = p^*(f)$. The proof can be outlined in the following way:

1. Establish that the cohomology of the complex with the classical piece of the BV differential $H(\mathcal{F}(V), Q)$ where $Q = Q_0 + Q_I$ is isomorphic to the cohomology of the complex with the full quantum interacting BV differential $H(\mathcal{F}(V), \delta)$ where $\delta = Q - i\hbar\Delta$.

2. Show that the cohomology $H(\mathcal{F}(V), Q)$ is isomorphic to the space of functionals of solutions which satisfy $\mathrm{EL}(\phi) = 0$.

3. Show that $p^* : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(V)$ is a one-to-one map when the image is restricted to functionals on solutions.

Then p^* is a one-to-one map between the space of functions $\mathcal{F}(\mathbb{R}^2)$ and the space of functionals of solutions which is isomorphic to the cohomology $H(\mathcal{F}(V), Q)$. Thus there is a one-to-one map between $\mathcal{F}(\mathbb{R}^2)$ and the cohomology $H(\mathcal{F}(V), Q)$. Let us now describe each of these steps in detail.

Step 1. We recall the homological perturbation lemma discussed in 1.3.2. The homological perturbation lemma ensures that given the original equivalence data $(\mathcal{F}(V), Q_0), H \to (\mathcal{F}(\mathbb{R}^2), 0)$, the perturbed data $(\mathcal{F}(V), \delta), H' \to (\mathcal{F}(\mathbb{R}^2), 0)$ is a homotopy equivalence data as defined in (1.68). Since the perturbed projections and inclusions are quasi-isomorphisms (just like the unperturbed ones), by definition they induce isomorphisms on cohomology. Therefore the cohomologies of the original and perturbed complexes must be isomorphic $H(\mathcal{F}(V), Q) \simeq H(\mathcal{F}(V), \delta)$.

Step 2. First, let us clarify what we mean by the space of functionals of solutions. Let us define the space of solutions, $\mathcal{E} \subseteq V$ defined as

$$\mathcal{E} = \{\phi \in V^0 | \mathrm{EL}(\phi(t)) = 0\}\,. \tag{4.55}$$

The functionals that take in the solutions as an input belong to the space $\mathcal{F}(\mathcal{E})$. This is defined in the following way. Since $\mathcal{E}$ is a subspace of V^0, any functional on V^0 can be restricted to a functional on $\mathcal{E}$, defining the projection,

$$r : \mathcal{F}(V^0) \to \mathcal{F}(\mathcal{E}) \,, \quad F \mapsto F|_{\mathcal{E}} \,. \tag{4.56}$$

We assume that any functional in $\mathcal{F}(\mathcal{E})$ can be obtained via r and that the map r has a right-inverse, so that any functional on $\mathcal{E}$ extends to a functional on V^0. In general, r is not a bijection since it may have a non-trivial kernel, but upon modding out the kernel we have an isomorphism:

$$\frac{\mathcal{F}(V^0)}{\ker r} \simeq \mathcal{F}(\mathcal{E}) \,. \tag{4.57}$$

Next, we show that the cohomology of $(\mathcal{F}(V), Q)$ in degree 0 is equal to the space $\mathcal{F}(\mathcal{E})$ and that the cohomology in degree -1 vanishes. We start with the functionals of degree 0. Since functionals of degree 0 contain no anti-fields, by recalling the definition of Q in (4.28) it follows that they are annihilated by Q. Therefore, $\ker Q_0 = \mathcal{F}(V^0)$ when restricted to degree 0 (here the subscript denotes the degree, i.e. $Q_i : \mathcal{F}(V)^i \to \mathcal{F}(V)^{i+1}$, and should not be confused with the free part of Q).

In degree 0, the image of Q consists of functionals of ϕ proportional to the equation of motion. To see this, consider a functional of degree -1,

$$G_{-1} = \int_{t_i}^{t_f} \mathrm{d}s \phi^*(s) g_{-1}[\phi, s] \,, \tag{4.58}$$

where $g_{-1}[\phi, s] = \int \mathrm{d}t_1 \cdots \mathrm{d}t_k f_{-1}(t_1, ..., t_k, s)\phi(t_1) \cdots \phi(t_k)$. By applying Q in (4.28) on G_{-1}, we obtain

$$Q(G_{-1}) = -\int_{t_i}^{t_f} \mathrm{d}t \, \mathrm{EL}(\phi(t)) g_{-1}[\phi, t] \,. \tag{4.59}$$

When restricting to $\mathcal{E}$, the above expression vanishes. Assuming that any functional in the kernel of r is of the form (4.59) (see appendix B in [71] for a discussion on this assumption), for a suitable function g_{-1}, the image of Q_{-1} is equal to the kernel of r, $\operatorname{im} Q_{-1} = \ker r$. Therefore, the cohomology in degree 0 is

$$H^0(Q) = \frac{\ker Q_0}{\operatorname{im} Q_{-1}} = \frac{\mathcal{F}(V^0)}{\ker r} \simeq \mathcal{F}(\mathcal{E}) \tag{4.60}$$

where we have used (4.57).

For the computation of the cohomology in degree -1, the kernel of Q consists of the functionals which satisfy $Q(G_{-1}) = 0$. We observe that the RHS of (4.59) has the form of the variation of the action. Setting the variation to zero, we can interpret the action to be invariant under the transformation $\delta\phi(t) \equiv g_{-1}[\phi, t]$. Computing Q acting on functionals of degree -2, which have the form

$$G_{-2} = \int \mathrm{d}s_1 \mathrm{d}s_2 \phi^*(s_1)\phi^*(s_2) g_{-2}[\phi, s_1, s_2] \tag{4.61}$$

$Q(G_{-2})$ yields functionals of the form G_1 above where

$$g_{-1}[\phi, t] = \int_{t_i}^{t_f} \mathrm{d}s\, \mathrm{EL}(\phi(s)) g_{-2}[\phi, s, t]\,, \quad g_{-2}[\phi, s, t] = -g_{-2}[\phi, t, s]\,. \tag{4.62}$$

Under the gauge symmetries interpretation, these functionals g_{-2} correspond to the trivial gauge symmetries. Therefore the cohomology $\ker Q_{-1}/\operatorname{im} Q_{-2}$ contain the non-trivial gauge transformations. Since we are only considering models without gauge symmetry, the cohomology in degree -1 vanishes. This argument can be applied to cohomologies in arbitrary negative degree. Therefore, the cohomology of $(\mathcal{F}(V), Q)$ is given by the cohomology in degree zero (4.60). This concludes the proof that the cohomology of the complex $(\mathcal{F}(V), Q)$ is equal to the space of functionals of on-shell functions $\mathcal{F}(\mathcal{E})$.

Step 3. Now we must prove that p^* is a one-to-one map when the image is restricted to $\mathcal{F}(\mathcal{E})$. p^* is defined in (4.44) as $p^*(f) = f \circ p$, so we must show that p in (4.34) is invertible on the space of solutions $\mathcal{E}$. This will imply that the pullback p^* is invertible on $\mathcal{F}(\mathcal{E})$.

The projection p to the boundary conditions evaluates a solution at the boundaries t_i and t_f:

$$p : \mathcal{E} \to \mathbb{R}^2\,, \quad p(\phi_p) := (\phi_p(t_i), \phi_p(t_f))\,. \tag{4.63}$$

The inclusion i to the space of solutions is

$$i : \mathbb{R}^2 \to \mathcal{E}\,, \quad i(x, y) := \phi_{x,y} \tag{4.64}$$

where $\phi_{x,y}$ is a solution, $\mathrm{EL}(\phi_{x,y}) = 0$, with the boundary values

$$\phi_{x,y}(t_i) = x\,, \quad \text{and} \quad \phi_{x,y}(t_f) = y\,. \tag{4.65}$$

It is easy to show that $p \circ i = \mathrm{id}$ and $i \circ p = \mathrm{id}$. First,

$$p \circ i(x,y) = p(\phi_{x,y}) = (\phi_{x,y}(t_i), \phi_{x,y}(t_f)) = (x,y)\,, \tag{4.66}$$

where the last equality follows from (4.65). To show that $i \circ p = \mathrm{id}$,

$$i \circ p(\phi_p) = i(\phi_p(t_i), \phi_p(t_f)) = \phi_{\phi_p(t_i),\phi_p(t_f)} = \phi_p\,, \tag{4.67}$$

since $\phi_{\phi_p(t_i),\phi_p(t_f)}$ is indeed the solution with the boundary values $\phi_{\phi_p(t_i),\phi_p(t_f)}(t_i) = \phi_p(t_i)$ and $\phi_{\phi_p(t_i),\phi_p(t_f)}(t_f) = \phi_p(t_f)$. Therefore p is a one-to-one map between the space of solutions $\mathcal{E}$ and $\mathbb{R}^2$.

From the identities $p \circ i = \mathrm{id}$ and $i \circ p = \mathrm{id}$ and the definitions (4.44) and (4.43), it follows that $p^* i^* = \mathrm{id}$ and $i^* p^* = \mathrm{id}$. Therefore, p^* is a one-to-one map when the image is restricted to $\mathcal{F}(\mathcal{E})$.

4.1.2 Homotopy Retract for Harmonic Oscillator

Here we give an example of the homotopy retract for the harmonic oscillator on which we will apply our methods and compare our results with the standard formulation. For the harmonic oscillator, the differential $\partial : V^0 \to V^1$ acts as

$$\partial\phi = \ddot{\phi} + \omega^2\phi\,. \tag{4.68}$$

We consider the initial conditions for solutions given by $(\phi(t_i), \dot{\phi}(t_i))$, which defines the projection from V to $\mathbb{R}^2$.

$$p : V \to \mathbb{R}^2\,, \qquad p(\phi) = (\phi(t_i), \dot{\phi}(t_i))\,. \tag{4.69}$$

The inclusion map takes a pair of boundary conditions in $\mathbb{R}^2$ and maps it into a solution in V^0 written as

$$i(q,p) = q\cos\omega(t-t_i) + \frac{p}{\omega}\sin\omega(t-t_i)\,. \tag{4.70}$$

For the boundary conditions given by p in (4.69), the homotopy map $h : V^1 \to V^0$ is

$$h(f)(t) = \int_{t_i}^{t_f} \mathrm{d}s K(t,s) f(s)\,, \tag{4.71}$$

where

$$K(t,s) := \theta(t-s)\frac{\sin\omega(t-s)}{\omega}\,, \tag{4.72}$$

where θ is the step function. The homotopy map can be extended to the full complex V by defining $h(\phi) = 0$ for $\phi \in V^0$.

Let us comment that the homotopy depends on the boundary conditions that are chosen for the projection. For instance, one can also choose the projection corresponding to Dirichlet boundary conditions,

$$p(\phi) = (\phi(t_i), \phi(t_f)), \tag{4.73}$$

and the corresponding inclusion is

$$i(x_i, x_f) = x_i \frac{\sin\omega(t_f - t)}{\sin\omega(t_f - t_i)} + x_f \frac{\sin\omega(t - t_i)}{\sin\omega(t_f - t_i)}. \tag{4.74}$$

The Green's function is then

$$K_{DD}(t, s) = \theta(t - s)\frac{\sin\omega(t - s)}{\omega} - \frac{\sin\omega(t - t_i)}{\sin\omega(t_f - t_i)} \frac{\sin\omega(t_f - s)}{\omega}, \tag{4.75}$$

which yields the homotopy map

$$h(f) = \int_{t_i}^{t} \mathrm{d}s\, f(s)\frac{\sin\omega(t - s)}{\omega} - \frac{\sin\omega(t - t_i)}{\sin\omega(t_f - t_i)} \int_{t_i}^{t_f} \mathrm{d}s\, f(s)\frac{\sin\omega(t_f - s)}{\omega}. \tag{4.76}$$

The Green's function satisfies the boundary conditions,

$$K_{DD}(t_i, s) = K_{DD}(t_f, s) = 0. \tag{4.77}$$

These ensure that there is a strong deformation retract (1.75).

We lift the homotopy retract from the space of functions to the space of functionals $(\mathcal{F}(V), Q_0)$, where

$$Q_0 = \int_{t_i}^{t_f} \mathrm{d}t(\ddot{\phi}(t) + \omega^2\phi(t))\frac{\delta}{\delta\phi^*(t)} \tag{4.78}$$

The projection and inclusion maps between the chain complex $(\mathcal{F}(V), Q_0)$ and the chain complex of functions $(\mathbb{R}^2, 0)$ as prescribed in (4.43) and (4.44) are

$$P(F)(q, p) = F[i(q, p), 0], \tag{4.79}$$

$$I(f)[\phi, \phi^*] = f(p(\phi)) = f(\phi(t_i), \dot{\phi}(t_i)), \tag{4.80}$$

where i is defined in (4.70). The homotopy map is defined on the functionals as

$$\begin{aligned} H(t)[\phi] &:= \int_{t_i}^{t} \mathrm{d}s \frac{\sin\omega(t-s)}{\omega}\phi^*(s) \equiv \int_{t_i}^{t_f} \mathrm{d}s K(t,s)\phi^*(s)\,, \\ H(t)[\phi^*] &:= 0\,, \end{aligned} \tag{4.81}$$

and is extended to the whole space $\mathcal{F}(V)$ by its action on products (4.48).

4.1.3 Perturbation Lemma

We would like to show that by using $P' : \mathcal{F}(V) \to \mathcal{F}(\mathbb{R}^2)$, $f = P'(F)$ computes the expectation value and compare our formalism to the path integral approach. We will apply the homological perturbation lemma in two steps, first considering the quantum part of the perturbation and then the interacting part, using the finite-dimensional cases in [66, 72, 73] as guides. We will see that the quantum perturbation will give rise to what are equivalent to Wick contractions, and by adding the interacting part we will reproduce the full path integral for the quantum interacting theory.

Without taking interactions into account, the perturbation to Q_0 is $\eta = -i\hbar\Delta$. There will neither be a induced differential on $\mathbb{R}^2$, nor a perturbation to the inclusion I, however the perturbation to the projection is

$$P_1 = P\sum_{n\geq 0}(i\hbar\Delta H)^n\,, \tag{4.82}$$

where we denote the perturbed projector by P_1 since later we will consider a second perturbation to be denoted P_2. In order to compare with the standard formulation, we would like to write out explicitly how this expression acts on $\mathcal{F}(V)$. Though we could in principle use the definition of H in (4.48), it turns out that we can obtain a more user-friendly version by first decomposing the space $\mathcal{F}(V)$ into on-shell and off-shell functionals.

Let us recall that the maps i and p defined in (4.33) and (4.34) relate the space of fields V to the phase space $\mathbb{R}^2$. We decompose the space of fields,

$$V = ip(V) \oplus (\mathrm{id} - ip)(V) =: V_p \oplus V_u\,. \tag{4.83}$$

where V_p is the "physical" space of on-shell fields which project to solutions to the equations of motion, and V_u is the "unphysical" space of off-shell fields. Any field can be written as $\phi = \phi_p + \phi_u$, where ϕ_p is a solution to the equations of motion. Applying the homotopy relation on degree zero elements in (4.38), one can see that ϕ_u depends on the choice of homotopy. Moreover, it satisfies the same boundary conditions as the Green's function used to construct the homotopy, e.g. for Dirichlet boundary conditions (4.77), $\phi_u(t_i) = \phi_u(t_f) = 0$. On the dual space of functionals, (4.83) induces the decomposition

$$\mathcal{F}(V) =: \mathcal{F}_p \oplus \mathcal{F}_u\,, \tag{4.84}$$

where $\mathcal{F}_p = \mathcal{F}(V_p)$ is the space of functionals depending on ϕ_p only and $\mathcal{F}_u$ is the space of functionals that contain at least one ϕ_u or at least one ϕ^*.

Given our decomposition, we also need to define how derivatives act on the functionals $F[\phi, \phi^*]$. In other words, we would like to decompose the functional derivative $\frac{\delta}{\delta\phi(t)}$ into the functional derivative in the direction of V_p and V_u (the functional derivative w.r.t. ϕ^* remains the same because we do not decompose the anti-fields). The functional derivative $\frac{\delta}{\delta\phi(t)}$ is defined as

$$\int \mathrm{d}t g(t) \frac{\delta F[\phi, \phi^*]}{\delta\phi(t)} := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} F[\phi + \epsilon g, \phi^*] \tag{4.85}$$

where $g \in V$. Since V_p is isomorphic to $\mathbb{R}^2$, let us label the on-shell fields as $\phi_{p;x,y}$ with $(x, y) \in \mathbb{R}^2$. Then the functional derivative in the direction of V_p is simply given by partial derivatives along directions in the phase space:

$$(\partial_x F)[\phi, \phi_*] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F[\phi + \phi_{p;x+\varepsilon,y}, \phi_*]\,, \quad (\partial_y F)[\phi] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F[\phi + \phi_{p;x,y+\varepsilon}, \phi_*]\,. \tag{4.86}$$

The functional derivative in the direction of V_u is written as

$$\int \mathrm{d}t\, g_u(t) \frac{\delta F[\phi, \phi^*]}{\delta\phi_u(t)} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F[\phi + \varepsilon g_u, \phi^*]\,, \tag{4.87}$$

where $g_u \in V_u$. We are ready to decompose $\frac{\delta}{\delta\phi(t)}$ into $\partial_x, \partial_y, \frac{\delta}{\delta\phi_u(t)}$. Let us briefly neglect the ϕ^* dependence and write $F[\phi]$ as a functional $F[\phi_u, x, y]$, where ϕ_u, x, y are seen as functions of ϕ:

$$F[\phi] = F[\phi_u(\phi), x(\phi), y(\phi)]\,. \tag{4.88}$$

ϕ_u takes the function ϕ and subtracts the part which is projected to V_p, and x and y map ϕ to its projected part in V_p and maps the result to the coordinate x and

y in the phase space $\mathbb{R}^2$. The functional derivative can then be expressed by the chain rule,

$$\frac{\delta}{\delta\phi(t)} = \int \mathrm{d}s\, \frac{\delta\phi_u(s)}{\delta\phi(t)} \frac{\delta}{\delta\phi_u(s)} + \frac{\delta x}{\delta\phi(t)} \frac{\partial}{\partial x} + \frac{\delta y}{\delta\phi(t)} \frac{\partial}{\partial y} . \tag{4.89}$$

To define how the homotopy acts on the functionals, we first define

$$V_h := \int \mathrm{d}t\, \mathrm{d}s\, \phi^*(t) K(t,s) \frac{\delta}{\delta\phi_u(s)} , \tag{4.90}$$

where K is a Green's function for the harmonic oscillator. We will see that, up to a rescaling, this implements the homotopy action on functionals in $\mathcal{F}_u$. We recall the differential Q_0 corresponding to the free theory,

$$Q_0 = \int \mathrm{d}t\, \left(\ddot{\phi}_u(t) + \omega^2\phi_u(t)\right) \frac{\delta}{\delta\phi^*(t)} , \tag{4.91}$$

where we used $\phi = \phi_u + \phi_p$ and that ϕ_p satisfies the equations of motion and thus drops out. We can compute the anticommutator of Q_0 and V_h,

$$\{Q_0, V_h\} = \int \mathrm{d}t\, \left\{\phi_u(t) \frac{\delta}{\delta\phi_u(t)} + \phi^*(t) \frac{\delta}{\delta\phi^*(t)}\right\} =: N , \tag{4.92}$$

which counts the total number of fields ϕ_u and anti-fields ϕ^* in the functional:

$$N(\phi_u(t_1)\cdots\phi_u(t_k)\phi^*(t_{k+1})\cdots\phi^*(t_n)) = n\phi_u(t_1)\cdots\phi_u(t_k)\phi^*(t_{k+1})\cdots\phi^*(t_n) . \tag{4.93}$$

We claim that on the space $\mathcal{F}_u$ of functionals that are at least linear in ϕ_u or ϕ^*, on which N is a positive operator, the homotopy map is implemented by

$$H_u := N^{-1} V_h , \tag{4.94}$$

where we have identified N with its eigenvalue (which is always positive on $\mathcal{F}_u$, so N^{-1} is well-defined). Indeed, the homotopy relation is then satisfied:

$$\{Q_0, H_u\} = N^{-1}\{Q_0, V_h\} = N^{-1}N = \mathrm{id} , \tag{4.95}$$

recalling that the subspace $\mathcal{F}_u$ is projected to 0. Finally, we can extend H_u to a homotopy H on the total space $\mathcal{F}(V) = \mathcal{F}_u \oplus \mathcal{F}_p$ by declaring H to be zero on

$\mathcal{F}_p \subseteq \mathcal{F}(V)$. We then have $\{Q_0, H\} = \mathrm{id} - IP$, which defines a strong deformation retract from $\mathcal{F}(V)$ to $\mathcal{F}_p$.

Having defined a more convenient lift of the homotopy map H we can now work out how to apply (4.82) on functionals. Since P_1 is of degree zero, it is zero on functionals with at least one anti-field. Thus we consider functionals of fields only, and with the definition of the Laplacian Δ in (4.28) and H in (4.94), ΔH acts as

$$\Delta H = \int \mathrm{d}t\, \mathrm{d}s\, K(t,s) \frac{\delta^2}{\delta\phi(t)\delta\phi_u(s)} \frac{1}{N} . \tag{4.96}$$

Since the action of $\frac{\delta}{\delta\phi(t)}$ reduces to $\frac{\delta}{\delta\phi_u(t)}$ when it is integrated against a function satisfying the boundary conditions of V_u, [3] as does $K(t,s)$, we can define

$$C := \int \mathrm{d}t\, \mathrm{d}s\, K(t,s) \frac{\delta^2}{\delta\phi_u(t)\delta\phi_u(s)} , \tag{4.97}$$

and rewrite (4.96) as

$$\Delta H = C \frac{1}{N} . \tag{4.98}$$

Let us now recall our decomposition of the functionals $F[\phi_u, x, y]$. Any functional in $\mathcal{F}(V)$ can be written as a superposition of functionals of the form

$$F[\phi_u, x, y] = \phi_u(t_1) \cdots \phi_u(t_m) f(x,y) , \tag{4.99}$$

for fixed $t_1, ..., t_m$ and for $f(x,y)$ polynomial in x and y. Thus the effect of P_1 in (4.82) on all functionals can be deduced from its effect on F by linearity. Since $P(\phi_u(t)) = 0$, the only non-zero contribution to P_1 in (4.82) comes from the term where $\sum_{n\geq 0}(i\hbar\Delta H)^n$ eliminates all fields ϕ_u. Recalling the action of N in (4.93), this can only happen when m is even, i.e., $m = 2k$, for which only the term with $n = k$ contributes. We then find with (4.98)

$$\begin{aligned} P_1(F) &= f(x,y)(i\hbar C)\frac{1}{2} \cdots (i\hbar C)\frac{1}{2k-2}(i\hbar C)\frac{1}{2k}\, \phi_u(t_1)\cdots\phi_u(t_{2k}) \\ &= f(x,y)\frac{1}{k!}\left(\frac{i\hbar}{2}C\right)^k \phi_u(t_1)\cdots\phi_u(t_{2k}) . \end{aligned} \tag{4.100}$$

[3] This follows from setting $g(t)$ in (4.85) to $g_u(t)$ and comparing the expression with the definition in (4.87).

This implies that on arbitrary functionals we have

$$P_1 = P \exp\left(\frac{i\hbar}{2} C\right) . \tag{4.101}$$

We can identify the operator C with the operator ∂_P in lemma 3.4.1 in [72] that generates Wick contractions. P can be interpreted as a normal ordering operation and P_1 implements Wick's theorem. In 4.2.2 we will compute the two-point function with respect to coherent states of the harmonic oscillator, where we will reconfirm that P_1 indeed results in Wick contractions.

We have seen that the quantum perturbation to the differential Q_0 gave us the Wick contractions. Now we will apply the perturbation lemma a second time to combine our result for P_1 with the interacting piece Q_I and compute the perturbed projection:

$$P_2 = P_1 \circ \sum_{n \geq 0} (-Q_I H)^n . \tag{4.102}$$

We would like to reproduce the full interacting path integral as defined in theorem 3 of [73] as

$$\tilde{P}_2(F) = \frac{P_1\left(F \exp\left(\frac{i}{\hbar} S_I\right)\right)}{Z} , \tag{4.103}$$

with normalization

$$Z = P_1 \exp\left(\frac{i}{\hbar} S_I\right) , \tag{4.104}$$

by following the proof in [73] for our model. The proof consists of three steps:

1. Show that
$$\tilde{P}_2 \circ I = \mathrm{id} . \tag{4.105}$$

2. Show that
$$\tilde{P}_2 - P_2 = \tilde{P}_2 H_2 \delta + \tilde{P}_2 \delta H_2 . \tag{4.106}$$

3. Show that the two terms on the RHS of (4.106) vanish.

Now let us navigate through each of these steps.

Proof of step 1. First, consider the numerator of (4.103) acting on an element $F = I(f)$, where $f \in \mathcal{F}(\mathbb{R}^2)$. Recalling the properties of a strong deformation retract (1.75), $HI = 0$. Applying the expression for P_1 in (4.82) on $I(f)\exp(iS_I/\hbar)$,

$$\begin{aligned}\tilde{P}_2(I(f))Z &= P_1\Big(I(f)e^{i\frac{S_I}{\hbar}}\Big) = P\Big(\sum_{n\geq 0}(i\hbar\Delta H)^n\Big(I(f)e^{i\frac{S_I}{\hbar}}\Big)\Big) \\ &= P\Big(I(f)\sum_{n\geq 0}(i\hbar\Delta H)^n e^{i\frac{S_I}{\hbar}}\Big) = PI(f)P\Big(\sum_{n\geq 0}(i\hbar\Delta H)^n e^{i\frac{S_I}{\hbar}}\Big) = PI(f)Z \\ &= fZ\,.\end{aligned} \tag{4.107}$$

For the third equality, since $HI = 0$ the action of the sum is $\sum_{n\geq 0}(i\hbar\Delta H)^n I(f) = I(f)$. Then for the fourth equality we used the fact that P is an algebra morphism, i.e. $P(FG) = P(F)P(G)$. Last we used the property $PI = \mathrm{id}$. Thus $\tilde{P}_2 \circ I = \mathrm{id}$.

Proof of step 2. We use the homotopy retract relation for P_2,

$$\mathrm{id} - I \circ P_2 = H_2 \circ \delta + \delta \circ H_2\,, \tag{4.108}$$

where $\delta = Q_0 + Q_I - i\hbar\Delta$ and $H_2 = H\sum_{n\geq 0}(i\hbar\Delta - Q_I)^n$. We apply $\tilde{P}_2$ to both sides of (4.108) and use the identity from step 1 (4.105). Then,

$$\tilde{P}_2 - P_2 = \tilde{P}_2 H_2 \delta + \tilde{P}_2 \delta H_2\,. \tag{4.109}$$

Proof of step 3. Since $\tilde{P}_2$ is non-zero only in degree zero, it follows that the first term on the RHS of (4.106) vanishes: $\tilde{P}_2 H_2 \delta = 0$. For the second term, we show that $Z\tilde{P}_2\delta = 0$, and with the invertibility of Z, $\tilde{P}_2\delta = 0$. Let us decompose the BV differential as $\delta = \delta_0 + Q_I$, where $\delta_0 = Q_0 - i\hbar\Delta$. We assume that S_I does not contain derivatives. We then write $Q_I = \{S_I, \cdot\}$ due to the absence of boundary terms. For a generic functional F, we compute

$$\begin{aligned}\delta_0\big(e^{\frac{i}{\hbar}S_I}F\big) &= \delta_0\big(e^{\frac{i}{\hbar}S_I}\big)F + e^{\frac{i}{\hbar}S_I}\delta_0 F - i\hbar\big\{e^{\frac{i}{\hbar}S_I}, F\big\} \\ &= e^{\frac{i}{\hbar}S_I}\delta_0 F + e^{\frac{i}{\hbar}S_I}\{S_I, F\} \\ &= e^{\frac{i}{\hbar}S_I}\delta F\,.\end{aligned} \tag{4.110}$$

Here we used that δ acts as a derivation of the product,

$$-i\hbar(-1)^F\{F, G\} = \delta(FG) - \delta F G - (-1)^F F\delta G\,. \tag{4.111}$$

The first term in the first line of the RHS of (4.110) vanishes because $\delta_0\big(e^{\frac{i}{\hbar}S_I}\big) = 0$, which follows because both S_0 and S_I contain no anti-fields. For the third term in the first line on the RHS we used $\{e^X, F\} = e^X\{X, F\}$ for any degree-zero object X, which follows from the graded Leibniz rule (1.81). In the last line we reconstructed δ from the decomposition $\delta = \delta_0 + \{S_I, \cdot\}$. Acting $Z\tilde{P}_2$ on δF, we have

$$Z\tilde{P}_2(\delta F) = P_1\big(\delta F e^{\frac{i}{\hbar}S_I}\big) = P_1\delta_0\big(e^{\frac{i}{\hbar}S_I}F\big) = 0\,, \tag{4.112}$$

where we used (4.110) in the second equality and for the last equality $P_1\delta_0 = 0$, i.e. that P_1 is a chain map with respect to $Q_0 - i\hbar\Delta$, as implied by the perturbation lemma. This concludes the proof of $P_2 = \tilde{P}_2$.

With this last proof, we have been able to reproduce the expression for the full path integral for the interacting theory. The quantum perturbation to the projection resulted in Wick contractions for the free theory, as one would expect. The non-linear interacting perturbation to the projection computes expectation values with respect to the free theory with functionals F weighted by the interacting part $e^{\frac{i}{\hbar}S_I}$. This is what we expect– this gives rise to the computation of Feynman diagrams.

4.1.4 Path Integral

In the previous proof, we showed that the projection we have defined P_2 is equal to the expression $\tilde{P}_2$ in (4.103) which was defined in [73], but without explicitly showing how (4.103) is equal to the path integral. To close this section, we motivate the expression by starting from the path integral and reorganizing it to compare with what P_2 computes:

$$\frac{\langle y; t_f|\, T(F[\phi])\, |x; t_i\rangle}{\langle y; t_f|x; t_i\rangle} = \frac{\int_{\phi(t_i)=x}^{\phi(t_f)=y} \mathcal{D}\phi\, F[\phi]\, e^{\frac{i}{\hbar}S[\phi]}}{\int_{\phi(t_i)=x}^{\phi(t_f)=y} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}} = \frac{P_1(Fe^{\frac{i}{\hbar}S_I})}{P_1(e^{\frac{i}{\hbar}S_I})} \equiv P_2(F)\,. \tag{4.113}$$

We first split the action as $S = S_0 + S_I$, where $S_0 = \int \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2$ and S_I is the non-linear piece. Instead of integrating over functions with boundary conditions (x, y), we can choose to integrate over functions with simpler boundary conditions $(0, 0)$, and add the contribution coming from the solution with the generic boundary conditions. Concretely, let us decompose the functions $\phi = \phi_u + \phi_p$ and choose the reference boundary conditions,

$$\phi_u(t_i) = \phi_u(t_f) = 0\,. \tag{4.114}$$

ϕ_p is the unique classical solution $\phi_p = i(x(t_i), y(t_f))$ with generic boundary conditions

$$\phi_p(t_i) = x, \quad \phi_p(t_f) = y\,, \tag{4.115}$$

Assuming that the integral measure is invariant under constant shifts, substituting $\phi = \phi_u + \phi_p$ yields $\mathcal{D}\phi = \mathcal{D}\phi_u$. We compute

$$\langle y; t_f|\, F[\phi]\, |x; t_i\rangle = \int_{\phi_u(t_i)=0}^{\phi_u(t_f)=0} \mathcal{D}\phi_u\, F[\phi_u + \phi_p]\, e^{\frac{i}{\hbar}S_I[\phi_u+\phi_p]} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \frac{1}{2}(\dot\phi_u^2 - \omega^2\phi_u^2)} e^{i\partial S(\phi_p)}\,, \tag{4.116}$$

where

$$\partial S = \frac{1}{2}\phi_p(t_f)\dot\phi_p(t_f) - \frac{1}{2}\phi_p(t_i)\dot\phi_p(t_i)\,. \tag{4.117}$$

Because we chose the boundary conditions (4.114), ∂S does not depend on ϕ_u and the phase $e^{i\partial S(\phi_p)}$ can be scaled out of the path integral:

$$\langle y; t_f|\, F[\phi]\, |x; t_i\rangle = e^{i\partial S(\phi_p)} \int_{\phi_u(t_i)=0}^{\phi_u(t_f)=0} \mathcal{D}\phi_u\, F[\phi_u + \phi_p]\, e^{\frac{i}{\hbar}S_I[\phi_u+\phi_p]} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \frac{1}{2}(\dot\phi_u^2 - \omega^2\phi_u^2)}\,. \tag{4.118}$$

The expression for the normalization factor is obtained with $F = 1$:

$$\langle y; t_f|x; t_i\rangle = e^{i\partial S(\phi_p)} \int_{\phi_u(t_i)=0}^{\phi_u(t_f)=0} \mathcal{D}\phi_u\, e^{\frac{i}{\hbar}S_I[\phi_u+\phi_p]} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \frac{1}{2}(\dot\phi_u^2 - \omega^2\phi_u^2)}\,. \tag{4.119}$$

We claim that the projection P_1 in (4.101) is equivalent to the integral

$$P_1(F) = \int_{\phi_u(t_i)=0}^{\phi_u(t_f)=0} \mathcal{D}\phi_u\, F[\phi_u + \phi_p]\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \frac{1}{2}(\dot\phi_u^2 - \omega^2\phi_u^2)}\,. \tag{4.120}$$

In the standard formulation, the integral on the RHS of (4.120) can be computed by using Wick contractions. Since we have shown that P_1 is the operation which generates Wick contractions, as long as we use the appropriate Green's function, $P_1(F)$ computes this integral. The result is a function on phase space $\mathbb{R}^2$ parametrized by the boundary conditions imposed on ϕ_p. If we accept the above claim, then it is easy to see that (4.118) and (4.119) can be written in terms of P_1, and consequently

$$\frac{\int_{\phi(t_i)=x}^{\phi(t_f)=y} \mathcal{D}\phi\, F[\phi]\, e^{\frac{i}{\hbar}S[\phi]}}{\int_{\phi(t_i)=x}^{\phi(t_f)=y} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}} = \frac{P_1(F e^{\frac{i}{\hbar}S_I})}{P_1(e^{\frac{i}{\hbar}S_I})}\,, \tag{4.121}$$

where the phases $e^{i\partial S(\phi_p)}$ in the numerator and denominator have cancelled each other out. This concludes the comparison of the homological formulation with the path integral.

4.2 Harmonic oscillator

In this section we apply the homological formulation to the one-dimensional harmonic oscillator. In particular we compute two-point functions with respect to position eigenstates and with respect to coherent states. Since for a free theory all expectation values can be computed in terms of the two-point function using Wick contractions, which can also be generated using our formulation, we only show the computation for two-point functions. We then check our results using the canonical formulation of quantum mechanics. Much of this section has been adapted from [19].

4.2.1 Homological Computation for Position Eigenstates

The BV differential for the quantum harmonic oscillator reads

$$\delta = \int_{t_i}^{t_f} \mathrm{d}t \left[\left(\ddot{\phi}(t) + \omega^2 \phi(t)\right) \frac{\delta}{\delta\phi^*(t)} + i\hbar \frac{\delta^2}{\delta\phi^*(t)\delta\phi(t)} \right] . \tag{4.122}$$

We want to compute the two-point function, meaning that we would like to compute the expectation value of the functional for fixed $t, s \in \mathbb{R}$,

$$F[\phi, \phi^*] = \phi(t)\phi(s) . \tag{4.123}$$

In order for $F' = f \circ p$ to be in the same cohomology as F there should be a G such that $F - F' = \delta G$, where G has degree minus one. Let us assume Dirichlet boundary conditions. The perturbation lemma immediately gives us the function f as follows:

$$f = P_1(F) , \qquad P_1 = i^* \exp\left(-\frac{i\hbar}{2} C\right) , \tag{4.124}$$

where C is (4.97) defined in terms of the Green's function with Dirichlet boundary conditions,

$$C = \int \mathrm{d}t\, \mathrm{d}s\, K_{DD}(t,s) \frac{\delta^2}{\delta\phi(t)\delta\phi(s)} , \tag{4.125}$$

where K_{DD} is the Green's function defined in (4.75). Note that here $P_2 = P_1$ since we are considering the free theory. We thus have

$$f = i^* \left(1 - \frac{i\hbar}{2} C\right) \phi(t)\phi(s) \,, \tag{4.126}$$

using that the higher-order terms in $\hbar$ vanish when acting on the functional (4.123) with two ϕ. For the second term on the right-hand side we need to use (4.125) to compute

$$C\big(\phi(t)\phi(s)\big) = 2\, K_{DD}(t,s) \,, \tag{4.127}$$

for which one uses that K_{DD} is symmetric. To evaluate then $f(x,y)$ the first term in (4.126) maps (x,y) via the inclusion map (4.74) to a solution ϕ_p with boundary conditions $\phi_p(t_i) = x$ and $\phi_p(t_f) = y$ and then evaluates the functional on this solution. Doing so yields

$$f(x,y) = \prod_{r=t,s} \left\{ \frac{\sin\omega(r-t_i)}{\sin\omega(t_f-t_i)} y + \frac{\sin\omega(t_f-r)}{\sin\omega(t_f-t_i)} x \right\} - i\hbar K_{DD}(t,s) \,. \tag{4.128}$$

Thus we have computed the normalized two-point function,

$$\frac{\langle y|T(F)|x\rangle}{\langle y|x\rangle} = f(x,y) \,. \tag{4.129}$$

4.2.2 Homological Computation for Coherent States

Here we will perform the computation of the two-point function with respect to coherent states. In the language of the canonical formulation, coherent states are the eigenstates $|z\rangle$ of the annihilation operator a. Let us use the convention

$$a\,|z\rangle = z\,|z\rangle \,, \quad \langle z|\, a^\dagger = \langle z|\, z \,, \quad \text{for all } z \in \mathbb{C} \,. \tag{4.130}$$

With this convention, $\langle \bar{z}|$ is the hermitian conjugate of $|z\rangle$, where $\bar{z}$ denotes the complex conjugate of z. For this computation we would like to use the Feynman propagator. This means that we want to derive an inclusion i_F and projection p_F, such that $\{\partial, h_f\} = \mathrm{id} - i_F \circ p_F$, with the homotopy map being the Feynman propagator

$$h_F(f)(t) = i \int_{t_i}^{t} \mathrm{d}s f(s) \frac{e^{-i\omega(t-s)}}{2\omega} + i \int_{t}^{t_f} \mathrm{d}s f(s) \frac{e^{i\omega(t-s)}}{2\omega} =: h_+(f)(t) + h_-(f)(t) \,. \tag{4.131}$$

Because $h_F(f)$ is complex, even when f is real, it is not sufficient to work with the field space V; we must work with the complexified field space $V \otimes \mathbb{C}$.

Let us now compute $\{\partial, h_f\}$. Since (4.131) is a Green's function it satisfies $(\partial_t^2 + \omega^2)h_F(f)(t) = f(t)$. On equations of motion $\ddot{\phi} + \omega^2\phi$ we find

$$h_+(\ddot{\phi} + \omega^2\phi)(t) = i\frac{\dot{\phi}(t)}{2\omega} - i\frac{\dot{\phi}(t_i)}{2\omega}e^{-i\omega(t-t_i)} + \frac{\phi(t)}{2} - \frac{\phi(t_i)}{2}e^{-i\omega(t-t_i)}, \tag{4.132}$$

$$h_-(\ddot{\phi} + \omega^2\phi)(t) = -i\frac{\dot{\phi}(t)}{2\omega} + i\frac{\dot{\phi}(t_f)}{2\omega}e^{i\omega(t-t_f)} + \frac{\phi(t)}{2} - \frac{\phi(t_f)}{2}e^{i\omega(t-t_f)}, \tag{4.133}$$

and so for the sum

$$\begin{aligned} h_F(\ddot{\phi} + \omega^2\phi)(t) =& \phi(t) - \frac{1}{2}\left(\phi(t_i) - \frac{\dot{f}(t_i)}{i\omega}\right)e^{-i\omega(t-t_i)} \\ & - \frac{1}{2}\left(f(t_f) + \frac{\dot{f}(t_f)}{i\omega}\right)e^{i\omega(t-t_f)}. \end{aligned} \tag{4.134}$$

We next define new (complex) functionals $a(t)$ and $a^\dagger(t)$ by

$$\phi(t) = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger(t) + a(t)), \qquad \dot{\phi}(t) = i\sqrt{\frac{\hbar\omega}{2}}(a^\dagger(t) - a(t)). \tag{4.135}$$

These expressions are motivated by the mode expansion of the harmonic oscillator, but we should emphasize that here these are just regular functions, not quantum operators. In particular, the function $a^\dagger$ is just the complex conjugate of the function a, with the notation just reminding us of the usual raising and lowering operators. In terms of these we have

$$h_F(\ddot{\phi} + \omega^2\phi)(t) = \phi(t) - \sqrt{\frac{\hbar}{2\omega}}\left(a(t_i)e^{-i\omega(t-t_i)} + a^\dagger(t_f)e^{i\omega(t-t_f)}\right). \tag{4.136}$$

This suggests that we define a projector $p_F : V \otimes \mathbb{C} \to \mathbb{C}^2$ by

$$\phi \mapsto (a(t_i), a^\dagger(t_f)), \quad \phi^* \mapsto 0, \tag{4.137}$$

and the inclusion $i_F : \mathbb{C}^2 \to V^0 \otimes \mathbb{C}$ by

$$(x, y) \mapsto \sqrt{\frac{\hbar}{2\omega}}\left(xe^{-i\omega(t-t_i)} + ye^{i\omega(t-t_f)}\right), \tag{4.138}$$

with zero image in V^1. With these definitions we have $p_F \circ i_F = \mathrm{id}$.

We can check whether it is reasonable that p_F gives rise to correlators with coherent states by looking again at the two-point function. By either applying the perturbation lemma or going through the same steps as in the previous section, we find that the representative F' of the cohomology of $F = \phi(t)\phi(s)$ satisfying $F' = f \circ p_F$ is given by

$$\begin{aligned} F' =& - i\hbar K_F(t,s) \\ &+ \frac{\hbar}{2\omega}\left(a(t_i)e^{-i\omega(t-t_i)} + a^\dagger(t_f)e^{i\omega(t-t_f)}\right)\left(a(t_i)e^{-i\omega(s-t_i)} + a^\dagger(t_f)e^{i\omega(s-t_f)}\right). \end{aligned} \tag{4.139}$$

We therefore claim that

$$f(w,z) = \frac{\langle w|\, T(\phi(t)\phi(s))\, |z\rangle}{\langle w|z\rangle}, \tag{4.140}$$

where

$$f(w,z) = -i\hbar K_F(t,s) + \frac{\hbar}{2\omega}\left(ze^{-i\omega(t-t_i)} + we^{i\omega(t-t_f)}\right)\left(ze^{-i\omega(s-t_i)} + we^{i\omega(s-t_f)}\right). \tag{4.141}$$

It is straightforward to verify equation (4.140) in the familiar operator language of quantum mechanics, as we do now. We first recall that Wick's theorem implies

$$T(\phi(t)\phi(s)) = -i\hbar K_F(t,s) + N(\phi(t)\phi(s)), \tag{4.142}$$

where N is the normal ordering operation. We have the operator relation

$$\hat{\phi}(t) = \sqrt{\frac{\hbar}{2\omega}}\left(a(t_i)e^{-i\omega(t-t_i)} + a^\dagger(t_f)e^{i\omega(t-t_f)}\right), \tag{4.143}$$

where a and $a^\dagger$ are now interpreted as the creation and annihilation operators of the harmonic oscillator, satisfying the familiar commutation relations. Usually the above expression appears in textbooks for $t_i = t_f = 0$ and normal ordering is defined with respect to $a := a(0)$ and $a^\dagger := a^\dagger(0)$. But this is the same as normal ordering $a(t_i)$ and $a^\dagger(t_f)$, since they only differ from a and $a^\dagger$ by phases. We can now compute

$$\langle w|\, N(\phi(t)\phi(s))\, |z\rangle \tag{4.144}$$

by evaluating $a(t_i)$ at z and $a^\dagger(t_f)$ at w. This follows because N moves all annihilation operators to the right, where we can then use $a(t_i)\,|z\rangle = z\,|z\rangle$. Similarly, when creation operators are on the right, we can use $\langle w| a^\dagger(t_f) = \langle w| w$. Therefore,

$$\langle w| N(\phi(t)\phi(s)) |z\rangle = \frac{\hbar}{2\omega}\left(ze^{-i\omega(t-t_i)} + we^{i\omega(t-t_f)}\right)\left(ze^{-i\omega(s-t_i)} + we^{i\omega(s-t_f)}\right) \langle z|w\rangle \ . \tag{4.145}$$

Combining this with (4.142) then proves (4.140).

In case of the Feynman propagator, this result explains why the perturbation lemma gives Wick's theorem via the projector P_1 (and P) in (4.101) can be interpreted as normal ordering. Recall that $P = i^*$ evaluates functionals on-shell, with boundary conditions specified by the inclusion map i. this is just what we did in (4.145). We evaluated $\phi(t)\phi(s)$ on the solution with $a(t) = z$ at $t = t_i$ and $a^\dagger(t) = w$ at $t = t_f$. Of course, there is nothing special about the two-point functions considered here, and so the perturbation lemma says that P_1 is really Wick's theorem squeezed between coherent states.

4.2.3 Comparison with the Canonical Formulation

In the previous two sections we applied the homological recipe to compute correlators with respect to position eigenstates and with respect to coherent states. The respective projectors were given by

$$p(\phi,\phi^*) = (\phi(t_i),\phi(t_f))\,, \qquad p_F(\phi,\phi^*) = (a(t_i),a^\dagger(t_f))\,. \tag{4.146}$$

Using Wick's theorem, for p_F it was straightforward to see that our approach agrees with the operator language. For p, however, it is harder to verify that f defined via $F' = f \circ p$ actually computes the correlator with respect to position eigenstates, although our formal path integral manipulations above suggest that this must be so. The general claim following from the homological approach is

$$f(x,y) = \frac{\langle y;t_f| T(\phi(t)\phi(s)) |x;t_i\rangle}{\langle y;t_f|x;t_i\rangle}\,, \tag{4.147}$$

where f is the function whose pullback $p^*(f)$ equals $F = \phi(t)\phi(s)$ in cohomology. Our goal in this subsection is to check this statement using the standard formalism of quantum mechanics. Since the operator computation is quite involved, for simplicity we set $x = y = 0$. Then, the result we have computed with the perturbation lemma (4.128) equals the Green's function $K_{DD}(t,s)$. Let us recompute the result, calling it $g(t,s)$, and try to produce the identity,

$$-i\hbar K_{DD}(t,s) = \frac{\langle y=0;t_f|\, T(\phi(t)\phi(s))\, |x=0;t_i\rangle}{\langle y=0;t_f|x=0;t_i\rangle} =: g(t,s)\,, \tag{4.148}$$

using the canonical formulation.

We will use the coherent states as a basis. Given a general state $|\psi\rangle$, its overlap with a coherent state $\langle z|$ gives a holomorphic function in z,

$$\psi(z) := \langle z|\psi\rangle\,. \tag{4.149}$$

The inner product of two such states is

$$\langle\psi_1|\psi_2\rangle = \frac{1}{\pi}\int \mathrm{d}^2z\, \bar{\psi}_1(\bar{z})\psi_2(z)e^{-|z|^2}\,. \tag{4.150}$$

The Hilbert space equipped with this inner product is called the Segal-Bargmann space. The identity can be written as

$$1 = \frac{1}{\pi}\int \mathrm{d}^2z e^{-|z|^2}\,|z\rangle\,\langle\bar{z}|\;. \tag{4.151}$$

The creation operator acts by multiplication since

$$\langle z|\,\hat{a}^\dagger\,|\psi\rangle = z\psi(z) \qquad \Rightarrow \qquad (\hat{a}^\dagger\psi)(z) = z\psi(z)\,. \tag{4.152}$$

We can then deduce from the inner product (4.150) that $\hat{a}$ acts by differentiation, i.e.

$$\langle z|\,\hat{a}\,|\psi\rangle = \frac{\partial}{\partial z}\psi(z) \qquad \Rightarrow \qquad (\hat{a}\psi)(z) = \frac{\partial}{\partial z}\psi(z)\,. \tag{4.153}$$

As a consistency check we note that $[\hat{a},\hat{a}^\dagger] = 1$ in this representation. Since the vacuum state $|0\rangle$ is annihilated by $\hat{a}$, the vacuum is represented by $\psi_0(z) = \langle z|0\rangle$ that is in fact constant (and equal to one if we normalize it). Likewise, the nth excited state is

$$\langle z|n\rangle = \frac{1}{\sqrt{n!}}z^n\,, \tag{4.154}$$

while a coherent state reads

$$\langle z|w\rangle = e^{zw}\;. \tag{4.155}$$

We now come back to the original goal of this section, i.e. establishing the identity (4.148) in the operator language. To do so, we use Wick's theorem

$$T(\phi(t)\phi(s)) = -i\hbar K_F(t,s) + N(\phi(t)\phi(s)) \,, \tag{4.156}$$

where K_F is the Feynman propagator and N denotes normal ordering. Using this in (4.148), we find

$$g(t,s) = -i\hbar K_F(t,s) + \frac{\langle y=0; t_f|\, N(\phi(t)\phi(s))\, |x=0; t_i\rangle}{\langle y=0; t_f|x=0; t_i\rangle} \,. \tag{4.157}$$

In order to compute the overlap involving the normal ordering, we express it in terms of coherent states using (4.151). We then need to express $|x=0\rangle$ in terms of coherent states. For an arbitrary state $|\psi\rangle$ we have

$$\langle x|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2 z\, e^{-\bar{z}z}\, \langle x|z\rangle\, \langle \bar{z}|\psi\rangle \;. \tag{4.158}$$

This formula can be reduced to an integral over the reals. For example, one can show that [74]

$$\langle x|\psi\rangle = \psi(x) = Ce^{-x^2/2}\int \mathrm{d}y\, e^{-y^2/2}\, \langle x+iy|\psi\rangle \;, \tag{4.159}$$

where C is some constant and $\langle x+iy|$ is a coherent state, and so

$$\langle x| = Ce^{-x^2/2}\int \mathrm{d}y e^{-y^2/2}\, \langle x+iy| \;. \tag{4.160}$$

In particular,

$$\langle x=0| = C\int \mathrm{d}y\, e^{-y^2/2}\, \langle iy| \;. \tag{4.161}$$

We now use this in (4.157) to compute $g(t,s)$. We first compute the denominator

$$Z := \langle y=0; t_f|x=0; t_i\rangle = \langle y=0|\, e^{i\frac{H}{\hbar}(t_f-t_i)}\, |x=0\rangle \;, \tag{4.162}$$

where we used the time evolution operator with respect to the Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ of the harmonic oscillator. Thus, using (4.161), we will need the time evolution of a coherent state. Defining $T = t_f - t_i$, we need to compute $e^{i\frac{H}{\hbar}T}\,|z\rangle$, which can be done by inserting a complete set of eigenstates $|n\rangle$ of the Hamiltonian and using the overlap (4.154). With this, we find $e^{i\frac{H}{\hbar}T}\,|z\rangle = e^{i\frac{\omega}{2}T}\,|e^{-i\omega T}z\rangle$. Defining $\lambda := e^{-i\omega T}$ we thus have $e^{i\frac{H}{\hbar}T}\,|z\rangle = \lambda^{-\frac{1}{2}}\,|\lambda z\rangle$. Using this together with

(4.161) we have:

$$\begin{aligned} Z &= C^2\lambda^{-\frac{1}{2}} \int \mathrm{d}y_1 \mathrm{d}y_2 e^{-(y_2^2+y_1^2)/2} \langle iy_2 | -i\lambda y_1 \rangle \\ &= C^2\lambda^{-\frac{1}{2}} \int \mathrm{d}y_1 \mathrm{d}y_2 e^{-(y_2^2+y_1^2)/2+\lambda y_1 y_2} \\ &= \frac{C_1}{\sqrt{\lambda-\lambda^3}}, \end{aligned} \tag{4.163}$$

where we performed the Gaussian integral, and $C_1 := 2\pi C^2$ is another constant that will cancel in the end. Next we turn to the numerator of (4.157). Expanding $\phi(t)$ in terms of ladder operators,

$$\phi(t) = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger e^{i\omega t} + a e^{-i\omega t}), \tag{4.164}$$

and using this in (4.157), we need to compute expectation values of operators quadratic in a and $a^\dagger$. For example, we find that

$$\begin{aligned} \langle y=0; t_f | \, a^\dagger a \, | x=0; t_i \rangle &= C^2\lambda^{-\frac{1}{2}} \int \mathrm{d}y_1 \mathrm{d}y_2 \, \lambda y_1 y_2 \, e^{-(y_2^2+y_1^2)/2+\lambda y_1 y_2} \\ &= \frac{C_1 \lambda^{\frac{3}{2}}}{(1-\lambda^2)^{\frac{3}{2}}} = Z\frac{\lambda^2}{1-\lambda^2}. \end{aligned} \tag{4.165}$$

Similarly, we have

$$\langle y=0; t_f | \, aa \, | x=0; t_i \rangle = -Z\frac{e^{i2\omega t_i}}{1-\lambda^2}, \tag{4.166}$$

$$\langle y=0; t_f | \, a^\dagger a^\dagger \, | x=0; t_i \rangle = -Z\frac{e^{-i2\omega t_f}}{1-\lambda^2}. \tag{4.167}$$

Since the operators are normal ordered we do not need to compute $\langle y=0; t_f | \, aa^\dagger \, | x=0; t_i \rangle$. We can now use the above to compute the normal ordered correlator in (4.157), for which we find after some algebra

$$\begin{aligned} g(t,s) = &- i\hbar K_F(t,s) - \hbar\frac{e^{-i2\omega t_f}}{1-\lambda^2}(2\omega)^{-1}e^{i\omega(t+s)} - \hbar\frac{e^{i2\omega t_i}}{1-\lambda^2}(2\omega)^{-1}e^{-i\omega(t+s)} \\ &+ \hbar\frac{\lambda^2}{1-\lambda^2}(2\omega)^{-1}(e^{i\omega(t-s)} + e^{-i\omega(t-s)}). \end{aligned} \tag{4.168}$$

In order to relate this to K_{DD} we rewrite the Feynman propagator,

$$\begin{aligned} -iK_F(t,s) &= (2\omega)^{-1}\theta(t-s)e^{-i\omega(t-s)} + (2\omega)^{-1}\theta(s-t)e^{i\omega(t-s)} \\ &= -iK_R(t,s) + (2\omega)^{-1}e^{i\omega(t-s)} , \end{aligned} \tag{4.169}$$

where $K_R(t,s) = \theta(t-s)\frac{\sin\omega(t-s)}{2\omega}$ is the retarded propagator. This yields

$$\begin{aligned} g(t,s) =& - i\hbar K_R(t,s) - \hbar\frac{e^{-i2\omega t_f}}{1-\lambda^2}(2\omega)^{-1}e^{i\omega(t+s)} - \hbar\frac{e^{i2\omega t_i}}{1-\lambda^2}(2\omega)^{-1}e^{-i\omega(t+s)} \\ &+ \hbar\frac{1}{1-\lambda^2}(2\omega)^{-1}e^{i\omega(t-s)} + \hbar\frac{\lambda^2}{1-\lambda^2}(2\omega)^{-1}e^{-i\omega(t-s)} \\ =& - i\hbar K_R(t,s) + i\hbar\frac{\cos\omega(t+s-t_i-t_f) - \cos\omega(t-s+t_f-t_i)}{\sin\omega(t_f-t_i)} , \end{aligned} \tag{4.170}$$

where we reintroduced t_i and t_f through $\lambda = e^{-i\omega(t_f-t_i)}$. We can now make use of the identity

$$\cos\omega((t-t_i)+(t_f-s)) - \cos\omega((t-t_i)-(t_f-s)) = -2\sin\omega(t-t_i)\sin\omega(t_f-s) , \tag{4.171}$$

to arrive at

$$\begin{aligned} \frac{\langle y=0;t_f|\, T(\phi(t)\phi(s))\, |x=0;t_i\rangle}{\langle y=0;t_f|x=0;t_i\rangle} &= -i\hbar K_R(t,s) + i\hbar\frac{\sin\omega(t_f-s)}{\omega}\frac{\sin\omega(t-t_i)}{\sin\omega(t_f-t_i)} \\ &= -i\hbar K_{DD}(t,s) , \end{aligned} \tag{4.172}$$

which is what we wanted to show.

4.2.4 General Boundary Conditions

In the previous subsection we exemplified our approach using two different projectors, which where given by $p_{DD}(\phi) = (\phi(t_i),\phi(t_f))$ and $p_F(\phi) = (a(t_i),a^\dagger(t_f))$. Our computations showed that these determine different types of correlation functions.

We now want to generalize to arbitrary linear boundary conditions. More precisely, we look at boundary conditions of the form

$$x = a\phi(t_i) + b\frac{\dot{\phi}(t_i)}{\omega}, \qquad y = c\phi(t_f) + d\frac{\dot{\phi}(t_f)}{\omega}, \tag{4.173}$$

where the numbers (a, b, c, d, x, y) can in general be complex. The numbers (x, y) parametrize solutions to the equations of motion. In this way, we obtain a projector

$$\begin{aligned} p : C^\infty([t_i, t_f]) \otimes \mathbb{C} &\longrightarrow \mathbb{C}^2, \\ \phi &\longmapsto (a\phi(t_i) + b\frac{\dot{\phi}(t_i)}{\omega}, c\phi(t_f) + d\frac{\dot{\phi}(t_f)}{\omega}). \end{aligned} \tag{4.174}$$

As usual, we extend p to $V^\bullet \otimes \mathbb{C}$ by setting $p|_{V^1} = 0$. We recover p_{DD} when $(a, b) = (c, d) = (1, 0)$, while p_F is given by $(a, b) = (\bar{c}, \bar{d}) = (\sqrt{\frac{\omega}{2\hbar}}, i\sqrt{\frac{\omega}{2\hbar}})$.

A solution with boundary conditions (4.173) is given by

$$\phi_{x,y}(t) = \frac{ya \sin\omega(t - t_i) - yb \cos\omega(t - t_i) + xc \sin\omega(t_f - t) + xd \cos\omega(t_f - t)}{(ad - bc)\cos\omega(t_f - t_i) + (ac + bd)\sin\omega(t_f - t_i)}, \tag{4.175}$$

This solution defines an inclusion

$$\begin{aligned} i : \mathbb{C}^2 &\longrightarrow V^0 \otimes \mathbb{C}, \\ (x, y) &\longmapsto \phi_{x,y}. \end{aligned} \tag{4.176}$$

which we extend to $V^\bullet$ via the inclusion $V^0 \hookrightarrow V^\bullet$. To find the homotopy h from the identity to $i \circ p$, we note that the homotopies h_{DD} and h_F satisfy the boundary conditions (4.173) with $x = y = 0$. So our ansatz for the homotopy $h(f)$ is the unique solution to $\ddot{\phi} + \omega^2\phi = f$ satisfying $p \circ h = 0$. It is given by

$$h(f)(t) = \int_{t_i}^{t} f(s)K_i(t, s) + \int_{t}^{t_f} K_f(t, s), \tag{4.177}$$

where

$$\begin{aligned} K_i(t, s) =& K_f(s, t) \\ =& \frac{(a \sin\omega(s - t_i) - b\cos\omega(s - t_i))(c \sin\omega(t - t_f) - d\cos\omega(t - t_f))}{(ad - bc)\omega\cos\omega(t_f - t_i) + (ac + bd)\omega\sin\omega(t_f - t_i)}. \end{aligned} \tag{4.178}$$

In kernel notation, we have

$$K(t,s) = \theta(t-s)K_i(t,s) + \theta(s-t)K_f(s,t)\,. \tag{4.179}$$

Note that $K(t,s)$ is manifestly symmetric in its arguments. A lengthy computation now shows that

$$\begin{aligned}
&h(\ddot{\phi}+\omega^2\phi)(t)\\
=&\phi(t) - \left(a\phi(t_i) + b\frac{\dot{\phi}(t_i)}{\omega}\right)\frac{c\sin\omega(t_f-t) + d\cos\omega(t_f-t)}{(ad-bc)\cos\omega(t_f-t_i) + (ac+bd)\sin\omega(t_f-t_i)}\\
&-\left(c\phi(t_f) + d\frac{\dot{\phi}(t_f)}{\omega}\right)\frac{a\sin\omega(t-t_i) - b\cos\omega(t-t_i)}{(ad-bc)\cos\omega(t_f-t_i) + (ac+bd)\sin\omega(t_f-t_i)}\,,
\end{aligned} \tag{4.180}$$

as well as $\ddot{h}(f) + \omega^2 h(f) = f$. Therefore, the homotopy relation $\{\partial, h\} = 1 - i \circ p$ is satisfied.

One application using these more general projectors and homotopies would be the computation of correlators with in- and out states living in different representations of the Hilbert space. For example, one could choose $(a,b) = (1,0)$ and $(c,d) = (0,1)$. In this case, the homotopy satisfies Dirichlet boundary conditions at $t = t_i$ and Neumann boundary conditions at $t = t_f$. The associated representative of the cohomology then uses position eigenstates $|x; t_i\rangle$ as in-states and momentum eigenstates $\langle p; t_f|$ as out-states.

4.3 Unruh Effect

In this section we present the first application of the homological formulation in the realm of genuine field theories. Specifically, we apply our homological method in the context of quantum field theory on curved spacetime by providing an alternative derivation of the Unruh effect: the quantum effect according to which the number of particles detected depends on the observer [75]. In the vacuum state an inertial observer in Minkowski space sees no particles, while in the same state a uniformly accelerated observer sees a thermal bath of particles.

4.3.1 Generalities and Homotopy Retract

Let us begin with a brief review of general features of uniformly accelerated observers in two-dimensional Minkowski spacetime with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2\,. \tag{4.181}$$

The trajectory of an observer is then parametrized by $x^\mu(\tau) = \big(t(\tau), x(\tau)\big)$, where τ is proper time, so that the 2-velocity $u^\mu(\tau) = dx(\tau)/d\tau$ satisfies the normalization condition

$$\eta_{\mu\nu}u^\mu u^\nu = 1\,. \tag{4.182}$$

The Lorentz-invariant condition for the acceleration being constant is expressed in terms of $a^\mu(\tau) = \dot{u}^\mu(\tau)$ as

$$\eta_{\mu\nu}a^\mu(\tau)a^\nu(\tau) = -a^2\,, \tag{4.183}$$

where a is a constant. The trajectory of a uniformly accelerated observer satisfying these two conditions can be written as

$$t(\tau) = \frac{1}{a}\sinh a\tau\,, \qquad x(\tau) = \frac{1}{a}\cosh a\tau\,. \tag{4.184}$$

Next, let us relate the inertial frame to a frame that is comoving with the observer. This means that denoting these coordinates by $(\tilde{t}, \tilde{x})$ the observer's worldline is a vertical line $\tilde{x} = 0$, so that the observer is indeed at rest in this frame. The Rindler coordinates having this property are defined by

$$t = a^{-1}e^{a\tilde{x}}\sinh a\tilde{t}\,, \tag{4.185}$$
$$x = a^{-1}e^{a\tilde{x}}\cosh a\tilde{t}\,, \tag{4.186}$$

and the inverse relation

$$\tilde{t} = \frac{1}{2a}\ln\frac{x+t}{x-t}\,, \tag{4.187}$$
$$\tilde{x} = \frac{1}{2a}\ln\left[a^2(x^2-t^2)\right]. \tag{4.188}$$

From these relations one finds the metric in Rindler coordinates,

$$\mathrm{d}s^2 = (\mathrm{d}t)^2 - (\mathrm{d}x)^2 = e^{2a\tilde{x}}\left[(\mathrm{d}\tilde{t})^2 - (\mathrm{d}\tilde{x})^2\right], \tag{4.189}$$

which is thus conformally equivalent to the Minkowski metric.

We now consider the action of a massless scalar field ϕ in a $1+1$ dimensional spacetime,

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^2x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\,, \tag{4.190}$$

where $g_{\mu\nu}$ is the metric and g is its determinant. In the inertial frame,

$$S[\phi] = \frac{1}{2}\int \mathrm{d}t\mathrm{d}x\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right]. \tag{4.191}$$

The action in the accelerated frame takes the same form:

$$S[\phi] = \frac{1}{2}\int \mathrm{d}\tilde{t}\mathrm{d}\tilde{x}\left[(\partial_{\tilde{t}}\phi)^2 - (\partial_{\tilde{x}}\phi)^2\right], \tag{4.192}$$

as a consequence of the conformal invariance of the action (4.190) in two dimensions and the Rindler metric (4.189) being conformally equivalent to the Minkowski metric. The equations of motion are

$$\ddot{\phi} - \partial_x^2\phi = 0\,, \tag{4.193}$$

$$\partial_{\tilde{t}}^2\phi - \partial_{\tilde{x}}^2\phi = 0\,, \tag{4.194}$$

where the dot denotes the partial derivative with respect to time t. Note that as a scalar we have for the coordinate-transformed field $\tilde{\phi}(\tilde{t},\tilde{x}) = \phi(t,x)$, so that in the second equation we could replace ϕ by $\tilde{\phi}$.

As a preparation for the homotopy retract we have to introduce the Fourier transform with respect to the spatial coordinate, both in inertial and Rindler coordinates:

$$\phi_k(t) := \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}}e^{-ikx}\phi(t,x)\,,\qquad \tilde{\phi}_l(\tilde{t}) := \int_{-\infty}^{+\infty}\frac{\mathrm{d}\tilde{x}}{\sqrt{2\pi}}e^{-il\tilde{x}}\tilde{\phi}(\tilde{t},\tilde{x})\,. \tag{4.195}$$

Note that even though in the second integral we could replace $\tilde{\phi}(\tilde{t},\tilde{x})$ by $\phi(t,x)$, the Fourier mode ϕ_k as a function of k of course differs from $\tilde{\phi}_l$ as a function of l. The inverse relations are

$$\phi(t,x) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{\sqrt{2\pi}}e^{ikx}\phi_k(t)\,,\qquad \tilde{\phi}(\tilde{t},\tilde{x}) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{\sqrt{2\pi}}\,e^{il\tilde{x}}\tilde{\phi}_l(\tilde{t})\,. \tag{4.196}$$

Since the scalar functions on the left-hand sides are equal (more precisely, we have $\phi(t,x) = \tilde{\phi}(\tilde{t}(t,x),\tilde{x}(t,x))$), we have two different expansions of the same ϕ

into Fourier modes:

$$\phi(t,x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx}\phi_k(t) \ = \int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{\sqrt{2\pi}}\, e^{il\tilde{x}(t,x)}\tilde{\phi}_l\big(\tilde{t}(t,x)\big)\,. \tag{4.197}$$

We will also use the following change of basis for the Fourier modes and their time derivatives:

$$\phi_k = \sqrt{\frac{\hbar}{2\omega_k}}\,(a^\dagger_{-k} + a_k)\,, \qquad \tilde{\phi}_l = \sqrt{\frac{\hbar}{2\Omega_l}}\,(b^\dagger_{-l} + b_l)\,, \tag{4.198}$$

$$\dot{\phi}_k = i\sqrt{\frac{\hbar\omega_k}{2}}\,(a^\dagger_{-k} - a_k)\,, \qquad \partial_{\tilde{t}}\tilde{\phi}_l = i\sqrt{\frac{\hbar\Omega_l}{2}}\,(b^\dagger_{-l} - b_l)\,, \tag{4.199}$$

where $\omega_k \equiv \sqrt{k^2}$, $\Omega_l \equiv \sqrt{l^2}$. The inverse relations read:

$$a_k = \sqrt{\frac{\omega_k}{2\hbar}}\left(\phi_k + \frac{i}{\omega_k}\dot{\phi}_k\right), \qquad a^\dagger_{-k} = \sqrt{\frac{\omega_k}{2\hbar}}\left(\phi_k - \frac{i}{\omega_k}\dot{\phi}_k\right), \tag{4.200}$$

$$b_l = \sqrt{\frac{\Omega_l}{2\hbar}}\left(\tilde{\phi}_l + \frac{i}{\Omega_l}\partial_{\tilde{t}}\tilde{\phi}_l\right), \qquad b^\dagger_{-l} = \sqrt{\frac{\Omega_l}{2\hbar}}\left(\tilde{\phi}_l - \frac{i}{\Omega_l}\partial_{\tilde{t}}\tilde{\phi}_l\right). \tag{4.201}$$

As for the harmonic oscillator these relations are motivated by the familiar definition of creation and annihilation operators, but we emphasize that also here these are just functions.

We now discuss the homotopy retract, beginning with the chain complex defining the theory:

$$0 \longrightarrow V^0 \xrightarrow{\ \partial\ } V^1 \longrightarrow 0\,. \tag{4.202}$$

Here the space of fields and the space of anti-fields are given by

$$V^0 = C^\infty([t_i,t_f]\times\mathbb{R})\,, \qquad V^1 = \Pi C^\infty([t_i,t_f]\times\mathbb{R})\,. \tag{4.203}$$

The notation indicates that the (anti-)fields depend on t, restricted to the interval $[t_i, t_f]$, and the space coordinate x living on the full real line $\mathbb{R}$. The differential is

$$\partial(\phi) = (\partial_t^2 - \partial_x^2)\phi\,. \tag{4.204}$$

The important new feature in field theory is that the projector $p : V^\bullet \to C^\infty(\mathbb{R})\times C^\infty(\mathbb{R})$ no longer maps to a finite-dimensional space like $\mathbb{R}^2$ but to infinite-dimensional functions spaces, however, with functions of one less coordinate. Specifically, the projector evaluates the functions a and $a^\dagger$ defined in (4.200) at t_i and t_f, respectively:

$$\phi \mapsto \left(a_k(t_i), a_l^\dagger(t_f)\right), \qquad \phi^* \mapsto 0\,. \tag{4.205}$$

Next, we need to define the inclusion map $i : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to V^0$ that takes two functions in momentum space, say $c(k)$ and $d(k)$, and produces a field in V^0 (i.e. in the present example a scalar field in two-dimensional Minkowski space). The proper inclusion map satisfying $p \circ i = 1$ is given by

$$(c,d) \mapsto \phi_{(c,d)}(t,x) := \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ikx} \sqrt{\frac{\hbar}{2\omega_k}} \left(d(-k)e^{i\omega_k(t-t_f)} + c(k)e^{-i\omega_k(t-t_i)}\right). \tag{4.206}$$

The homotopy map $h : V^1 \to V^0$ is defined, for any $f \in V^1$, in terms of the Green's function of the operator $\partial_t^2 - \partial_x^2$:

$$h(f)(t,x) = \int_{t_i}^{t_f} \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}y\, K(t-s, x-y) f(s,y)\,, \tag{4.207}$$

where the kernel is explicitly given by

$$K(t-s,x-y) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{4\pi\omega_l}\, i\left(\Theta(t-s)e^{-i\omega_l(t-s)+il(x-y)} + \Theta(s-t)e^{i\omega_l(t-s)-il(x-y)}\right). \tag{4.208}$$

Indeed, one can verify that with the above definitions for projector, inclusion and homotopy the homotopy relation $\partial h + h\partial = 1 - ip$ is satisfied. To this end one needs to assume that $\phi(t,x)$ and $\partial_x\phi(t,x)$ vanish at $x = -\infty$ and $x = +\infty$.

For completeness we also display the important operations of the dual space of functionals on which the BV algebra is defined. The BV complex $\mathcal{F}(V^\bullet)$ is equipped with the differential,

$$Q = \int_{t_i}^{t_f} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\, \left(\ddot{\phi}(t,x) + \partial_x^2\phi(t,x)\right) \frac{\delta}{\delta\phi^*(t,x)}\,. \tag{4.209}$$

In addition, the BV-differential is defined as

$$\delta \equiv Q - i\hbar\Delta\,, \qquad \Delta \equiv -\int_{t_i}^{t_f} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\delta}{\delta\phi^*(t,x)} \frac{\delta}{\delta\phi(t,x)}\,. \tag{4.210}$$

For a functional $F[\phi,\phi^*]$ in $\mathcal{F}(V^\bullet)$, we obtain the pull-back functional in $\mathcal{F}[C^\infty(\mathbb{R})\times C^\infty(\mathbb{R})]$ defined by

$$i^*(F)(c,d) = (F\circ i)(c,d)\,. \tag{4.211}$$

Similarly, the pullback of a functional f in $\mathcal{F}[C^\infty(\mathbb{R})\times C^\infty(\mathbb{R})]$ with respect to the projection is the functional in $\mathcal{F}(V^\bullet)$ given by

$$p^*(f)(\phi,\phi^*) = (f\circ p)(\phi,\phi^*)\,. \tag{4.212}$$

4.3.2 Number Expectation Value

To derive the Unruh effect, one assumes that the number of particles measured by an accelerated observer is given by the expectation value of the number operator with respect to Rindler space, i.e., with respect to creation and annihilation operators defined with the Fourier modes in Rindler space. More precisely, one computes

$$\mathcal{N}_k := \langle N_k\rangle \equiv \langle 0|\,\hat{b}_k^\dagger\hat{b}_k\,|0\rangle\ , \tag{4.213}$$

where $\hat{b}_k$ and $\hat{b}_k^\dagger$ are the Rindler space annihilation and creation operators defined in analogy to (4.201), and $|0\rangle$ is the Minkowski vacuum state. This state is defined so that it is annihilated by the inertial frame operator $\hat{a}_k$:

$$\hat{a}_k\,|0\rangle = 0\,. \tag{4.214}$$

For definiteness we take the Heisenberg picture operators b and $b^\dagger$ to be at time $\tilde{t} = 0$ (which is equivalent to $t = 0$ for all x). The usual textbook computation involves relating the creation and annihilation operators of the accelerated and inertial frames through Bogolyubov transformations. We provide an alternative approach which does not require finding the Bogolyubov transformations. Instead, our strategy is to define the functional $F[\phi]$ of the massless scalar field ϕ to be given by $b_k^\dagger b_k$, with $b_k^\dagger$ and b_k being defined in terms of the classical field ϕ via (4.201). Following our approach for the harmonic oscillator in sec. 4, we then find $f(c,d)$ such that $F' = f\circ p$ is in the same cohomology class as $F[\phi]$. Then $f(c,d)$ computes the expectation value

$$f(c,d) = \lim_{\tilde{t}\to 0}\frac{\langle d|\,T\big(\hat{b}_k^\dagger(\tilde{t})\hat{b}_k(0)\big)\,|c\rangle}{\langle d|c\rangle}\ , \tag{4.215}$$

where $|c\rangle$ and $|d\rangle$ are coherent states with respect to a_k, i.e.,

$$a_k\,|c\rangle = c(k)\,|c\rangle\ , \tag{4.216}$$

and analogously for $|d\rangle$. Here we take the limit $\tilde{t} \to 0$ after performing the computation, as opposed to setting $\tilde{t} = 0$ from the beginning, since some care is needed in order to deal with the step functions entering the Green's function. Note that the result does not depend on whether one takes the limit from above or from below, which follows from the symmetry of the Green's function. Finally, in order to find the expectation value of the Rindler number operator with respect to the Minkowski vacuum, we set $c = d = 0$, i.e.,

$$\mathcal{N}_k = f(0,0)\,. \tag{4.217}$$

The choice $c = d = 0$ is the analog of the equation (4.214) specifying the Minkowski vacuum.

We begin by expressing the functional $b_k^\dagger(\tilde{t})b_k(0)$ in terms of $\phi(t,x)$. By taking the Fourier transform of (4.201), one obtains b_k and $b_k^\dagger$ in terms of ϕ and $\partial_{\tilde{t}}\phi$:

$$b_k(\tilde{t}) = \int \mathrm{d}\tilde{x}\, e^{-ik\tilde{x}} \sqrt{\frac{\Omega_k}{4\pi\hbar}} \left(\phi + \frac{i}{\Omega_k}\partial_{\tilde{t}}\phi\right), \tag{4.218}$$

$$b_k^\dagger(\tilde{t}) = \int \mathrm{d}\tilde{x}\, e^{ik\tilde{x}} \sqrt{\frac{\Omega_k}{4\pi\hbar}} \left(\phi - \frac{i}{\Omega_k}\partial_{\tilde{t}}\phi\right). \tag{4.219}$$

For the second equation we use the chain rule to obtain

$$\partial_{\tilde{t}}\phi = \frac{\partial t}{\partial \tilde{t}}\,\dot{\phi} + \frac{\partial x}{\partial \tilde{t}}\,\partial_x\phi = e^{a\tilde{x}}\cosh(a\tilde{t})\dot{\phi} + e^{a\tilde{x}}\sinh(a\tilde{t})\partial_x\phi\ . \tag{4.220}$$

Note that this is only valid when $x > |t|$ since the Rindler coordinates only cover this part of the Minkowski spacetime. With (4.218) – (4.220), we can explicitly write out the functional $F[\phi] = b_k^\dagger(\tilde{t})b_k(0)$:

$$\begin{aligned} F[\phi] = \int \mathrm{d}\tilde{x} \int \mathrm{d}\tilde{y}\, e^{ik(\tilde{x}-\tilde{y})}\frac{\Omega_k}{4\pi\hbar}\Big(&\phi(t,x)\phi(0,y) + \frac{i}{\Omega_k}e^{a\tilde{y}}\phi(t,x)\dot{\phi}(0,y) \\ &+ \frac{1}{\Omega_k^2}e^{a(\tilde{x}+\tilde{y})}\big(\cosh(a\tilde{t})\dot{\phi}(0,y)\dot{\phi}(t,x) + \sinh(a\tilde{t})\dot{\phi}(0,y)\partial_x\phi(t,x)\big) \\ &- \frac{i}{\Omega_k}e^{a\tilde{x}}\big(\cosh(a\tilde{t})\phi(0,y)\dot{\phi}(t,x) + \sinh(a\tilde{t})\phi(0,y)\partial_x\phi(t,x)\big)\Big)\,, \end{aligned} \tag{4.221}$$

where of course t and x on the right-hand side must be viewed as functions of $(\tilde{t}, \tilde{x})$.

We apply the perturbation lemma to find $f(c,d)$, by using P_1 in (4.101),

$$P_1 = i^* \exp\left(-\frac{i\hbar}{2}C\right) , \tag{4.222}$$

where the functional derivatives in the C operator are now with respect to $\phi(t,x)$:

$$C = \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}y\, K(t-s, x-y)\, \frac{\delta^2}{\delta\phi(t,x)\delta\phi(s,y)} , \tag{4.223}$$

and $K(t-s, x-y)$ is given in (4.208). Applying P_1 on $F[\phi]$,

$$\begin{aligned} P_1(F)(c,d) \;=\; & i^* F(c,d) \\ & - i\hbar \int \mathrm{d}\tilde{x} \int \mathrm{d}\tilde{y}\; e^{ik(\tilde{x}-\tilde{y})} \frac{\Omega_k}{4\pi\hbar} \Bigg(K(t, x-y) - \frac{i}{\Omega_k} e^{a\tilde{y}} \partial_t K(t, x-y) \\ & - \frac{1}{\Omega_k^2} e^{a(\tilde{x}+\tilde{y})} \left[\cosh(a\tilde{t})\, \partial_t \partial_t K(t, x-y) + \sinh(a\tilde{t}) \partial_t \partial_x K(t, x-y) \right] \\ & - \frac{i}{\Omega_k} e^{a\tilde{x}} \left[\cosh(a\tilde{t}) \partial_t K(t, x-y) + \sinh(a\tilde{t}) \partial_x K(t, x-y) \right] \Bigg) . \end{aligned} \tag{4.224}$$

There are no further terms in the expansion of $\exp\left(-\frac{i\hbar}{2}C\right)$ because $F[\phi]$ only contains two ϕs. Let us start by treating the first term on the right-hand side of (4.224). Since we set $c = d = 0$, the inclusion to the space of fields (4.206) is $i(0,0) = 0$. Therefore, with (4.211), the first term on the right-hand side of (4.224) vanishes:

$$i^* F(0,0) = 0 . \tag{4.225}$$

Next, we take the limit $t = \tilde{t} = 0$. After inserting the derivatives of $K(t,s)$, using (4.208), and writing these in terms of Rindler coordinates, we obtain

$$\begin{aligned} f(0,0) &= P_1(F)(0,0) \\ &= \int \mathrm{d}\tilde{x} \int \mathrm{d}\tilde{y} \int \mathrm{d}l\, \frac{\Omega_k}{16\pi^2 \omega_l} e^{ik(\tilde{x}-\tilde{y})} e^{ila^{-1}(e^{a\tilde{x}} - e^{a\tilde{y}})} \times \\ & \qquad \left(1 + \frac{1}{\Omega_k^2} e^{a(\tilde{x}+\tilde{y})} \omega_l^2 - \frac{\omega_l}{\Omega_k} e^{a\tilde{x}} - \frac{\omega_l}{\Omega_k} e^{a\tilde{y}} \right) . \end{aligned} \tag{4.226}$$

We now perform the change of variables:

$$u = e^{a\tilde{x}}\,, \quad \frac{1}{au}du = d\tilde{x}\,, \tag{4.227}$$

$$v = e^{a\tilde{y}}\,, \quad \frac{1}{av}du = d\tilde{y}\,. \tag{4.228}$$

Then $f(0,0)$ takes the form

$$\begin{aligned} f(0,0) = \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \int_{-\infty}^\infty \mathrm{d}l\, e^{ika^{-1}(\ln u - \ln v)} \frac{\Omega_k}{16\pi^2 a^2 \omega_l} e^{ila^{-1}(u-v)} \times \\ \left(\frac{1}{uv} + \frac{\omega_l^2}{\Omega_k^2} - \frac{\omega_l}{\Omega_k}\frac{1}{v} - \frac{\omega_l}{\Omega_k}\frac{1}{u} \right). \end{aligned} \tag{4.229}$$

Evaluating the integrals over u and v,[4]

$$f(0,0) = \int_{-\infty}^\infty \mathrm{d}l\, \frac{\Omega_k}{4\pi^2 a^2 \omega_l} \Gamma\left(-\frac{ik}{a}\right)\Gamma\left(\frac{ik}{a}\right)(-1)^{ik/a}. \tag{4.231}$$

As a consistency check, one may verify that the integrand in (4.231) coincides with the expression in equation (8.43) of [76]. By using the identity for Gamma functions,

$$|\Gamma(ik/a)|^2 = \frac{\pi a}{k \sinh(\pi k/a)} = \frac{2\pi a}{|k|} \frac{e^{\pi |k|/a}}{(e^{2\pi|k|/a} - 1)}\,, \tag{4.232}$$

we obtain

$$f(0,0) = (e^{2\pi|k|/a} - 1)^{-1} \int_{-\infty}^\infty \mathrm{d}l\, \frac{1}{2\pi a \omega_l}\,, \tag{4.233}$$

as long as we choose $(-1)^{ik/a} = e^{-\pi|k|/a}$. The expectation value of the number of particles observed by an accelerated observer is a Bose-Einstein distribution with the Unruh temperature

$$T = \frac{\hbar a}{2\pi k_B}\,, \tag{4.234}$$

where k_B is the Boltzmann constant. The divergent integral in (4.233) is also present in the conventional derivation of the Unruh effect (see, e.g., chapter 8

[4]For this computation we used the integral identities:

$$\int_0^\infty dx\, e^{iA\ln(x)} e^{iBx} x^{-1} = (-iB)^{-iA}\Gamma(iA)\,, \qquad \int_0^\infty dx\, e^{iA\ln(x)} e^{iBx} = (iA)(-iB)^{-1-iA}\Gamma(iA)\,. \tag{4.230}$$

in [76]) and is interpreted as the infinite volume of the entire space.

Chapter 5

Outlook

Having investigated the applications of homotopy algebras to gauge theories and quantum mechanics, let us mention some questions that might branch out from this work.

In chapter 2 we have constructed a procedure to derive an action of a gauge theory in terms of gauge invariant variables. However, the procedure is far from complete; as we have seen in section 2.5, it can still be a challenge to identify which gauge invariant variables are dynamical. When reducing the scalar part of the quadratic action on FLRW backgrounds, it took a considerable amount of field redefinitions in order to arrive at the Mukhanov-Sasaki action. Within the L_∞ algebraic framework there seems to be no systematic procedure to finish the analysis. Perhaps it may be feasible to define a projection to a complex of solutions to the constraints, in the same way that we can project to a theory of gauge invariant variables or to the homology of a chain complex. Then one would be left to find the field redefinitions needed to reorganize the action in a systematic way. If successful, one could apply this new method to the gauge invariant quadratic expansion of double field theory developed in chapter 3 towards the process of computing observables.

In chapter 3 we have given a gauge invariant quadratic expansion of double field theory on time-dependent backgrounds. As mentioned before, the next step in the program of applying double field theory to cosmology would be to compute correlation functions from both the quadratic and the cubic theory. By comparing these with the cosmic microwave background data, one could eventually test different cosmological scenarios. Our findings show that in contrast to general relativity, in double field theory the degrees of freedom are distributed differently among scalar, vector and tensor modes. This suggests that observational signatures would be different from those predicted by general relativity coupled to ordinary matter. It would be helpful to develop more systematic techniques to investigate this.

In addition, in the long run, it would be desirable to construct a more realistic cosmological model. For instance, it would be important to add further matter, perhaps initially in the form of a generic duality invariant energy-momentum tensor. Eventually one would need to incorporate real observational parameters.

In chapter 4 we have presented a partial reformulation of quantum mechanics. We have demonstrated that our approach works for the harmonic oscillator and its perturbations. However, we have yet to apply our approach to theories with gauge symmetries, such as Yang-Mills theory. We expect to find an extension to theories with gauge symmetries, since the BV formalism was invented for gauge theories in the first place.

Furthermore, our homological formulation is not just defined for perturbation theories but can in principle be applied to non-perturbative problems. Since perturbation theory is only applicable for theories with weak coupling, but not for strongly coupled physics such as quantum chromodynamics, it would be worthwhile to investigate to apply the homological formulation to non-perturbative problems.

An additional follow-up project for the homological approach would be to consider systems with spin. Since spin is a purely quantum quantity and does not have a classical equivalent, our method of setting up the classical theory with the chain complex and upgrading to quantum theory might not be sufficient. In order to have a complete reformulation of quantum mechanics, it would be necessary to be able to describe systems with spin.

As a final remark, it might be interesting to map out the connections between homotopy Lie algebras and Hopf algebras in the context of quantum field theory. For instance we have seen that gauge theories have L_∞ algebraic structure, and upon quantization, the BV formalism and the highly related BV algebras become important structures. Beyond this, the Hopf algebras dictate the algebraic structure of Feynman diagrams and play a key role in renormalization [25, 26]. It would be fascinating to study the possibility of bridging together all of these algebraic structures for a more mathematically sound foundation of quantum field theory.

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