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Drag on spheres in micropolar fluids with non-zero boundary conditions for microrotations

KARL-HEINZ HOFFMANN[1], DAVID MARX[2] AND NIKOLAI D. BOTKIN[1]

[1]Department of Applied Mathematics, Technical University of Munich, Boltzmann Street 3, 85747 Garching/Munich, Germany

[2]DFG Research Center Matheon, Humboldt-University of Berlin, Unter den Linden 6, 10099 Berlin, Germany

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The Stokes formula for the resistance force exerted on a sphere moving with constant velocity in a fluid is extended to the case of micropolar fluids. A non-homogeneous boundary condition for the micro-rotation vector is used: the micro-rotation on the boundary of the sphere is assumed proportional to the rotation rate of the velocity field on the boundary.

1. Introduction

The hydrodynamics of classical fluids is based on the assumption that the fluid particles do not have any internal structure. This results in the well-known Navier–Stokes equations which describe many hydrodynamical phenomena. Nevertheless, fluid particles may exhibit some microscopical effects such as rotation, shrinking etc. for some fluids such as polymeric suspensions, animal blood etc. Therefore, the internal structure should be taken into account for fluids whose particles have complex shapes. Moreover, the internal structure plays a role even for ordinary fluids such as water in models with small scales (see e.g. Papautsky *et al.* 1999). A well-accepted theory which accounts for internal structures of fluids is *micropolar fluid theory* by Eringen (see Eringen 1964, 1966; Stokes 1984; Straughan 2004). Here, individual particles can rotate independently from the rotation and movement of the fluid as whole. Therefore, new variables which represent angular velocities of fluid particles and new equations governing this variables should be added to the conventional model.

The aim of this paper is to calculate the resistant force exerted on a sphere moving with a constant velocity in a micropolar fluid and to compare the result with the conventional Stokes force derived from classical hydrodynamics (see e.g. Landau & Lifshitz 1995) and with similar results from authors who have obtained such a formula in the case of homogeneous boundary conditions for the micro-rotation (cf. Lakshmana Rao & Bhujanga Rao 1970; Erdogan 1972; Ramkissoon & Majumdar 1976; Ramkissoon 1985; Hayakawa 2000). This allows us to estimate the influence of the micro-rotation on the motion of rigid bodies in micropolar fluids for various boundary conditions posed on the variables describing micro-rotations.

The paper is structured as follows. First, a mathematical model of micropolar fluids is discussed and its most important features are outlined. Then, a formula for the resistant force is derived. Finally, the comparison with the conventional Stokes force is given for water and blood. Comparisons with results of other authors are given in the conclusion.

2. Micropolar field equation

The most important feature of micropolar fluid theory is the use of a non-symmetric stress tensor so that the conservation of angular momentum results in new equations describing rotation of fluid particles on the micro-scale. Such a stress tensor is given as follows (see e.g. Lukaszewicz 1999):

$$T_{ij} = (-p + \zeta\, v_{k,k})\delta_{ij} + \nu(v_{i,j} + v_{j,i}) + \nu_R(v_{j,i} - v_{i,j}) - 2\nu_R \varepsilon_{mij}\omega_m. \tag{2.1}$$

In some papers, micropolar fluids are defined as being governed by such a tensor. Here, commas followed by indices denote differentiation with respect to the corresponding coordinates, δ_{ij} and ε_{mij} are the Christoffel and Levi–Civita symbols, respectively, and the summation over repeating indices is assumed. The meaning of the other variables and constants is explained below.

In the most general form, the micropolar field equations for incompressible and viscous fluids are (see Lukaszewicz 1999):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho = 0, \tag{2.2a}$$

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) = (\nu+\nu_r)\Delta\,\boldsymbol{v} - \nabla p + 2\nu_r \operatorname{curl}\boldsymbol{\omega} + \rho\boldsymbol{f}, \tag{2.2b}$$

$$\rho I\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\omega}\right) = (c_a + c_d)\Delta\boldsymbol{\omega} + (c_0 + c_d - c_a)\nabla \operatorname{div}\boldsymbol{\omega} + 2\nu_r(\operatorname{curl}\boldsymbol{v} - 2\boldsymbol{\omega}) + \rho\boldsymbol{g}, \tag{2.2c}$$

$$\operatorname{div}\boldsymbol{v} = 0, \tag{2.2d}$$

where ρ is the density, $\boldsymbol{v}$ the velocity field, $\boldsymbol{\omega}$ the micro-rotation field, I the micro-inertia coefficient, $\boldsymbol{f}$ body forces per unit mass, $\boldsymbol{g}$ micro-rotation driving forces per unit mass, p the hydrostatical pressure, ν the classical viscosity coefficient, ν_r the vortex viscosity coefficient, c_a, c_d, c_0 are spin gradient viscosity coefficients.

Equation (2.2*a*) represents the conservation of mass, (2.2*b*) and (2.2*c*) describe the conservation of impulse and angular momentum, respectively. Equation (2.2*d*) accounts for the incompressibility of the fluid. If $\nu_r = 0$, the conservation of impulse becomes independent of the micro-rotation. The system reduces to the classical Navier–Stokes equation, if ν_r, c_0, c_a, c_d and g vanish. Note that the choice of boundary conditions for micropolar fluids is not obvious. The boundary condition for the velocity field is the same as in the classical case. As for the micro-rotation, there is no general agreement in the literature (see § 1.5(3) of Lukaszewicz (1999) and papers cited therein). The Dirichlet boundary condition $\boldsymbol{\omega} = 0$ is often used. Some authors propose the following dynamic boundary condition: $\boldsymbol{\omega} = (\alpha/2)\operatorname{curl}\boldsymbol{v}$ with $0 \leqslant \alpha \leqslant 1$ (see Lukaszewicz 1999). This boundary condition for the micro-rotation and the no-slip boundary condition for the velocity field will be used here.

3. Calculation of the resistance force

Assume equivalently that a sphere of radius R is immovable, whereas the fluid exhibits a steady-state flow with velocity $\boldsymbol{u}$ at infinity. The velocities and micro-rotations are assumed to be small so that the field equations become linear:

$$-(\nu+\nu_r)\Delta\,\boldsymbol{v}+\nabla p=2\nu_r\mathrm{curl}\,\boldsymbol{\omega}, \tag{3.1}$$

$$-(c_a+c_d)\Delta\boldsymbol{\omega}-(c_0+c_d-c_a)\nabla\,\mathrm{div}\,\boldsymbol{\omega}+4\nu_r\boldsymbol{\omega}=2\nu_r\mathrm{curl}\,\boldsymbol{v}, \tag{3.2}$$

$$\mathrm{div}\,\boldsymbol{v}=0, \tag{3.3}$$

Conditions at infinity are:

$$\boldsymbol{v}=\boldsymbol{u}, \tag{3.4}$$

$$\boldsymbol{\omega}=0. \tag{3.5}$$

The boundary conditions for the sphere are:

$$\boldsymbol{v}=0, \tag{3.6}$$

$$\boldsymbol{\omega}=\frac{\alpha}{2}\mathrm{curl}\,\boldsymbol{v}\quad\text{with}\quad 0\leqq\alpha\leqq 1\quad\text{at}\quad |x|=R. \tag{3.7}$$

The calculation of the resistance force is based on the explicit analytical representation of solutions to (3.1)–(3.3).

3.1. *Calculation of the velocity, micro-rotation and pressure*

Equation (3.3) implies that $\mathrm{div}\,(\boldsymbol{v}-\boldsymbol{u})=0$. Hence, $\boldsymbol{v}$ can be expressed as follows:

$$\boldsymbol{v}=\boldsymbol{u}+\mathrm{curl}\,\boldsymbol{A},$$

where $\boldsymbol{A}$ is a vector field such that $\mathrm{curl}\,\boldsymbol{A}$ vanishes at infinity. Analogously to Landau & Lifshitz (1995), observe that $\boldsymbol{A}$ is a polar vector and take into account the symmetry of the sphere to conclude that $\boldsymbol{A}=f'(r)\boldsymbol{n}\times\boldsymbol{u}$, where f is a function of $r=(x^2+y^2+z^2)^{1/2}$. Therefore, the velocity $\boldsymbol{v}$ is of the form:

$$\boldsymbol{v}=\boldsymbol{u}+\mathrm{curl}(\nabla f\times\boldsymbol{u})=\boldsymbol{u}+\mathrm{curl}\,\mathrm{curl}\,f\boldsymbol{u}. \tag{3.8}$$

Taking the curl of $\boldsymbol{v}$ yields:

$$\mathrm{curl}\,\boldsymbol{v}=\mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,f\boldsymbol{u}=(\nabla\,\mathrm{div}-\Delta)\mathrm{curl}\,f\boldsymbol{u}=-\Delta\mathrm{curl}\,f\boldsymbol{u}. \tag{3.9}$$

Applying the curl operator to (3.2) and using the well-known formula $\mathrm{curl}\,\mathrm{curl}=\nabla\mathrm{div}-\Delta$ gives:

$$-(c_a+c_d)\Delta\mathrm{curl}\,\boldsymbol{\omega}+4\nu_r\mathrm{curl}\,\boldsymbol{\omega}=2\nu_r\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}. \tag{3.10}$$

Combining (3.1) and (3.10) results in:

$$\frac{(c_a+c_d)(\nu+\nu_r)}{2\nu_r}\Delta^2\boldsymbol{v}-\frac{(c_a+c_d)}{2\nu_r}\Delta\nabla p+2(-(\nu+\nu_r)\Delta\,\boldsymbol{v}+\nabla p)=2\nu_r\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}.$$

Applying now the curl operator to the last equation yields:

$$\frac{(c_a+c_d)(\nu+\nu_r)}{2\nu_r}\Delta^2\mathrm{curl}\,\boldsymbol{v}-2(\nu+\nu_r)\Delta\mathrm{curl}\,\boldsymbol{v}=2\nu_r\mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}.$$

Using again the formula $\mathrm{curl}\,\mathrm{curl}=\nabla\mathrm{div}-\Delta$ results in:

$$\frac{(c_a+c_d)(\nu+\nu_r)}{2\nu_r}\Delta^2\mathrm{curl}\,\boldsymbol{v}-2\nu\Delta\mathrm{curl}\,\boldsymbol{v}=0.$$

Introducing the notation $c_1=(c_a+c_d)(\nu+\nu_r)/2\nu_r$, $c_2=2\nu$ and combining the last equations with (3.9) gives:

$$c_1\Delta^3\mathrm{curl}\,f\boldsymbol{u}-c_2\Delta^2\mathrm{curl}\,f\boldsymbol{u}=0, \tag{3.11}$$

which can be rewritten as

$$(c_1\Delta - c_2\mathrm{Id})\Delta^2(\nabla f \times \boldsymbol{u}) = 0,$$

or

$$(c_1\Delta - c_2\mathrm{Id})\Delta^2\nabla f \equiv \nabla(c_1\Delta - c_2\mathrm{Id})\Delta^2 f = \lambda u,$$

where λ is a scalar function, Id denotes the identity operator. Considering spherical reference system r, θ, ϵ whose polar axis has the direction of $\boldsymbol{u}$, and taking into account that the function $\psi = (c_1\Delta - c_2\mathrm{Id})\Delta^2 f$ depends only on r, we conclude that $(\nabla\psi)_\theta = (\nabla\psi)_\epsilon = 0$ at each point (r, θ, ϵ) in the local reference system formed by the tangents to the coordinate lines. On the other hand, $(\lambda\boldsymbol{u})_\theta = -\lambda|\boldsymbol{u}|\sin\theta$ in this local reference system, which implies that $\lambda \equiv 0$. Thus,

$$(c_1\Delta - c_2\mathrm{Id})\Delta^2 f = \mathrm{const.} \tag{3.12}$$

Equation (3.8) considered in the spherical reference system shows that only the second derivatives of f in r describe $\boldsymbol{v} - u$ at infinity and vanish there because $\boldsymbol{v} - \boldsymbol{u} \to 0$ as $r \to \infty$. Therefore, all higher derivatives of f are expected to vanish at infinity too. Thus, the constant on the right-hand side may be assumed to be zero. Note that this argumentation is not a strict proof, but some physically reasonable consideration. It is shown below that this assumption leads to a unique solution of the problem.

Thus, (3.12) reduces to

$$c_1\Delta g - c_2 g = 0,$$

where $g := \Delta^2 f$. Since $\Delta g = r^{-2}(\mathrm{d}/\mathrm{d}r)(r^2(\mathrm{d}/\mathrm{d}r)g)$ in spherical coordinates, the general solution of the last equation is of the form:

$$g(r) = \frac{\mathscr{A}\mathrm{e}^{kr} + B\mathrm{e}^{-kr}}{r},$$

where $k = \sqrt{c_2/c_1}$. Choose $\mathscr{A} = 0$ because g vanishes at infinity and integrate the equation

$$\Delta^2 f = \frac{B\mathrm{e}^{-kr}}{r},$$

bearing in mind that the Laplace operator is considered in spherical coordinates. This yields:

$$f = \frac{1}{k^4}\frac{B\mathrm{e}^{-kr}}{r} + ar + \frac{b}{r}. \tag{3.13}$$

Substitution of (3.13) into (3.8) yields:

$$\begin{aligned}\boldsymbol{v} = \boldsymbol{u}\left(1 - \frac{a}{r} - \frac{b}{r^3} - B\mathrm{e}^{-kr}\left(\frac{1}{k^4r^3} + \frac{1}{k^3r^2} + \frac{1}{k^2r}\right)\right)\\ + \boldsymbol{n}(\boldsymbol{un})\left(-\frac{a}{r} + \frac{3b}{r^3} + B\mathrm{e}^{-kr}\left(\frac{3}{k^4r^3} + \frac{3}{k^3r^2} + \frac{1}{k^2r}\right)\right).\end{aligned} \tag{3.14}$$

It is easily seen that the components of v in spherical coordinates (the polar axis has the direction of $\boldsymbol{u}$) are:

$$v_r = u\cos\theta\left(1 - \frac{2a}{r} + \frac{2b}{r^3} + B\mathrm{e}^{-kr}\left(\frac{2}{k^4r^3} + \frac{2}{k^3r^2}\right)\right), \tag{3.15}$$

$$v_\theta = -u\sin\theta\left(1 - \frac{a}{r} - \frac{b}{r^3} - B\mathrm{e}^{-kr}\left(\frac{1}{k^4r^3} + \frac{1}{k^3r^2} + \frac{1}{k^2r}\right)\right), \tag{3.16}$$

$$v_\epsilon = 0. \tag{3.17}$$

We can prove immediately (see also Landau & Lifshitz 1995) that only the ϵ-component of curl $\boldsymbol{v}$ is different from zero. Applying the curl operator to (3.1) yields:

$$-(\nu+\nu_r)\Delta \mathrm{curl}\,\boldsymbol{v} = 2\nu_r \mathrm{curl}\,\mathrm{curl}\,\omega, \tag{3.18}$$

which implies that only the ϵ-component of curl curl $\boldsymbol{\omega}$ can be different from zero. Because of the symmetry of the sphere, $\boldsymbol{\omega}$ depends only on r and θ. Taking this into account and performing calculations in polar coordinates on (3.2) shows that only the ϵ-component of $\boldsymbol{\omega}$ can be non-zero.

Setting now $\overline{\omega} = ((\nu+\nu_r)\mathrm{curl}\,\boldsymbol{v} - 2\nu_r\boldsymbol{\omega}$, we obtain from (3.18):

$$\mathrm{curl}\,\mathrm{curl}\,\overline{\boldsymbol{\omega}} = 0.$$

The solution of such an equation under the condition that $\overline{\boldsymbol{\omega}}$ has just one non-zero component (the ϵ-one) and depends only on r and θ, is well-known (see e.g. Loitsyanskii 1996):

$$\overline{\omega}_\epsilon = \frac{A\sin\theta}{r^2}. \tag{3.19}$$

Computing the ϵ-component of curl $\boldsymbol{v}$ in spherical coordinates, we obtain:

$$\omega_\epsilon = -\frac{A\sin\theta}{2\nu_r r^2} - \frac{\nu+\nu_r}{2\nu_r}\frac{u\sin\theta(2ak^2 + B\mathrm{e}^{-kr}(1+kr))}{k^2r^2}. \tag{3.20}$$

Consider first the conditions on the boundary of the sphere to determine the unknown constants.

$$\boldsymbol{v}|_{r=R} = 0, \tag{3.21}$$

$$\boldsymbol{\omega}|_{r=R} = \frac{\alpha}{2}\mathrm{curl}\,\boldsymbol{v}|_{r=R} \quad \text{with} \quad 0 \leqq \alpha \leqq 1. \tag{3.22}$$

Since $\boldsymbol{n}$ is arbitrary, (3.14) and (3.21) imply:

$$1 - \frac{a}{R} - \frac{b}{R^3} - B\mathrm{e}^{-kR}\left(\frac{1}{k^4R^3} + \frac{1}{k^3R^2} + \frac{1}{k^2R}\right) = 0, \tag{3.23}$$

$$-\frac{a}{R} + \frac{3b}{R^3} + B\mathrm{e}^{-kR}\left(\frac{3}{k^4R^3} + \frac{3}{k^3R^2} + \frac{1}{k^2R}\right) = 0. \tag{3.24}$$

Moreover, (3.20) and (3.22) yield:

$$-\frac{A\sin\theta}{2\nu_r R^2} - \frac{\nu+\nu_r}{2\nu_r}\frac{u\sin\theta(2ak^2 + B\mathrm{e}^{-kR}(1+kR))}{k^2R^2} = \frac{\alpha}{2}\frac{u\sin\theta(2ak^2 + B\mathrm{e}^{-kR}(1+kR))}{k^2R^2}. \tag{3.25}$$

The system (3.23)–(3.25) defines a, b and A as functions of B as follows:

$$a = \tfrac{3}{4}R - \frac{B\mathrm{e}^{-kR}}{2k^2},$$

$$b = \tfrac{1}{4}R^3 - \frac{B\mathrm{e}^{-kR}}{4k^4}(2k^2R^2 + 4kR + 4),$$

$$A = -\frac{uR}{2k}(3k + 2B\mathrm{e}^{-kR})(\nu + (1-\alpha)\nu_r).$$

To determine B, we substitute these expressions into (3.15)–(3.17) and (3.20) and then apply the result to (3.2). The value

$$B = \frac{3k^2 R\nu_r(1-\alpha)}{2\mathrm{e}^{-kR}[kR((\alpha-1)\nu_r-\nu)-\nu]}$$

satisfies the resulting relation independently of the variables r, θ and ϵ, which means that the desired solution is constructed.

Compute now the pressure. Equations (3.1), the definition of $\overline{\boldsymbol{\omega}}$, and the identity $\operatorname{curl}\operatorname{curl} = \nabla\operatorname{div} - \Delta$ imply that

$$\nabla p = -\operatorname{curl}\overline{\boldsymbol{\omega}}.$$

Considering the last relation in spherical coordinates and taking into account (3.19), we obtain

$$p = p_0 + \frac{A}{r^2}\cos\theta. \tag{3.26}$$

3.2. *Calculation of the resistance force*

The force is given by the formula (see Landau & Lifshitz 1995)

$$F = \oint (-p\cos\theta + T'_{rr}\cos\theta - T'_{r\theta}\sin\theta)\,\mathrm{d}f, \tag{3.27}$$

where $T' = T + p\delta_{ij}$, and $\boldsymbol{T}$ is the tensor introduced in (2.1). Expressing the components of this tensor through the calculated velocity and micro-rotation fields leads to the formulae:

$$T'_{rr} = 2\nu\frac{\partial v_r}{\partial r}, \quad T'_{r\theta} = \nu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) + \nu_r\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right) - 2\nu_r\omega_\epsilon.$$

On the boundary of the sphere, we have:

$$T'_{rr} = 0, \qquad T'_{r\theta} = \frac{A}{R^2}\sin\theta, \quad p = p_0 + \frac{A}{R^2}\cos\theta.$$

Therefore, the integral (3.27) reduces to

$$F = -\frac{A}{R^2}\oint \mathrm{d}f.$$

Finally, we obtain

$$F = -4\pi A = 6\pi\nu u R + \frac{6\pi\nu u R\nu_r(\alpha-1)}{kR\,((\alpha-1)\nu_r - nu) - \nu}. \tag{3.28}$$

It is interesting to compare the calculated value with the classical Stokes force given by $F_S = 6\pi\nu u R$. We have

$$F = F_S\left(1 + \frac{(1-\alpha)\nu_r}{(1-\alpha)kR\nu_r + (1+kR)\nu}\right). \tag{3.29}$$

Remembering that $k = \sqrt{4\nu_r\nu/(\nu+\nu_r)(c_a+c_d)}$, we obtain:

$$F = F_S\left(1 + \frac{(1-\alpha)(c_a+c_d)\nu_r}{(1-\alpha)\sqrt{\dfrac{\nu\nu_r(c_a+c_d)}{\nu+\nu_r}}\nu_r R + \nu\left(c_a + c_d + 2\sqrt{\dfrac{\nu\nu_r(c_a+c_d)}{\nu+\nu_r}}R\right)}\right),$$

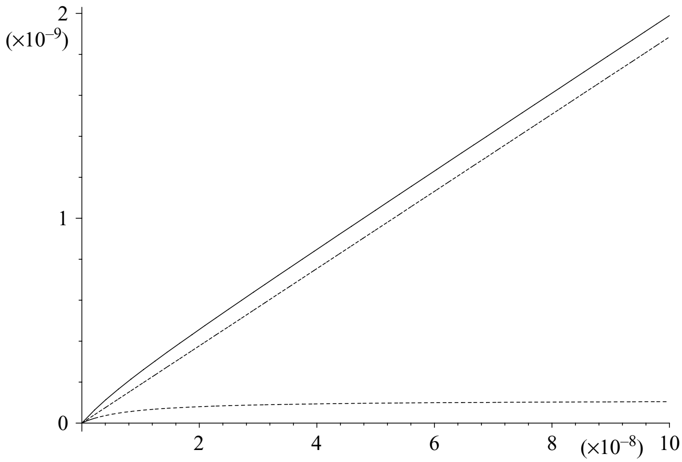

FIGURE 1. The modified force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) versus R in the case of water.

which implies:

(i) $F = F_s$, if $\alpha = 1$ (non-symmetric part of the stress tensor vanishes),
(ii) $F = F_s$, if $\nu_r \longrightarrow 0$, whereas $c_a + c_d$ remains bounded,
(iii) $F = F_s$, if $c_a + c_d \longrightarrow 0$, whereas ν_r remains bounded.

These results are in agreement with the expected behaviour of the force when varying material parameters related to the micro-rotation.

4. Numerical results

The following examples present the calculation of the modified resistance force for water and blood. The results are compared with the classical Stokes force. Unfortunately, there is little information concerning the values of micropolar viscosity coefficients and boundary constant α in the literature. We refer to Kolpashchikov, Mingun & Prokhorenko (1983) where formulae for the calculation of material constants for water on the base of experimental data are given. Unfortunately, results for a simplified model (two-dimensional-Poiseuille flow), for which only two material constants are required, are given. To calculate the other material constants, we guessed that $\alpha = 0.5$. In this way, the following values for the micropolar viscosity constants were found: $\nu_r = 1.448275862 \times 10^{-3}$, $(c_a + c_d) = 4.828973844 \times 10^{-19}$. Figure 1 shows the dependence of the modified resistance force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) on the radius R. Figure 2 shows F/F_S versus R. The velocity $u = 1\,\mathrm{m\,s^{-1}}$ was used. The difference $F - F_S$ is very small for macro- and mesoscopic values of R. The difference is observable for very small radii only. This is not surprising because water is a classical Newtonian fluid so that effects of the inner structure of its particles are important only on very small scales.

As for blood, some values can be found in the literature (see e.g. Papautsky *et al.* 1999). The values $\nu = 2.9 \times 10^{-3}$, $\nu_r = 2.32 \times 10^{-4}$, $(c_a + c_d) = 10^{-6}$ are declared there for the Dirichlet boundary condition ($\alpha = 0$). Figures 3 and 4 show the same curves for blood as in the case of water. As expected, a higher influence of the micro-rotation is present.

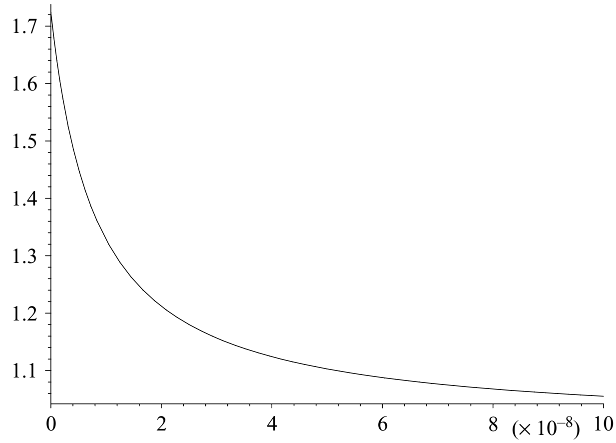

FIGURE 2. The ratio F/F_S versus R in the case of water.

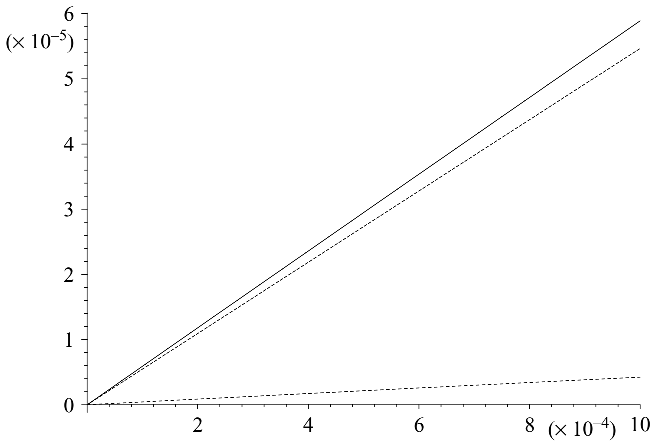

FIGURE 3. The modified force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) versus R in the case of blood.

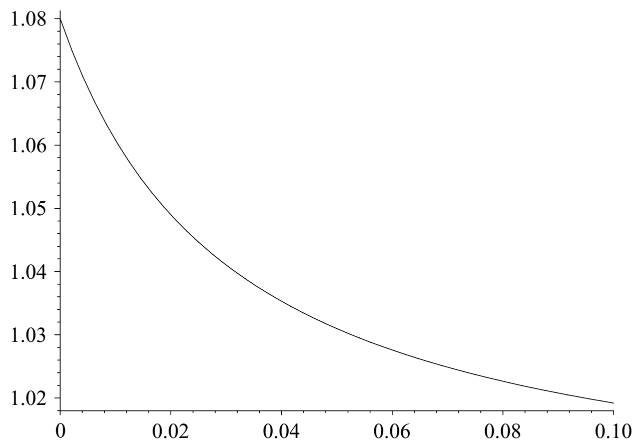

FIGURE 4. The ratio F/F_S versus R in the case of blood.

5. Results of other authors

In this section, results obtained by Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), Ramkissoon & Majumdar (1976), Ramkissoon (1985) and Hayakawa (2000) on the calculation of the drag force exerted on a sphere by a moving micropolar fluid are compared with our results. The comparison can be summarized as follows.

1. All off the above cited papers consider the same governing equations.
2. All papers except for the present one consider the homogeneous boundary condition for the micro-rotation field (see (3.7)).
3. The results of Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), and Ramkissoon & Majumdar (1976) are identical to the present results, if α is equal to zero in (3.7).
4. The computation of the velocity and micro-rotation fields in Ramkissoon (1985) is not quite correct. This is discussed in Hayakawa (2000).
5. The velocity, pressure and micro-rotation fields are found correctly in Hayakawa (2000). They are identical with our results for the case $\alpha = 0$. Nevertheless, the formula for the drag force is not derived correctly in Hayakawa (2000). It is quite different from the formula obtained in Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), Ramkissoon & Majumdar (1976), and in this paper, although all of these papers consider the same problem.
6. In the sequence of papers: Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), Ramkissoon & Majumdar (1976) and Hayakawa (2000), no paper cites any preceding one.

5.1. *Comparison with Lakshmana Rao & Bhujanga Rao* (1970), *Erdogan* (1972) *and Ramkissoon & Majumdar* (1976)

Remember that the drag force in this paper is given by (3.29), i.e.

$$F = 6\pi\nu R u\left(1 + \frac{(1-\alpha)\nu_r}{(1-\alpha)kR\nu_r + (1+kR)\nu}\right) \text{ where } k = \sqrt{\frac{4\nu_r\nu}{(\nu+\nu_r)(c_a+c_d)}}. \tag{5.1}$$

The formulae for the drag force obtained in Lakshmana Rao & Bhujanga Rao (1970) and Ramkissoon & Majumdar (1976) are identical. They use the same notation and are:

$$F_{L\&R} = \frac{6\pi a u(2\mu+\kappa)(\mu+\kappa)(1+a\chi)}{\kappa+2\mu+2a\chi\mu+2a\chi\kappa} \text{ with } \chi = \sqrt{\frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}}. \tag{5.2}$$

It holds in the notation of our paper:

$$\kappa = 2\nu_r \quad \gamma = c_a + c_d, \quad \mu = \nu - \nu_r, \quad a = R. \tag{5.3}$$

The formula for the drag force obtained in Erdogan (1972) is:

$$F_E = 6\pi\nu R u\left[1 - \frac{N}{L}\frac{K_{1/2}(NL)}{K_{3/2}(NL)}\right]^{-1}, \tag{5.4}$$

where $K_{n+1/2}$, $n = 0, 1$ are the modified spherical Bessel functions of the second kind. Remember that

$$K_{1/2}(x) = \sqrt{\pi/2}\frac{e^{-x}}{\sqrt{x}}, \quad K_{3/2}(x) = \sqrt{\pi/2}\frac{e^{-x}(1+1/x)}{\sqrt{x}}.$$

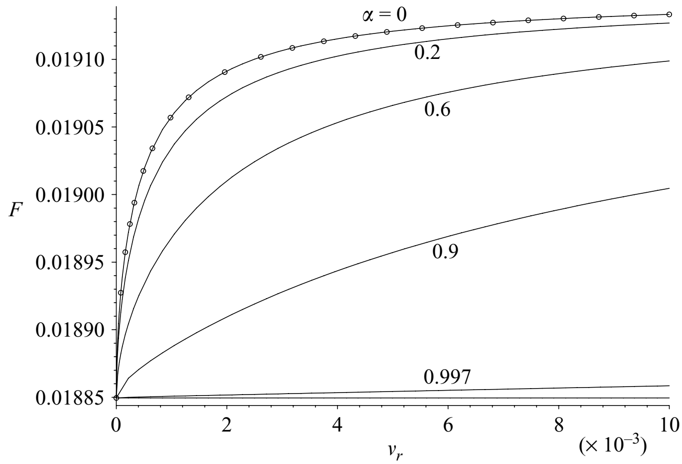

FIGURE 5. The dependence of the drag force F on ν_r for $\nu = 10^{-3}$, $c_a + c_d = 10^{-6}$, $u = 1$ and $R = 1$. Thereby, the parameter α assumes the values 0, 0.2, 0.6, 0.9 and 0.997. The curves are computed using (5.1). The horizontal line represents the classical Stokes drag force that is, of course, independent of ν_r. The circle markers are computed using (5.2) (or (5.4)) to emphasis the identity of (5.1), (5.2) and (5.4) as $\alpha = 0$. As expected, the drag force tends to the classical Stokes one as α tends to 1.

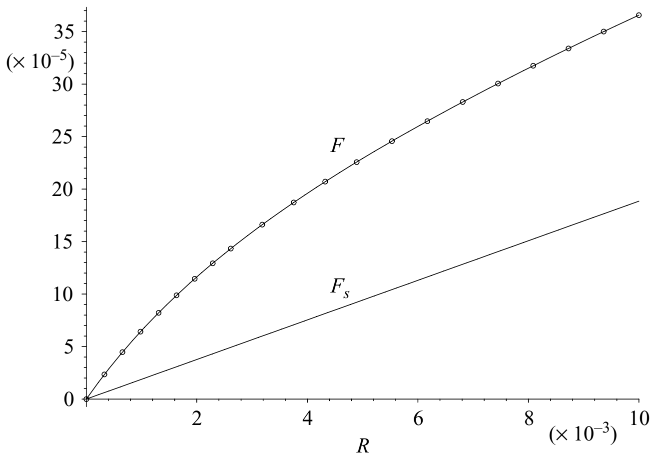

FIGURE 6. The marked curve shows the dependence of the drag force F on R for $\nu = 10^{-3}$, $\nu_r = 3 \times 10^{-3}$, $c_a + c_d = 10^{-6}$, $u = 1$ and $\alpha = 0$. The circle markers are computed using (5.2) (or (5.4)) to emphasis the identity of (5.1), (5.2) and (5.4) as $\alpha = 0$. The straight line represents the classical Stokes force that is, of course, linear in R.

The quantities N and L are defined as follows:

$$N = \left(\frac{\nu_r}{\nu + \nu_r}\right)^{1/2}, \quad L = R\left(\frac{c_a + c_d}{4\nu}\right)^{-1/2}.$$

Setting $\alpha = 0$ in (5.1) and comparing (5.1) with (5.2) and (5.4) under notation (5.3) shows the identity of F, $F_{L\&R}$ and F_E. The comparison is done using the MAPLE software package for symbolic calculations. Figures 5 and 6 illustrate the dependence of the drag force on some parameters.

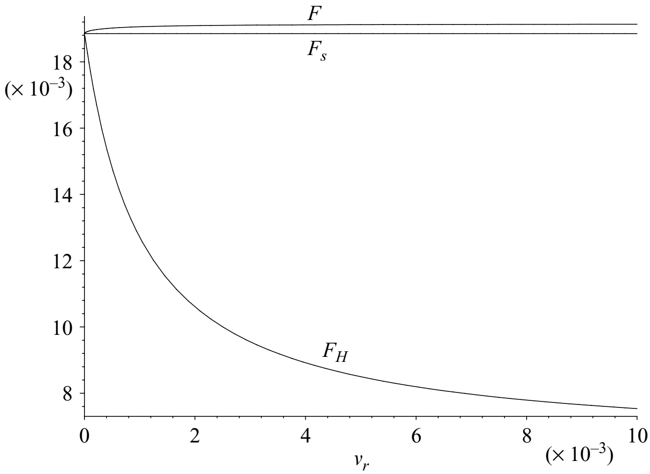

FIGURE 7. The dependence of the function F_H on ν_r for $\nu = 10^{-3}$, $c_a + c_d = 10^{-6}$, $u = 1$ and $R = 1$. The horizontal line represents the classical Stokes force that is of course independent on ν_r. The graph of the correct drag force F (see also figure 5) is given for comparison.

5.2. *Comparison with Hayakawa* (2000)

The result obtained in Hayakawa (2000) is:

$$F_H = \frac{2\pi(\eta + \eta_r)au(1+\xi)(2-\mu_r)(3-\mu_r)}{2(1+\xi) - \mu_r\xi} \quad \text{where} \quad \xi = \sqrt{\frac{\mu_B}{2-\mu_r}}. \tag{5.5}$$

In our notation, the parameters are:

$$\eta = \nu, \quad \eta_r = \nu_r, \quad \mu_r = \frac{2\nu_r}{\nu + \nu_r}, \quad \mu_B = \frac{c_a + c_d}{2\nu_r}, \quad a = R. \tag{5.6}$$

Under (5.6), the formulae obtained in Hayakawa (2000) and in this paper (with $\alpha = 0$) for the velocity, pressure and micro-rotation fields are identical, which is verified using MAPLE. Nevertheless, final formula (5.5) seems not to be true. First of all, the dependence on the radius R (i.e. on a) is linear, which contradicts the results of Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), Ramkissoon & Majumdar (1976) and this paper. The linearity means in particular that $(F_H - F_S)/F_S$, where F_S is the Stokes drag force, does not depend on R. This is not correct because the relative effect of the micro-rotation must decrease as $R \to \infty$. Thus, it would be expected that $(F_H - F_S)/F_S \to 0$ as $R \to \infty$. Such a behaviour holds for the dependence given by (5.1), (5.2) and (5.4). Moreover, figure 7 shows a strange behaviour of F_H when varying the parameter ν_r: F_H decreases as ν_r increases.

6. Conclusion

This paper can be considered as the extension of the result of Lakshmana Rao & Bhujanga Rao (1970), Erdogan (1972), and Ramkissoon & Majumdar (1976) to the case of nonhomogeneous boundary conditions for the micro-rotation field. Examples show that sufficient deviations from classical results are being observed for very small radii only when considering classical Newtonian fluids such as water. This should be expected because the effects of the inner structure of their molecules are important on very small scales only. In the case of fluids with large molecules (e.g. blood), the modified formula shows results that differ from the classical case even for macroscopic radii.

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