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The Lovász Number of Random Graphs

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We study the Lovász number ϑ along with two related SDP relaxations $\vartheta_{1/2}$, ϑ_2 of the independence number and the corresponding relaxations $\bar{\vartheta}$, $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}_2$ of the chromatic number on random graphs $G_{n,p}$. We prove that $\vartheta, \vartheta_{1/2}, \vartheta_2(G_{n,p})$ are concentrated about their means, and that $\bar{\vartheta}, \bar{\vartheta}_{1/2}, \bar{\vartheta}_2(G_{n,p})$ in the case $p < n^{-1/2-\varepsilon}$ are concentrated in intervals of constant length. Moreover, extending a result of Juhász [28], we estimate the probable value of $\vartheta, \vartheta_{1/2}, \vartheta_2(G_{n,p})$ for edge probabilities $c_0/n \leqslant p \leqslant 1 - c_0/n$, where $c_0 > 0$ is a constant. As an application, we give improved algorithms for approximating the independence number of $G_{n,p}$ and for deciding k-colourability in polynomial expected time.

1. Introduction and results

Given a graph $G = (V, E)$, let $\alpha(G)$ be the independence number, let $\omega(G)$ be the clique number, and let $\chi(G)$ be the chromatic number of G. Further, let $\bar{G}$ signify the complement of G. Since it is NP-hard to compute any of $\alpha(G)$, $\omega(G)$ or $\chi(G)$, it is remarkable that there exists an efficiently computable function $\vartheta(G)$ that is 'sandwiched' between $\alpha(G)$ and $\chi(\bar{G})$, *i.e.*, $\alpha(G) \leqslant \vartheta(G) \leqslant \chi(\bar{G})$. Passing to complements, and letting $\bar{\vartheta}(G) = \vartheta(\bar{G})$, we have $\omega(G) \leqslant \bar{\vartheta}(G) \leqslant \chi(G)$. The function ϑ was introduced by Lovász [37], and is called the *Lovász number* of G (*cf.* also [30]). The Lovász number can be seen as a semidefinite programming ('SDP') relaxation of the independence number, and is therefore computable in polynomial time within any precision [25].

Though $\vartheta(G)$ is sandwiched between $\alpha(G)$ and $\chi(\bar{G})$, Feige [12] proved that the gap between $\alpha(G)$ and $\vartheta(G)$ or between $\chi(\bar{G})$ and $\vartheta(G)$ can be as large as $n^{1-\varepsilon}$, $\varepsilon > 0$. Indeed, unless NP=coRP, none of $\alpha(G)$, $\omega(G)$, $\chi(G)$ can be approximated within a factor of $n^{1-\varepsilon}$, $\varepsilon > 0$, in polynomial time [26, 13]. However, though there exist graphs G such that $\vartheta(G)$ is not a good approximation of $\alpha(G)$ (or $\bar{\vartheta}(G)$ of $\chi(G)$), it might be the case that the Lovász number performs well on 'average' instances. In fact, several algorithms for random and

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semirandom graph problems are based on computing ϑ [6, 8, 7, 14, 15]. Therefore, the aim of this paper is to study the Lovász number of random graphs more thoroughly.

The standard model of a random graph is the binomial model $G_{n,p}$, pioneered by Erdős and Rényi. We let $0 < p = p(n) < 1$ be a number that may depend on n. Let $V = \{1, \ldots, n\}$. Then the random graph $G_{n,p}$ is obtained by including each of the $\binom{n}{2}$ possible edges $\{v, w\}$, $v, w \in V$, with probability p independently. Though $G_{n,p}$ may fail to model some types of input instances appropriately, both the combinatorial structure and the algorithmic theory of $G_{n,p}$ are of fundamental interest [4, 27, 20]. We say that $G_{n,p}$ has some property A *with high probability* (w.h.p.), if $\lim_{n\to\infty} \mathrm{P}(G_{n,p}$ has property $A) = 1$.

In addition to the Lovász number, we also address two further natural SDP relaxations $\vartheta_{1/2}$, ϑ_2 of α (*cf.* [41]) on random graphs. These relaxations satisfy

$$\alpha(G) \leqslant \vartheta_{1/2}(G) \leqslant \vartheta(G) \leqslant \vartheta_2(G) \leqslant \chi(\bar{G}),$$

for all G, *i.e.*, $\vartheta_{1/2}$ is the strongest relaxation of α among $\vartheta_{1/2}, \vartheta, \vartheta_2$. Passing to complements, and setting $\bar{\vartheta}_i(G) = \vartheta_i(\bar{G})$ $(i = 1/2, 2)$, one obtains

$$\omega(G) \leqslant \bar{\vartheta}_{1/2}(G) \leqslant \bar{\vartheta}(G) \leqslant \bar{\vartheta}_2(G) \leqslant \chi(G),$$

i.e., $\bar{\vartheta}_2$ is the strongest relaxation of χ. The relaxation $\bar{\vartheta}_{1/2}(G)$ coincides with the well-known *vector chromatic number* $\vec{\chi}(G)$ of Karger, Motwani and Sudan [29]. For general background on ϑ, $\vartheta_{1/2}$, and ϑ_2 the reader is referred to [5, 17, 25, 29, 41].

1.1. The concentration of ϑ, $\bar{\vartheta}$, *etc.*

Facing a real-valued random variable $X(G_{n,p})$, there are two obvious questions to ask:

(1) What is the mean of $X(G_{n,p})$?

(2) Is $X(G_{n,p})$ concentrated about its mean?

The main contributions of this paper are concentration results on ϑ, $\bar{\vartheta}$ *etc.*, that is, they concern the second question. Such results are important for instance in the design of algorithms with polynomial expected running time. First, we show that the probability that $\vartheta(G_{n,p})$, $\vartheta_{1/2}(G_{n,p})$, or $\vartheta_2(G_{n,p})$ is far from its median is exponentially small.

Theorem 1.1. *Suppose that $p \leqslant 0.99$ and that $n \geqslant n_0$ for a certain constant $n_0 > 0$. Let m be a median of $\vartheta(G_{n,p})$, i.e.,* $\mathrm{P}(\vartheta(G_{n,p}) \leqslant m) \geqslant 1/2$ *and* $\mathrm{P}(\vartheta(G_{n,p}) \geqslant m) \geqslant 1/2$.

(i) *Let* $\xi \geqslant \max\{10, m^{1/2}\}$. *Then* $\mathrm{P}(\vartheta(G_{n,p}) \geqslant m + \xi) \leqslant 30 \exp(-\xi^2/(5m + 10\xi))$.

(ii) *Let* $\xi > 10$. *Then* $\mathrm{P}(\vartheta(G_{n,p}) \leqslant m - \xi) \leqslant 3 \exp(-\xi^2/10m)$.

The same holds with ϑ replaced by $\vartheta_{1/2}$ or by ϑ_2.

Up to the constants involved, the right-hand sides of the bounds in Theorem 1.1 are similar to well-known bounds on the tails of the binomial distribution (*e.g.*, [27, p. 26]). The assumption $p \leqslant 0.99$ is made to ensure that the median m of $\vartheta(G_{n,p})$ is not 'too small'; for the case $p > 0.99$ a bound on the lower tail will be given in Theorem 1.4 below. The proof of Theorem 1.1 is based on Talagrand's inequality: see Section 3.1 for details.

A remarkable fact concerning the chromatic number of sparse random graphs $G_{n,p}$, $p \leqslant n^{-\varepsilon-1/2}$, is that $\chi(G_{n,p})$ is concentrated in an interval of constant length. Indeed,

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Shamir and Spencer [40] proved that there is a function $u = u(n, p)$ such that in the case $p = n^{-\beta}$, $1/2 < \beta < 1$, we have $\mathrm{P}(u \leqslant \chi(G_{n,p}) \leqslant u + \lceil (2\beta + 1)/(2\beta - 1) \rceil) = 1 - o(1)$. Furthermore, Łuczak [38] has shown that in the case $5/6 < \beta < 1$, the chromatic number is concentrated in width one, which is best possible. In fact, Alon and Krivelevich [2] have proved that two-point concentration holds for the entire range $p = n^{-\beta}$, $1/2 < \beta < 1$. The two following theorems show that similar results as given by Shamir and Spencer and by Łuczak for the chromatic number also hold for the relaxations $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, and $\bar{\vartheta}_2(G_{n,p})$.

Theorem 1.2. *Suppose that $c_0/n \leqslant p \leqslant n^{-\beta}$ for some large constant $c_0 > 0$ and some number $1/2 < \beta < 1$. Then $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ are concentrated in width $s = \frac{2}{2\beta - 1} + o(1)$, i.e., there exist numbers u, u', u'' depending on n and p such that, w.h.p.,*

$$u \leqslant \bar{\vartheta}_{1/2}(G_{n,p}) \leqslant u + s, \quad u' \leqslant \bar{\vartheta}(G_{n,p}) \leqslant u' + s, \quad \text{and } u'' \leqslant \bar{\vartheta}_2(G_{n,p}) \leqslant u'' + s.$$

Theorem 1.3. *Suppose that $c_0/n < p \leqslant n^{-5/6-\delta}$ for some large constant c_0 and some $\delta > 0$. Then $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, and $\bar{\vartheta}_2(G_{n,p})$ are concentrated in width 1.*

In contrast to the chromatic number, $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$ need not be integral. Therefore, the above results do *not* imply that $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ are concentrated on a constant number of points. The proofs of Theorems 1.2 and 1.3 are given in Section 3.2.

1.2. The probable value of $\vartheta(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, *etc.*

Concerning the probable value of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$, Juhász [28] has given the following partial answer. If $\ln(n)^6/n \ll p \leqslant 1/2$, then with high probability we have $\vartheta(G_{n,p}) = \Theta(\sqrt{n/p})$ and $\bar{\vartheta}(G_{n,p}) = \Theta(\sqrt{np})$. However, we shall indicate in Section 4 that Juhász's proof fails in the case of sparse random graphs (*e.g.*, $np = O(1)$). Making use of the above concentration results on ϑ, $\bar{\vartheta}$ *etc.*, we can compute the probable value not only of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$, but also of $\vartheta_i(G_{n,p})$ and $\bar{\vartheta}_i(G_{n,p})$, $i = 1/2, 2$, for essentially the entire range of edge probabilities p. To the best of the author's knowledge, no previous results concerning $\vartheta_i(G_{n,p})$ and $\bar{\vartheta}_i(G_{n,p})$, $i = 1/2, 2$, occur in the literature. Note that we only need to consider edge probabilities $p \leqslant 1/2$, because $G_{n,1-p} = \bar{G}_{n,p}$.

Theorem 1.4. *Suppose that $c_0/n \leqslant p \leqslant 1/2$ for some large constant $c_0 > 0$. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$c_1\sqrt{n/p} \leqslant \vartheta_{1/2}(G_{n,p}) \leqslant \vartheta(G_{n,p}) \leqslant \vartheta_2(G_{n,p}) \leqslant c_2\sqrt{n/p} \tag{1.1}$$

$$\text{and} \quad c_3\sqrt{np} \leqslant \bar{\vartheta}_{1/2}(G_{n,p}) \leqslant \bar{\vartheta}(G_{n,p}) \leqslant \bar{\vartheta}_2(G_{n,p}) \leqslant c_4\sqrt{np}$$

with high probability. More precisely,

$$\mathrm{P}(c_3\sqrt{np} \leqslant \bar{\vartheta}_{1/2}(G_{n,p}) \leqslant \bar{\vartheta}(G_{n,p}) \leqslant \bar{\vartheta}_2(G_{n,p})) \geqslant 1 - \exp(-n). \tag{1.2}$$

Assume that $c_0/n \leqslant p = o(1)$. Then $\alpha(G_{n,p}) \sim \frac{2\ln(np)}{p}$ and $\chi(G_{n,p}) \sim \frac{np}{2\ln(np)}$ w.h.p. (*cf.* [27]). Hence, Theorem 1.4 shows that $\vartheta_2(G_{n,p})$ (resp. $\bar{\vartheta}_{1/2}(G_{n,p})$) approximates $\alpha(G_{n,p})$ (resp. $\chi(G_{n,p})$)

within a factor of $O(\sqrt{np})$. In fact, if $np = O(1)$, then we get a constant factor approximation. On the other hand, as $\alpha(G_{n,1/2}) \sim 2\log_2(n)$ and $\chi(G_{n,1/2}) \sim \frac{n}{2\log_2(n)}$, in the random graph $G = G_{n,1/2}$ the gap between $\vartheta_{1/2}(G)$ (resp. $\bar{\vartheta}_2(G)$) and $\alpha(G)$ (resp. $\chi(G)$) is as large as $n^{1/2-\varepsilon}$ w.h.p. Our estimate on the probable value of the vector chromatic number $\bar{\vartheta}_{1/2}(G_{n,p})$ in Theorem 1.4 answers a question of Krivelevich [31]. For $\vartheta, \bar{\vartheta}(G_{n,p})$ related results have also been obtained by Feige and Ofek [16]; *cf.* the remark in Section 4.4. Indeed, the proof of Theorem 1.1 builds on spectral considerations from [16].

The large deviation result (1.2) is in a sense stronger than Theorem 1.1, as the probability of a deviation of order $\sqrt{np}$ is already $\leqslant \exp(-n)$. Moreover, (1.2) also applies to the regime of p not covered by Theorem 1.1. It is not true that a bound similar to (1.2) holds for the upper tail; *cf.* Remark 4.1.

As a consequence of the upper bound on $\bar{\vartheta}_2(G_{n,p})$ in Theorem 1.4, we obtain a lower bound on the probable value of the SDP relaxation SDP_k of MAX k-CUT due to Frieze and Jerrum [19].

Corollary 1.5. *Let $k \geqslant 2$ be an integer. Suppose that $c_0k^2/n \leqslant p \leqslant 1/2$. Then w.h.p. we have* $\mathrm{SDP}_k(G_{n,p}) \geqslant \left(1-\frac{1}{k}\right)\binom{n}{2}p + c_1n^{3/2}p^{1/2}$ *for some constant* $c_1 > 0$.

Corollary 1.5 complements an upper bound on $\mathrm{SDP}_k(G_{n,p})$ due to Coja-Oghlan, Moore and Sanwalani [10], who proved that there is a constant $c_2 > 0$ such that $\mathrm{SDP}_k(G_{n,p}) \leqslant (1-1/k)\binom{n}{2}p + c_2n^{3/2}p^{1/2}$ w.h.p. In contrast, the weight $\mathrm{MC}_k(G_{n,p})$ of a MAX k-CUT of $G_{n,p}$ is at most

$$\left(1-\frac{1}{k}\right)\binom{n}{2}p + \sqrt{\frac{\ln(k)}{k}} \cdot n^{3/2}p^{1/2}$$

w.h.p. (*cf.* [10]). Thus, Corollary 1.5 shows that for large k there is a moderate gap between $\mathrm{SDP}_k(G_{n,p})$ and $\mathrm{MC}_k(G_{n,p})$. The proofs of Theorem 1.4 and Corollary 1.5 are given in Section 4.

Finally, let us consider the *random regular graph* $G_{n,r}$, *i.e.*, an r-regular graph of order n chosen uniformly at random. The proof of the following theorem will be given in Section 5.

Theorem 1.6. *Let c_0 be a sufficiently large constant, and let $c_0 \leqslant r = o(n^{1/4})$. There are constants $c_1, c_2 > 0$ such that w.h.p. the random regular graph $G_{n,r}$ satisfies*

$$c_1nr^{-1/2} \leqslant \vartheta_{1/2}(G_{n,r}) \leqslant \vartheta(G_{n,r}) \leqslant \vartheta_2(G_{n,r}) \leqslant c_2nr^{-1/2}.$$

Moreover, there is a constant $c_3 > 0$ such that in the case $c_0 \leqslant r = o(n^{1/2})$ we have

$$\mathrm{P}(c_3\sqrt{r} \leqslant \bar{\vartheta}_{1/2}(G_{n,r}) \leqslant \bar{\vartheta}(G_{n,r}) \leqslant \bar{\vartheta}_2(G_{n,r})) \geqslant 1 - \exp(-n). \tag{1.3}$$

Finally, there is a constant $c_4 > 0$ such that if $c_0 \leqslant r = O(1)$, then

$$\bar{\vartheta}_{1/2}(G_{n,r}) \leqslant \bar{\vartheta}(G_{n,r}) \leqslant \bar{\vartheta}_2(G_{n,r}) \leqslant c_4\sqrt{r}.$$

1.3. Algorithmic applications

There are two types of efficient algorithms for NP-hard random graph problems. First, there are *heuristics* that *always* run in polynomial time, and *almost always* output a good solution. On the other hand, there are algorithms that guarantee some approximation ratio on *any* input instance, and which have polynomial *expected* running time when applied to $G_{n,p}$ (*cf.* [11]). Here we say that an algorithm A runs in polynomial expected time if there is a constant $l > 0$ such that $\sum_G R_A(G)\,\mathrm{P}(G_{n,p} = G) = O(n^l)$, where $R_A(G)$ is the running time of A on input G and the sum ranges over all graphs G of order n. In this paper, we are concerned with algorithms with polynomial expected running time.

First, we consider the maximum independent set problem in random graphs. Krivelevich and Vu [34] have given an algorithm that in the case $p \gg n^{-1/2}$ approximates the independence number of $G_{n,p}$ in polynomial expected time within a factor of $O(\sqrt{np}/\ln(np))$. Moreover, they ask whether a similar algorithm exists for smaller values of p. A first answer has been obtained by Coja-Oghlan and Taraz [8], who gave an $O(\sqrt{np}/\ln(np))$-approximative algorithm for the case $p \gg \ln(n)^6/n$. Using Theorems 1.1 and 1.4, we can improve on the analysis given in [8], thereby answering the question of Krivelevich and Vu in the affirmative.

Theorem 1.7. *Suppose that $c_0/n \leqslant p \leqslant 1/2$. There exists an algorithm* `ApproxMIS` *that for any input graph G outputs an independent set of size at least $c_1^{-1}\alpha(G)(\ln np)(np)^{-1/2}$, and which applied to $G_{n,p}$ runs in polynomial expected time. Here $c_0, c_1 > 0$ denote constants.*

As a second application, we give an algorithm for deciding within polynomial expected time whether the input graph is k-colourable. Instead of $G_{n,p}$, we shall even consider the *semirandom model* $G^+_{n,p}$ that allows for an adversary to add edges to the random graph. More precisely, the semirandom graph $G^+_{n,p}$ is constructed in two steps as follows. First, a random graph $G_0 = G_{n,p}$ is chosen. Then, an adversary completes the instance $G = G^+_{n,p}$ by adding arbitrary edges to G_0. We say that *the expected running time of an algorithm $\mathscr{A}$ is polynomial over $G^+_{n,p}$* if there is some constant l such that the expected running time of $\mathscr{A}$ is $O(n^l)$ regardless of the behaviour of the adversary.

Theorem 1.8. *Suppose that $k = o(\sqrt{n})$, and that $p \geqslant c_0 k^2/n$, for some constant $c_0 > 0$. There exists an algorithm* $\texttt{Decide}_k$ *that for any input graph G decides whether G is k-colourable, and that applied to $G^+_{n,p}$ has polynomial expected running time.*

The algorithm $\texttt{Decide}_k$ is essentially identical to Krivelevich's algorithm for deciding k-colourability in polynomial expected time [31]. However, the analysis given in [31] requires that $np \geqslant \exp(\Omega(k))$, whereas Theorem 1.8 only requires that np is quadratic in k. The improvement results from the fact that the analysis given in this paper relies on the asymptotics for $\bar{\vartheta}_{1/2}(G_{n,p})$ derived in Theorem 1.4 (instead of the concept of semicolourings). Finally, we prove that our algorithm $\texttt{Decide}_k$ also applies to random regular graphs $G_{n,r}$.

Theorem 1.9. *Suppose that $c_0 k^2 \leqslant r = o(n^{1/2})$ for some constant $c_0 > 0$. Then, applied to $G_{n,r}$, the algorithm* $\texttt{Decide}_k$ *has polynomial expected running time.*

1.4. Organization of the paper

First we recall the definitions of ϑ, $\bar{\vartheta}$ *etc.* and prove some elementary facts in Section 2. Section 3 deals with the concentration results, and Section 4 contains the proofs of Theorem 1.4 and Corollary 1.5. In Section 5 we prove Theorem 1.6, and Section 6 is devoted to the algorithms. Finally, Section 7 contains some concluding remarks. Some of the material appeared in extended abstracts in *Proc. STACS 2003* and *Proc. RANDOM 2003*. Most of the proofs were omitted from the conference papers.

1.5. Notation

Throughout we let $V = \{1, \ldots, n\}$. If $G = (V, E)$ is a graph, and $U \subset V$, then $N(U)$ is the neighbourhood of U, *i.e.*, the set of all $v \in V$ such that there is $w \in U$ satisfying $\{v, w\} \in E$. Moreover, $A(G)$ is the adjacency matrix of G. By $\vec{1}$ we denote the vector with all entries equal to one in any dimension. Furthermore, J denotes a square matrix of any size with all entries equal to one. If M is a real symmetric $n \times n$-matrix, then $\lambda_1(M) \geqslant \cdots \geqslant \lambda_n(M)$ signify the eigenvalues of M, and $\|M\| = \max\{\lambda_1(M), -\lambda_n(M)\}$ is the spectral radius of M. We let $\langle \cdot, \cdot \rangle$ denote the scalar product of vectors. By $c_0, c_1, \ldots$ we denote constants, *i.e.*, numbers that are independent of n and p.

2. Preliminaries

In this section we recall the definitions of ϑ, $\vartheta_{1/2}$, ϑ_2, and provide some elementary facts which will be useful later. We let $G = (V, E)$ be a graph, and let $\bar{G}$ be the complement of G. Let $(v_1, \ldots, v_n)$ be an n-tuple of unit vectors in $\mathbb{R}^n$, and let $k > 1$. Then $(v_1, \ldots, v_n)$ is a *vector k-colouring* of G if $\langle v_i, v_j \rangle \leqslant -1/(k-1)$ for all edges $\{i, j\} \in E$. Furthermore, $(v_1, \ldots, v_n)$ is a *strict* vector k-colouring if $\langle v_i, v_j \rangle = -1/(k-1)$ for all $\{i, j\} \in E$. Finally, we say that $(v_1, \ldots, v_n)$ is a *rigid* vector k-colouring if $\langle v_i, v_j \rangle = -1/(k-1)$ for all $\{i, j\} \in E$ and $\langle v_i, v_j \rangle \geqslant -1/(k-1)$ for all $\{i, j\} \notin E$. Following [29, 22, 5], we define

$$\begin{aligned} \bar{\vartheta}_{1/2}(G) &= \inf\{k > 1 \mid G \text{ admits a vector } k\text{-colouring}\}, \\ \bar{\vartheta}(G) = \bar{\vartheta}_1(G) &= \inf\{k > 1 \mid G \text{ admits a strict vector } k\text{-colouring}\}, \\ \bar{\vartheta}_2(G) &= \inf\{k > 1 \mid G \text{ admits a rigid vector } k\text{-colouring}\}. \end{aligned} \tag{2.1}$$

Observe that $\bar{\vartheta}_{1/2}(G)$ is precisely the *vector chromatic number* introduced by Karger, Motwani and Sudan [29]; $\bar{\vartheta}_2$ occurs in [22, 41]. Further, we let $\vartheta_{1/2}(G) = \bar{\vartheta}_{1/2}(\bar{G})$, $\vartheta(G) = \vartheta_1(G) = \bar{\vartheta}(\bar{G})$, and $\vartheta_2(G) = \bar{\vartheta}_2(\bar{G})$. It is shown in [29] that the above definition of ϑ is equivalent with Lovász's original definition [37].

Proposition 2.1. *Let $G = (V, E)$ be a graph of order n, and let $S \subset V$. Let $G[S]$ denote the subgraph of G induced on S. Then $\vartheta_i(G) \leqslant \vartheta_i(G[S]) + \vartheta_i(G[V \setminus S])$, $i \in \{1/2, 1, 2\}$.* □

Although this observation may be known to specialists in the area, to the best of the author's knowledge it is not explicitly stated (or proved) in the literature. (For $i = 1$, *i.e.*, $\vartheta(G)$, the result is an easy consequence of [30, Section 18]; however, this proof does not apply directly to $\vartheta_{1/2}$, ϑ_2.) Therefore, we give a simple proof for $\bar{\vartheta}_2$ which applies to $\bar{\vartheta}_{1/2}$ and $\bar{\vartheta}$ as well.

Proof of Proposition 2.1. Let $k > \bar{\vartheta}_2(G[S])$ and let $l > \bar{\vartheta}_2(G[V \setminus S])$. Further, let $(a_v)_{v\in S}$ be a rigid vector k-colouring of $G[S]$, and let $(b_v)_{v\in V\setminus S}$ be a rigid vector l-colouring of $G[V\setminus S]$. Set

$$\alpha = \left(\frac{l}{(k+l)(k-1)}\right)^{1/2} \text{ and } \beta = \left(\frac{k}{(k+l)(l-1)}\right)^{1/2}.$$

Embedding the a_vs and b_ws into a high-dimensional space, we may assume that $a_v \perp b_w$ for all $v \in S$, $w \in V \setminus S$, and that there is a unit vector z such that $z \perp a_v$, $z \perp b_w$ for all v, w. Let

$$x_v = (1+\alpha^2)^{-1/2}(a_v + \alpha z) \text{ and } x_w = (1+\beta^2)^{-1/2}(b_w - \beta z) \qquad (v \in S, w \in V \setminus S).$$

Then $\|x_v\| = \|x_w\| = 1$. Moreover, if two vertices $v, v' \in S$ are adjacent, then

$$\langle x_v, x_{v'}\rangle = (1+\alpha^2)^{-1}\big(\langle a_v, a_{v'}\rangle + \alpha^2\big) = (1+\alpha^2)^{-1}\left(-\frac{1}{k-1} + \alpha^2\right) = -\frac{1}{k+l-1}.$$

Likewise, if $v, v' \in S$ are non-adjacent, then $\langle x_v, x_{v'}\rangle \geqslant -1/(k+l-1)$. Consequently, $(x_v)_{v\in S}$ is a rigid vector $(k+l)$-colouring of $G[S]$. Similarly, $(x_w)_{w\in V\setminus S}$ is a rigid vector $(k+l)$-colouring of $G[V \setminus S]$. Since $\langle x_v, x_w\rangle = -1/(k+l-1)$ for all $v \in S$, $w \in V \setminus S$, $(x_v)_{v\in V}$ is a rigid vector $(k+l)$-colouring of the entire graph G, thereby proving $\bar{\vartheta}_2(G) \leqslant k + l$. □

In addition to Proposition 2.1, we will frequently make use of the well-known fact that

$$\omega(G) \leqslant \bar{\vartheta}_{1/2}(G) \leqslant \bar{\vartheta}(G) \leqslant \bar{\vartheta}_2(G) \leqslant \chi(G) \text{ and } \alpha(G) \leqslant \vartheta_{1/2}(G) \leqslant \vartheta(G) \leqslant \vartheta_2(G) \tag{2.2}$$

for all graphs $G = (V, E)$. The lower bounds $\omega(G) \leqslant \bar{\vartheta}_{1/2}(G)$, $\alpha(G) \leqslant \vartheta_{1/2}(G)$ are established in [29]. The upper bound $\bar{\vartheta}_2(G) \leqslant \chi(G)$ can be proved *e.g.*, by decomposing V into $\chi(G)$ disjoint independent sets and applying Proposition 2.1. Moreover, it is obvious from the definitions that for any weak subgraph H of G we have

$$\bar{\vartheta}_i(H) \leqslant \bar{\vartheta}_i(G) \qquad (i \in \{1/2, 1, 2\}). \tag{2.3}$$

In addition to ϑ, $\vartheta_{1/2}$, and ϑ_2, we consider the following semidefinite relaxation of MAX k-CUT, due to Frieze and Jerrum [19]. Let G be a graph with adjacency matrix $A = A(G) = (a_{ij})_{i,j=1,\dots,n}$, and let $k \geqslant 2$. Then

$$\mathrm{SDP}_k(G) = \max \sum_{i<j} a_{ij} \frac{k-1}{k}\big(1 - \langle v_i, v_j\rangle\big) \text{ s.t. } \|v_i\| = 1,\ \langle v_i, v_j\rangle \geqslant -\frac{1}{k-1}, \tag{2.4}$$

where the max is taken over $v_1, \dots, v_n \in \mathbb{R}^n$, is an upper bound on the weight of a MAX k-CUT of G. In the case $k = 2$, we obtain the semidefinite relaxation $\mathrm{SMC} = \mathrm{SDP}_2$ of MAX CUT invented by Goemans and Williamson [23]. In this case, the constraint $\langle v_i, v_j\rangle \geqslant -1/(2-1) = -1$ is void.

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3. The concentration results

3.1. Proof of Theorem 1.1

3.1.1. The large deviation result for ϑ. In order to bound the probability that the Lovász number $\vartheta(G_{n,p})$ is far from its median, we shall apply the following version of *Talagrand's inequality* (*cf.* [27, p. 44]).

Theorem 3.1. *Let $\Lambda_1, \ldots, \Lambda_N$ be probability spaces. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N$. Let $A, B \subset \Lambda$ be measurable sets such that for some $t \geqslant 0$ the following condition is satisfied: For every $b \in B$ there is $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N \setminus \{0\}$ such that for all $a \in A$ we have*

$$\sum_{i:\, a_i \neq b_i} \alpha_i \geqslant t \left(\sum_{i=1}^{N} \alpha_i^2 \right)^{1/2},$$

where a_i (resp. b_i) denotes the ith coordinate of a (resp. b). Then $\mathrm{P}(A)\mathrm{P}(B) \leqslant \exp(-t^2/4)$.

Let $G = (V, E)$ be a graph. We need the following equivalent characterization of $\vartheta(G)$. A tuple $(v_1, \ldots, v_n)$ of vectors $v_i \in \mathbb{R}^d$ is called an *orthogonal labelling* of $\bar{G}$ if, for any two vertices $i, j \in V$, $i \neq j$, with $\{i, j\} \in E$, we have $v_i \perp v_j$ (*cf.* [30]). Here $d > 0$ is any integer. Furthermore, the *cost* of a d-dimensional vector $a = (a_1, \ldots, a_d)^T$ is

$$c(a) = \begin{cases} a_1^2 \|a\|^{-2} & \text{if } a \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leqslant c(a) \leqslant 1$, and we have

$$\vartheta(G) = \max \left\{ \sum_{i=1}^{n} c(v_i) \,|\, (v_1, \ldots, v_n) \text{ is an orthogonal labelling of } \bar{G} \right\} \quad (cf.\ [29, 30]). \tag{3.1}$$

The proof of Theorem 1.1 relies on the following lemma.

Lemma 3.2. *Let m be a median of $\vartheta(G_{n,p})$. Let $\vartheta_0 > 0$ be any number, and let $\xi \geqslant 10$. Then*

$$\mathrm{P}(m + \xi \leqslant \vartheta(G_{n,p}) \leqslant \vartheta_0) \leqslant 2\exp(-\xi^2/(5\vartheta_0)).$$

Proof. For $i \geqslant 2$, let $\Lambda_i \in \{0,1\}^{i-1}$ consist of the first $i-1$ entries of the ith row of the adjacency matrix of $G_{n,p}$. Then $\Lambda_2, \ldots, \Lambda_n$ are independent random variables, and Λ_i determines to which of the $i-1$ vertices $1, \ldots, i-1$ vertex i is adjacent. Therefore, we can identify $G_{n,p}$ with the product space $\Lambda_2 \times \cdots \times \Lambda_n$. Let $\pi_i : G_{n,p} = \Lambda_2 \times \cdots \times \Lambda_n \to \Lambda_i$ be the ith projection. Let $A = \{G \in G_{n,p} |\ \vartheta(G) \leqslant m\}$ and $B = \{H \in G_{n,p} |\ m + \xi \leqslant \vartheta(H) \leqslant \vartheta_0\}$.

Let $H \in B$, and let $(b_1, \ldots, b_n)$ be an orthogonal labelling of $\bar{H}$ such that

$$m + \xi \leqslant \vartheta(H) = \sum_{i=1}^{n} c(b_i). \tag{3.2}$$

Set $\alpha_i = c(b_i)$, and $\alpha = (\alpha_2, \ldots, \alpha_n)$. As $0 \leqslant \alpha_i \leqslant 1$ for all i, we have

$$\sum_{i=2}^{n} \alpha_i^2 \leqslant \sum_{i=1}^{n} \alpha_i = \vartheta(H) \leqslant \vartheta_0; \tag{3.3}$$

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observe that the first sum starts at $i = 2$, because $G_{n,p} = \Lambda_2 \times \cdots \times \Lambda_n$. Now let $G \in A$, set $a_1 = 0$, and let

$$a_i = \begin{cases} b_i \text{ if } \pi_i(G) = \pi_i(H) \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } i = 2, \ldots, n.$$

We claim that $(a_1, \ldots, a_n)$ is an orthogonal labelling of $\bar{G}$. For if $i, j \in V$ are adjacent in G, and $i < j$, then we have either $\pi_j(G) = \pi_j(H)$ or $a_j = 0$. In the first case, i and j are adjacent in H, whence $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$. Moreover, if $a_j = 0$, then obviously $a_i \perp a_j$. Thus, as $(a_1, \ldots, a_n)$ is an orthogonal labelling of $\bar{G}$, we have $\sum_{i=1}^{n} c(a_i) \leqslant \vartheta(G) \leqslant m$. Hence, equation (3.2) yields

$$\xi \leqslant c(b_1) + \sum_{i=2}^{n} c(b_i) - c(a_i) \leqslant 1 + \sum_{i:\pi_i(G) \neq \pi_i(H)} c(b_i) = 1 + \sum_{i:\pi_i(G) \neq \pi_i(H)} \alpha_i. \tag{3.4}$$

Set $t = (\xi - 1)/\sqrt{\vartheta_0}$. Then, by (3.3) and (3.4), for all $G \in A$ we have

$$\sum_{i:\pi_i(G) \neq \pi_i(H)} \alpha_i \geqslant t \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2}.$$

Consequently, Theorem 3.1 entails

$$\mathrm{P}(A)\mathrm{P}(m + \xi \leqslant \vartheta(G_{n,p}) \leqslant \vartheta_0) \leqslant \exp(-t^2/4) = \exp\left(-\frac{(\xi - 1)^2}{4\vartheta_0} \right) \leqslant \exp\left(-\frac{\xi^2}{5\vartheta_0} \right),$$

where the last inequality follows from the assumption that $\xi \geqslant 10$. Hence, the assertion follows from the fact that $\mathrm{P}(A) \geqslant 1/2$. □

Proof of Theorem 1.1. As for the upper tail bound (i), by Lemma 3.2, for all $l \geqslant 1$, we have

$$\mathrm{P}(m + l\xi \leqslant \vartheta \leqslant m + (l+1)\xi) \leqslant 2\exp\left(-\frac{l^2\xi^2}{5(m + (l+1)\xi)} \right) \leqslant 2\exp\left(-\frac{l\xi^2}{5(m + 2\xi)} \right).$$

Therefore, our assumptions that $\xi \geqslant \sqrt{m}$ and $\xi \geqslant 10$ imply that

$$\begin{aligned} \mathrm{P}(m + \xi \leqslant \vartheta) &\leqslant \sum_{l=1}^{\infty} \mathrm{P}(m + l\xi \leqslant \vartheta \leqslant m + (l+1)\xi) \leqslant 2 \sum_{l=1}^{\infty} \exp\left(-\frac{\xi^2}{5(m + 2\xi)} \right)^l \\ &\leqslant 2\exp\left(-\frac{\xi^2}{5m + 10\xi} \right) \left(1 - \exp\left(-\frac{\xi^2}{5m + 10\xi} \right) \right)^{-1} \\ &\leqslant 2\exp\left(-\frac{\xi^2}{5m + 10\xi} \right) \left(1 - \exp\left(-\frac{1}{5(m/\xi^2) + (10/\xi)} \right) \right)^{-1} \\ &\leqslant 2\exp\left(-\frac{\xi^2}{5m + 10\xi} \right) (1 - \exp(-1/6))^{-1} \leqslant 30\exp\left(-\frac{\xi^2}{5m + 10\xi} \right), \end{aligned}$$

as desired.

By our assumption that $p \leqslant 0.99$, we have that $\vartheta(G_{n,p}) \geqslant \alpha(G_{n,p}) = \Omega(\ln(n))$ w.h.p. Hence, we can choose n_0 large enough such that any median m of $\vartheta(G_{n,p})$ satisfies $m \geqslant C$ for some large constant C. In order to prove the lower tail bound (ii), observe that

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by (i) we can choose C large enough such that $\mathrm{P}(\vartheta(G_{n,p}) \geqslant 2m) < 1/6$, say. Consequently, $\mathrm{P}(m \leqslant \vartheta(G_{n,p}) \leqslant 2m) \geqslant 1/3$. Let $A = \{G | \ \vartheta(G) \leqslant m - \xi\}$ and $B = \{H | \ m \leqslant \vartheta(H) \leqslant 2m\}$. Then, a similar argument as in the proof of Lemma 3.2 yields

$$\mathrm{P}(A)\mathrm{P}(B) \leqslant \exp(-t^2/4) = \exp\left(-\frac{(\xi-1)^2}{8m}\right) \leqslant \exp\left(-\frac{\xi^2}{10m}\right).$$

Thus, our assertion follows from the fact that $\mathrm{P}(B) \geqslant 1/3$. □

3.1.2. The large deviation result for $\vartheta_{1/2}$. To prove the bounds in Theorem 1.1 for $\vartheta_{1/2}$, we make use of a characterization of $\vartheta_{1/2}$ established in [24]. Let $(v_1, \ldots, v_n)$ be an assignment of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ to the vertices of $G = (V, E)$, where $d > 0$ is any integer. We call an orthogonal labelling $(v_1, \ldots, v_n)$ of $\bar{G}$ a *strong* orthogonal labelling if $\langle v_i, v_j \rangle \geqslant 0$ for all $i, j \in \{1, \ldots, n\}$. As shown in [24, pp. 51ff],

$$\vartheta_{1/2}(G) = \max\left\{\sum_{i=1}^{n} c(v_i) | \ (v_1, \ldots, v_n) \text{ is a strong orthogonal labelling of } \bar{G}\right\}; \tag{3.5}$$

the proof goes along the lines of [30]. Using (3.5), the argument given for ϑ above carries over without essential changes and yields the proof of the tail bounds for $\vartheta_{1/2}$.

3.1.3. The large deviation result for ϑ_2. We shall establish a characterization of ϑ_2 that corresponds to the characterization (3.1) of ϑ, and which may be of independent interest. Let $G = (V, E)$ be a graph. If $x, y \in \mathbb{R}^d$, then we let $c(x, y) = \langle x, y\rangle^2 \|x\|^{-2}\|y\|^{-2}$, if $x, y \neq 0$, and $c(x, y) = 0$ otherwise. Moreover, we call a family $(v_0, \ldots, v_n)$ of vectors $v_i \in \mathbb{R}^d$ a *weak orthogonal labelling* of $\bar{G}$ if $\langle v_0, v_i \rangle \geqslant 0$ for all i, and $\langle v_i, v_j \rangle \leqslant 0$ if $\{i, j\} \in E$, $i, j = 1, \ldots, n$. Here d is any positive integer. Note that a weak orthogonal labelling consists of $n + 1 = \#V + 1$ vectors. We define

$$\vartheta_2'(G) = \max\left\{\sum_{i=1}^{n} c(v_i, v_0) | \ (v_0, \ldots, v_n) \text{ is a weak orthogonal labelling of } G\right\}. \tag{3.6}$$

Lemma 3.3. *We have $\vartheta_2(G) = \vartheta_2'(G)$ for all graphs $G = (V, E)$.*

Proof. The following formulation of ϑ_2 as a semidefinite program is given in [41]:

$$\vartheta_2(G) = \max \sum_{i,j=1}^{n} b_{ij} \text{ s.t. } b_{ij} \leqslant 0 \text{ for all } \{i, j\} \in E \text{ and } \sum_{i=1}^{n} b_{ii} = 1, \tag{3.7}$$

where the max is taken over all positive semidefinite matrices $B = (b_{ij})_{i,j}$. (One can prove equation (3.7) *e.g.*, using a similar argument as given in [29] to prove that (2.1) is equivalent to Lovász's original definition of $\bar{\vartheta}$.) To prove that $\vartheta_2(G) \leqslant \vartheta_2'(G)$, let $B = (b_{ij})$ be a feasible matrix that maximizes (3.7). Since B is positive semidefinite, there are vectors $b_1, \ldots, b_n \in \mathbb{R}^n$ such that $b_{ij} = \langle b_i, b_j \rangle$. Let $b = \sum_{i=1}^{n} b_i$. Then $\langle b_k, b \rangle \geqslant 0$ for all k. For assume otherwise, and consider the matrix $B' = (b'_{ij})$, where $b'_{ij} = b_{ij}$ for $i, j \neq k$, $b'_{ki} = b'_{ik} = 0$ for $i \neq k$, and $b'_{kk} = b_{kk}$. Then B' is positive semidefinite, and is a feasible

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solution to (3.7). Consequently, our assumption $0 > \langle b_k, b\rangle = \sum_j b_{kj} = \sum_j b_{jk}$ entails

$$\sum_{i,j} b_{ij} = \vartheta_2(G) \geqslant \sum_{i,j} b'_{ij} = \sum_{i,j} b_{ij} - 2\sum_{i\neq k} b_{ik} > \sum_{i,j} b_{ij},$$

a contradiction. Hence, $\langle b_k, b\rangle \geqslant 0$ for all k. Letting $v_0 = b/\|b\|$, $v_i = b_i/\|b_i\|$ if $b_i \neq 0$, and $v_i = 0$ otherwise ($i = 1,\ldots,n$), we obtain a weak orthogonal labelling of $\bar{G}$ satisfying $\sum_{i=1}^n c(v_i, v_0) \geqslant \sum_{i,j} b_{ij} = \vartheta_2(G)$ (*cf.* the proof of Theorem 5 in [37]).

Conversely, let $(v_0,\ldots,v_n)$ be a weak orthogonal labelling of $\bar{G}$ such that

$$\sum_{i=1}^n c(v_i, v_0) = \vartheta'_2(G).$$

We may assume that v_i either is a unit vector or is equal to zero for all i. Let

$$b_i = \vartheta'_2(G)^{-1/2}\langle v_0, v_i\rangle v_i,$$

set $b_{ij} = \langle b_i, b_j\rangle$, and $B = (b_{ij})_{i,j}$. Then B is positive semidefinite, and if $\{i,j\} \in E$, then $b_{ij} = \langle v_0, v_i\rangle\langle v_0, v_j\rangle\langle v_i, v_j\rangle/\vartheta'_2(G) \leqslant 0$, because $\langle v_0, v_i\rangle, \langle v_0, v_j\rangle \geqslant 0 \geqslant \langle v_i, v_j\rangle$. Moreover, $\sum_{i=1}^n b_{ii} = \sum_{i=1}^n c(v_0, v_i)/\vartheta'_2(G) = 1$, whence B is a feasible solution to (3.7). Finally, to show that $\sum_{i,j} b_{ij} \geqslant \vartheta'_2(G)$, we adapt the argument used in [24] to prove (3.5). Let $M = \sum_{i=1}^n v_i {v_i}^T$. Then $\vartheta'_2(G) = \sum_i \langle v_0, v_i\rangle^2 = \langle Mv_0, v_0\rangle \leqslant \|Mv_0\|$. Consequently,

$$\vartheta'_2(G) \leqslant \|M(\vartheta'_2(G)^{-1/2}v_0)\|^2 = \left\|\sum_{i=1}^n \vartheta'_2(G)^{-1/2}\langle v_0, v_i\rangle v_i\right\|^2 = \sum_{i,j} b_{ij} \leqslant \vartheta_2(G),$$

thereby proving the lemma. □

Using the characterization (3.6) of ϑ_2, the arguments used to prove Theorem 1.1 for $\vartheta(G_{n,p})$ also apply to $\vartheta_2(G_{n,p})$.

3.2. Concentration of $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$ in intervals of constant length

Though the proofs of Theorems 1.2 and 1.3 go along the lines of [38, 40], we have to replace arguments concerning the chromatic number by arguments that apply to $\bar{\vartheta}_{1/2}$, $\bar{\vartheta}$, and $\bar{\vartheta}_2$. We shall demonstrate the proofs for $\bar{\vartheta}_2$, as this turns out to be the most demanding case. All arguments carry over to $\bar{\vartheta}_{1/2}$ and $\bar{\vartheta}$ immediately. We adapt a simplification of the argument given in [40] attributed to Frieze in [38].

Proof of Theorem 1.2. Let p and β be as in Theorem 1.2. The proof is based on the following large deviation result, which is a consequence of Azuma's inequality (*cf.* [27, p. 37]).

Lemma 3.4. *Suppose that $X : G_{n,p} \to \mathbb{R}$ is a random variable that satisfies the following conditions for all graphs $G = (V, E)$.*

- *For all $v \in V$ the following holds. Let $G^* = G + \{\{v,w\} \mid w \in V,\ w < v\}$, and let $G_* = G - \{\{v,w\} \mid w \in V,\ w < v\}$. Then $|X(G^*) - X(G_*)| \leqslant 1$.*
- *If H is a weak subgraph of G, then $X(H) \leqslant X(G)$.*

Then $\mathrm{P}(|X - \mathrm{E}(X)| > t\sqrt{n}) \leqslant 2\exp(-t^2/2)$.

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Let $\omega = \omega(n)$ be a sequence tending to infinity slowly, *e.g.*, $\omega(n) = \ln\ln(n)$. Furthermore, let

$$k = k(n,p) = \inf\{x > 0 |\ \mathrm{P}(\bar{\vartheta}_2(G_{n,p}) \leqslant x) \geqslant \omega^{-1}\}. \tag{3.8}$$

For any graph $G = (V, E)$ let

$$Y(G) = Y_k(G) = \min\{\#U |\ U \subset V,\ \bar{\vartheta}_2(G - U) \leqslant k\}.$$

Then $\bar{\vartheta}_2(G) \leqslant k$ if and only if $Y(G) = 0$. Hence, $\mathrm{P}(Y = 0) \geqslant \omega^{-1}$. Moreover, by Proposition 2.1 and (2.3), the random variable Y satisfies the assumptions of Lemma 3.4. Therefore, letting $\mu = \mathrm{E}(Y)$, for any $\lambda > 0$ we have

$$\mathrm{P}(|Y(G_{n,p}) - \mu| \geqslant \lambda\sqrt{n}) \leqslant 2\exp(-\lambda^2/2). \tag{3.9}$$

We claim that $\mu \leqslant \sqrt{n}\omega$. For if $\mu > \sqrt{n}\omega$, then (3.9) yields

$$\omega^{-1} \leqslant \mathrm{P}(Y = 0) \leqslant \mathrm{P}(Y \leqslant \mu - \sqrt{n}\omega) \leqslant 2\exp(-\omega^2/2),$$

a contradiction. Thus, again by Lemma 3.4, $Y \leqslant 2\sqrt{n}\omega$ with high probability. The following lemma is implicit in [40] (*cf.* the proof of Lemma 8 in [40]).

Lemma 3.5. *Let $\delta > 0$. With high probability the random graph $G = G_{n,p}$ enjoys the following property. If $U \subset V$, $\#U \leqslant 2\sqrt{n}\omega$, then $\#E(G[U]) < \#Us/2$, where $s = \frac{2}{2\beta-1} + \delta$. Consequently, $\chi(G[U]) \leqslant s$,*

To conclude the proof of Theorem 1.2, let $G = G_{n,p}$, and suppose that there is some $U \subset V, \#U \leqslant 2\sqrt{n}\omega$, such that $\bar{\vartheta}_2(G - U) \leqslant k \leqslant \bar{\vartheta}_2(G)$. Since by (2.2) we have $\bar{\vartheta}_2(G[U]) \leqslant \chi(G[U])$, and since by Lemma 3.5 w.h.p. $\chi(G[U]) \leqslant s$ holds, we conclude that $\bar{\vartheta}_2(G[U]) \leqslant s$ w.h.p. Hence, Proposition 2.1 entails that $k \leqslant \bar{\vartheta}_2(G) \leqslant k + s$ w.h.p., thereby proving Theorem 1.2. □

Proof of Theorem 1.3. Let $\omega = \omega(n) = (\ln\ln n)^{1/3}$ be a sequence tending to infinity slowly. By Lemma 3.5 and the assumption $\beta > 5/6 + \delta$, the random graph $G = G_{n,p}$ admits no $U \subset V$, $\#U \leqslant \omega^3\sqrt{n}$, spanning more than $(3 - \varepsilon)\#U/2$ edges w.h.p., where $\varepsilon > 0$ is a small constant. Let k be defined as in (3.8). As shown in the proof of Theorem 1.2, w.h.p. there is a set $U \subset V, \#U \leqslant \omega\sqrt{n}$, such that $\bar{\vartheta}_2(G - U) \leqslant k$. Following Łuczak [38], we let $U = U_0$, and construct a sequence $U_0, \ldots, U_m$ as follows. If there is no edge $\{v, w\} \in E$ with $v, w \in N(U_i) \setminus U_i$, then we let $m = i$ and finish. Otherwise, we let $U_{i+1} = U_i \cup \{v, w\}$ and continue. Then $m \leqslant m_0 = \omega^2\sqrt{n}$, because otherwise $\#U_{m_0} = (2 + o(1))\omega^2\sqrt{n}$ and $\#E(G[U_{m_0}]) \geqslant 3(1 - o(1))\#U_{m_0}/2$. Let $R = U_m$.

By Lemma 3.5, $\bar{\vartheta}_2(G[R]) \leqslant \chi(G[R]) \leqslant 3$. Furthermore, $I = N(R) \setminus R$ is an independent set. Let $G_1 = G[R \cup I]$, $S = V \setminus (R \cup I)$, and $G_2 = G[S \cup I]$. Then $\bar{\vartheta}_2(G_2) \leqslant k$, and $\bar{\vartheta}_2(G_1) \leqslant 4$. In order to prove that $\bar{\vartheta}_2(G) \leqslant k + 1$, we shall first construct a rigid vector $k + 1$-colouring of G_2 that assigns the same vector to all vertices in I. Thus, let $(x_v)_{v \in S \cup I}$ be a rigid vector k-colouring of G_2. Let x be a unit vector perpendicular to x_v for all $v \in S$.

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Moreover, let $\alpha = (k^2-1)^{-1/2}$, and set

$$y_v = \begin{cases} (\alpha^2+1)^{-1/2}(x_v - \alpha x) & \text{for } v \in S, \\ x & \text{for } v \in I. \end{cases}$$

Then all y_v are unit vectors, and if $v \in S$, $w \in I$, then $\langle y_v, y_w\rangle = \langle y_v, x\rangle = -1/k$. Further, if $v, w \in S$ are adjacent in G_2, then

$$\langle y_v, y_w\rangle = \frac{1}{\alpha^2+1}\left(\langle x_v, x_w\rangle + \alpha^2\right) = \frac{1}{\alpha^2+1}\left(-\frac{1}{k-1} + \alpha^2\right) = -\frac{1}{k}.$$

Likewise, if $v, w \in S$ are non-adjacent in G_2, then $\langle y_v, y_w\rangle \geqslant -1/k$, thereby proving that $(y_v)_{v\in S\cup I}$ is a rigid vector $(k+1)$-colouring of G_2. In a similar manner, we can construct a rigid vector $(k+1)$-colouring $(y'_v)_{v\in R\cup I}$ of G_1 that assigns the same vector x' to all vertices in I.

Applying a suitable orthogonal transformation to the vectors $(y'_v)_{v\in R\cup I}$ if necessary, we may assume that $x = x'$ and that $\langle y'_v, y_w\rangle = k^{-2}$ for all $v \in R$, $w \in S$. Let $l = \max\{4, k+1\}$. Since $N(R) \subset R \cup I$, we obtain a rigid vector l-colouring $(z_v)_{v\in V}$ of G, where $z_v = y_v$ if $v \in S \cup I$, and $z_v = y'_v$ if $v \in R$. By the lower bound on $\bar{\vartheta}_2(G_{n,p})$ in Theorem 1.4 (which does not rely on Theorem 1.3 of course), choosing c_0 large enough we may assume that $k \geqslant 4$, whence $k \leqslant \bar{\vartheta}_2(G) \leqslant k+1$. □

4. The probable value of $\vartheta(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, *etc.*

In Section 4.1 we prove the lower bounds asserted in Theorem 1.4. These follow from results on the SDP relaxation of MAX CUT on random graphs given in [10], and do not depend on the concentration results in the previous section. In Sections 4.3 and 4.4 we prove the upper bounds on ϑ_2 and $\bar{\vartheta}_2$. Using the concentration results Theorem 1.1 and Theorem 1.3, we get rather clean proofs in Sections 4.3 and 4.4. In addition, the proofs of the upper bounds rely on a lemma on the spectrum of a certain auxiliary matrix given in Section 4.2. Finally, in Section 4.5 we prove Corollary 1.5.

4.1. The lower bound on $\bar{\vartheta}_{1/2}(G_{n,p})$

To bound $\bar{\vartheta}_{1/2}(G_{n,p})$ from below, we make use of an estimate on the probable value of the SDP relaxation SMC = SDP_2 of MAX CUT (*cf.* Section 2 for the definition). Combining Theorems 4 and 5 of [10] instantly yields the following bound on the probable value of $\mathrm{SMC}(G_{n,p})$.

Lemma 4.1. *Suppose that $c_0/n \leqslant p \leqslant 1 - c_0/n$ for some large constant $c_0 > 0$. There is a constant $\lambda > 0$ (independent of n, p) such that*

$$\mathrm{P}\left(\mathrm{SMC}(G_{n,p}) > \frac{1}{2}\binom{n}{2}p + \lambda n^{3/2}p^{1/2}(1-p)^{1/2}\right) \leqslant \exp(-2n). \tag{4.1}$$

Let $G = (V, E)$ be a graph with adjacency matrix $A = (a_{ij})_{i,j=1,\dots,n}$. Let $v_1, \dots, v_n$ be a vector k-colouring of G, where $k = \bar{\vartheta}_{1/2}(G) \geqslant 2$. Then $\|v_i\| = 1$ for all i, and $\langle v_i, v_j\rangle \leqslant -1/(k-1)$ whenever $\{i, j\} \in E$. Therefore, we can interpret $v_1, \dots, v_n$ as a feasible solution to SMC, whence

$$\mathrm{SMC}(G) \geqslant \sum_{i<j} \frac{a_{ij}}{2}(1 - \langle v_i, v_j \rangle) \geqslant \frac{\#E}{2}\left(1 + \frac{1}{k-1}\right) = \frac{\#E}{2}\left(1 + \frac{1}{\bar{\vartheta}_{1/2}(G) - 1}\right).$$

Consequently,

$$\bar{\vartheta}_{1/2}(G) \geqslant \bar{\vartheta}_{1/2}(G) - 1 \geqslant \frac{\#E}{2\mathrm{SMC}(G) - \#E}. \tag{4.2}$$

Let $c_0/n \leqslant p \leqslant 1 - c_0/n$ for some large constant $c_0 > 0$. By Chernoff's bounds (*cf.* [27, p. 26]),

$$\mathrm{P}\left(\#E(G_{n,p}) < \binom{n}{2}p - 8n^{3/2}p^{1/2}(1-p)^{1/2}\right) \leqslant \exp(-2n). \tag{4.3}$$

Combining (4.1), (4.2), and (4.3), we conclude that

$$\bar{\vartheta}_{1/2}(G_{n,p}) \geqslant \bar{\vartheta}_{1/2}(G_{n,p}) - 1 \geqslant \frac{\binom{n}{2}p - 8n^{3/2}p^{1/2}(1-p)^{1/2}}{(2\lambda + 8)n^{3/2}p^{1/2}(1-p)^{1/2}} \geqslant \frac{1}{8\lambda + 32}\sqrt{\frac{np}{1-p}}$$

holds with probability at least $1 - \exp(-n)$. As $\bar{G}_{n,p} = G_{n,1-p}$, this proves (1.2) and the lower bounds in (1.1) in Theorem 1.4.

Remark. Suppose $np = O(1)$. Then (1.2) shows that the probability that $\bar{\vartheta}_{1/2}(G_{n,p})$ is less than $c\sqrt{np}$ is exponentially small, for some constant $c > 0$. It is easily seen that no similar statement holds for the upper tail, *i.e.*, for the event that $\bar{\vartheta}_{1/2}(G_{n,p}) > \zeta\sqrt{np}$ for some large ζ: the probability that $\omega(G_{n,p}) > \zeta\sqrt{np}$ is bounded from below by the probability that $\zeta\sqrt{np}$ fixed vertices form a clique, which is at least $p^{\zeta^2 np} \geqslant \exp(-O(\ln(n))$. Since $\bar{\vartheta}_{1/2}(G) \geqslant \omega(G)$, we conclude $\mathrm{P}(\bar{\vartheta}_{1/2}(G) \geqslant \zeta\sqrt{np}) \geqslant \exp(-O(\ln(n)).$

4.2. Spectral considerations

Let us briefly recall Juhász's proof that $\vartheta(G_{n,p}) \leqslant (2 + o(1))\sqrt{n(1-p)/p}$ for constant values of p, say. Given a graph $G = (V, E)$, we consider the matrix $M = M(G) = (m_{ij})_{i,j=1,\ldots,n}$, where

$$m_{ij} = 1 \text{ if } i \neq j \text{ and } \{i,j\} \notin E, m_{ij} = (p-1)/p \text{ if } \{i,j\} \in E, \text{ and } m_{ii} = 1 \text{ for all } i. \tag{4.4}$$

Then $\lambda_1(M) \geqslant \vartheta(G)$. Moreover, as p is constant, the result of Füredi and Komlós [21] on the eigenvalues of random matrices applies and yields that $\vartheta(G_{n,p}) \leqslant \lambda_1(M) \leqslant (2 + o(1))(n(1-p)/p)^{1/2}$ w.h.p. This argument carries over to the case $\ln(n)^7/n \leqslant p \leqslant 1/2$.

Lemma 4.2. *Suppose that* $\ln(n)^7/n \leqslant p \leqslant 1/2$. *Then* $\|M(G_{n,p})\| \leqslant 3\sqrt{n/p}$ *w.h.p.*

Proof. As it is assumed in [21] that the variance of the matrix entries is independent of n, the proof of Füredi and Komlós [21] needs some minor adaptations to prove the lemma; all details have been carried out in [8, Section 4]. □

However, it is easily seen that in the sparse case, *e.g.*, if $np = O(1)$, we have $\lambda_1(M) \gg n$ w.h.p. The reason is that in the case $np \geqslant \ln(n)^7$ the random graph $G_{n,p}$ is 'almost regular',

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which is not true if $np = O(1)$ (*cf.* [33]). We will get around this problem by chopping off all vertices of degree considerably larger than np, as first proposed in [1]. Thus, let $\varepsilon > 0$ be a small constant, and consider the graph $G' = (V', E')$ obtained from $G = G_{n,p}$ by deleting all vertices of degree greater than $(1+\varepsilon)np$.

Lemma 4.3. *Suppose that $c_0/n \leqslant p \leqslant \ln(n)^7/n$ for some large constant c_0. Let $G = G_{n,p}$, and let $M' = M(G')$. Then $\mathrm{P}(\|M'\| \leqslant c_1\sqrt{n/p}) \geqslant 9/10$, where $c_1 > 0$ denotes some constant.*

To prove Lemma 4.3, we make use of the following lemma from [16, Sections 2 and 3].

Lemma 4.4. *Let $G = G_{n,p}$ be a random graph, where $c_0/n \leqslant p \leqslant \ln(n)^7/n$ for some large constant $c_0 > 0$. Let $n' = \#V(G')$, $e = n'^{-1/2}\vec{1} \in \mathbb{R}^{n'}$, and $A' = A(G')$. For each $\delta > 0$ there is a constant $C(\delta) > 0$ such that in the case $np \geqslant C(\delta)$ with probability $\geqslant 1-\delta$ we have*

$$\max\{|\langle A'v, e\rangle|, |\langle A'v, w\rangle|\} \leqslant c_1\sqrt{np} \text{ for all } v, w \perp \vec{1},\ \|v\| = \|w\| = 1. \tag{4.5}$$

Here $c_1 > 0$ denotes a certain constant.

Lemma 4.5. *Let c_1 be a large constant. The probability that in $G = G_{n,p}$ there exists a set $U \subset V$, $\#U \geqslant n/2$, such that $|\#E(G[U]) - \#U^2 p/2| \geqslant c_1(\#U)^{3/2}p^{1/2}$ is less than $\exp(-n)$.*

Proof. There are at most 2^n sets U. By Chernoff's bounds (*cf.* [27, p. 26]), for a fixed U the probability that $|\#E(G[U]) - \#U^2 p/2| \geqslant c_1(\#U)^{3/2}p^{1/2}$ is at most $\exp(-2n)$, provided that c_0, c_1 are large enough. □

Proof of Lemma 4.3. Let $G = G_{n,p}$, let $n' = \#V(G')$, and let A', e be as in Lemma 4.4. Without loss of generality, we may assume that $V' = V(G') = \{1, \ldots, n'\}$. Let $c_1 > 0$ be a sufficiently large constant. Letting $\delta > 0$ be sufficiently small and $c_0 \geqslant C(\delta)$, we assume hereafter that (4.5) holds, and that G has the property stated in Lemma 4.5. Let $z \in \mathbb{R}^{n'}$, $\|z\| = 1$. Then we have a decomposition $z = \alpha e + \beta v$, $\|v\| = 1$, $v \perp \vec{1}$, $\alpha^2 + \beta^2 = 1$. Since $\|M'z\| \leqslant \|M'e\| + \|M'v\|$, it suffices to bound $\max_{v \perp e, \|v\|=1} \|M'v\|$ and $\|M'e\|$.

Let $\rho : \mathbb{R}^{n'} \to \mathbb{R}^{n'}$ be the projection onto the space $\vec{1}^{\perp}$. Then $A'v = \rho A'v + \langle A'v, e\rangle e$, whence $\|A'v\| \leqslant \|\rho A'v\| + c_1\sqrt{np}$, for all unit vectors $v \perp \vec{1}$. In order to bound $\|\rho A'v\|$, we estimate $\|\rho A'\rho\|$ via (4.5):

$$\|\rho A'\rho\| = \sup_{\|y\|=1} |\langle \rho A'\rho y, y\rangle| = \sup_{\|y\|=1} |\langle A'\rho y, \rho y\rangle| = \sup_{\|y\|=1,\ \vec{1}\perp y} |\langle A'y, y\rangle| \leqslant c_1\sqrt{np}.$$

Consequently, $\|M'v\| = \|(J - \frac{1}{p}A')v\| = \frac{1}{p}\|A'v\| \leqslant 2c_1\sqrt{n/p}$ for all unit vectors $v \perp \vec{1}$.

To bound $\|M'e\|$, note that $-pM' = A' - pJ$. Let $\bar{d} = 2\#E(G')/n'$, and $x = A'e - (\bar{d}/n')Je$. Since $\langle A'\vec{1} - (\bar{d}/n')J\vec{1}, \vec{1}\rangle = \langle A'\vec{1}, \vec{1}\rangle - \frac{\bar{d}}{n'}\langle J\vec{1}, \vec{1}\rangle = 2\#E(G') - \bar{d}n' = 0$, we obtain $x \perp \vec{1}$. Thus, by (4.5) we have $\|x\|^2 = \langle A'e, x\rangle - \langle(\bar{d}/n')Je, x\rangle = \langle A'e, x\rangle \leqslant c_1\sqrt{np}\|x\|$, whence $\|x\| \leqslant c_1\sqrt{np}$. By Lemma 4.5, $|\bar{d} - n'p| \leqslant c_1\sqrt{np}$. As a consequence, $\|(\bar{d}/n')Je - pJe\| \leqslant c_1\sqrt{np}$. Therefore, $\|pM'e\| \leqslant \|x\| + \|(\bar{d}/n')Je - pJe\| \leqslant 2c_1\sqrt{np}$, i.e., $\|M'e\| \leqslant 2c_1\sqrt{n/p}$. □

4.3. Bounding $\vartheta_2(G_{n,p})$ from above

Let $c_0/n \leqslant p \leqslant 1/2$ *for some large constant* $c_0 > 0$. First we observe that the largest eigenvalue of the matrix $M(G)$ considered in the previous section provides an upper bound on $\vartheta_2(G)$. (Actually this follows from the characterization of $\bar{\vartheta}_2$ as an eigenvalue minimization problem given in [41]. However, as [41] does not contain the proof, we show a brief *ad hoc* argument.)

Lemma 4.6. *Let G be any graph. Let* $M = M(G)$. *Then* $\lambda_1(M) \geqslant \vartheta_2(G)$.

Proof. Let $\lambda > \lambda_1(M)$. Then the matrix $\lambda E_n - M$ is positive definite, whence there exist vectors $b_1, \ldots, b_n \in \mathbb{R}^n$ such that $m_{ij} = -\langle b_i, b_j\rangle$ for $i \neq j$, and $\lambda - 1 = \lambda - m_{ii} = \langle b_i, b_i\rangle = \|b_i\|^2 > 0$. Let $a_i = (\lambda-1)^{-1/2} b_i$. Then $\|a_i\| = 1$ for all i. Moreover, if $i \neq j$ and $\{i,j\} \notin E$, then $\langle a_i, a_j\rangle = m_{ij}/(\lambda - 1) = -1/(\lambda-1)$. If $\{i,j\} \in E$, then $\langle a_i, a_j\rangle \geqslant 0$. Hence $(a_1, \ldots, a_n)$ is a rigid vector λ-colouring of $\bar{G}$. Therefore, $\vartheta_2(G_{n,p}) \leqslant \lambda$ for all $\lambda > \lambda_1(M)$. □

In the case $\ln(n)^7/n \leqslant p \leqslant 1/2$, combining Lemmas 4.2 and 4.6 yields that $\vartheta_2(G_{n,p}) \leqslant c_2\sqrt{n/p}$ w.h.p. for some constant $c_2 > 0$, as desired. Thus, let us assume that $c_0/n \leqslant p \leqslant \ln(n)^7/n$ hereafter. Let $\varepsilon > 0$ be a small constant.

Lemma 4.7. *With probability at least* $9/10$ *the random* $G_{n,p}$ *has at most* $1/p$ *vertices of degree greater than* $(1+\varepsilon)np$.

Proof. For each vertex v of $G_{n,p}$, the degree $d(v)$ is binomially distributed with mean $(n-1)p$. By Chernoff's bounds (*cf.* [27, p. 26]), the probability that $d(v) > (1+\varepsilon)np$ is at most $\exp(-\varepsilon^2 np/100)$. Hence, the expected number of vertices v such that $d(v) > (1+\varepsilon)np$ is at most $n\exp(-\varepsilon^2 np/100) < 1/(10p)$, provided $np \geqslant c_0$ for some large constant $c_0 > 0$. Therefore, the assertion follows from Markov's inequality. □

Let $G = G_{n,p}$, and let $G' = (V', E')$ be the graph obtained from G by deleting all vertices of degree greater than $(1+\varepsilon)np$. Let $V'' = V \setminus V'$, and $G'' = G[V'']$. Combining Lemmas 4.7 and 4.3, we obtain that $\mathrm{P}(\vartheta_2(G') \leqslant c_2\sqrt{n/p}$ and $\vartheta_2(G'') \leqslant \#V(G'') \leqslant 1/p \leqslant \sqrt{n/p}) > 1/2$, where c_2 denotes a suitable constant. Consequently, Proposition 2.1 yields that $\mathrm{P}(\vartheta_2(G_{n,p}) \leqslant (c_2+1)\sqrt{n/p}) > 1/2$. Let $\mu = (c_2+1)\sqrt{n/p}$ and $t = \ln(n)\sqrt{n}$. Then, on the one hand $t = o(\sqrt{n/p})$, and on the other hand t is bounded from below by the square root of any median of $\vartheta_2(G_{n,p})$, since $\vartheta_2(G_{n,p}) \leqslant n$. Therefore, by Theorem 1.1, $\mathrm{P}(\vartheta_2(G_{n,p}) > \mu + t) \leqslant 30\exp\left(-\Omega(\ln(n)^2)\right) = o(1)$. Since $t < \sqrt{n/p}$, we get that $\vartheta_2(G_{n,p}) \leqslant (c_2+2)\sqrt{n/p}$ w.h.p.

4.4. Bounding $\bar{\vartheta}_2(G_{n,p})$ from above

Let us first assume that $\ln(n)^7/n \leqslant p \leqslant 1/2$. Let $G = (V,E) = G_{n,p}$ be a random graph, and consider the matrix $\bar{M} = \frac{1}{1-p}E_n - \frac{p}{1-p}M(G)$, where E_n is the $n \times n$-unit matrix, and $M(G)$ is the matrix defined in (4.4). Combining Lemmas 4.2 and 4.6, we have

$$\bar{\vartheta}_2(G) = \vartheta_2(\bar{G}) \leqslant \lambda_1(\bar{M}) \leqslant \left\|\frac{1}{1-p}E_n - \frac{p}{1-p}M\right\| \leqslant \frac{p}{1-p}\|M\| + 2 \leqslant c_4\sqrt{np}$$

w.h.p., where $c_4 > 0$ is a certain constant.

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Now let $c_0/n \leqslant p \leqslant \ln(n)^7/n$ for some large constant $c_0 > 0$. In this case, the proof of our upper bound on $\bar{\vartheta}_2(G_{n,p})$ makes use of the concentration result Theorem 1.3.

Lemma 4.8. *With high probability the random graph $G = G_{n,p}$ admits no set $U \subset V$, $\#U \leqslant 1/p$, such that $\chi(G[U]) > \sqrt{np}$.*

Proof. We shall prove that for all $U \subset V$, $\#U = v \leqslant 1/p$, we have $\#E(G[U]) < v\sqrt{np}/2$. Then each subgraph $G[U]$ has a vertex of degree $< \sqrt{np}$, a fact which immediately implies our assertion. Thus, let $v \leqslant 1/p$. The probability that there exists some $U \subset V$, $\#U = v$, $\#E(G[U]) \geqslant v\sqrt{np}/2$, is at most

$$\binom{n}{v}\binom{\binom{v}{2}}{v\sqrt{np}/2}p^{v\sqrt{np}/2} \leqslant \left(\frac{\mathrm{e}n}{v}\left(\frac{\mathrm{e}v^2 p}{v\sqrt{np}}\right)^{\sqrt{np}/2}\right)^v = \left(\frac{\mathrm{e}n}{v}\left(\frac{\mathrm{e}v\sqrt{p}}{\sqrt{n}}\right)^{\sqrt{np}/2}\right)^v.$$

Let $b_v = (\mathrm{e}n/v)(\mathrm{e}v\sqrt{p}/\sqrt{n})^{\sqrt{np}/2}$. Observe that the sequence $(b_v)_{v=1,\ldots,n}$ is monotone increasing, and that $b_{1/p} = \mathrm{e}np(\mathrm{e}/\sqrt{np})^{\sqrt{np}/2} \leqslant \exp(-2)$. Therefore, $\sum_{v=\ln(n)}^{1/p} b_v^v \leqslant b_{1/p}^{\ln(n)}/p \leqslant n^{-2}p^{-1} = o(1)$. Moreover, if $v \leqslant \ln(n)$, then $b_v = \mathrm{e}nv^{-1}(\mathrm{e}v\sqrt{p}/\sqrt{n})^{\sqrt{np}/2} \leqslant 1/n$, whence $\sum_{v=1}^{\ln n} b_v^v = o(1)$. Thus, $\sum_{v=1}^{1/p} b_v^v = o(1)$, thereby proving the lemma. □

Let $G = (V, E) = G_{n,p}$ be a random graph, and let $G' = (V', E')$ be the graph obtained from G by removing all vertices of degree greater than $(1 + \varepsilon)np$, where $\varepsilon > 0$ is small but constant. Let $V'' = V \setminus V'$, and let $G'' = G[V'']$. By Lemma 4.7, with probability at least $9/10$ we have $\#V'' \leqslant 1/p$. Therefore, by Lemma 4.8, $\mathrm{P}(\bar{\vartheta}_2(G'') \leqslant \sqrt{np}) \geqslant \mathrm{P}(\chi(G'') \leqslant \sqrt{np}) \geqslant 9/11$. To bound $\bar{\vartheta}_2(G')$, we consider the matrix $\bar{M} = \frac{1}{1-p}E_{n'} - \frac{p}{1-p}M(G')$, where $E_{n'}$ is the $\#V' \times \#V'$-unit matrix, and $M(G')$ the matrix (4.4). By Lemma 4.6, $\bar{\vartheta}_2(G') \leqslant \lambda_1(\bar{M})$. Moreover, by Lemma 4.3, with probability $\geqslant 9/10$ we have

$$\bar{\vartheta}_2(G') \leqslant \lambda_1(\bar{M}) \leqslant \left\|\frac{1}{1-p}E - \frac{p}{1-p}M'\right\| \leqslant \frac{p}{1-p}\|M'\| + 2 \leqslant c_4\sqrt{np},$$

for some constant $c_4 > 0$. Proposition 2.1 implies that $\bar{\vartheta}_2(G) \leqslant \bar{\vartheta}_2(G') + \bar{\vartheta}_2(G'')$, whence we conclude that $\mathrm{P}(\bar{\vartheta}_2(G_{n,p}) \leqslant (c_4 + 1)\sqrt{np}) > 1/2$. Since Theorem 1.3 shows that $\bar{\vartheta}_2(G_{n,p})$ is concentrated in width one, we have

$$\mathrm{P}\big(\bar{\vartheta}_{1/2}(G_{n,p}) \leqslant \bar{\vartheta}(G_{n,p}) \leqslant \bar{\vartheta}_2(G_{n,p}) \leqslant (c_4 + 1)\sqrt{np} + 1\big) = 1 - o(1),$$

thereby completing the proof of Theorem 1.4.

Remark. One can prove slightly weaker results on the probable value of $\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$ than provided by Theorem 1.4 without applying any concentration results, or bounds on the SDP relaxation SMC of MAX CUT. Indeed, using only Lemmas 4.3, 4.7 and 4.8 (thus implicitly [16]) and the estimates proposed in [28], one can show that for each $\delta > 0$ there is $C(\delta) > 0$ such the following holds. If $C(\delta) \leqslant np \leqslant 1/2$, then

$$\mathrm{P}(c_1\sqrt{n/p} \leqslant \vartheta(G_{n,p}) \leqslant c_2\sqrt{n/p}) \geqslant 1 - \delta, \;\; \mathrm{P}(c_3\sqrt{np} \leqslant \bar{\vartheta}(G_{n,p}) \leqslant c_4\sqrt{np}) \geqslant 1 - \delta. \quad (4.6)$$

Such an approach is mentioned independently (without explicit proofs) in the current version of [16]. However, Theorem 1.4 is a bit stronger than (4.6), as the bounds (1.1) on

$\vartheta(G_{n,p})$ and $\bar{\vartheta}(G_{n,p})$ hold with probability $1-o(1)$ as $n\to\infty$ even if np remains bounded. Though it might be possible to improve the eigenvalue bounds in [16] to get a result that holds with probability $1-o(1)$ as $n\to\infty$ as well (*cf.* [35] for a discussion), it seems hard to obtain exponentially small probabilities as in (1.2). (In a preliminary version of this paper, a weaker spectral bound than provided by Lemma 4.3 via [16] was used, which gave a bound $\vartheta(G_{n,p})\leqslant(n\ln(np)/p)^{1/2}$.)

4.5. The lower bound on $\mathrm{SDP}_k(G_{n,p})$

Having established Theorem 1.4, we know that there exist constants $c_0,c_1,c_2>0$ such that in the case $c_0/n\leqslant p\leqslant 1/2$ we have

$$c_1\sqrt{np}\leqslant\bar{\vartheta}_2(G_{n,p})\leqslant c_2\sqrt{np} \tag{4.7}$$

with high probability. Let $k\geqslant 2$ be a fixed integer, and let us assume that $c_3k^2/n\leqslant p\leqslant 1/2$, where $c_3=\max\{c_0,c_1^{-1}\}$. Let $G=G_{n,p}$ satisfy (4.7), and consider any rigid vector $\bar{\vartheta}_2(G)$-colouring $(v_1,\ldots,v_n)$ of G. Then

$$\langle v_i,v_j\rangle\geqslant-\frac{1}{c_1\sqrt{np}-1}\geqslant-\frac{1}{k-1}$$

for all i,j, whence $(v_1,\ldots,v_n)$ is a feasible solution to SDP_k. Furthermore, if $\{i,j\}\in E$, then

$$\langle v_i,v_j\rangle\leqslant-\frac{1}{c_2\sqrt{np}-1}.$$

Consequently, letting $A=(a_{ij})_{i,j=1,\ldots,n}$ be the adjacency matrix of G, we have

$$\mathrm{SDP}_k(G)\geqslant\sum_{i<j}a_{ij}\frac{k-1}{k}(1-\langle v_i,v_j\rangle)\geqslant\left(1-\frac{1}{k}\right)\#E(G)+\frac{\#E(G)}{2c_2\sqrt{np}}.$$

As $\#E(G_{n,p})$ is concentrated about its mean $\binom{n}{2}p$, we conclude that

$$\mathrm{SDP}_k(G_{n,p})\geqslant\left(1-\frac{1}{k}\right)\binom{n}{2}p+\frac{n^{3/2}p^{1/2}}{3c_2}$$

with high probability, thereby proving Corollary 1.5.

Remark. Consider the following relaxation SDP_k' of SDP_k:

$$\mathrm{SDP}_k'(G)=\max\sum_{i<j}a_{ij}\frac{k-1}{k}\big(1-\langle v_i,v_j\rangle\big)\ \text{s.t.}\ \|v_i\|=1,$$

where $A=(a_{ij})_{i,j=1,\ldots,n}$ is the adjacency matrix of G and the max is taken over all families $v_1,\ldots,v_n$ of unit vectors in $\mathbb{R}^n$; the difference between SDP_k' and SDP_k is that in SDP_k' we omit the constraint $\langle v_i,v_j\rangle\geqslant-1/(k-1)$. Then $\mathrm{SDP}_k'(G)=(2(k-1)/k)\mathrm{SMC}(G)$. Consequently, Lemma 4.1 shows that $\mathrm{SDP}_k'(G_{n,p})\leqslant(1-1/k)\binom{n}{2}p+c_3n^{3/2}p^{1/2}$ w.h.p., where $c_3>0$ is some constant. Thus, Corollary 1.5 implies that w.h.p. both SDP_k and SDP_k' overestimate the weight $\mathrm{MC}_k(G_{n,p})$ of a MAX k-CUT by at least $c_4n^{3/2}p^{1/2}$, for some constant $c_4>0$. Thus, in the case of random graphs the additional constraints $\langle v_i,v_j\rangle\geqslant-1/(k-1)$ only affect the precise constant in front of the second-order term $n^{3/2}p^{1/2}$.

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5. Random regular graphs

We show how to adapt the arguments given in the previous section to cover the case of random regular graphs. Throughout we assume that $r \geqslant c_0$ for some large constant $c_0 > 0$.

The proof of (1.3) relies on the upper bound on $\mathrm{SMC}(G_{n,r})$ [10, Theorem 15], and is similar to the proof of (1.2). To bound $\vartheta_2(G_{n,r})$, $c_0 \leqslant r = o(n^{1/4})$, we switch to the *configuration model* (*cf.* [39]). Let $W = V \times \{1,\ldots,r\}$. The elements of W are called *half-edges*. A *configuration* ρ is a partition of W into $m = rn/2$ pairs, where we assume that rn is even. Thus, to each half-edge (u,v), ρ assigns another half-edge $\rho(u,v) \neq (u,v)$ such that $\rho^2 = \mathrm{id}$. We say that (u,v) and $\rho(u,v)$ form an *edge*. By $\mathscr{C} = \mathscr{C}(r)$ we denote the set of all configurations. Then $\#\mathscr{C}(r) = (2m-1)!!$.

To each $\rho \in \mathscr{C}$ the canonical map $\pi : W \to V$ assigns an r-regular multigraph $\pi(\rho)$. If we equip $\mathscr{C}$ with the uniform distribution, then, conditional on $G_{n,r}$, π induces the uniform distribution. By $\sigma(\rho)$ we denote the simple graph obtained from the multigraph $\pi(\rho)$ by deleting all loops and turning all multiple edges into single edges. We define the *adjacency matrix* $A = A(\rho) = (a_{ij})_{i,j}$ of $\rho \in \mathscr{C}$ to be the matrix with entries

$$a_{ij} = \text{number of edges joining } i \text{ and } j \text{ in } \pi(\rho)$$

if $i \neq j$, and let a_{ii} be twice the number of loops at vertex i in $\pi(\rho)$.

Lemma 5.1. *Let $i \in \{1/2,1,2\}$. Let μ be the expectation of $\vartheta_i \circ \sigma$ over $\mathscr{C}$ (where $\vartheta_i \circ \sigma(\rho) = \vartheta_i(\sigma(\rho))$). Then, for any $t > 0$, we have $\mathrm{P}(|\vartheta_i \circ \sigma - \mu| > t) \leqslant 2\exp(-t^2/(128m))$.*

Proof. The proof is based on a martingale argument that is very similar to arguments used in [18] and in the proof of Lemma 14 in [10]. For each $j \in \{0,\ldots,rn\}$ we define an equivalence relation $\equiv_j$ on $\mathscr{C}$ as follows. For $v,v' \in V$, $t,t' \in \{1,\ldots,r\}$ we let $(v,t) < (v',t')$ if and only if $v < v'$ or $v = v'$ and $t < t'$. We write $j \sim (v,t)$ if (v,t) is the jth smallest element w.r.t. $<$. For $\rho,\rho' \in \mathscr{C}$ we let $\rho \equiv_j \rho'$ if and only if ρ and ρ' coincide on the first j half-edges w.r.t. $<$. Then $\equiv_0$ has only one equivalence class, whereas in $\equiv_{2m}$ all equivalence classes are singletons. Let F_j denote the σ-algebra corresponding to $\equiv_j$. Then $F_j \subset F_{j+1}$. Hence, letting $X = \vartheta_i \circ \sigma$ and $X_j = \mathrm{E}(X|F_j)$, we obtain a Doob martingale $(X_0,\ldots,X_{2m})$. Let $Z_j = X_j - X_{j-1} = \mathrm{E}(X|F_j) - \mathrm{E}(X|F_{j-1})$ be the martingale difference $(j = 1,\ldots,rn)$. We shall prove that $|Z_j| \leqslant 4$ for all j. The assertion is then an immediate consequence of Azuma's inequality (*e.g.*, [27, p. 37]).

In order to prove that $|Z_j(\rho)| \leqslant 4$ for all $\rho \in \mathscr{C}$, let $j \sim (u,t)$, and let $\rho(u,t) = (v,s)$. In the case $(v,s) < (u,t)$ there is nothing to prove, because $X_j(\rho) = X_{j-1}(\rho)$. Therefore, assume that $(v,s) > (u,t)$. Let $\mathscr{J}_0 = \{(u',t')|\ (u',t') < (u,t)\}$ and $\mathscr{J} = \mathscr{J}_0 \cup \rho(\mathscr{J}_0) \cup \{(u,t)\}$. If (v,s) is the only element of $W \setminus \mathscr{J}$, then again there is nothing to prove because $X_j(\rho) = X_{j-1}(\rho)$. Since $\#\mathscr{J}$ is odd, we assume that $\#W \setminus \mathscr{J} \geqslant 3$ and consider two distinct half-edges $(v',s'),(v'',s'') \notin \mathscr{J} \cup \{(v,s)\}$. Let $\mathscr{A}$ be the $\equiv_{j-1}$-class of ρ. Let

$$\mathscr{A}(v,s,v',s',v'',s'') = \{\rho' \in \mathscr{A}|\ \rho'(u,t) = (v,s),\ \rho'(v',s') = (v'',s'')\}$$

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and $\mathscr{A}(v',s') = \{\rho' \in \mathscr{A} \mid \rho'(u,t) = (v',s')\}$. Then there is a bijection

$$\mathscr{A}(v,s,v',s',v'',s'') \to \mathscr{A}(v',s',v,s,v'',s''),\ \tau \mapsto \tau', \tag{5.1}$$

where τ' coincides with τ except that $\tau'(u,t) = (v',s')$, $\tau'(v,s) = (v'',s'')$. Let $\beta = \#W \setminus J$ and $\alpha = \#\mathscr{A}(v,s,v',s',v'',s'')$. Then α is independent of the choice of (v',s') and (v'',s''). Using the bijection (5.1), we can expand the martingale difference

$$Z_k(\rho) = \frac{1}{\alpha\beta(\beta-2)} \sum_{(v',s')} \sum_{(v'',s'')} \sum_{\tau \in \mathscr{A}(v,s,v',s',v'',s'')} X(\tau) - X(\tau'), \tag{5.2}$$

where $(v',s'),(v'',s'') \notin J \cup \{v,s\}$ are distinct (*cf.* the proof of Lemma 14 in [10] for a detailed computation).

Let $\rho \in \mathscr{A}(v,s,v',s',v'',s'')$. Let H_1 be the (simple) graph obtained from $\sigma(\rho)$ by adding the edges $\{u,v'\}$, $\{v,v''\}$. Then both $\sigma(\rho)$ and $\sigma(\rho')$ are (weak) subgraphs of H_1. Conversely, let H_2 be the graph obtained from $\sigma(\rho)$ by deleting the edges $\{u,v\}$ and $\{v',v''\}$. Then H_2 is a subgraph of both $\sigma(\rho)$ and $\sigma(\rho')$. Thus,

$$\vartheta_i(H_2) \leqslant \min\{X(\rho), X(\rho')\} \leqslant \max\{X(\rho), X(\rho')\} \leqslant \vartheta_i(H_1),$$

whence $|X(\rho) - X(\rho')| \leqslant \vartheta_i(H_1) - \vartheta_i(H_2)$. Since H_1 can be obtained from H_2 by adding at most four edges, $\vartheta_i(H_1) - \vartheta_i(H_2) \leqslant 4$. Thus, by (5.2), $|Z_k(\sigma)| \leqslant 4$. □

The proof of the following lemma goes along the lines of [18]. However, as we work with the configuration model and consider also the case that the degree r tends to infinity, some adaptions are necessary; these have been carried out in [10].

Lemma 5.2. *There is a constant $\gamma > 0$ such that with high probability the adjacency matrix $A = A(\pi(\rho))$, $\rho \in \mathscr{C}$, satisfies $|\langle Ax, y\rangle| \leqslant \gamma\sqrt{r}$, for all unit vectors $x \perp \vec{1}$, $y \perp \vec{1}$.*

Given a configuration $\rho \in \mathscr{C}$, we let $M = M(\rho) = J - \frac{n}{r}A(\pi\rho)$. Then M is a symmetric $n \times n$ matrix.

Lemma 5.3. *There is a constant $c > 0$ such that w.h.p. we have $\|M(\rho)\| \leqslant cnr^{-1/2}$.*

Proof. Let $M = M(\rho)$. As $M\vec{1} = 0$, it suffices to prove that $\langle Mv, w\rangle \leqslant cnr^{-1/2}$ for all unit vectors $v, w \perp \vec{1}$ w.h.p. But this follows from Lemma 5.2 easily. □

Lemma 5.4. *The expected number of loops and multiple edges in $\pi(\rho)$, $\rho \in \mathscr{C}$, is at most $9r^2$. Hence, with probability $\geqslant 1/2$ there are at most $18r^2$ loops or multiple edges.*

Proof. Let $(u,t),(u,s),(v,r),(v,r') \in W$ be distinct half-edges. The probability that $\rho(u,t) = (v,r)$ and $\rho(u,s) = (v,r')$ is $\sim (2m)^{-2}$. There are $2m$ choices of (v,r), and then at most r choices of (v,r'). Further, given (u,t), there are r possible choices of s. Hence, the expected number of half-edges that participate in multiple edges is at most $2(2m)^2r^2(2m)^{-2} \leqslant 8r^2$.

As for loops, let $(u,t) \in W$, and let $s \in \{1,\ldots,r\}\setminus\{t\}$. The probability that $\rho(u,t) = (u,s)$ is $\sim (2m)^{-1}$. Since there are at most $2m$ possible choices of (u,t), and then at most r choices of s, the expected number of loops is at most $2(2m)r(2m)^{-1} \leqslant 4r < r^2$. $\square$

Lemma 5.5. *There are constants $c_1, c_2 > 0$ such that w.h.p. a random configuration $\rho \in \mathscr{C}$ satisfies $c_1 nr^{-1/2} \leqslant \vartheta_{1/2}(\sigma\rho) \leqslant \vartheta(\sigma\rho) \leqslant \vartheta_2(\sigma\rho) \leqslant c_2 nr^{-1/2}$.*

Proof. Let $\mathscr{B}$ be the event that the number of multiple edges and loops in $\pi\rho$, $\rho \in \mathscr{C}$, is at most $20r^2$. By Lemma 5.4, $\mathrm{P}(\mathscr{B}) \geqslant 1/2$. Consequently, by Lemma 5.3, there is a constant $c_1 > 0$ such that $\mathrm{P}(\|M(\rho)\| \leqslant c_1 nr^{-1/2} | \mathscr{B}) \geqslant 1/2$. Indeed, $\mathrm{P}(\|M(\rho)\| > c_1 nr^{-1/2} | \mathscr{B}) > 1/2$ would imply that

$$\mathrm{P}(\|M(\rho)\| > c_1 nr^{-1/2}) \geqslant \mathrm{P}(\|M(\rho)\| > c_1 nr^{-1/2} | \mathscr{B})\mathrm{P}(\mathscr{B}) > 1/4,$$

which contradicts Lemma 5.3. Hence, $\mathrm{P}(\rho \in \mathscr{B} \text{ and } \|M(\rho)\| \leqslant c_1 nr^{-1/2}) \geqslant 1/4$.

We claim that if $\rho \in \mathscr{B}$ satisfies $\|M(\rho)\| \leqslant c_1 nr^{-1/2}$, then $\vartheta_2(\sigma\rho) \leqslant 2c_1 nr^{-1/2}$. For let Y be the set of all vertices $v \in V$ that participate in a multiple edge or a loop. Then $y = \#Y \leqslant 40r^2$. Relabelling the vertices if necessary, we may assume that $Y = \{n-y+1,\ldots,n\}$. Let $M = M(\rho) = (m_{ij})_{i,j=1,\ldots,n}$, and set $M' = (m_{ij})_{i,j=1,\ldots,n-y}$. Then

$$m_{ij} = \begin{cases} 1 & \text{if } i,j \text{ are non-adjacent in } \sigma\rho \\ (r-n)/r & \text{otherwise} \end{cases} \qquad (1 \leqslant i < j \leqslant n-y),$$

and $m_{ii} = 1$ for all i. Let H be the simple graph on $V(H) = \{1,\ldots,n-y\}$ induced by $\sigma\rho$. Then $\vartheta_2(H) \leqslant \lambda_1(M') \leqslant \|M\| \leqslant c_1 nr^{-1/2}$. Since the graph $\sigma\rho$ can be obtained from H by adding y vertices, and since $y \leqslant nr^{-1/2}$, we conclude that $\vartheta_2(\sigma\rho) \leqslant \vartheta_2(H) + y \leqslant 2c_1 nr^{-1/2}$. Hence, $\mathrm{P}(\vartheta_2(\sigma\rho) \leqslant 2c_1 nr^{-1/2}) \geqslant 1/4$. Invoking Lemma 5.1 completes the proof of the upper bound.

As for the lower bound, let c_2 be a sufficiently large constant, and let $\rho \in \mathscr{B}$ be such that the adjacency matrix $A = A(\pi(\rho))$ satisfies $|\langle A\xi,\eta\rangle| \leqslant c_2 r^{1/2}$ for all unit vectors $\xi,\eta \perp \vec{1}$. Let Y, y, and H be as before, $y \leqslant 40r^2$. Moreover, let $\bar{A}$ be the adjacency matrix of $\overline{\sigma(\rho)}[V \setminus Y]$, and let E_{n-y} and E_n be unit matrices of size $n-y$ and n. Since $\vec{1}$ is an eigenvector of $J - A$, we have

$$\lambda_n(J - A - E_n) = -1 - \lambda_2(A) = -1 - \max_{\xi \perp \vec{1},\ \|\xi\|=1} \langle A\xi,\xi\rangle \geqslant -2c_2 r^{1/2}.$$

Hence, $\lambda_n(\bar{A}) \geqslant -2c_2 r^{1/2}$, because $\bar{A} = J - A(\sigma(\rho)[V \setminus Y]) - E_{n-y}$ is a principal minor of $J - A - E_n$. Let $B = (n-y)^{-1}\big(E_{n-y} - \lambda_n(\bar{A})^{-1}\bar{A}\big)$. Then the matrix $B = (b_{ij})_{i,j=1,\ldots,n-y}$ is positive semidefinite, and we have $\sum_i b_{ii} = 1$. Moreover, $b_{ij} \geqslant 0$ for all i,j, and if $i,j \in V \setminus Y$ are adjacent in $\sigma(\rho)$, then $b_{ij} = 0$. It is shown in [24, pp. 51ff] that such a matrix B satisfies $\vartheta_{1/2}(\sigma(\rho)[V \setminus Y]) \geqslant \sum_{i,j} b_{ij}$ (the proof goes along the lines of [30, Section 7–9]). Hence,

$$\vartheta_{1/2}(\sigma(\rho)) \geqslant \vartheta_{1/2}(\sigma(\rho)[V \setminus Y]) \geqslant \sum_{i,j} b_{ij} \geqslant 1 - \frac{(n-y)^2 - (n-y)r}{(n-y)\lambda_n(\bar{A})} \geqslant \frac{n}{4r^{1/2}c_2}.$$

Finally, applying Lemma 5.1 once more yields our assertion. $\square$

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It is shown in [39] that in the case $r = o(n^{1/4})$ we have $\mathrm{P}(\pi(\rho)$ is a simple graph$) \geqslant \exp(-o(n^{1/2}))$. Therefore, letting $t = \Omega(nr^{-1/2})$ and applying Lemma 5.1 we conclude that there is some constant $c > 0$ such that $\mathrm{P}(\vartheta_2(G_{n,r}) \leqslant cnr^{-1/2}) = 1 - o(1)$. Similarly, $\mathrm{P}(\vartheta_{1/2}(G_{n,r}) \geqslant c'nr^{-1/2}) = 1 - o(1)$ for some constant c'.

Finally, assume that $r = O(1)$. Then $\mathrm{P}(\pi(\rho)$ is a simple graph$) = \Omega(1)$ by [39]. Hence, by Lemma 5.2 w.h.p. the adjacency matrix A of $G = G_{n,r}$ satisfies $\max\{\lambda_2(A), -\lambda_n(A)\} \leqslant c_4\sqrt{r}$, where $c_4 > 0$ is a constant. Let $\tilde{M} = (\tilde{m}_{ij})_{i,j}$ be the matrix with entries $\tilde{m}_{ij} = 1$ if $\{i, j\} \in E(G)$ or $i = j$, and $\tilde{m}_{ij} = \frac{r}{r-n}$ otherwise $(i, j = 1, \ldots, n)$. Then $\tilde{M} = \frac{r}{r-n}J + \frac{n}{n-r}(A + E_n)$. Since $\tilde{M}\vec{1} = \vec{1}$, we have $\|\tilde{M}\| \leqslant c_4\sqrt{r}$. The same argument as in the proof of Lemma 4.6 shows that $\bar{\vartheta}_2(G) \leqslant \|\tilde{M}\|$, thereby proving that $\bar{\vartheta}_2(G_{n,r}) \leqslant c_4\sqrt{r}$ w.h.p.

6. Approximating the independence number and deciding *k*-colourability

In this section we present the algorithms required for Theorems 1.7, 1.8 and 1.9. The algorithm for the independent set problem is essentially identical to that proposed in [8], and the algorithm for deciding k-colourability resembles that given in [31]. Thus, our contribution is that using our new results on the Lovász number of random graphs and the vector chromatic number, we can improve on the *analyses* given in [8, 31].

6.1. Approximating the independence number

The algorithm `ApproxMIS` for approximating the independence number consists of two parts. First, we employ a certain greedy procedure that on input $G = G_{n,p}$ most probably finds an independent set of size at least $\ln(np)/(2p)$, thereby providing a lower bound on $\alpha(G)$. Secondly, we compute $\vartheta(G)$ to bound $\alpha(G)$ from above. Throughout, we assume that $np \geqslant c_0$ for some large constant c_0.

Following [34], to find a large independent set of $G = G_{n,p}$, we run the greedy algorithm for graph colouring and pick the largest colour class it produces. Remember that the greedy algorithm goes through the vertices $v = 1, \ldots, n$ of G, and assigns to v the least colour among $\{1, \ldots, n\}$ that is not occupied by a neighbour $w < v$ of v.

Lemma 6.1. *The probability that the largest colour class produced by the greedy colouring algorithm contains* $< \ln(np)/(2p)$ *vertices is at most* $\exp(-n)$.

Proof. The proof given in [34] for the case that $p \geqslant n^{\varepsilon - 1/2}$ carries over also for $p \geqslant c_0/n$. □

Thus, with probability $\geqslant 1 - \exp(-n)$, the greedy algorithm shows that $\alpha(G) \geqslant \ln(np)/(2p)$, where $G = G_{n,p}$. To obtain an $O(\sqrt{np}/\ln(np))$ approximation as needed for Theorem 1.7, we employ the following procedure that tries to certify that $\alpha(G) \leqslant C\sqrt{n/p}$ for a sufficiently large constant $C > 0$. This procedure is essentially identical to the one given in [8]. (The difference between the algorithm proposed in [34] and the one below is that our algorithm uses the Lovász number as an upper bound on $\alpha(G)$ instead of the largest eigenvalue of the matrix (4.4).)

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Algorithm 1. `BoundAlpha`(G)
Input: A graph $G = (V, E)$ of order n.
Question: Is $\alpha(G) < C\sqrt{n/p}$?
Output: Either 'yes' or 'no'.

(1) Let $a := C(n/p)^{1/2}$ and $b := 25\ln(np)/p$.
(2) Compute $\vartheta(G)$. If $\vartheta(G) < a$, then output 'yes' and halt.
(3) Check whether there exists a subset S of V, $\#S = b$, such that $\#V \setminus (N(S) \cup S) \geqslant a - b$. If no such set exists, then output 'yes' and halt.
(4) Check all sets of size a. If none of them is independent, then output 'yes' and halt. Otherwise, output 'no'.

Lemma 6.2. *For any graph G,* `BoundAlpha`(G) *answers correctly. If $p \geqslant c_0/n$ for a certain constant $c_0 > 0$, then* `BoundAlpha`$(G_{n,p})$ *runs in expected polynomial time. Finally, the probability that* `BoundAlpha` *answers 'no' is at most* $\exp(-n)$.

Proof. We adapt the argument given in the proof of Lemma 14 of [8]. Since $\alpha(G) \leqslant \vartheta(G)$, it is clear that the answer of `BoundAlpha` is correct if it halts after step (2) or step (4). Moreover, if there is an independent set T in G of cardinality $\geqslant a$, then step (3) eventually encounters a set $S \subset T$. As $T \setminus S \subset V \setminus (N(S) \cup S)$, step (3) will *not* output 'yes'. Consequently, `BoundAlpha`(G) answers correctly on any input.

To prove that the expected running time on input $G_{n,p}$ is polynomial, we just have to estimate the probability that `BoundAlpha` runs step (3) or (4). On the one hand, running step (3) consumes time

$$\leqslant n^{O(1)}\binom{n}{b} \leqslant n^{O(1)} \exp\left(\frac{25\ln(np)^2}{p}\right).$$

On the other hand, if $C > c_2 + C''$, where c_2 is the constant from Theorem 1.4 and C'' is a further sufficiently large constant, then Theorem 1.1 entails that

$$\mathrm{P}\left(\vartheta(G_{n,p}) > C\sqrt{\frac{n}{p}} = (c_2 + C'')\sqrt{\frac{n}{p}}\right) \leqslant 30\exp\left(-\frac{C''^2}{5c_2 + 10C''}\sqrt{\frac{n}{p}}\right) < \exp\left(\frac{-25\ln(np)^2}{p}\right),$$

so that the expected time spent on executing step (3) is polynomial.

Similarly, step (4) consumes time

$$\leqslant n^{O(1)}\binom{n}{a} \leqslant n^{O(1)}\left(\frac{\mathrm{e}\sqrt{np}}{C}\right)^{a} \leqslant n^{O(1)}\exp\bigl(\ln(np)a\bigr).$$

As the probability that there is a set S of cardinality b such that $\#V \setminus (N(S) \cup S) > a - b$ is

$$\leqslant \binom{n}{a}\binom{n}{b}(1-p)^{b(a-b)} \leqslant \exp\bigl(\ln(np)(a+b) - p(a-b)b\bigr) \leqslant \exp(-20\ln(np)a),$$

the expected running time of step (4) is polynomial. Finally,

$$\mathrm{P}\bigl(\alpha(G_{n,p}) \geqslant a\bigr) \leqslant \binom{n}{a}(1-p)^{\binom{a}{2}} \leqslant \exp\left(\ln(np)a - \frac{a^2 p}{4}\right) \leqslant \exp(-n),$$

thereby proving the lemma. □

Combining the greedy algorithm with the procedure `BoundAlpha` yields the algorithm for Theorem 1.7.

Algorithm 2. `ApproxMIS`(G)
Input: A graph $G = (V, E)$. *Output:* An independent set of G.

(1) Run the greedy algorithm for graph colouring on input G. Let I be the largest resulting colour class. If $\#I < \ln(np)/(2p)$, then go to (3).
(2) If `BoundAlpha`(G) answers '$\alpha(G) \leqslant C\sqrt{n/p}$', then output I and halt.
(3) Enumerate all subsets of V and output a maximum independent set.

Proof of Theorem 1.7. Since the probability that `ApproxMIS`$(G_{n,p})$ runs step (3) is $\leqslant 2\exp(-n)$, by Lemmas 6.1 and 6.2, and since `BoundAlpha`$(G_{n,p})$ runs in expected polynomial time, the expected running time of `ApproxMIS`$(G_{n,p})$ is polynomial. Moreover, if `ApproxMIS`(G) halts after step (2), then $\frac{\#I}{\alpha(G)} \geqslant \frac{\ln(np)\sqrt{p}}{2Cp\sqrt{n}} \geqslant \frac{\ln(np)}{2C\sqrt{np}}$, as desired. □

Remark. The lower bounds on $\vartheta_{1/2}(G_{n,p})$ in Theorem 1.4 show that we could not achieve an approximation ratio of $(np)^{\beta}$, $\beta < 1/2$, even if we were to use the relaxation $\vartheta_{1/2}$ instead of ϑ to upper-bound the independence number in our algorithm `ApproxMIS`.

6.2. Deciding k-colourability

Following [31], we decide k-colourability by computing the vector chromatic number of the input graph. Let $k = k(n)$ be a sequence of positive integers such that $k(n) = o(\sqrt{n})$. Since the vector chromatic number is always a lower bound on the chromatic number, the answer of the following algorithm is correct for all input graphs G.

Algorithm 3. `Decide`$_k(G)$
Input: A graph $G = (V, E)$. *Output:* Either '$\chi(G) \leqslant k$' or '$\chi(G) > k$'.

(1) If $\bar{\vartheta}_{1/2}(G) > k$ then terminate with output '$\chi(G) > k$'.
(2) Otherwise, compute $\chi(G)$ in time $o(\exp(n))$ using Lawler's algorithm [36], and answer correctly.

Theorem 1.8 is a consequence of the following lemma.

Lemma 6.3. *Suppose that $p \geqslant Ck^2/n$ for some large constant C. Then the expected running time of* `Decide`$_k(G^+_{n,p})$ *is polynomial.*

Proof. In [29] it is shown that $\bar{\vartheta}_{1/2}$ can be computed in polynomial time (we disregard rounding issues). Since the second step consumes time $o(\exp(n))$, inequality (1.2) shows that the expected running time of `Decide`$_k$ on input $G_{n,p}$ is polynomial. Consequently, by (2.3), we conclude that the expected running time of `Decide`$_k(G^+_{n,p})$ is polynomial. □

The analysis of `Decide`$_k$ on input $G_{n,r}$, $r \geqslant Ck^2$, is based on (1.3) and yields the proof of Theorem 1.9.

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Remark. The analysis of $\texttt{Decide}_k$ shows that we can decide in polynomial expected time whether $G^+_{n,p}$ is k-colourable, provided $np \geqslant c_0k^2$. Conversely, the upper bounds on $\bar{\vartheta}_{1/2}(G_{n,p})$, $\bar{\vartheta}(G_{n,p})$, $\bar{\vartheta}_2(G_{n,p})$ in Theorem 1.4 show that even if we were to use the relaxation $\bar{\vartheta}_2$ instead of the vector chromatic number $\bar{\vartheta}_{1/2}$ in our algorithm $\texttt{Decide}_k$, we would still have to assume that $np = \Omega(k^2)$.

7. Conclusion

The results presented in this paper show that the Lovász number and other SDP relaxations of the independence number or the chromatic number provide powerful tools in the design of algorithms with polynomial expected running time. Indeed, in addition to the algorithmic applications given in this paper, the results were used by Coja-Oghlan, Goerdt, Lanka, and Schädlich to obtain an algorithm for deciding in polynomial expected time whether a random $2k$-SAT formula is satisfiable [9]. Furthermore, using Theorems 1.1 and 1.4, Coja-Oghlan obtained an algorithm that finds a large independent set hidden in a semirandom graph in polynomial expected time [7]. Therefore, the author expects that the general idea of combining large deviation techniques such as Talagrand's inequality with SDP relaxations will lead to further contributions to the algorithmic theory of random structures (*cf.* also [10]).

In comparison with purely combinatorial techniques or computing eigenvalues, semidefinite programming requires rather heavy machinery (*cf.* [25]). However, compared to the eigenvalues of the adjacency matrix, semidefinite programs such as the Lovász number seem to be rather 'robust' (*cf.* the discussion in [14]). In fact, this robustness may be the reason why for the Lovász number we can derive a large deviation result such as Theorem 1.1. By contrast, for 'small' values of p, no similarly strong tail bounds on the eigenvalues of the auxiliary matrix $M(G_{n,p})$ are known (*cf.* [3, 34]) – although we used the eigenvalues of $M(G_{n,p})$ to bound the mean of $\vartheta(G_{n,p})$.

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