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Zero forcing sets and the minimum rank of graphs ☆

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Abstract

The minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose *ij*th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. This paper introduces a new graph parameter, $Z(G)$, that is the minimum size of a zero forcing set of vertices and uses it to bound the minimum rank for numerous families of graphs, often enabling computation of the minimum rank.

AMS classification: 05C50; 15A18; 15A03

Keywords: Minimum rank; Rank; Graph; Symmetric matrix; Matrix

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1. Introduction

A *graph* is a pair $G = (V, E)$, where V is the set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and E is the set of edges (an edge is a two-element subset of vertices); what we call a graph is sometimes called a simple undirected graph. In this paper each graph is finite and has nonempty vertex set. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

If F is a field, the set of symmetric matrices over F will be denoted by $S_n(F)$. For such a matrix, the *graph* of A, denoted $\mathscr{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\}: a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$. Note that the diagonal of A is ignored in determining $\mathscr{G}(A)$.

The *set of symmetric matrices* of graph G (over $\mathbb{R}$) is defined to be

$$\mathscr{S}(G) = \{A \in S_n(\mathbb{R}): \mathscr{G}(A) = G\}.$$

More generally, the set of symmetric matrices over F of G is $\mathscr{S}(F, G) = \{A \in S_n(F): \mathscr{G}(A) = G\}$.

The *minimum rank* of a graph G (over $\mathbb{R}$) is defined to be

$$\operatorname{mr}(G) = \min\{\operatorname{rank}(A): A \in \mathscr{S}(G)\}.$$

More generally, the minimum rank over F is $\operatorname{mr}^F(G) = \min\{\operatorname{rank}(A): A \in \mathscr{S}(F, G)\}$. Over $\mathbb{R}$, the *positive semidefinite minimum rank* of G is defined to be

$$\operatorname{mr}_+(G) = \min\{\operatorname{rank}(A): A \in \mathscr{S}(G), A \text{ positive semidefinite}\}.$$

Clearly

$$\operatorname{mr}(G) \leqslant \operatorname{mr}_+(G).$$

For $A \in \mathbb{R}^{n\times n}$, the *corank* of A is the nullity of A and the *maximum nullity* (or *maximum corank*) of a graph G (over $\mathbb{R}$) is defined to be

$$M(G) = \max\{\operatorname{corank}(A): A \in \mathscr{S}(G)\}.$$

More generally, the maximum nullity over F is $M^F(G) = \max\{\operatorname{corank}(A): A \in \mathscr{S}(F, G)\}$. Clearly

$$\operatorname{mr}^F(G) + M^F(G) = |G|.$$

The *minimum rank problem* (of a graph) is to determine $\operatorname{mr}(G)$ (or $\operatorname{mr}^F(G)$) for any graph G. See [8] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. In Section 3 of this paper we establish the minimum rank/maximum nullity of several families of graphs; see Table 1 for a list. As far as we know all of these results are new with the exception of 3.17 which was established earlier by one of the coauthors of this paper, but had not been published. The information in this table is also available on-line in the form of a minimum rank graph catalog [1], and will be updated routinely. In Section 2, we discuss the use of zero forcing sets to bound $M(G)$ from above and introduce the graph parameter $Z(G)$. Section 4 contains a discussion of graphs for which $Z(G) = M(G)$ and an example where $Z(G) > M^F(G)$ for all F.

A *path* is a graph $P_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\}: i = 1, \ldots, n-1\}$. A *cycle* is a graph $C_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\}: i = 1, \ldots, n-1\} \cup \{\{v_n, v_1\}\}$. The *length* of a path or cycle is the number of edges. A *complete graph* is a graph $K_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_j\}: 1 \leqslant i < j \leqslant n\}$. A graph (V, E) is *bipartite* if the vertex set V can be partitioned into two nonempty subsets U, W, such that every edge of E has one endpoint in U and one in W. A *complete bipartite graph* is a bipartite graph $K_{p,q} = (U \cup W, E)$ such that $|U| = p, |W| = q$ and $E = \{\{u, w\}: u \in U, w \in W\}$.

Table 1
Summary of minimum rank and maximum nullity results established in this paper

Result #	G	Order	$M(G)$	$\mathrm{mr}(G)$
3.1	Q_n (hypercube)	2^n	2^{n-1}	2^{n-1}
3.2	T_n (supertriangle)	$\frac{1}{2}n(n+1)$	n	$\frac{1}{2}n(n-1)$
3.3	$K_s \,\square\, P_t$	st	s	$s(t-1)$
3.7	$P_s \,\square\, P_t$	st	$\min\{s,t\}$	$st - \min\{s,t\}$
3.13	$P_s \boxtimes P_t$	st	$s+t-1$	$(s-1)(t-1)$
3.8	$C_s \,\square\, P_t$	st	$\min\{s,2t\}$	$st - \min\{s,2t\}$
3.9	Möbius ladder	$2n$	4	$2n-4$
3.11	$K_s \,\square\, K_t$	st	$st-s-t+2$	$s+t-2$
3.12	$C_s \,\square\, K_t, s \geqslant 4$	st	$2t$	$(s-2)t$
3.14	$K_t \circ K_s, t \geqslant 2$	$st+t$	$st-1$	$t+1$
3.15	$\overline{C_n}, n \geqslant 5$	n	$n-3$	3
3.17	$\overline{T}$, T a tree (with $\|T\|=n$), $n \geqslant 4, T \neq K_{1,n-1}$	n	$n-3$	3
3.18	$L(K_n)$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}(n^2-3n+4)$	$n-2$
3.20 3.21	$L(G)$ (with $\|G\|=n$) if G has a Hamiltonian path or contains $K_{k,n-k}$ as a subgraph $(1<k<n-1)$			$n-2$
3.24	$L(T)$, T a tree and ℓ= # pendent vertices of T	$\|T\|-1$	$\ell-1$	$\|T\|-\ell$
3.26	Petersen	10	5	5
3.28	4-Antiprism	8	4	4

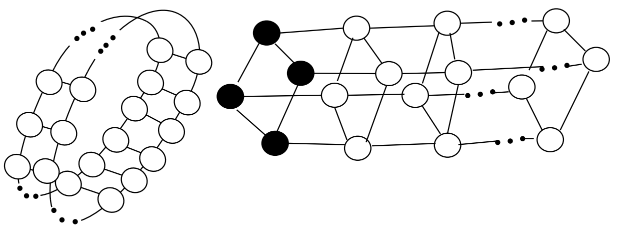

Fig. 1. $C_s \,\square\, P_2$ and $C_4 \,\square\, P_t$.

The following graph operations are used to construct families of graphs:

- The *complement* of a graph $G=(V,E)$ is the graph $\overline{G}=(V,\overline{E})$, where $\overline{E}$ consists of all two-element sets from V that are not in E.
- The *line graph* of a graph $G=(V,E)$, denoted $L(G)$, is the graph having vertex set E, with two vertices in $L(G)$ adjacent if and only if the corresponding edges share an endpoint in G. Since we require a graph to have a nonempty set of vertices, the line graph $L(G)$ is defined only for a graph G that has at least one edge. See Fig. 7 in Section 3 for a picture of a line graph of a tree.
- The *Cartesian product* of two graphs G and H, denoted $G \,\square\, H$, is the graph with vertex set $V(G)\times V(H)$ such that (u,v) is adjacent to (u',v') if and only if (1) $u=u'$ and $vv' \in E(H)$, or (2) $v=v'$ and $uu' \in E(G)$. In $G \,\square\, P_t$ with the vertex v_0 being an endpoint of the path P_t, the subgraph induced by the vertices $\{(u,v_0): u \in V(G)\}$ is called an *endpoint copy of* G. Fig. 1 shows examples of $C_s \,\square\, P_2$ and $C_4 \square P_t$; the latter has an endpoint copy of C_4 colored black.

- The *strong product* of two graphs G and H, denoted $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $uu' \in E(G)$ and $vv' \in E(H)$, or (2) $u = u'$ and $vv' \in E(H)$, or (3) $v = v'$ and $uu' \in E(G)$. See Fig. 5 in Section 2 for a picture of $P_s \boxtimes P_t$.
- The *corona* of G with H, denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H, and joining all the vertices in the ith copy of H to the ith vertex of G. See Fig. 4 in Section 2 for a picture of $C_5 \circ K_2$. Note that $G \circ H$ and $H \circ G$ are usually not isomorphic (in fact, if $|G| \neq |H|$, then $|G \circ H| \neq |H \circ G|$).

The nth *hypercube*, Q_n, is defined inductively by $Q_1 = K_2$ and $Q_{n+1} = Q_n \,\square\, K_2$. Clearly $|Q_n| = 2^n$. The nth *supertriangle*, T_n, is an equilateral triangular grid with n vertices on each side (see Fig. 4 in Section 2 for a picture). The order of T_n is $\frac{1}{2}n(n+1)$. The *Möbius ladder* is obtained from $C_n \,\square\, P_2$ by replacing one pair of parallel cycle edges with a crossed pair (see Fig. 6 in Section 3).

We need a few additional definitions. A graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$. The subgraph $G[R]$ of $G = (V, E)$ *induced* by $R \subseteq V$ is the subgraph with vertex set R and edge set $\{\{i, j\} \in E \mid i, j \in R\}$. The result $G[V \setminus \{v\}]$ of deleting a vertex v is also denoted by $G - v$.

An induced subgraph G' of a graph G is a *clique* if G' has an edge between every pair of vertices of G' (i.e., G' is isomorphic to $K_{|G'|}$). A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a *clique covering* of G. The *clique covering number* of G, denoted by $\mathrm{cc}(G)$, is the smallest number of cliques in a clique covering of G. We have:

Observation 1.1 [6,8]. Since a matrix obtained from a clique covering as a sum of rank 1 matrices is positive semidefinite

$$\mathrm{mr}(G) \leqslant \mathrm{mr}_+(G) \leqslant \mathrm{cc}(G).$$

If F is an infinite field then $\mathrm{mr}^F(G) \leqslant \mathrm{cc}(G)$, and this is true for every field if every pair of distinct cliques in a minimal clique covering intersect in at most one vertex.

Furthermore, it is known [6] that if G is chordal, then $\mathrm{mr}_+(G) = \mathrm{cc}(G)$, whereas $\mathrm{mr}(G)$ is often less than $\mathrm{cc}(G)$ for chordal graphs.

The matrix $\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = [g_{ij}] \in \mathbb{R}^{n\times n}$ defined by $g_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$, $i, j \in \{1, 2, \ldots, n\}$ is called the *Gram matrix* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$. Note that any Gram matrix is positive semidefinite.

The Colin de Verdière-type parameter ξ can be useful in computing minimum rank or maximum nullity (over the real numbers). A symmetric real matrix M is said to satisfy the *Strong Arnold Hypothesis* provided there does not exist a nonzero symmetric matrix X satisfying:

- $MX = 0$.
- $M \circ X = 0$.
- $I \circ X = 0$,

where $\circ$ denotes the Hadamard (entrywise) product and I is the identity matrix. For a graph G, $\xi(G)$ is the maximum nullity among matrices $A \in \mathscr{S}(G)$ that satisfy the Strong Arnold Hypothesis. It follows that $\xi(G) \leqslant M(G)$.

A *contraction* of G is obtained by identifying two adjacent vertices of G, and suppressing any loops or multiple edges that arise in this process. A *minor* of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges. A graph parameter ζ is *minor monotone* if for any minor G' of G, $\zeta(G') \leqslant \zeta(G)$. The parameter ξ was introduced in [3], where it was shown that ξ is minor monotone. It was also established that $\xi(K_n) = n - 1$ and $\xi(K_{p,q}) = p + 1$ (under the assumptions that $p \leqslant q, 3 \leqslant q$).

The main goal of this paper is the calculation of $M(G)$ for many families of graphs. Prior to this work $M(G)$ was known for a very limited number of graphs on an arbitrary number of vertices. Our technique is to establish tight upper and lower bounds on $M(G)$.

In Section 2, we introduce the new graph parameter $Z(G)$, the minimum size of a zero forcing set. We show that $Z(G)$ is an upper bound for $M^F(G)$ for any field F. Somewhat surprisingly, $M(G) = Z(G)$ for most graphs for which $M(G)$ is known, for example for all graphs with fewer than seven vertices. Moreover, for the families of graphs in Table 1, $Z(G)$ is easily found.

In Section 3, we establish tight lower bounds for $M(G)$. Our main tools are explicit constructions of matrices A in $\mathscr{S}(G)$ with $\operatorname{corank}(A) = M(G)$, the lower bound $\xi(G) \leqslant M(G)$ coupled with minor monotonicity, and the lower bound obtained via Observation 1.1. The bound $\xi(G) \leqslant M(G)$ is for the real field only, and some of the other techniques used rely on properties of the real numbers. Consequently, the results in Table 1 are stated just for the real field, although a few of the actual results are established in more general settings.

In Section 4, we give an example of a graph for which $M^F(G) < Z(G)$ for every field F, introduce the parameter $\operatorname{mz}(G) = |G| - Z(G)$ and make a few observations that are more conveniently expressed in terms of $\operatorname{mz}(G)$, and establish $Z(G) = M(G)$ for a few additional graphs.

In Section 5, we give some extensions to combinatorially symmetric matrices, and in Section 6 we make concluding remarks.

2. Zero forcing sets and the graph parameter $Z(G)$

What we now call zero forcing sets have been used previously on an ad hoc basis to bound $M(G)$ from above (see for example [11]). Here we discuss the use of this technique, including exhibiting zero forcing sets for several families of graphs, and introduce the graph parameter $Z(G)$ as the minimum size of a zero forcing set.

Definition 2.1

- *Color-change rule*:
  If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.
- Given a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more changes are possible.
- A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

For example, an endpoint of a path is a zero forcing set for the path. In a cycle, any set of two adjacent vertices is a zero forcing set. More examples of zero forcing sets are given below.

The derived coloring (of a specific coloring) is in fact unique, since any vertex that turns black under one sequence of applications of the color-change rule can always be turned black regardless of the order of color changes. This can be proved by an induction on the number of color changes necessary to turn the vertex black, but since for our purposes the uniqueness of the derived coloring is not necessary, we do not supply the details.

The underlying idea is that a black vertex is associated with a coordinate in a vector that is required to be zero, while a white vertex indicates a coordinate that can be either zero or nonzero. Changing a vertex from white to black is essentially noting that the corresponding coordinate is forced to be zero if the vector is in the kernel of a matrix in $\mathscr{S}(G)$ and all black vertices indicate coordinates assumed to be or previously forced to be 0 (cf. Proposition 2.3). Hence the use of the term "zero forcing set".

The *support* of a vector $\mathbf{x} = [x_i]$, denoted $\operatorname{supp}(\mathbf{x})$, is the set of indices i such that $x_i \neq 0$.

Proposition 2.2. *If F is a field, $A \in F^{n\times n}$, and $\operatorname{corank}(A) > k$, then there is a nonzero vector $\mathbf{x} \in \ker(A)$ vanishing at any k specified positions. In other words, if W is a set of k indices, then there is a nonzero vector $\mathbf{x} \in \ker(A)$ such that $\operatorname{supp}(\mathbf{x}) \cap W = \emptyset$.*

Proof. Let $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and let

$$V_k = \{\mathbf{x} \in F^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}.$$

Then $\dim V_k = n - k$. Let $N = \ker(A)$. Then

$$\dim(V_k \cap N) = \dim V_k + \dim N - \dim(V_k + N) > n - k + k - n = 0,$$

since $\dim N = \operatorname{corank}(A) > k$ and $\dim(V_k + N) \leqslant \dim(F^n) = n$. Therefore, $V_k \cap N \neq \{0\}$. □

Let G be a graph on n vertices, and let u be a vertex of G. Write $v \sim u$ if v is adjacent to u, and $v \nsim u$ if $v \neq u$ and v is not adjacent to u. Then if $A \in \mathscr{S}(F, G)$ and $\mathbf{x} \in F^n$

$$(A\mathbf{x})_u = a_{uu}x_u + \sum_{v\sim u} a_{uv}x_v + \sum_{v\nsim u} a_{uv}x_v = a_{uu}x_u + \sum_{v\sim u} a_{uv}x_v.$$

Proposition 2.3. *Let Z be a zero forcing set of $G = (V, E)$ and $A \in \mathscr{S}(F, G)$. If $\mathbf{x} \in \ker(A)$ and $\operatorname{supp}(\mathbf{x}) \cap Z = \emptyset$, then $\mathbf{x} = 0$.*

Proof. If $Z = V$, there is nothing to do, so suppose $Z \neq V$. Since Z is a zero forcing set we must be able to perform a color change. That is, there exists a vertex u colored black (x_u is required to be 0) with exactly one neighbor v colored white (so x_v is not yet required to be 0). Upon examination, the equation $(A\mathbf{x})_u = 0$ reduces to $a_{uv}x_v = 0$, which implies that $x_v = 0$. Similarly each color change corresponds to requiring another entry in $\mathbf{x}$ to be zero. Thus $\mathbf{x} = 0$. □

Proposition 2.4. *Let $G = (V, E)$ be a graph and let $Z \subseteq V$ be a zero forcing set. Then $M^F(G) \leqslant |Z|$, and thus $M^F(G) \leqslant Z(G)$ for any field F.*

Proof. Assume $M^F(G) > |Z|$, and let $A \in \mathscr{S}(G)$ with $\operatorname{corank}(A) > |Z|$. By Proposition 2.2, there is a nonzero vector $\mathbf{x} \in \ker(A)$ that vanishes on all vertices in Z. By Proposition 2.3, $\mathbf{x} = 0$, a contradiction. □

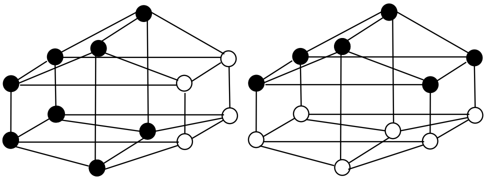

Fig. 2. Two types of zero forcing sets shown on $K_3 \,\Box\, C_4$.

The next proposition provides an upper bound for the parameter Z for any Cartesian product. Fig. 2 illustrates Proposition 2.5 for $K_3 \,\Box\, C_4$.

Proposition 2.5. *For any graphs* G, H, $Z(G \,\Box\, H) \leqslant \min\{Z(G)|H|, Z(H)|G|\}$.

Proof. The set of vertices associated with (the same) zero forcing set in each copy of G is a zero forcing set for $G \,\Box\, H$, so $Z(G \,\Box\, H) \leqslant Z(G)|H|$. Similarly, $Z(G \,\Box\, H) \leqslant Z(H)|G|$. $\square$

Corollary 2.6. $Z(G \,\Box\, P_t) \leqslant \min\{|G|, Z(G)t\}$.

Corollary 2.7. $Z(Q_n) \leqslant 2^{n-1}$.

Proof. This follows from the fact that $Q_n = Q_{n-1} \,\Box\, K_2$ and Corollary 2.6. $\square$

Corollary 2.8. $Z(G \,\Box\, C_t) \leqslant \min\{Z(G)t, 2|G|\}$.

Corollary 2.9. $Z(G \,\Box\, K_t) \leqslant \min\{Z(G)t, |G|(t-1)\}$.

In the case of $K_s \,\Box\, K_t$ there is a better bound than that in Corollary 2.9.

Proposition 2.10. $Z(K_s \,\Box\, K_t) \leqslant st - s - t + 2$.

Proof. The set of all vertices of one copy of K_s and zero forcing sets for all but one of the remaining copies of K_s form a zero forcing set of size $s + (s-1)(t-2) = st - s - t + 2$ for $K_s \,\Box\, K_t$. This is illustrated in Fig. 3. $\square$

Observation 2.11. The n vertices on one edge of T_n are a zero forcing set for T_n and thus $M^F(T_n) \leqslant Z(T_n) \leqslant n$ for any field F. See Fig. 4.

Proposition 2.12. $Z(G \circ H) \leqslant Z(H)|G| + Z(G)|H| - Z(G)Z(H)$. *In particular, for* $t \geqslant 2$, $Z(K_t \circ K_s) \leqslant st - 1$.

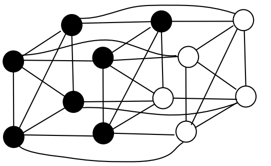

Fig. 3. Zero forcing set for $K_4 \,\Box\, K_3$.

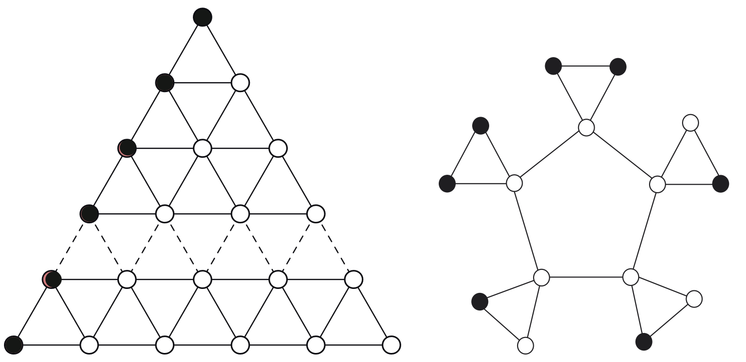

Fig. 4. Zero forcing sets for supertriangle T_n and corona $C_5 \circ K_2$.

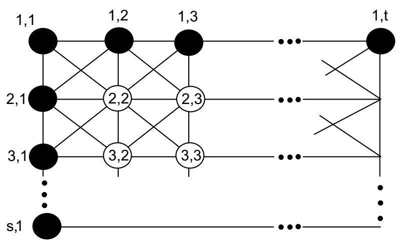

Fig. 5. Zero forcing set for $P_s \boxtimes P_t$.

Proof. Consider the corona $G \circ H$. Choose a minimal zero forcing set Z_G for G. Construct a zero forcing set for $G \circ H$ (that consists entirely of vertices of copies of H) as follows: Let Z consist of all the vertices in the copies of H associated with the vertices in Z_G, and for each of the $|G| - Z(G)$ remaining copies of H, choose a zero forcing set of size $Z(H)$. This is illustrated in Fig. 4, where $G = C_5$, $Z(C_5) = 2$, $H = K_2$, and $Z(H) = 1$. Clearly the order of Z is $Z(G)|H| + (|G| - Z(G))Z(H)$. The copies of H that are all black will change the vertices in Z_G black. This zero forcing set then turns at least one more vertex v in G black. Then all the vertices of the copy of H adjacent to v can be turned black by the zero forcing set in this copy of H. Repeat this process as needed (i.e., change a vertex of G to black, then change its copy of H to black, etc.). Thus

$$Z(G \circ H) \leqslant Z(H)|G| + Z(G)|H| - Z(G)Z(H).$$

The statement $Z(K_t \circ K_s) \leqslant st - 1$ is immediate for $t \geqslant 2$ unless $s = 1$, in which case the bound $Z(K_1)|K_t| + Z(K_t)|K_1| - Z(K_t)Z(K_1) = 1(t-1) + t(1) - (t-1)1 = t$ rather than $t - 1$. In this case, a zero forcing set can be obtained by using all but one of the copies of K_1, so in fact, $Z(K_t \circ K_1) \leqslant t - 1$. $\square$

Observation 2.13. The graph $P_s \boxtimes P_t$ is shown in Fig. 5 and $Z = \{(1, j){:}\ 1 \leqslant j \leqslant t\} \cup \{(i, 1){:}\ 1 \leqslant i \leqslant s\}$ is a zero forcing set. Thus $Z(P_s \boxtimes P_t) \leqslant s + t - 1$.

3. Minimum rank and maximum nullity of graphs

In this section, we determine the minimum rank of several families of graphs and several regular graphs.

Theorem 3.1. *For the hypercube, $M(Q_n) = 2^{n-1} = Z(Q_n)$. This is the value of maximum nullity over any field of characteristic not 2 that contains $\sqrt{2}$ or any field of characteristic 2.*

Proof. Let F be a field that contains $\sqrt{2}$. We recursively define two sequences of matrices. Let

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Given L_{n-1}, define

$$H_n = \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix} \quad \text{and} \quad L_n = \frac{1}{\sqrt{2}} \begin{bmatrix} L_{n-1} & I \\ I & -L_{n-1} \end{bmatrix}.$$

Then $\mathscr{G}(H_n) = Q_n$. By induction, $L_n^2 = I$. Since

$$\begin{bmatrix} I & 0 \\ -L_{n-1} & I \end{bmatrix} \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & I \\ 0 & 0 \end{bmatrix},$$

$\operatorname{rank}(H_n) = 2^{n-1}$.

For a field of characteristic 2, we recursively define one sequence of matrices. Let $H_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Given H_{n-1}, define

$$H_n = \begin{bmatrix} H_{n-1} + I & I \\ I & H_{n-1} + I \end{bmatrix}.$$

Then $\mathscr{G}(H_n) = Q_n$. By induction, $H_n^2 = 0$. Since

$$\begin{bmatrix} I & 0 \\ H_{n-1} + I & I \end{bmatrix} \begin{bmatrix} H_{n-1} + I & I \\ I & H_{n-1} + I \end{bmatrix} = \begin{bmatrix} H_{n-1} + I & I \\ 0 & 0 \end{bmatrix},$$

$\operatorname{rank}(H_n) = 2^{n-1}$.

Therefore, in either case, $\operatorname{mr}^F(Q_n) \leqslant 2^{n-1}$, and thus $M^F(Q_n) \geqslant 2^{n-1}$. Then

$$2^{n-1} \leqslant M^F(Q_n) \leqslant Z(Q_n) \leqslant 2^{n-1},$$

by Corollary 2.7 (and Proposition 2.4). □

Proposition 3.2. *For the supertriangle T_n, $M(T_n) = n = Z(T_n)$ and $\operatorname{mr}(T_n) = \frac{1}{2}n(n-1) = \operatorname{cc}(T_n)$.*

Proof. By Observation 2.11, $M(T_n) \leqslant Z(T_n) \leqslant n$. We can cover T_n by $\frac{1}{2}n(n-1)$ copies of K_3, so by Observation 1.1, $\operatorname{mr}(T_n) \leqslant \operatorname{cc}(T_n) \leqslant \frac{1}{2}n(n-1)$. Since $M(T_n) + \operatorname{mr}(T_n) = \frac{1}{2}n(n+1)$, all inequalities are equalities. □

Note that in the proof of Proposition 3.2 we have also shown that $\operatorname{mr}(T_n) = \operatorname{mr}_+(T_n)$.

3.1. The minimum rank of products

Proposition 3.3. $M(K_s \,\square\, P_t) = s = Z(K_s \,\square\, P_t)$.

Proof. From Corollary 2.6, $M(K_s \,\square\, P_t) \leqslant Z(K_s \square P_t) \leqslant s$. Note that K_{s+1} is a minor of $K_s \,\square\, P_t$ (contract all vertices except the vertices of one endpoint copy of K_s into one vertex). Thus, $s = \xi(K_{s+1}) \leqslant \xi(K_s \,\square\, P_t) \leqslant M(K_s \,\square\, P_t)$. □

Proposition 3.3 need not be valid over the field $\mathbb{Z}_2$, as the next example shows.

Example 3.4. With appropriate ordering of the vertices, any matrix in $\mathscr{S}^{\mathbb{Z}_2}(K_3 \,\square\, K_2)$ is of the form $\begin{bmatrix} d_1 & 1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 0 & 1 & 0 \\ 1 & 1 & d_3 & 0 & 0 & 1 \\ 1 & 0 & 0 & d_4 & 1 & 1 \\ 0 & 1 & 0 & 1 & d_5 & 1 \\ 0 & 0 & 1 & 1 & 1 & d_6 \end{bmatrix}$ and computation using all 64 possible $(d_1, \ldots, d_6)$ shows the minimum rank is 4. This follows also from Theorem 32 in [4].

We will use a technique involving Kronecker products to construct matrices with the desired corank for several graphs (cf. [9, Section 9.7]). This technique is particularly well-suited to graphs that are formed from Cartesian products.

If A is an $s \times s$ real matrix and B is a $t \times t$ real matrix, then $A \otimes B$ is the $s \times s$ block matrix whose ijth block is the $t \times t$ matrix $a_{ij}B$. The following results are standard.

Observation 3.5. Let G be a graph on s vertices, let H be a graph on t vertices, let $A \in \mathscr{S}(G)$ and $B \in \mathscr{S}(H)$. Then $A \otimes I_t + I_s \otimes B \in \mathscr{S}(G \,\square\, H)$.

If $\mathbf{x}$ is an eigenvector of A for eigenvalue λ and $\mathbf{y}$ is an eigenvector of B for eigenvalue μ, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector $A \otimes I_t + I_s \otimes B$ for eigenvalue $\lambda + \mu$.

Theorem 3.6. *If* $|G| \leqslant t$, *then* $M(G \,\square\, P_t) = |G| = Z(G \,\square\, P_t)$.

Proof. Let $|G| = s$. From Corollary 2.6, $M(G \,\square\, P_t) \leqslant Z(G \,\square\, P_t) \leqslant s$.

Choose $A \in \mathscr{S}(G)$ with s distinct eigenvalues, denoted $\lambda_1, \ldots, \lambda_s$ with associated eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_s$ (such an A exists by Gershgorin's Theorem). Then there exists $B \in \mathscr{S}(P_t)$ having eigenvalues $-\lambda_1, \ldots, -\lambda_s, \mu_{s+1}, \ldots, \mu_t$ (see [10] and the references therein). Denote eigenvectors for these eigenvalues by $\mathbf{y}_1, \ldots, \mathbf{y}_t$. Then $A \otimes I_t + I_s \otimes B$ has at least s eigenvectors, namely $\mathbf{x}_i \otimes \mathbf{y}_i$, $i = 1, \ldots, s$, for eigenvalue $0 = \lambda_i + (-\lambda_i)$, so $M(G \,\square\, P_t) \geqslant s$. $\square$

Corollary 3.7. $M(P_s \,\square\, P_t) = \min\{s, t\} = Z(P_s \,\square\, P_t)$.

Theorem 3.8. $M(C_s \,\square\, P_t) = \min\{s, 2t\} = Z(C_s \,\square\, P_t)$.

Proof. That $M(C_s \,\square\, P_t) \leqslant Z(C_s \,\square\, P_t) \leqslant \min\{s, 2t\}$ follows from Corollary 2.6.

Let $k = \lceil \frac{s}{2} \rceil$. Let A be the matrix obtained from the adjacency matrix of C_s by changing the sign on two symmetrically placed ones. Then the (distinct) eigenvalues of A are $\lambda_i = 2\cos\frac{\pi(2i-1)}{s}$, $i = 1, \ldots, k$, each with multiplicity 2, except that if s is odd, $\lambda_k = -2$ has multiplicity 1. Since A is a real symmetric matrix, each eigenvalue of multiplicity 2 has 2 independent eigenvectors; for eigenvalue λ_i, denote these vectors by $\mathbf{x}_i, \mathbf{z}_i$ (if s is odd there is no $\mathbf{z}_k$).

For any distinct real numbers $\mu_1, \ldots, \mu_t$, we can choose $B \in \mathscr{S}(P_t)$ having eigenvalues $\mu_1, \ldots, \mu_t$. Let $r = \min\{k, t\}$, and choose $B \in \mathscr{S}(P_t)$ having eigenvalues $\mu_i = -\lambda_i$, $i = 1, \ldots, r$ with eigenvectors $\mathbf{y}_i$. Then $A \otimes I_t + I_s \otimes B$ has at least $\min\{s, 2t\}$ eigenvectors for eigenvalue 0, namely $\mathbf{x}_i \otimes \mathbf{y}_i$, $\mathbf{z}_i \otimes \mathbf{y}_i$, $i = 1, \ldots, r$ (if $s = 2k - 1 < 2t$, so $r = k$, the eigenvectors are $\mathbf{x}_i \otimes \mathbf{y}_i$, $i = 1, \ldots, k$ and $\mathbf{z}_i \otimes \mathbf{y}_i$, $i = 1, \ldots, k-1$).

Thus $M(C_s \,\square\, P_t) \geqslant \min\{s, 2t\}$. $\square$

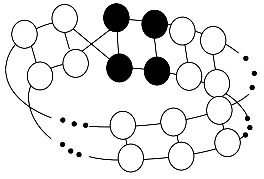

Fig. 6. Zero forcing set for Möbius ladder.

Proposition 3.9. *If G is the Möbius ladder on $2n$ vertices where $n \geqslant 3$, then $M(G) = 4 = Z(G)$.*

Proof. A zero forcing set of four vertices for the Möbius ladder G is shown in Fig. 6, so $M(G) \leqslant 4$. For $n = 3$, $G = K_{3,3}$, and more generally, $K_{3,3}$ is a minor of G. Since $\xi(K_{3,3}) = 4$, $M(G) \geqslant 4$. □

Theorem 3.10. *For any graph G with at least one edge and any $t \geqslant 2$, $M(G \,\square\, K_t) \geqslant M(G)(t-1) + \zeta$, where ζ is the maximum multiplicity of a nonzero eigenvalue in a matrix $A \in \mathscr{S}(G)$ such that* $\operatorname{rank}(A) = \operatorname{mr}(G)$.

Proof. Choose $A \in \mathscr{S}(G)$ with eigenvalue 0 of multiplicity $M(G)$ and $\lambda \neq 0$ of multiplicity ζ. Since A is a real symmetric matrix, eigenvalue 0 has independent eigenvectors $\mathbf{x}_i$, $i = 1, \ldots, M(G)$, and eigenvalue λ has independent eigenvectors $\mathbf{z}_j$, $j = 1, \ldots, \zeta$. We can choose $B \in \mathscr{S}(K_t)$ having eigenvalues 0 with multiplicity $t-1$ with independent eigenvectors $\mathbf{y}_k$, $k = 1, \ldots, t-1$ and $-\lambda$ of multiplicity 1 with eigenvector $\mathbf{w}$. Then $A \otimes I_t + I_s \otimes B$ has at least $M(G)(t-1) + \zeta$ eigenvectors for eigenvalue 0, namely $\mathbf{x}_i \otimes \mathbf{y}_k$, $i = 1, \ldots, M(G)$; $k = 1, \ldots, t-1$, and $\mathbf{z}_j \otimes \mathbf{w}$, $j = 1, \ldots \zeta$, so $M(G \,\square\, K_t) \geqslant M(G)(t-1) + \zeta$. □

Corollary 3.11. *For $s, t \geqslant 2$, $M(K_s \,\square\, K_t) = st - s - t + 2 = Z(K_s \,\square\, K_t)$, and* $\operatorname{mr}(K_s \,\square\, K_t) = s + t - 2$.

Proof. From Theorem 3.10 and Proposition 2.10

$$st - s - t + 2 = (s-1)(t-1) + 1 \leqslant M(K_s \,\square\, K_t) \leqslant Z(K_s \square K_t) \leqslant st - s - t + 2. \qquad \square$$

Corollary 3.12. *For $s \geqslant 4$, $M(C_s \,\square\, K_t) = 2t = Z(C_s \,\square\, K_t)$.*

Proof. For C_s, $s \geqslant 4$, $\zeta = 2$, so

$$2(t-1) + 2 = 2t \leqslant M(C_s \,\square\, K_t) \leqslant Z(C_s \,\square\, K_t) \leqslant Z(C_s)t = 2t. \qquad \square$$

Proposition 3.13. *$M(P_s \boxtimes P_t) = s + t - 1 = Z(P_s \boxtimes P_t)$ and* $\operatorname{mr}(P_s \boxtimes P_t) = (s-1)(t-1) = \operatorname{cc}(P_s \boxtimes P_t)$.

Proof. By Observation 2.13, $M(P_s \boxtimes P_t) \leqslant Z(P_s \boxtimes P_t) \leqslant s + t - 1$. We can cover $P_s \boxtimes P_t$ by $(s-1)(t-1)$ copies of K_4 so by Observation 1.1, $\operatorname{mr}(P_s \boxtimes P_t) \leqslant \operatorname{cc}(P_s \boxtimes P_t) \leqslant (s-1)(t-1)$. Since $M(P_s \boxtimes P_t) + \operatorname{mr}(P_s \boxtimes P_t) = st$, all inequalities are equalities. □

Note that in the proof of Proposition 3.13 we have also shown that $\operatorname{mr}(P_s \boxtimes P_t) = \operatorname{mr}_+(P_s \boxtimes P_t)$.

Proposition 3.14. *For $t \geqslant 2$, $M(K_t \circ K_s) = st - 1 = Z(K_t \circ K_s)$ and $\mathrm{mr}(K_t \circ K_s) = t + 1 = \mathrm{cc}(K_t \circ K_s)$.*

Proof. From Proposition 2.12, $M(K_t \circ K_s) \leqslant Z(K_t \circ K_s) \leqslant st - 1$. The K_t and the t copies of K_{s+1} consisting of each K_s and its neighbor form a clique cover, so $\mathrm{mr}(G) \leqslant \mathrm{cc}(K_t \circ K_s) \leqslant t + 1$. Since $M(K_t \circ K_s) + \mathrm{mr}(K_t \circ K_s) = st + t$, all inequalities are equalities. $\square$

Note that in the proof of Proposition 3.14 we have also shown that $\mathrm{mr}(K_t \circ K_s) = \mathrm{mr}_+(K_t \circ K_s)$.

3.2. The minimum rank of complements

Proposition 3.15. *If $n \geqslant 5$, then $\mathrm{mr}(\overline{C_n}) = 3$.*

Proof. If $n \geqslant 5$, then C_n contains an induced P_4, and therefore, $\overline{C_n}$ does too. So $\mathrm{mr}(\overline{C_n}) \geqslant 3$.

Embed C_n as a regular polygon on the unit circle in $\mathbb{R}^2$ and let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the vectors representing the vertices. Let B be the Gram matrix of these vectors. Then $b_{i,i+1} = \cos(2\pi/n)$ and if $1 < |i - j| < n - 1$ then $b_{i,j} < b_{i,i+1}$. Now $\mathrm{rank}(B) = 2$ so $B - \cos(2\pi/n)J$ has rank at most three, and $\mathscr{G}(B - \cos(2\pi/n)J) = \overline{C_n}$. Thus $\mathrm{mr}(\overline{C_n}) \leqslant 3$. $\square$

An *orthogonal representation* of a graph $G = (V, E)$ in $\mathbb{R}^d$ is a function $\varphi\colon V \to \mathbb{R}^d$ such that $\varphi(u)$ and $\varphi(v)$ are orthogonal if and only if u and v are nonadjacent vertices. If $\varphi\colon V \to \mathbb{R}^d$ is an orthogonal representation, then the Gram matrix of the vectors $\varphi(u)$ is a positive semidefinite matrix in $\mathscr{S}(G)$. Hence if $\varphi\colon V \to \mathbb{R}^d$ is an orthogonal representation of G, then $\mathrm{mr}_+(G)$ is less than or equal to d (in fact, $\mathrm{mr}_+(G)$ is equal to the smallest d such there exists an orthogonal representation $\varphi\colon V \to \mathbb{R}^d$).

Theorem 3.16. *For any tree T, $\mathrm{mr}_+(\overline{T}) \leqslant 3$.*

Proof. We prove by induction on the order of T the following statement: $\overline{T} = (V, E)$ has an orthogonal representation $\varphi\colon V(\overline{T}) \to \mathbb{R}^3$ such that $\varphi(v)$ and $\varphi(w)$ are linearly independent for any pair of distinct vertices v, w of T. The case where T has only one vertex is clear.

Assume now that the statement holds for every tree with at most $n - 1$ vertices. Let T be a tree with n vertices. Let v be a leaf of T. Since $T - v$ has $n - 1$ vertices, there is an orthogonal representation $\varphi\colon V(\overline{T} - v) \to \mathbb{R}^3$ such that $\varphi(u)$ and $\varphi(w)$ are linearly independent for every two distinct vertices u, w. For each vertex u of $\overline{T} - v$, let L_u be the plane orthogonal to $\varphi(u)$. Let w be the vertex adjacent to v in T. Choose a vector $\mathbf{x}$ in L_w which is not in L_u for all $u \in V(\overline{T} - v - w)$ and not a multiple of $\varphi(u)$ for $u \in V(\overline{T} - v - w)$. Extend φ to $V(\overline{T})$ by defining $\varphi(v) = \mathbf{x}$. Then $\varphi\colon V(\overline{T}) \to \mathbb{R}^3$ is an orthogonal representation of $\overline{T}$ such that $\varphi(u)$ and $\varphi(z)$ are linearly independent for any distinct vertices u, z of $\overline{T}$. $\square$

Corollary 3.17. *Let T be a tree of order $n \geqslant 3$. Then*

$$\mathrm{mr}(\overline{T}) = \begin{cases} 3, & \text{if } P_4 \text{ is an induced subgraph of } T; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. For any tree T, $\mathrm{mr}(\overline{T}) \leqslant 3$, since $\mathrm{mr}(T) \leqslant \mathrm{mr}_+(T)$.

Let $|T| = n$. If T contains an induced P_4, $\overline{T}$ does too. So $\mathrm{mr}(\overline{T}) \geqslant 3$. If P_4 is not induced in T, any two vertices are connected by a path of length at most two, and so $T = K_{1,n-1}$. Since $\overline{K_{1,n-1}} = K_{n-1} \cup K_1$, $\mathrm{mr}(\overline{K_{1,n-1}}) = 1$. $\square$

3.3. The minimum rank of line graphs

Given a graph $G = (V, E)$, an orientation G^τ assigns to each edge $\{u, v\}$ exactly one of the two arcs (u, v), (v, u). The *incidence matrix* of an orientation G^τ is the $|V| \times |E|\{0, \pm 1\}$-matrix $D(G^\tau) = [d_{ve}]$ having rows indexed by the vertices and columns indexed by the oriented edges of G and

$$d_{ve} = \begin{cases} 0, & \text{if } v \notin e, \\ 1, & \text{if } \exists w, e = (w, v), \\ -1, & \text{if } \exists w, e = (v, w). \end{cases}$$

If G is connected, $\mathrm{rank}(D(G^\tau)) = |G| - 1$ [9, Theorem 8.3.1].

Theorem 3.18. $\mathrm{mr}(L(K_n)) = n - 2$.

Proof. For $n = 2$, $L(K_2) = K_1$ and $\mathrm{mr}(K_1) = 0 = n - 2$. For $n = 3$, $L(K_3) = K_3$ and $\mathrm{mr}(K_3) = 1 = n - 2$. For $n = 4$, $L(K_4) = K_{2,2,2}$ and $\mathrm{mr}(K_{2,2,2}) = 2 = n - 2$ [5].

So now assume $n \geqslant 5$. The vertices of $L(K_n)$ will be the unordered pairs from $\{1, \ldots, n\}$. The subgraph induced by a neighborhood of a vertex in $L(K_n)$ is isomorphic to $K_{n-2} \square P_2$, which has minimum rank $n - 2$ by Proposition 3.3. Thus $\mathrm{mr}(L(K_n)) \geqslant n - 2$.

For the upper bound, let D denote the incidence matrix of an orientation of K_{n-1}. Then $\mathrm{rank}(D) = n - 2$. Consider the matrix

$$M = \begin{bmatrix} I_{n-1} - \frac{1}{n-1} J_{n-1} & D \\ D^{\mathrm{T}} & D^{\mathrm{T}} D \end{bmatrix}.$$

The matrix partition corresponds to the pairs (edges) that contain 1, and those that do not; it is straightforward to check that $M \in L(K_n)$. Since $D^{\mathrm{T}} J_{n-1} = 0$

$$\begin{bmatrix} I & 0 \\ -D^{\mathrm{T}} & I \end{bmatrix} \begin{bmatrix} I_{n-1} - \frac{1}{n-1} J_{n-1} & D \\ D^{\mathrm{T}} & D^{\mathrm{T}} D \end{bmatrix} = \begin{bmatrix} I_{n-1} - \frac{1}{n-1} J_{n-1} & D \\ 0 & 0 \end{bmatrix}.$$

Since all the columns of $I_{n-1} - \frac{1}{n-1} J_{n-1}$ and of D are orthogonal to the all 1s vector

$$\mathrm{rank}(M) = \mathrm{rank}\left(\left[I_{n-1} - \frac{1}{n-1} J_{n-1} D \right]\right) \leqslant n - 2,$$

so $\mathrm{mr}(L(K_n)) \leqslant n - 2$. $\square$

It is well known, and straightforward that if H is a subgraph of G (not-necessarily induced), then $L(H)$ (the line graph of H) is an induced subgraph of $L(G)$. If G has n vertices, then G is obviously a subgraph of K_n, hence $L(G)$ is an induced subgraph of $L(K_n)$. By Theorem 3.18 we have:

Corollary 3.19. $\mathrm{mr}(L(G)) \leqslant n-2$.

On the other hand, if G contains P_n as a subgraph (in other words, G has a Hamiltonian path) then $L(G)$ contains $L(P_n) = P_{n-1}$ as an induced subgraph. Since $\mathrm{mr}(P_{n-1}) = n-2$ we have:

Corollary 3.20. *If G has $n \geqslant 2$ vertices and contains a Hamiltonian path, then* $\mathrm{mr}(L(G)) = n-2$.

Since the majority of graphs on n vertices have a Hamiltonian path (if n is large enough), Corollary 3.20 provides a large class of graphs with known minimum rank.

For the complete bipartite graph $K_{k,n-k}$ with $1 < k < n-1$, the minimum rank of the line graph also attains the maximum value $n-2$, because $L(K_{k,n-k})$ is isomorphic to $K_k \,\square\, K_{n-k}$, which has minimum rank $n-2$ by Corollary 3.11. Thus we have:

Corollary 3.21. *If G contains $K_{k,n-k}$ as a subgraph (with $1 < k < n-1$), then* $\mathrm{mr}(L(G)) = n-2$.

Note that this corollary also implies that $\mathrm{mr}(L(G)) = n-2$ if G is a complete multipartite graph with more than two classes.

We now turn our attention to line graphs of trees; for such line graphs Corollary 3.20 gives the actual value only if $G = P_n$. If T is the star $K_{1,n-1}$ then $L(T) = K_{n-1}$, hence $\mathrm{mr}(L(T)) = 1$. In fact, for a tree T it follows from Corollary 3.24 below that $\mathrm{mr}(L(T)) = n-2$ if and only if $T = P_n$ (with $n \geqslant 2$).

An example of a tree and its corresponding line graph is shown in Fig. 7. In this example $L(T)$ consists of four cliques, one clique for each non-pendent vertex of T. Furthermore, these cliques intersect only at vertices. This holds in general.

A connected graph is *nonseparable* if it does not have a cut-vertex. A *block* of a graph is a maximal nonseparable subgraph. A graph is *block-clique* (also called a 1-*chordal*) if every block is a clique. A block-clique graph can be built by adding one block at a time via union, where the intersection consists of a single vertex. Clearly the clique cover number of a block-clique graph is the number of blocks. A *pendent clique* of a block-clique graph G such that $\mathrm{cc}(G) \geqslant 2$ is a clique containing exactly one cut-vertex of G.

Observation 3.22. A graph is the line graph of a tree if and only if it is block-clique and no vertex is contained in more than 2 blocks. The number of blocks is the number of non-pendent vertices of the tree.

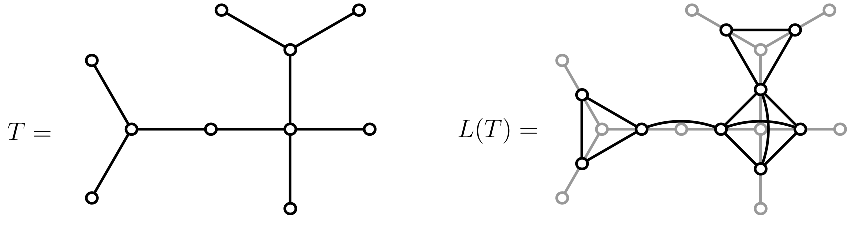

Fig. 7. A tree T and its line graph $L(T)$.

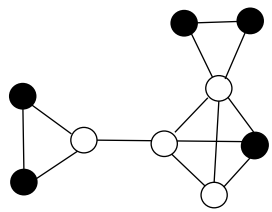

Fig. 8. A zero forcing set for $L(T)$.

Proposition 3.23. *Let F be a field, and let G be a block-clique graph of order at least 2 such that no vertex is contained in more than 2 blocks. Then* $\mathrm{mr}^F(G) = \mathrm{cc}(G)$ *and* $M^F(G) = Z(G)$.

Proof. Since the blocks intersect only in vertices, $\mathrm{mr}^F(G) \leqslant \mathrm{cc}(G)$. We establish the following two statements by induction on $\mathrm{cc}(G)$:

1. If W_G is the set of vertices of G that are not cut-vertices, then $|W_G| = |G| - \mathrm{cc}(G) + 1$.
2. A zero forcing set Z for G can be obtained by choosing all but one of the vertices of W_G (see Fig. 8).

Both statements are clearly true for $\mathrm{cc}(G) = 1$ (since $|G| \geqslant 2$). Assume true for all graphs H such that $\mathrm{cc}(H) < \mathrm{cc}(G)$. Choose a pendent clique K of G and denote the cut-vertex of K by v. The subgraph H of G induced by $V(G) \setminus V(K) \cup \{v\}$ is a block-clique graph with $\mathrm{cc}(H) = \mathrm{cc}(G) - 1$. Note that $v \in W_H$, since v is in only one clique of H.

Then by hypothesis, $|W_H| = |H| - \mathrm{cc}(H) + 1$, and

$$|W_G| = |W_H| - 1 + |K| - 1 = |H| - \mathrm{cc}(H) + 1 - 1 + |K| - 1 = |G| - \mathrm{cc}(G) + 1.$$

To obtain a zero forcing set for G consisting of all but one of the vertices in W_G, select the zero forcing set Z_H consisting of all non-cut-vertices of H except v. Then any set consisting of Z_H and all vertices of K except v and one other vertex of K is a zero forcing set for G, because by applying the color-change rule to the vertices in H, v can be changed to black, and then the last vertex of K can be changed to black.

Since $\mathrm{mr}^F(G) \leqslant \mathrm{cc}(G) = |G| - |W_G| + 1$, $M^F(G) \leqslant Z(G) \leqslant |W_G| - 1$, and $\mathrm{mr}^F(G) + M^F(G) = |G|$, all inequalities are equalities. □

A zero forcing set for the line graph of the tree T in Fig. 7 is shown in Fig. 8.

Corollary 3.24. *Let F be a field, let T be a tree on n vertices with ℓ pendent vertices, and let $L(T)$ be the line graph of T. Then* $\mathrm{mr}^F(L(T)) = n - \ell$ *and* $M^F(L(T)) = \ell - 1 = Z(L(T))$.

3.4. The minimum rank of certain regular graphs

Next we determine the minimum rank/maximum nullity of some well-known regular graphs. A graph G is *strongly regular* with parameters (n, k, a, c) if $|G| = n$, G is k-regular, every pair of

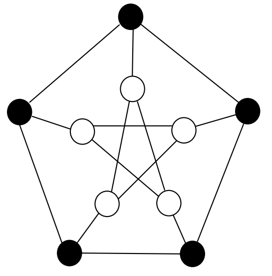

Fig. 9. Zero forcing set for the Petersen graph.

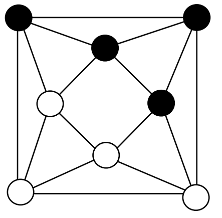

Fig. 10. Zero forcing set for the 4-antiprism.

adjacent vertices has a common neighbors, and every pair of nonadjacent vertices has c common neighbors.

Proposition 3.25. *Let G be a strongly regular graph. Then* $M(G) \geqslant \left\lfloor \frac{|G|}{2} \right\rfloor$.

Proof. The adjacency matrix A_G of a strongly regular graph G has exactly three eigenvalues, one of which is k and has multiplicity 1 [9, Section 10.2]. For λ the eigenvalue of maximal multiplicity m, $A_G - \lambda I$ has corank m, and clearly $m \geqslant \left\lceil \frac{|G|-1}{2} \right\rceil = \left\lfloor \frac{|G|}{2} \right\rfloor$. $\square$

Note that C_5 is strongly regular with parameters $(5, 2, 0, 1)$ and $K_3 \square K_3$ is strongly regular with parameters $(9, 4, 1, 2)$ (these are both Paley graphs). Since $M(C_5) = 2$, C_5 achieves equality of the bound in Proposition 3.25, which implies that a translation of the adjacency matrix realizes minimum rank/maximum nullity. However, $K_3 \square K_3$ does not, since by Corollary 3.11, $M(K_3 \square K_3) = 5 > 4 = \left\lfloor \frac{9}{2} \right\rfloor$.

Proposition 3.26. *Let P denote the Petersen graph shown in Fig. 9. Then $M(P) = 5 = Z(P)$ and* $\operatorname{mr}(P) = 5$.

Proof. The five vertices on the outer cycle form a zero forcing set, so $M(P) \leqslant Z(P) \leqslant 5$. The Petersen graph is strongly regular with parameters $(10, 3, 0, 1)$, so by Proposition 3.25, $M(P) \geqslant 5$. Thus we have $M(P) = 5$ and $\operatorname{mr}(P) = 5$. $\square$

Lemma 3.27 [12]. $\xi(Q_3) = 4$.

Proposition 3.28. *Let G_8 denote the 4-antiprism shown in Fig. 10. Then $M(G_8) = 4 = Z(G_8)$ and* $\operatorname{mr}(G_8) = 4$.

Proof. A zero forcing set of four vertices for the 4-antiprism is shown in Fig. 10, so $M(G_8) \leqslant Z(G_8) \leqslant 4$. Note that Q_3 is a minor of the 4-antiprism G_8 (by deleting four edges), and $\xi(Q_3) = 4$. So $4 \leqslant \xi(G_8) \leqslant M(G_8)$. □

4. Graphs for which $Z(G) = M(G)$

In the previous sections we have shown that $M(G) = Z(G)$ for most of the graphs in Table 1, and we will establish this equality for the remaining graphs listed there. We noted in Section 2 that $M(G) = Z(G)$ for $G = P_n$ and $G = C_n$, and this equality is also true for $G = K_n$ and $G = K_{p,q}$ (use any set of $n - 1$ vertices and any set omitting exactly one vertex from each of the bipartition sets as zero forcing sets). However, not every graph satisfies $M(G) = Z(G)$. For a graph, such as $K_{3,3,3}$, where $M(G) < M(\mathscr{S}(F, G))$ for some field F ($\mathbb{Z}_2$ in the case of $K_{3,3,3}$), necessarily $M(G) < Z(G)$. The next example shows $Z(G)$ can be strictly greater than $M(G)$ even when $M(G)$ is field independent.

Example 4.1. Consider the corona $C_5 \circ K_1$ (sometimes also called the penta-sun) shown in Fig. 11. The set $\{6, 7, 8\}$ (shown) is a zero forcing set (as is $\{6, 7, 9\}$ and others), but there is no smaller zero forcing set, so $Z(C_5 \circ K_1) = 3$, but $M(C_5 \circ K_1) = 2$ by cut-vertex reduction (over any field); see [2] for details.

We now establish $M(G) = Z(G)$ for several additional families of graphs. A *path cover* of a tree T is a set of vertex disjoint paths occurring as (induced) subgraphs of T that cover all the vertices of T. A *minimum path cover* of T is a path cover having the fewest possible paths among all path covers of T. The *path cover number* of T, $P(T)$, is the number of paths in a minimum path cover of T. For any tree T, $M(T) = P(T)$ [13]. Note that there are algorithms for finding a minimum path cover (and hence $P(T)$ and $M(T)$), e.g., [8]. As shown in [7], for any field F, $M^F(T) = M(T)$.

Proposition 4.2. *For any tree T, $M^F(T) = Z(T)$.*

Proof. A zero forcing set Z for T can be obtained by choosing a minimum path cover and selecting one endpoint of each path in the minimum path cover. That such a Z is a zero forcing set can be shown by induction on $P(T)$. It is clearly true for $P(T) = 1$. Assume true for all trees T such that $P(T) < P(T_1)$. Choose a minimum path cover for T_1, let Z be a set consisting of one end point of each path in the minimum path cover (hereafter called *black endpoints*) and identify a path P_1 in the minimum path cover that is joined to the rest of T_1 by only one edge uv not in P_1, and say $v \in V(P_1)$. Then by applying the color-change rule repeatedly starting at the black end-point of P_1, all vertices from the black endpoint through v are colored black. Now the path P_1 is

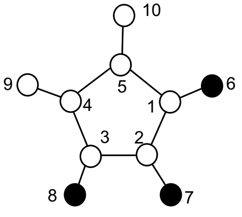

Fig. 11. Zero forcing set for the corona $C_5 \circ K_1$.

irrelevant to the analysis of the tree $T_1 - V(P_1)$, so by the induction hypothesis, the black endpoints of the remaining paths are a zero forcing set for $T_1 - V(P_1)$, and all vertices not in P_1, including u, can be colored black. Hence the remainder of path P_1 can also be colored black and Z is a zero forcing set for T_1. □

We have verified the following by direct computation (the values of $M(G) = Z(G)$ are listed in the on-line catalog [1]).

Proposition 4.3. *If* $|G| \leqslant 6$, *then* $Z(G) = M(G)$.

For a graph $G = (V, E)$, define $\mathrm{mz}(G) = |G| - Z(G)$. Notice that $\mathrm{mz}(G) \leqslant \mathrm{mr}^F(G)$ for every graph G and every field F, and $\mathrm{mz}(G) = \mathrm{mr}^F(G)$ is equivalent to $Z(G) = M^F(G)$.

Proposition 4.4. *If H is an induced subgraph of G, then* $\mathrm{mz}(H) \leqslant \mathrm{mz}(G)$.

Proof. Let Z be a zero forcing set of H with $|Z| = Z(H)$. Then $Z \cup (V(G) \setminus V(H))$ is a zero forcing set for G. Hence $Z(G) \leqslant |Z| + |G| - |H|$. From this it follows that $|H| - Z(H) \leqslant |G| - Z(G)$. Hence $\mathrm{mz}(H) \leqslant \mathrm{mz}(G)$. □

The class of graphs G with $\mathrm{mz}(G) \leqslant k$ can therefore be characterized by a collection (possibly infinite) of forbidden induced subgraphs. Note that Z itself is not monotone on induced subgraphs, as can be seen trivially by deleting a vertex of degree 2 from a path, or in the next example, where $G - v$ remains connected.

Example 4.5. Consider the graph G with zero forcing set of size 2 shown in Fig. 12. The deletion of vertex v leaves a tree, and so $Z(G - v) = M(G - v) = P(G - v) = 3$.

Proposition 4.6. *Let H be an induced subgraph of G. If* $\mathrm{mr}^F(H) = \mathrm{mr}^F(G)$ *and* $\mathrm{mr}^F(H) = \mathrm{mz}(H)$, *then* $\mathrm{mz}(G) = \mathrm{mr}^F(G)$.

Proof. This follows from $\mathrm{mr}^F(G) = \mathrm{mr}^F(H) = \mathrm{mz}(H) \leqslant \mathrm{mz}(G) \leqslant \mathrm{mr}^F(G)$. □

Proposition 4.7. *For any tree T*, $\mathrm{mr}(\overline{T}) = \mathrm{mz}(\overline{T})$.

Proof. Let $n = |T|$. Suppose P_4 is an induced subgraph of T. Since $\mathrm{mr}(P_4) = \mathrm{mz}(P_4) = 3$ and $\mathrm{mr}(\overline{T}) = 3$, Proposition 4.6 tells us that $\mathrm{mr}(\overline{T}) = \mathrm{mz}(\overline{T})$. If P_4 is not an induced subgraph and $n \geqslant 3$, then $T = K_{1,n-1}$ and $\mathrm{mr}(\overline{T}) = 1 = \mathrm{mz}(\overline{T})$. If $n \leqslant 2$, the result follows by direct computation (Proposition 4.3). □

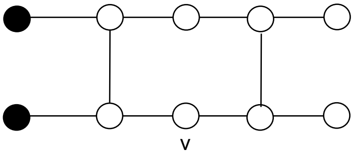

Fig. 12. A graph G having $Z(G - v) > Z(G)$.

Proposition 4.8. *For any cycle C_n,* $\mathrm{mr}(\overline{C_n}) = \mathrm{mz}(\overline{C_n})$.

Proof. By Proposition 4.3, it suffices to consider the case that $n \geqslant 6$. Because $\mathrm{mr}(\overline{C_n}) = 3$ and $\overline{C_n}$ contains a P_4, Proposition 4.6 tells us that $\mathrm{mr}(\overline{C_n}) = \mathrm{mz}(\overline{C_n})$. $\square$

Proposition 4.9. *If G has $n \geqslant 3$ vertices and contains a Hamiltonian path, contains a subgraph $K_{k,n-k}$ with $1 < k < n-1$, or $G = K_n$, then* $\mathrm{mr}(L(G)) = \mathrm{mz}(L(G))$.

Proof. In each of these three cases, $L(G)$ contains an induced subgraph H such that $\mathrm{mz}(H) = \mathrm{mr}(H) = \mathrm{mr}(L(G))$. If $G = K_n$, $H = K_{n-2} \,\square\, K_2$; if G contains a Hamiltonian path, $H = P_{n-1}$; if G contains $K_{k,n-k}$, $H = K_k \square K_{n-k}$. Thus by Proposition 4.6, $\mathrm{mr}(L(G)) = \mathrm{mz}(L(G))$. $\square$

The following theorem has now been established.

Theorem 4.10. *For each of the following families of graphs, $Z(G) = M(G)$:*

1. *Any graph G such that $|G| \leqslant 6$.*
2. *K_n, P_n, C_n.*
3. *Any tree T.*
4. *All the graphs listed in Table* 1.

5. Maximum corank of not-necessarily symmetric matrices

A matrix A is *combinatorially symmetric* if $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. A combinatorially symmetric matrix has a symmetric zero–nonzero pattern. For such a matrix, the *graph* of A, denoted $\mathscr{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\}: a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$. (Whenever we write $\mathscr{G}(A)$, we are assuming A is combinatorially symmetric.) Let

$$N^F(G) = \max\{\mathrm{corank}(A): A \in F^{n\times n}, \mathscr{G}(A) = G\}.$$

The proofs of Propositions 2.2–2.4 did not use the symmetry of the matrix, so they remain valid for all matrices (not-necessarily symmetric) that have a given graph.

Proposition 5.1. *Let $A \in F^{n\times n}$, $\mathscr{G}(A) = G$, and $Z \subseteq V(G)$ be a zero forcing set for G. If $\mathbf{x} \in \ker(A)$ and $\mathrm{supp}(\mathbf{x}) \cap Z = \emptyset$, then $\mathbf{x} = 0$.*

Proposition 5.2. *Let $G = (V, E)$ be a graph and let $Z \subseteq V$ be a zero forcing set. Then for any $A \in F^{n\times n}$ such that $\mathscr{G}(A) = G$, $\mathrm{corank}(A) \leqslant |Z|$, and thus $N^F(G) \leqslant Z(G)$ for any field F.*

Given a graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, let $\mathscr{H}(G)$ be the set of all Hermitian $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E$. There is no restriction on the diagonal entries of A. We define

$$\mathrm{hmr}(G) = \min\{\mathrm{rank}(A) | A \in \mathscr{H}(G)\}.$$

As is the case for symmetric matrices, the sum of the minimum rank and maximum nullity is the order of the graph:

$$\min\{\mathrm{rank}(A): A \in \mathbb{C}^{n\times n}, \mathscr{G}(A) = G\} + N^{\mathbb{C}}(G) = |G|.$$

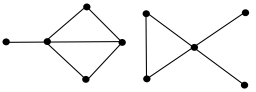

Fig. 13. The graphs dart and $\ltimes$.

Since any matrix $A \in \mathscr{H}(G)$ has $\mathscr{G}(A) = G$, $\mathrm{hmr}(G) \geqslant \min\{\mathrm{rank}(A): A \in \mathbb{C}^{n\times n}, \mathscr{G}(A) = G\}$. Thus $N^{\mathbb{C}}(G) \geqslant |G| - \mathrm{hmr}(G)$ and so $\mathrm{mz}(G) \leqslant \mathrm{hmr}(G)$. In [5], the following theorem is proved ($\ltimes$ and dart are shown in Fig. 13).

Theorem 5.3. *Let G be a graph. Then the following are equivalent*:

1. $\mathrm{hmr}(G) \leqslant 2$.
2. *G is* (P_4, $\ltimes$, dart, $P_3 \cup K_2$, $3K_2$)*-free*.

Theorem 5.4. *A graph G has* $\mathrm{mz}(G) \leqslant 2$ *if and only if G is* (P_4, $\ltimes$, dart, $P_3 \cup K_2$, $3K_2$)*-free*.

Proof. Since $\mathrm{mz}(P_4) = 3$, $\mathrm{mz}(\ltimes) = 3$, $\mathrm{mz}(\text{dart}) = 3$, $\mathrm{mz}(P_3 \cup K_2) = 3$, and $\mathrm{mz}(3K_2) = 3$, a graph G with $\mathrm{mz}(G) \leqslant 2$ is (P_4, $\ltimes$, dart, $P_3 \cup K_2$, $3K_2$)-free.

Conversely, if G is (P_4, $\ltimes$, dart, $P_3 \cup K_2$, $3K_2$)-free, then $\mathrm{mz}(G) \leqslant \mathrm{hmr}(G) \leqslant 2$. □

6. Conclusion and open questions

We consider the following to be the main results of this paper:

- The introduction of $Z(G)$ and its systematic application to many families of graphs to obtain upper bounds for $M^F(G)$ for any field F.
- Obtaining sharp lower bounds for $M(G)$ (over the real field) for many families of graphs, thereby establishing the results in Table 1.

We conclude with the following questions:

Question 1. What is the class of graphs G for which $M^F(G) = Z(G)$ for some field F?

As Question 1 is surely difficult, we list the following sub-questions:

Question 1a. For the class of graphs for which $M^F(G)$ is field independent, what is the subclass of graphs with $M(G) = Z(G)$?

Question 1b. What is the class of graphs G for which $M(G) = Z(G)$?

Question 1c. What are sufficient conditions in order that $M^F(G) = Z(G)$ for some field F?

Question 1d. What are sufficient conditions in order that $M^F(G) < Z(G)$ for every field F?

Question 2. For those graphs with $M^F(G) < Z(G)$ for every field F (or for a subclass of these graphs), is there a graph theoretic parameter Y such that $M^F(G) \leqslant Y(G) < Z(G)$?

It would also be of interest to develop additional techniques for establishing lower bounds for $M^F(G)$ that are independent of the real field and apply them to the classes of graphs in Table

1, and to determine for which of these classes of graphs $M^F(G)$ is field independent. Note that Example 3.4 shows that $M^F(K_3 \square K_2)$ depends on the field.

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