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Vortex pinning in super-conductivity as a rate-independent process†

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1 Introduction

For superconductors of type II the phenomenon of vortex pinning plays an important role in technological applications. Several models have been proposed for this effect (Kim *et al*., 1963; Bean, 1964; Bossavit, 1994). In Du *et al*. (1999) and Prigozhin (1996), some of these models are analyzed. In this work we want to contribute to the analysis for the two-dimensional, rate-independent model proposed in Chapman (2000), which has the special feature that vortex movement and creation is an activated process occurring only when a threshold value of the magnetic field is reached. For analytical studies of related rate-dependent models we refer to Chapman *et al*. (1996), Schätzle & Styles (1999) and Elliott & Styles (2000).

For our model let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain (see Barnett & Prigozhin (2005) for the case that Ω has holes which needs different boundary conditions). Denote by $\tilde{H} : \Omega \to \mathbb{R}$ the magnetic field perpendicular to the plane. The vortex tube density $\omega : \Omega \to \mathbb{R}$ is related to $\tilde{H}$ via the constitutive relation

$$\omega = \tilde{A}(\tilde{H}) := \alpha\tilde{H} - \mathrm{div}(\beta\nabla\tilde{H}),$$

where α and β are material parameters and $\lambda = \sqrt{\beta/\alpha}$ is called the penetration depth. In the classical Bean model (Bean, 1964) one has $\beta = 0$, however, our approach does not work for this case. The modeling assumption in Chapman (2000) is now that the vortex tubes will not move if the modulus of the induced current $J = (\nabla\tilde{H})^{\perp} = (-\partial_2\tilde{H}, \partial_1\tilde{H}, 0)^{\top} \in \mathbb{R}^3$ is smaller than a critical value J_c and that they move immediately if $|J| = J_c$. The movement is then described by a mobility function $m : [0, T]\times\Omega \to \mathbb{R}$ which plays the role of a

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Lagrange multiplier. The full problem has then the following form:

$$\left.\begin{array}{c}\partial_t\omega = \operatorname{div}(m\nabla\tilde{H}) \quad \text{with } \omega = \tilde{A}(\tilde{H}),\\ m \geqslant 0, \quad J_{\mathrm{c}} - |\nabla\tilde{H}| \geqslant 0, \quad (J_{\mathrm{c}} - |\nabla\tilde{H}|)m = 0\end{array}\right\} \quad \text{in } [0,T]\times\Omega, \tag{1.1}$$
$$\tilde{H}(0,\cdot) = H_0 \quad \text{on } \Omega \quad \text{and} \quad \tilde{H}(t,x) = H_{\mathrm{ext}}(t) \quad \text{on } [0,T]\times\partial\Omega.$$

The first equation expresses the conservation of the vortex-tube density which is driven by the current J. The second line contains the variational inequalities which model the pinning as an activated process. The magnetic field outside of Ω is assumed to be constant, since the external current is 0, i.e. $0 = J_{\mathrm{ext}} = (-\partial_2 H_{\mathrm{ext}}, \partial_1 H_{\mathrm{ext}}, 0)^\top$.

The aim of this work is to rewrite the problem in an energetic formulation which provides a much easier approach to the existence and uniqueness theory. As the main unknown, we use $H = \tilde{H} - GH_{\mathrm{ext}}(t)$, where $G : \Omega \to \mathbb{R}$ is defined in (2.1), and choose the state space $X = \mathrm{H}^1_0(\Omega)$. We define the energy functional $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ via

$$\mathscr{E}(t,H) = \int_\Omega \frac{1}{2}A(H)(x)H(x) - \alpha H_{\mathrm{ext}}(t)H(x)\,\mathrm{d}x$$

and a dissipation functional for $v = \partial_t H$ via

$$\Psi(v) = \sup\left\{ \int_\Omega A(\hat{H})(x)v(x)\,\mathrm{d}x \;\middle|\; \hat{H} \in \mathrm{H}^1_0(\Omega), |\nabla\hat{H}| \leqslant J_{\mathrm{c}} \right\}. \tag{1.2}$$

Here A denotes the self-adjoint operator $A : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega), H \mapsto \tilde{A}(H)$. By definition Ψ is 1-homogeneous, i.e.

$$\forall\,\lambda \geqslant 0 \;\; \forall\,v \in X: \quad \Psi(\lambda v) = \lambda\Psi(v), \tag{1.3}$$

and convex. This implies the triangle inequality

$$\forall\,v_1, v_2 \in X: \quad \Psi(v_1 + v_2) \leqslant \Psi(v_1) + \Psi(v_2). \tag{1.4}$$

Note that $\Psi(H_1 - H_0)$ has the physical dimension of an energy and can be interpreted as the minimal amount of energy dissipated due to vortex movement when changing the state from H_0 to H_1.

We show that (1.1) is formally equivalent to the differential inclusion

$$0 \in \partial\Psi(\partial_t H) + \mathrm{D}\mathscr{E}(t,H) \subset X^*, \tag{1.5}$$

where $\partial\Psi(v)$ is the set-valued subdifferential defined in (2.6). Moreover, the differential inclusion is equivalent to the following *energetic formulation*:

For all $t \in [0,T]$ we have

$$\text{(S)} \quad \mathscr{E}(t,H(t)) \leqslant \mathscr{E}(t,\hat{H}) + \Psi(\hat{H} - H(t)) \quad \text{for all } \hat{H} \in X \tag{1.6}$$
$$\text{(E)} \quad \mathscr{E}(t,H(t)) + \int_0^t \Psi(\partial_t H(t))\,\mathrm{d}t = \mathscr{E}(0,H(0)) - \int_0^t\int_\Omega \partial_\tau H_{\mathrm{ext}}(\tau)H(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Under the simple assumption $H_{\mathrm{ext}} \in \mathrm{C}^1([0,T],\mathbb{R})$ we show that (1.5) and (1.6) have, for each $H(0) = H_0 \in \mathrm{H}^1_0(\Omega)$ which satisfies (S) at time 0, a unique solution $H \in \mathrm{C}^{\mathrm{Lip}}([0,T],X)$. The reformulation of problem (1.1) into (1.5) and (1.6) will be discussed in § 2. Note that the variational inequality stated in Du *et al.* (1999, Theorem 4.1) is different from our energetic formulation, which has a much more direct physical interpretation, see the discussion.

In § 3 we provide a self-contained existence and uniqueness proof which is a slight generalization of the theory in Mielke & Theil (2004). It is based on time-discretization and the incremental minimization problem

$$\mathscr{E}(t_k,H) + \Psi(H - H_{k-1}) \to \underset{H\in X}{\mathrm{minimum}}.$$

We believe that the simplicity of the approach will allow for several generalizations such that more general models in super-conductivity can be studied.

2 Reformulation of the model

We denote by ${}_*\langle\cdot,\cdot\rangle_X$ the duality between the dual $X^* = \mathrm{H}^{-1}(\Omega)$ and $X = \mathrm{H}^1_0(\Omega)$. By the general assumption that $\alpha,\beta \in (0,\infty)$ are fixed, we see that $\tilde{A}(H) = \alpha H - \nabla\cdot(\beta\nabla H)$ defines a self-adjoint operator

$$A : \begin{cases} X & \to & X^*, \\ H & \mapsto & \tilde{A}(H), \end{cases}$$

i.e. ${}_*\langle A(H_2),H_1\rangle_X = {}_*\langle A(H_1),H_2\rangle_X$. In fact, we may also assume $\alpha \in \mathrm{L}^\infty(\Omega)$ and $\beta \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with $\alpha,\beta \geqslant \delta > 0$ for some $\delta > 0$. We also define the auxiliary function $G \in \mathrm{H}^1(\Omega)$ via

$$\tilde{A}(G) = 0 \text{ in } \Omega \quad \text{and} \quad G|_{\partial\Omega} \equiv 1. \tag{2.1}$$

The choice was done such that for $H = \tilde{H} - GH_{\mathrm{ext}}$ with $\tilde{H}(t,x) = H_{\mathrm{ext}}(t)$ for $x \in \partial\Omega$ we have

$$H(t,x) = 0 \quad \text{for } (t,x) \in [0,T]\times\partial\Omega \quad \text{and} \quad \tilde{A}(\partial_t\tilde{H}) = \tilde{A}(\partial_t H).$$

With this definition the first equation in (1.1) can be written in weak form as

$$-{}_*\langle A(\partial_t H),\hat{H}\rangle_X = \int_\Omega m\nabla(H + GH_{\mathrm{ext}})\cdot\nabla\hat{H}\,\mathrm{d}x \quad \text{for all } \hat{H} \in X. \tag{2.2}$$

The conditions involving the Lagrange multiplier (or mobility factor) can be written more precisely in terms of convex analysis. For this introduce the set

$$\mathscr{C} = \{\, \hat{H} \in X \mid |\nabla\hat{H}| \leqslant J_{\mathrm{c}} \text{ a.e. in } \Omega \,\} \subset X. \tag{2.3}$$

Obviously, $\mathscr{C}$ is closed, convex and bounded. Note that $0 \in \mathscr{C}$, but $\mathscr{C}$ has empty interior in X. We define the set-valued normal cone $\mathrm{N}_{\mathscr{C}}$ via

$$\mathrm{N}_{\mathscr{C}}(H) := \begin{cases} \{\, v^* \in X^* \mid {}_*\langle v^*, H - \hat{H}\rangle_X \geqslant 0 \quad \text{for all} \quad \hat{H} \in \mathscr{C} \,\} & \text{for } H \in \mathscr{C}, \\ \emptyset & \text{for } H \notin \mathscr{C}. \end{cases}$$

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With this we postulate the following differential inclusion:

$$-A(\partial_t H) \in \mathrm{N}_{\mathscr{C}}(H + (G-1)H_{\mathrm{ext}}(t)) \subset X^* \quad \text{for a.e. } t \in [0, T]. \tag{2.4}$$

Proposition 2.1 *If the pair* $(\tilde{H}, m)$*, with* $\tilde{H} \in \mathrm{W}^{1,1}([0,T], \mathrm{H}^1(\Omega))$ *and* $m \in \mathrm{L}^1([0,T], \mathrm{L}^2(\Omega))$*, is a solution of* (1.1)*, then* $H = \tilde{H} - GH_{\mathrm{ext}}$ *solves* (2.4).

Proof We first eliminate the Lagrange multiplicator m in (1.1). For $\tilde{H} \in \mathscr{C}$ we set

$$\mathscr{M}(\tilde{H}) := \left\{ v^* \in \mathrm{H}^{-1}(\Omega) \middle| \begin{array}{l} \exists\, m \in \mathrm{L}^2(\Omega) : m \geqslant 0 \text{ and } (J_{\mathrm{c}} - |\nabla \tilde{H}|)m = 0 \text{ a.e.,} \\ {}_*\langle v^*, \varphi \rangle_X = \int_\Omega m \nabla \tilde{H} \cdot \nabla \varphi \, \mathrm{d}x \text{ for all } \varphi \in X \end{array} \right\} \tag{2.5}$$

and $\mathscr{M}(\tilde{H}) := \emptyset$ for $\tilde{H} \notin \mathscr{C}$. For each constant h we have $\mathscr{M}(\tilde{H}) = \mathscr{M}(\tilde{H} - h)$. With this definition (1.1) takes the form $-\tilde{A}(\partial_t \tilde{H}) \in \mathscr{M}(\tilde{H} - H_{\mathrm{ext}}(t))$ for a.e. $t \in [0, T]$.

Using $\tilde{H} = H + GH_{\mathrm{ext}}$ and $\tilde{A}(G) = 0$, we see that the assertion holds if we are able to show that $\mathscr{M}(H) \subset \mathrm{N}_{\mathscr{C}}(H)$ for all $H \in X$. For $H \notin \mathscr{C}$ we have $\mathscr{M}(H) = \mathrm{N}_{\mathscr{C}}(H) = \emptyset$. Thus, assume $H \in \mathscr{C}$ and take $v^* \in \mathscr{M}(H)$, we then have to show

$${}_*\langle v^*, H - \hat{H} \rangle_X \geqslant 0 \quad \text{for all} \quad \hat{H} \in \mathscr{C}.$$

By the definition of $\mathscr{M}(H)$ there exists $m \in \mathrm{L}^2(\Omega)$ with $m \geqslant 0$, $(J_{\mathrm{c}} - |\nabla H|)m = 0$ and

$${}_*\langle v^*, H - \hat{H} \rangle_X = \int_\Omega m \nabla H \cdot (\nabla H - \nabla \hat{H}) \, \mathrm{d}x.$$

In the last integral the integrand is in fact pointwise nonnegative a.e.. In fact, if $m(x) = 0$ this is obvious, and if $m(x) > 0$ then $|\nabla H| = J_{\mathrm{c}}$ which implies

$$\nabla H \cdot (\nabla H - \nabla \hat{H}) = |\nabla H|^2 - \nabla H \cdot \nabla \hat{H} \geqslant (J_{\mathrm{c}})^2 - J_{\mathrm{c}} |\nabla \hat{H}| \geqslant 0,$$

since $\hat{H} \in \mathscr{C}$. Thus, we have ${}_*\langle v^*, H - \hat{H} \rangle_X \geqslant 0$ as desired. □

In fact, we believe that the problems (1.1) and (2.4) are equivalent. However, so far we were unable to prove $\mathscr{M}(H) = \mathrm{N}_{\mathscr{C}}(H)$ in general.

It is now easy to reformulate (2.4) in several ways by using the Legendre transform (see Visintin (1994), Monteiro Marques (1993) and Mielke & Theil (1999)). Introduce the convex characteristic function $\mathscr{X}_{\mathscr{C}}$ via $\mathscr{X}_{\mathscr{C}}(H) = 0$ for $H \in \mathscr{C}$ and ∞ else and its Legendre–Fenchel transform $\mathscr{X}^*_{\mathscr{C}} = \mathscr{L}\mathscr{X}_{\mathscr{C}}$ via

$$(\mathscr{L}\mathscr{X}_{\mathscr{C}})(v^*) = \sup\{ {}_*\langle v^*, \varphi \rangle_X - \mathscr{X}_{\mathscr{C}}(\varphi) \mid \varphi \in X \}.$$

Moreover, define the subdifferential ∂f for any convex function $f : Y \to \mathbb{R} \cup \{\infty\}$ via

$$\partial f(y) = \{ v^* \in Y^* \mid \forall \hat{y} \in Y : f(\hat{y}) \geqslant f(y) + \langle v^*, \hat{y} - y \rangle \}, \tag{2.6}$$

where Y will be either X or X^*. Then, the following standard relations hold:

$$(a) \quad \mathrm{N}_{\mathscr{C}}(H) = \partial\mathscr{X}_{\mathscr{C}}(H),$$
$$(b) \quad v^* \in \partial\mathscr{X}_{\mathscr{C}}(H) \Leftrightarrow H \in \partial\mathscr{X}^*_{\mathscr{C}}(v^*).$$

Using (a) and (b) we see that (2.4) is equivalent to $H + (G-1)H_{\mathrm{ext}} \in \partial\mathscr{X}^*_{\mathscr{C}}(-A\partial_t H)$: exploiting the symmetry $\mathscr{C} = -\mathscr{C}$ and applying A we arrive at

$$-(AH - \alpha H_{\mathrm{ext}}) \in A(\partial\mathscr{X}^*_{\mathscr{C}}(A\partial_t H)) \subset X^*, \tag{2.7}$$

where we have used $\tilde{A}G = 0$ and $\tilde{A}1 = \alpha$.

Lemma 2.2 *Let $\Psi : X \to [0,\infty)$ be defined via $\Psi(v) = \sup\{\, {}_*\langle AH, v\rangle_X \mid H \in \mathscr{C}\}$, then $\Psi(v) = \mathscr{X}^*_{\mathscr{C}}(Av)$ and $\partial\Psi(v) = A\partial\mathscr{X}^*_{\mathscr{C}}(Av)$ for all $v \in X$.*

Proof By this definition we easily find $\mathscr{X}^*_{\mathscr{C}}(v^*) = \sup\{\, {}_*\langle v^*, H\rangle_X \mid H \in \mathscr{C}\}$. Thus we have $\Psi(v) = \mathscr{X}^*_{\mathscr{C}}(Av)$ and the result for the subdifferential follows from the chain rule and $A = A^*$. □

Finally, we define the energy functional

$$\mathscr{E}(t, H) = \frac{1}{2}\, {}_*\langle AH, H\rangle_X - \int_\Omega \alpha H(x) H_{\mathrm{ext}}(t)\mathrm{d}x$$

and obtain the main result of this section, since $\mathrm{D}\mathscr{E}(t,H) = AH - \alpha H_{\mathrm{ext}}$.

Proposition 2.3 *Equation (2.4) is equivalent to*

$$0 \in \partial\Psi(\partial_t H) + \mathrm{D}\mathscr{E}(t,H) \quad \textit{for a.e. } t \in [0,T]. \tag{2.8}$$

Such equations are called "doubly nonlinear" in Colli & Visintin (1990), where also a general existence theory is developed for the rate-dependent case.

Using the rate-independence of our model, which is the same as the 1-homogeneity of Ψ (see (1.3)), and the triangle inequality for Ψ in (1.4) it is easy to see that (2.8) is equivalent to the two conditions

$$\left.\begin{array}{ll} (\mathrm{S})_{\mathrm{loc}} & {}_*\langle \mathrm{D}\mathscr{E}(t,H), v\rangle_X + \Psi(v) \geqslant 0 \quad \text{for all } v \in X, \\ (\mathrm{E})_{\mathrm{loc}} & {}_*\langle \mathrm{D}\mathscr{E}(t,H), \partial_t H\rangle_X + \Psi(\partial_t H) = 0. \end{array}\right\} \tag{2.9}$$

Since $\mathscr{E}(t,\cdot) : X \to \mathbb{R}$ is also convex, we arrive at the energetic formulation

$$(\mathrm{S}) \quad \mathscr{E}(t, H(t)) \leqslant \mathscr{E}(t, \hat{H}) + \Psi(\hat{H} - H(t)) \quad \text{for all } \hat{H} \in X,$$
$$(\mathrm{E}) \quad \mathscr{E}(t, H(t)) + \textstyle\int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau = \mathscr{E}(0, H(0)) - \int_0^t \int_\Omega \partial_t H_{\mathrm{ext}}(\tau)\alpha H(\tau, x)\,\mathrm{d}x\,\mathrm{d}\tau$$

The stability condition (S) has the obvious interpretation, that a state $H(t)$ can only occur if for no other state $\hat{H}$ we can release more energy than is dissipated by the moving vortices. Obviously, $(\mathrm{S})_{\mathrm{loc}}$ is the same as $0 \in \partial\Psi(0) + \mathrm{D}\mathscr{E}(t, H(t))$. Using Lemma 2.2 we find

$$\partial\Psi(0) = A\mathscr{C} = \{\, AH \mid H \in \mathscr{C} \,\} \subset X^* \tag{2.10}$$

and thus, $(\mathrm{S})_{\mathrm{loc}}$, and hence (S), is equivalent to $A^{-1}\mathrm{D}\mathscr{E}(t, H(t)) = A^{-1}(AH - \alpha H_{\mathrm{ext}}) = H + (G-1)H_{\mathrm{ext}} \in \mathscr{C}$. This is of course the condition $|\nabla\tilde{H}| \leqslant J_{\mathrm{c}}$.

The energy balance (E) just states that the total stored energy $\mathscr{E}(t, H(t))$ at time t is the initial energy plus the work of the boundary conditions through the external field H_{ext} minus the dissipated energy.

For more exact proofs of these equivalences we refer to Mielke & Theil (1999).

3 Existence and uniqueness

To formulate the main result most conveniently we recall $\partial\Psi(0) = A\mathscr{C}$.

Theorem 3.1 *Let $H_{\mathrm{ext}} \in \mathrm{C}^1([0,T])$ and H_0 be given with $H_0 + (G-1)H_{\mathrm{ext}}(0) \in \mathscr{C}$. Then,* (2.8) *has a unique solution $H \in \mathrm{C}^{\mathrm{Lip}}([0,T], X)$ with $H(0) = H_0$.*

This result is a special case of several well-established theories. In fact, we simplified the problem by assuming C^1 smoothness of H_{ext} which would not be necessary. However, in rate-independent systems we may always rescale time to gain smoothness. For instance, combining Theorem 3.1 and Proposition 3.5 in Krejčí (1999) proves our result. Moreover, in Visintin (1994) or Monteiro Marques (1993), corresponding results can be found. Nevertheless, we find it worthwhile to provide an independent short proof which is based on the energetic formulation (S) and (E), and thus is closer to the underlying physics. We follow the more general approach in Mielke & Theil (1999, 2004), however we have to work around their hypothesis $\Psi(v) \geqslant c\|v\|$ which is not true in our situation.

We introduce the set $\mathscr{S}(t)$ of stable states at time t via

$$\mathscr{S}(t) = \{\, H \in X \mid \mathscr{E}(t, H) \leqslant \mathscr{E}(t, \hat{H}) + \Psi(\hat{H} - H) \text{ for all } \hat{H} \in X \,\}.$$

The condition (S) is equivalent to $H(t) \in \mathscr{S}(t)$. As seen at the end of § 2 we have

$$\mathscr{S}(t) = (1-G)H_{\mathrm{ext}}(t) + \mathscr{C},$$

which shows that $\mathscr{S}(t)$ is a closed, convex, bounded set depending smoothly on $t \in [0, T]$.

Proof of Theorem 3.1 The proof relies on time discretization. For $n \in \mathbb{N}$ subdivide $[0, T]$ equidistantly into 2^n intervals via $t_k^n = kT/2^n$ for $k = 0, 1, \ldots, 2^n$. We let $H_0^n = H_0$ and define H_k^n iteratively via

$$H_{k+1}^n = \arg\min \{\mathscr{E}(t_{k+1}^n, H) + \Psi(H - H_k^n) \mid H \in X\}. \tag{3.1}$$

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Since $\mathscr{E}$ is strictly convex, the minimizer exists and is unique. Moreover, we have

$$\text{(A)}\quad H_k^n \in \mathscr{S}(t_k^n) \quad \text{for } n \in \mathbb{N} \text{ and } k \in \{0, 1, \ldots, 2^n\},$$
$$\text{(B)}\quad \mathscr{E}(t_k^n, H_k^n) + \Psi(H_k^n - H_{k-1}^n) \leqslant \mathscr{E}(t_{k-1}^n, H_{k-1}^n) + \int_{t_{k-1}^n}^{t_k^n} \partial_s \mathscr{E}(s, H_{k-1}^n)\,\mathrm{d}s$$

For (A) use that (i) H_k^n is a minimizer and that (ii) Ψ satisfies the triangle inequality:

$$\begin{aligned}\mathscr{E}(t_k^n, \hat{H}) + \Psi(\hat{H}-H_k^n) \;&=\; \mathscr{E}(t_k^n, \hat{H}) + \Psi(\hat{H}-H_{k-1}^n) + \Psi(\hat{H}-H_k^n) - \Psi(\hat{H}-H_{k-1}^n) \\ \overset{\text{(i)}}{\geqslant}\; \mathscr{E}(t_k^n, H_k^n) + \Psi(H_k^n-H_{k-1}^n) &+ \Psi(\hat{H}-H_k^n) - \Psi(\hat{H}-H_{k-1}^n) \;\overset{\text{(ii)}}{\geqslant}\; \mathscr{E}(t_k^n, H_k^n).\end{aligned}$$

For (B) we again use that H_k^n is a minimizer

$$\mathscr{E}(t_k^n, H_k^n) + \Psi(H_k^n-H_{k-1}^n) \leqslant \mathscr{E}(t_k^n, H_{k-1}^n) = \mathscr{E}(t_{k-1}^n, H_{k-1}^n) + \int_{t_{k-1}^n}^{t_k^n} \partial_s \mathscr{E}(s, H_{k-1}^n)\,\mathrm{d}s.$$

The stability in (A) is equivalent to

$$_*\langle \mathrm{D}\mathscr{E}(t_k^n, H_k^n), v\rangle_X + \Psi(v) \geqslant 0 \quad \text{for all } v \in X, \tag{3.2}$$

and the minimization property shows that for $v = H_k^n - H_{k-1}^n$ equality holds. Thus, we have

$$\begin{aligned} {}_*\langle A(H_k^n-H_{k-1}^n), H_k^n-H_{k-1}^n\rangle_X \;&=\; {}_*\langle \mathrm{D}\mathscr{E}(t_k^n, H_k^n)-\mathrm{D}\mathscr{E}(t_k^n, H_{k-1}^n), H_k^n-H_{k-1}^n\rangle_X \\ &\overset{(3.2)}{=} -\Psi(H_k^n-H_{k-1}^n) - {}_*\langle \mathrm{D}\mathscr{E}(t_{k-1}^n, H_{k-1}^n), H_k^n-H_{k-1}^n\rangle_X \\ &\quad - \int_{t_{k-1}^n}^{t_k^n} {}_*\langle \partial_s \mathrm{D}\mathscr{E}(s, H_{k-1}^n), H_k^n-H_{k-1}^n\rangle_X\,\mathrm{d}s \\ &\overset{(3.2)}{\leqslant} 0 + \|H_k^n-H_{k-1}^n\|_X \|\partial_t H_{\mathrm{ext}}\|_{\mathrm{C}^0} \|\alpha\|_{X^*}(t_k^n-t_{k-1}^n). \end{aligned}$$

Since the operator A is positive definite, we obtain the a priori Lipschitz bound

$$\|H_k^n - H_{k-1}^n\|_X \leqslant \mathrm{C}_1 |t_k^n - t_{k-1}^n|.$$

We now define the piecewise linear interpolants $H^n : [0, T] \to X$ with $H^n(t_k^n) = H_k^n$, then we know $\| \partial_t H^n(t)\|_X \leqslant \mathrm{C}_1$ for a.a. $t \in [0, T]$. Thus, the Arzelà-Ascoli theorem for $\mathrm{C}^0([0, T], X)$ yields a subsequence (not renumbered) and a limit function $H : [0, T] \to X$ such that for all $t \in [0, T]$ we have $H^n(t) \rightharpoonup H(t)$ in X as $n \to \infty$, where $\rightharpoonup$ denotes weak convergence. Moreover H is Lipschitz continuous with $\|\partial_t H(t)\| \leqslant \mathrm{C}_1$ a.e. in [0,T].

Keeping $t^* = k^* T/2^{n_*}$ fixed, then for all $n \geqslant n_*$ we have $H^n(t^*) \in \mathscr{S}(t^*)$. Since $\mathscr{S}(t^*)$ is closed and convex we conclude $H(t^*) \in \mathscr{S}(t^*)$. Since $\{k^* T/2^{n_*} \in [0, T] \mid n_* \in \mathbb{N} \text{ and } k^* \in \{0, \ldots, 2^{n_*}\}\ \}$ is dense in $[0, T]$, since $H : [0, T] \to X$ is Lipschitz continuous and since $\mathscr{S}(t)$ depends continuously on t, we conclude $H(t) \in \mathscr{S}(t)$ for all $t \in [0, T]$.

Finally, we consider the energy equation. Let t^* be as above and add the discrete energy estimates (B) for $n = n_*$ over $k = 1, \ldots, k^*$. Note that in the case $k = 1$ we use the fact that $H_0 = H_0^n$ lies in $\mathscr{S}(0)$. We find

$$\mathscr{E}(t^*, H^{n_*}(t^*)) + \int_0^{t^*} \Psi(\partial_t H^{n_*}(\tau))\,\mathrm{d}\tau \leqslant \mathscr{E}(0, H_0) - \int_0^{t^*} \partial_t H_{\mathrm{ext}}(\tau) \int_\Omega \alpha\, \overline{H}^{n_*}(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \tag{3.3}$$

where H^{n_*} is the piecewise linear interpolant from above while $\overline{H}^{n_*}$ is the piecewise constant interpolant with $\overline{H}^{n_*}(t) = H^n_{k-1}$ for $t \in [t^n_{k-1}, t^n_k)$. The right-hand side is weakly continuous and on the left-hand side $\mathscr{E}(t, \cdot)$ is convex and continuous and hence weakly lower semi-continuous. It remains us to show the following lemma.

Lemma 3.2 *Assume the sequence $(H^n)_{n\in\mathbb{N}}$ as above, then*

$$\int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau \leqslant \liminf_{n\to\infty} \int_0^t \Psi(\partial_t H^n(\tau))\,\mathrm{d}\tau\,.$$

Proof The sequence $(H^n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C^{Lip}}([0,T],X) = \mathrm{W}^{1,\infty}([0,T],\mathrm{H}^1(\Omega))$, which is continuously embedded into the Hilbert space $\mathscr{H} = \mathrm{H}^1([0,T],X)$. Thus, the sequence converges weakly in $\mathscr{H}$ to the limit H constructed above. For this note, that the sequence is also bounded in $\mathscr{H}$ and hence it has a weakly converging subsequence. Since $\mathscr{H}$ is compactly embedded in $\mathscr{Y} = \mathrm{L}^2([0,T],\mathrm{L}^2(\Omega)) = \mathrm{L}^2([0,T]\times\Omega)$ this subsequence converges strongly in $\mathscr{Y}$. However, the convergence invoked from the Arzelà-Ascoli theorem also implies strong converge in $\mathscr{Y}$. Thus, the weak limit in $\mathscr{H}$ is unique and equal to H.

We now define the functional $\mathscr{I} : \mathscr{H} \to \mathbb{R}$ via $\mathscr{I}(H) := \int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau$. Since $\Psi : \mathrm{H}^1 \to [0,\infty)$ is convex we get immediately the convexity of $\mathscr{I}$. Further the upper estimate $\Psi(v) \leqslant C\|v\|_{\mathrm{H}^1}$ implies the strong continuity of $\mathscr{I}$. Together with convexity this implies sequential weak lower semi-continuity of $\mathscr{I}$ on $\mathscr{H}$, which is the desired result. □

Hence we can go to the limit in (3.3) and find

$$\begin{aligned}&0 \geqslant \mu(t) \quad \text{where}\\ &\mu(t) := \mathscr{E}(t,H(t)) + \textstyle\int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau - \mathscr{E}(0,H_0) + \int_0^t \partial_t H_{\mathrm{ext}}(\tau)\int_\Omega \alpha H(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

This provides one side of the energy balance.

As H is Lipschitz, we can differentiate μ and obtain, after a cancellation, $\dot\mu(t) = {}_*\langle \mathrm{D}\mathscr{E}(t,H(t)), \partial_t H\rangle_X + \Psi(\partial_t H(t))$ which is nonnegative by the stability of $H(t)$. Thus, $\mu(t) \leqslant 0$, $\mu(0) = 0$ and $\dot\mu(t) \geqslant 0$ imply $\mu \equiv 0$. Hence, we have established (E) as well.

Finally we have to show uniqueness which follows again from the variational inequalities (2.9). Let H_j, $j = 1,2$ be two solutions, then for each v by subtracting $(\mathrm{S})_{\mathrm{loc}}$ from $(\mathrm{E})_{\mathrm{loc}}$ we have ${}_*\langle \mathrm{D}\mathscr{E}(t,H_j), \partial_t H_j - v\rangle_X + \Psi(\partial_t H_j) - \Psi(v) \leqslant 0$. Testing with $v = \partial_t H_{3-j}$ and adding both inequalities gives

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\ {}_*\langle A(H_1-H_2), H_1-H_2\rangle_X = {}_*\langle \mathrm{D}\mathscr{E}(t,H_1)-\mathrm{D}\mathscr{E}(t,H_2), \partial_t(H_1-H_2)\rangle_X \leqslant 0.$$

If $H_1(0) = H_2(0)$, this implies $H_1(t) = H_2(t)$ and uniqueness is established. □

4 Discussion

We have reformulated the mean-field model (1.1) for vortex pinning in superconductivity which was formulated in terms of the magnetic field $\tilde{H}$ and the mobility m, the latter being a Lagrange multiplier for the unilateral constraint $|\nabla\tilde{H}| \leqslant J_c$. The reformulations involve

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either the differential inclusion (2.4) are the doubly nonlinear inclusion (2.8). However, we want to emphasize that the *energetic formulation* via (S) and (E) is physically most relevant. First it uses the energy functional $\mathscr{E}$ which denotes the energy stored into the system. Moreover, it involves the dissipation potential Ψ which measures the energy dissipation through changes of H, i.e. through the movement and nucleation of vortices. The stability condition (S) expresses the fact that vortices will move immediately, if the energy dissipated via Ψ is less than the gain in the energy $\mathscr{E}$. This is the easiest way to describe systems with activation thresholds. The energy balance (E) is the usual energy conservation. The present energy plus the dissipated energy equals the initial energy plus the work done by the external forces.

Note that the subdifferential equation (2.8) can also be written as the variational inequality

$$\forall\, v \in X: \quad {}_*\langle AH(t) - \alpha H_{\text{ext}}(t), v - \partial_t H(t)\rangle_X + \Psi(v) - \Psi(\partial_t H(t)) \geqslant 0. \tag{4.1}$$

To see this, just subtract $(\text{E})_{\text{loc}}$ from $(\text{S})_{\text{loc}}$, see (2.9). Variational inequalities of this type where also derived in [4] but the physical interpretation of stability (S) and the energy balance (E) are not highlighted there. Our variational inequality is different from the one stated in Du *et al.* (1999, Theorem 4.1), which reads in our notation

$$\begin{aligned}&\int_0^s {}_*\langle A\partial_t\phi(t), \phi(t) - H(t)\rangle_X + {}_*\langle \partial_t H_{\text{ext}}(t), \phi(t) - H(t)\rangle_X \,\mathrm{d}t\\ &\leqslant \tfrac{1}{2}\, {}_*\langle A(\phi(s) - H(s)), \phi(s) - H(s)\rangle_X - \tfrac{1}{2}\, {}_*\langle A(\phi(0) - H(0)), \phi(0) - H(0)\rangle_X\end{aligned}$$

for all $\phi \in \mathrm{H}^1([0,T],X)$ with $|\nabla\phi(t,x)| \leqslant J_{\mathrm{c}}$ a.e. For a proof of the equivalence of these two variational inequalities we refer to Mielke & Theil (2004) and Mielke (2005).

It should be noted that the theory in section 7 of Mielke & Theil (2004) can be generalized to prove strong convergence with

$$\|H^n(t) - H(t)\|_X \leqslant C(\tau_n)^{1/2} \quad \text{with}\tau_n = T/2^n.$$

Moreover, the time-incremental minimization problems (3.1) can be used to introduce spatial discretization by replacing X by a finite-dimensional subspace X_h – see Du *et al.* (1999, Section 6). We expect that the related convergence results for space-time discretizations obtained for elastoplasticity (see Han & Reddy (1999) and Alberty & Carstensen (2000)) also hold in the present situation.

References

[1] Alberty, J. & Carstensen, C (2000) Numerical analysis of time-depending primal elastoplasticity with hardening. *SIAM J. Numer. Anal.* **37**, 1271–1294.

[2] Bean, C. P. (1964) Magnetization of high-field superconductors. *Rev. Mod. Phys.* **36**(1), 31–39.

[3] Bossavit, B. (1994) Numerical modelling of superconductors in three dimensions: a model and a finite element method. *IEEE Trans. Magn.* **30**, 3363–3366.

[4] Barnett, J. W. & Prigozhin, L. (2005) Sandpiles and superconductors: dual variational formulations for critical-state problems. `http://arxiv.org/abs/cond-mat/0412648`, 2004. (Submitted to *22nd IFIP TC 7 Conference on System Modeling and Optimization*, Turin, Italy, July 2005.)

https://doi.org/10.1017/S0956792505006352 Published online by Cambridge University Press

[5] Chapman, S. (2000) A hierachy of models for type-II superconductors. *SIAM Rev.* **42**(4), 555–598.

[6] Chapman, S. J., Rubinstein, J. & Schatzman, M. (1996) A mean-field model of superconducting vortices. *Euro. J. Appl. Math.* **7**, 97–111.

[7] Colli, P. & Visintin, R. (1990) On a class of doubly nonlinear evolution equations. *Comm. Partial Diff. Eq.* **15**(5), 737–756.

[8] Du, Q., Gunzburger, M. D. & Lee, H. K. (1999) Analysis and computation of a mean-field model for superconductivity. *Numer. Math.* **81**(4), 539–560.

[9] Elliott, C. M. & Styles, V. (2000) Flux pinning and boundary nucleation of vorticity in a mean-field model of superconducting vortices. *Interfaces Free Bound.* **2**, 143–180.

[10] Han, W. & Reddy, B. D. (1999) *Plasticity (Mathematical theory and numerical analysis)*, volume 9 of *Interdisciplinary Applied Mathematics.* Springer-Verlag.

[11] Kim, Y., Hempstead, C. & Strand, A. (1963) Magnetization and critical supercurrents. *Phys. Rev.* **129**(2), 528–535.

[12] Krejčí, P. (1999) Evolution variational inequalities and multidimensional hysteresis operators. *Nonlinear Differential Equations (Chvalatice, 1998)*, volume 404 of *Res. Notes Math.*, pp. 47–110. Chapman & Hall/CRC.

[13] Mielke, A. (2005) Evolution in rate-independent systems. In *Handbook of Differential Equations II* Dafermos, C. and Feireisl, E. (eds). North-Holland. In press

[14] Monteiro Marques, M. D. P. (1993) *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and dry friction.* Birkhäuser Verlag.

[15] Mielke, A. & Theil, F. (1999) A mathematical model for rate–independent phase transformations with hysteresis. In: H.-D. Alber, R. Balean and R. Farwig, editors, *Proceedings of the Workshop on 'Models of Continuum Mechanics in Analysis and Engineering'*, pp. 117–129. Shaker–Verlag.

[16] Mielke, A. & Theil, F. (2004) On rate–independent hysteresis models. *Nonl. Diff. Eq. Appl. (NoDEA)* **11**, 151–189. (Accepted July 2001).

[17] Prigozhin, L. (1996) On the bean critical-state model in superconductivity. *Euro. J. Appl. Math.* **7**, 237–248.

[18] Schätzle, R. & Styles, V. (1999) Analysis of a mean-field model of superconducting vortices. *Euro. J. Appl. Math.* **10**, 319–352.

[19] Visintin, A. (1994) *Differential Models of Hysteresis.* Springer-Verlag, Berlin, 1994.