Working paper

2023-13

Statistics and Econometrics

ISSN 2387-0303

# Observability Analysis for Structural System Identification based on State Estimation

Ahmad Alahmad, Roberto Mínguez, Rocío Porras, Jose Antonio Lozano-Galant, Jose Turmo

uc3m | Universidad Carlos III de Madrid

# Observability Analysis for Structural System Identification based on State Estimation

Ahmad Alahmad[a], Roberto Mínguez[b], Rocío Porras[c], Jose Antonio Lozano-Galant[d], Jose Turmo[a]

[a]*Dept. of Civil and Environmental Engineering, Polytechnic University of Catalonia, Barcelona, 08034, Spain*
[b]*Dept. of Statistics, Univ. Carlos III of Madrid, Getafe, 28903, Spain*
[c]*Dept. of Applied Mechanics and Project Engineering, Univ. of Castilla-La Mancha, Ciudad Real, 13001, Spain*
[d]*Dept. of Civil Engineering, Univ. of Castilla-La Mancha, Ciudad Real, 13001, Spain*

**Abstract**

The concept of observability analysis (OA) has garnered substantial attention in the field of Structural System Identification. Its primary aim is to identify a specific set of structural characteristics, such as Young's modulus, area, inertia, and possibly their combinations (e.g., flexural or axial stiffness). These characteristics can be uniquely determined when provided with a suitable subset of deflections, forces, and/or moments at the nodes of the structure. This problem is particularly intricate within the realm of Structural System Identification, mainly due to the presence of nonlinear unknown variables, such as the product of vertical deflection and flexural stiffness, in accordance with modern methodologies. Consequently, the mechanical and geometrical properties of the structure are intricately linked with node deflections and/or rotations. The paper at hand serves a dual purpose: firstly, it introduces the concept of State Estimation (SE), specially tailored for the identification of structural systems; and secondly, it presents a novel OA method grounded in SE principles, designed to overcome the aforementioned challenges. Computational experiments shed light on the algorithm's potential for practical Structural System Identification applications, demonstrating significant advantages over the existing state-of-the-art methods found in the literature. It is noteworthy that these advantages could potentially be further amplified by addressing the SE problem, which constitutes a subject for future research. Solving this problem would help address the additional challenge of developing efficient techniques that can accommodate redundancy and uncertainty when estimating the current state of the structure.

*Keywords:* Structural Health Monitoring, Structural System Identification, State Estimation, Observability Analysis,

## 1. Introduction

In recent years, the field of Structural Health Monitoring (SHM) has experienced significant progress, marked by innovative methodologies such as the pioneering work of Liu et al. [1] in utilizing a multi-task deep neural network for simultaneous defect identification in tunnels. This momentum continued with Moghadam et al. [2], introducing a groundbreaking multiple-presence Influence Line (MP-IL) technique for bridge health monitoring, and Parisi et al. [3]'s study on automated damage location in a steel truss bridge using machine learning.

Despite these technological leaps in SHM, the fundamental role of Structural System Identification remains indispensable. Traditionally, SHM tasks, such as damage detection , assume a priori knowledge of a structure's properties, a presumption that can be inaccurate due to uncertainties in materials, construction methods, or stress states [4, 5, 6]. Structural System Identification, as a valuable tool, enables the extraction of essential information about a structure's dynamic and mechanical properties. This involves capturing the structure's response to loading scenarios and applying various analysis techniques [7, 8, 9].

The existing literature on Structural System Identification encompasses a wide range of methods, which can be classified into two categories [10]: Parametric and Nonparametric. Parametric methods follow a direct approach where actual system model parameters, such as structural stiffness, are used to represent physical properties. These methods explicitly incorporate the physical meaning of the parameters. On the other hand, nonparametric methods characterize the input-output relation using a set of equations that may not have explicit physical interpretations. These methods focus on establishing the relation between the input and output without directly using physical parameters. For example, nonparametric methods are used for movable bridge component monitoring [11], providing insights into structural integrity. Another instance is the dynamic time-delay fuzzy wavelet neural network model [12], effectively removing noise and capturing data dynamics for structural identification.

Observability Analysis (OA) is a pivotal aspect within various scientific and engineering disciplines, serving as a critical tool for gaining insights into the internal workings of systems. Rooted in control theory, observability is defined as the measure of a system's ability to have its internal states revealed through the examination of its outputs [13]. This principle has been extensively explored in the context of dynamic systems and networks [14], where the key focus in OA lies in the assessment of matrices that encapsulate the relations between a system's states and its outputs. These matrices, such as the observability matrix or the measurement Jacobian, play a crucial role in guiding the analysis [15]. The fundamental question addressed is whether a system can be effectively observed or if additional measurements and adjustments are necessary [16].

Lozano-Galant et al. [6] introduced the Observability Method (OM) as a novel parametric approach in the field of Structural System Identification. This method was pioneering in its utilization of OA for identifying unknown parameters of the stiffness matrix of a structural system when only a subset of deflections and nodal forces is measurable.

The Observability Method leverages OA on the stiffness matrix of a structural system, which poses several challenges. One prominent challenge involves the necessity for symbolic analysis prior to examining the observability of the system. This analysis is performed to derive a modified stiffness matrix that links known variables with unknown variables. The symbolic analysis results in a system of nonlinear equations, wherein some unknown variables are interconnected, thereby intensifying the problem's complexity. To mitigate the issue of nonlinearity, OM employs a recursive algorithm. The problem is tackled through a series of recursive steps, with the output of each preceding step serving as input for the next. This approach can be computationally intensive, potentially impractical for large systems, and may necessitate additional measurements to effectively handle the interconnected unknowns [6].

In the paper by Nogal et al. [17], an innovative algorithm rooted in the OM is introduced, which integrates both symbolic and numerical techniques. This algorithm is designed for numerical damage identification in structural systems. The symbolic approach is applied to OA to mitigate numerical errors, addressing issues linked to the quantity of observed parameters and error accumulation in recursive steps. Concurrently, the numerical approach is employed to guarantee precise parameter estimation. In a separate investigation, the OM is augmented with the consideration of the effect of shear lag to enhance the accuracy of mechanical parameter estimation for wide-flange box girder bridges, as discussed by [18].

However, the methods proposed by Nogal et al. [17] and Sun et al. [18] have not adequately addressed the handling of measurement errors and uncertainties [19]. It is crucial to acknowledge that measured data obtained from sensors inherently contains errors and uncertainties. These errors can arise from several sources, including limitations of the measurement devices themselves and environmental or structural conditions. For example, sensor noise, calibration inaccuracies, temperature variations, and ambient vibrations can all introduce uncertainties into the measurement data [20]. These measurement errors and uncertainties can significantly impact the reliability of the Structural System Identification results. They can lead to inaccurate estimations of structural properties and potentially distort the detection of structural damage. Therefore, careful consideration and handling of these errors are necessary to ensure the effectiveness and reliability of the outcomes [21, 22, 23].

In a separate study conducted by Lei et al. [24], the research introduced the Constrained Observability Method (COM) to address the issue of partial observability and to enhance the configuration of measurements in the context of the OM. Notably, the study unearthed critical insights into the factors contributing to partial observability, specifically identifying two key issues: (a) the occurrence of premature termination of recursive steps, and (b) the presence of redundant measurements. The findings from this research underscore the limitations of OM in handling redundant measurements and capitalizing on the surplus of sensors. This observation is particularly relevant in the contemporary context, where there is a growing trend toward utilizing low-cost sensors. Moreover, the study highlights that the recursive nature of OM introduces additional complexity to the problem, emphasizing the need for further optimization and refinement in the pursuit of more efficient structural system identification techniques. Subsequently, Peng et al. [25] introduced a novel decision-support tool aimed at establishing the optimal SHM+SSI strategy and enhancing estimation accuracy using COM. However, it is important to note that this tool did not address the inherent issues within COM, such as its incapability to handle redundant measurements.

In various engineering domains like traffic networks [26] and water distribution systems [27], challenges related to OM have been effectively addressed using OA based on SE techniques. The success in these fields highlights the potential for cross-disciplinary knowledge transfer and the adaptation of strategies for improved problem-solving and system analysis [28].

Originally developed in the 1970s to characterize the electric state of complex power systems [29], State Estimation techniques have found application in water distribution systems as well, as demonstrated by Carpentier and Cohen [27]. Generally speaking, a state estimator is an algorithm that computes the current state of a system through the combination of the information provided by on-line measurements and network flow equations. Importantly, SE take into account measurement errors and uncertainties associated with sensor data, ensuring that the inherent inaccuracies are appropriately handled. Moreover, SE techniques leverage redundant measurements or sensors present within the system. This redundancy plays a pivotal role in enhancing the accuracy and reliability of estimates for the system's state variables [30].

However, for any state estimator to function correctly, the measurement set should at least provide estimation of the *state variables*, which is the minimal set of variables that allows the status of the system to be fully characterized. Notably, not all configurations of measurement devices are valid for achieving full system characterization. The measurement set must ensure that all variables within the system can be infered from the system equations, i.e., the system must be observable. This explains, in general, the necessity of carrying out OA before using SE [31].

There are additional reasons to make use of OA. State Estimation procedures enable the estimation of variables that are not directly measured by utilizing the relations among variables dictated by the physical equations governing the system. OA is a previous analysis of which variables are observable from the available measurement set which is monitored by the telemetry system, thereby enabling those regions of the system where SE would provide reliable results to be identified. Moreover, OA is especially required if iterative methods based on least-squares are used, because those methods only work for observable systems, i.e. if any of the state variables are not observable according to the measurement configuration, then it is not possible to obtain the estimate of the system [32]. The problem is even more critical if mathematical programming or heuristic techniques, such as genetic algorithms, are used for minimizing the SE errors, because those procedures provide a solution for the SE problem even when the system might be unobservable and this might go unnoticed. For this reason, OA is quite established in power systems, where sensor placement problems are to be dealt with while conceiving and operating the network.

Observability Analysis in SE can be assessed by checking whether the measurement Jacobian matrix is full rank [33]. This simple "yes or no" approach is applied to all possible subsets of measurements to ensure the system's observability. It serves as a foundational and essential method for evaluating the capability to determine the system's state based on the available measurements. Researchers have explored different approaches to observability analysis in state estimation using mathematical programming techniques. Notably, one approach, as referenced by Caro et al. [34], involved solving a well-behaved integer linear programming problem to address observability issues. Additionally, Habiballah and Irving [35] applied linear programming techniques to the problem of observability analysis in the context of state estimation. Of all the available contributions in the technical literature, the algebraic proposal by Pruneda et al. [33] is especially suitable for Structural System Identification due to the possibility of simultaneously analyzing the observability of a set of available measurements and the remaining potential measurements in the system. This approach starts from the analysis of the full Jacobian matrix of possible measurements within the network and transfers columns to rows using a Gauss-based elimination technique to progressively express state variables as functions of available measurements. This method basically analyzes how the incorporation of any measurement affects the observability of both state and network variables. Therefore, the algorithm allows to check observability for the given subset, but also to identify critical and redundant measurements, thereby enabling identification of observable variables and islands if the system is not fully observable.

In summary, implementing OA as a previous and complementary step to SE in a Structural System Identification problem answers the following questions: i) whether any set of measurements is enough to appropriately carry out SE, ii) how robust is that measurement set in the face of the potential loss of measurements, iii) which variables are observable and unobservable, and iv) how to locate new sensors in order to increase resilience against the loss of one or several assets.

The primary objective of this study is to advance the field of Structural System Identification by introducing an innovative concept of OA grounded in the principles of SE. The study places a special emphasis on OA, addressing the intricate challenge of determining the observability of specific structural characteristics even in the presence of nonlinear variables and their complex interrelations. While the OM relies on symbolic analysis, the proposed method utilizes a numerical approach, which is more computationally efficient. This increased computational efficiency substantially extends the applicability of the method to real large-scale structures. Furthermore, this study highlights the prospects for future research in solving the SE problem in the Structural System Identification field, which could pave the way for efficient techniques accommodating redundancy and uncertainty, thereby enhancing the accuracy of estimating the current state of the structure.

The rest of the paper is organized as follows: in the first section an overview of the SE and OA problems is set out. Then, the structure of the measurement Jacobian matrix of the system for Structural System Identification is explored. Note that this matrix is the starting point for application of the OA method. The algorithm for OA is outlined in the following section. Subsequently, an illustrative example is presented to explore in detail what possible applications the methodology offers, followed by a case study on how the developed methodology could be applied to real structures with a comparison to other available methods. Finally, relevant conclusions are duly drawn.

## 2. STATE ESTIMATION IN STRUCTURAL SYSTEMS : a general overview

For a 2D structure loaded in its plane, modeled with Bernouilli beam elements, the vector of measurements, , is considered within $\mathbb{R}^m$, encompassing displacements or/and forces values at nodes. Similarly, a vector of state variables, , is accommodated within $\mathbb{R}^n$, where these state variables are a minimal set of variables in a system that, once their values are known, provide all the essential information needed to comprehend and forecast the system's state at a specific moment. These variables enable the calculation of all other system variables at that same moment, facilitating a comprehensive understanding of the system's condition. Furthermore, there exists a relation, denoted as $\mathbb{R}^n$ $\mathbb{R}^m$, between these measurements and state variables for a specific system. In this particular case, this relation is a result of the application of the stiffness method, and it can be expressed mathematically as follows:

(1)

which represents a system of linear and/or nonlinear equations, where denotes the errors associated with measurements. These errors are conventionally assumed to follow a Gaussian distribution with zero mean, implying unbiasedness , and a variance-covariance matrix [36].

SE consists in finding the most likely values of the state variables by solving the following Weighted Least Squares (WLS) problem:

$$\text{Minimum} \qquad {}^T\mathbf{R}^- \qquad {}^T\mathbf{R}^- \tag{2}$$

In this context, corresponds to the optimal solution of problem (2). It is important to note that errors are scaled by the inverse of the variance-covariance matrix associated with error measurements, which, due to their typical independence, results in a diagonal matrix. As a result, the objective function is formulated to minimize the sum of squared errors, as defined by equation (1). A key advantage of this approach is the prioritization of measurements with lower standard deviation errors, facilitating the integration of sensors with varying levels of quality into the estimation process.

In the context of addressing the problem outlined in (2), the normal equations method, as described in [37], proves to be a valuable approach. This method allows for the determination of the optimal solution for state variables at each iteration, denoted by The solution is obtained through the iterative solution of the following linear system of equations:

$$\mathbf{J}^T_{\nu}\,\mathbf{R}^{-}\,\mathbf{J}_{\nu} \quad {}_{\nu} \qquad \mathbf{J}^T_{\nu}\,\mathbf{R}^{-} \qquad\qquad {}_{\nu} \tag{3}$$

where is utilized as an iteration counter, ${}_{\nu}$ represents the estimated state variable solution at iteration , $\mathbf{J}_{\nu}$ $\mathbb{R}^{m\times n}$ stands for the Jacobian measurement matrix at the state estimate ${}_{\nu}$, and ${}_{\nu}$ corresponds to the vector of measurements predicted by the system model based on the state estimate ${}_{\nu}$.

According to equation (3), a theoretical and sufficient condition for the existence of a unique solution to the SE problem (2) is that the system is determined and compatible. This implies that the $\mathbf{J}$ matrix must have full rank, denoted as .

The full rank Jacobian condition, which makes matrix $\mathbf{J}^T_{\nu}\,\mathbf{R}^{-}\,\mathbf{J}_{\nu}$ invertible, identifies the system as observable or unobservable. The measurement Jacobian matrix plays a crucial role for the system to be observable. Besides, the matrix maintains the structural relations among measurements and state variables even if the equation (1) is linearized around any point by differentiating:

$$\mathbf{J} \tag{4}$$

In this context, represents the differential measurement residual vector, signifies the differential change in the system state, and corresponds to the differential change in errors, Observability Analysis serves a dual purpose. It not only helps resolve the question of whether observability is achieved but also enables the exploration of vital inquiries essential for effectively monitoring any given structure.

The information about the interdependencies among measurements and other variables for OA purposes is consolidated in the measurement Jacobian matrix at any given state for the structural system. Thus, this analysis is independent with respect to the uncertainty associated with measurements, since it is only based on the relations among variables due to the system topology. Therefore, it is standard practice to define the state variables and explain how the measurement Jacobian matrix can be computed for any structural system associated with status .

## 3. STRUCTURAL SYSTEM MODEL

In the case of static structural analysis, a statement of the equilibrium conditions together with strength of materials theory leads to a relation between forces and displacements that has the form of a system of equations:

$$(5)$$

where is the stiffness matrix, and and are the column matrices of nodal displacements and forces, respectively, in which the stiffness matrix is a singular matrix that leads to a system with infinitely many solutions. It is required to add some boundary conditions for obtaining a new system with a unique solution. Note that from the physical point of view this means that the structure is not a mechanism, i.e., it can not translate or rotate in the space, the only movements allowed are due to the deformation induced by the forces acting on the structure.

Continuing, the critical aspect of selecting state variables will be examined, emphasizing its pivotal role in ensuring precision and effectiveness in the analysis of State Estimation (SE).

### 3.1. Selection of the state variables

A set of state variables represents the minimal set of variables necessary to compute the values of all other system variables using the structural model defined by equation (5). Consequently, the selection of state variables is not unique. In structural systems, the state variables are typically chosen as the displacement conditions associated with nodal boundary conditions and nodal external forces that differ from those imposed by the boundary conditions, i.e., $^{n\times}$ $^{n-r\times}$ $^{r\times}$, where the superscripts indicate vector dimensions and is the number of degrees of freedom associated with the boundary conditions.

### 3.2. Relations among non-state and state variables

Equation (1) states that there is a functional relation among measurements and state variables. This linear relation is derived from equation (5). Assuming that state variables are known, the rest of variables can be obtained from equation (5) by partitioning the system of equations as follows:

$$^{r\times n-r} \quad ^{r\times r} \quad ^{n-r\times} \quad ^{r\times}$$
$$^{n-r\times n-r} \quad ^{n-r\times r} \quad ^{r\times} \quad ^{n-r\times} \qquad (6)$$

In Equation (6), the subindex correspond to the unknown variables within the system, while subindex represents known variables. The variables are defined as follows:

$^{r\times\, n-r}$ : A submatrix of the stiffness matrix that relates unknown displacements to unknown forces.

$^{r\times r}$ : A submatrix of the stiffness matrix that connects unknown displacements to known forces.

$^{n-r\,\times\, n-r}$ : A submatrix of the stiffness matrix , linking known displacements to unknown forces.

$^{n-r\,\times r}$ : A submatrix of the stiffness matrix , establishing connections between known displacements and known forces.

$^{n-r\,\times}$ : A vector representing unknown displacements.

$^{r\times}$ : A vector representing known displacements.

$^{r\times}$ : A vector representing unknown forces.

$^{n-r\,\times}$ : A vector representing known forces.

In order to split the unknowns and the state variables (known variables) the system can be written equivalently as follows:

$$
\begin{matrix} {}^{n-r\times n-r} & {}^{n-r\times r} & {}^{n-r\times} & {}^{n-r\times n-r} & {}^{n-r\times r} & {}^{n-r\times} \\ {}^{r\times n-r} & {}^{r\times r} & {}^{r\times} & {}^{r\times n-r} & {}^{r\times r} & {}^{r\times} \end{matrix} \tag{7}
$$

In Equation (7):

: Null matrix.

: Identity matrix.

where matrices on the left and right side of the equality correspond to and , respectively. These matrices play a crucial role in defining the relations between non-state variables and state variables in the context of the structural system. Matrices and are used to compute the non-state variables as a function of the state variables by solving the system of equations:

$$
\begin{matrix} {}^{n-r\times} \\ {}^{r\times} \end{matrix} \qquad {}^{-} \tag{8}
$$

Note that the inverse $^{-}$ exist if boundary conditions are properly chosen to avoid mechanisms.

### 3.3. Definition of the measurements

From a SE perspective, it is essential to consider various types of measurements related to the variables defined earlier. In this work, four measurement types are associated with the previously defined variables:

1. Measurements of external forces associated with degrees of freedom not constrained by boundary conditions .
2. Displacements and rotations associated with degrees of freedom constrained by boundary conditions , usually this value is null but it might have some uncertainty associated with structure foundations.
3. Displacements and rotations associated with degrees of freedom not constrained by boundary conditions .
4. Measurements of external forces associated with degrees of freedom constrained by boundary conditions .

Thus the vector including all possible measurements in the structural system corresponds to:

$$ ^{n\times} \tag{9} $$

where the tilde refers to measurements.

Obviously, in practical cases, not all measurement types are encompassed within the vector of measurements. This raises the fundamental question of whether a measurement subset is adequate for the estimation of state variables. In other words, observability analysis within the realm of SE precisely aims to determine whether problem (2) can be effectively solved given a measurement subset $^{m\times}$ $^{n\times}$.

### 3.4. *The measurement Jacobian matrix*

The measurement Jacobian matrix includes the first-order partial derivatives of all the variables that can be measured in the system with respect to the nodal heads, i.e. state variables. The structure of the Jacobian matrix for a generic structural system is as follows: wrong

$$\mathbf{J}^{n\times n} \quad {}^{n-r\times} \quad {}^{r\times} \quad {}^{n-r\times} \quad {}^{r\times} \quad {}^{n-r\times} \quad {}^{r\times} \tag{10}$$

In these equation, represents the total number of degrees of freedom within the structural system, while represents the number of degrees of freedom associated with boundary conditions. The upper block of the Jacobian associated with state variable measurements and state variables correspond to the identity matrix , while the lower block corresponds to the partial derivatives of the unknown variables with respect to the state variables, which can be obtained differentiating expression from (8), being the corresponding derivatives equal to $-$ . The Jacobian results in:

$$\mathbf{J}^{n\times n} \quad {}^{n\times} \quad {}^{n\times} \quad {}^{n\times} \tag{11}$$

The primary objective has been to acquire information about the current state of the system, particularly in terms of forces and displacements. However, in the context of structural identification it is more relevant to infer the stiffness of the elements, which are implicitly used to compute the stiffness matrix . In this paper, it is assumed that all structural elements have the axial and flexural stiffness and . The relation between state variables and the rest of variables in the structural system given by expression (8) presumes the knowledge of the element stiffness, Therefore, in this context, the parameter vector ${}^{p\times}$ , where represents the number of parameters considered, will be incorporated into the SE problem. In fact, these parameters might also be considered as state variables. As a result, equation (1) is modified to:

(12)

while problem (2) becomes:

$$\text{Minimum} \quad {}^{T}\mathbf{R}^{-} \quad {}^{T}\mathbf{R}^{-} \tag{13}$$

where parameter constitutes a decision variable in the optimization/estimation problem. In the context of SE this problem is typically known as *calibration* [26]. Finally, the Jacobian (11) requires to be updated as follows:

$$\mathbf{J}^{\,n\times n\ p} \quad n\times \quad n\times \quad p\times \quad n\times \tag{14}$$

where matrix $\frac{\partial}{\partial}$ corresponds to the partial derivatives of non-state variables with respect to parameters, which has to be computed. In this particular case, a nonlinear system is encountered because forces are equal to the product of stiffness and displacements. The differentiation of the two expressions obtained from system (6) is performed as follows:

(15)

By rearranging terms, it is possible to express unknown variables as functions of state variables and stiffness parameters:

(16)

From (16), the required elements in the Jacobian matrix correspond to:

$$_{n\times n} \quad \begin{matrix} _{n-r\times n-r} & _{n-r\times r} \\ _{r\times n-r} & _{r\times r} \end{matrix} \quad \begin{matrix} _{n-r\times n-r} & _{n-r\times r} \\ _{r\times n-r} & _{r\times r} \end{matrix} \quad - \tag{17}$$

and

$$_{n\times p} \quad \begin{matrix} _{n-r\times n-r} & _{n-r\times r} \\ _{r\times n-r} & _{r\times r} \end{matrix} \quad \begin{matrix} _{n-r\times p} \\ _{r\times p} \end{matrix} \quad - \tag{18}$$

To conclude, it is essential to emphasize that in order to apply the proposed technique, a numerical instance of the Jacobian matrix J is required to particularize (14) for any likely and realistic physical status of the system as in equation (4). Besides, since the aim of this work is to focus on OA, it is also possible to perform the analysis assuming that the current values of parameter vector are all equal to a known value (e.g. the unity). This strategy reduces numerical errors derived of the application of the observability algorithm. It should be noted that the use of an algebraic method analyzes not only topological but also numerical observability. Nevertheless, it is unlikely to detect unobservable numerical systems that are, at the same time, topologically observable.

### *3.5. Elementary element stiffness matrix and derivatives*

In the context of 2D structures loaded in their plane modelled by Bernoulli beam elements, the element stiffness matrix (as depicted in Figure 1) corresponds to the following matrix:

$$\left[\begin{array}{ccc|ccc} \frac{EA}{L} & & & \frac{EA}{L} & & \\ & \frac{EI}{L^3} & \frac{EI}{L^2} & & \frac{EI}{L^3} & \frac{EI}{L^2} \\ & \frac{EI}{L^2} & \frac{EI}{L} & & \frac{EI}{L^2} & \frac{EI}{L} \\ \hline \frac{EA}{L} & & & \frac{EA}{L} & & \\ & \frac{EI}{L^3} & \frac{EI}{L^2} & & \frac{EI}{L^3} & \frac{EI}{L^2} \\ & \frac{EI}{L^2} & \frac{EI}{L} & & \frac{EI}{L^2} & \frac{EI}{L} \end{array}\right] \tag{19}$$

In this context, the variables are defined as follows:

and represent horizontal and vertical displacements respectively.

represents rotations.

is the element stiffness matrix, and its components are associated with the displacements and rotations at both nodes.

represents the axial stiffness of the element.

represents the flexural stiffness of the element.

The derivatives with respect to the products and are, respectively, as follows:

$$\frac{\quad}{\quad} \left[\begin{array}{cc|cc} \frac{}{L} & & \frac{}{L} & \\ & & & \\ \hline \frac{}{L} & & \frac{}{L} & \\ & & & \end{array}\right] \tag{20}$$

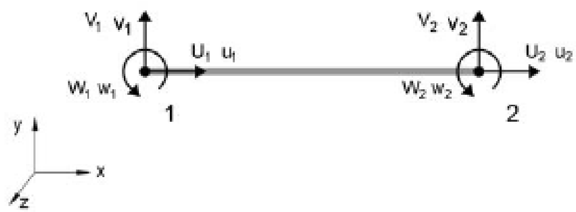


**Figure 1:** Node forces, moments, displacements and rotations for an elementary horizontal bar.

and

$$\underline{\qquad}\quad \left[\begin{array}{cc|cc} & & & \\ \overline{L^3} & \overline{L^2} & \overline{L^3} & \overline{L^2} \\ \overline{L^2} & \overline{L} & \overline{L^2} & \overline{L} \\ \hline & & & \\ \overline{L^3} & \overline{L^2} & \overline{L^3} & \overline{L^2} \\ \overline{L^2} & \overline{L} & \overline{L^2} & \overline{L} \end{array}\right] \tag{21}$$

The calculation of the stiffness matrix derivatives with respect to the stiffness parameters , as needed in expression (18), becomes straightforward when using these pre-defined matrices in combination with the typical assembly process for constructing the complete structural stiffness matrix. It is important to note that the assembly process involves integrating the relevant coordinate transformations for the individual elements within the structure. Furthermore, it is worth highlighting that the force labels correspond to the corresponding displacement labels, all represented in capital letters.

## 4. Algebraic observability analysis

The general solution of the system of linear equations given by equation (4) is:

$$ \tag{22} $$

where is a null-space matrix and a parameter vector. Equation (22) provides a unique solution if and only if null-space matrix is nil. Thus, problem (13) is well posed if and only if is null. Row-wise, (22) is:

$$ {}_{i} \quad {}_{i} \quad \sum_{j}^{k} {}_{ij} \, {}_{j} \tag{23} $$

thus, variable $_i$ (or state variable $_i$) is observable if and only if $_{ij}$ ; otherwise it is not.

The observability of the remaining variables is a key consideration. Once the determination has been made regarding the observability of state variables, as assessed by the Jacobian matrix, the observability status of other variables can also be established. If the Jacobian matrix's row components associated with a particular variable depend on any non-observable variables (i.e., the corresponding columns are not null), then that variable is deemed non-observable. However, if the values associated with non-observable variables are all null, the variable is considered observable because it only depends on observable state variables.

Additional algorithms, as proposed by Pruneda et al. [33], are available, which not only facilitate the computation of observability associated with state and non-state variables but also support the identification of all observable variables for a given measurement configuration and the categorization of measurements into different types (essential, redundant, critical, etc.). It is important to consider that in the Jacobian definition provided in this manuscript, as illustrated in equation (14), all possible measurements (i.e., all rows) are taken into account. However, for the OA, only rows associated with variables belonging to the measurement set should be included in the computations.

## 5. Illustrative example

The example under consideration was originally introduced by Lozano-Galant et al. [38]. Figure 2 provides a visual representation of the system's layout, which comprises three nodes and two elementary beams. The first beam element 1 is oriented vertically, with a length of 6 meters and mechanical properties represented by and , while the second beam element 2 is oriented horizontally, with a length of 4 meters and mechanical properties represented by and . All mechanical properties are set to unity to reduce numerical errors since their values do not impact the overall results of the analysis.

In this context, the variables are defined as follows:

- and represent horizontal and vertical displacements, respectively.
- represents rotations.
- and represent horizontal and vertical forces, respectively.
- represents moments.

Imposed boundary conditions require specific displacements to be set to zero, namely . Furthermore, the forces corresponding to degrees of freedom not constrained by these boundary conditions are known. Specifically, , , and .

This information about boundary conditions could be considered as a measurement with its associated uncertainty, or it could even be considered an error free measurement. The second assumption requires an special treatment during the SE inference process but, since from the observability point of view measurement uncertainty does not affect results, it does not really matter how those measurements are treated for the OA. In this example, there are two additional sensors which provide displacements and . The two beams have different mechanical characteristics, so the vector of parameters corresponds to

State variables encompass a collection of variables associated with the boundary conditions and loading case of the system, represented as:

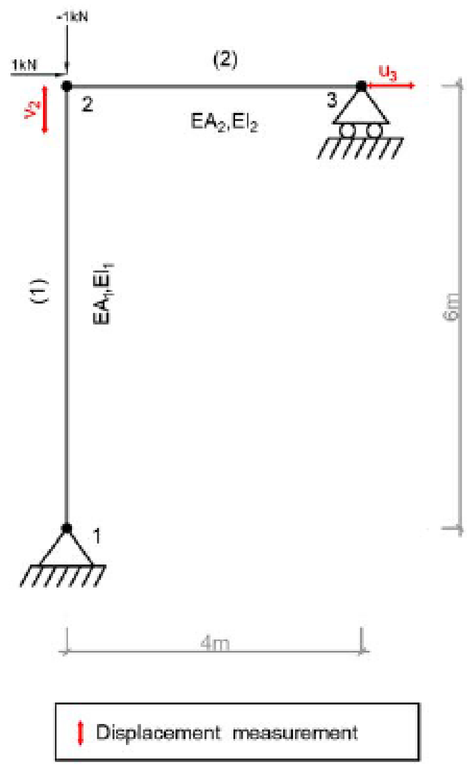


**Figure 2:** Structure with applied load and measured displacements.

Observable state variables and parameters are the ones whose values can be calculated based on the given boundary conditions, loading case, and the measured data available. The vector of measurements corresponds to:

The methodology involves the following steps:

1. Compute the stiffness matrix and the derivatives with respect to parame-

ters. The corresponding matrices are given below:

$$\mathbf{K} = \begin{array}{c|ccc|ccc|ccc|} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 & u_3 & v_3 & w_3 \\ \hline u_1 & \frac{1}{18} & 0 & -\frac{1}{6} & -\frac{1}{18} & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ v_1 & 0 & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ w_1 & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \hdashline u_2 & -\frac{1}{18} & 0 & \frac{1}{6} & \frac{11}{36} & 0 & \frac{1}{6} & -\frac{1}{4} & 0 & 0 \\ v_2 & 0 & -\frac{1}{6} & 0 & 0 & \frac{17}{48} & \frac{3}{8} & 0 & -\frac{3}{16} & \frac{3}{8} \\ w_2 & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} & \frac{3}{8} & \frac{5}{3} & 0 & -\frac{3}{8} & \frac{1}{2} \\ \hdashline u_3 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & -\frac{3}{16} & -\frac{3}{8} & 0 & \frac{3}{16} & -\frac{3}{8} \\ w_3 & 0 & 0 & 0 & 0 & \frac{3}{8} & \frac{1}{2} & 0 & -\frac{3}{8} & 1 \end{array} \tag{24}$$

$$\frac{\partial \boldsymbol{K}}{\partial (EA_1)} = \begin{array}{c|ccc|ccc|ccc|} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 & u_3 & v_3 & w_3 \\ \hline u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & 0 & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} ; \quad \frac{\partial \boldsymbol{K}}{\partial (EA_2)} = \begin{array}{c|ccc|ccc|ccc|} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 & u_3 & v_3 & w_3 \\ \hline u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline u_2 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline u_3 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{25}$$

$$\frac{\partial \boldsymbol{K}}{\partial (EI_1)} = \begin{array}{c|ccc|ccc|ccc|} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 & u_3 & v_3 & w_3 \\ \hline u_1 & \frac{1}{18} & 0 & -\frac{1}{6} & -\frac{1}{18} & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1 & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \hdashline u_2 & -\frac{1}{18} & 0 & \frac{1}{6} & \frac{1}{18} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ \hdashline u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} ; \quad \frac{\partial \boldsymbol{K}}{\partial (EI_2)} = \begin{array}{c|ccc|ccc|ccc|} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 & u_3 & v_3 & w_3 \\ \hline u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & \frac{3}{16} & \frac{3}{8} & 0 & -\frac{3}{16} & \frac{3}{8} \\ w_2 & 0 & 0 & 0 & 0 & \frac{3}{8} & 1 & 0 & -\frac{3}{8} & \frac{1}{2} \\ \hdashline u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & -\frac{3}{16} & -\frac{3}{8} & 0 & \frac{3}{16} & -\frac{3}{8} \\ w_3 & 0 & 0 & 0 & 0 & \frac{3}{8} & \frac{1}{2} & 0 & -\frac{3}{8} & 1 \end{array} \tag{26}$$

2. Using the previous matrices, the information about state variables and expressions (17) and (18), compute the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ used to compose the Jacobian (14). The corresponding matrices are given below:

$$
\boldsymbol{U} = \begin{array}{c} \\ \bar{w}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{w}_2 \\ \bar{u}_3 \\ \bar{w}_3 \\ \bar{U}_1 \\ \bar{V}_1 \\ \bar{V}_3 \end{array}
\!\!\begin{array}{c} \begin{array}{ccccccccc} W_1 & U_2 & V_2 & W_2 & U_3 & W_3 & u_1 & v_1 & v_3 \end{array} \\
\begin{bmatrix}
\frac{185}{24} & -\frac{113}{4} & -\frac{3}{2} & \frac{41}{24} & -\frac{113}{4} & -\frac{7}{24} & 0 & -\frac{1}{4} & \frac{1}{4} \\
-\frac{113}{4} & \frac{267}{2} & 9 & -\frac{41}{4} & \frac{267}{2} & \frac{7}{4} & 1 & \frac{3}{2} & -\frac{3}{2} \\
-\frac{3}{2} & 9 & 6 & -\frac{3}{2} & 9 & -\frac{3}{2} & 0 & 1 & 0 \\
\frac{41}{24} & -\frac{41}{4} & -\frac{3}{2} & \frac{41}{24} & -\frac{41}{4} & -\frac{7}{24} & 0 & -\frac{1}{4} & \frac{1}{4} \\
-\frac{113}{4} & \frac{267}{2} & 9 & -\frac{41}{4} & \frac{275}{2} & \frac{7}{4} & 1 & \frac{3}{2} & -\frac{3}{2} \\
-\frac{7}{24} & \frac{7}{4} & -\frac{3}{2} & -\frac{7}{24} & \frac{7}{4} & \frac{41}{24} & 0 & -\frac{1}{4} & \frac{1}{4} \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & -\frac{3}{2} & -1 & \frac{1}{4} & -\frac{3}{2} & \frac{1}{4} & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{3}{2} & 0 & -\frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 0 & 0
\end{bmatrix} \end{array};\;
\boldsymbol{V} = \begin{array}{c} \\ \bar{w}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{w}_2 \\ \bar{u}_3 \\ \bar{w}_3 \\ \bar{U}_1 \\ \bar{V}_1 \\ \bar{V}_3 \end{array}
\!\!\begin{array}{c} \begin{array}{cccc} EA_1 & EI_1 & EA_2 & EI_2 \end{array} \\
\begin{bmatrix}
\frac{3}{4} & 18 & 0 & 8 \\
-\frac{9}{2} & -72 & 0 & -48 \\
-3 & 0 & 0 & 0 \\
\frac{3}{4} & 0 & 0 & 8 \\
-\frac{9}{2} & -72 & 0 & -48 \\
\frac{3}{4} & 0 & 0 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \end{array}
\tag{27}
$$

3. The full Jacobian (14) of this example corresponds to the following matrix:

$$
\boldsymbol{J} = \begin{array}{c} \\ \bar{W}_1 \\ \bar{U}_2 \\ \bar{V}_2 \\ \bar{W}_2 \\ \bar{U}_3 \\ \bar{W}_3 \\ \bar{u}_1 \\ \bar{v}_1 \\ \bar{v}_3 \\ \bar{\mathbf{w}}_1 \\ \bar{\mathbf{u}}_2 \\ \bar{v}_2 \\ \bar{\mathbf{w}}_2 \\ \bar{u}_3 \\ \bar{\mathbf{w}}_3 \\ \bar{\mathbf{U}}_1 \\ \bar{\mathbf{V}}_1 \\ \bar{\mathbf{V}}_3 \end{array}
\!\!\begin{array}{c} \begin{array}{ccccccccccccc} W_1 & U_2 & V_2 & W_2 & U_3 & W_3 & u_1 & v_1 & v_3 & EA_1 & EI_1 & EA_2 & EI_2 \end{array} \\
\left[\begin{array}{ccccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
\mathbf{\frac{185}{24}} & \mathbf{-\frac{113}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{41}{24}} & \mathbf{-\frac{113}{4}} & \mathbf{-\frac{7}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{\frac{3}{4}} & \mathbf{18} & \mathbf{0} & \mathbf{8} \\
\mathbf{-\frac{113}{4}} & \mathbf{\frac{267}{2}} & \mathbf{9} & \mathbf{-\frac{41}{4}} & \mathbf{\frac{267}{2}} & \mathbf{\frac{7}{4}} & \mathbf{1} & \mathbf{\frac{3}{2}} & \mathbf{-\frac{3}{2}} & \mathbf{-\frac{9}{2}} & \mathbf{-72} & \mathbf{0} & \mathbf{-48} \\
-\frac{3}{2} & 9 & 6 & -\frac{3}{2} & 9 & -\frac{3}{2} & 0 & 1 & 0 & -3 & 0 & 0 & 0 \\
\mathbf{\frac{41}{24}} & \mathbf{-\frac{41}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{41}{24}} & \mathbf{-\frac{41}{4}} & \mathbf{-\frac{7}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{8} \\
-\frac{113}{4} & \frac{267}{2} & 9 & -\frac{41}{4} & \frac{275}{2} & \frac{7}{4} & 1 & \frac{3}{2} & -\frac{3}{2} & -\frac{9}{2} & -72 & 0 & -48 \\
\mathbf{-\frac{7}{24}} & \mathbf{\frac{7}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{-\frac{7}{24}} & \mathbf{\frac{7}{4}} & \mathbf{\frac{41}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{\frac{3}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{-4} \\
\mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{\frac{1}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{-1} & \mathbf{\frac{1}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{1}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{-\frac{1}{4}} & \mathbf{\frac{3}{2}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{3}{2}} & \mathbf{-\frac{1}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right] \end{array}
\tag{28}
$$

where boldfaced rows correspond to rows which should be removed because they correspond to variables not measured in the system.

4. Computation of the null space associated with the measurement Jacobian, i.e. expression (28) removing the boldfaced rows, the following result is obtained:

(29)

\- - - - - - - - - - - -

Note that according to (29), all state variables are observable but parameters , because the corresponding rows contain elements different from zero (boldfaced elements in matrix ). Therefore, observable state variables correspond to boldfaced variables in the following list:

5. The remaining variables not included in the state variable list, once the observable state variables have been determined, can be used for identifying additional observable variables by employing the Jacobian (14) (or expression (28) for the example). For instance:
   (a) Variable , , and are **not observable**, because their corresponding Jacobian's rows contains non-null elements associated with non-observable state variables and .
   (b) Variable and are **observable** because they are measured.

(c) Variable , and are **observable**, because their corresponding Jacobian's rows contain null elements associated with non-observable state variables, i.e. this variable do not depend on non-observable state variables.

In summary, the boldfaced variables in the list are additional observable variables:

This result is totally consistent with that obtained by Lozano-Galant et al. [38].

To delve deeper into the proposed methodology, an analysis is conducted regarding the enhancement of observability for stiffness parameters , , and by introducing supplementary measurements. This aspect can also be addressed using the previously discussed Jacobian matrix:

1. It should be noted that, under the present measurement configuration and load case, it is not feasible to establish the observability of the axial stiffness . This is evident from the fact that the corresponding column of the Jacobian matrix is null. Even if all nodal variables were measured, it would still be impossible to deduce the value of . The underlying issue lies in the inadequacy of the loading case.
2. For the flexural stiffness , measuring or would make the variable observable. It is worth noting that the elements related to the variables in (28) are only non-null for and , with values of 8 and -4, respectively.
3. For the flexural stiffness , it is not possible to make it observable without making also observable. Measurements of variables , and are possible candidates for making the variable observable, however, these measurements also contain non-null elements related to , which implies that two additional measurements must be incorporated simultaneously.

Nonetheless, adherence to the same approach as outlined by Lozano-Galant et al. [38] will be followed. Firstly, a modification is needed in the structural model, transitioning from the configuration presented in Figure 2 to the revised layout depicted in Figure 3. This adaptation introduces an intermediate node in the beam element 2 and divides it into two equal beam elements with identical mechanical properties represented by and . the observability of the flexural stiffness, , can be achieved when measurements are taken for both the vertical deflections and at the node 3.

Secondly, the axial stiffness, , can only be observed when measurements of are available. Moreover, the load case must be adjusted to activate the axial

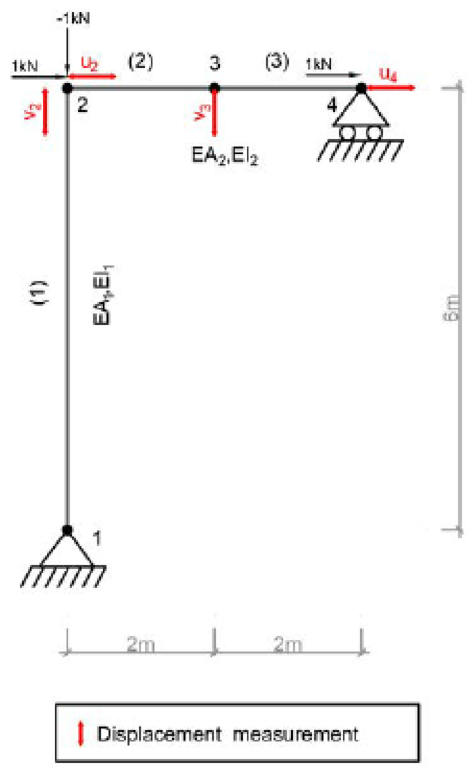


**Figure 3:** Modified structure with applied load and measured displacements.

stiffness of the beam element. This can be achieved by applying a horizontal load at node 4 of Figure 3 with a non-zero value. Consequently, the displacement is also measured and will differ from .

1. In this particular case, the full Jacobian (11) of this example corresponds to

the following matrix:

$$
J = \begin{array}{c|cccccccccccc|cccc|}
 & W_1 & U_2 & V_2 & W_2 & U_3 & V_3 & W_3 & U_4 & W_4 & u_1 & v_1 & v_4 & EA_1 & EI_1 & EA_2 & EI_2 \\
\tilde{W}_1 & \frac{1}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{U}_2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{V}_2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{W}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{U}_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{V}_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{W}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{U}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{W}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{u}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{v}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\tilde{v}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
\tilde{\mathbf{w}}_1 & \mathbf{\frac{185}{24}} & \mathbf{-\frac{113}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{41}{24}} & \mathbf{-\frac{113}{4}} & \mathbf{\frac{1}{4}} & \mathbf{\frac{5}{24}} & \mathbf{-\frac{113}{4}} & \mathbf{-\frac{7}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{3} & \mathbf{36} & \mathbf{0} & \mathbf{16} \\
\tilde{u}_2 & -\frac{113}{4} & \frac{267}{2} & 9 & -\frac{41}{4} & \frac{267}{2} & -\frac{3}{2} & -\frac{5}{4} & \frac{267}{2} & \frac{7}{4} & 1 & \frac{3}{2} & -\frac{3}{2} & -18 & -144 & 0 & -96 \\
\tilde{v}_2 & -\frac{3}{2} & 9 & 6 & -\frac{3}{2} & 9 & 3 & -\frac{3}{2} & 9 & -\frac{3}{2} & 0 & 1 & 0 & -\frac{12}{1} & 0 & 0 & 0 \\
\tilde{\mathbf{w}}_2 & \mathbf{\frac{41}{24}} & \mathbf{-\frac{41}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{41}{24}} & \mathbf{-\frac{41}{4}} & \mathbf{\frac{1}{4}} & \mathbf{\frac{5}{24}} & \mathbf{-\frac{41}{4}} & \mathbf{-\frac{7}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{16} \\
\tilde{\mathbf{u}}_3 & \mathbf{-\frac{113}{4}} & \mathbf{\frac{267}{2}} & \mathbf{9} & \mathbf{-\frac{41}{4}} & \mathbf{\frac{271}{2}} & \mathbf{-\frac{3}{2}} & \mathbf{-\frac{5}{4}} & \mathbf{\frac{271}{2}} & \mathbf{\frac{7}{4}} & \mathbf{1} & \mathbf{\frac{3}{2}} & \mathbf{-\frac{3}{2}} & \mathbf{-18} & \mathbf{-144} & \mathbf{-2} & \mathbf{-96} \\
\tilde{v}_3 & \frac{1}{4} & -\frac{3}{2} & 3 & \frac{1}{4} & -\frac{3}{2} & \frac{17}{6} & -\frac{3}{4} & -\frac{3}{2} & -\frac{7}{4} & 0 & \frac{1}{2} & \frac{1}{2} & -6 & 0 & 0 & 12 \\
\tilde{\mathbf{w}}_3 & \mathbf{\frac{5}{24}} & \mathbf{-\frac{5}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{5}{24}} & \mathbf{-\frac{5}{4}} & \mathbf{-\frac{3}{4}} & \mathbf{\frac{17}{24}} & \mathbf{-\frac{5}{4}} & \mathbf{\frac{5}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{-2} \\
\tilde{u}_4 & -\frac{113}{4} & \frac{267}{2} & 9 & -\frac{41}{4} & \frac{271}{2} & -\frac{3}{2} & -\frac{5}{4} & \frac{275}{2} & \frac{7}{4} & 1 & \frac{3}{2} & -\frac{3}{2} & -18 & -144 & -4 & -96 \\
\tilde{\mathbf{w}}_4 & \mathbf{-\frac{7}{24}} & \mathbf{\frac{7}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{-\frac{7}{24}} & \mathbf{\frac{7}{4}} & \mathbf{-\frac{7}{4}} & \mathbf{\frac{5}{24}} & \mathbf{\frac{7}{4}} & \mathbf{\frac{41}{24}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{1}{4}} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{-8} \\
\tilde{\mathbf{U}}_1 & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\tilde{\mathbf{V}}_1 & \mathbf{\frac{1}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{-1} & \mathbf{\frac{1}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{-\frac{1}{2}} & \mathbf{\frac{1}{4}} & \mathbf{-\frac{3}{2}} & \mathbf{\frac{1}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\tilde{\mathbf{V}}_4 & \mathbf{-\frac{1}{4}} & \mathbf{\frac{3}{2}} & \mathbf{0} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{3}{2}} & \mathbf{-\frac{1}{2}} & \mathbf{-\frac{1}{4}} & \mathbf{\frac{3}{2}} & \mathbf{-\frac{1}{4}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array} \quad (30)
$$

where boldfaced rows correspond to rows which should be removed because they correspond to variables not measured in the system.

2. Computation of the null space associated with the measurement Jacobian (30) removing the boldfaced rows, provides the following result :

$$
N = \emptyset \qquad (31)
$$

This implies that all state variables are indeed observable. It is important to emphasize that, according to the definition of *state variables*, having knowledge of both the state variables and the stiffness parameters of the model enables deducing the values of all other variables.

In the MATLAB-based evaluation of both methods, while this result is entirely consistent with the findings presented in Lozano-Galant et al. [38], it is important to note that OM required two recursive steps to finalize the analysis as shown in Figure 4.

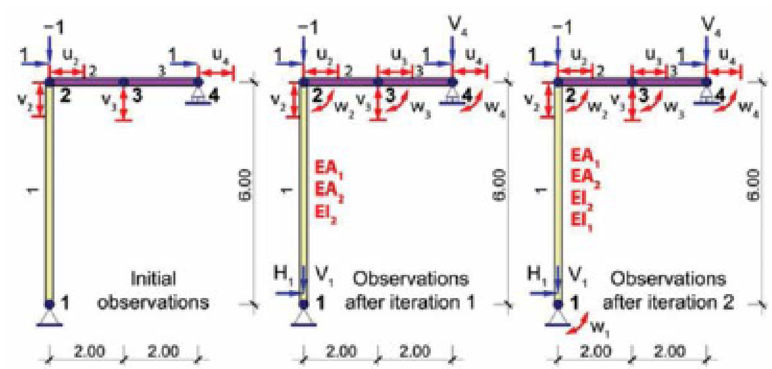


**Figure 4:** Modified structure: OM recursive steps with corresponding observed variables.

## 6. Case study

This example is drawn from a two-span continuous beam previously examined by Nogal et al. [17] using OM, as illustrated in Figure 5.a. The girder possesses a cross-sectional area of m and a moment of inertia of m . Its material properties encompass a modulus of elasticity , set at GN/m , and a density equal to kg/m . The Finite Element Model (FEM) adopted for this structure comprises 61 nodes and 60 beam elements. These beam elements are characterized by flexural stiffness labeled through and axial stiffness identified as through , as depicted in Figure 5.b. It is worth emphasizing that, within the context of the OA, these mechanical properties are considered to be unknown.

The measurement setup replicated the configuration utilized by Nogal et al. [17], encompassing the acquisition of 60 measurements involving both deflections and rotations. Specifically, this setup includes 58 common vertical deflections, spanning from to and to , in addition to two separate rotations: and . The load case entails the application of a concentrated load of 100 kN

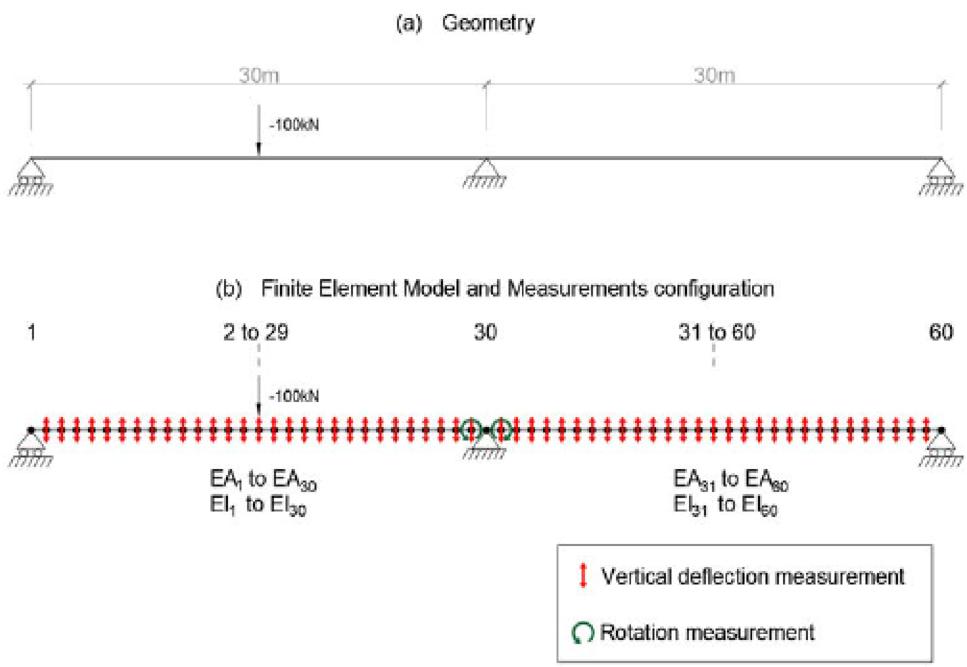


**Figure 5:** Case study: (a) Geometric representation of the structure analyzed in the case study and (b) Finite Element Model (FEM) of the structure along with the measurement configuration.

at node 16. From the State Estimation perspective, the boundary conditions and known nodal forces were considered as measurements.

By employing the proposed method, the Jacobian matrix was calculated. Subsequently, an examination of the null space of this Jacobian indicates that all the flexural stiffness variables within the range from to for the beam elements are indeed observable. This finding is consistent with the results obtained by Nogal et al. [17].

### *6.1. Comparison among methods*

In the evaluation of OM and the proposed method, using MATLAB and applied to the same case study, the severe contrast in computational time became evident. OM demanded 40.24 seconds to complete the analysis, while the proposed method accomplished the task in a mere 0.11 seconds, signifying a substantial efficiency improvement and a 99.73% reduction in computation time.

Table 1 shows that OM required 31 recursive steps to analyze the observability of all variables in this case study, with the number of steps aligning with the number of unknowns. In contrast, the proposed method efficiently obtained the results by directly computing the null space of the measurement Jacobian. The use of the measurement Jacobian provides a direct and more efficient approach to establishing the relation between state variables and measurements, eliminating the need for symbolic analysis, which introduces coupled variables. The proposed approach simplifies the identification of observable state variables while avoiding complications associated with the recursive process required to address coupled variables in OM.

Table 2 provides a comprehensive comparison between the two methods. The analysis of this table reveals that the proposed method outperforms OM in several key aspects, including computational efficiency, and practicality. It streamlines the process of determining observable state variables, and potentially reduces the computational burden involved in the system analysis.

**Table 1:** Case Study: OM's recursive process with flexural stiffness variables observed at each step.

| Step | Observed flexural stiffness variables |
| --- | --- |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| 12 | |
| 13 | |
| 14 | |
| 15 | |
| 16 | |
| 17 | |
| 18 | |
| 19 | |
| 20 | |
| 21 | |
| 22 | |
| 23 | |
| 24 | |
| 25 | |
| 26 | |
| 27 | |
| 28 | |
| 29 | |
| 30 | |
| 31 | |

**Table 2:** Comparison of OA based on SE and OM.

| Aspect | Proposed Method | OM |
|---|---|---|
| Symbolic Analysis | No | Yes |
| Recursive Analysis | No | Yes |
| Coupled variables | No | Yes |
| Computational Complexity | Lower | Higher |
| Computational Efficiency | Superior | Lower |
| Practicality | Higher | Lower |

## 7. Conclusions

This manuscript introduces a groundbreaking application of OA based on SE within the realm of Structural System Identification, with a strong focus on the OA aspect. The study showcases the development of a specialized SE model designed for structural systems, establishing a clear connection between the system's measurements and the underlying state variables and parameters through the Jacobian matrix.

By harnessing the Jacobian matrix, the identification of observable state variables becomes a straightforward and efficient process, eliminating the need for recursive analysis, which stands in huge contrast to the conventional OM. Instead of employing recursive process, this approach directly computes the null space of the measurement Jacobian to identify observable state variables. What sets this method apart is its numerical approach, which not only dispenses with the symbolic analysis required by OM but also reduces computational complexity. This feature streamlines the process, rendering it well-suited for real-world applications.

The efficacy of the method is demonstrated through an illustrative example and a case study involving a bridge structure. In both instances, it excels over OM in terms of computational efficiency and practicality. In the illustrative example, the proposed method effortlessly identifies observable state variables in a structural system, bypassing the need for recursion or symbolic analysis, as seen in the case of OM. The efficiency of the proposed method becomes even more evident in the case study, where OM necessitates significant computation time, primarily due to the escalating recursion steps required as the structural complexity increases. In contrast, the proposed method swiftly determines the observability of all relevant variables, resulting in a significant improvement in computational efficiency and a remarkable 99.73% reduction in computation time for this particular case study.

In summary, this methodology stands out as a practical and efficient tool for evaluating observability in structural systems, offering substantial value to engineers and researchers grappling with complex structures. Notably, the proposed method has demonstrated its capability to effectively tackle numerous challenges that have traditionally hindered existing methods in the field of Structural System Identification. Its potential to streamline the analysis process and reduce computational overhead holds great promise for real-world applications of Structural System Identification.

## 8. Acknowledgements

The authors would like to express their gratitude for the funding provided by the following projects: Project PID2021-126405OB-C31: ”Development of low-cost modular sensors for use in the structural identification of bridges subjected to quasi-static loads.” Project PID2021-126405OB-C32: ”Alarm system for bridge management systems with BIM digital twins using artificial intelligence.” These projects have been generously funded by MICIN (Ministry of Science and Innovation), AEI (State Research Agency), and the ”A way to make Europe” FEDER funds. Project CNS2022-135472, supported by the ”Grants to Encourage Research Consolidation” program, which is part of the ”State Program to Develop, Attract, and Retain Talent” within the State Plan for Scientific and Technical Research and Innovation for the period 2021-2023. Finally, scholarship funding from L’Agència de Gestió d’Ajuts Universitaris i de Recerca (AGAUR) through the ”Ajuts de suport a departaments i unitats de recerca universitaris per a la contractació de personal investigador predoctoral en formació (FI SDUR 2022)”.

## References

[1] B. Liu, J. Zhang, M. Lei, S. Yang, Z. Wang, Simultaneous tunnel defects and lining thickness identification based on multi-tasks deep neural network from ground penetrating radar images, Automation in Construction 145 (2023) 104633. doi:https://doi.org/10.1016/j.autcon.2022.104633.

[2] A. Moghadam, M. AlHamaydeh, R. Sarlo, Dual-purpose procedure for bridge health monitoring and weigh-in-motion used for multiple-vehicle events, Automation in Construction 148 (2023) 104768. doi:https://doi.org/10.1016/j.autcon.2023.104768.

[3] F. Parisi, A. Mangini, M. Fanti, J. M. Adam, Automated location of steel truss bridge damage using machine learning and raw strain sensor data, Automation in Construction 138 (2022) 104249. doi:https://doi.org/10.1016/j.autcon.2022.104249.

[4] T. Stepinski, T. Uhl, W. Staszewski, Advanced structural damage detection: from theory to engineering applications (2013).

[5] G. S. Urgessa, Vibration properties of beams using frequency-domain system identification methods, Journal of Vibration and Control 17 (2011) 1287–1294.

[6] J. A. Lozano-Galant, M. Nogal, E. Castillo, J. Turmo, Application of observability techniques to structural system identification, Computer-Aided Civil and Infrastructure Engineering 28 (2013) 434–450.

[7] R. Ghanem, M. Shinozuka, Structural-system identification. i: Theory, Journal of Engineering Mechanics 121 (1995) 255–264.

[8] S. Emadi, Y. Sun, J. A. Lozano-Galant, J. Turmo, Observing material properties in composite structures from actual rotations, Applied Sciences 13 (2023). doi:10.3390/app132011456.

[9] S. Quqa, L. Landi, P. P. Diotallevi, Automatic identification of dense damage-sensitive features in civil infrastructure using sparse sensor networks, Automation in Construction 128 (2021) 103740. doi:https://doi.org/10.1016/j.autcon.2021.103740.

[10] G. Sirca, H. Adeli, System identification in structural engineering, Scientia Iranica 19 (2012) 1355–1364. doi:https://doi.org/10.1016/j.scient.2012.09.002.

[11] F. N. Catbas, M. Malekzadeh, A machine learning-based algorithm for processing massive data collected from the mechanical components of movable bridges, Automation in Construction 72 (2016) 269–278. doi:https://doi.org/10.1016/j.autcon.2016.02.008.

[12] H. Adeli, X. Jiang, Dynamic fuzzy wavelet neural network model for structural system identification, Journal of Structural Engineering 132 (2006) 102–111. doi:10.1061/(ASCE)0733-9445(2006)132:1(102).

[13] R. Kalman, On the general theory of control systems, IFAC Proceedings Volumes 1 (1960) 491–502. doi:https://doi.org/10.1016/S1474-6670(17)70094-8, 1st International IFAC Congress on Automatic and Remote Control, Moscow, USSR, 1960.

[14] L. A. Aguirre, L. L. Portes, C. Letellier, Structural, dynamical and symbolic observability: From dynamical systems to networks, PLOS ONE 13 (2018) 1–21. doi:10.1371/journal.pone.0206180.

[15] A. Rouhani, A. Abur, Observability analysis for dynamic state estimation of synchronous machines, IEEE Transactions on Power Systems 32 (2017) 3168–3175. doi:10.1109/TPWRS.2016.2614879.

[16] A. N. Montanari, L. A. Aguirre, Observability of network systems: A critical review of recent results, Journal of Control, Automation and Electrical Systems 31 (2020) 1348–1374.

[17] M. Nogal, J. A. Lozano-Galant, J. Turmo, E. Castillo, Numerical damage identification of structures by observability techniques based on static loading tests, Structure and Infrastructure Engineering 12 (2016) 1216–1227.

[18] Y. Sun, Y. Xu, J. A. Lozano-Galant, X. Wang, J. Turmo, Analytical observability method for the structural system identification of wide-flange box girder bridges with the effect of shear lag, Automation in Construction 131 (2021) 103879. doi:https://doi.org/10.1016/j.autcon.2021.103879.

[19] J. Lei, D. Xu, J. Turmo, Static structural system identification for beam-like structures using compatibility conditions, Structural Control and Health Monitoring 25 (2018) e2062.

[20] Y. Fu, C. Peng, F. Gomez, Y. Narazaki, B. F. Spencer Jr, Sensor fault management techniques for wireless smart sensor networks in structural health monitoring, Structural Control and Health Monitoring 26 (2019) e2362.

[21] J. Lei, D. Xu, J. A. Lozano Galant, M. Nogal Macho, J. Turmo Coderque, Error analysis of structural system identification by observability method, in: IABSE Symposium: Vancouver, 2017: Engineering the Future, International Association for Bridge and Structural Engineers (IABSE), 2017, pp. 3261–3268.

[22] M. S. Agbabian, S. F. Masri, R. Miller, T. K. Caughey, System identification approach to detection of structural changes, Journal of Engineering Mechanics 117 (1991) 370–390.

[23] J.-H. Jang, I. Yeo, S. Shin, S.-P. Chang, Experimental investigation of system-identification-based damage assessment on structures, Journal of Structural Engineering 128 (2002) 673–682.

[24] J. Lei, M. Nogal, J. A. Lozano-Galant, D. Xu, J. Turmo, Constrained observability method in static structural system identification, Structural Control and Health Monitoring 25 (2018) e2040. doi:https://doi.org/10.1002/stc.2040, e2040 STC-16-0194.R4.

[25] T. Peng, M. Nogal, J. R. Casas, J. Turmo, Planning low-error shm strategy by constrained observability method, Automation in Construction 127 (2021) 103707. doi:https://doi.org/10.1016/j.autcon.2021.103707.

[26] T. Seo, A. M. Bayen, T. Kusakabe, Y. Asakura, Traffic state estimation on highway: A comprehensive survey, Annual Reviews in Control 43 (2017) 128–151. doi:https://doi.org/10.1016/j.arcontrol.2017.03.005.

[27] P. Carpentier, G. Cohen, State estimation and leak detection in water distribution networks, Civil Engineering Systems 8 (1991) 247–257. doi:10.1080/02630259108970634.

[28] X.-B. Jin, R. J. Robert Jeremiah, T.-L. Su, Y.-T. Bai, J.-L. Kong, The new trend of state estimation: From model-driven to hybrid-driven methods, Sensors 21 (2021). doi:10.3390/s21062085.

[29] F. C. Schweppe, J. Wildes, Power system static state estimation, part I: Exact model, IEEE Trans Power App Syst 89 (1970) 120–125.

[30] S. K. Kotha, B. Rajpathak, Power system state estimation using non-iterative weighted least square method based on wide area measurements with maximum redundancy, Electric Power Systems Research 206 (2022) 107794. doi:https://doi.org/10.1016/j.epsr.2022.107794.

[31] M. N. Chatzis, E. N. Chatzi, A. W. Smyth, On the observability and identifiability of nonlinear structural and mechanical systems, Structural control and health monitoring. 22 (2015).

[32] A. Abur, A. Expósito, Power System State Estimation: Theory and Implementation, Marcel Dekker, New York, USA, 2004.

[33] R. E. Pruneda, C. Solares, A. J. Conejo, E. Castillo, An efficient algebraic approach to observability analysis in state estimation, Electr Power Syst Res 80 (2010) 277–286.

[34] E. Caro, I. Arévalo, C. García-Martos, A. J. Conejo, Power system observability via optimization, Electr Power Syst Res 104 (2013) 207–215.

[35] I. O. Habiballah, M. R. Irving, Observability analysis for state estimation using linear programming, Generation, Transmission and Distribution, IEE Proceedings- 148 (2001) 142–145. doi:10.1049/ip-gtd:20010061.

[36] A. Abur, A. G. Exposito, Power system state estimation: theory and implementation, CRC press, 2004.

[37] A. G. Exposito, A. Abur, Generalized observability analysis and measurement classification, IEEE Trans Power Syst 13 (1998) 1090–1095.

[38] J. A. Lozano-Galant, M. Nogal, E. Castillo, J. Turmo, Application of observability techniques to structural system identification, Computer-Aided Civil and Infrastructure Engineering 28 (2013) 434–450.