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Energy Scaling Law for a Singularly Perturbed Four-Gradient Problem in Helimagnetism

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Abstract
We study pattern formation in magnetic compounds near the helimagnetic/ferromagnetic transition point in case of Dirichlet boundary conditions on the spin field. The energy functional is a continuum approximation of a $J_1 - J_3$ model and was recently derived in Cicalese et al. (SIAM J Math Anal 51: 4848–4893, 2019). It contains two parameters, one measuring the incompatibility of the boundary conditions and the other measuring the cost of changes between different chiralities. We prove the scaling law of the minimal energy in terms of these two parameters. The constructions from the upper bound indicate that in some regimes branching-type patterns form close to the boundary of the sample.

Keywords Microstructure · Energy scaling · Chirality transitions · Frustrated spin system

Mathematics Subject Classification 49J40 · 82B21 · 82B24

1 Introduction

We study a continuum variational problem that arises from a statistical mechanics description of magnetic compounds and describes pattern formation in case of incompatible boundary conditions. We prove a scaling law for the minimal energy in terms of the problem parameters. Such scaling results have proven useful in a huge variety of singularly perturbed non-(quasi-)convex models for pattern forming systems

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where explicit minimizers cannot be easily determined analytically or numerically, see, e.g., (Kohn 2007) for some examples. Pattern formation is then often related to competing terms in the energy functional, favoring rather uniform or highly oscillatory structures, respectively. The proofs of the scaling laws often involve branching-type constructions where structures oscillate on refining scales near the boundary, among many others see for instance the scaling laws in Kohn and Müller (1994), Conti (2000, 2006), Capella and Otto (2009, 2012), Chan and Conti (2015), Knüpfer et al. (2013), Bella and Goldman (2015), Conti and Zwicknagl (2016), Conti et al. (2020), Rüland and Tribuzio (2022)) for martensitic microstructure, (Kohn and Wirth 2014, 2016) for compliance minimization, (Choksi et al. 2004, 2008; Conti et al. 2016) for type-I-superconductors, (Ben Belgacem et al. 2002; Bella and Kohn 2014; Bourne et al. 2017; Conti et al. 2005) for compressed thin elastic films, (Conti and Ortiz 2016; Conti and Zwicknagl 2016) for dislocation patterns, and (Brancolini and Wirth 2017; Brancolini et al. 2018) for transport networks.

We point out that in particular a variety of magnetization patterns (including branching structures) have been successfully explained via scaling laws of continuum micromagnetic energies, see, e.g., Choksi and Kohn (1998), Choksi et al. (1998), Dabade et al. (2019), DeSimone et al. (2006a), DeSimone et al. (2006b), Knüpfer and Muratov (2011), Otto and Steiner (2010), Otto and Viehmann (2010), Venkatraman et al. (2020)). While these models typically contain local and non-local terms, we will focus on a purely local model that arises - at least heuristically - from a frustrated spin system, see Diep (2013), Diep (2015) for the general context and Cicalese and Solombrino (2015), Cicalese et al. (2019) for the specific setting considered here. Precisely, starting from a 2-dimensional square lattice $\varepsilon\mathbb{Z}^2$ with lattice width $\varepsilon > 0$, we consider spin fields $v : \varepsilon\mathbb{Z}^2 \to S^1$ and a configurational energy of the form (also called $J_1 - J_3$-model)

$$E_\varepsilon(v) := -\alpha \sum_{|i-j|=1} v(\varepsilon i) \cdot v(j\varepsilon) + \sum_{|i-j|=2} v(\varepsilon i) \cdot v(j\varepsilon)$$

with some positive parameter $\alpha > 0$, where the summation is taken over indices $i, j \in \mathbb{Z}^2 \cap \frac{1}{\varepsilon}\Omega$. While the first term favors nearest neighbors to have aligned spins, the second term favors next-to-nearest neighbors (horizontally and vertically) to have opposite spins. Note that the model considered here does not take into account diagonal interactions, for a recent analysis in that case see (Cicalese et al. 2021) and the references therein. Depending on the size of the parameter α, different minimizers arise. Precisely, the analysis in Cicalese and Solombrino (2015), Cicalese et al. (2019) shows that (at least locally) the energy is minimized by ferromagnetic configurations, i.e., constant spin fields, if $\alpha \geq 4$, while for small $\alpha < 4$, the energy is minimized by helimagnetic configurations, i.e., spin fields in which spins rotate at a fixed angle $\phi = \pm \arccos(\alpha/4)$ between horizontal and vertical nearest neighbors, respectively. Such helical structures have recently been observed experimentally, see, e.g., Schoenherr et al. 2018; Uchida et al. 2006.
Of particular interest is the transition point $\alpha \nearrow 4$ where the ground state changes from a helimagnetic to a ferromagnetic structure. Mathematically, an asymptotic analysis in the sense of Γ-convergence in this transition regime in the limit of vanishing lattice

spacing has been performed in Cicalese et al. (2019). In the sequel, we briefly sketch the idea as outlined there, for details and references see Cicalese et al. (2019). Heuristically, it can be shown that the appropriately rescaled normalized energy $E_\varepsilon - \min E_\varepsilon$ can be rewritten in terms of an appropriately rescaled version u of the angular lifting $\tilde{u}$, given by $v = (\cos\tilde{u}, \sin\tilde{u})$, as an energy of the form

$$I_\tau(u) = \int \tau \left(|\,\partial_1\partial_1 u\,|^2 + |\,\partial_2\partial_2 u\,|^2 \right) + \frac{1}{\tau} \left((1-|\,\partial_1 u\,|^2)^2 + (1-|\,\partial_2 u\,|^2)^2 \right) dx \tag{1}$$

with $\tau = \frac{\sqrt{2}\varepsilon}{\sqrt{4-\alpha}}$. Here, the preferred derivatives $\partial_1 u = \pm 1$ and $\partial_2 u = \pm 1$ are the order parameters describing the chiralities, i.e., they correspond to helical structures rotating clockwise (-1) and counterclockwise $(+1)$ between horizontally and vertically adjacent spins, respectively. We note that this is a heuristic simplification where in particular discrete derivatives are approximated by continuous ones and we assume that there are no vortices in the spin field. However, the rigorous analysis of Cicalese et al. (2019), Cicalese and Solombrino (2015) supports such a perspective, at least in certain parameter regimes. Roughly speaking, by the classical Modica–Mortola result, one expects that the functional I_τ converges in the sense of convergence as $\tau \to 0$ to a functional that is finite only on fields $\nabla u \in BV$ satisfying the differential inclusion $\nabla u \in \{(\pm 1, \pm 1)\}$, see Cicalese et al. (2019) for the rigorous derivation. A respective rigorous result in terms of Γ-convergence in the regime $\tau \to \tau_0 \in (0, \infty)$, relating the discrete $J_1 - J_3$-spin model with boundary conditions to a continuum functional of the form (1) holds also true (see Cicalese and Solombrino 2015 for a one-dimensional local result and Ginster et al. (in preparation) for the two-dimensional setting considered here).

While the analysis in Cicalese et al. (2019) focuses on the local behavior, we study the system under Dirichlet boundary conditions on the spin field. More precisely, we start from the continuum model (1) on a square domain $\Omega = (0, 1)^2$, and derive the scaling law of the minimal energy among configurations satisfying affine boundary conditions $u(0, y) = (1-2\theta)y$ at the left boundary. Here, the parameter $\theta \in (0, 1/2)$ is a compatibility parameter, where for $\theta = 0$, the boundary condition is compatible with the helical structures $(\pm 1, 1)$, $\theta = 1/2$ corresponds to a ferromagnetic configuration (at least in vertical direction), and $\theta \in (0, 1/2)$ indicates that the spin field on the boundary rotates in vertical direction with an angle that is smaller than the optimal angle $\phi = \arccos(\alpha/4)$.

For the ease of notation, we present the proof of the scaling law for a slightly simplified functional, namely

$$J_\sigma(u) := \int_{(0,1)^2} \operatorname{dist}^2(\nabla u, \{(\pm 1, \pm 1)\})\, d\mathcal{L}^2 + \sigma\,|\,D^2 u\,|\,((0, 1)^2)$$

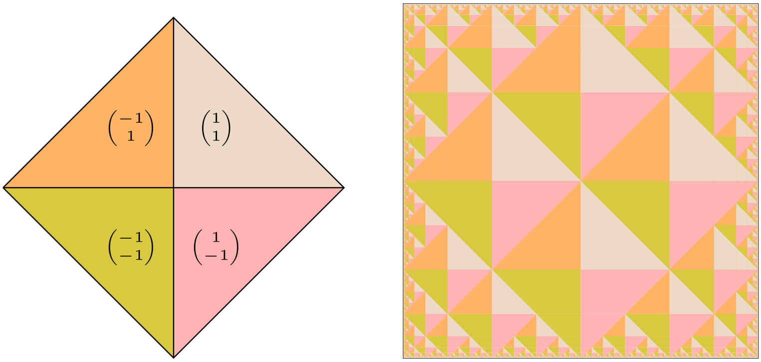

Fig. 1 Left: The four preferred gradients can be combined so that the corresponding function is zero outside the rotated square. Right: Rescaled versions of the rotated square can be used to cover $(0, 1)^2$ so that the resulting function u satisfies $u = 0$ on $\partial(0, 1)^2$ and $\nabla u \in \{(\pm 1, \pm 1)\}$ a.e

with affine boundary conditions on one boundary. Precisely, we show that there are two scaling regimes for the minimal energy,

$$\min_{u(0,y)=(1-2\theta)y} J_\sigma(u) \sim \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right), \theta^2\right\}.$$

The second scaling is attained, for example, by the affine functions $u(x, y) = (1 - 2\theta)y \pm x$, while the first one, which is relevant for small σ, is attained by a branching-type construction. It turns out that in contrast to many other branching-type constructions, the length scale on which patterns form, depends only on the compatibility parameter θ but not on σ. Also, our upper bound construction does not show equi-partition of energy but indicates that the surface term plays a major role. We note that the second scaling implies that minimizers in this regime just fail to have gradients in BV, see Rüland et al. (2019).
Let us briefly comment on the differences of J_σ compared to I_τ. First, the double-well potential penalizing deviations from the preferred gradients $(\pm 1, \pm 1)$ is different. However, the main difference lies in the growth for large arguments which play no role in our estimates, and we can easily transfer our results to the original double-well potential, see Sect. 4.2. Next, the higher-order term in I_τ does not control the full Hessian but only the two diagonal components. This also does not influence the scaling properties, see Remark 3. Finally, we work in J_σ with a BV-type regularization while I_τ contains a quadratic regularization term. As is well known for related problems (see, e.g., Schreiber 1994; Zwicknagl 2014) this usually does not qualitatively change the scaling regimes of the minimal energy, see also Remark 2.

The functional J_σ is a four-gradient functional, and hence formally lies "in between" very well-studied problems, namely scalar models for martensitic microstructures (preferred gradients $(1, \pm 1)$), see, e.g., Kohn and Müller (1992), Kohn and Müller (1994), and the Aviles–Giga functional (preferred gradients in S^1), see, e.g., Aviles et al. (1987). While for the two-gradient problem, the minimal energy scales as

$\min\{\sigma^{2/3}\theta^{2/3},\ \theta^2\}$ (this follows by a change of variables[1] directly e.g. from Zwicknagl 2014), the minimal energy for the Aviles–Giga functional in our setting is 0.[2] Thus the scaling we prove here for the four-gradient setting indeed lies "in between." However, while any test function for the two-gradient problem yields a test function for J_σ, this functional has much more flexibility which comes from the high compatibility of the four gradients, see Fig. 1. Roughly speaking, this allows to construct test functions with low energy by covering the domain with building blocks using only the preferred gradients in the spirit of a simplified convex integration for differential inclusions, see, e.g., Conti (2008), Müller and Šverák (1999), Pompe (2010), Rüland et al. (2018), Rüland et al. (2020), Rüland and Tribuzio (2022). This relation will be explored in Sect. 5. We remark that a similar functional with corresponding four preferred magnetizations, in which patterns form due to non-local terms, has been studied in Dabade et al. (2019), Venkatraman et al. (2020).

The rest of the article is structured as follows: After briefly collecting the notation in Sect. 2, the main result will be proven in Sect. 3. We state the energy scaling law, discuss the regimes and prove the upper bound in Sect. 3.1 and the lower bound in Sect. 3.2. In Sect. 4, several generalizations are considered, including p-growth, different double-well potentials, boundary conditions on the full boundary, and rectangles. Finally, in Sect. 5, consequences for solutions of the related differential inclusion as derived in Cicalese et al. (2019) are collected.

2 Notation and Preliminaries

We will write C or c for generic constants that may change from line to line but do not depend on the problem parameters. The notation c_i with an index i indicates that these are fixed constants which do not change within a proof. We write log to denote the natural logarithm. For the ease of notation, we always identify vectors with their transposes.

For a measurable set $B \subset \mathbb{R}^n$ with $n = 1, 2$, we use the notation $|\ B\ |$ or $\mathcal{L}^n(B)$ to denote its n-dimensional Lebesgue measure.

For $\sigma > 0$ and $\theta \in (0, 1/2]$, we set

$$\mathcal{A}_\theta := \left\{ u \in W^{1,2}((0,1)^2) :\ \nabla u \in BV((0,1)^2),\ u(0, x_2) = (1-2\theta)x_2 \right\},$$

and consider the functional $E_{\sigma,\theta} : \mathcal{A}_\theta \to [0, \infty)$ by

$$E_{\sigma,\theta}(u) = \int_{(0,1)^2} \operatorname{dist}^2(\nabla u, K)\, dx + \sigma\ |\ D^2 u\ |\ (\Omega)$$

where

$$K := \{(\pm 1, \pm 1)\}.$$

[1] Set $v(x, y) = \frac{1}{2}(u(x, y) - x - (1-2\theta)y)$.

[2] Take $u(x, y) = (1-2\theta)y + 2(\theta(1-\theta))^{1/2}x$.

The expression $| D^2u | (\Omega)$ in the second term of the functional $E_{\sigma,\theta}$ denotes the total variation of the vector measure D^2u. Note that $u \in \mathcal{A}_\theta$ in particular implies that $u \in W^{1,1}((0,1)^2)$ and $\nabla u \in BV$. Hence, u has a continuous representative on the closed square $[0,1]^2$, see, e.g., (Conti and Ortiz 2016, Lemma 9). We will always identify such functions with their continuous representatives.
For a Borel set $B \subset \mathbb{R}^2$ and $u \in W^{1,2}(B)$ with $\nabla u \in BV$, we use the notation $E_{\sigma,\theta}(u; B)$ for the energy on B, i.e.,

$$E_{\sigma,\theta}(u; B) = \int_B \operatorname{dist}^2(\nabla u, K)\, dx + \sigma \,| D^2u | (B). \tag{5}$$

3 Energy Scaling on the Square $(0, 1)^2$

Our main result is the following scaling law for the minimal energy.

Theorem 1 *There exists a constant $C_T > 0$ such that for all $\sigma > 0$ and all $\theta \in (0, \frac{1}{2}]$,*

$$\frac{1}{C_T} \min \left\{ \sigma \left(\frac{| \log \sigma |}{| \log \theta |} + 1 \right), \theta^2 \right\} \leq \min_{u \in \mathcal{A}_\theta} E_{\sigma,\theta}(u) \leq C_T \min \left\{ \sigma \left(\frac{| \log \sigma |}{| \log \theta |} + 1 \right), \theta^2 \right\}.$$

We will prove the upper bound in Proposition 2 and the lower bound in Proposition 3.

Remark 1 We note some properties of the scaling regimes in Theorem 1.

(i) If $\sigma \geq \theta^2$ then

$$\sigma \left(\frac{| \log \sigma |}{| \log \theta |} + 1 \right) \geq \sigma \geq \theta^2.$$

(ii) If $\sigma \in [\theta^{k+1}, \theta^k)$ for some $k \in \mathbb{N}$, then

$$\sigma k \leq \sigma(k+1) \leq \sigma \left(\frac{| \log \sigma |}{| \log \theta |} + 1 \right) \leq \sigma(k+2) \leq 3\sigma k.$$

3.1 The Upper Bound

In this section, we provide test functions to prove the upper bound in Theorem 1. Before we start the proof, let us briefly explain the heuristics of the construction of the test function in the regime $\sigma < \theta^2$ in which the affine function does not yield the optimal scaling. Instead, we provide a branching construction which (up to a small interpolation layer) only uses the four preferred gradients. In particular, in the y-variable the function is a saw-tooth function with slope ± 1, where the volume fraction of slope $+1$ is $1-\theta$ and of slope -1 is θ to match the slope $1-2\theta = (1-\theta)\cdot 1 + \theta \cdot (-1)$

on the boundary, see Fig. 2. If we assume that $\partial_1 u = \pm 1$, we observe that from the boundary condition, we have $| u(x, y) - (1-2\theta)y | \leq x$. On the other hand, assuming that $\partial_2 u(x, y) = \pm 1$, one obtains that the number of jumps of the y-derivative on the slice $\{x\} \times (0, 1)$ is of order $\frac{\theta}{x}$, see Fig. 4. Following these estimates, we present a self-similar construction that refines in the k-th step from $x \approx \theta^k$ approximately θ^{-k+1} jumps of the y-derivative into approximately θ^{-k} jumps at $x \approx \theta^{k+1}$. If $\theta = \frac{1}{m}$ for some $m \in \mathbb{N}$ (i.e., $\theta^{-k} \in \mathbb{N}$) this can be done in an exact manner, see Fig. 2, for other θ a modification is needed leading to slightly more complicated branching patterns, see Fig. 3. Moreover, we note that although other branching constructions are in principle possible, the construction presented below yields in every construction step an approximate balance between the occurring horizontal and vertical interfaces. The proof of the lower bound (see Sect. 3.2) indicates that this is essential for a function providing the optimal scaling.

Proposition 2 *There exists a constant* $C_U > 0$ *such that for all* $\sigma > 0$ *and all* $\theta \in (0, \frac{1}{2}]$ *there exists* $u \in \mathcal{A}_\theta$ *such that*

$$E_{\sigma,\theta}(u) \leq C_U \min\left\{\sigma\left(\frac{| \log \sigma |}{| \log \theta |} + 1\right), \theta^2\right\}.$$

Proof *Step 1: Preparation.* We first note that the affine function $u_{\text{aff}}(x, y) = (1 - 2\theta)y + x$ satisfies

$$E_{\sigma,\theta}(u_{\text{aff}}) \leq 4\theta^2.$$

In view of Remark 1(i), it hence suffices to consider the case $\sigma < \theta^2$ and to construct a function u such that $E_{\sigma,\theta}(u) \leq C_U \sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1\right)$ with a constant C_U chosen below. Let $k \in \mathbb{N}$ be such that $\theta^{k+1} \leq \sigma < \theta^k$. To simplify notation, set

$$m := \lceil \frac{1}{\theta} \rceil < \frac{1}{\theta} + 1 \qquad \text{and} \qquad \delta := \frac{1}{m} \leq \theta.$$

We note that we always have $1/\delta \in \mathbb{N}$ and $\delta = 1/m > \theta/(\theta + 1)$, and hence

$$\delta\theta - (\theta - \delta) = \delta(\theta + 1) - \theta > 0.$$

We point out that many of the expressions below simplify if $\theta^{-1} \in \mathbb{N}$ since then $\delta = \theta$. For an illustration of the construction described in the next steps for $\theta = \frac{1}{3}$ see Fig. 2 (the case $\theta = 1/2$ is sketched in Fig. 6).

Finally, we fix some $N \in \mathbb{N}$ to be chosen later (see (14)).

Step 2: Construction of the building block.

As in many branching constructions (see, e.g., Kohn and Müller 1994), we first construct an auxiliary function that acts as a building block for the construction of u. For the ease of notation, we describe an admissible function via its gradient field. By the boundary condition $u(0, x_2) = (1 - 2\theta)x_2$, this uniquely determines the function u.

Precisely, we define $V : (\delta\theta, \theta] \times \mathbb{R} \to \mathbb{R}^2$ as the function which is 1-periodic in y-direction and satisfies the following (see Fig. 3):

(i) If $(x, y) \in [\delta\theta, \theta] \times [1-\delta, 1)$ then

$$V(x, y) = \begin{cases} (-1,-1) & \text{if } y \geq 1-\delta-(x-\theta) \text{ and } x \geq \delta\theta+\theta-\delta, \\ (1,-1) & \text{if } y \geq 1-\delta\theta \text{ and } x \leq \delta\theta+\theta-\delta, \\ (1,1) & \text{else.} \end{cases}$$

(ii) If $(x, y) \in [\delta\theta, \theta] \times [1-2\delta, 1-\delta)$ then

$$V(x, y) = \begin{cases} (1,-1) & \text{if } y \geq \max\{1-(1+\theta)\delta, 1-\theta+(x-\theta)\}, \\ (1,1) & \text{if } y \leq 1-(1+\theta)\delta \ \text{ and } \delta\theta \leq x \leq 2\theta-\delta-\delta\theta, \\ (-1,1) & \text{else.} \end{cases}$$

(iii) If $(x, y) \in [\delta\theta, \theta] \times [(\ell-1)\delta, \ell\delta)$ for $1 \leq \ell \leq \frac{1}{\delta} - 2$ then

$$V(x, y) = \begin{cases} (1,-1) & \text{if } y \geq \max\{(\ell-\theta)\delta,\ (\ell-\theta)\delta + x - (\delta\theta+\theta-\delta+(\ell-1)\delta\theta)\}, \\ (1,1) & \text{if } y \leq (\ell-\theta)\delta \text{ and } x \leq \delta\theta+\theta-\delta+(\ell-1)\delta\theta, \\ (-1,1) & \text{else.} \end{cases}$$

Note that V is curl-free on $(\delta\theta, \theta) \times \mathbb{R}$ as it is piecewise constant and $\nu \parallel (V^- - V^+)$ on its jump set J_V, where ν is the measure-theoretic normal to J_V, see also Fig. 3. Consequently, V is a gradient field on $(\delta\theta, \theta) \times \mathbb{R}$, and additionally,
$V(x, y) \in K$ for almost all (x, y), and

$$|\, DV \,|\, ((\delta\theta, \theta) \times (0, 1)) \leq 2(1-\delta)\theta m\, (\sqrt{2}+2) \leq 8m\theta \leq 16.$$

We will use in the next step that for the second component $V^{(2)}$ of V, we have

$$V^{(2)}(\theta, y) = V^{(2)}(\delta\theta, \delta y) \qquad \text{for all } y \in \mathbb{R}. \tag{8}$$

Step 3: Branching construction.
We now set $V_N : (\delta^N\theta, 1) \times (0, 1) \to \mathbb{R}^2$ for the fixed number $N \in \mathbb{N}$ by

$$V_N(x, y) = \begin{cases} (1,-1) & \text{if } x \geq \theta \text{ and } y \geq 1-\theta, \\ (1,1) & \text{if } x \geq \theta \text{ and } y \leq 1-\theta, \\ V(\delta^{-k+1}x, \delta^{-k+1}y) & \text{if } x \in [\delta^k\theta, \delta^{k-1}\theta) \text{ for some } 1 \leq k \leq N. \end{cases}$$

We note that V_N is curl-free as $\nu \parallel (V_N^- - V_N^+)$ on J_{V_N}, where ν is the measure-theoretic normal to J_{V_N}, see also (8) and Fig. 3. Moreover, $V_N(x, y) \in K$ for almost all $(x, y) \in (\delta^N\theta, 1)$, and

$$|\, DV_N \,|\, \Big((\delta^N\theta, 1) \times (0, 1)\Big) \leq (16+2)N + 2 \leq 20N. \tag{9}$$

Let $\tilde{u}_N : (\delta^N\theta, 1) \times (0, 1) \to \mathbb{R}$ be a potential, i.e., $\nabla\tilde{u}_N = V_N$, such that $\tilde{u}_N(\delta^N\theta, 0) = 0$. Then notice that (see Fig. 4)

$$|\tilde{u}(\delta^N\theta, y) - (1 - 2\theta)y| \leq \delta^N 2\theta. \tag{10}$$

Finally, we interpolate linearly in x to satisfy the boundary condition, and eventually define $u_N : (0, 1)^2 \to \mathbb{R}$ by

$$u_N(x, y) = \begin{cases} \tilde{u}_N(x, y) & \text{if } x \geq \delta^N\theta, \\ (1 - 2\theta)y + \delta^{-N}\theta^{-1}x\left(\tilde{u}_N(\delta^N\theta, y) - (1 - 2\theta)y\right) & \text{else.} \end{cases} \tag{11}$$

Step 4: Estimate for the energy.
By (11) and (10), we have for a.e. $(x, y) \in (0, \delta^N\theta) \times (0, 1)$ that

$$| \partial_1 u_N(x, y) | \leq 2 \quad \text{and} \quad | \partial_2 u_N(x, y) | \leq (1 - 2\theta) + | \partial_2\tilde{u}_N(\delta^N\theta, y) - (1 - 2\theta) | \leq 3.$$

In particular, we have

$$\int_{(0,1)^2} \operatorname{dist}(\nabla u_N, K)^2\, dxdy \leq 5\delta^N\theta. \tag{12}$$

Next, by (9), and since $\mathcal{L}^1\left(\left\{y \in (0, 1) : \partial_2 u_N(\delta^N\theta, y) = 1\right\}\right) = 1 - \theta$ and $\mathcal{L}^1\left(\left\{y \in (0, 1) : \partial_2 u_N(\delta^N\theta, y) = -1\right\}\right) = \theta$, we have

$$\begin{aligned} | D^2 u_N | ((0, 1)^2) &\leq 20N + | D^2 u_N | ((0, \delta^N\theta] \times (0, 1)) \\ &\leq 20N + \delta^N\theta|\partial_2\partial_2 u_N(\delta^N\theta, \cdot)|((0, 1)) + 2\int_0^1 | \partial_2\tilde{u}_N(\delta^N\theta, y) - (1 - 2\theta) |\, dy \\ &\leq 20N + \delta^N\theta 2\delta^{-N} + 2((1 - \theta)2\theta + (2 - 2\theta)\theta) \\ &\leq 20N + 10\theta \leq 30N. \end{aligned} \tag{13}$$

Now fix

$$N := \left\lceil \frac{\log\frac{\sigma}{\theta}}{\log\delta} \right\rceil. \tag{14}$$

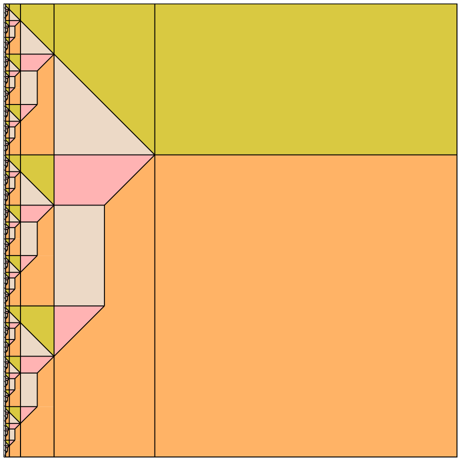

Fig. 2 Sketch of the construction for $\theta = 1/3$ and $N = 4$. The regions of the four different gradients are color-coded as in Fig. 1

Since $0 < \sigma < \theta^2 < 1$ we have $\frac{\log \frac{\sigma}{\theta}}{\log \delta} > 0$ and consequently $N \geq 1$. Combining (12) and (13) and using that $| \log \delta | \geq | \log \theta |$, we obtain

$$
\begin{aligned}
E_{\sigma,\theta}(u_N) \leq 5\theta\delta^N + 30\sigma N &\leq 5\theta\delta^{\frac{\log \frac{\sigma}{\theta}}{\log \delta}} + 30\sigma \left(\frac{\log \frac{\sigma}{\theta}}{\log \delta} + 1 \right) \\
&\leq 5\sigma + 30\sigma \left(\frac{\log \sigma}{\log \delta} + 1 \right) \\
&\leq 5\sigma + 30\sigma \left(\frac{| \log \sigma |}{| \log \theta |} + 1 \right).
\end{aligned}
$$

This concludes the proof of the upper bound with $C_U :=35$. □

Remark 2 If we replace the BV-type regularization $\sigma \mid D^2 u \mid$ by the smoother one $\sigma^2 \int_\Omega (D^2 u)^2 \, d\mathcal{L}^2$, we can use slight modifications of the above-constructed test functions to obtain the same upper bound on the energy scaling. Clearly, the function $u = (1 - 2\theta)y \pm x$ produces again an energy of order θ^2. Hence, it remains to consider the branching regime $\sigma < \theta^2$. Starting with the branching construction $u_N : (0, 1)^2 \to \mathbb{R}$ (N chosen as in (14)) which we can extend in y-direction so that ∇u_N is 1-periodic, we set

$$
\tilde{u}(x, y) := \begin{cases} u_N(x - 2\sigma, y) & \text{if } x \in (2\sigma, 1), \\ (1 - 2\theta)y & \text{if } x \in (0, 2\sigma) \end{cases}
$$

and smooth this function with a symmetric mollifier of support $B_\sigma(0)$. For the smoothed function a straightforward computation shows that one can estimate the term $\sigma^2 \int_\Omega (D^2 u)^2 \, d\mathcal{L}^2$, up to a constant, by σN. On the other hand, in addition to the region $(0, 4\sigma) \times (0, 1)$ (outside of $(0, 3\sigma) \times (0, 1)$ we have $\nabla \tilde{u} \in \mathcal{K}$) the gradient of the mollified function agrees with one of the preferred gradients except for a tube

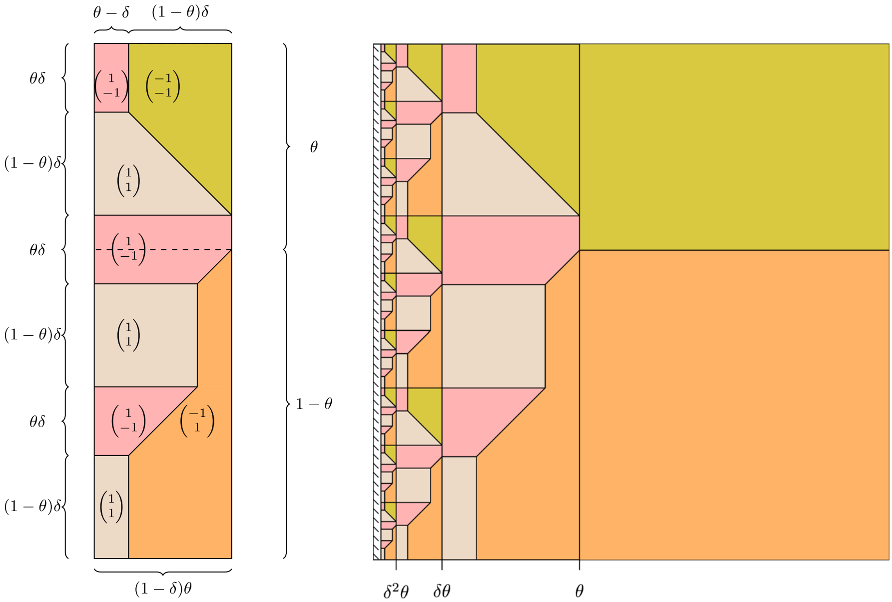

Fig. 3 Left: Gradient field V of the building block described in Step 2 for $\theta = 2/5$ and $m = 3$. Note that the left region of size $\theta - \delta$ is not needed in the construction but is rather added for the sake of an easier notation in the proof. On the other hand, due to Proposition 3 deleting this region from the construction cannot lead to an improved energy scaling. Right: The corresponding branching construction for $N = 3$

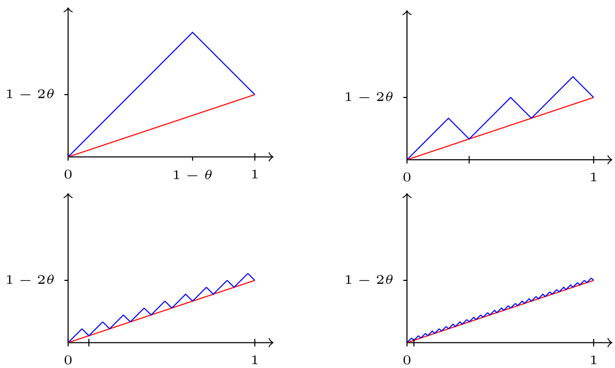

Fig. 4 Sketch of the function $u(\delta^k\theta, \cdot)$ for $\delta = \theta = \frac{1}{3}$ and $k = 0, 1, 2, 3$

with width 2σ around the jump set of $J_{\nabla\tilde{u}}$. Thus, we can estimate the second term in the energy, up to a constant, by $\sigma N + 4\sigma$. Recalling the computation at the end of the proof of Proposition 2 leads to the claimed energy scaling.

3.2 The Lower Bound

In this section, we prove the ansatz-free lower bound in Theorem 1. Precisely, we show the following statement.

Proposition 3 *There exists a constant $C_L > 0$ such that for all $\sigma > 0$, all $\theta \in (0, \frac{1}{2}]$, and all $u \in \mathcal{A}_\theta$*

$$E_{\sigma,\theta}(u) \geq C_L \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right), \theta^2\right\}.$$

Remark 3 A careful inspection of the upcoming proof shows that the same lower bound holds true even if the term $\sigma \mid D^2u \mid (\Omega)$ is replaced by the term $\sigma(\mid \partial_1\partial_1 u \mid (\Omega)+ \mid \partial_2\partial_2 u \mid (\Omega))$.

Proof The proof is split in several steps. By Lemma 4, for fixed $\theta_0 \in (0, 1/2]$, there exists a constant $c_A > 0$ such that for all $\theta \in [\theta_0, 1/2]$, all $\sigma > 0$ and all $u \in \mathcal{A}_\theta$,

$$E_{\sigma,\theta}(u) \geq c_A \min\{\sigma(|\log\sigma|+1), 1\}.$$

On the other hand, Corollary 7 shows that there exist $\theta_0 \in (0, 1/2]$, $k_0 \in \mathbb{N}$ with $k_0 \geq 2$ and $c_B > 0$ (depending only on k_0) such that for all $\sigma > 0$ and all $\theta \in (0, \theta_0]$, we have

$$\min_{u\in\mathcal{A}_\theta} E_{\sigma,\theta}(u) \geq c_B \begin{cases} \min\{\sigma, \theta^2\} & \text{if } \sigma \geq \theta^{k_0}, \\ k\sigma & \text{if } \sigma \in [\theta^{k+1}, \theta^k) \text{for some } k_0 \leq k. \end{cases} \tag{15}$$

Note that by Remark 1, this indeed implies the assertion:

(i) If $\sigma \geq 1$ then

$$\min_{u\in\mathcal{A}_\theta} E_{\sigma,\theta}(u) \geq c_B \min\{\sigma, \theta^2\} = c_B\theta^2 \geq c_B \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right), \theta^2\right\}.$$

(ii) If $\sigma \in [\theta^{\ell+1}, \theta^\ell)$ for some $0 \leq \ell < k_0$ then by Remark 1(ii), we have

$$(k_0+2)\sigma \geq (\ell+2)\sigma \geq \sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right),$$

and consequently, by (15),

$$\min_{u\in\mathcal{A}_\theta} E_{\sigma,\theta}(u) \geq c_B \frac{k_0+2}{k_0+2}\min\{\sigma, \theta^2\} \geq \frac{c_B}{k_0+2}\min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right), \theta^2\right\}.$$

(iii) If $\sigma \in (\theta^{k+1}, \theta^k)$ for some $2 \leq k_0 \leq k$ then again by Remark 1(ii), we have

$$2k\sigma \geq (k+2)\sigma \geq \sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right) \geq \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right), \theta^2\right\}$$

and consequently, by (15),

$$\min_{u\in\mathcal{A}_\theta} E_{\sigma,\theta}(u) \ge c_B k\sigma \ge \frac{c_B}{2}\min\left\{\sigma\left(\frac{|\log\sigma|}{|\log\theta|}+1\right),\theta^2\right\}.$$

Hence, choosing $c_L = \min\{c_A, \frac{c_B}{k_0+2}\}$ concludes the proof. □

As outlined above, we will prove the lower bound (15) separately for large and small θ, respectively. We build on some techniques that have been used for example in the derivation of scaling laws for martensitic microstructures, see, e.g., Conti (2006), Conti et al. (2017), Conti et al. (2020), Zwicknagl (2014).

Lemma 4 *Let $\theta_0 \in (0, 1/2]$. There exists a constant $c_A > 0$ (depending only on θ_0) such that for all $\sigma > 0$, all $\theta \in [\theta_0, 1/2]$, and all $u \in \mathcal{A}_\theta$*

$$E_{\sigma,\theta}(u) \ge c_A \min\{\sigma(|\log\sigma|+1), 1\}.$$

Proof Let u be an admissible function such that

$$E_{\sigma,\theta}(u) \le \min\{\sigma(|\log\sigma|+1), 1\}. \tag{16}$$

(Otherwise we are done.) Note that there exist measurable functions $\rho_x, \rho_y : (0,1)^2 \to \{\pm 1\}$ such that almost everywhere in $(0,1)^2$ it holds

$$\min\{|\partial_1 u - 1|, |\partial_1 u + 1|\} = |\partial_1 u + \rho_x| \quad \text{and}$$
$$\min\{|\partial_2 u - 1|, |\partial_2 u + 1|\} = |\partial_2 u + \rho_y|.$$

Step 1: Comparison of u on vertical slices to the boundary data
For almost every $x \in (0,1)$ and almost every $y \in (0,1)$ we have by the fundamental theorem of calculus that

$$\begin{aligned} u(x,y) - (1-2\theta)y &= u(x,y) - u(0,y) \\ &= \int_0^x \partial_1 u(t,y)\,\mathrm{d}t \\ &= \int_0^x (\partial_1 u(t,y) + \rho_x(t,y))\,\mathrm{d}t - \int_0^x \rho_x(t,y)\,\mathrm{d}t. \end{aligned}$$

Consequently, it holds for almost every $x \in (0,1)$ that

$$\begin{aligned} \int_0^1 |u(x,y) - (1-2\theta)y|\,\mathrm{d}y &\le \int_0^1\int_0^x |\partial_1 u(t,y) + \rho_x(t,y)|\,\mathrm{d}t\,\mathrm{d}y + x \\ &\le x^{\frac12}\left(\int_0^1\int_0^x |\partial_1 u(t,y) + \rho_x(t,y)|^2\,\mathrm{d}t\,\mathrm{d}y\right)^{\frac12} + x \\ &\le x^{\frac12} E_{\sigma,\theta}(u)^{\frac12} + x. \end{aligned}$$

Step 2: A lower bound on the energy on many vertical slices.
We fix a constant

$$0 < c_1 \leq \frac{\theta_0^3}{4 \cdot 64 \cdot 72^2}. \tag{17}$$

Let $\tilde{x} \in (0, 1)$ be such that

1. $| \partial_2 \partial_2 u | (\{\tilde{x}\} \times (0, 1)) \leq \frac{c_1}{\tilde{x}}$, and
2. $\int_0^1 \min\{| \partial_2 u(\tilde{x}, s) - 1 |, | \partial_2 u(\tilde{x}, s) + 1 |\}^2 \, \mathrm{d}s \leq \frac{\theta_0^2}{12}$.

We claim that this implies that $\tilde{x} \leq \frac{c_1}{\theta_0} \min \{\sigma (| \log \sigma | +1), 1\}$ or $\tilde{x} \geq \frac{2c_1}{\theta_0}$, so that in particular for almost all $\tilde{x} \in (\frac{c_1}{\theta_0} \min \{\sigma (| \log \sigma | +1), 1\}, \frac{2c_1}{\theta_0})$ at least one of the two properties fails. Note that by (17), this is an interval of length at least $\frac{c_1}{\theta_0}$ that is completely contained in $(0, 1)$.
To see the claim, we proceed similarly to Conti et al. (2017) and subdivide the interval $(0, 1)$ into the three subsets

$$\begin{aligned} M_1 &:= \{y \in (0, 1) : \partial_2 u(\tilde{x}, y) \geq 1 - \theta_0\}, \\ M_2 &:= \{y \in (0, 1) : \partial_2 u(\tilde{x}, y) \leq -1 + \theta_0\}, \\ \text{and } M_3 &:= \{y \in (0, 1) : -1 + \theta_0 < \partial_2 u(\tilde{x}, y) < 1 - \theta_0\}. \end{aligned}$$

Since the three sets form a partition of the interval $(0, 1)$, one of them has measure at least $\frac{1}{3}$. From property 2. it follows immediately that $| M_3 | \leq \frac{1}{12} < \frac{1}{3}$.
We consider the remaining two cases separately.

(a) Consider first the case that $| M_1 | \geq \frac{1}{3}$. By the coarea formula, we have (using property 1.)

$$\begin{aligned} &\int_{1-\frac{3\theta_0}{2}}^{1-\theta_0} \mathcal{H}^0(\partial\{y \in (0, 1) : \partial_2 u(\tilde{x}, y) > s\}) \, \mathrm{d}s \\ &\leq \int_{\mathbb{R}} \mathcal{H}^0(\partial\{y \in (0, 1) : \partial_2 u(\tilde{x}, y) > s\}) \, \mathrm{d}s \\ &= | \partial_2 \partial_2 u(\tilde{x}, \cdot) | (0, 1) \leq \frac{c_1}{\tilde{x}}. \end{aligned}$$

Therefore, there is some $s \in (1 - \frac{3\theta_0}{2}, 1 - \theta_0)$ such that $\mathcal{H}^0(\partial\{y \in (0, 1) : \partial_2 u(\tilde{x}, y) > s\}) \leq \frac{2c_1}{\theta_0 \tilde{x}}$. Since $|\{y : \partial_2 u(\tilde{x}, y) > s\}| \geq | M_1 | \geq \frac{1}{3}$, there exists a family of disjoint open intervals $(I_k)_{k=1}^K$ such that

$$K \leq \left\lceil \frac{2c_1}{\theta_0 \tilde{x}} \right\rceil, \qquad \sum_{k=1}^K | I_k | \geq \frac{1}{3} \qquad \text{and} \qquad \bigcup_{k=1}^K I_k \subseteq \{y \in (0, 1) : \partial_2 u(\tilde{x}, y) \geq s\}.$$

On each interval I_k, we have $\partial_2 u(\tilde{x}, \cdot) \geq 1 - \frac{3\theta_0}{2}$, which implies that for all $y \in I_k$ (recall that $1 - \frac{3\theta_0}{2} = 1 - 2\theta_0 + \theta_0/2 \geq 1 - 2\theta + \theta_0/2$)

$$
\begin{aligned}
|u(\tilde{x}, y) - (1-2\theta)y| &= \left| u(\tilde{x}, 0) + \int_0^y (\partial_2 u(\tilde{x}, t) - (1-2\theta))\, dt \right| \\
&\geq \left| |u(\tilde{x}, 0)| - \left| \int_0^y (\partial_2 u(\tilde{x}, t) - (1-2\theta))\, dt \right| \right| \\
&\geq \min_{\alpha \in \mathbb{R}} |\alpha - \frac{\theta_0}{2} y|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\int_{I_k} |\ u(\tilde{x}, y) - (1-2\theta)y\ |\ \mathrm{d}y &\geq \min_{\alpha \in \mathbb{R}} \int_{I_k} |\alpha - \frac{\theta_0}{2} y|\, \mathrm{d}y \\
&= 2\frac{\theta_0}{2} \int_0^{|I_k|/2} y\, dy = \frac{\theta_0}{8}\ |\ I_k\ |^2 . \qquad (18)
\end{aligned}
$$

Summing estimate (18) over k and using *Step 1* yields by assumption (16)

$$
\begin{aligned}
\sum_{k=1}^{K} \frac{\theta_0}{8}\ |\ I_k\ |^2 &\leq \int_0^1 |u(\tilde{x}, y) - (1-2\theta)y|\, \mathrm{d}y \leq \tilde{x}^{\frac{1}{2}} E_{\sigma,\theta}(u)^{\frac{1}{2}} + \tilde{x} \\
&\leq \tilde{x}^{\frac{1}{2}} \min\{\sigma(|\log\sigma\ | + 1), 1\}^{\frac{1}{2}} + \tilde{x}. \qquad (19)
\end{aligned}
$$

There are two possibilities: If $K = 1 \geq \frac{2c_1}{\theta_0 \tilde{x}}$ then $\tilde{x} \geq \frac{2c_1}{\theta_0}$. Otherwise, we have $1 < K \leq \left\lceil \frac{2c_1}{\theta_0 \tilde{x}} \right\rceil \leq \frac{4c_1}{\theta_0 \tilde{x}}$ and from $\sum_{k=1}^{K} |\ I_k\ | \geq \frac{1}{3}$ we deduce by convexity and (19) that

$$
\frac{\theta_0^2}{72} \cdot \frac{\tilde{x}}{4c_1} \leq \frac{\theta_0}{8} \frac{K}{(3K)^2} \leq \sum_{k=1}^{K} \frac{\theta_0}{8}\ |\ I_k\ |^2 \leq \tilde{x}^{\frac{1}{2}} \min\{\sigma(|\log\sigma\ | + 1), 1\}^{\frac{1}{2}} + \tilde{x}. \qquad (20)
$$

Note that for $t \geq \frac{4 \cdot 64 \cdot 72^2 \cdot c_1^2}{\theta_0^4} \min\{\sigma(|\log\sigma\ | + 1), 1\}$,

$$
t^{-1/2} \min\{\sigma(|\log\sigma\ | + 1), 1\}^{1/2} + 1 \leq \frac{\theta_0^2}{8 \cdot 72 \cdot c_1},
$$

which implies that (20) can only hold for $\tilde{x} < \frac{4 \cdot 64 \cdot 72^2 c_1^2}{\theta_0^4} \min\{\sigma(|\log\sigma\ | + 1), 1\}$. Note that $\frac{4 \cdot 64 \cdot 72^2 \cdot c_1^2}{\theta_0^4} \leq \frac{c_1}{\theta_0}$ which concludes the proof of the claim in this case.

(b) Consider now the case $| M_2 | \geq \frac{1}{3}$. We proceed along the lines of (a), and find that there is some $t \in (-1, -1+\theta_0)$ such that $\mathcal{H}^0(\partial\{y \in (0,1) : \partial_2 u(\tilde{x}, \cdot) < t\}) \leq \frac{c_1}{\theta_0 \tilde{x}}$. Consequently, there exists a family of disjoint open intervals $(I_k)_{k=1}^K$ such that

$$K \leq \left\lceil \frac{c_1}{\theta_0 \tilde{x}} \right\rceil, \qquad \sum_{k=1}^K | I_k | \geq \frac{1}{3} \qquad \text{and} \qquad \bigcup_{k=1}^K I_k \subseteq \{y \in (0,1) : \partial_2 u(\tilde{x}, y) \leq t\}.$$

In this case, we obtain that in each interval I_k,

$$\begin{aligned}\int_{I_k} | u(\tilde{x}, y) - (1-2\theta)y | \, \mathrm{d}y &\geq \min_{\alpha \in \mathbb{R}} \int_{I_k} |\alpha - (1-\theta_0)y| \, \mathrm{d}y \\ &= \frac{1-\theta_0}{4} | I_k |^2 \geq \frac{\theta_0}{8} | I_k |^2,\end{aligned}$$

and the rest follows as in case (a).

Step 3. Conclusion.

By *Step 2*, for almost all $\tilde{x} \in (\frac{c_1}{\theta_0} \min\{\sigma(| \log \sigma | + 1), 1\}, \frac{2c_1}{\theta_0})$ at least one of the properties 1. or 2. is not true. Consequently,

$$E_{\sigma,\theta}(u) \geq \int_{\frac{c_1}{\theta_0} \min\{\sigma(|\log\sigma|+1),1\}}^{\frac{2c_1}{\theta_0}} \min\left\{\frac{\sigma c_1}{x}, \frac{\theta_0^2}{12}\right\} \mathrm{d}x. \tag{21}$$

We now consider two cases separately.

(i) If $\sigma \leq \frac{1}{2}$ then we find for $x \geq \frac{c_1}{\theta_0} \min\{\sigma(| \log \sigma | + 1), 1\} = \frac{c_1}{\theta_0} \sigma(| \log \sigma | + 1)$ that

$$\frac{\sigma c_1}{x} \leq \frac{\sigma c_1 \theta_0}{c_1 \sigma(| \log \sigma | + 1)} \leq \theta_0.$$

Hence, $\min\{\frac{\sigma c_1}{x}, \frac{\theta_0^2}{12}\} \geq \frac{\theta_0}{12} \frac{\sigma c_1}{x}$ for all $x \in (\frac{c_1}{\theta_0} \min\{\sigma(| \log \sigma | + 1), 1\}, \frac{2c_1}{\theta_0})$, and since $\log(| \log \sigma | + 1) \leq \max\left\{\frac{9}{10} \log(2), \frac{3}{4} | \log(\sigma) |\right\}$, we deduce from (21)

$$\begin{aligned}E_{\sigma,\theta}(u) &\geq \frac{\theta_0}{12} \int_{\frac{c_1}{\theta_0}\sigma(|\log\sigma|+1)}^{\frac{2c_1}{\theta_0}} \frac{\sigma c_1}{x} \, \mathrm{d}x = \frac{\theta_0}{12} c_1 \sigma \left(\log(2) - \log(\sigma(| \log \sigma | + 1))\right) \\ &\geq \frac{\theta_0}{12} c_1 \sigma \left(\frac{1}{10} \log(2) + \frac{1}{4} | \log(\sigma) |\right) \geq \frac{\theta_0 c_1}{12} \frac{\log(2)}{10} \sigma \left(| \log(\sigma) | + 1\right),\end{aligned} \tag{23}$$

which concludes the proof in this case.

(ii) If $\sigma > 1/2$, we have for all $x \leq \frac{2c_1}{\theta_0}$ that $\frac{\sigma c_1}{x} \geq \frac{\sigma\theta_0}{2} \geq \frac{\theta_0}{4} \geq \frac{\theta_0^2}{12}$, and hence

$$E_{\sigma,\theta}(u) \geq \int_{\frac{c_1}{\theta_0}}^{\frac{2c_1}{\theta_0}} \frac{\theta_0^2}{12} = \frac{c_1\theta_0}{12}. \tag{24}$$

If we choose $c_A := \frac{\theta_0 c_1}{12}\frac{\log(2)}{10}$, the assertion follows from (23) and (24). □

The proof of the lower bound in the case $\theta \ll 1$ is split into two lemmata which combined lead to the estimate in Corollary 7. We start with a general estimate which will be relevant for the lower bound only if $\sigma \geq \theta^{32}$.

Lemma 5 *There exists $c_B^{(1)} > 0$ such that for all $\theta \in (0, 1/2]$, all $\sigma > 0$, and all $u \in \mathcal{A}_\theta$, we have*

$$E_{\sigma,\theta}(u) \geq c_B^{(1)} \min\{\theta^2, \sigma\}.$$

Proof Let u be an admissible function such that $E_{\sigma,\theta}(u) \leq \frac{1}{2\cdot 24^2}\min\{\theta^2,\sigma\}$ (otherwise there is nothing to prove).

Step 1: Choice of representative vertical and horizontal slices.

By Fubini and slicing, we find $\tilde{x} \in (\frac{3}{4}, 1)$ satisfying

$$\begin{aligned}
&\int_0^1 \min\{|\,\partial_2 u(\tilde{x},t) - 1\,|, |\,\partial_2 u(\tilde{x},y) + 1\,|\}^2 \,\mathrm{d}y + \sigma|\partial_2\partial_2 u(\tilde{x},\cdot)|(0,1) \\
&\quad \leq 4\Big(\int_{3/4}^1\int_0^1 \min\{|\,\partial_2 u(x,y) - 1\,|, |\,\partial_2 u(x,y) + 1\,|\}^2 \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \sigma|D^2 u|((3/4,1)\times(0,1))\Big) \\
&\quad \leq 4E_{\sigma,\theta}(u) \leq \frac{2}{24^2}\min\{\theta^2,\sigma\}.
\end{aligned} \tag{25}$$

Similarly, there are $y_1 \in (1/8, 1/4)$ and $y_2 \in (3/4, 7/8)$ such that

$$\begin{aligned}
&\int_0^1 \min\{|\,\partial_1 u(x,y_1) - 1\,|, |\,\partial_1 u(x,y_1) + 1\,|\}^2 \,\mathrm{d}x + \sigma|\partial_1\partial_1 u(\cdot,y_1)|(0,1) \\
&\quad \leq \frac{4}{24^2}\min\{\theta^2,\sigma\} \quad \text{and} \\
&\int_0^1 \min\{|\,\partial_1 u(x,y_2) - 1\,|, |\,\partial_1 u(x,y_2) + 1\,|\}^2 \,\mathrm{d}x + \sigma|\partial_1\partial_1 u(\cdot,y_2)|(0,1) \\
&\quad \leq \frac{4}{24^2}\min\{\theta^2,\sigma\}.
\end{aligned} \tag{26}$$

Step 2: Partial derivatives do not jump between the wells on the chosen slices. By the coarea formula,

$$\int_{-1/2}^{1/2} \mathcal{H}^0\left(\partial\left\{x \in (0,1) : \partial_1 u(x, y_i) > s\right\}\right) \mathrm{d}s \leq |\partial_1\partial_1 u(\cdot, y_i)|(0,1) \quad \text{for } i = 1, 2.$$

Hence, there exists $\tilde{s} \in (-\frac{1}{2}, \frac{1}{2})$ such that

$$\begin{aligned}\mathcal{H}^0(\partial\{x : \partial_1 u(x, y_1) > \tilde{s}\}) &+ \mathcal{H}^0(\partial\{x : \partial_1 u(x, y_2) > \tilde{s}\}) \\ &\leq |\partial_1\partial_1 u(\cdot, y_1)|\,((0,1)) + |\partial_1\partial_1 u(\cdot, y_2)|(0,1) \leq \frac{8}{24^2} < 1,\end{aligned}$$

and therefore

$$\mathcal{H}^0(\partial\{x : \partial_1 u(x, y_1) > \tilde{s}\}) = \mathcal{H}^0(\partial\{x : \partial_1 u(x, y_2) > \tilde{s}\}) = 0.$$

In particular, we have for $i = 1, 2$ either $\partial_1 u(x, y_i) \geq \tilde{s}$ for almost every $x \in (0, 1)$ or $\partial_1 u(x, y_i) \leq \tilde{s}$ for almost every $x \in (0, 1)$. Without loss of generality, we may assume that

$$\partial_1 u(x, y_1) \geq \tilde{s} \quad \text{for almost every } x \in (0,1), \tag{27}$$

the other case can be treated analogously. Note that we will consider the two possibilities for y_2 separately in the sequel.
Proceeding analogously, we also find some $\tilde{t} \in (-\frac{3}{4}, -\frac{1}{2})$ such that

$$\mathcal{H}^0\left(\partial\{y \in (0,1) : \partial_2 u(\tilde{x}, y) > \tilde{t}\}\right) = 0. \tag{28}$$

Step 3: A lower bound for the energy.
By (27), for almost every $x \in (0, 1)$ we have $\partial_1 u(x, y_1) \geq \tilde{s} > -1/2$, which implies that

$$|\partial_1 u(x, y_1) - 1| < 3|\partial_1 u(x, y_1) + 1|.$$

Hence, by Hölder's inequality, we have for every $x \in (0, 1)$, using the choice of y_1 (see (26))

$$\begin{aligned}\left|\int_0^x \partial_1 u(t, y_1)\,\mathrm{d}t - x\right|^2 &\leq \left(\int_0^x |\partial_1 u(t, y_1) - 1|\,\mathrm{d}t\right)^2 \leq \int_0^x |\,\partial_1 u(t, y_1) - 1\,|^2 \,\mathrm{d}t \\ &\leq 9\int_0^x \min\{|\,\partial_1 u(t, y_1) - 1\,|, |\,\partial_1 u(t, y_1) + 1\,|\}^2 \,\mathrm{d}t \\ &\leq \left(\frac{\theta}{4}\right)^2.\end{aligned}$$

Consequently, by the fundamental theorem of calculus,

$$(u(x, y_1) - u(0, y_1)) - x = \int_0^x \partial_1 u(t, y_1)\,\mathrm{d}t - x \in \left(-\frac{\theta}{4}, \frac{\theta}{4}\right). \tag{29}$$

We proceed similarly for y_2 where we consider the two cases from *Step 2* separately. If $\partial_1 u(x, y_2) \geq \tilde{s}$ for almost every $x \in (0, 1)$ then as above for y_1,

$$(u(x, y_2) - u(0, y_2)) - x \in \left(-\frac{\theta}{4}, \frac{\theta}{4}\right) \qquad \text{for all } x \in (0, 1).$$

In the other case, i.e., if $\partial_1 u(x, y_2) \leq \tilde{s}$ for almost every $x \in (0, 1)$ then we find similarly

$$(u(x, y_2) - u(0, y_2)) + x \in \left(-\frac{\theta}{4}, \frac{\theta}{4}\right) \qquad \text{for all } x \in (0, 1).$$

We consider the two cases separately.

Case 1: Suppose that $u(x, y_2) - u(0, y_2) - x \in (-\frac{\theta}{4}, \frac{\theta}{4})$ for all $x \in (0, 1)$. Recalling (29), in this case, we have that $x - \frac{\theta}{4} < u(x, y_i) - u(0, y_i) < x + \frac{\theta}{4}$ for $i \in \{1, 2\}$. Hence, using $y_2 - y_1 \geq \frac{1}{2}$ and the boundary condition at $x = 0$, we obtain

$$\begin{aligned}
&(1 - 3\theta)(y_2 - y_1) \\
&\quad \leq (1 - 2\theta)(y_2 - y_1) - \frac{1}{2}\theta = u(0, y_2) - u(0, y_1) - \frac{\theta}{2} \\
&\quad = u(x, y_2) - (u(x, y_2) - u(0, y_2)) - \frac{\theta}{4} - u(x, y_1) + (u(x, y_1) - u(0, y_1)) - \frac{\theta}{4} \\
&\quad \leq u(x, y_2) - u(x, y_1) \\
&\quad = u(0, y_2) + (u(x, y_2) - u(0, y_2)) - u(0, y_1) - (u(x, y_1) - u(0, y_1)) \\
&\quad \leq (1 - 2\theta)(y_2 - y_1) + \frac{\theta}{2} \leq (1 - \theta)(y_2 - y_1),
\end{aligned}$$

which implies that

$$-\frac{1}{2} \leq 1 - 3\theta \leq \frac{u(x, y_2) - u(x, y_1)}{y_2 - y_1} \leq 1 - \theta \qquad \text{for all } x \in (0, 1), \tag{30}$$

and hence in particular

$$\left| \frac{u(x, y_2) - u(x, y_1)}{y_2 - y_1} - 1 \right| \geq \theta \qquad \text{for all } x \in (0, 1).$$

We now consider $\tilde{x}$ as chosen in *Step 1*. Using the lower bound in (30) we deduce from (28) that for almost all $y \in (0, 1)$ we have $\partial_2 u(\tilde{x}, y) > \tilde{t} > -\frac{3}{4}$, and hence for almost all $y \in (0, 1)$

$$| \partial_2 u(\tilde{x}, y) - 1 | \leq 7 \min\{| \partial_2 u(\tilde{x}, y) - 1 |, | \partial_2 u(\tilde{x}, y) + 1 |\}.$$

Putting things together, we obtain by Hölder's inequality (recall that $| y_2 - y_1 | \geq 1/2$), using (25) in the last step,

$$\begin{aligned} \theta^2 &\leq \left| \frac{u(\tilde{x}, y_2) - u(\tilde{x}, y_1)}{y_2 - y_1} - 1 \right|^2 = \left| \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \partial_2 u(\tilde{x}, y) \, \mathrm{d}y - 1 \right|^2 \\ &\leq \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} | \partial_2 u(\tilde{x}, y) - 1 |^2 \, \mathrm{d}y \leq 2 \int_0^1 | \partial_2 u(\tilde{x}, y) - 1 |^2 \, \mathrm{d}y \\ &\leq 98 \int_0^1 \min\{| \partial_2 u(\tilde{x}, y) - 1 |, | \partial_2 u(\tilde{x}, y) + 1 |\}^2 \, \mathrm{d}y \leq 392 E_{\sigma,\theta}(u). \end{aligned} \tag{31}$$

This concludes the proof of the lower bound in this case if $c_B \leq \frac{1}{392}$.

Case 2: Suppose that $u(x, y_2) - u(0, y_2) + x \in (-\frac{\theta}{4}, \frac{\theta}{4})$ for all $x \in (0, 1)$. In this case, we have (recall (29))

$$x - \frac{\theta}{4} < u(x, y_1) - u(0, y_1) < x + \frac{\theta}{4} \quad \text{and} \quad -x - \frac{\theta}{4} < u(x, y_2) - u(0, y_2) < -x + \frac{\theta}{4}.$$

Proceeding similarly to Case 1, we apply this with $x = \tilde{x}$, and we obtain using that $\tilde{x} \in (3/4, 1)$, $\frac{3}{4} \geq y_2 - y_1 \geq 1/2$, and the boundary condition at $x = 0$,

$$\begin{aligned} &(-3 - 3\theta)(y_2 - y_1) \\ &\leq (1 - 2\theta)(y_2 - y_1) - 2 - \frac{\theta}{2} \leq (1 - 2\theta)(y_2 - y_1) - 2\tilde{x} - \frac{\theta}{2} \\ &\leq u(0, y_2) - u(0, y_1) + (u(\tilde{x}, y_2) - u(0, y_2)) - (u(\tilde{x}, y_1) - u(0, y_1)) \\ &= u(\tilde{x}, y_2) - u(\tilde{x}, y_1) \\ &\leq (1 - 2\theta)(y_2 - y_1) - 2\tilde{x} + \frac{\theta}{2} \leq (1 - 2\theta)(y_2 - y_1) - \frac{3}{2} + \frac{\theta}{2} \\ &\leq (-1 - \theta)(y_2 - y_1). \end{aligned}$$

This yields in particular

$$-3 - 3\theta \leq \frac{u(\tilde{x}, y_2) - u(\tilde{x}, y_1)}{y_2 - y_1} \leq -1 - \theta \leq \tilde{t},$$

and we deduce from (28) that $\partial_2 u(\tilde{x}, y) \leq \tilde{t}$ for almost all $y \in (0, 1)$. Thus,

$$| \partial_2 u(\tilde{x}, y) + 1 | = \min\{| \partial_2 u(\tilde{x}, y) - 1 |, | \partial_2 u(\tilde{x}, y) + 1 |\} \qquad \text{for almost all } y \in (0, 1).$$

Since

$$\left| \frac{u(\tilde{x}, y_2) - u(\tilde{x}, y_1)}{y_2 - y_1} + 1 \right| \geq \theta,$$

the claimed lower bound on $E_{\sigma,\theta}(u)$ follows as in (31) (with an even better constant). This concludes the proof of the lemma. □

Lemma 6 *There exists a constant* $c_B^{(2)} > 0$ *with the following property. For all* $\theta \in (0, 1/2]$ *and all* $\sigma \in (0, \theta^k)$ *for some* $k \in \mathbb{N}$ *with* $k \geq 32$*, we have*

$$E_{\sigma,\theta}(u) \geq c_B^{(2)} k\sigma.$$

Proof It suffices to consider $\sigma \in [\theta^{k+1}, \theta^k)$ for some $k \geq 32$. Let $u \in \mathcal{A}_\theta$ be such that $E_{\sigma,\theta}(u) \leq k\sigma$ (otherwise there is nothing to prove). Let $K = \lfloor \frac{k}{8} \rfloor$. We fix $c_2 := \frac{1}{2000}$.

Step 1: Choice of representative vertical slices and reduction to auxiliary statement. By Fubini's theorem and standard slicing arguments we can find points $x_i \in (\frac{1}{2}\theta^{2i}, \frac{3}{2}\theta^{2i})$, $i = 1, \ldots, K$, such that $u(x_i, \cdot) \in H^1(0, 1)$, $\partial_2 u(x_i, \cdot) \in BV(0, 1)$ and

$$\begin{aligned} &\int_0^1 \min\{| \partial_2 u(x_i, y) - 1 |, | \partial_2 u(x_i, y) + 1 |\}^2 \, dy + \sigma \mid \partial_2 \partial_2 u(x_i, \cdot) \mid (0, 1) \\ &\leq \theta^{-2i} E_{\sigma,\theta} \left(u, (\frac{1}{2}\theta^{2i}, \frac{3}{2}\theta^{2i}) \times (0, 1) \right). \end{aligned}$$

For an illustration, see Fig. 5. Note that by the assumption $\theta \leq 1/2$, the intervals $(\frac{1}{2}\theta^{2i}, \frac{3}{2}\theta^{2i})$ for $1 = 1, \ldots, K$ are pairwise disjoint and contained in $(0, 1)$.
In the subsequent steps, we will prove the following result: There exists a constant $c_3 > 0$ (not depending on σ and θ) such that for each $1 \leq i \leq K - 1$ we have (recall that we chose c_2 above)

$$\int_0^1 \min\{| \partial_2 u(x_i, y) - 1 |, | \partial_2 u(x_i, y) + 1 |\}^2 dy + \sigma \mid \partial_2 \partial_2 u(x_i, \cdot) \mid (0, 1) \geq c_2 \theta^{-2i} \sigma \quad (32)$$

or

$$E_{\sigma,\theta}(u; (x_{i+1}, x_i) \times (0, 1)) \geq c_3 \sigma. \quad (33)$$

Note that this indeed implies the assertion since (using the choice of x_i)

$$E_{\sigma,\theta}(u) \geq \frac{1}{2}\sum_{i=1}^{K-1}\left(E_{\sigma,\theta}(u,(\frac{1}{2}\theta^{2i},\frac{3}{2}\theta^{2i})\times(0,1)) + E_{\sigma,\theta}(u,(x_{i+1},x_i)\times(0,1))\right)$$
$$\geq \frac{1}{2}\sum_{i=1}^{K-1}\Bigg(\theta^{2i}\int_0^1 \min\{|\,\partial_2 u(x_i,y)-1\,|,|\,\partial_2 u(x_i,y)+1\,|\}^2\,dy$$
$$+\theta^{2i}\sigma\,|\,\partial_2\partial_2 u(x_i,\cdot)\,|\,(0,1) + E_{\sigma,\theta}(u,(x_{i+1},x_i)\times(0,1))\Bigg)$$
$$\geq \frac{1}{2}\sum_{i=1}^{K-1}\min\{c_2,c_3\}\sigma \geq \frac{\min\{c_2,c_3\}}{32}k\sigma.$$

Therefore, from now on, we fix some $1 \leq i \leq K-1$ and assume that (32) does not hold, i.e., we have

$$\int_0^1 \min\{|\,\partial_2 u(x_i,y)-1\,|,|\,\partial_2 u(x_i,y)+1\,|\}^2\,dy + \sigma\,|\,\partial_2\partial_2 u(x_i,\cdot)\,|\,(0,1) \leq c_2\theta^{-2i}\sigma \quad (34)$$

In the rest of the proof, we will show that (33) holds for a constant $c_3 > 0$ chosen below.

Step 2: Choice of a large representative portion with an almost constant derivative. We note that inequality (34) implies that $|\,\partial_2\partial_2 u(x_i,\cdot)\,|\,(0,1) \leq c_2\theta^{-2i}$ and

$$\int_0^1 \min\{|\,\partial_2 u(x_i,y)-1\,|,|\,\partial_2 u(x_i,y)+1\,|\}^2\,dy$$
$$\leq c_2\theta^{-2i}\sigma$$
$$\leq c_2\theta^{-2K+k} \leq c_2\theta^3. \quad (35)$$

We proceed similarly to the proof of Lemma 5. By the choice of c_2, the set

$$P_1 := \{y \in (0,1) : \partial_2 u(x_i,y) \in (-1+\theta, 1-\theta) \text{ or } \partial_2 u(x_i,y) \leq -1-\theta$$
$$\text{or } \partial_2 u(x_i,y) \geq 1+\theta\}$$

has measure (much) less than $1/3$. Next we show that also the set $P_1 := \{y \in (0,1) : -1-\theta \leq \partial_2 u(x_i,y) \leq -1+\theta\}$ has measure less than $1/3$. For a contradiction, let us assume that $\mathcal{L}^1(P_1) \geq 1/3$. We find $y_2, y_1 \in (0,1)$ such that $y_2 - y_1 \geq 1 - \frac{1}{12}$ and for $j = 1, 2$

$$\int_0^1 \min\{|\,\partial_1 u(x,y_j)-1\,|,|\,\partial_1 u(x,y_j)+1\,|\}^2\,dx \leq 24E_{\sigma,\theta}(u).$$

Then we estimate, using that $E_{\sigma,\theta}(u) \leq k\sigma \leq \theta^2$,

$$\int_{y_1}^{y_2} \partial_2 u(x_i, t)\, dt = u(x_i, y_2) - u(x_i, y_1)$$
$$\geq u(0, y_2) - u(0, y_1) - 2x_i - 2x_i^{\frac{1}{2}} \left(24E_{\sigma,\theta}(u)\right)^{\frac{1}{2}}$$
$$\geq (1-2\theta)(1-\frac{1}{12}) - 3\theta^2 - 2\left(36\theta^4\right)^{\frac{1}{2}}.$$

Since $\mathcal{L}^1\left((y_2, y_1) \cap P_1\right) \geq \mathcal{L}^1(P_1) - \mathcal{L}^1\left((0,1)\setminus(y_1, y_2)\right) \geq \frac{1}{3} - \frac{1}{12}$, we have for $\theta_0 \leq \frac{1}{16}$

$$\begin{aligned}&\int_{(y_2,y_1)\cap\{\partial_2 u(x_i,\cdot)\geq 0\}} \partial_2 u(x_i, t)\, dt \\ &\geq (1-2\theta)(1-\frac{1}{12}) - 3\theta^2 - 12\theta^2 - (-1+\theta)(\frac{1}{3} - \frac{1}{12}) \\ &\geq \frac{14}{12} - \frac{1}{16}\left(2 + \frac{3}{16} + \frac{3}{4} + \frac{1}{4}\right) \\ &\geq \frac{9}{10}.\end{aligned} \tag{36}$$

On the other hand, we have by (35) and the choice of c_2

$$\begin{aligned}&\int_{(y_2,y_1)\cap\{\partial_2 u(x_i,\cdot)\geq 0\}} \partial_2 u(x_i, t)\, dt \\ &\leq \left(\int_{(y_2,y_1)\cap\{\partial_2 u(x_i,\cdot)\geq 0\}} (\partial_2 u(x_i,t) - 1)^2\, dt\right)^{\frac{1}{2}} + |\,\{\partial_2 u(x_i,\cdot) \geq 0\}\,| \\ &\leq c_2{}^{\frac{1}{2}}\theta + \frac{2}{3} < 0.67.\end{aligned} \tag{37}$$

Combining (36) and (37) yields a contradiction. Consequently, the set

$$P_2 := \{y \in (0,1) :\ 1-\theta \leq \partial_2 u(x_i, y) \leq 1+\theta\}$$

has measure at least $1/3$. Hence, using the coarea formula, we derive that there exists $\tilde{\theta} \in (\theta, 3/2\theta)$ such that $A_i = \{\partial_2 u(x_i,\cdot) \in (1-\tilde{\theta}, 1+\tilde{\theta})\}$ is of finite perimeter and such that

$$\mathcal{L}^1(A_i) \geq \frac{1}{3} \text{ and } \quad \mathcal{H}^0(\partial^* A_i) \leq 2c_2\theta^{-2i-1}.$$

Let us now consider the disjoint intervals $I_l = (\frac{l}{9c_2}\theta^{2i+1}, \frac{(l+1)}{9c_2}\theta^{2i+1})$ for $l \in \mathbb{N}_0$. For θ_0 so small that $c_2\theta_0^{-3} \geq 2$ it follows that at least $\frac{c_2}{2}\theta^{-2i-1}$ of those intervals are (up to a set of measure 0) contained in A_i. Indeed, by volume considerations the number of intervals I_l intersecting A_i is larger than $3c_2\theta^{-2i-1}$. In addition, the number of intervals that contain a point of $\partial^* A_i$ is bounded by the number of points in $\partial^* A_i$. Eventually, there might be an interval intersecting A_i, which does not contain a point from $\partial^* A_i$ but is not fully contained in (0, 1). Hence, the number of intervals that are fully contained in A_i is at least $3c_2\theta^{-2i-1} - c_2 2\theta^{-2i-1} - 1 \geq \frac{c_2}{2}\theta^{-2i-1}$. In particular, we can find an interval $I_{\bar{l}} \subseteq A_i$ such that

$$\begin{aligned} &E_{\sigma,\theta}(u; (0,1) \times I_{\bar{l}}) \leq 40\,\mathcal{L}^1(I_{\bar{l}})\, E_{\sigma,\theta}(u) \\ \text{and}\quad &E_{\sigma,\theta}(u; (x_{i+1}, x_i) \times I_{\bar{l}}) \leq 40\,\mathcal{L}^1(I_{\bar{l}})\, E_{\sigma,\theta}(u; (x_{i+1}, x_i) \times (0,1)) \end{aligned} \tag{38}$$

Step 3: Estimate on horizontal difference quotients between $\{x_i\} \times (0, 1)$ *and* $\{x_{i+1}\} \times (0, 1)$.

Let us write $I_{\bar{l}} = (a_{\bar{l}}, b_{\bar{l}})$ and estimate for $t \in (a_{\bar{l}}, a_{\bar{l}}+ \mid I_{\bar{l}} \mid /2)$

$$\begin{aligned} &|u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) + u(x_{i+1}, t)| \\ &\quad \geq |u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - (1-2\theta) \mid I_{\bar{l}} \mid /2| - |u(x_{i+1}, t) - (1-2\theta)t| \\ &\quad\quad - |u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) - (1-2\theta)(t+ \mid I_{\bar{l}} \mid /2)|. \end{aligned} \tag{39}$$

By the definition of A_i and $I_{\bar{l}}$, we find for the first term of the right-hand side (using $\tilde{\theta} \in (\theta, 3/2\theta)$)

$$|u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - (1-2\theta) \mid I_{\bar{l}} \mid /2|\, dt \geq \frac{\mid I_{\bar{l}} \mid \theta}{4}.$$

For the second term, we follow the argument of *Step 1* in the proof of Lemma 4 and find for almost every $t \in (a_{\bar{l}}, a_{\bar{l}}+ \mid I_{\bar{l}} \mid /2)$

$$\begin{aligned} &|u(x_{i+1}, t) - (1-2\theta)t| \\ &\quad = |u(x_{i+1}, t) - u(0, t)| \\ &\quad \leq x_{i+1}^{\frac{1}{2}} \left(\int_0^1 \min\{\mid \partial_1 u(s,t) - 1 \mid, \mid \partial_1 u(s,t) + 1 \mid\}^2\, ds \right)^{\frac{1}{2}} ds + x_{i+1}. \end{aligned}$$

The third term in (39) can be treated similarly. Putting things together, (39) yields for almost every t

$$\begin{aligned} &|u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) + u(x_{i+1}, t)| \\ &\quad \geq \frac{\mid I_{\bar{l}} \mid \theta}{4} - 2x_{i+1} - x_{i+1}^{\frac{1}{2}} \left(\int_0^1 \min\{\mid \partial_1 u(s,t) - 1 \mid, \mid \partial_1 u(s,t) + 1 \mid\}^2\, ds \right)^{\frac{1}{2}} \\ &\quad\quad - x_{i+1}^{\frac{1}{2}} \left(\int_0^1 \min\{\mid \partial_1 u(s, t + \frac{\mid I_{\bar{l}} \mid}{2}) - 1 \mid, \mid \partial_1 u(s, t + \frac{\mid I_{\bar{l}} \mid}{2}) + 1 \mid\}^2\, ds \right)^{\frac{1}{2}}. \end{aligned} \tag{40}$$

On the other hand, we may estimate similarly to above for almost every t

$$
\begin{aligned}
&|u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) + u(x_{i+1}, t)| \\
&\quad \le |u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - (1-2\theta) \mid I_{\bar{l}} \mid /2| \\
&\qquad + |u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) - u(x_{i+1}, t) - (1-2\theta) \mid I_{\bar{l}} \mid /2| \\
&\quad \le 2\theta |I_{\bar{l}}| + 2x_{i+1} + x_{i+1}^{\frac{1}{2}} \left(\int_0^1 \min\{\mid \partial_1 u(s,t) - 1 \mid, \mid \partial_1 u(s,t) + 1 \mid\}^2 \, ds \right)^{\frac{1}{2}} \\
&\qquad + x_{i+1}^{\frac{1}{2}} \left(\int_0^1 \min\{\mid \partial_1 u(s, t + \frac{\mid I_l \mid}{2}) - 1 \mid, \mid \partial_1 u(s, t + \frac{\mid I_l \mid}{2}) + 1 \mid\}^2 \, ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{41}
$$

Next, we notice that by the choice of $I_{\bar{l}}$ (see (38))

$$
\begin{aligned}
&\int_{a_{\bar{l}}}^{a_{\bar{l}} + |I_{\bar{l}}|/2} \int_0^1 \min\{\mid \partial_1 u(s,t) - 1 \mid, \mid \partial_1 u(s,t) + 1 \mid\}^2 \, ds \, dt \\
&\quad \le E_{\sigma,\theta}(u; (0,1) \times I_{\bar{l}}) \le 40 \mid I_{\bar{l}} \mid \; E_{\sigma,\theta}(u) \quad \text{and} \\
&\int_{a_{\bar{l}}}^{a_{\bar{l}} + |I_{\bar{l}}|/2} \int_0^1 \min\{\mid \partial_1 u(s, t+ \mid I_{\bar{l}} \mid /2) - 1 \mid, \mid \partial_1 u(s, t+ \mid I_{\bar{l}} \mid /2) + 1 \mid\}^2 \, ds \\
&\quad \le E_{\sigma,\theta}(u; (0,1) \times I_{\bar{l}}) \le 40 \mid I_{\bar{l}} \mid \; E_{\sigma,\theta}(u).
\end{aligned}
$$

Hence, there exists a subset of $(a_{\bar{l}}, a_{\bar{l}}+ \mid I_{\bar{l}} \mid /2)$ whose measure is at least $\frac{1}{4} \mid I_{\bar{l}} \mid$ such that for all its elements t it holds

$$
\begin{aligned}
&\int_0^1 \min\{\mid \partial_1 u(s,t) - 1 \mid, \mid \partial_1 u(s,t) + 1 \mid\}^2 \, ds \le 320 E_{\sigma,\theta}(u) \text{ and} \\
&\int_0^1 \min\{\mid \partial_1 u(s, t + \frac{\mid I_{\bar{l}} \mid}{2}) - 1 \mid, \mid \partial_1 u(s, t + \frac{\mid I_{\bar{l}} \mid}{2}) + 1 \mid\}^2 \, ds \le 320 \, E_{\sigma,\theta}(u).
\end{aligned} \tag{42}
$$

Next, we note that it holds by assumption that (recall that $\sigma < \theta^k$, $k \ge 32$, $i \le \frac{k}{8}$, and $\theta \le \frac{1}{2}$)

$$
E_{\sigma,\theta}(u) \le k\sigma \le k\theta^k \le \theta^{\frac{k}{2}} \le \theta^{2i+2}.
$$

Hence, for all the t from above we obtain from (40) and (42)

$$\begin{aligned}
&| u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) + u(x_{i+1}, t) | \\
&\geq \frac{\mid I_{\bar{l}} \mid \theta}{4} - 2x_{i+1} - 2x_{i+1}^{\frac{1}{2}}\sqrt{320}\sqrt{E_{\sigma,\theta}(u)} \\
&\geq \frac{1}{36c_2}\theta^{2i+2} - 3\theta^{2i+2} - 2\sqrt{480}\theta^{2i+2} \\
&\geq \theta^{2i+2}(50 - 3 - 22) \\
&\geq \theta^{2i+2}.
\end{aligned} \tag{43}$$

On the other hand, we obtain similarly for the same $t \in I_{\bar{l}}$ from (41) and (42)

$$\begin{aligned}
&|u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_i, t) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2) + u(x_{i+1}, t)| \\
&\leq 2\theta\frac{1}{9c_2}\theta^{2i+1} + 2x_{i+1} + 2x_{i+1}^{\frac{1}{2}}\sqrt{320E_{\sigma,\theta}(u)} \\
&\leq (500 + 3 + 2\sqrt{480})\theta^{2i+2}.
\end{aligned} \tag{44}$$

By definition of x_i and x_{i+1} we have $\frac{1}{8}\theta^{2i} \leq x_i - x_{i+1} \leq \frac{3}{2}\theta^{2i}$. Together with (43) and (44) this yields for the $t \in I_{\bar{l}}$ from above that

$$\begin{aligned}
\frac{2}{3}\theta^2 &\leq \left| \frac{u(x_i, t+ \mid I_{\bar{l}} \mid /2) - u(x_{i+1}, t+ \mid I_{\bar{l}} \mid /2)}{x_i - x_{i+1}} - \frac{u(x_i, t) - u(x_{i+1}, t)}{x_i - x_{i+1}} \right| \\
&\leq 8(500 + 3 + 2\sqrt{480})\theta^2.
\end{aligned} \tag{45}$$

We now choose $\theta_0 \in (0, 1/2]$ small enough such that $8(500 + 3 + 2\sqrt{480})\theta_0^2 \leq 1/2$. Hence, roughly speaking, at most one of the difference quotients occurring in (45) can be close (at the order of θ^2) to $\{\pm 1\}$. Precisely, summarizing the results of this step, there is a universal constant $c_4 > 0$ and a subset of $I_{\bar{l}}$ whose measure is at least $\frac{1}{4} \mid I_{\bar{l}} \mid$ such that for all t in this subset it holds

$$\left| \left| \frac{u(x_i, t) - u(x_{i+1}, t)}{x_i - x_{i+1}} \right| - 1 \right| \geq c_4\theta^2.$$

Step 4: Conclusion of estimate (33).
Let us now assume that for a point t from *Step 2* it holds $\mid \partial_1\partial_1 u(\cdot, t) \mid (x_{i+1}, x_i) < \frac{1}{2}$. Then we may assume without loss of generality for almost all $s \in (x_{i+1}, x_i)$ that

$| \partial_1 u(s,t) - 1 | \leq 3 \min\{| \partial_1 u(s,t) - 1 |, | \partial_1 u(s,t) + 1 |\}$ and thus

$$\begin{aligned}
&\int_{x_{i+1}}^{x_i} \min\{| \partial_1 u(s,t) - 1 |, | \partial_1 u(s,t) + 1 |\}^2 \, ds \\
&\geq \frac{1}{9} \int_{x_{i+1}}^{x_i} | \partial_1 u(s,t) - 1 |^2 \, ds \\
&\geq \frac{1}{9(x_i - x_{i+1})} \left(\int_{x_{i+1}}^{x_i} \partial_1 u(s,t) - 1 \, ds \right)^2 \\
&= \frac{1}{9}(x_i - x_{i+1}) \left(\frac{u(x_i,t) - u(x_{i+1},t)}{x_i - x_{i+1}} - 1 \right)^2 \\
&\geq \frac{{c_4}^2}{72} \theta^{2i} \theta^4 \\
&\geq \frac{{c_4}^2}{72} \sigma.
\end{aligned}$$

For the last estimate, we used again that $2i + 4 \leq \frac{k}{4} + 4 \leq k$ as $k \geq 32$. Consequently,

$$E_{\sigma,\theta}(u, (x_{i+1}, x_i) \times I_{\bar{l}}) \geq \sigma \mid I_{\bar{l}} \mid \min\{\frac{1}{72}{c_4}^2, \frac{1}{4}\}.$$

This concludes the proof of (33) for $\theta_0 \leq \min\left\{ \sqrt[3]{c_2/2}, 1/16, \sqrt{\frac{1}{16(500+3+2\sqrt{480})}} \right\}$ with $c_3 \leq \min\{\frac{1}{72}{c_4}^2, \frac{1}{4}\}$, and hence the assertion is proven. □

Combining the estimates of Lemma 5 and 6, we obtain the following lower bound.

Corollary 7 *There exist $c_B > 0$, $k_0 \in \mathbb{N}$, and $\theta_0 \in (0, 1/2]$ such that for all $\theta \in (0, \theta_0]$, all $\sigma > 0$ and all $u \in \mathcal{A}_\theta$,*

$$E_{\sigma,\theta}(u) \geq c_B \begin{cases} \min\{\theta^2, \sigma\} & \text{if } \sigma \geq \theta^{k_0}, \\ k\sigma & \text{if } \sigma \in [\theta^{k+1}, \theta^k) \text{ for some } k_0 \leq k. \end{cases}$$

4 Generalizations of the Scaling Result

4.1 General *p*

We consider for $1 \leq p < \infty$ the energy $E_{\sigma,\theta} : \mathcal{A}_\theta \to [0, \infty)$

$$E^p_{\sigma,\theta}(u) = \int_{(0,1)^2} \operatorname{dist}^p(\nabla u, K) \, dx + \sigma \mid D^2 u \mid (\Omega).$$

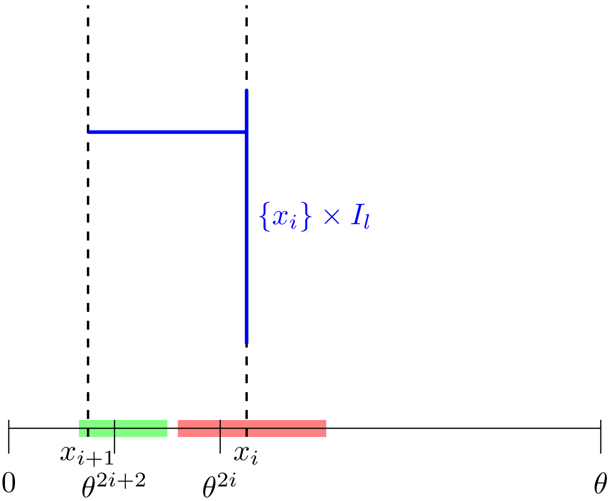

Fig. 5 In the proof of the lower bound representative vertical slices (dashed lines) at x_i (x_{i+1}) are chosen in a neighborhood around θ^{2i} (green) (θ^{2i+2} (red)). On a representative vertical slice $\{x_i\} \times (0, 1)$ intervals $\{x_i\} \times I_l$ (vertical blue line) are identified in which u is almost affine. Then difference quotients of u between x_{i+1} and x_i are estimated along horizontal lines (horizontal blue line)

Corollary 8 *Let $p \in (1, \infty)$. There exists a constant $C > 0$ such that for all $\sigma > 0$ and all $\theta \in (0, \frac{1}{2}]$,*

$$\frac{1}{C} \min \left\{ \sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1 \right), \theta^p \right\} \leq \min_{u \in \mathcal{A}_\theta} E^p_{\sigma,\theta}(u) \leq C \min \left\{ \sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1 \right), \theta^p \right\}.$$

Proof Fix $p \in (1, \infty)$. For an upper bound one can use the constructions for $p = 2$ from Proposition 2. Clearly, the function $u(x, y) = (1 - 2\theta)y \pm x$ produces an energy of order θ^p. On the other hand, it can be seen from the proof of Proposition 2 that the function constructed via branching u_N satisfies $\nabla u_N \in K$ except for an interpolation region of size θ^N on which it holds $|\nabla u_N| \leq 5$. Hence, one obtains again

$$\int_{(0,1)^2} \mathrm{dist}(\nabla u_N, K)^p \, dxdy \leq C\theta^N.$$

and $|D^2 u_N|((0, 1)^2) \leq CN$. Setting $N = \lceil \frac{|\log \sigma|}{|\log \theta|} \rceil$ yields as in the proof of Proposition 2, $E^p_{\sigma,\theta}(u_N) \leq C\sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1 \right)$.

The lower bounds can be shown analogously to the case $p = 2$. □

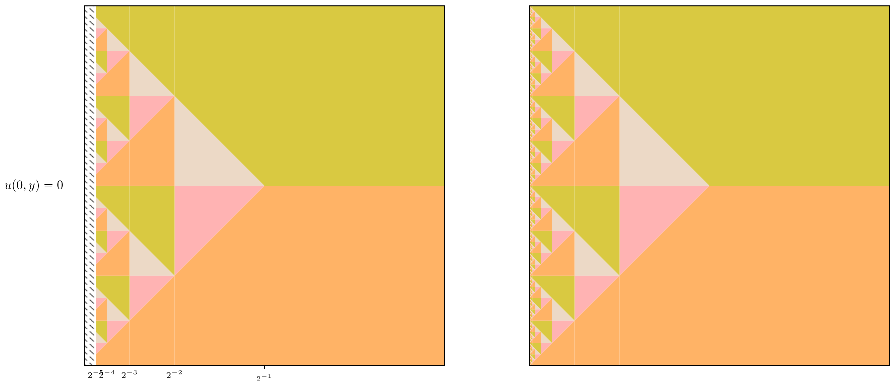

Fig. 6 Left: Sketch of the function u_N as constructed in Proposition 2 for $\theta = \frac{1}{2}$ and $N = 4$. The regions of constant gradients are color-coded, the ruled region indicates the necessary interpolation to meet the boundary values. For larger N the gradient of u_N is only changed in the interpolation region. Right: Sketch of the function $u = \lim_{N\to\infty} u_N$ as discussed in the proof of Corollary 10

4.2 Classical Double-Well Potential

We define the function $W : \mathbb{R}^2 \to \mathbb{R}$, $W(x, y) = (1 - x^2)^2 + (1 - y^2)^2$, and the energy $F_{\sigma,\theta} : \mathcal{A}_\theta \to [0, \infty)$,

$$F_{\sigma,\theta}(u) = \int_{(0,1)^2} W(\partial_1 u, \partial_2 u)\, dx + \sigma \mid D^2 u \mid (\Omega).$$

The following corollary shows that the scaling law for min $E_{\sigma,\theta}$ and min $F_{\sigma,\theta}$ are the same.

Corollary 9 *There exists a constant $C > 0$ such that for all $\sigma > 0$ and all $\theta \in (0, \frac{1}{2}]$ it holds*

$$\frac{1}{C} \min \left\{ \sigma \left(\frac{\mid \log \sigma \mid}{\mid \log \theta \mid} + 1 \right), \theta^2 \right\} \le \min_{u \in \mathcal{A}_\theta} F_{\sigma,\theta}(u) \le C \min \left\{ \sigma \left(\frac{\mid \log \sigma \mid}{\mid \log \theta \mid} + 1 \right), \theta^2 \right\}.$$

Proof First note that

$$W(x, y) = (1 - x)^2 (1 + x)^2 + (1 - y)^2 (1 + y)^2 \ge \operatorname{dist}((x, y), K)^2.$$

Consequently, inf $E_{\sigma,\theta} \le$ inf $F_{\sigma,\theta}$ and the lower bound follows from Theorem 1.

Again, it can be easily checked that the competitors from the proof of Proposition 2 produce a corresponding upper bound for min $F_{\sigma,\theta}$. □

4.3 Boundary Conditions on Whole Boundary

In this section we show that in the symmetric case $\theta = \frac{1}{2}$, we can replace the boundary condition $u(0, \cdot) = 0$ by the more restrictive boundary condition $u = 0$ on $\partial(0, 1)^2$.

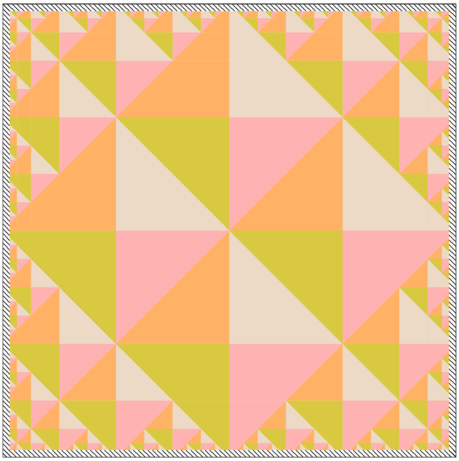

Fig. 7 Sketch of a competitor such that $u = 0$ on $\partial(0,1)^2$. The construction is obtained by subdividing the square $(0,1)^2$ through diagonals into four triangles. On each of those triangles, a version of the construction sketched in Fig. 6 is used

Corollary 10 *Let $\sigma > 0$. Then there exists $u \in W^{1,\infty}((0,1)^2)$ such that $u = 0$ on $\partial(0,1)^2$ such that*

$$E_{\sigma,1/2} \leq C \min\{1, \sigma(|\log\sigma| + 1)\}.$$

Proof Clearly $u = 0$ meets the more restrictive boundary conditions and satisfies $E_{\sigma,1/2}(u) = 2$.

In the branching regime, $\sigma < 1/4$, we recall that in the proof of Proposition 2 we constructed functions u_N such that $u_N(0,\cdot) = 0$ and $E_{\sigma,1/2}(u_N) \leq C(\sigma N + 2^{-N})$, see Fig. 6. It is easy to see that the limit $u = \lim u_N$ exists in L^1, c.f. the proof of Proposition 12. The function u belongs to $W^{1,\infty}((0,1)^2)$ and satisfies $\nabla u \in K$ almost everywhere and $u(0,\cdot) = 0$. Moreover, $u(x,x) = u(x, 1-x) = 0$ for all $0 \leq x \leq \frac{1}{2}$. In addition, $\nabla u \in BV_{loc}((0,1)^2)$ with $|D\nabla u|((2^{-N}, 1-2^{-N})^2) \leq CN$. Then define $\tilde{u} : (0,1)^2 \to \mathbb{R}$ in the following way: For $(x,y) \in (2^{-N}, 1-2^{-N})^2$ we set

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } x \leq y \leq 1-x, \\ u(1-x,y) & \text{if } 1-x \leq y \leq x, \\ u(y,x) & \text{if } y \leq x \leq 1-y, \\ u(1-y, 1-x) & \text{if } 1-y \leq x \leq y. \end{cases}$$

One checks that $|\tilde{u}(x,y)| \leq 2^{-N}$ for all $(x,y) \in \partial(2^{-N}, 1-2^{-N})^2$. Then one interpolates on $(0,1)^2 \setminus (2^{-N}, 1-2^{-N})^2$ so that $\tilde{u} = 0$ on $\partial(0,1)^2$. See Fig. 7 for an illustration of $\tilde{u}$. The energy estimates on the interpolation layer are analogous to the computations in the proof of Proposition 2. Choosing as in Proposition 2 $N \approx |\log\sigma|$ leads to $E_{\sigma,1/2}(\tilde{u}) \leq C\sigma(|\log\sigma| + 1)$. □

4.4 Scaling Law on Rectangles

Proposition 11 *There is a constant $C > 0$ with the following property: For $L > 0$ consider the rectangle $\Omega_L := (0, L) \times (0, 1)$ and set*

$$\mathcal{A}_\theta^{(L)} := \left\{ u \in W^{1,2}(\Omega_L) : \nabla u \in BV(\Omega_L),\ u(0, y) = (1 - 2\theta) y \right\}.$$

We define

$$s(L, \theta, \sigma) := \begin{cases} \min\left\{ L\theta^2,\ \sigma\left(\frac{|\log\sigma|}{|\log\theta|} + 1\right)\right\} & \text{if } L \geq \theta \\ \min\left\{ L\theta^2,\ \sigma\left(1 + \frac{\log(L/\sigma)}{|\log\theta|}\right)\right\} & \text{if } \sigma \leq L < \theta \\ L\theta^2 & \text{if } L \leq \min\{\theta,\ \sigma\}. \end{cases}$$

Then for all σ, $L > 0$ and $\theta \in (0, 1/2)$,

$$\frac{1}{C} s(L, \theta, \sigma) \leq \min_{u \in \mathcal{A}_\theta^{(L)}} E_{\sigma,\theta}(u; \Omega_L) \leq C s(L, \theta, \sigma),$$

where we use the notation $E_{\sigma,\theta}(u; \Omega_L)$ as defined in (5).

Proof *Upper Bound.* The affine function $u(x, y) = (1-2\theta)y$ satisfies $E_{\sigma,\theta}(u; ((0, L) \times (0, 1)) \leq L\theta^2$. This in particular concludes the proof of the upper bound if $L \leq \min\{\theta, \sigma\}$.
For the other two regimes, we use the test function $u_N : (0, 1)^2 \to \mathbb{R}$ constructed in the proof of Proposition 2 and note that

$$u_N(\theta, y) = \begin{cases} \alpha + y & \text{if } y \in (0, 1 - \theta) \\ \alpha + (2 - 2\theta) - y & \text{if } y \in (1 - \theta, 1) \end{cases}$$

for some value $\alpha \in \mathbb{R}$. We now define the auxiliary function $u : (0, \infty) \times (0, 1) \to \mathbb{R}$ via

$$u(x, y) := \begin{cases} u_N(x, y) & \text{if } x \in (0, \theta),\ y \in (0, 1) \\ \alpha + y - (x - \theta) & \text{if } x \in (\theta, \infty),\ y \in (0, \min\{1,\ x + 1 - 2\theta\}) \\ \alpha + (2 - 2\theta) - y + (x - \theta) & \text{if } x \in (\theta, 2\theta),\ y \in (x + 1 - 2\theta, 1). \end{cases}$$

Note that this construction resembles the *truncated branching* construction used for martensitic microstructures (Conti and Zwicknagl 2016; Conti et al. 2020; Zwicknagl 2014). We claim that restrictions of the so-defined function u yield the respective energy scalings. We consider the cases from the definition of $s(L, \theta, \sigma)$ separately.

(i) If $L \geq \theta$ then by Proposition 2 we have

$$E_{\sigma,\theta}(u; \Omega_L) \leq E_{\sigma,\theta}(u_N; \Omega_\theta) + \sigma + \sigma\theta\sqrt{2} \leq c\sigma\left(\frac{|\log\sigma|}{|\log\theta|} + 1\right).$$

(ii) If $\sigma \leq L < \theta$ then there exists $m \in \mathbb{N}$ such that $\delta^{m+1} \leq L/\theta < \delta^m$, where $\delta = \frac{1}{\lceil \theta^{-1} \rceil} \leq \theta$ as in the proof of Proposition 2. We note that $m < \frac{|\log L/\theta|}{|\log \delta|} \leq \frac{|\log \sigma/\theta|}{|\log \delta|} \leq N$ with $N \in \mathbb{N}$ as defined in the proof of Proposition 2, see (14).We estimate using the computations from the proof of Proposition 2

$$\begin{aligned}
&E_{\sigma,\theta}(u; (0, L) \times (0, 1)) \\
&\quad \leq E_{\sigma,\theta}(u_N; (0, \theta\delta^m) \times (0, 1)) \leq c \left(\theta\delta^N + \sigma(N - m)\right) \\
&\quad \leq c\sigma \left(1 + \frac{|\log \sigma/\theta|}{|\log \delta|} - \frac{|\log L/\theta|}{|\log \delta|}\right) \leq c\sigma \left(1 + \frac{\log(L/\sigma)}{|\log \theta|}\right).
\end{aligned}$$

Lower bound. Step 1: $L \geq 1$. In this case, the proof follows from the lower bound for $E_{\sigma,\theta}$ on $(0, 1)^2$. We will from now on focus on the case $L < 1$.

Step 2: $\sigma \geq \theta_0^{k_0}$. An argument along the lines of the proof of Lemma 5 shows that for all admissible functions u,

$$E_{\sigma,\theta}(u, \Omega_L) \geq c \min\{L\theta^2, \sigma\}.$$

This in particular concludes the proof of the lower bound in the case $L < \min\{\theta, \sigma\}$.

Step 3: $L \in [\theta, 1)$. In this case, the proof of the lower bound follows as for $L = 1$. More precisely, for large $\theta \geq \theta_0$ both, θ and L are of order one, and we can proceed similarly to the proof of Lemma 4 to obtain a lower bound $E_{\sigma,\theta}(u; \Omega_L) \geq c \min\{\sigma(|\log \sigma| + 1), 1\}$. For $\sigma \geq \theta^k$ for some $k \geq 33$, we can directly use Lemma 6 since this proof uses only the energy on a domain that is contained in $(0, \theta) \times (0, 1) \subseteq \Omega_L$.

Step 4: $L \in [\sigma, \theta)$. First note that by the argument in step 2 it always holds $E_{\sigma,\theta}(u, \Omega_L) \geq c \min\{L\theta^2, \sigma\}$. This shows the desired lower bound as long as $\frac{\log(L/\sigma)}{|\log \theta|} \leq 33$.

If $\sigma \in [\theta^{k+1}, \theta^k)$ for some k and $L \in [\theta^{m+1}, \theta^m)$ for some $m \in \mathbb{N}$ such that $k - m \geq 32$ define $K := \left\lfloor \frac{k-m}{8} \right\rfloor$. Then one can find for $i = 1, \ldots, K$ points $x_i \in (\frac{1}{2}L\theta^{2i}, \frac{3}{2}L\theta^{2i})$ such that $u(x_i, \cdot) \in H^1(0, 1)$, $\partial_2 u(x_i, \cdot) \in BV(0, 1)$ and

$$\begin{aligned}
&\int_0^1 \min\{|\partial_2 u(x_i, y) - 1|, |\partial_2 u(x_i, y) + 1|\}^2 \, dy + \sigma \, |\partial_2 \partial_2 u(x_i, \cdot)| \, (0, 1) \\
&\quad \leq L^{-1}\theta^{-2i} E_{\sigma,\theta}\left(u, (\frac{1}{2}L\theta^{2i}, \frac{3}{2}L\theta^{2i} \times (0, 1)\right).
\end{aligned}$$

With this notation a lower bound can then be proven with minor modifications along the lines of part (B) in the proof of Proposition 3. ◻

5 Regularity of Solutions to the Differential Inclusion

Refining the construction in the proof of Proposition 2, yields a solution to the differential inclusion problem derived in Cicalese et al. (2019) subject to boundary conditions.

While our scaling shows that the resulting gradient is not in BV (cf. also Rüland et al. 2019), we can in the spirit of Rüland et al. (2018), Rüland et al. (2019), Rüland et al. (2020) exploit regularity properties of it.

Proposition 12 *There exists a function* $u \in W^{1,\infty}((0,1)^2)$ *such that* $u(0,y) = (1-2\theta)y$, $\nabla u \in K$ *almost everywhere, and* $\nabla u \in W^{s,q}$ *for all* $0 < s < 1$ *and* $q \in (1,\infty)$ *such that* $\frac{1}{q} > s$. *Moreover,* $\nabla u \in BV_{loc}((0,1)^2)$ *and consequently* $\dim_{\mathcal{H}}(J_u) = 1$.

Proof Fix $s \in (0,1)$ and $q \in (1,\infty)$ such that $\frac{1}{q} > s$. Then there exists $p \in (1,\infty)$ such that $\frac{1}{q} = \frac{1-s}{p} + s$.

Now, recall the branching construction of the function $u_N : (0,1)^2 \to \mathbb{R}$ in the proof of Proposition 2, see Fig. 6. In particular, we note that for all $N \in \mathbb{N}$ we have

(i) $u_N(0,y) = (1-2\theta)y$ in $L^1(0,1)$,
(ii) $\nabla u_N \in K$ for almost all $x \in (\theta^N, 1) \times (0,1)$,
(iii) if $M > N$ then $\nabla u_M = \nabla u_N$ for almost every $x \in (\theta^N, 1) \times (0,1)$,
(iv) $\|u_{N+1} - u_N\|_{L^1} \leq C\theta^N$,
(v) $\|\nabla u_N\|_{L^\infty} \leq C$,
(vi) $\|\nabla u_N\|_{BV} \leq CN$,
(vii) $\|\nabla u_N - \nabla u_{N+1}\|_{L^p} \leq C\theta^N$,
(viii) $\|\nabla u_N - \nabla u_{N+1}\|_{BV} \leq C$.

First note for $M > N$ that by (iv)

$$\|u_M - u_N\|_{L^1} \leq \sum_{k=N}^{M-1} \|u_{k+1} - u_k\|_{L^1} \leq C \sum_{k=N}^{\infty} \theta^N.$$

Hence, $(u_N)_N$ forms a Cauchy sequence in L^1. Similarly, one shows using (vii) that $(\nabla u_N)_N$ is Cauchy in L^p. Consequently, $(u_N)_N$ is a Cauchy sequence in $W^{1,1}$ and converges strongly in $W^{1,1}$ to some $u \in W^{1,1}((0,1)^2)$. As the trace is continuous with respect to strong convergence in $W^{1,1}$ we obtain from (i) that $u(0,y) = (1-2\theta)y$ in $L^1(0,1)$. Moreover, it follows from (ii) and (iii) that $\nabla u \in K$. In particular, $u \in W^{1,\infty}$.

Next, we apply an interpolation inequality between L^p and BV, see (Rüland et al. 2018, Corollary 2.1) to $\nabla u_{N+1} - \nabla u_N$ which yields using (v), (vii) and (viii)

$$\|\nabla u_{N+1} - \nabla u_N\|_{W^{s,q}} \leq C\|\nabla u_{N+1} - \nabla u_N\|_{L^p}^{1-s} \|\nabla u_{N+1} - \nabla u_N\|_{BV}^{s} \leq C\theta^{(1-s)N}.$$

Hence, we obtain for $M > N$ that

$$\|\nabla u_M - \nabla u_N\|_{W^{s,q}} \leq \sum_{k=N}^{M-1} \|\nabla u_{k+1} - \nabla u_k\|_{W^{s,q}} \leq C \sum_{k=N}^{\infty} \left(\theta^{1-s}\right)^k$$

In particular, $(\nabla u_N)_N$ is a Cauchy sequence in $W^{s,q}$. Its limit is already identified to be ∇u. Consequently, $\nabla u \in W^{s,q}$. Eventually, we remark that (iii) and (vi) imply that $(\nabla u_N)_N$ is bounded in $BV_{loc}((0,1)^2)$. By BV-compactness, it follows $\nabla u \in BV_{loc}((0,1)^2)$. It remains to show that $\dim_{\mathcal{H}}(J_{\nabla u}) = 1$. Since $\nabla u \in BV_{loc}((0,1)^2)$, we have for $s > 1$ that $\mathcal{H}^s(J_{\nabla u}) \leq \sum_k \mathcal{H}^s(J_{\nabla u} \cap (1/k, 1-1/k)^2) = 0$. On the other hand, it follows from the energy scaling result Theorem 1 for that $\mathcal{H}^1(J_{\nabla u}) = +\infty$ which implies $\dim_{\mathcal{H}}(J_{\nabla u}) \geq 1$. $\square$

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References

Aviles, P., Giga, Y., et al.: A mathematical problem related to the physical theory of liquid crystal configurations. In: Miniconference on Geometry/partial Differential Equations, vol. 2, pp. 1–16 (1987). Centre for Mathematics and its Applications, Mathematical Sciences Institute

Bella, P., Goldman, M.: Nucleation barriers at corners for cubic-to-tetragonal phase transformation. Proc. R. Soc. Edinburgh A **145**, 715–724 (2015)

Bella, P., Kohn, R.V.: Wrinkles as the result of compressive stresses in an annular thin film. Commun. Pure Appl. Math. **67**(5), 693–747 (2014)

Ben Belgacem, H., Conti, S., DeSimone, A., Müller, S.: Energy scaling of compressed elastic films. Arch. Rat. Mech. Anal. **164**(1), 1–37 (2002)

Bourne, D., Conti, S., Müller, S.: Energy bounds for a compressed elastic film on a substrate. J. Nonlinear Sci. **27**, 453–494 (2017). https://doi.org/10.1007/s00332-016-9339-0

Brancolini, A., Wirth, B.: Optimal energy scaling for micropatterns in transport networks. SIAM J. Math. Anal. **49**(1), 311–359 (2017). https://doi.org/10.1137/15M1050227

Brancolini, A., Rossmanith, C., Wirth, B.: Optimal Micropatterns in 2D Transport Networks and Their Relation to Image Inpainting. Arch. Rational Mech. Anal. **228**, 279–308 (2018)

Capella, A., Otto, F.: A rigidity result for a perturbation of the geometrically linear three-well problem. Commun. Pure Appl. Math. **62**(12), 1632–1669 (2009). https://doi.org/10.1002/cpa.20297

Capella, A., Otto, F.: A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy. Proc. R. Soc. Edinburgh Sect. A **142**(2), 273–327 (2012). https://doi.org/10.1017/S0308210510000478

Chan, A., Conti, S.: Energy scaling and branched microstructures in a model for shape-memory alloys with SO(2) invariance. Math. Models Methods Appl. Sci. **25**, 1091–1124 (2015). https://doi.org/10.1142/S0218202515500281

Choksi, R., Kohn, R.V.: Bounds on the micromagnetic energy of a uniaxial ferromagnet. Commun. Pure Appl. Math. **51**(3), 259–289 (1998). https://doi.org/10.1002/(SICI)1097-0312(199803)51:3

Choksi, R., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. **201**(1), 61–79 (1998)

Choksi, R., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. **14**(2), 119–171 (2004). https://doi.org/10.1007/s00332-004-0568-2

Choksi, R., Conti, S., Kohn, R.V., Otto, F.: Ground state energy scaling laws during the onset and destruction of the intermediate state in a type I superconductor. Commun. Pure Appl. Math. **61**(5), 595–626 (2008)

Cicalese, M., Forster, M., Orlando, G.: Variational analysis of the J_1-J_2-J_3 model: a non-linear lattice version of the Aviles-Giga functional. https://cvgmt.sns.it/paper/5241/ (2021)

Cicalese, M., Solombrino, F.: Frustrated ferromagnetic spin chains: a variational approach to chirality transitions. J. Nonlinear Sci. **25**(2), 291–313 (2015). https://doi.org/10.1007/s00332-015-9230-4
Cicalese, M., Forster, M., Orlando, G.: Variational analysis of a two-dimensional frustrated spin system: emergence and rigidity of chirality transitions. SIAM J. Math. Anal. **51**, 4848–4893 (2019). arXiv:1904.07792
Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. **53**, 1448–1474 (2000)
Conti, S.: A lower bound for a variational model for pattern formation in shape-memory alloys. Cont. Mech. Thermodyn. **17**(6), 469–476 (2006)
Conti, S.: Quasiconvex functions incorporating volumetric constraints are rank-one convex. Journal de mathématiques pures et appliquées **90**(1), 15–30 (2008)
Conti, S., Ortiz, M.: Optimal Scaling in Solids Undergoing Ductile Fracture by Crazing. Arch. Rat. Mech. Anal. **219**, 607–636 (2016)
Conti, S., Zwicknagl, B.: Low volume-fraction microstructures in martensites and crystal plasticity. Math. Models Methods Appl. Sci. **26**, 1319–1355 (2016)
Conti, S., DeSimone, A., Müller, S.: Self-similar folding patterns and energy scaling in compressed elastic sheets. Comput. Methods Appl. Mech. Eng. **194**(21–24), 2534–2549 (2005). https://doi.org/10.1016/j.cma.2004.07.044
Conti, S., Otto, F., Serfaty, S.: Branched microstructures in the Ginzburg-Landau model of type-I superconductors. SIAM J. Math. Anal. **48**, 2994–3034 (2016). https://doi.org/10.1137/15M1028960
Conti, S., Diermeier, J., Zwicknagl, B.: Deformation concentration for martensitic microstructures in the limit of low volume fraction. Calc. Var. PDE **56**, 16 (2017)
Conti, S., Diermeier, J., Melching, C., Zwicknagl, B.: Energy scaling laws for geometrically linear elasticity models for microstructures in shape memory alloys. ESAIM COCV **26**, 115–164 (2020). https://doi.org/10.1051/cocv/2020020
Dabade, V., Venkatraman, R., James, R.D.: Micromagnetics of Galfenol. J. Nonlinear Sci. **29**, 415–460 (2019)
DeSimone, A., Kohn, R.V., Müller, S., Otto, F.: Recent analytical developments in micromagnetics. In: Bertotti, G., Mayergoyz, I.D. (eds.) Vol II: Physical Modelling, Micromagnetics, and Magnetization Dynamics. The Science of Hysteresis, pp. 269–381. Academic Press, Oxford (2006)
DeSimone, A., Knüpfer, H., Otto, F.: 2-d stability of the Néel wall. Calc. Var. Partial Differ. Equ. **27**(2), 233–253 (2006). https://doi.org/10.1007/s00526-006-0019-z
Diep, H.T. (ed.): Frustrated Spin Systems. World Scientific, Singapore (2013)
Diep, H.T.: Quantum theory of helimagnetic thin films. Phys. Rev. B **91**, 014436 (2015). https://doi.org/10.1103/PhysRevB.91.014436
Ginster, J., Koser, M., Zwicknagl, B.: Microstructures in a two dimensional frustrated spin system: scaling regimes and a discrete-to-continuum limit (in preparation)
Knüpfer, H., Muratov, C.B.: Domain structure of bulk ferromagnetic crystals in applied fields near saturation. J. Nonlin. Sc. **21**, 921–962 (2011)
Knüpfer, H., Kohn, R.V., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transformation. Commun. Pure Appl. Math. **66**(6), 867–904 (2013). https://doi.org/10.1002/cpa.21448
Kohn, R.: Energy-driven pattern formation. In: International Congress of Mathematicians. Vol. I, pp. 359–383. Eur. Math. Soc., Zürich (2007). https://doi.org/10.4171/022-1/15
Kohn, R.V., Müller, S.: Branching of twins near an austenite-twinned martensite interface. Philos. Mag. A **66**, 697–715 (1992)
Kohn, R.V., Müller, S.: Surface energy and microstructure in coherent phase transitions. Commun. Pure Appl. Math. **47**(4), 405–435 (1994). https://doi.org/10.1002/cpa.3160470402
Kohn, R.V., Müller, S.: Surface energy and microstructure in coherent phase transitions. Commun. Pure Appl. Math. **XLVII**, 405–435 (1994)
Kohn, R.V., Wirth, B.: Optimal fine-scale structures in compliance minimization for a uniaxial load. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci **470**(2170), 20140432–16 (2014). https://doi.org/10.1098/rspa.2014.0432
Kohn, R.V., Wirth, B.: Optimal fine-scale structures in compliance minimization for a shear load. Commun. Pure Appl. Math. **69**(8), 1572–1610 (2016). https://doi.org/10.1002/cpa.21589
Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transitions and partial differential equations. J. Eur. Math. Soc. **1**, 393–422 (1999). https://doi.org/10.1007/s100970050012
Otto, F., Steiner, J.: The concertina pattern. Calc. Var. **39**, 139–181 (2010)

Otto, F., Viehmann, T.: Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy. Calc. Var. PDE **38**(1–2), 135–181 (2010)

Pompe, W.: Explicit construction of piecewise affine mappings with constraints. Bull. Pol. Acad. Sci. Math. **58**(3), 209–220 (2010)

Rüland, A., Tribuzio, A.: On the energy scaling behaviour of a singularly perturbed tartar square. Arch. Rational Mech. Anal. **243**, 401–431 (2022). https://doi.org/10.1007/s00205-021-01729-1

Rüland, A., Zillinger, C., Zwicknagl, B.: Higher Sobolev regularity of convex integration solutions in elasticity: The Dirichlet problem with affine data in int (K^{lc}). SIAM J. Math. Anal. **50**(4), 3791–3841 (2018). https://doi.org/10.1137/17M1149687

Rüland, A., Taylor, J.M., Zillinger, C.: Shape-memory alloys: Some remarks on rigidity, flexibility and some numerical implementations. J. Nonlinear Sci. **29**, 2137–2184 (2019)

Rüland, A., Zillinger, C., Zwicknagl, B.: Higher Sobolev regularity of convex integration solutions in elasticity: the planar geometrically linearized hexagonal-to-rhombic phase transformation. J. Elasticity **138**(1), 1–76 (2020). https://doi.org/10.1007/s10659-018-09719-3

Schoenherr, P., Müller, J., Köhler, L., Rosch, A., Kanazawa, N., Tokura, Y., Garst, M., Meier, D.: Topological domain walls in helimagnets. Nat. Phys. **14**(5), 465–468 (2018)

Schreiber, C.: Rapport de Stage. D.E.A. Freiburg, Freiburg (1994)

Uchida, M., Onose, Y., Matsui, Y., Tokura, Y.: Real-space observation of helical spin order. Science **311**(5759), 359–361 (2006)

Venkatraman, R., Dabade, V., James, R.D.: Bounds on the energy of a soft cubic ferromagnet with large magnetostriction. J. Nonlinear Sci. **30**(6), 3367–3388 (2020). https://doi.org/10.1007/s00332-020-09653-6

Zwicknagl, B.: Microstructures in low-hysteresis shape memory alloys: Scaling regimes and optimal needle shapes. Arch. Rat. Mech. Anal. **213**(2), 355–421 (2014)