Aeronautics and Astronautics - AIDAA XXVII International Congress Materials Research Forum LLC
Materials Research Proceedings 37 (2023) 487-494 https://doi.org/10.21741/9781644902813-107

A revisited and general Kane's formulation applied to very flexible multibody spacecraft

D.P. Madonna[1,a], P. Gasbarri[2,b], M. Pontani[3,c], F. Gennari[4,d], L. Scialanga[4,e] and A. Marchetti[4,f]

[1]Department of Mechanical and Aerospace Engineering, Sapienza University of Rome, via Eudossiana 18, 00184 Rome, Italy

[2]School of Aerospace Engineering, Sapienza University of Rome, via Salaria 851, 00138 Rome, Italy

[3]Department of Astronautical, Electrical, and Energy Engineering, Sapienza University of Rome, via Salaria 851, 00138 Rome, Italy

[4]Thales Alenia Space Italia, via Saccomuro 24, 00131 Rome, Italy

[a]davidpaolo.madonna@uniroma1.it, [b]paolo.gasbarri@uniroma1.it, [c]mauro.pontani@uniroma1.it, [d]fabrizio.gennari@thalesaleniaspace.com, [e]luigi.scialanga@thalesaleniaspace.com, [f]andrea.marchetti@thalesaleniaspace.com

Keywords: Multibody Spacecraft Dynamics, Kane's Method, Flexible Spacecraft

Abstract. Current space missions require predicting the spacecraft dynamics with considerable reliability. Among the various components of a spacecraft, subsystems like payload, structures, and power depend heavily on the dynamic behavior of the satellite during its operational life. Therefore, to ensure that the results obtained through numerical simulations correspond to the actual behavior, an accurate dynamical model must be developed. In this context, an implementation of Kane's method is presented to derive the dynamical equations of a spacecraft composed of both rigid and flexible bodies connected via joints in tree topology. Starting from the kinematics of two generic interconnected bodies, a systematic approach is derived and the recursive structure of the equations is investigated. The Kane's formulation allows a relatively simple derivation of the equation of motion while obtaining the minimum set of differential equations, which implies lower computational time. On the other hand, this formulation excludes reaction forces and torques from the dynamical equations. Nevertheless, in this work a strategy to compute them a posteriori without further numerical integrations is presented. Flexibility is introduced through the standard modal decomposition technique, so that modal shapes obtained by FEA software can be directly utilized to characterize the elastic motion of the flexible bodies. A spacecraft composed of a rigid bus and several flexible appendages is modeled and numerical simulations point out that this systematic method is very effective for this illustrative example.

Introduction

The advanced level of technology in space missions and the substantial economic investment they require necessitate a high level of predictability in all aspects of the mission. The performance of critical subsystems, such as the payload, structures, and power subsystem, is directly influenced by the dynamic behavior of the satellite throughout its operational lifetime. Therefore, it is crucial to develop an accurate dynamical model that ensures the correspondence between numerical simulations and the actual behavior of the satellite. While it may be acceptable in some cases to model the spacecraft as a single rigid body, typically it is necessary to consider the satellite as a multibody system comprising both rigid and flexible elements. Various approaches exist to derive the dynamical equations of a multibody system. However, in this work, Kane's formulation is exclusively adopted due to its distinct advantages in terms of both algebraic and computational aspects [1]. A practical version of Kane's equation, as provided in [2], has been extended in this research to encompass spacecraft consisting of both flexible and rigid bodies. The outcome is a concise matrix formulation that is also compatible with the results of modal analysis obtained through a finite element code such as NASTRAN.

Kinematics

Before providing the general form of Kane's equation, it is necessary to outline the kinematic quantities of interest.

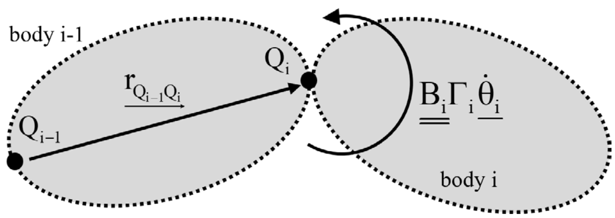

Fig. 1: two bodies connected via rotary joint

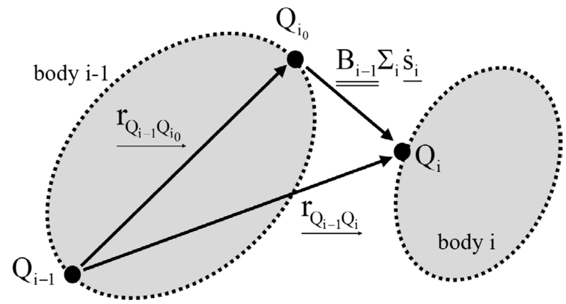

Fig. 2: two bodies connected via prismatic joint

Considering Figs. 1-2, the linear velocity of the connection point Q_i between two flexible bodies and the angular velocity of body i, both evaluated with respect to the inertial frame N, are

$$\text{rotary joint}: \begin{cases} {}^N\vec{v}^{Q_i} = {}^N\vec{v}^{Q_{i-1}} - \underline{\underline{B}}_{i-1}\underline{\tilde{r}}_{Q_{i-1}Q_i}\underline{\omega}_{i-1} + \underline{\underline{B}}_{i-1}\sum_{k=1}^{n_F}\underline{\Phi_k^{(i-1)}(Q_i)\dot{q}_k} \\ {}^N\vec{\omega}^{B_i} = {}^N\vec{\omega}^{B_{i-1}} + \underline{\underline{B}}_i\Gamma_i\underline{\dot{\theta}}_i \end{cases} \tag{1}$$

$$\text{prismatic joint}: \begin{cases} {}^N\vec{v}^{Q_i} = {}^N\vec{v}^{Q_{i-1}} - \underline{\underline{B}}_{i-1}\underline{\tilde{r}}_{Q_{i-1}Q_i}\underline{\omega}_{i-1} + \underline{\underline{B}}_{i-1}\sum_{k=1}^{n_F}\underline{\Phi_k^{(i-1)}(Q_{i_0})}\dot{q}_k + \underline{\underline{B}}_{i-1}\Sigma_i\underline{\dot{s}}_i \\ {}^N\vec{\omega}^{B_i} = {}^N\vec{\omega}^{B_{i-1}} \end{cases} \tag{2}$$

where $\vec{\square}$ denotes an invariant physical vector, $\underline{\square}$ denotes the components of a vector, $\underline{\underline{B}}_i = \begin{bmatrix} \vec{b}_1^i & \vec{b}_2^i & \vec{b}_3^i \end{bmatrix}$ is the vectrix associated with the i-th body frame [3], superscript "$\tilde{\square}$" denotes the skew matrix associated with a vector, Γ_i and Σ_i are the i-th rotary and prismatic "joint partial" respectively, i.e. the $3\times n_j$ matrices (n_j is the number of degrees of freedom allowed by the joint) that, if post multiplied by the i-th joint velocity vector (angular $\underline{\dot{\theta}}_i$ or linear $\underline{\dot{s}}_i$), provide the relative velocity of the i-th body with respect to the (i-1)-th body [2]. Moreover, following the standard modal decomposition approach [4], $\underline{\Phi_k^{(i)}(P_i)}$ is the k-th modal shape associated with the i-th body and evaluated in the generic point P_i of body i, while q_k is the k-th modal amplitude and n_F is the total number of elastic modes. For the sake of clarity, it is important to notice that $\underline{q} = \begin{bmatrix} q_1 & \cdots & q_k & \cdots & q_{n_F} \end{bmatrix}$ contains the concatenation of the elastic modes of all the flexible bodies that compose the structure, so $\underline{\Phi_k^{(i)}(P_i)}$ is a zero vector when k corresponds to the elastic mode of a body different from the i-th one. The "Eulerian" velocities of Eqs. 1-2 are a function of the generalized velocities, i.e. the minimum-dimension set of velocities that completely describe the system dynamics. Considering a typical spacecraft topology and calling the bus "body 1", the vector of generalized velocities shows the following structure:

$$\underline{u} = \begin{bmatrix} \underline{v}_{Q_1}{}^{T} & \underline{\omega}_1{}^{T} & \left\{\underline{\dot{\theta}}_i{}^{T}\right\}_{N_{RJ}} & \left\{\underline{\dot{s}}_i{}^{T}\right\}_{N_{PJ}} & \dot{q}_1 & \cdots & \dot{q}_{n_F} \end{bmatrix} \tag{3}$$

where $\underline{v}_1$ and $\underline{\omega}_1$ are the components of linear and angular velocity of the bus respectively (being the root body, Q_1 can be any point of body 1), written with respect to $\underline{\underline{B}}_1$; the terms in parentheses refer to N_{RJ} revolute joints and N_{PJ} prismatic joints, respectively. To pass from generalized velocities to the Eulerian ones, it is necessary to introduce the partial velocity matrices, which play a crucial role in the Kane's formulation [5]. Specifically, the $3N_B \times 1$ vector (N_B is the number of bodies) containing the velocities of all points Q_i written with respect to $\underline{\underline{B}}_i$ is obtained by pre-multiplying $\underline{u}$ by the matrix of linear partial velocities V, while the angular velocities are provided by the use of the matrix of angular partial velocities Ω. Both V and Ω have dimensions $3N_B \times n$, where n is the total number of degrees of freedom of the structure. Each $3 \times n$ block is associated with a body, while each column is associated with a single degree of freedom of the system. Thanks to recursion, in Eqs. 1-2, it is possible to identify a repeating structure even in the partial velocities. Specifically, the i-th $3 \times n_{DOF}$ block shows the following structure:

$$V_i = \left[\underset{i \leftarrow 1}{R} \quad -\underset{i \leftarrow 1}{R}\, \widetilde{\underline{r}}^{(1)}_{Q_1Q_i} \quad \left\{ -\underset{i \leftarrow j}{R}\, \widetilde{\underline{r}}^{(j)}_{Q_jQ_i} \Gamma_j \right\}_{N_{RJ}} \quad \left\{ \underset{i \leftarrow j}{R}\, \Sigma_{j+1} \right\}_{N_{PJ}} \quad \left\{ \sum_{m=1}^{i-1} \underset{i \leftarrow m}{R}\, \underline{\Phi_k^{(m)}\left(Q_{m+1}\right)} \right\}_{n_F} \right] \tag{4}$$

$$\Omega_i = \left[0_{3\times3} \quad \underset{i \leftarrow 1}{R} \quad \left\{ \underset{i \leftarrow j}{R}\, \Gamma_j \right\}_{N_{RJ}} \quad 0_{3\times n_{PJ}} \quad 0_{3\times n_F} \right] \tag{5}$$

where $\underset{i \leftarrow j}{R}$ is the rotation matrix from frame $\underline{\underline{B}}_j$ to frame $\underline{\underline{B}}_i$, superscript (i) in vectors specifies the frame with respect the components are written to, j refers to the body downstream of the joint whose degrees of freedom are being considered, n_{PJ} is the total number of degrees of freedom associated with prismatic joints. Moreover, the last component in Eq. 4 needs the introduction of the concept of "kinematic chain" to be explained. The kinematic chain can be seen as a branch of the tree topology of the multibody spacecraft. Every kinematic chain starts from the root body (body 1) and branches out to one of the terminal bodies: the number of kinematic chains of a structure corresponds to the number of end bodies. Hence, the index m in the last term of Eq. 4 proceeds only along bodies belonging to the same kinematic chain. Furthermore, all the terms in parentheses of Eqs. 4-5 must be replaced by blocks of zeros (with consistent dimensions) if the two considered bodies do not belong to the same kinematic chain.

To complete the kinematic description, accelerations must be derived. In Kane's formulation, it is important to identify the terms of the accelerations that do not depend on the time derivative of the generalized velocities. These terms are called "remainder accelerations" and, with reference to the building blocks in Figs. 1-2, show the following structure:

$$RJ: \begin{cases} \underline{a}_i^{(R)} = \underset{i \leftarrow i-1}{R}\, \underline{a}_{i-1}^{(R)}\Big|_{Q_i} \\ \underline{a}_i^{(R)}\Big|_{Q_{i+1}} = \underset{i \leftarrow i-1}{R}\, \underline{a}_{i-1}^{(R)}\Big|_{Q_i} - \widetilde{\underline{r}}_{Q_iQ_{i+1}} \underline{\omega}_i^{*} - \widetilde{\underline{\omega}}_i \widetilde{\underline{r}}_{Q_iQ_{i+1}} \underline{\omega}_i + 2\widetilde{\underline{\omega}}_i \sum_{k=1}^{n_F} \underline{\Phi_k^{(i)}\left(Q_{i+1}\right)}\, \dot{q}_k \\ \underline{\alpha}_i^{(R)} = \underset{i \leftarrow i-1}{R}\, \underline{\alpha}_{i-1}^{(R)} + \dot{\Gamma}_i \dot{\theta}_i + \widetilde{\underline{\omega}}_i \Gamma_i \dot{\theta}_i \end{cases} \tag{6}$$

$$PJ:\begin{cases} \underline{a}_i^{(R)} = \underset{i\leftarrow i-1}{R}\,\underline{a}_{i-1}^{(R)}\Big|_{Q_i} \\ \underline{a}_i^{(R)}\Big|_{Q_{i+1}} = \underset{i\leftarrow i-1}{R}\,\underline{a}_{i-1}^{(R)}\Big|_{Q_i} - \widetilde{\underline{r}_{Q_iQ_{i+1}}}\,\underline{\omega}_i^* - \left(\dot{\Sigma}_{i+1}s_{i+1} + 2\Sigma_{i+1}\dot{s}_{i+1}\right)^{\sim}\underline{\omega}_i \\ \qquad -\widetilde{\underline{\omega}}_i\widetilde{\underline{r}_{Q_iQ_{i+1}}}\,\underline{\omega}_i + 2\widetilde{\underline{\omega}}_i\sum_{k=1}^{n_F}\underline{\Phi_k^{(i)}\left(Q_{i+1}\right)}\dot{q}_k + \dot{\Sigma}_{i+1}\dot{s}_{i+1} \\ \underline{\alpha}_i^{(R)} = \underset{i\leftarrow i-1}{R}\,\underline{\alpha}_{i-1}^{(R)} \end{cases} \tag{7}$$

$$\underline{\omega}_i^* = \underset{i\leftarrow 1}{\dot{R}}\,\underline{\omega}_1 + \sum_{m=2}^{i-1}\left(\underset{i\leftarrow m}{\dot{R}}\,\Gamma_m\dot{\theta}_m + \underset{i\leftarrow m}{R}\,\dot{\Gamma}_m\dot{\theta}_m\right) + \dot{\Gamma}_i\dot{\theta}_i \tag{8}$$

Actually, even in this case the structure must follow the scheme of kinematic chains: the passage from body i-1 to body i must be intended as a passage between two consecutive bodies on the same kinematic chain, not as a passage between two bodies with consecutive numeration. As for the last term of Eq. 4, the index m in Eq. 8 proceeds along the kinematic chains, not following the consecutive numeration.

Kane's equations

Applying the Kane's procedure to derive the dynamics of a multibody structure, the following expression is obtained:

$$\begin{aligned}&\left\langle V^T\left\{MV - S\Omega + B\Delta\right\} + \Omega^T\left\{SV - J\Omega + C\Delta\right\} + \Delta^T\left\{B^TV_F - C^T\Omega_F + Y\Delta\right\}\right\rangle \underline{\dot{u}} = \\ &= V^T\left\{-M\underline{a}^{(R)} + S\underline{\alpha}^{(R)} + \left[\tilde{\underline{\omega}}S\underline{\omega}\right] - 2\left[\tilde{\underline{\omega}}B\right]\right\} + \Omega^T\left\{-S\underline{a}^{(R)} - J\underline{\alpha}^{(R)} - \left[\tilde{\underline{\omega}}J\underline{\omega}\right] - 2\left[N\underline{\omega}\right]\right\} + \\ &\quad + \Delta^T\left\{-B^T\underline{a}_F^{(R)} - C^T\underline{\alpha}_F^{(R)} + \left[\underline{\omega}^TL\underline{\omega}\right] + 2\left[\underline{\omega}^T\underline{d}\right]\right\} - K\underline{q} - Z\underline{\dot{q}} + \hat{\underline{f}}\end{aligned} \tag{9}$$

where $\Delta = \left[0_{n_F\times n_R} \quad I_{n_F\times n_F}\right]$, being n_R and n_F the rigid and flexible degrees of freedom respectively of the whole system, M, S and J are matrices containing masses, static moments and inertia moments respectively, B and C are matrices containing translation and rotation modal participation factors respectively of all the flexible bodies of the structure; Y is the modal mass matrix, $\underline{a}^{(R)}$ and $\underline{\alpha}^{(R)}$ are $3N_B\times 1$ vectors containing respectively linear and angular remainder accelerations of all the bodies, subscript F in vectors and matrices indicates that only rows associated with flexible bodies must be retained, $\underline{\omega}$ is the $3N_B\times 1$ vector containing the angular velocities of all the bodies, N, L and D are three other modal integrals (in addition to B, C and G), K is the stiffness matrix, Z is the damping matrix, $\hat{f}$ is the vector containing the generalized active forces, i.e. the projection of external and interface forces and torques along the directions of partial velocities [5]. The structure of the terms appearing in Eq. 9 are reported in the Appendix.

Extraction of constraint reactions

The unavailability of constraint reactions is a significant limitation in Kane's formulation. However, for spacecraft with a tree topology configuration, it is possible to reconstruct the time histories of constraint reactions quite easily through a post-processing approach that relies on Newton/Euler equations, following the numerical integration of Kane's equations. In fact, unlike Kane's method, the Newton/Euler formulation for multibody structures incorporates constraint reactions in the state vector (however, this inclusion leads to longer computational times) [1, 6]. The procedure follows the subsequent steps: after the numerical solution of Kane's equations, one obtains $\underline{u}(t)$ and $\underline{\dot{u}}(t)$. Then, by utilizing the partial velocities and remainder accelerations, it is possible to reconstruct the temporal profiles of velocities and accelerations for all bodies within the structure. As a result, the constraint reactions become the only unknowns in the Newton-Euler equations that can be resolved through a post-processing module. During this operation, a top-down approach is necessary, starting from the bodies at the end of the kinematic chains. This is because each of these bodies has only a single joint, and one can then proceed backward along the kinematic chain toward the root body. In the example of Fig. 4,

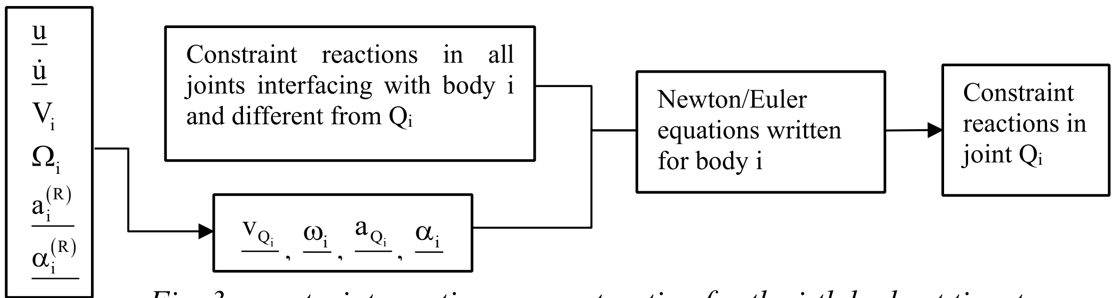

Fig. 3: constraint reactions reconstruction for the i-th body at time t

the constraint reactions must be computed first in joints Q_3, Q_4 and Q_6, and then in joints Q_2 and Q_5. The order of computing reactions for joints with the same subordination ranking can be any.

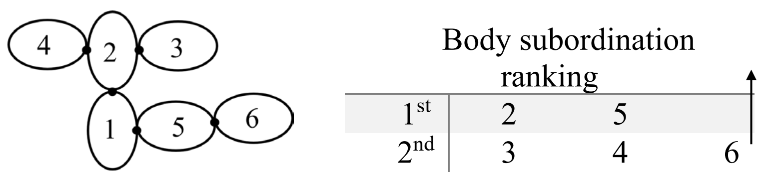

Fig. 4: example of top-down logic in deriving the constraint

Illustratuive simulation results

The presented formulation has been implemented in a numerical code to simulate the dynamic behavior of Explorer I, which is the same case studied in Reference [2]. The spacecraft consists of a cylindrical rigid bus and four appendages connected to the bus, as depicted in Figure 5. Similarly to the study in Reference [2], this investigation focuses on the spontaneous transition from a minor-axis to a major-axis spin caused by damping effects in the structure. However, there is a difference in the approach: while the analysis reported in [2] considered the appendages as rigid and introduced flexibility by incorporating a torsional spring-damper system at the interfaces between the appendages and the bus, in this work, the appendages are directly treated as flexible beams attached to the central the body of the spacecraft. Figure 6 illustrates the expected behavior of the bus angular velocity components, which exhibit the previously mentioned transition of the rotational behavior.

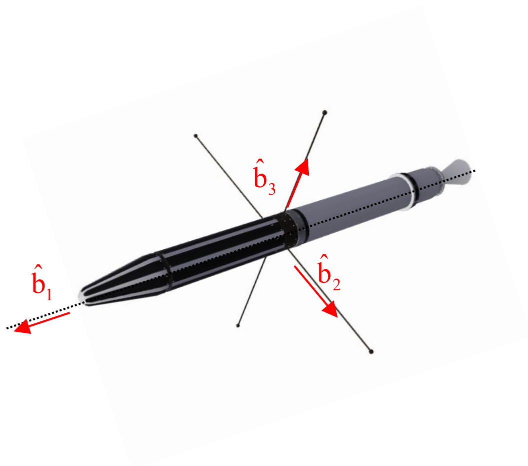

Fig. 5: sketch of the Explorer I [7]

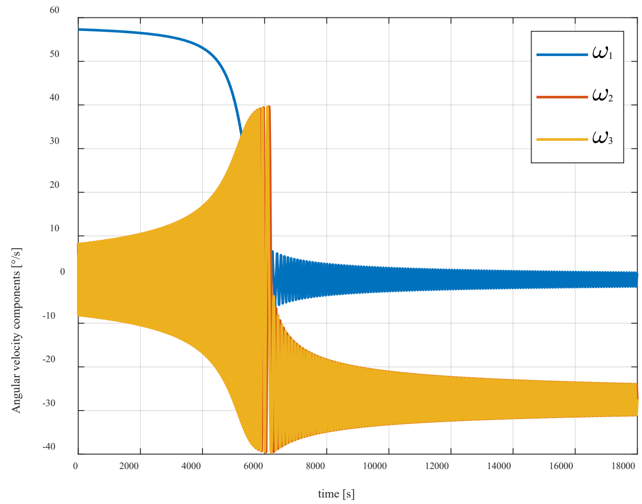

Fig. 6: time histories of the angular velocity components

Concluding remarks

A revised formulation of Kane's dynamical equations for a flexible multibody spacecraft is presented. By conducting a kinematic analysis, the expressions for partial velocities and remainder accelerations are derived, while emphasizing their recursive nature. The complete matrix formulation is provided, along with a proposed strategy for determining constraint reactions. Additionally, a numerical implementation of the formulation at hand is presented using the case of the Explorer I mission.

References

[1] A. Pisculli, L. Felicetti, M. Sabatini, P. Gasbarri and G. B. Palmerini, "A Hybrid Formulation for Modelling Multibody Spacecraft", Aerotecnica Missili & Spazio, no. 94, pp. 91-101, 2015. https://doi.org/10.1007/BF03404692

[2] E. T. Stoneking, "Implementation of Kane's Method for a Spacecraft", in AIAA Guidance, Navigation, and Control (GNC) Conference, Boston, 2013. https://doi.org/10.2514/6.2013-4649

[3] P. C. Hughes, "Spacecraft Attitude Dynamics", Wiley, 1986. https://doi.org/10.1017/S0001924000015578

[4] S. S. Rao, "Vibration of Continuous Systems", Wiley, 2007. https://doi.org/10.1002/9780470117866

[5] C. M. Roithmayr and D. H. Hodges , "Dynamics: Theory and Application of Kane's Method", Cambridge, 2016. https://doi.org/10.1115/1.4034731

[6] P. Santini and P. Gasbarri, "Dynamics of multibody systems in space environment; Lagrangian vs. Eulerian approach". Acta Astronautica, vol. 54, no. 1, pp. 1-24, 2004. https://doi.org/10.1016/S0094-5765(02)00277-1

[7] "Explorer 1 - Overview", NASA JPL, [Online]. Available: https://explorer1.jpl.nasa.gov/about/.

Materials Research Proceedings 37 (2023) 487-494 https://doi.org/10.21741/9781644902813-107

APPENDIX

A1. mass distribution and modal integrals

In the following expressions, the index "j" indicates the body, while the indices "k" and "l" identify the elastic mode.

- $m_j = \int_{B_j} dm$
- $\underline{b}_k^j = \int_{B_j} \underline{\Phi}_k^j \, dm$
- $\underline{s}_j = \int_{B_j} \underline{r}_{Q_jP_j} \, dm + \sum_{k=1}^{n_F} \underline{b}_k^j q_k$
- $\underline{c}_k^j = \int_{B_j} \underline{\widetilde{r}_{Q_jP_j}} \, \underline{\Phi}_k^j \, dm$
- $\underline{d}_{kl}^j = \int_{B_j} \underline{\widetilde{\Phi}_k^j} \, \underline{\Phi}_l^j \, dm$
- $\underline{g}_k^j = \underline{c}_k^j - \sum_{l=1}^{n_F} \underline{d}_{kl}^j q_k$
- $y_k^j = \int_{B_j} \left(\underline{\Phi}_k^j\right)^2 dm$
- $\underline{N}_k^j = -\int_{B_j} \underline{\widetilde{\Phi}_k^j} \, \underline{\widetilde{r}_{Q_jP_j}} \, dm$
- $J_j = -\int_{B_j} \underline{\widetilde{r}_{Q_jP_j}} \, \underline{\widetilde{r}_{Q_jP_j}} \, dm + \sum_{k=1}^{n_F} \left(\underline{N}_k^j + \underline{N}_k^{j\,T}\right) q_k$
- $\underline{L}_k^j = \underline{N}_k^j - \sum_{l=1}^{n_F} q_l \int_{B_j} \underline{\widetilde{\Phi}_l^j} \, \underline{\widetilde{\Phi}_k^j} \, dm$

A2. description of the terms of eq. 9

$$
M = \begin{bmatrix} m_1 I_{3x3} & & & 0 \\ & m_2 I_{3x3} & & \\ & & \ddots & \\ 0 & & & m_{N_B} I_{3x3} \end{bmatrix}
\quad
S = \begin{bmatrix} \widetilde{\underline{s}}_1 & & & 0 \\ & \widetilde{\underline{s}}_2 & & \\ & & \ddots & \\ 0 & & & \widetilde{\underline{s}}_{N_B} \end{bmatrix}
\quad
J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_{N_B} \end{bmatrix}
$$

$$
B = \begin{bmatrix} \underline{b}_1^1 & \underline{b}_2^1 & \cdots & \underline{b}_{n_F}^1 \\ \underline{b}_1^2 & \underline{b}_2^2 & \cdots & \underline{b}_{n_F}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \underline{b}_1^{N_{B,F}} & \underline{b}_2^{N_{B,F}} & \cdots & \underline{b}_{n_F}^{N_{B,F}} \end{bmatrix}
\quad
C = \begin{bmatrix} \underline{c}_1^1 & \underline{c}_2^1 & \cdots & \underline{c}_{n_F}^1 \\ \underline{c}_1^2 & \underline{c}_2^2 & \cdots & \underline{c}_{n_F}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \underline{c}_1^{N_{B,F}} & \underline{c}_2^{N_{B,F}} & \cdots & \underline{c}_{n_F}^{N_{B,F}} \end{bmatrix}
\quad
G = \begin{bmatrix} \underline{g}_1^1 & \underline{g}_2^1 & \cdots & \underline{g}_{n_F}^1 \\ \underline{g}_1^2 & \underline{g}_2^2 & \cdots & \underline{g}_{n_F}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \underline{g}_1^{N_{B,F}} & \underline{g}_2^{N_{B,F}} & \cdots & \underline{g}_{n_F}^{N_{B,F}} \end{bmatrix}
$$

$$Y=\begin{bmatrix} Y^1 & & & 0 \\ & Y^2 & & \\ & & \ddots & \\ 0 & & & Y^{N_{B,F}} \end{bmatrix} \quad \text{where} \quad Y^j=\begin{bmatrix} y_1^j & & \\ & \ddots & \\ & & y_{n_{F,j}}^j \end{bmatrix}$$

$N_{B,F}$ is the number of flexible bodies in the multibody spacecraft

$n_{F,j}$ is the number of elastic modes associated with the j-th body

$$\left[\tilde{\underline{\omega}} S \underline{\omega}\right]=\begin{bmatrix} \tilde{\underline{\omega}}_1 \underline{s}_1 \underline{\omega}_1 \\ \tilde{\underline{\omega}}_2 \underline{s}_2 \underline{\omega}_2 \\ \vdots \\ \tilde{\underline{\omega}}_{N_B} \underline{s}_{N_B} \underline{\omega}_{N_B} \end{bmatrix} \qquad \left[\tilde{\underline{\omega}} B\right]=\begin{bmatrix} \tilde{\underline{\omega}}_1 \sum_{k=1}^{n_F} \underline{b}_k^1 \dot{q}_k \\ \tilde{\underline{\omega}}_2 \sum_{k=1}^{n_F} \underline{b}_k^2 \dot{q}_k \\ \vdots \\ \tilde{\underline{\omega}}_{N_B} \sum_{k=1}^{n_F} \underline{b}_k^{N_{B,F}} \dot{q}_k \end{bmatrix}$$

$$\left[\tilde{\underline{\omega}} J \underline{\omega}\right]=\begin{bmatrix} \tilde{\underline{\omega}}_1 \underline{J}_1 \underline{\omega}_1 \\ \tilde{\underline{\omega}}_2 \underline{J}_2 \underline{\omega}_2 \\ \vdots \\ \tilde{\underline{\omega}}_{N_B} \underline{J}_{N_B} \underline{\omega}_{N_B} \end{bmatrix} \qquad \left[N^T \underline{\omega}\right]=\begin{bmatrix} \sum_{k=1}^{n_F} \underline{N_k^1}^T \dot{q}_k \underline{\omega}_1 \\ \sum_{k=1}^{n_F} \underline{N_k^2}^T \dot{q}_k \underline{\omega}_2 \\ \vdots \\ \sum_{k=1}^{n_F} \underline{N_k^{N_{B,F}}}^T \dot{q}_k \underline{\omega}_{N_B} \end{bmatrix}$$

$$\left[\underline{\omega}^T L \underline{\omega}\right]=\begin{bmatrix} \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \underline{L}_1^j \underline{\omega}_j \\ \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \underline{L}_2^j \underline{\omega}_j \\ \vdots \\ \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \underline{L}_{n_F}^j \underline{\omega}_j \end{bmatrix} \qquad \left[\underline{\omega}^T \underline{d}\right]=\begin{bmatrix} \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \left(\sum_{k=1}^{n_F} \underline{d}_{1k}^j \dot{q}_k\right) \\ \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \left(\sum_{k=1}^{n_F} \underline{d}_{2k}^j \dot{q}_k\right) \\ \vdots \\ \sum_{j=1}^{N_{B,F}} \underline{\omega}_j^T \left(\sum_{k=1}^{n_F} \underline{d}_{n_F k}^j \dot{q}_k\right) \end{bmatrix}$$

$$K=\begin{bmatrix} 0_{n_R \times n_R} & & & \\ & (\lambda_1)^2 & 0 & 0 \\ & 0 & \ddots & 0 \\ & 0 & 0 & (\lambda_{n_F})^2 \end{bmatrix} \qquad Z=\begin{bmatrix} 0_{n_R \times n_R} & & & \\ & 2\zeta_1\lambda_1 & 0 & 0 \\ & 0 & \ddots & 0 \\ & 0 & 0 & 2\zeta_{n_F}\lambda_{n_F} \end{bmatrix}$$

λ_k is the natural frequency of the k-th elastic mode

ζ_k is the damping factor of the k-th mode