An algorithm and codes for fast computations of the opposition effects in a semi-infinite discrete random medium

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We present an algorithm and FORTRAN codes to compute the opposition effects in the reflection of light from a semi-infinite discrete random medium at normal incidence to the boundary of the medium. It is assumed that the medium is sparse enough that the waves propagating between the scatterers are spherical. In this case, the reflection matrix is determined only by contributions of the incoherent (diffuse) and coherent components. When calculating the coherent component, the contribution of the doubly scattered radiation to the reflection matrix is rigorously taken into account, while the contributions of the higher orders are calculated approximately. To be more specific, the multiply scattered radiation coming to some point of the medium "from above" is calculated exactly, but the radiation coming "from below", approximately. Under this supposition, the solution of the system of integral equations is reduced to that of the system of linear algebraic equations. The matrix of this system is calculated with the recurrent relation, which radically speeds up the computations as compared to the direct procedure. This allows the opposition effect characteristics to be computed rapidly enough so that the codes may be used in interpretation of the remotely measured intensity and polarization of light reflected by different media to estimate, at least at a qualitative level, their properties.

Keywords: multiple scattering; coherent backscattering; opposition effects; discreate random medium; reflection matrix

Highlights:

A fast algorithm to calculate the opposition effects is presented

FORTRAN codes to compute the opposition effects are described

Errors introduced by the algorithm are concisely considered

1. INTRODUCTION

In the radiation reflected by many discrete media of natural and artificial origin, the so-called photometric and polarimetric opposition effects are often observed. These phenomena are usually associated with the weak localization effect, which has been under active experimental and theoretical examination for the last years (see, e.g., [1–3] and references therein). The interference nature of this effect suggests that its characteristics essentially depend in the properties of the scattering medium, which is of key importance for interpretation of remote-sensing data of different objects. Of particular value is this effect for retrieving the properties of atmosphereless celestial bodies of the Solar system from the photometric and polarimetric observational data [4, 5].

In recent years, considerable progress has been made in computations of the weak localization characteristics for a plane-parallel layer of a medium containing scatterers, the sizes of which are much larger than the wavelength of the incident radiation. The algorithm described in [3] makes it possible to calculate the characteristics of this effect (the contribution of cyclical diagrams into radiation scattered by a medium) with accounting for the near field and the correlation in particle positions, specifically, under oblique radiation incidence onto the layer. However, to calculate the characteristics of radiation reflected by random media containing densely packed scatterers, the sizes of which are of the order of the wavelength, remains an open problem. The contribution of the other diagrams, particularly, the diagrams responsible for the mutual shielding or correlation of waves propagating in the medium along neighbor pathways, is difficult to analyze (see, [2] and references therein).

More substantial advances have been made in the description of the weak localization effect for sparse media, more precisely, in the frames of the model assuming that the waves propagating between the scatterers in a medium are spherical. In this case the near-field effects [6] and correlation in particle positions may be ignored, and the scattering characteristics of the medium are determined only by contributions of the ladder and cyclical diagrams (i.e., the diffuse and coherent components, respectively) [1–3]. To take into account the dense packing of particles, at least approximately, it was proposed that randomly oriented clusters should be used in this model as a volume element of the medium [7]. In this case, all effects connected with the dense packing of scatterers are automatically accounted for, at least on the scale of clusters. The comparison of the reflectance characteristics calculated with this model of a semi-infinite medium to those measured in the laboratory shows that they agree well for some samples [7]. This gives grounds to suppose that this model of a medium can be successfully used to estimate, at least approximately, the parameters of natural and artificial media investigated remotely.

In this paper we describe the efficient algorithm to compute the reflectance characteristics of a semi-infinite discrete random medium under normal radiation incidence. This algorithm works rapidly, which is of high importance for the remote-sensing data analysis. The single scattering matrix of randomly oriented scatterers (in particular, clusters of scatterers) obtained from calculations or measurements serve as characteristics of a so-called “elementary (or differential) volume element” of the discrete scattering medium, i.e., as input data for the procedure.

2. BASIC RELATIONSHIPS

The described algorithm is based on the approximate method of solving the equation for weak localization of waves reflected from a semi-infinite discrete random medium under normal radiation incidence [8]. To calculate a sum of cyclical diagrams with this method, the radiation coming to some point of the medium “from above” is determined exactly, while the radiation coming “from below” is taken into account approximately (Fig. 1). Namely, it is assumed that the intensity of multiply (more than twice) scattered radiation decreases exponentially with depth below the considered point of the medium, while the rate of the decrease can be determined from some independent relationship. In this case, the system of integral equations, describing the weak localization effect, is reduced to the system of linear algebraic equations. Though all of the basic equations were introduced in papers [7, 8], we describe them here with some comments and explanations.

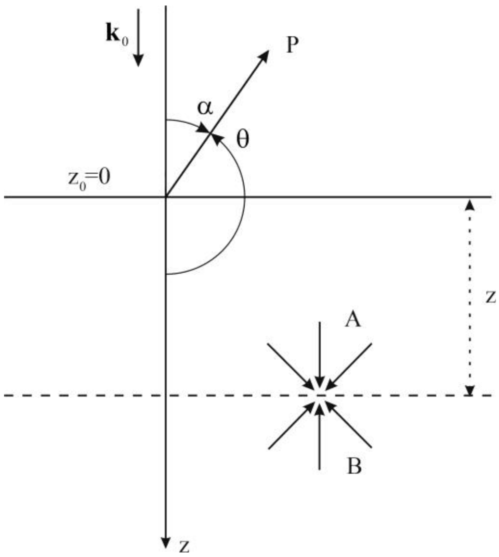

Fig. 1. Geometry of scattering by a semi-infinite particulate medium. The incident light propagates normally to the boundary of the medium (z_0 = 0). The incidence direction is indicated by the wave vector $\mathbf{k_0}$, α and ϑ are the phase and scattering angles, respectively, “P” shows the direction of observations, and letters “A” and “B” denote the scattered radiation that comes to some point at a depth z from above and below, respectively.

The reflection matrix $\mathbf{S}^{(C)}$, describing the weak localization effect in the circular-polarization (CP) representation, is [7, 8]

$$S^{(C)}_{pn\nu\mu} = \frac{\pi\eta^2}{k_0^4\,\mathrm{Re}(\varepsilon)} \sum_{qq_1LM} (-1)^L \zeta^{*(q_1\mu)(qp)}_{LM} \gamma^{(qn)(q_1\nu)}_{LM}, \qquad (1)$$

where $p,n,\mu,\nu,q,q_1 = \pm 1$, the range of indices L,M is specified below (see Section3), η is the particle number density, $k_0 = 2\pi/\lambda$, λ is the wavelength of the incident radiation, the asterisk denotes complex conjugation,

$$\varepsilon = \mathrm{Im}(m_{eff})\left(1 - \frac{1}{\cos\vartheta}\right) + \mathrm{i}(1+\cos\vartheta)\left(\frac{\mathrm{Re}(m_{eff})-1}{\cos\vartheta} + 1\right), \qquad (2)$$

m_{eff} is the complex effective refractive index of the medium, ϑ is the scattering angle, and $\mathrm{i} = \sqrt{-1}$. For sparse media, it can be assumed that $m_{eff} = 1 + i\eta C_{ext}/2k_0$, where C_{ext} is the mean extinction cross-section of scatterers in the medium.

The coefficients $\gamma^{(qn)(q_1\nu)}_{LM}$ are determined from the following system of linear algebraic equations:

$$\gamma^{(pn)(\mu\nu)}_{LM} = Q^{(pn)(\mu\nu)}_{LM} + \frac{2\pi\eta}{k_0^3} \sum_{qq_1lm} \chi_l^{(pq)(\mu q_1)} \gamma_{lm}^{(qn)(q_1\nu)} G^{(N_0)}_{LMlm}, \qquad (3)$$

where

$$Q^{(pn)(\mu\nu)}_{LM} = \sum_{lm} \zeta^{(pn)(\mu\nu)}_{lm} \int_0^{\pi} d^L_{MN_0}(\omega) d^l_{mN_0}(\omega) I_N(c,f) \sin\omega\, d\omega, \qquad (4)$$

$$G^{(N_0)}_{LMlm} = \int_0^{\frac{\pi}{2}} \left[I_N(c,f) d^L_{MN_0}(\omega) d^l_{mN_0}(\omega) + (-1)^{L+l+M+m} I_N(c,g) d^L_{M,-N_0}(\omega) d^l_{m,-N_0}(\omega) \right] \sin\omega\, d\omega. \qquad (5)$$

Here, $N = |M-m|$, d with indices are the Wigner d-functions [9], $N_0 = \mu - p$,

$$I_N(c,x) = \mathrm{i}^{-N} \frac{c^N}{\sqrt{c^2+x^2}\left(x + \sqrt{c^2+x^2}\right)^N}, \qquad (6)$$

$$c = \sin\vartheta \sin\omega, \qquad (7)$$

$$\begin{aligned} f &= 2\,\mathrm{Im}(m_{eff}) + |\cos\omega|\,\mathrm{Im}(m_{eff})\left(1 - \frac{1}{\cos\vartheta}\right) + \\ &+ \mathrm{i}\cos\omega(1+\cos\vartheta)\left(\frac{\mathrm{Re}(m_{eff})-1}{\cos\vartheta} + 1\right), \end{aligned} \qquad (8)$$

$$g = 2\,\mathrm{Im}(m_{eff})(1 + \sigma|\cos\omega|). \qquad (9)$$

By χ with indices, we denote the coefficients of the Wigner *d*-function expansions of the single scattering matrix averaged over orientations and properties of scatterers in the CP-representation (the scattering matrix for a volume element in the medium). The analogous coefficients in the linear-polarization (LP) representation, though in the basis of the generalized spherical function closely related to the Wigner functions, are widely used in the radiative transfer theory to calculate the diffuse component of the reflection matrix for different media [10]. Their explicit form for spherical and randomly oriented irregular scatterers and the formulas to convert from the CP-basis to the LP-basis and back can be found in a paper [13].

The quantities ζ with indices are coefficients that have not appeared in the scattering theory before. For spherical scatterers, the formulas required for calculating these coefficients are given in papers [2, 7, 11], where their physical content is considered in detail. For arbitrary scatterers, a calculation technique to obtain the coefficients ζ is still lacking. It was shown in papers [2, 11] that, for spherical particles and at $\vartheta \approx \pi$, the following approximation is valid

$$\zeta_{lm}^{(pn)(\mu\nu)} \approx \delta_{m,\nu-n}\chi_l^{(pn)(\mu\nu)}, \qquad (10)$$

where $\delta_{m,\nu-n}$ is the Kronecker delta. This approximation made it possible to generalize the weak-localization equation obtained for a medium of spherical particles to the case of arbitrary scatterers, for which the coefficients χ can be calculated. Specifically, this is true for a medium of the scatterers, the scattering matrix of which is measured in laboratory, if the examination is thorough enough that the coefficients χ can be calculated. In the following, we assume that approximation (10) is obeyed.

As has been already mentioned, when deriving the above equations, we suppose that the radiation coming from below and scattered more than twice decreases exponentially with growing the depth z in the layer. We assume that this dependence takes the form $\exp(-2\sigma z\,\mathrm{Im}(m_{eff}))$, where the coefficient σ characterizes the rate of the decrease of the radiation intensity with depth. This coefficient is determined from the relationship that connects the reflection matrix elements of the incoherent component with those for weak localization in the exactly backscattering direction ($\vartheta = \pi$). These relationships for the diagonal matrix elements are given in papers [10] and [11] in the LP and CP representations, respectively. For a semi-infinite medium, an analogous approach to deriving the contribution of radiation coming from below is proposed in a paper [3]; however, a more complicated nonlinear dependence of the exponent on the depth *z* is introduced in that study. The fast algorithm proposed here for computing the characteristics of the weak localization effect can be applied to both the linear and nonlinear dependences of the exponent on *z*.

Let us point to the following property of Eqs. (1)–(5). In papers [3, 12], the weak localization equations are written in such a way that they contain the single-scattering contribution, which is then removed. In Eqs. (1)–(5) above, the single-scattering contribution is already absent. Coefficients (4) correspond to the double scattering, and they are calculated in a rigorous manner, so that the approximation mentioned at the beginning of this section is used only for the higher orders of the scattered radiation coming from below.

In actual practice, the reflection matrix in the LP representation $\mathbf{R}$ is used

$$\mathbf{R} = \mathbf{R}^{(L)} + \mathbf{R}^{(C)} . \tag{11}$$

Here, $\mathbf{R}^{(L)}$ is the reflection matrix of the diffuse component, while the elements of the reflection matrix of the coherent component in the LP basis $\mathbf{R}^{(C)}$ take the form

$$\begin{aligned} &R_{11}^{(C)} = U\sum_{pn} S_{pnpn}^{(C)}, \qquad R_{21}^{(C)} = R_{12}^{(C)} = -U\sum_{pn} S_{pn-pn}^{(C)}, \qquad R_{22}^{(C)} = U\sum_{pn} S_{pn-p-n}^{(C)}, \\ &R_{33}^{(C)} = U\sum_{pn} S_{pn-p-n}^{(C)} \mathrm{i}^{p-n}, \qquad R_{44}^{(C)} = U\sum_{pn} S_{pnpn}^{(C)} \mathrm{i}^{p-n}, \qquad R_{34}^{(C)} = -R_{43}^{(C)} = \mathrm{i}U\sum_{pn} S_{pnp-n}^{(C)} \mathrm{i}^{p-n}, \end{aligned} \tag{12}$$

where $U = -\pi / 2k_0^2 \cos\vartheta$ and $n, p = \pm 1$.

3. A FAST ALGORITHM TO COMPUTE THE COHERENT REFLECTION MATRIX

The computing speed in solving system (3) is mainly determined by the computing speed for matrix (5). The dimension of the matrix depends on the sizes of scatterers and the scattering angle. The highest value of the index L depends on the sizes of scatterers and is determined by the required accuracy of calculations of the coefficients χ For example, for spherical particles, the highest value of the index L is equal to the double value of the highest summation index l_m in the Mie formulas (see, e.g., [13]). In general, $-L \le M \le L$. Because of this, the dimension of matrix (5) is $[2l_m(2l_m+1)]\times[2l_m(2l_m+1)]$; consequently, for the medium composed of scatterers larger than the wavelength in size, the matrix dimension may be very large. For example, for the medium composed of spherical scatterers with the size parameter $x_0 = 5$ ($x_0 = k_0 a$, where a is the scatterer radius), for which the maximal summation index can be determined as $l_m = x_0 + 4.05 x_0^{1/3} + 8$ [13], the dimension of matrix (5) is 1640 × 1640. When approximation (10) is used, the matrix dimension falls dramatically to roughly 200 × 200 at $\vartheta = \pi$ or to somewhat larger dimensions at smaller scattering angles.

The computations of so large matrices can be made much faster with recurrent relation (A.4) given in the Appendix. To use it, it is convenient to rewrite the integral in Eq. (4) as follows:

$$\int_0^{\frac{\pi}{2}}\left[I_N(c,f)d^L_{MN_0}(\omega)d^l_{mN_0}(\omega)+(-1)^{L+l+M+m}I_N(c,f^*)d^L_{M-N_0}(\omega)d^l_{m-N_0}(\omega)\right]\sin\omega d\omega\,. \quad (13)$$

As is seen from Eqs. (4), (5), and (13), calculations of the matrix of system (5) and coefficients (4) are reduced to calculations of the integrals such as

$$F^{lmN_1}_{LMN_1}=\int_0^{\frac{\pi}{2}} I_N(c,x)d^L_{MN_1}(\omega)d^l_{mN_1}(\omega)\sin\omega d\omega\,, \qquad (14)$$

where $N_1=\pm N_0$.

Though the recurrent relation (A.4) is applicable directly to integral (14), a more efficient way is to expand the products of the Wigner functions to a series in the Wigner functions, i.e.,

$$F^{lmN_1}_{LMN_1}=(-1)^{m+N_1}\sum_{L_1=|L-l|}^{L+l} C^{L_1M_1}_{LMl-m}C^{L_10}_{LN_1l-N_1}\int_0^{\frac{\pi}{2}} I_{|M_1|}(c,x)d^{L_1}_{M_10}(\omega)\sin\omega d\omega\,. \qquad (15)$$

Here the C's with indices are the Clebsch–Gordan coefficients [9], and $M_1=M-m$. Relationship (15) completely corresponds to (A.7) from the Appendix, which formula (A.4) can be applied to.

Since the symmetry relation $d^L_{M0}(\omega)=(-1)^M d^L_{-M0}(\omega)$ [9] is obeyed, the integral in the right side of Eq. (15) may be calculated only for positive and zero values of M_1. The reflection matrix may be computed substantially faster also by using the following symmetry relations [7]:

$$F^{lmN_1}_{LMN_1}=F^{LMN_1}_{lmN_1}=(-1)^{M+m}F^{l,-m,-N_1}_{L,-M,-N_1} \qquad G^{(N_0)}_{LMlm}=(-1)^{M+m}G^{(-N_0)}_{L-Ml-m}$$

$$Q^{(pn)(\mu\nu)}_{LMlm}=(-1)^{M+m}Q^{(-p-n)(-\mu-\nu)}_{L-Ml-m} \qquad \gamma^{(pn)(\mu\nu)}_{LM}=(-1)^M\gamma^{(-p-n)(-\mu-\nu)}_{L-M}$$

Thus, the matrix of system (5) and coefficients (4) may be computed according to the following algorithm. The arrays of values of the integrals in the right side of Eq. (15) are calculated for $x=f$, $x=g$, and the scattering angle $\vartheta=\pi$. With the recurrent formula (A.4) (see also relationship (A.7) in the Appendix), the integrals of type (15) are calculated; then, coefficients (4) and matrix (5) are found. System (3) is solved, and the value of σ is determined with some method chosen. The computational procedure is repeated with the obtained value of σ to find coefficients (4) and matrix (5) and to solve system (3) for the other scattering angles specified.

4. COMPUTATIONAL CODES

The described procedure was realized in the software package [14], the codes of which are written in the Fortran-90 programming language. The coherent component of the reflection matrix $\mathbf{R}^{(C)}$ is calculated on the base of approximation (10). The coefficient σ is found from the equation

$$R_{11}^{(C)} = (R_{11}^{(M)} + R_{22}^{(M)} - R_{33}^{(M)} + R_{44}^{(M)})/2, \quad (16)$$

which is valid for $\vartheta = \pi$ [13] and solved with the simple bisection method. Here the matrix $\mathbf{R}^{(M)} = \mathbf{R}^{(L)} - \mathbf{R}^{(1)}$, and $\mathbf{R}^{(1)}$ corresponds to the single scattering.

To solve system (3), the iteration Bi-CGSTAB method [15] is used. The main program *opp_eff.f90* operates in conjunction with the file of input parameters *data_inp.txt* and other files specified there. The *data_inp.txt* file contains the following data:

Number of integration intervals: this parameter specifies the number of identical subintervals K, which the entire integration interval ($\omega = \pi/2$) in integral (15) is divided into. The Chebyshev quadrature formula is used for integration in each of the subintervals. As a rule, $K = 6$ is sufficient for calculating the integrals of type (15) with a high accuracy even for very large scatterers.

alp0, dalp1, alp1, dalp2, alp2, dalp3, alp3: these are the boundaries of three predetermined intervals for changing the phase angles α ($\alpha = \pi - \vartheta$)---from 0 to α_1, from α_1 to α_2, and from α_2 to α_3 (*alp0, alp1, alp2, alp3*); and each of them may have its own step of changes in α --- $\Delta\alpha_1$, $\Delta\alpha_2$, and $\Delta\alpha_3$ (*dalp1, dalp2, dalp3)*, respectively.

Relative particle concentration: $\xi = \dfrac{4\pi a_v^3}{3}\eta$, where a_v is the radius of the volume-equivalent sphere of a scatterer in the medium.

Output file: the name of the file containing the result of computations of the $\mathbf{R}^{(C)}$ matrix elements.

Input file of the single-scattering matrix expansion coefficients: the name of the file containing the characteristics of a volume elements of the medium. This file should be prepared beforehand. Its structure is described below.

Input file of the incoherent reflection matrix: the name of the file containing the $\mathbf{R}^{(L)}$ matrix elements obtained in calculations of the incoherent component of the reflection matrix. As the previous file, it should be prepared beforehand. Its structure is also described below.

The first line of the *Input file of the single-scattering matrix expansion coefficients* should contain the values of the following characteristics of a volume element of the medium: the single scattering albedo $\tilde{\omega} = Q_{sca}/Q_{ext}$ (where Q_{sca} and Q_{ext} are the scattering and extinction efficiency factors, respectively), the maximal value of the index *L*, Q_{sca}, the asymmetry parameter

$<\cos\vartheta>$, and the equivalent size parameter $x_v = k_0 a_v$. The efficiency factors are defined with respect to the volume-mean radius a_v of the volume element. The other lines contain the expansion coefficients of the normalized scattering matrix of the volume element in generalized spherical functions in the LP basis. The elements should be in the following succession in the line: α_1^s, $\alpha_2^s+\alpha_3^s$, $\alpha_2^s-\alpha_3^s$, α_4^s, β_1^s, and β_2^s, where α_1^s, α_2^s, α_3^s, α_4^s, β_1^s, and β_2^s are the expansion coefficients of the scattering matrix elements $a_{11}(\vartheta)$, $a_{22}(\vartheta)$, $a_{33}(\vartheta)$, $a_{44}(\vartheta)$, $a_{21}(\vartheta)$, and $a_{34}(\vartheta)$, respectively; and $0 \le s \le \max(L)$. In this file, the scattering matrix in the LP representation is expanded in the basis of the generalized spherical functions, since the latter is traditional in the theory of light scattering by small particles and widely used in the radiative transfer theory [10, 13].

The $\mathbf{R}^{(L)}$ matrix elements should be in the *Input file of the incoherent reflection matrix*; the sequence of the quantities in each of the lines in this file is: ϑ ($90^0 \le \vartheta \le 180^0$), $R_{11}^{(L)}$, $R_{12}^{(L)}$, $R_{21}^{(L)}$, $R_{22}^{(L)}$, $R_{33}^{(L)}$, and $R_{44}^{(L)}$. The scattering angle step in calculations of this matrix should be small enough near the backscattering direction for further interpolating to those specified for the $\mathbf{R}^{(C)}$ matrix. The last line should contain the matrix elements for $\vartheta = 180^0$. The values from this line are used to solve equation (16) and find the coefficient σ. The $\mathbf{R}^{(L)}$ matrix may be computed with any code available. For example, the code described in a paper [16] produces the diffuse matrix elements in the format applicable to the present procedure.

To find the reflection matrix $\mathbf{R}$ (Eq. (11)), the $\mathbf{R}^{(L)}$ matrix elements are interpolated according to the phase angle values, which were specified in calculations of the coherent component $\mathbf{R}^{(C)}$ in the *data_inp.txt* file. The obtained elements of the $\mathbf{R}$ matrix are written to the file called REFL_*Output file.txt*. Its first line contains the heading; and x_v, $1/(2k_0 l_{tr})$, σ, ξ, and $\mathrm{Im}(m_{eff})$ are in the second line. Here, $l_{tr} = \mathrm{Im}(m_{eff})(1-<\cos\vartheta>)$ is the transport free-path. The third line contains the diagonal elements of the incoherent matrix $\mathbf{R}^{(L)}$ taken from the *Input file of the incoherent reflection matrix,* while the same elements, but calculated from the elements of the coherent matrix $\mathbf{R}^{(C)}$ with the equations connecting the coherent and incoherent components at $\alpha = 0$ ($\vartheta = \pi$), are in the fourth line. Since the latter are determined for the obtained value of σ, they show how correct the approximation works for these matrix elements and the specified parameters of the medium. In the fifth line, there are values of the left and right parts of Eq. (16), respectively, under the obtained σ. The other lines contain the two elements of the reflection matrix $\mathbf{R}$ --- $R_{11}(\alpha)$ and $R_{21}(\alpha)$---and the derived quantities: the normalized first element $R_{11}(\alpha)/R_{11}(0)$, the linear polarization degree $-R_{21}(\alpha)/R_{11}(\alpha)\times 100$, and the enhancement factor $R_{11}(\alpha)/R_{11}^{(L)}(\alpha)$.

5. APPLICATION EXAMPLES

In many applications, for example, when analyzing the remote sensing data, there is a need to simulate numerically the reflectance characteristics of discrete random media rather rapidly. In this paper, we propose the algorithm and codes for fast computations of the weak localization characteristics (the opposition effects). This algorithm uses the approximation developed in a paper [8] for sparse semi-infinite discrete random medium under the normal radiation incidence onto the medium boundary. The algorithm is based on introducing the recurrent formula (A.4) into calculations of the matrix of system (3), which substantially diminishes the processing time as compared to that of direct computations of the matrix elements. For example, to calculate the coherent reflection matrix for the medium containing randomly oriented bispheres with a size parameter of monomers $x_0 = 2$ and the refractive index $\tilde{m} = 1.5 + \mathrm{i}0.1$, the use of formula (A.4) requires the processing time an order of magnitude smaller than the direct computations. For larger scatterers, the time saving may be much more effective.

The results of computations with the described codes are presented in Fig. 2. The enhancement factor and the linear polarization degree for the above described bispheres and Chebyshev polydisperse particles are shown versus the phase angle in the backscattering domain. The parameters of Chebyshev particles are: the effective size parameter corresponding to the equal-volume sphere $x_v = 2.5$, the effective variance of the power-law distribution $v_{eff} = 0.05$ and $\tilde{m} = 1.51 + \mathrm{i}0.0$; their shape is specified by $r(\vartheta) = r_0(1 - 0.1T_8(\vartheta))$, where r_0 is the radius of the unperturbed sphere and $T_8(\vartheta)$ is the Chebyshev polynomial. The relative particle concentration ξ is assumed to be 0.01 for the both media. It is worth noting that, for bispheres, this value corresponds to the concentration of particles rather than bispheres (see [7]).

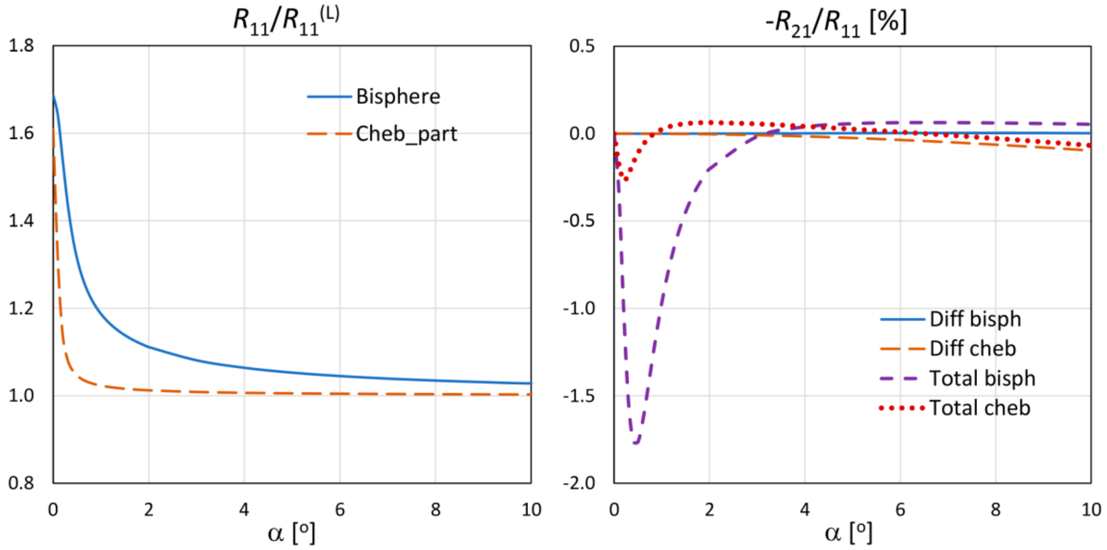

Fig. 2. The enhancement factor (left) and the linear polarization degree (right) for bispheres and Chebyshev particles (see the text for details) in dependence on the phase angle in the backscattering domain. The quantities obtained from the total reflection matrix elements and the diffuse (incoherent) contribution are shown for polarization.

6. CONCLUDING REMARKS

Let us analyze the inaccuracy of the proposed method. First, errors may be caused by the usage of the approximation, where the double-scattering contribution is rigorously accounted for while the higher-order contributions to the radiation coming from below is calculated under the assumption of the exponential decrease of the intensity with depth [8]. With increasing the absorption in the medium, the assumption on the radiation exponentially weakening with depth becomes closer to the truth and the calculations of the reflection matrix yield more accurate results. The second source of errors is the use of approximation (10). The weaker the dependence of the scattering matrix of a volume element on the scattering angle near $\vartheta = \pi$, the smaller the inaccuracy caused by this approximation.

Of course, these are only inaccuracies introduced by the approximations assumed when solving the initial equation [8, 11, 12]. The latter, as the other equations describing the weak localization effect (see, e.g., [3] and references therein), was itself obtained under some suppositions. The limits of applicability of these assumptions could be estimated from the reflectance characteristics of discrete medium samples measured in the laboratory with completely controlled samples. In particular, this would make it possible to define the limits of applicability of the far-zone approximation, within which the waves propagating between the scatterers are assumed to be spherical. Consequently, this would allow us to estimate the highest concentration of scatterers in the medium appropriate for this approximation. The results of numerical solution of the vector radiative transfer equation were compared to the reflection matrix elements of suspensions of submicron latex particles in water measured in the laboratory, and it was found that the far-zone approximation can be applied to the media with packing densities smaller than ~5% [17]. This suggests that, for the coherent component of the reflected radiation, this approximation is also valid up to the same packing densities of particles. Unfortunately, measurements with completely controlled samples are still lacking, which does not allow us to determine more precisely the applicability limits of the far-field approximation and other assumptions of the described algorithm. However, at a qualitative level, it was successfully tested by the laboratory data for some samples [7, 18]. Consequently, the proposed algorithm may be used at least to estimate the parameters of the media when interpreting the remote sensing data, particularly, the data of ground-based observations of the Solar system bodies [18, 19].

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APPENDIX. RECURRENT RELATION: DERIVATION AND APPLICATION SCHEME

We derive here a recurrent relation, which is used to calculate the matrix elements, and describe a scheme of its application.

Let us consider the integral

$$V_{LMN}^{l+1,mn} = \int f(\omega) d_{MN}^{L}(\omega) d_{mn}^{l+1}(\omega) \sin\omega d\omega , \tag{A.1}$$

where $f(\omega)$ is an arbitrary integrable function. The following recurrent relation [9] will also be used

$$\cos\omega d_{mn}^{l}(\omega) = a_{lmn} d_{mn}^{l+1}(\omega) + c_{lmn} d_{mn}^{l}(\omega) + b_{lmn} d_{mn}^{l-1}(\omega) , \tag{A.2}$$

where

$$a_{lmn} = \frac{\sqrt{\left[(l+1)^2 - m^2\right]\left[(l+1)^2 - n^2\right]}}{(l+1)(2l+1)}, \quad b_{lmn} = \frac{\sqrt{\left(l^2 - m^2\right)\left(l^2 - n^2\right)}}{l(2l+1)}, \quad c_{lmn} = \frac{nm}{l(l+1)} \quad . \text{ (A.3)}$$

Let us multiply (A.1) to a_{lmn} and apply relationship (A.2) to (A.1), namely, sequentially to $a_{lmn} d^{\,l+1}_{mn}(\omega)$ and $\cos\omega d^{\,L}_{MN}(\omega)$. This yields the following recurrent relation for the integral V^{lmn}_{LMN}:

$$a_{lmn} V^{l+1,mn}_{LMN} + (c_{lmn} - c_{LMN}) V^{lmn}_{LMN} + b_{lmn} V^{l-1,mn}_{LMN} - a_{LMN} V^{lmn}_{L+1,MN} - b_{LMN} V^{lmn}_{L-1,MN} = 0 . \quad \text{(A.4)}$$

Relationship (A.4) is multipurpose. It can be applied to some functions that are frequent in the light scattering theory. For example, it can be applied to the product of two Clebsch–Gordan coefficients. To demonstrate this, we will use the expansion of the product of two Wigner functions [9]

$$d^{\,L}_{MN}(\omega) d^{\,l}_{mn}(\omega) = \sum_{k=|L-l|}^{L+l} C^{kM_1}_{LMlm} C^{kN_1}_{LNln} d^{\,k}_{M_1N_1}(\omega) . \qquad \text{(A.5)}$$

Here, $M_1 = M + m$ and $N_1 = N + n$. Let us apply formula (A.2) to the left part of (A.5) in the same way as in deriving relationship (A.4) above. Expand the obtained products of two Wigner functions into series of type (A.5) and introduce

$$W^{lmn}_{LMN}(k) = C^{kM_1}_{LMlm} C^{kN_1}_{LNln} . \qquad \text{(A.6)}$$

Taking into account the orthogonal property of the Wigner functions [9], we obtain a recurrent relation analogous to (A.4) for the coefficients $W^{lmn}_{LMN}(k)$ under fixed *k*. By multiplying relation (A.6) to an arbitrary function $f^{\,k}_{M_1N_1}$, which is independent of the indices *L* and *l*, and summing up the product over all possible k, we obtain a recurrent formula analogous to (A.4) for the coefficients F^{lmn}_{LMN}:

$$F^{lmn}_{LMN} = \sum_{k=|L-l|}^{L+l} C^{kM_1}_{LMlm} C^{kN_1}_{LNln} f^{\,k}_{M_1N_1} . \qquad \text{(A.7)}$$

In particular, the recurrent relation (A.4) is valid for the coefficients of the translational addition theorem for spherical vector wave functions (see, e.g., [2, 3, 19, 20])

$$H_{LMlm} = \sum_{k=|L-l|}^{L+l} C^{kM_1}_{LMl-m} C^{k0_1}_{Lql-q} z_k(k_0 r) D^{k}_{M_1 0}(\varphi, \vartheta, 0) . \qquad \text{(A.8)}$$

Here, $q = \pm 1$, $z_k(k_0 r)$ is the Bessel or Hankel spherical function, $D^{k}_{M_1 0}(\varphi, \vartheta, 0)$ is the Wigner function [9]. The recurrent relation (A.4) for the coefficients of the translational addition theorem for spherical vector wave functions was obtained earlier in [20] and used in [21] to calculate the scattering matrix of aggregates.

Relationship (A.4) may be used according to the following procedure. As it appears from the properties of the Wigner functions [9], the coefficients V^{lmn}_{LMN} (as well as coefficients

(A.6)–(A.8)) are zero for $L < \max(|M|,|N|)$ or $l < \max(|m|,|n|)$. Assume that, in the matrix of the coefficients V_{LMN}^{lmn}, the elements of the first nonzero line $(l = \max(|m|,|n|))$ and the last column ($L = L_{mzx}$) are known. Then the elements of the second line are calculated with formula (A.4), and all coefficients $V_{LMN}^{l-1mn} = 0$ for this line. Since the matrix of the coefficients V_{LMN}^{lmn} is symmetric relative to the main diagonal, it will suffice to calculate only the elements of the upper triangular matrix. Analogously, only the elements of the lower triangular matrix of the coefficients V_{LMN}^{lmn} may be calculated, if the elements of the first nonzero column of the matrix $L = \max(|M|,|N|)$ and the last line are precalculated.

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FIGURE CAPTIONS

Fig. 1. Geometry of scattering by a semi-infinite particulate medium. The incident light propagates normally to the boundary of the medium ($z_0 = 0$). The incidence direction is indicated by the wave vector $\mathbf{k_0}$, α and ϑ are the phase and scattering angles, respectively, "P" shows the direction of observations, and letters "A" and "B" denote the scattered radiation that comes to some point at a depth z from above and below, respectively.

Fig. 2. The enhancement factor (left) and the linear polarization degree (right) for bispheres and Chebyshev particles (see the text for details) in dependence on the phase angle in the backscattering domain. The quantities obtained from the total reflection matrix elements and the diffuse (incoherent) contribution are shown for polarization.