Exact controllability for quasi-linear perturbations of KdV

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Abstract. We prove that the KdV equation on the circle remains exactly controllable in arbitrary time with localized control, for sufficiently small data, also in presence of quasi-linear perturbations, namely nonlinearities containing up to three space derivatives, having a Hamiltonian structure at the highest orders. We use a procedure of reduction to constant coefficients up to order zero (adapting [6]), classical Ingham inequality and HUM method to prove the controllability of the linearized operator. Then we prove and apply a modified version of the Nash-Moser implicit function theorems by Hörmander [27, 28]. *MSC2010:* 35Q53, 35Q93.

Contents.

1 Introduction

A question in control theory for PDEs regards the persistence of controllability under perturbations. In this paper we study the effect of *quasi-linear* perturbations (namely nonlinearities containing derivatives of the highest order) on the controllability of the KdV equation. We consider equations of the form

$$u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{1.1}$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, with $t \in \mathbb{R}$, where $u = u(t, x)$ is real-valued, and $\mathcal{N}$ is a given real-valued nonlinear function which is at least quadratic around $u = 0$. For solutions of small amplitude, (1.1) is a quasi-linear perturbation of the Airy equation $u_t + u_{xxx} = 0$, which is the linear part of KdV; then the KdV nonlinear term uu_x can be included in $\mathcal{N}$.

Motivated by a question, which was posed in [31], about the possibility of including the dependence on higher derivatives in nonlinear perturbations of KdV, equations of the form (1.1) have recently been studied in [6, 7, 8] in the context of KAM theory. In this paper we study (1.1) from the point of view of control theory, proving its exact controllability by means of an internal control, in arbitrary time, for sufficiently small data (Theorem 1.1).

Most of the known results about controllability of quasi-linear PDEs deal with first order quasi-linear hyperbolic systems of the form $u_t + A(u)u_x = 0$ (including quasi-linear wave, shallow water, and Euler equations), see for example Li and Zhang [37], Coron [18] (chapter 6.2, and see also the many references therein), Li and Rao [36], Coron, Glass and Wang [19], and recently Alabau-Boussouira, Coron and Olive [1]. Recent results for different kinds of quasi-linear PDEs are contained in Alazard, Baldi and Han-Kwan [3] on the internal controllability of 2D gravity-capillary water waves equations, and Alazard [2] on the boundary observability of 2D and 3D (fully nonlinear) gravity water waves. For a general introduction to the theory of control for PDEs see, for example, Lions [38], Micu and Zuazua [39], Coron [18], while for important results in control for hyperbolic PDEs see, for example, Bardos, Lebeau and Rauch [9], Burq and Gérard [16], Burq and Zworski [17].

Regarding the KdV equation, the first controllability results are due to Zhang [49] and Russell [45]. Among recent results, we mention the work by Laurent, Rosier and Zhang [35] for large data. A beautiful review on the literature on control for KdV can be found in [44]. For more on KdV, see the rich survey [24] by Guan and Kuksin, and the many references therein.

1.1 Main result

We assume that the nonlinearity $\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx})$ is at least quadratic around $u = 0$, namely the real-valued function $\mathcal{N} : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ satisfies

$$|\mathcal{N}(x, z_0, z_1, z_2, z_3)| \le C|z|^2 \quad \forall z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4, \ |z| \le 1. \tag{1.2}$$

We assume that the dependence of $\mathcal{N}$ on u_{xx}, u_{xxx} is Hamiltonian, while no structure is required on its dependence on u, u_x. More precisely, we assume that

$$\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) + \mathcal{N}_0(x, u, u_x) \tag{1.3}$$

where

$$\begin{aligned} &\mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) = \partial_x\{(\partial_u \mathcal{F})(x, u, u_x)\} - \partial_{xx}\{(\partial_{u_x} \mathcal{F})(x, u, u_x)\} \\ &\text{for some function } \mathcal{F} : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}. \end{aligned} \tag{1.4}$$

Note that the case $\mathcal{N} = \mathcal{N}_1$, $\mathcal{N}_0 = 0$ corresponds to the Hamiltonian equation $\partial_t u = \partial_x \nabla H(u)$ where the Hamiltonian is

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 \, dx + \int_{\mathbb{T}} \mathcal{F}(x, u, u_x) \, dx \tag{1.5}$$

and ∇ denotes the $L^2(\mathbb{T})$-gradient. The unperturbed KdV is the case $\mathcal{F} = -\frac{1}{6}u^3$.

Notations. For periodic functions $u(x)$, $x \in \mathbb{T}$, we expand $u(x) = \sum_{n \in \mathbb{Z}} u_n e^{inx}$, and, for $s \in \mathbb{R}$, we consider the standard Sobolev space of periodic functions

$$H^s_x := H^s(\mathbb{T}, \mathbb{R}) := \big\{ u : \mathbb{T} \to \mathbb{R} : \|u\|_s < \infty \big\}, \quad \|u\|_s^2 := \sum_{n \in \mathbb{Z}} |u_n|^2 \langle n \rangle^{2s}, \tag{1.6}$$

where $\langle n \rangle := (1 + n^2)^{\frac{1}{2}}$. We consider the space $C([0, T], H^s_x)$ of functions $u(t, x)$ that are continuous in time with values in H^s_x. We will use the following notation for the standard norm in $C([0, T], H^s_x)$:

$$\|u\|_{T,s} := \|u\|_{C([0,T], H^s_x)} := \sup_{t \in [0,T]} \|u(t)\|_s. \tag{1.7}$$

For continuous functions $a : [0, T] \to \mathbb{R}$, we will denote

$$|a|_T := \sup\{|a(t)| : t \in [0, T]\}. \tag{1.8}$$

Theorem 1.1 (Exact controllability). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. There exist positive universal constants r, s_1 such that, if $\mathcal{N}$ in (1.1) is of class C^r in its arguments and satisfies (1.2), (1.3), (1.4), then there exists a positive constant δ_* depending on $T, \omega, \mathcal{N}$ with the following property.*

Let $u_{in}, u_{end} \in H^{s_1}(\mathbb{T}, \mathbb{R})$ with

$$\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1} \leq \delta_*.$$

Then there exists a function $f(t,x)$ satisfying

$$f(t,x) = 0 \quad \text{for all } x \notin \omega, \text{ for all } t \in [0,T],$$

belonging to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x) \cap C^2([0,T], H^{s-6}_x)$ for all $s < s_1$, such that the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f & \forall (t,x) \in [0,T] \times \mathbb{T} \\ u(0,x) = u_{in}(x) \end{cases} \tag{1.9}$$

has a unique solution $u(t,x)$ belonging to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x) \cap C^2([0,T], H^{s-6}_x)$ for all $s < s_1$, which satisfies

$$u(T,x) = u_{end}(x). \tag{1.10}$$

Moreover, for all $s < s_1$,

$$\|u, f\|_{C([0,T], H^s_x)} + \|\partial_t u, \partial_t f\|_{C([0,T], H^{s-3}_x)} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0,T], H^{s-6}_x)} \leq C_s (\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1}) \tag{1.11}$$

for some $C_s > 0$ depending on $s, T, \omega, \mathcal{N}$.

Remark 1.2. In Theorem 1.1 there is an arbitrarily small loss of regularity: if the initial and final data u_{in}, u_{end} have Sobolev regularity $H^{s_1}_x$, then the control f and the solution u are continuous in time with values in H^s_x for all $s < s_1$. Such loss of regularity is in some sense fictitious: it is due to our choice of working with standard Sobolev spaces, but it could be avoided by working with the (slightly "worse-looking") weak spaces E'_a introduced by Hörmander in [28] (see Section 7). What we actually prove is that, if the initial and final data are in the weak space $(H^{s_1}_x)'$ (i.e. the weak version *à la* Hörmander [28] of the Sobolev space $H^{s_1}_x$), then f and u are continuous in time with values in *the same* space $(H^{s_1}_x)'$. □

Remark 1.3. Our proof of Theorem 1.1 does not use results of existence and uniqueness for the Cauchy problem (1.9). On the contrary, our method directly proves local existence and uniqueness for (1.9) (see Theorem 1.4). This situation occurs quite often in control problems (see Remark 4.12 in [18]). □

1.2 Description of the proof

It would be natural to try to solve the control problem (1.9)-(1.10) using a fixed point argument or the usual implicit function theorem. However, this seems to be impossible because of the presence of three derivatives in the nonlinear term. A similar difficulty was overcome in [3] by using a suitable nonlinear iteration scheme adapted to quasi-linear problems. Such a nonlinear scheme requires to solve a linear control problem with variable coefficients at each step of the iteration, with no loss of regularity with respect to the coefficients (i.e., the solution must have the same regularity as the coefficients). In [3] this is achieved by means of para-differential calculus, together with linear transformations, Ingham-type inequalities and the Hilbert uniqueness method.

As an alternative method, in this paper we use a Nash-Moser implicit function theorem. The Nash-Moser approach also demands to solve a linear control problem with variable coefficients, but it has the advantage of requiring weaker estimates, allowing losses of regularity. The proof of such weaker estimates is easier to obtain, and it does not require the use of powerful techniques like para-differential calculus. In this sense our Nash-Moser method is alternative to the method in [3] (for a discussion about pseudo- and para-differential calculus in connection with the Nash-Moser theorem, see, for example, Hörmander [29], Alinhac and Gérard [4]). On the other hand, the result that we obtain with the Nash-Moser method is slightly weaker than the one in [3] regarding the regularity of the solution of the nonlinear control problem with respect to the regularity of the data: the arbitrarily small loss of regularity in Theorem 1.1 is discussed in Remark 1.2, while Theorem 1.1 of [3] has no loss of regularity also in the standard Sobolev spaces.

Nash-Moser schemes in control problems for PDEs have been used by Beauchard, Coron, Alabau-Boussouira, Olive in [10, 12, 11, 1]. A discussion about Nash-Moser as a method to overcome the problem of the loss of derivatives in the context of controllability for PDEs can be found in [18, section 4.2.2]. In [13] Beauchard and Laurent were able to avoid the use of the Nash-Moser theorem in semilinear control problems thanks to some regularizing effect. We remark that Theorem 1.1 could also be proved without Nash-Moser (for example, by adapting the method of [3]).

Now we describe our method in more detail. Given a nonempty open set $\omega \subset \mathbb{T}$, we first fix a C^∞ function $\chi_\omega(x)$ with values in the interval $[0,1]$ which vanishes outside ω, and takes value $\chi_\omega = 1$ on a nonempty open subset of ω. Thus, given initial and final data u_{in}, u_{end}, we look for u, f that solve

$$\begin{cases} P(u) = \chi_\omega f \\ u(0) = u_{in} \\ u(T) = u_{end} \end{cases} \tag{1.12}$$

where

$$P(u) := u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}). \tag{1.13}$$

We define

$$\Phi(u,f) := \begin{pmatrix} P(u) - \chi_\omega f \\ u(0) \\ u(T) \end{pmatrix} \tag{1.14}$$

so that problem (1.12) is written as

$$\Phi(u,f) = (0, u_{in}, u_{end}).$$

The crucial assumption to verify in order to apply any Nash-Moser theorem is the existence of a right inverse of the linearized operator. The linearized operator $\Phi'(u,f)[h,\varphi]$ at the point (u,f) in the direction (h,φ) is

$$\Phi'(u,f)[h,\varphi] := \begin{pmatrix} P'(u)[h] - \chi_\omega \varphi \\ h(0) \\ h(T) \end{pmatrix}. \tag{1.15}$$

Thus we have to prove that, given any (u, f) and any $g := (g_1, g_2, g_3)$ in suitable function spaces, there exists (h, φ) such that

$$\Phi'(u, f)[h, \varphi] = g. \tag{1.16}$$

Moreover we have to estimate (h, φ) in terms of u, f, g in a "tame" way (an estimate is said to be tame when it is linear in the highest norms: see (7.13) and (4.41)).

Problem (1.16) is a linear control problem. We observe that the linearized operator $P'(u)[h]$ is a differential operator having variable coefficients also at the highest order (which is a consequence of linearizing a *quasi-linear* PDE). Explicitly, it has the form

$$P'(u)[h] = \partial_t h + (1 + a_3(t, x))\partial_{xxx} h + a_2(t, x)\partial_{xx} h + a_1(t, x)\partial_x h + a_0(t, x)h.$$

We solve (1.16) in Theorem 4.5. Note that the choice of the function spaces is not given a priori: to fix a suitable functional setting is part of the problem.

Theorem 4.5 is proved by adapting a procedure of reduction to constant coefficients developed in [6, 7]. Such a procedure conjugates $P'(u)$ to an operator $\mathcal{L}_5$ (see (2.57)) having constant coefficients up to a bounded remainder. This conjugation is achieved by means of changes of the space variable, reparametrization of time, multiplication operators, and Fourier multipliers. Using Ingham inequality and a perturbation argument we prove the observability of $\mathcal{L}_5$. Then we prove the observability of $P'(u)$ exploiting the explicit formulas of the transformations that conjugate $P'(u)$ to $\mathcal{L}_5$. The linear control problem (1.16) is solved in L^2_x by the HUM (Hilbert uniqueness method). Then further regularity of the solution (h, φ) of (1.16) is proved by adapting an argument used by Dehman-Lebeau [20], Laurent [34], and [3].

To conclude the proof of Theorem 1.1 we apply Theorem 7.1, which is a modified version of two Nash-Moser implicit function theorems by Hörmander (Theorem 2.2.2 in [27] and main theorem in [28]; see also Alinhac-Gérard [4]). With respect to the abstract theorem in [28], our Theorem 7.1 assumes slightly stronger hypotheses on the nonlinear operator, and it removes two conditions that are assumed in [28], which are the compact embeddings in the codomain scale of Banach spaces and the continuity of the approximate right inverse of the linearized operator with respect to the approximate linearization point. This improvement is obtained by adapting the iteration scheme introduced in [27]. On the other hand, the Nash-Moser implicit function theorem in [27] holds for Hölder spaces with noninteger indices, and it does not apply to Sobolev spaces (in particular, Theorem A.11 of [27] does not hold for Sobolev spaces).

This method is not confined to KdV, and it could be applied to prove controllability of other quasi-linear evolution PDEs.

The use of Ingham-type inequalities and HUM is classical in control theory (see, for example, [26, 39, 33, 30] for Ingham and [38, 39, 18, 32] for HUM). As mentioned above, the Nash-Moser theorem has also been used in control theory (see, for example, [10, 12, 11, 1]). It was first introduced by Nash [42], then several refinements were developed afterwards, see for example Moser [40], Zehnder [48], Hamilton [25], Gromov [23], Hörmander [27, 28, 29], and, recently, Berti, Bolle, Corsi and Procesi [14, 15], Ekeland and Séré [21, 22]. For our problem, Hörmander's versions [27, 28] seem to be the best ones concerning the loss of regularity of the solution with respect to the regularity of the data (see also Remark 1.2). As already said, the theorems in [27, 28] cannot be applied directly, but they can be adapted to our goal. This is the content of Section 7.

1.3 Byproduct: a local existence and uniqueness result

As a byproduct, with the same technique and no extra work, we have the following existence and uniqueness theorem for the Cauchy problem of the quasi-linear PDE (1.1).

Theorem 1.4 (Local existence and uniqueness). *There exist positive universal constants r, s_0 such that, if $\mathcal{N}$ in (1.1) is of class C^r in its arguments and satisfies (1.2), (1.3), (1.4), then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{in} \in H^{s_0}_x$, $f \in C([0,T], H^{s_0}_x) \cap C^1([0,T], H^{s_0-6}_x)$ (possibly $f = 0$) satisfying*

$$\|u_{in}\|_{s_0} + \|f\|_{T,s_0} + \|\partial_t f\|_{T,s_0-6} \leq \delta_* , \tag{1.17}$$

the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f, & (t,x) \in [0,T] \times \mathbb{T} \\ u(0,x) = u_{in}(x) \end{cases} \tag{1.18}$$

has one and only one solution $u \in C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x) \cap C^2([0,T], H^{s-6}_x)$ for all $s < s_0$. Moreover, for all $s < s_0$,

$$\begin{aligned} &\|u\|_{C([0,T],H^s_x)} + \|\partial_t u\|_{C([0,T],H^{s-3}_x)} + \|\partial_{tt} u\|_{C([0,T],H^{s-6}_x)} \\ &\qquad \leq C_s \Big(\|u_{in}\|_{s_0} + \|f\|_{C([0,T],H^{s_0}_x)} + \|\partial_t f\|_{C([0,T],H^{s_0-6}_x)} \Big) \end{aligned} \tag{1.19}$$

for some $C_s > 0$ depending on $s, T, \mathcal{N}$.

Remark 1.5. Theorem 1.4 is not sharp: we expect that better results for the Cauchy problem (1.18) can be proved by using a para-differential approach. □

Remark 1.6. The loss of regularity in Theorem 1.4 is of the same type as the one in Theorem 1.1, see the discussion in Remark 1.2. □

1.4 Organization of the paper

In Section 2 we describe the transformations that conjugate the linearized operator $P'(u)$ to constant coefficients up to a bounded remainder, and we give quantitative estimates on these transformations. In Section 3 we exploit these results to prove the observability of $P'(u)$. In Section 4 we use observability to solve the linear control problem (1.16) via HUM (Theorem 4.5) and we fix suitable function spaces (4.36)-(4.37). In Section 5 we prove Theorems 1.1 and 1.4 by applying Theorem 7.1. In Section 6 we prove well-posedness with tame estimates for all the linear operators involved in the reduction procedure. These well-posedness results are used many times along the Sections 3, 4, 5. In Section 7 we prove Nash-Moser Theorem 7.1. In Section 8 we recall standard tame estimates that are used in the rest of the paper.

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2 Reduction of the linearized operator to constant coefficients

In this section we consider some changes of variables that conjugate the linearized operator to constant coefficients up to a bounded remainder. This reduction procedure closely follows the analysis in [6] and [7], with some adaptations.

The linearized operator $P'(u)$ is

$$P'(u)[h] = \partial_t h + (1 + a_3)\partial_{xxx} h + a_2 \partial_{xx} h + a_1 \partial_x h + a_0 h, \tag{2.1}$$

where the coefficients $a_i = a_i(t, x)$, $i = 0, \dots, 3$ are real-valued functions of $(t, x) \in [0, T] \times \mathbb{T}$, depending on u by

$$a_i = a_i(u) := (\partial_{z_i} \mathcal{N})(x, u, u_x, u_{xx}, u_{xxx}), \quad i = 0, \dots, 3 \tag{2.2}$$

(recall the notation $\mathcal{N} = \mathcal{N}(x, z_0, z_1, z_2, z_3)$). Note that $a_2 = 2\partial_x a_3$ because of the Hamiltonian structure of the component $\mathcal{N}_1$ of the nonlinearity (see (1.3)-(1.4)).

Lemma 2.1. *Let $\mathcal{N} \in C^r(\mathbb{T} \times \mathbb{R}^4, \mathbb{R})$ satisfying* (1.2). *For all $1 \leq s \leq r - 3$, and for all $u \in C^2([0, T], H_x^{s+3})$ such that $\|u, \partial_t u, \partial_{tt} u\|_{T,4} \leq 1$, the coefficients $a_i(u)$ satisfy*

$$\|a_i(u), \partial_t a_i(u), \partial_{tt} a_i(u)\|_{T,s} \leq C \|u, \partial_t u, \partial_{tt} u\|_{T,s+3}, \qquad i = 0, 1, 2, 3. \tag{2.3}$$

Proof. Apply standard tame estimates for composition of functions, see Lemma 8.2. □

Now we apply the reduction procedure to any linear operator of the form (2.1) where

$$a_2(t, x) = c\partial_x a_3(t, x) \tag{2.4}$$

for some constant $c \in \mathbb{R}$ (note that $P'(u)$ has $c = 2$ because of the Hamiltonian structure of $\mathcal{N}_1$). Regarding the loss of regularity with respect to the space variable x, the estimates in the sequel will be not sharp. In the whole section we consider $T > 0$ fixed, and, unless otherwise specified, all the constants may depend on T.

Remark 2.2. Given a linear operator $\mathcal{L}_0$ of the form (2.1), define the operator $\mathcal{L}_0^*$ as

$$\mathcal{L}_0^* h := -\partial_t h - \partial_{xxx}\{(1 + a_3)h\} + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h\ . \tag{2.5}$$

Note that $-\mathcal{L}_0^*$ is still an operator of the form (2.1), namely

$$-\mathcal{L}_0^* = \partial_t + (1 + a_3^*)\partial_{xxx} + a_2^* \partial_{xx} + a_1^* \partial_x + a_0^* \tag{2.6}$$

with

$$\begin{aligned} a_3^* &:= a_3, & a_2^* &:= 3(a_3)_x - a_2, \\ a_1^* &:= 3(a_3)_{xx} - 2(a_2)_x + a_1, & a_0^* &:= (a_3)_{xxx} - (a_2)_{xx} + (a_1)_x - a_0. \end{aligned} \tag{2.7}$$

It follows from (2.6), (2.7) that if $\mathcal{L}_0$ satisfies (2.4), then also $-\mathcal{L}_0^*$ satisfies (2.4) (with a different constant), namely $a_2^* = (3 - c)\partial_x a_3^*$. In particular, if $\mathcal{L}_0$ satisfies (2.4) with $c = 2$ (which is the case if $\mathcal{L}_0 = P'(u)$), then $-\mathcal{L}_0^*$ satisfies (2.4) with $c = 1$. □

2.1 Step 1. Change of the space variable

We consider a t-dependent family of diffeomorphisms of the circle $\mathbb{T}$ of the form

$$y = x + \beta(t,x), \tag{2.8}$$

where β is a real-valued function, 2π periodic in x, defined for $t \in [0,T]$, with $|\beta_x(t,x)| \leq 1/2$ for all $(t,x) \in [0,T] \times \mathbb{T}$. We define the linear operator

$$(\mathcal{A}h)(t,x) := h(t, x + \beta(t,x)). \tag{2.9}$$

The operator $\mathcal{A}$ is invertible, with inverse $\mathcal{A}^{-1}$, transpose $\mathcal{A}^T$ (transpose with respect to the usual L^2_x-scalar product) and inverse transpose $\mathcal{A}^{-T}$ given by

$$\begin{aligned} &(\mathcal{A}^{-1}v)(t,y) = v(t, y + \tilde\beta(t,y)), \quad (\mathcal{A}^T v)(t,y) = (1 + \tilde\beta_y(t,y))\, v(t, y + \tilde\beta(t,y)), \\ &(\mathcal{A}^{-T}h)(t,x) = (1 + \beta_x(t,x))\, h(t, x + \beta(t,x)) \end{aligned} \tag{2.10}$$

where $y \mapsto y + \tilde\beta(t,y)$ is the inverse diffeomorphism of (2.8), namely

$$x = y + \tilde\beta(t,y) \quad \Longleftrightarrow \quad y = x + \beta(t,x). \tag{2.11}$$

Given the operator

$$\mathcal{L}_0 := \partial_t + (1 + a_3(t,x))\partial_{xxx} + a_2(t,x)\partial_{xx} + a_1(t,x)\partial_x + a_0(t,x) \ , \tag{2.12}$$

with $a_2(t,x) = c\partial_x a_3(t,x)$ we calculate the conjugate $\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}$. The conjugate $\mathcal{A}^{-1}a\mathcal{A}$ of any multiplication operator $a : h(t,x) \mapsto a(t,x)h(t,x)$ is the multiplication operator $(\mathcal{A}^{-1}a)$ that maps $v(t,y) \mapsto (\mathcal{A}^{-1}a)(t,y)\, v(t,y)$. By conjugation, the differential operators become

$$\mathcal{A}^{-1}\partial_t\mathcal{A} = \partial_t + (\mathcal{A}^{-1}\beta_t)\partial_y \qquad \mathcal{A}^{-1}\partial_x\mathcal{A} = \{\mathcal{A}^{-1}(1+\beta_x)\}\,\partial_y$$

then $\mathcal{A}^{-1}\partial_{xx}\mathcal{A} = (\mathcal{A}^{-1}\partial_x\mathcal{A})(\mathcal{A}^{-1}\partial_x\mathcal{A})$, and similarly for the conjugate of ∂_{xxx}. We calculate

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \partial_t + a_4(t,y)\partial_{yyy} + a_5(t,y)\partial_{yy} + a_6(t,y)\partial_y + a_7(t,y) \tag{2.13}$$

where

$$\begin{aligned} &a_4 = \mathcal{A}^{-1}\{(1+a_3)(1+\beta_x)^3\}, \qquad a_5 = \mathcal{A}^{-1}\{a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)\}, \\ &a_6 = \mathcal{A}^{-1}\{\beta_t + (1+a_3)\beta_{xxx} + a_2\beta_{xx} + a_1(1+\beta_x)\}, \qquad a_7 = \mathcal{A}^{-1}a_0. \end{aligned} \tag{2.14}$$

We look for $\beta(t,x)$ such that the coefficient $a_4(t,y)$ of the highest order derivative ∂_{yyy} in (2.13) does not depend on y, namely $a_4(t,y) = b(t)$ for some function $b(t)$ of t only. This is equivalent to

$$\big(1 + a_3(t,x)\big)\big(1 + \beta_x(t,x)\big)^3 = b(t), \tag{2.15}$$

namely

$$\beta_x = \rho_0, \qquad \rho_0(t,x) := b(t)^{1/3}\big(1 + a_3(t,x)\big)^{-1/3} - 1. \tag{2.16}$$

The equation (2.16) has a solution β, periodic in x, if and only if $\int_{\mathbb{T}} \rho_0(t,x)\,dx = 0$ for all t. This condition uniquely determines

$$b(t) = \left(\frac{1}{2\pi}\int_{\mathbb{T}} \big(1 + a_3(t,x)\big)^{-\frac{1}{3}}\,dx\right)^{-3}. \tag{2.17}$$

Then we fix the solution (with zero average) of (2.16),

$$\beta(t,x) := (\partial_x^{-1}\rho_0)(t,x), \tag{2.18}$$

where $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier). We have conjugated $\mathcal{L}_0$ to

$$\mathcal{L}_1 = \mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \partial_t + a_4(t)\partial_{yyy} + a_5(t,y)\partial_{yy} + a_6(t,y)\partial_y + a_7(t,y), \tag{2.19}$$

where $a_4(t) := b(t)$ is defined in (2.17).

We prove here some bounds that will be used later.

Lemma 2.3. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, and let $a_3(t,x), a_2(t,x), a_1(t,x), a_0(t,x)$ be four functions with $a_2 = c\partial_x a_3$ for some $c \in \mathbb{R}$. Moreover, assume $\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0 \in C([0,T], H_x^{s+\sigma})$. Let*

$$\delta(\mu) := \|\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{2.20}$$

If $\delta(0) \leq \delta_*$, then the operator $\mathcal{A}$ defined in (2.9), (2.18), (2.16), (2.17) *belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0,s]$ and satisfies*

$$\|\mathcal{A}h\|_{T,\mu} \leq C_\mu\big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H_x^\mu), \tag{2.21}$$

for some positive C_μ depending on μ. The inverse operator $\mathcal{A}^{-1}$, the transpose $\mathcal{A}^T$ and the inverse transpose $\mathcal{A}^{-T}$ all satisfy the same estimate (2.21) *as $\mathcal{A}$.*

The functions $a_4(t) = b(t)$, $a_5(t,y)$, $a_6(t,y)$, $a_7(t,y)$, $\beta(t,x)$, $\tilde\beta(t,y)$ defined in (2.17), (2.16), (2.18), (2.14), (2.11) *belong to $C([0,T], H_x^\mu)$ for all $\mu \in [0,s]$ and satisfy*

$$\|\beta, \tilde\beta, a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\mu} + |a_4 - 1, a_4'|_T \leq C_\mu \delta(\mu) \ . \tag{2.22}$$

Finally, the coefficient $a_5(t,y)$ satisfies

$$\int_{\mathbb{T}} a_5(t,y)\,dy = 0 \qquad \forall t \in [0,T] \ . \tag{2.23}$$

Proof. The proof of (2.21) and (2.22) is a straightforward application of the standard tame estimates for products, composition of functions and changes of variable, see section 8.

To prove (2.23), we use the definition of $b(t)$ in (2.17), the equality $a_2 = c\partial_x a_3$, and the change of variables (2.11), and we compute

$$\begin{aligned}
\int_{\mathbb{T}} a_5(t,y)\,dy &= \int_{\mathbb{T}} [a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)](1+\beta_x)\,dx \\
&= b(t)\left\{c\int_{\mathbb{T}} \frac{\partial_x a_3(t,x)}{1+a_3(t,x)}\,dx + 3\int_{\mathbb{T}} \frac{\beta_{xx}(t,x)}{1+\beta_x(t,x)}\,dx\right\} \\
&= b(t)\left\{c\int_{\mathbb{T}} \partial_x \log(1+a_3(t,x))\,dx + 3\int_{\mathbb{T}} \partial_x \log(1+\beta_x(t,x))\,dx\right\} = 0. \ \square
\end{aligned}$$

2.2 Step 2. Time reparametrization

The goal of this section is to obtain a constant coefficient instead of $a_4(t)$. We consider a diffeomorphism $\psi : [0,T] \to [0,T]$ which gives the change of the time variable

$$\psi(t) = \tau \quad \Leftrightarrow \quad t = \psi^{-1}(\tau), \tag{2.24}$$

with $\psi(0) = 0$ and $\psi(T) = T$. We define

$$(\mathcal{B}h)(t,y) := h(\psi(t),y), \quad (\mathcal{B}^{-1}v)(\tau,y) := v(\psi^{-1}(\tau),y). \tag{2.25}$$

By conjugation, the differential operators become

$$\mathcal{B}^{-1}\partial_t\mathcal{B} = \rho(\tau)\partial_\tau, \quad \mathcal{B}^{-1}\partial_y\mathcal{B} = \partial_y, \quad \rho := \mathcal{B}^{-1}(\psi'), \tag{2.26}$$

and therefore (2.19) is conjugated to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\partial_\tau + (\mathcal{B}^{-1}a_4)\partial_{yyy} + (\mathcal{B}^{-1}a_5)\partial_{yy} + (\mathcal{B}^{-1}a_6)\partial_y + (\mathcal{B}^{-1}a_7). \tag{2.27}$$

We look for ψ such that the (variable) coefficients of the highest order derivatives (∂_τ and ∂_{yyy}) are proportional, namely

$$(\mathcal{B}^{-1}a_4)(\tau) = m\rho(\tau) = m(\mathcal{B}^{-1}(\psi'))(\tau) \tag{2.28}$$

for some constant $m \in \mathbb{R}$. Since $\mathcal{B}$ is invertible, this is equivalent to requiring that

$$a_4(t) = m\psi'(t). \tag{2.29}$$

Integrating on $[0,T]$ determines the value of the constant m, and then we fix ψ:

$$m := \frac{1}{T}\int_0^T a_4(t)\,dt, \quad \psi(t) := \frac{1}{m}\int_0^t a_4(s)\,ds. \tag{2.30}$$

With this choice of ψ we get

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\mathcal{L}_2, \qquad \mathcal{L}_2 := \partial_\tau + m\partial_{yyy} + a_8(\tau,y)\,\partial_{yy} + a_9(\tau,y)\,\partial_y + a_{10}(\tau,y), \tag{2.31}$$

where

$$a_8(\tau,y) := \frac{1}{\rho(\tau)}\,(\mathcal{B}^{-1}a_5)(\tau,y), \qquad a_9(\tau,y) := \frac{1}{\rho(\tau)}\,(\mathcal{B}^{-1}a_6)(\tau,y), \tag{2.32}$$
$$a_{10}(\tau,y) := \frac{1}{\rho(\tau)}\,(\mathcal{B}^{-1}a_7)(\tau,y).$$

Note that for all $\tau \in [0,T]$ one has

$$\int_{\mathbb{T}} a_8(\tau,y)\,dy = \frac{1}{(\mathcal{B}^{-1}\psi')(\tau)}\int_{\mathbb{T}}(\mathcal{B}^{-1}a_5)(\tau,y)\,dy = \frac{1}{\psi'(t)}\int_{\mathbb{T}} a_5(t,y)\,dy = 0\ . \tag{2.33}$$

By straightforward calculations, we prove the following lemma.

Lemma 2.4. *There exists $\delta_* > 0$ with the following properties. Let $a_4 \in C([0,T],\mathbb{R})$ with $|a_4(t) - 1| \le \delta_*$ for all $t \in [0,T]$. Then the operator $\mathcal{B}$ defined in* (2.25), (2.30) *is an invertible isometry of $C([0,T],H^s_x)$ for all $s \ge 0$, namely*

$$\|\mathcal{B}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0,T],H^s_x), \quad s \ge 0. \tag{2.34}$$

Moreover there exists a positive constant σ with the following property. Let $a_4 \in C^1([0,T],\mathbb{R})$, with $|a_4(t) - 1| \le \delta_*$ and $|a_4'(t)| \le 1$ for all $t \in [0,T]$. Let $s \ge 0$, and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T],H^s_x)$ with $\int_{\mathbb{T}} a_5(t,y)\,dy = 0$ for all $t \in [0,T]$. Then the functions $a_8(t,x)$, $a_9(t,x)$, $a_{10}(t,x)$, $\psi(t)$, $\rho(t)$ and the constant m defined in (2.32), (2.30), (2.26) *satisfy*

$$|m-1| + |\psi' - 1, \rho - 1|_T + \|a_8, \partial_\tau a_8, a_9, \partial_\tau a_9, a_{10}\|_{T,s} \le C\|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,s} \tag{2.35}$$

where C is independent of s. Moreover one has

$$\int_{\mathbb{T}} a_8(\tau,y)\,dy = 0 \qquad \forall \tau \in [0,T] \ . \tag{2.36}$$

2.3 Step 3. Multiplication

In this section we eliminate the term $a_8(\tau,y)\partial_{yy}$ from the operator $\mathcal{L}_2$ defined in (2.31). To this end, we consider the multiplication operator $\mathcal{M}$ defined as

$$\mathcal{M}h(\tau,y) := q(\tau,y)h(\tau,y) \tag{2.37}$$

with $q : [0,T] \times \mathbb{T} \to \mathbb{R}$. We compute

$$\mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \partial_\tau + m\partial_{yyy} + a_{11}(\tau,y)\partial_{yy} + a_{12}(\tau,y)\partial_y + a_{13}(\tau,y) \tag{2.38}$$

with

$$a_{11} := a_8 + \frac{3mq_y}{q}, \qquad a_{12} := a_9 + \frac{2a_8q_y + 3mq_{yy}}{q}, \qquad a_{13} := \frac{\mathcal{L}_2 q}{q}. \tag{2.39}$$

We want to choose q such that $a_{11} = 0$, which is equivalent to

$$3mq_y + a_8 q = 0 \ . \tag{2.40}$$

Thanks to (2.36), equation (2.40) admits the space-periodic solution

$$q(\tau,y) := \exp\Big\{ -\frac{1}{3m}(\partial_y^{-1}a_8)(\tau,y)\Big\} \ . \tag{2.41}$$

As a consequence, we get

$$\mathcal{L}_3 := \mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \partial_\tau + m\partial_{yyy} + a_{12}(\tau,y)\partial_y + a_{13}(\tau,y) \ . \tag{2.42}$$

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $s \ge 0$ and let $a_8 \in C([0,T],H^s_x)$ with $\int_{\mathbb{T}} a_8(\tau,y)\,dy = 0$ for all $\tau \in [0,T]$. Then for all $\mu \in [0,s]$, the operator $\mathcal{M}$ defined in* (2.37), (2.41) *and its inverse $\mathcal{M}^{-1}$ belong to $C([0,T],\mathcal{L}(H^\mu_x))$. Note that $\mathcal{M} = \mathcal{M}^T$.*

Furthermore, there exist two positive constants δ_*, σ with the following properties. Assume that $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H_x^{s+\sigma})$ and let

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma} \ . \tag{2.43}$$

Then if $\delta(0) \leq \delta_*$, for all $\mu \in [0,s]$ the operator $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ satisfy

$$\|\mathcal{M}^{\pm 1} h\|_{T,\mu} \leq C_\mu \big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0} \big) \quad \forall h \in C([0,T], H_x^\mu), \tag{2.44}$$

for some positive C_μ depending on μ. Moreover, the functions $a_{12}(\tau,y), a_{13}(\tau,y), q(\tau,y)$ defined in (2.39), (2.41) *satisfy*

$$\|q-1, a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu} \leq C_\mu \delta(\mu) \ . \tag{2.45}$$

2.4 Step 4. Translation of the space variable

We consider the change of the space variable $z = y + p(\tau)$ and the operators

$$\mathcal{T} h(\tau, y) := h(\tau, y + p(\tau)), \quad \mathcal{T}^{-1} v(\tau, z) := v(\tau, z - p(\tau)) \tag{2.46}$$

where p is a function $p : [0,T] \to \mathbb{R}$. The differential operators become $\mathcal{T}^{-1}\partial_y \mathcal{T} = \partial_z$ and $\mathcal{T}^{-1}\partial_\tau \mathcal{T} = \partial_\tau + \{\partial_\tau p(\vartheta)\}\,\partial_z$. This is a special, simple case of the transformation $\mathcal{A}$ of section 2.1. Thus

$$\mathcal{L}_4 := \mathcal{T}^{-1}\mathcal{L}_3\mathcal{T} = \partial_\tau + m\partial_{zzz} + a_{14}(\tau,z)\partial_z + a_{15}(\tau,z) \tag{2.47}$$

where

$$a_{14}(\tau,z) := p'(\tau) + (\mathcal{T}^{-1}a_{12})(\tau,z), \quad a_{15}(\tau,z) := (\mathcal{T}^{-1}a_{13})(\tau,z). \tag{2.48}$$

Now we look for $p(\tau)$ such that a_{14} has zero space average. We fix

$$p(\tau) := -\frac{1}{2\pi}\int_0^\tau \int_{\mathbb{T}} a_{12}(s,y)\,dyds. \tag{2.49}$$

With this choice of p, after renaming the space-time variables $z = x$ and $\tau = t$, we have

$$\mathcal{L}_4 = \partial_t + m\partial_{xxx} + a_{14}(t,x)\partial_x + a_{15}(t,x), \qquad \int_{\mathbb{T}} a_{14}(t,x)\,dx = 0 \quad \forall t \in [0,T]. \tag{2.50}$$

With direct calculations we prove the following estimates.

Lemma 2.6. *Let $a_{12} \in C([0,T], L_x^2)$. Then the operator $\mathcal{T}$ defined in* (2.46), (2.49) *belongs to $C([0,T], \mathcal{L}(H_x^s))$ for all $s \in [0,+\infty)$. In fact $\mathcal{T}$ is an isometry, namely*

$$\|\mathcal{T}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0,T], H_x^s). \tag{2.51}$$

Moreover, $\mathcal{T}$ is invertible and its transpose is $\mathcal{T}^T = \mathcal{T}^{-1}$.

Let $s \geq 0$, and let $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H_x^{s+1})$ with $\|a_{12}\|_{T,0} \leq 1$. Then the functions a_{14}, a_{15}, p defined in (2.48), (2.49) *satisfy*

$$\sup_{t\in[0,T]} |p(t)| + \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,s} \leq C\|a_{12}, \partial_t a_{12}, a_{13}\|_{T,s+1} \tag{2.52}$$

where C is independent of s.

2.5 Step 5. Elimination of the order one

The goal of this section is to eliminate the term $a_{14}(t,x)\partial_x$. Consider an operator $\mathcal{S}$ of the form

$$\mathcal{S}h := h + \gamma(t,x)\partial_x^{-1}h \tag{2.53}$$

where $\gamma(t,x)$ is a function to be determined. Note that $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = \pi_0$ where $\pi_0 h := h - \frac{1}{2\pi}\int_{\mathbb{T}} h\,dx$. We directly calculate

$$\mathcal{L}_4\mathcal{S} - \mathcal{S}(\partial_t + m\partial_{xxx}) = a_{16}\partial_x + a_{17} + a_{18}\partial_x^{-1} \tag{2.54}$$

where

$$\begin{aligned} &a_{16} := 3m\gamma_x + a_{14}, \qquad a_{17} := a_{15} + (3m\gamma_{xx} + a_{14}\gamma)\pi_0, \\ &a_{18} := \gamma_t + m\gamma_{xxx} + a_{14}\gamma_x + a_{15}\gamma. \end{aligned} \tag{2.55}$$

We fix γ as

$$\gamma := -\frac{1}{3m}\,\partial_x^{-1}a_{14}, \tag{2.56}$$

so that $a_{16} = 0$. By the following Lemma 2.7, $\mathcal{S}$ is invertible, and we obtain

$$\mathcal{L}_5 := \mathcal{S}^{-1}\mathcal{L}_4\mathcal{S} = \partial_t + m\partial_{xxx} + \mathcal{R}, \qquad \mathcal{R} := \mathcal{S}^{-1}(a_{17} + a_{18}\partial_x^{-1}). \tag{2.57}$$

Lemma 2.7. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, let a_{14}, a_{15} be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{2.58}$$

If $\delta(0) \leq \delta_*$, then the operator $\mathcal{S}$ defined in (2.53), (2.56) *belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0,s]$ and satisfies*

$$\|\mathcal{S}h\|_{T,\mu} \leq C_\mu\big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H_x^\mu), \tag{2.59}$$

for some positive C_μ depending on μ. The operator $\mathcal{S}$ is invertible, and its inverse $\mathcal{S}^{-1}$, its transpose $\mathcal{S}^T$ and its inverse transpose $\mathcal{S}^{-T}$ all satisfy the same estimate (2.59) *as $\mathcal{S}$.*

The operator $\mathcal{R}$ defined in (2.57) *belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0,s]$ and it satisfies*

$$\|\mathcal{R}h\|_{T,\mu} \leq C_\mu\big(\delta(0)\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H_x^\mu). \tag{2.60}$$

The transpose $\mathcal{R}^T$ belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ and satisfies the same estimate (2.60) *as $\mathcal{R}$.*

Proof. Estimate $\|\gamma\partial_x^{-1}h\|_{T,\mu}$ by the usual tame estimates for the product of two functions (Lemma 8.1), then use Neumann series in its tame version. $\square$

3 Observability

In this section we prove the observability of linear operators of the form (2.12). Such observability property will be used in Section 4 in order to prove controllability of the linearized problem. We split the proof into several simple lemmas, starting with a direct consequence of Ingham inequality. Since we actually need observability of a Cauchy problem flowing backwards in time (see Lemma 4.2) with datum at time T, we will accordingly state our lemmas.

Lemma 3.1 (Ingham inequality for $\partial_t + m\partial_{xxx}$)**.** *For every $T > 0$ there exists a positive constant $C_1(T)$ such that, for all $(w_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C})$, all $m \geq 1/2$,*

$$\int_0^T \Big| \sum_{n\in\mathbb{Z}} w_n e^{imn^3 t} \Big|^2 \, dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n|^2.$$

Proof. See, for example, Theorem 4.3 in Section 4.1 of [39]. The fact that the constant $C_1(T)$ does not depend on m is obtained by closely following the proof in [39], and taking into account the lower bound for the distance between two different eigenvalues $|mn^3 - mk^3| \geq m \geq \frac{1}{2}$, for all $n, k \in \mathbb{Z}$, $n \neq k$. □

The following observability result is classical (see, e.g., [46] for a closely related result); for completeness, we also give here its proof.

Lemma 3.2 (Observability for $\partial_t + m\partial_{xxx}$)**.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $v_T \in L^2(\mathbb{T})$, $m \geq 1/2$, and let v satisfy*

$$\partial_t v + m\partial_{xxx} v = 0, \quad v(T) = v_T. \tag{3.1}$$

Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_2 \|v_T\|^2_{L^2_x} \tag{3.2}$$

with $C_2 := C_1(T)|\omega|$, where $C_1(T)$ is the constant of Proposition 3.1, and $|\omega|$ is the Lebesgue measure of ω.

Proof. Let $v_T(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, so that $v(t,x) = \sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t}$ where $w_n(x) := a_n e^{i(nx - mn^3 T)}$. By Lemma 3.1, for each $x \in \mathbb{T}$ we have

$$\int_0^T \Big| \sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t} \Big|^2 dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n(x)|^2 = C_1(T) \sum_{n\in\mathbb{Z}} |a_n|^2 = C_1(T) \|v_T\|^2_{L^2(\mathbb{T})},$$

then we integrate over $x \in \omega$. □

Lemma 3.3 (Observability of $\mathcal{L}_5 := \partial_t + m\partial_{xxx} + \mathcal{R}$)**.** *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $\mathcal{R} \in C([0,T], \mathcal{L}(L^2_x))$, with $\|\mathcal{R}(t)h\|_0 \leq r_0 \|h\|_0$ for all $h \in L^2_x$, all $t \in [0,T]$, where r_0 is a positive constant. Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem*

$$\partial_t v + m\partial_{xxx} v + \mathcal{R} v = 0, \quad v(T) = v_T, \tag{3.3}$$

which is globally wellposed by Lemma 6.2(iii). Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_3 \|v_T\|^2_{L^2_x}$$

with $C_3 := C_2/4$, provided that r_0 is small enough (more precisely, r_0 smaller than a constant depending only on T, C_2 where C_2 is the constant in Lemma 3.2).

Proof. Let v_1 be the solution of $\partial_t v_1 + m\partial_{xxx} v_1 = 0$, $v_1(T) = v_T$, and let $v_2 := v - v_1$. Then v_2 solves

$$(\partial_t + m\partial_{xxx} + \mathcal{R})v_2 = -\mathcal{R}v_1, \quad v_2(T) = 0. \tag{3.4}$$

By (6.10), applied for $s = 0$, $\alpha = 0$, $f = -\mathcal{R}v_1$, we get

$$\|v_2\|_{T,0} \le 2^{4Tr_0} 4T \|\mathcal{R}v_1\|_{T,0} \le 2^{4Tr_0} 4Tr_0 \|v_T\|_0. \tag{3.5}$$

Using the elementary inequality $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ for all $a, b \in \mathbb{R}$,

$$\int_0^T \int_\omega |v|^2 dx dt \ge \frac{1}{2} \int_0^T \int_\omega |v_1|^2 dx dt - \int_0^T \int_\omega |v_2|^2 dx dt.$$

The integral of $|v_1|^2$ is estimated from below by (3.2). The integral of $|v_2|^2$ is bounded by $T\|v_2\|_{T,0}^2$, then use (3.5). $\square$

Lemma 3.4 (Observability of $\mathcal{L}_4 := \partial_t + m\partial_{xxx} + a_{14}(t,x)\partial_x + a_{15}(t,x)$, a_{14} with zero mean)**.** *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $m \ge 1/2$ and let $a_{14}(t,x)$, $a_{15}(t,x)$ be two functions, with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H_x^\sigma)$,*

$$\int_{\mathbb{T}} a_{14}(t,x)\, dx = 0 \quad \forall t \in [0,T], \qquad \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\sigma} \le \delta. \tag{3.6}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_4 v = 0, \quad v(T) = v_T, \tag{3.7}$$

which is globally wellposed by Lemma 6.3. Then

$$\int_0^T \int_\omega |v(t,x)|^2\, dx\, dt \ge C_4 \|v_T\|^2_{L^2_x}$$

with $C_4 := C_3/16$, provided that δ is small enough (more precisely, δ smaller than a constant depending only on T, C_3).

Proof. Following the procedure of Section 2.5, we consider the transformation $\mathcal{S}$ in (2.53), (2.56), which conjugates $\mathcal{L}_4$ to

$$\mathcal{L}_5 := \mathcal{S}^{-1}\mathcal{L}_4\mathcal{S} = \partial_t + m\partial_{xxx} + \mathcal{R},$$

where the operator $\mathcal{R}$ is defined in (2.57), (2.55), it belongs to $C([0,T], \mathcal{L}(L^2_x))$, and satisfies the bounds in Lemma 2.7. Let v be the solution of (3.7), and define $\tilde v := \mathcal{S}^{-1}v$. Then $\tilde v$ solves $\mathcal{L}_5\tilde v = 0$, $\tilde v(T) = \tilde v_T$ where $\tilde v_T := \mathcal{S}^{-1}(T)v_T$, and therefore Lemma 3.3 applies to $\tilde v$ if δ is sufficiently small. By Lemmas 2.7, 6.3 and Remark 6.8 we get

$$\int_0^T \int_\omega |(\mathcal{S}^{-1} - I)v|^2\, dx\, dt \le T\|(\mathcal{S}^{-1} - I)v\|^2_{T,0} \le C\delta^2 \|v\|^2_{T,0} \le C'\delta^2\|v_T\|_0^2$$

for some constant C' depending on T. We split $\tilde v = v + (\mathcal{S}^{-1} - I)v$, and we get

$$\int_0^T \int_\omega |\tilde v|^2\, dx\, dt \le 2\int_0^T \int_\omega |v|^2\, dx\, dt + 2C'\delta^2 \|v_T\|_0^2.$$

Moreover $\|v_T\|_0 = \|\mathcal{S}(T)\tilde v_T\|_0 \le 2\|\tilde v_T\|_0$, and the thesis follows for δ small enough. $\square$

Lemma 3.5 (Observability of $\mathcal{L}_3 := \partial_t + m\partial_{xxx} + a_{12}(t,x)\partial_x + a_{13}(t,x)$)**.** *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $a_{12}(t,x)$, $a_{13}(t,x)$ be two functions, with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H^\sigma_x)$,*

$$\|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\sigma} \leq \delta. \tag{3.8}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_3 v = 0, \quad v(T) = v_T, \tag{3.9}$$

which is globally wellposed by Lemma 6.4. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_5 \|v_T\|^2_{L^2_x} \tag{3.10}$$

for some $C_5 > 0$ depending on T, ω, provided that δ in (3.8) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_4).

Proof. Following the procedure of Section 2.4, we consider the transformation $\mathcal{T}$ defined in (2.46), (2.49), which conjugates $\mathcal{L}_3$ to

$$\mathcal{L}_4 := \mathcal{T}^{-1}\mathcal{L}_3\mathcal{T} = \partial_t + m\partial_{xxx} + a_{14}(t,x)\partial_x + a_{15}(t,x),$$

where a_{14}, a_{15} are defined in (2.48), and $\int_{\mathbb{T}} a_{14}(t,x)\, dx = 0$. By (2.52), the function p defined in (2.49) satisfies $|p(t)| \leq C\delta$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3.9). Then $\tilde{v} := \mathcal{T}^{-1} v$ solves $\mathcal{L}_4 \tilde{v} = 0$, $\tilde{v}(T) = \mathcal{T}^{-1}(T) v_T$. Let $\omega_1 = [\alpha_1, \beta_1]$ be an interval contained in ω. For δ small enough, one has

$$[\alpha_1 - p(t), \beta_1 - p(t)] \subseteq [\alpha_1 - \delta, \beta_1 + \delta] \subset \omega \quad \forall t \in [0,T].$$

The change of variable $x - p(t) = y$, $dx = dy$ gives

$$\int_0^T \int_{\omega_1} |\tilde{v}(t,x)|^2 \, dx \, dt = \int_0^T \int_{\alpha_1 - p(t)}^{\beta_1 - p(t)} |v(t,y)|^2 \, dy \, dt \leq \int_0^T \int_\omega |v(t,y)|^2 \, dy \, dt.$$

By (2.52), for δ small enough, Lemma 3.4 can be applied to $\tilde{v}$ on the interval ω_1 and the thesis follows, since $\|\tilde{v}(T)\|_0 = \|\mathcal{T}^{-1}(T) v_T\|_0 = \|v_T\|_0$. ☐

Lemma 3.6 (Observability of $\mathcal{L}_2 := \partial_t + m\partial_{xxx} + a_8(t,x)\partial_{xx} + a_9(t,x)\partial_x + a_{10}(t,x)$)**.** *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $a_8(t,x)$, $a_9(t,x)$, $a_{10}(t,x)$ be three functions, with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^\sigma_x)$,*

$$\int_{\mathbb{T}} a_8(t,x)\, dx = 0 \quad \forall t \in [0,T], \qquad \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\sigma} \leq \delta. \tag{3.11}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_2 v = 0, \quad v(T) = v_T, \tag{3.12}$$

which is globally wellposed by Lemma 6.5. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_6 \|v_T\|^2_{L^2_x} \tag{3.13}$$

for some $C_6 > 0$ depending on T, ω, provided that δ in (3.11) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_5).

Proof. Following the procedure of Section 2.3, we consider the multiplication operator $\mathcal{M}$ defined in (2.37), (2.41), which conjugates $\mathcal{L}_2$ to

$$\mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \mathcal{L}_3, \quad \mathcal{L}_3 = \partial_t + m\partial_{xxx} + a_{12}(t,x)\partial_x + a_{13}(t,x),$$

where a_{12}, a_{13} are defined in (2.39). Let v be the solution of the Cauchy problem (3.12). Then $\tilde{v} := \mathcal{M}^{-1}v$ solves $\mathcal{L}_3\tilde{v} = 0$, $\tilde{v}(T) = \mathcal{M}^{-1}(T)v_T$. Using (2.45), we have

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx\, dt = \int_0^T \int_\omega |\tilde{v}|^2 \, dx\, dt + \int_0^T \int_\omega |\tilde{v}|^2(|q|^2 - 1) \, dx\, dt \geq (C_5 - C\delta)\|v_T\|_0^2.$$

The first of the two integrals has been estimated from below by applying Lemma 3.5 to $\mathcal{L}_3$ (by Lemma 2.5, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2.45), since $|q(t) - 1| \leq C\|q - 1\|_{T,1} \leq C'\delta$. Moreover, we have used the inequality $\|\tilde{v}\|_{T,0} \leq C\|\tilde{v}_T\|_0$ from Lemma 6.4. The thesis follows with $C_6 := C_5/2$ by choosing δ small enough. ☐

Lemma 3.7 (Observability of $\mathcal{L}_1 := \partial_t + a_4(t)\partial_{xxx} + a_5(t,x)\partial_{xx} + a_6(t,x)\partial_x + a_7(t,x)$)**.** *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_4, a_5, a_6, a_7 be four functions, with $a_4 \in C^1([0,T],\mathbb{R})$, $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^\sigma_x)$, satisfying*

$$\int_{\mathbb{T}} a_5(t,x)\, dx = 0 \quad \forall t \in [0,T], \qquad \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\sigma} + |a_4 - 1, a_4'|_T \leq \delta. \tag{3.14}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_1 v = 0, \quad v(T) = v_T, \tag{3.15}$$

which is globally wellposed by Lemma 6.6. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx\, dt \geq C_7\|v_T\|^2_{L^2_x} \tag{3.16}$$

for some $C_7 > 0$ depending on T, ω, provided that δ in (3.14) *is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_6).*

Proof. Following the procedure of Section 2.2, we consider the re-parametrization of time $\mathcal{B}$ defined in (2.25), (2.30), which conjugates $\mathcal{L}_1$ to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\mathcal{L}_2, \quad \mathcal{L}_2 = \partial_\tau + m\partial_{xxx} + a_8(\tau,x)\partial_{xx} + a_9(\tau,x)\partial_x + a_{10}(\tau,x),$$

where ρ, a_8, a_9, a_10 are defined in (2.28), (2.32) and $\int_{\mathbb{T}} a_8(\tau,x) = 0$ for all $\tau \in [0,T]$. Let v be the solution of the Cauchy problem (3.15). Then $\tilde{v} := \mathcal{B}^{-1}v$ solves $\mathcal{L}_2\tilde{v} = 0$, $\tilde{v}(T) = \mathcal{B}^{-1}(T)v_T$. Using (2.35), we have

$$\begin{aligned}
\int_0^T \int_\omega |v(t,x)|^2 \, dx\, dt &= \int_0^T \int_\omega |\tilde{v}(\psi(t),x)|^2 \, dx\, dt \\
&= \int_0^T \int_\omega |\tilde{v}(\psi(t),x)|^2[\psi'(t) + (1 - \psi'(t))] \, dx\, dt \\
&= \int_0^T \int_\omega |\tilde{v}(\tau,x)|^2 \, dx\, d\tau + \int_0^T \int_\omega |\tilde{v}(\psi(t),x)|^2(1 - \psi'(t)) \, dx\, dt \\
&\geq (C_6 - C\delta)\|v_T\|_0^2.
\end{aligned}$$

The first of the two integrals has been estimated from below by applying Lemma 3.6 to $\mathcal{L}_2$ (by Lemma 2.4, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2.35) for $|\psi'(t) - 1|$ and also the inequality $\|\tilde{v}\|_{T,0} \leq C\|\tilde{v}_T\|_0$ from Lemma 6.5. The thesis follows with $C_7 := C_6/2$ by choosing δ small enough, since $\|\tilde{v}_T\|_0 = \|\mathcal{B}^{-1}(T)v_T\|_0 = \|v_T\|_0$. $\square$

Lemma 3.8 (Observability of $\mathcal{L}_0 := \partial_t + (1+a_3)\partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0$)**.** *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $c \in \mathbb{R}$ and $a_3(t,x), a_2(t,x), a_1(t,x), a_0(t,x)$ be four functions with $a_2 = c\partial_x a_3$,*

$$\|\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T,\sigma} \leq \delta. \tag{3.17}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_0 v = 0, \quad v(T) = v_T, \tag{3.18}$$

which is globally wellposed by Lemma 6.7. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_8 \|v_T\|^2_{L^2_x} \tag{3.19}$$

for some $C_8 > 0$ depending on T, ω, provided that δ in (3.17) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_7).

Proof. Following the procedure of Section 2.1, we consider the transformation $\mathcal{A}$ defined in (2.9), (2.16), (2.17), (2.18), which conjugates $\mathcal{L}_0$ to

$$\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \mathcal{L}_1 = \partial_t + a_4(t)\partial_{xxx} + a_5(t,x)\partial_{xx} + a_6(t,x)\partial_x + a_7(t,x)$$

(see (2.19)), where a_4, a_5, a_6, a_7 are defined in (2.14) and $\int_{\mathbb{T}} a_5(t,x) = 0$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3.18). Then $\tilde{v} := \mathcal{A}^{-1}v$ solves $\mathcal{L}_1\tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde{v}_0 := \mathcal{A}^{-1}(0)v_0$. Let $\omega_1 = [\alpha_1, \beta_1] \subset \omega$. By (2.22) in Lemma 2.3, for δ sufficiently small Lemma 3.7 applies to $\tilde{v}$ on ω_1, and

$$\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt \geq C_7 \|\tilde{v}_T\|_0^2.$$

By Lemma 2.3, $\|v_T\|_0 = \|\mathcal{A}(T)\tilde{v}_T\|_0 \leq C\|\tilde{v}_T\|_0$. The change of integration variable $y = x + \beta(t,x)$, $dy = (1 + \beta_x(t,x))dx$ gives

$$\begin{aligned}\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt &= \int_0^T \int_{\omega_1} |(\mathcal{A}^{-1}v)(t,y)|^2 \, dy \, dt \\ &= \int_0^T \int_{\omega_2(t)} \frac{|v(t,x)|^2}{1 + \beta_x(t,x)} \, dx \, dt \leq 2 \int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt,\end{aligned}$$

where $\omega_2(t) := \{x : x + \beta(t,x) \in \omega_1\}$. We have used the fact that, for δ small enough, $\omega_2(t) \subset \omega$, and the bound (2.22) for $|\beta_x(t,x)| \leq C\|\beta\|_{T,2} \leq C'\delta$. $\square$

4 Controllability

In this section we prove the controllability of the linearized operator $\mathcal{L}_0$, using its observability (Lemma 3.8), by means of the HUM method. We also prove higher regularity of the control.

Lemma 4.1 (Controllability of $\mathcal{L}_0$)**.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_3, a_2, a_1, a_0 be four functions of (t, x) with $a_2 = 2\partial_x a_3$ satisfying* (3.17)*. Let $\mathcal{L}_0$ be the linear operator*

$$\mathcal{L}_0 := \partial_t + (1 + a_3)\partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0. \tag{4.1}$$

(i) Existence. *There exist constants δ_0, C such that, if δ in* (3.17) *is smaller than δ_0, then the following property holds. Given any three functions $g_1(t, x)$, $g_2(x)$, $g_3(x)$, with $g_1 \in C([0, T], L^2_x)$, $g_2, g_3 \in L^2_x$, there exists a function $\varphi \in C([0, T], L^2_x)$ such that the solution h of the Cauchy problem*

$$\mathcal{L}_0 h = g_1 + \chi_\omega \varphi, \quad h(0) = g_2 \tag{4.2}$$

satisfies $h(T) = g_3$. (Note that the Cauchy problem (4.2) *is globally well-posed by Lemma 6.7). Moreover*

$$\|\varphi\|_{T,0} \leq C(\|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0). \tag{4.3}$$

(ii) Uniqueness. *Let $\mathcal{L}_0^*$ be the linear operator*

$$\mathcal{L}_0^* \psi := -\partial_t \psi - \partial_{xxx}\{(1 + a_3)\psi\} + \partial_{xx}(a_2\psi) - \partial_x(a_1\psi) + a_0\psi. \tag{4.4}$$

The control φ in (i) *is the unique solution of the equation $\mathcal{L}_0^*\varphi = 0$ such that the solution h of the Cauchy problem* (4.2) *satisfies $h(T) = g_3$.*

The proof of Lemma 4.1 is given below, and it is based on the following classical lemma. In this section we use the standard notation $\langle u, v\rangle := \int_{\mathbb{T}} uv\, dx$.

Lemma 4.2. *Let a_3, a_2, a_1, a_0 be functions satisfying* (3.17) *and $a_2 = 2\partial_x a_3$. Let $\mathcal{L}_0^*$ be the operator defined in* (4.4)*. For every (g_1, g_2, g_3) with $g_1 \in C([0, T], L^2_x)$, $g_2, g_3 \in L^2_x$ there exists a unique $\varphi_1 \in L^2_x$ such that for all $\psi_1 \in L^2_x$, the solutions $\varphi, \psi \in C([0, T], L^2_x)$ of the Cauchy problems*

$$\begin{cases} \mathcal{L}_0^*\varphi = 0 \\ \varphi(T) = \varphi_1 \end{cases} \qquad \begin{cases} \mathcal{L}_0^*\psi = 0 \\ \psi(T) = \psi_1 \end{cases} \tag{4.5}$$

satisfy

$$\int_0^T \langle g_1 + \chi_\omega\varphi, \psi\rangle\, dt + \langle g_2, \psi(0)\rangle - \langle g_3, \psi(T)\rangle = 0 \tag{4.6}$$

(note that the global well-posedness of the Cauchy problems (4.5) *follows from Lemma 6.7 and Remark 6.8). Moreover φ satisfies* (4.3)*.*

Proof. Given $\varphi_1, \psi_1 \in L^2_x$, let φ, ψ be the solutions of the Cauchy problems (4.5), and define

$$B(\varphi_1, \psi_1) := \int_0^T \langle \chi_\omega\varphi, \psi\rangle\, dt, \qquad \Lambda(\psi_1) := \langle g_3, \psi(T)\rangle - \langle g_2, \psi(0)\rangle - \int_0^T \langle g_1, \psi\rangle\, dt. \tag{4.7}$$

The bilinear map $B : L^2_x \times L^2_x \to \mathbb{R}$ is well defined and continuous because $|\chi_\omega(x)| \leq 1$ and, by Lemma 6.7 and Remark 6.8, $\|\varphi\|_{T,0} \leq C\|\varphi_1\|_0$, and similarly for ψ. Moreover B is coercive by Lemma 3.8 and Remark 2.2. The linear functional Λ is bounded, with

$$|\Lambda(\psi_1)| \leq C\|g\|_{T,0}\|\psi_1\|_0 \quad \forall \psi_1 \in L^2_x, \qquad \|g\|_{T,0} := \|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0.$$

Thus, by Riesz representation theorem (or Lax-Milgram), there exists a unique $\varphi_1 \in L^2_x$ such that

$$B(\varphi_1, \psi_1) = \Lambda(\psi_1) \quad \forall \psi_1 \in L^2_x. \tag{4.8}$$

Moreover $\|\varphi_1\|_0 \leq C\|\Lambda\|_{\mathcal{L}(L^2_x,\mathbb{R})} \leq C'\|g\|_{T,0}$. Since $\|\varphi\|_{T,0} \leq C\|\varphi_1\|_0$, we get (4.3). □

Proof of Lemma 4.1. (*i*). Let $\varphi_1 \in L^2_x$ be the unique solution of (4.8) given by Lemma 4.2. Consider any $\psi_1 \in L^2_x$, and let $\varphi, \psi \in C([0,T], L^2_x)$ be the unique solutions of the Cauchy problems (4.5). Recalling (4.6), (4.2) and integrating by parts, we have

$$\begin{aligned}
0 &= \int_0^T \langle g_1 + \chi_\omega \varphi, \psi\rangle\, dt + \langle g_2, \psi(0)\rangle - \langle g_3, \psi(T)\rangle \\
&= \int_0^T \langle \mathcal{L}_0 h, \psi\rangle\, dt + \langle g_2, \psi(0)\rangle - \langle g_3, \psi(T)\rangle \\
&= \langle h(T), \psi(T)\rangle - \langle h(0), \psi(0)\rangle + \int_0^T \langle h, \mathcal{L}_0^* \psi\rangle\, dt + \langle g_2, \psi(0)\rangle - \langle g_3, \psi(T)\rangle \\
&= \langle h(T), \psi(T)\rangle - \langle g_3, \psi(T)\rangle \\
&= \langle h(T) - g_3, \psi_1\rangle,
\end{aligned}$$

from which it follows that $h(T) = g_3$.

(*ii*). Assume that $\tilde\varphi \in C([0,T], L^2_x)$ satisfies $\mathcal{L}_0^*\tilde\varphi = 0$ and it has the property that the solution h of the Cauchy problem (4.2) satisfies $h(T) = g_3$. Let $\tilde\varphi_1 := \tilde\varphi(T)$. The same integration by parts as above shows that $B(\tilde\varphi_1, \psi_1) = \Lambda(\psi_1)$ for all $\psi_1 \in L^2_x$. By the uniqueness in Lemma 4.2, $\tilde\varphi_1 = \varphi_1$. □

Lemma 4.3 (Higher regularity)**.** *Let $T, \omega, a_3, a_2, a_1, a_0, \mathcal{L}_0, g_1, g_2, g_3$ be as in Lemma 4.1. There exist two positive constants δ_*, σ with the following property. Let $s > 0$ be given. Assume that $a_0, a_1, a_2, a_3 \in C^2([0,T], H^{s+\sigma}_x)$. Let*

$$\delta(\mu) := \sum_{k=0,1,2,\ i=0,1,2,3} \|\partial_t^k a_i\|_{T,\mu+\sigma}, \quad \mu \in [0,s].$$

Let $\|g\|_{T,s} := \|g_1\|_{T,s} + \|g_2\|_s + \|g_3\|_s < \infty$. If $\delta(0) \leq \delta_*$, then the control φ constructed in Lemma 4.1 and the solution h of (4.2) *satisfy*

$$\|\varphi, h\|_{T,s} \leq C_s(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}) \tag{4.9}$$

for some positive C_s depending on s, T, ω. Moreover, if $g_1 \in C^1([0,T], H^s_x)$, then

$$\|\partial_t\varphi, \partial_t h\|_{T,s+3} + \|\partial_{tt}\varphi, \partial_{tt}h\|_{T,s} \leq C_s\{\|g\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \delta(s)\|g\|_{T,6}\}. \tag{4.10}$$

Proof. Let $g_1 \in C([0,T], H^s_x)$, $g_2, g_3 \in H^s_x$. Let $\varphi, h \in C([0,T], L^2_x)$ be the solution of the control problem constructed in Lemma 4.1, namely

$$\mathcal{L}_0^*\varphi = 0, \quad \mathcal{L}_0 h = \chi_\omega\varphi + g_1, \quad h(0) = g_2, \quad h(T) = g_3. \tag{4.11}$$

To prove that $h, \varphi \in C([0,T], H^s_x)$, it is convenient to use the transformations of Section 2, to prove higher regularity for the solution $\tilde h, \tilde\varphi$ of the transformed control problem, and then to go back to h, φ proving their higher regularity. Recall that

$$\mathcal{L}_0 = \mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S}\mathcal{L}_5\mathcal{S}^{-1}\mathcal{T}^{-1}\mathcal{M}^{-1}\mathcal{B}^{-1}\mathcal{A}^{-1}, \tag{4.12}$$

where $\mathcal{L}_5 = \partial_t + m\partial_{xxx} + \mathcal{R}$ and $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ are defined in Section 2. In particular,

- $\mathcal{A}$ is the change of the space variable $(\mathcal{A}h)(t,x) = h(t, x + \beta(t,x))$ (see (2.9)), where β is defined in (2.18), (2.16), (2.17);
- $\mathcal{B}$ is the reparametrization of time $(\mathcal{B}h)(t,x) = h(\psi(t), x)$ (see (2.25)), where ψ is defined in (2.30);
- $\rho(t)$ is the function defined in (2.26);
- $\mathcal{M}$ is the multiplication operator $(\mathcal{M}h)(t,x) = q(t,x)h(t,x)$ (see (2.37)), where q is defined in (2.41);
- $\mathcal{T}$ is the translation of the space variable $(\mathcal{T}h)(t,x) = h(t, x + p(t))$ (see (2.46)), where p is defined in (2.49);
- $\mathcal{S}$ is the pseudo-differential operator $(\mathcal{S}h)(t,x) = h(t,x) + \gamma(t,x)\partial_x^{-1}h(t,x)$ (see (2.53)), where γ is defined in (2.56) and $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier);
- $\mathcal{R}$ is the bounded operator defined in (2.57).

Let

$$\mathcal{L}_5^* := -\partial_t - m\partial_{xxx} + \mathcal{R}^T, \tag{4.13}$$

where $\mathcal{R}^T$ is the L^2_x-adjoint of $\mathcal{R}$. Let

$$\begin{aligned}
&\tilde h := (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1}h, && \tilde g_1 := (\mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S})^{-1}g_1,\\
&\tilde g_2 := (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1}|_{t=0}\, g_2, && \tilde g_3 := (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1}|_{t=T}\, g_3,\\
&\tilde\varphi := \mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T\varphi, && K\tilde\varphi := (\mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S})^{-1}(\chi_\omega(\mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T)^{-1}\tilde\varphi).
\end{aligned} \tag{4.14}$$

Note that, except for $\mathcal{S}^{-1}, \mathcal{S}^{-T}$, the operator K is a multiplication operator, namely

$$K\tilde\varphi = \mathcal{S}^{-1}(\zeta\mathcal{S}^{-T}\tilde\varphi), \quad \text{where} \quad \zeta(t,x) := \rho^{-1}\mathcal{T}^{-1}\mathcal{M}^{-2}\mathcal{B}^{-1}\mathcal{A}^{-1}[(1+\beta_x)\chi_\omega]. \tag{4.15}$$

Since $h, \varphi \in C([0,T], L^2_x)$, and $g_1 \in C([0,T], H^s_x)$, $g_2, g_3 \in H^s_x$, by (4.14) and the estimates for $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in Section 2, one has

$$\tilde h, \tilde\varphi, K\tilde\varphi \in C([0,T], L^2_x), \quad \tilde g_1 \in C([0,T], H^s_x), \quad \tilde g_2, \tilde g_3 \in H^s_x.$$

Since h, φ satisfy (4.11), one proves that $\tilde h, \tilde\varphi$ satisfy

$$\mathcal{L}_5^*\tilde\varphi = 0, \quad \mathcal{L}_5\tilde h = K\tilde\varphi + \tilde g_1, \quad \tilde h(0) = \tilde g_2, \quad \tilde h(T) = \tilde g_3. \tag{4.16}$$

The last three equations in (4.16) are straightforward. To prove that $\mathcal{L}_5^*\tilde\varphi = 0$, we start from the equality

$$\langle\varphi(T), v(T)\rangle - \langle\varphi(0), v(0)\rangle = \int_0^T \langle\varphi, \mathcal{L}_0 v\rangle dt \quad \forall v \in C^\infty([0,T]\times\mathbb{T})$$

(which is a weak form of $\mathcal{L}_0^*\varphi = 0$), we recall (4.12), and apply all the changes of variables $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in the integral. Thus $\tilde{h}, \tilde{\varphi}$ solve this control problem:

$$\begin{cases} \text{Given } \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \text{ find } \tilde{\varphi} \text{ such that the solution } \tilde{h} \\ \text{of the Cauchy problem } \mathcal{L}_5\tilde{h} = K\tilde{\varphi} + \tilde{g}_1,\ \tilde{h}(0) = \tilde{g}_2 \\ \text{satisfies } \tilde{h}(T) = \tilde{g}_3, \text{ and moreover } \tilde{\varphi} \text{ solves } \mathcal{L}_5^*\tilde{\varphi} = 0. \end{cases} \tag{4.17}$$

The function $\tilde{\varphi}$ is the unique solution of (4.17). To prove it, assume that $\tilde{\varphi}_{bis} \in C([0,T], L^2_x)$ solves (4.17), and let $\tilde{h}_{bis}$ be the solution of the corresponding Cauchy problem $\mathcal{L}_5\tilde{h}_{bis} = K\tilde{\varphi}_{bis} + \tilde{g}_1$, $\tilde{h}_{bis}(0) = \tilde{g}_2$. Define

$$h_{bis} := \mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S}\tilde{h}_{bis}, \quad \varphi_{bis} := \mathcal{A}^{-T}\mathcal{B}\mathcal{M}^{-T}\mathcal{T}^{-T}\mathcal{S}^{-T}\tilde{\varphi}_{bis}.$$

Then h_{bis}, φ_{bis} solve (4.11). By the uniqueness in Lemma 4.1(ii) it follows that $\varphi_{bis} = \varphi$, $h_{bis} = h$. Therefore $\tilde{\varphi}_{bis} = \tilde{\varphi}$ and $\tilde{h}_{bis} = \tilde{h}$.

Now we prove that $\tilde{h}, \tilde{\varphi} \in C([0,T], H^s_x)$. We follow an argument used by Dehman-Lebeau [20, Lemma 4.2], Laurent [34, Lemma 3.1], and [3, Proposition 8.1]. First, we prove the thesis for $\tilde{g}_1 = 0$, $\tilde{g}_3 = 0$. Consider the map

$$S\colon L^2_x \to L^2_x, \quad S\tilde{\varphi}_1 = \tilde{h}(0) \tag{4.18}$$

obtained by the composition $\tilde{\varphi}_1 \mapsto \tilde{\varphi} \mapsto \tilde{h} \mapsto \tilde{h}(0)$, where $\tilde{\varphi}, \tilde{h}$ are the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_5^*\tilde{\varphi} = 0 \\ \tilde{\varphi}(T) = \tilde{\varphi}_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5\tilde{h} = K\tilde{\varphi} \\ \tilde{h}(T) = 0. \end{cases} \tag{4.19}$$

From the existence and uniqueness of $\tilde{\varphi}_1 \in L^2_x$ such that $\tilde{\varphi}$ solves (4.17) it follows that S is an isomorphism of L^2_x. The initial datum $\tilde{g}_2$ is given, so we fix $\tilde{\varphi}_1 \in L^2_x$ such that $S\tilde{\varphi}_1 = \tilde{g}_2$. We have to estimate $\|\Lambda^s\tilde{\varphi}_1\|_0 \leq C\|S\Lambda^s\tilde{\varphi}_1\|_0$, where Λ^s is the Fourier multiplier of symbol $\langle\xi\rangle^s := (1+\xi^2)^{s/2}$, $s > 0$. To study the commutator $[S, \Lambda^s]$, we compare $(\Lambda^s\tilde{\varphi}, \Lambda^s\tilde{h})$ with $(\bar{\varphi}, \bar{h})$ defined by

$$\begin{cases} \mathcal{L}_5^*\bar{\varphi} = 0 \\ \bar{\varphi}(T) = \Lambda^s\varphi_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5\bar{h} = K\bar{\varphi} \\ \bar{h}(T) = 0. \end{cases} \tag{4.20}$$

The difference $\Lambda^s\tilde{\varphi} - \bar{\varphi}$ satisfies

$$\begin{cases} \mathcal{L}_5^*(\Lambda^s\tilde{\varphi} - \bar{\varphi}) = \mathcal{F}_1, \\ (\Lambda^s\tilde{\varphi} - \bar{\varphi})(T) = 0 \end{cases} \quad \text{where} \quad \mathcal{F}_1 := [\mathcal{L}_5^*, \Lambda^s]\tilde{\varphi} = [\mathcal{R}^T, \Lambda^s]\tilde{\varphi}. \tag{4.21}$$

From Lemma 6.2 and Remark 6.8, $\|\Lambda^s\tilde{\varphi} - \bar{\varphi}\|_{T,0} \leq C\|\mathcal{F}_1\|_{T,0}$. We recall the classical estimate for the commutator of Λ^s and any multiplication operator $h \mapsto ah$:

$$\|[\Lambda^s, a]h\|_0 \leq C_s(\|a\|_2\|h\|_{s-1} + \|a\|_{s+1}\|h\|_0). \tag{4.22}$$

By (4.22) and formulas (2.53), (2.56), (2.57), the commutator $\mathcal{F}_1 = [\mathcal{R}^T, \Lambda^s]\tilde{\varphi}$ satisfies

$$\begin{aligned} \|\mathcal{F}_1\|_{T,0} &\leq C_s(\|a_{14}, a_{17}, a_{18}\|_{T,\sigma}\|\tilde{\varphi}\|_{T,s-1} + \|a_{14}, a_{17}, a_{18}\|_{T,s+\sigma}\|\tilde{\varphi}\|_{T,0}) \\ &\leq C_s(\delta(0)\|\tilde{\varphi}\|_{T,s-1} + \delta(s)\|\tilde{\varphi}\|_{T,0}). \end{aligned} \tag{4.23}$$

The difference $\Lambda^s\tilde{h} - \bar{h}$ satisfies

$$\begin{cases} \mathcal{L}_5(\Lambda^s\tilde{h} - \bar{h}) = K(\Lambda^s\tilde{\varphi} - \bar{\varphi}) + \mathcal{F}_2, \\ (\Lambda^s\tilde{h} - \bar{h})(T) = 0, \end{cases} \quad \text{where} \quad \mathcal{F}_2 := [\mathcal{R}^T, \Lambda^s]\tilde{h} + [\Lambda^s, K]\tilde{\varphi}. \tag{4.24}$$

We have $\|K(\Lambda^s\tilde{\varphi} - \bar{\varphi})\|_{T,0} \leq C\|\Lambda^s\tilde{\varphi} - \bar{\varphi}\|_{T,0} \leq C\|\mathcal{F}_1\|_{T,0}$, and therefore, by Lemma 6.2,

$$\|\Lambda^s\tilde{h} - \bar{h}\|_{T,0} \leq C(\|\mathcal{F}_1\|_{T,0} + \|\mathcal{F}_2\|_{T,0}). \tag{4.25}$$

Using (4.22) and (4.15), we get

$$\|\mathcal{F}_2\|_{T,0} \leq C_s(\|\tilde{h}, \tilde{\varphi}\|_{T,s-1} + \delta(s)\|\tilde{h}, \tilde{\varphi}\|_{T,0}). \tag{4.26}$$

By (4.23), (4.25) and (4.26) we deduce that

$$\|\Lambda^s\tilde{h} - \bar{h}\|_{T,0} \leq C_s(\|\tilde{h}, \tilde{\varphi}\|_{T,s-1} + \delta(s)\|\tilde{h}, \tilde{\varphi}\|_{T,0}).$$

By (4.19), Lemma 6.2 and Remark 6.8,

$$\|\tilde{h}, \tilde{\varphi}\|_{T,\mu} \leq C_\mu\big(\|\tilde{\varphi}\|_{T,\mu} + \delta(\mu)\|\tilde{\varphi}\|_{T,0}\big) \leq C_\mu\big(\|\tilde{\varphi}_1\|_\mu + \delta(\mu)\|\tilde{\varphi}_1\|_0\big), \quad \mu \geq 0. \tag{4.27}$$

Therefore

$$\|(\Lambda^s\tilde{h} - \bar{h})(0)\|_0 \leq \|\Lambda^s\tilde{h} - \bar{h}\|_{T,0} \leq C_s(\|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0). \tag{4.28}$$

Since $S\tilde{\varphi}_1 = \tilde{h}(0) = \tilde{g}_2$, we have $\Lambda^s\tilde{h}(0) = \Lambda^s g_2$. Moreover, by the definition of S in (4.18)-(4.19), $\bar{h}(0) = S\Lambda^s\tilde{\varphi}_1$. Thus

$$\|S\Lambda^s\tilde{\varphi}_1\|_0 \leq \|(\Lambda^s\tilde{h} - \bar{h})(0)\|_0 + \|\Lambda^s\tilde{h}(0)\|_0 \leq C_s(\|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0) + \|\tilde{g}_2\|_s. \tag{4.29}$$

Since S is an isomorphism of L^2_x, $\|\Lambda^s\tilde{\varphi}_1\|_0 \leq C\|S\Lambda^s\tilde{\varphi}_1\|_0$, whence

$$\|\tilde{\varphi}_1\|_s \leq C_s(\|\tilde{g}_2\|_s + \|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0). \tag{4.30}$$

Since $\|\tilde{\varphi}_1\|_0 \leq C\|\tilde{g}_2\|_0$, by induction we deduce that

$$\|\tilde{\varphi}_1\|_s \leq C_s(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0). \tag{4.31}$$

By (4.27), we obtain

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \leq C_s(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0), \tag{4.32}$$

which is the thesis in the case $\tilde{g}_1 = 0$, $\tilde{g}_3 = 0$.

Now we prove the higher regularity of $\tilde{h}, \tilde{\varphi}$ removing the assumption $\tilde{g}_1 = 0$, $\tilde{g}_3 = 0$. Let $\tilde{g}_1 \in C([0,T], H^s_x)$, $\tilde{g}_2, \tilde{g}_3 \in H^s_x$, and let $\tilde{h}, \tilde{\varphi}$ be the solution of (4.17). Let w be the solution of the problem

$$\mathcal{L}_5 w = \tilde{g}_1, \quad w(T) = \tilde{g}_3.$$

By Lemma 6.2, $w \in C([0,T], H^s_x)$, with

$$\|w\|_{T,s} \leq C_s\{\|\tilde{g}_1\|_{T,s} + \|\tilde{g}_3\|_s + \delta(s)(\|\tilde{g}_1\|_{T,0} + \|\tilde{g}_3\|_0)\}. \tag{4.33}$$

Let $v := \tilde{h} - w$. Then

$$\mathcal{L}_5 v = K\tilde{\varphi}, \quad v(0) = \tilde{g}_2 - w(0), \quad v(T) = 0.$$

This means that $v, \tilde\varphi$ solve (4.17) where $(\tilde g_1, \tilde g_2, \tilde g_3)$ are replaced by $(0, \tilde g_2 - w(0), 0)$. Hence (4.32) applies to $v, \tilde\varphi$, and we get

$$\|v, \tilde\varphi\|_{T,s} \le C_s(\|\tilde g_2 - w(0)\|_s + \delta(s)\|\tilde g_2 - w(0)\|_0). \tag{4.34}$$

We estimate $\|\tilde g_2 - w(0)\|_s \le \|\tilde g_2\|_s + \|w\|_{T,s}$, we use (4.33) and $\|\tilde h\|_{T,s} \le \|v\|_{T,s} + \|w\|_{T,s}$ to conclude that

$$\|\tilde h, \tilde\varphi\|_{T,s} \le C_s\{\|\tilde g\|_{T,s} + \delta(s)\|\tilde g\|_{T,0}\} \tag{4.35}$$

where we have denoted, in short, $\|\tilde g\|_{T,s} := \|\tilde g_1\|_{T,s} + \|\tilde g_2\|_s + \|\tilde g_3\|_s$. This proves the higher regularity for the transformed control problem (4.17). By the definitions in (4.14),

$$\|\varphi\|_{T,s} \le C_s(\|\tilde\varphi\|_{T,s} + \delta(s)\|\tilde\varphi\|_{T,0}), \qquad \|h\|_{T,s} \le C_s(\|\tilde h\|_{T,s} + \delta(s)\|\tilde h\|_{T,0}),$$
$$\|\tilde g\|_{T,s} \le C_s(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}),$$

and the proof of (4.9) is complete.

The bound (4.10) is deduced in a classical way from the fact that h, φ solve the equations $\mathcal{L}_0^*\varphi = 0$, $\mathcal{L}_0 h = \chi_\omega \varphi + g_1$. □

Remark 4.4. Another possible way to prove higher regularity for h, φ is to apply the argument of [20, 34, 3] directly to the control problem for $\mathcal{L}_0$, instead of passing to the transformed problem (4.17), applying that argument, and then going back to h, φ. Such a more direct method adapted to the present case would require the construction of two operators A_s, B_s such that

(i) $C_1\|v\|_s \le \|A_s v\|_0 \le C_2\|v\|_s$ (equivalent norm in H^s),

(ii) the commutator $[\mathcal{L}_0, A_s]$ is an operator of order $s-1$,

(iii) the difference $B_s\mathcal{L}_0^* - \mathcal{L}_0^* A_s$ is also of order $s-1$.

The construction of such A_s, B_s is possible, but probably the proof given above is more straighforward, and it fully exploits the advantages of conjugating $\mathcal{L}_0$ to $\mathcal{L}_5$ (Section 2). The main point is that the commutator $[\mathcal{L}_5, \Lambda^s]$ is of order $s-1$ (because $\mathcal{L}_5$ has constant coefficients up to a *bounded* remainder), while $[\mathcal{L}_0, \Lambda^s]$ is of order $s+2$ (because $\mathcal{L}_0$, which was obtained by linearizing a *quasi-linear* PDE, has variable coefficients also at the highest order), so that a modified version A_s of Λ^s is needed. □

In view of the application of Nash-Moser theorem in section 5, we define the spaces

$$E_s := X_s \times X_s, \quad X_s := C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^{s+3}) \cap C^2([0,T], H_x^s) \tag{4.36}$$

and

$$F_s := \{g = (g_1, g_2, g_3) : g_1 \in C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^s), g_2, g_3 \in H_x^{s+6}\} \tag{4.37}$$

equipped with the norms

$$\|u, f\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s} \tag{4.38}$$

and

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2, g_3\|_{s+6}. \tag{4.39}$$

With this notation, we have proved the following linear inversion result.

Theorem 4.5 (Right inverse of the linearized operator)**.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. There exist two universal constants $\tau, \sigma \geq 3$ and a positive constant δ_* depending on T, ω with the following property.*

Let $s \in [0, r - \tau]$, where r is the regularity of the nonlinearity $\mathcal{N}$ (see Lemma 2.1). Let $g = (g_1, g_2, g_3) \in F_s$, and let $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \leq \delta_*$. Then there exists $(h, \varphi) := \Psi(u, f)[g] \in E_s$ such that

$$P'(u)[h] - \chi_\omega \varphi = g_1, \quad h(0) = g_2, \quad h(T) = g_3, \tag{4.40}$$

and

$$\|h, \varphi\|_{E_s} \leq C_s \big(\|g\|_{F_s} + \|u\|_{X_{s+\sigma}} \|g\|_{F_0} \big) \tag{4.41}$$

where C_s depends on s, T, ω.

5 Proofs

In this section we prove Theorems 1.1 and 1.4.

5.1 Proof of Theorem 1.1

The spaces defined in (4.36)-(4.39), with $s \geq 0$, form scales of Banach spaces. We define smoothing operators S_θ in the following way. We fix a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ with $0 \leq \varphi \leq 1$,

$$\varphi(\xi) = 1 \quad \forall |\xi| \leq 1 \qquad \text{and} \qquad \varphi(\xi) = 0 \quad \forall |\xi| \geq 2.$$

For any real number $\theta \geq 1$, let S_θ be the Fourier multiplier with symbol $\varphi(\xi/\theta)$, namely

$$S_\theta u(x) := \sum_{k \in \mathbb{Z}} \hat{u}_k \, \varphi(k/\theta) \, e^{ikx} \qquad \text{where} \quad u(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{ikx} \in L^2(\mathbb{T}). \tag{5.1}$$

The definition of S_θ extends to functions $u(t, x) = \sum_{k \in \mathbb{Z}} \hat{u}_k(t) \, e^{ikx}$ depending on time in the obvious way. Since S_θ and ∂_t commute, the smoothing operators S_θ are defined on the spaces E_s, F_s defined in (4.36)-(4.37) by setting $S_\theta(u, f) := (S_\theta u, S_\theta f)$ and similarly on $g = (g_1, g_2, g_3)$. One easily verifies that S_θ satisfies (7.1)-(7.4) on E_s and F_s. We define the spaces E'_a with norm $\| \, \|'_a$ and F'_b with $\| \, \|'_b$ as constructed in section 7.

We observe that $\Phi(u, f) := (P(u) - \chi_\omega f, \, u(0), \, u(T))$ defined in (1.13)-(1.14) belongs to F_s when $(u, f) \in E_{s+3}$, $s \in [0, r - 6]$, with $\|u\|_{T,4} \leq 1$. Its second derivative is

$$\Phi''(u, f)[(h_1, \varphi_1), (h_2, \varphi_2)] = \begin{pmatrix} P''(u)[h_1, h_2] \\ 0 \\ 0 \end{pmatrix}.$$

For u in a fixed ball $\|u\|_{X_1} \leq \delta_0$, with δ_0 small enough, we estimate

$$\|P''(u)[h, w]\|_{F_s} \leq C_s \big(\|h\|_{X_1} \|w\|_{X_{s+3}} + \|h\|_{X_{s+3}} \|w\|_{X_1} + \|u\|_{X_{s+3}} \|h\|_{X_1} \|w\|_{X_1} \big) \tag{5.2}$$

for all $s \in [0, r - 6]$. We fix $V = \{(u, f) \in E_3 : \|(u, f)\|_{E_3} \leq \delta_0\}$, $\delta_1 = \delta_*$,

$$a_0 = 1, \quad \mu = 3, \quad a_1 = \sigma, \quad \alpha = \beta = 2\sigma, \quad a_2 \in (3\sigma, r - \tau] \tag{5.3}$$

where δ_*, σ, τ are given by Theorem 4.5, and r is the regularity of $\mathcal{N}$ in Theorem 1.1. The right inverse Ψ in Theorem 4.5 satisfies the assumptions of Theorem 7.1. Thus by

Theorem 7.1 we obtain that, if $g = (0, u_{in}, u_{end}) \in F'_\beta$ with $\|g\|'_{F_\beta} \leq \delta$, then there exists a solution $(u, f) \in E'_\alpha$ of the equation $\Phi(u, f) = g$, with $\|u, f\|'_{E_\alpha} \leq C\|g\|'_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 6$, and (1.11) is proved. In fact, we have proved slightly more than (1.11), because $\|g\|'_{F_\beta} \leq C\|g\|_{F_\beta}$ and $\|u, f\|_{E_a} \leq C_a\|u, f\|'_{E_\alpha}$ for all $a < \alpha$.

We have found a solution (u, f) of the control problem (1.9)-(1.10). Now we prove that u is the unique solution of the Cauchy problem (1.9), with that given f. Let u, v be two solutions of (1.9) in E_{s-6} for all $s < s_1$. We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u - v))[u - v]\, d\lambda =: \widetilde{\mathcal{L}}_0[u - v]$$

where

$$\widetilde{\mathcal{L}}_0 := \partial_t + (1 + \tilde{a}_3(t, x))\partial_{xxx} + \tilde{a}_2(t, x)\partial_{xx} + \tilde{a}_1(t, x)\partial_x + \tilde{a}_0(t, x),$$
$$\tilde{a}_i(t, x) := \int_0^1 a_i(v + \lambda(u - v))(t, x)\, d\lambda, \quad i = 0, 1, 2, 3,$$

and $a_i(u)$ is defined in (2.2). Note that $\tilde{a}_2 = 2\partial_x \tilde{a}_3$ because $a_2(v + \lambda(u - v)) = 2\partial_x a_3(v + \lambda(u - v))$ for all $\lambda \in [0, 1]$. The difference $u - v$ satisfies $\widetilde{\mathcal{L}}_0(u - v) = 0$, $(u - v)(0) = 0$. Hence, by Lemma 6.7, $u - v = 0$. The proof of Theorem 1.1 is complete. □

5.2 Proof of Theorem 1.4

We define

$$E_s := C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^{s+3}) \cap C^2([0, T], H_x^s), \tag{5.4}$$
$$F_s := \{g = (g_1, g_2) : g_1 \in C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^s), g_2 \in H_x^{s+6}\} \tag{5.5}$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s} \tag{5.6}$$
$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2\|_{s+6}, \tag{5.7}$$

and $\Phi(u) := (P(u), u(0))$. Given $g := (f, u_{in}) \in F_{s_0}$, the Cauchy problem (1.18) writes $\Phi(u) = g$. We fix $V, \delta_1, a_0, \mu, a_1, \alpha, \beta, a_2$ like in (5.3), where the constants σ, δ_* are now given in Lemma 6.7 and $\tau = \sigma + 9$ by Lemma 2.1 combined with Lemma 6.7 and the definition of the spaces E_s, F_s. Assumption (7.13) about the right inverse of the linearized operator is satisfied by Lemmas 6.7 and 2.1. We fix $s_0 := \alpha + 6$. Then Theorem 7.1 applies, giving the existence part of Theorem 1.4. The uniqueness of the solution is proved exactly as in the proof of Theorem 1.1. □

6 Appendix A. Well-posedness of linear operators

Lemma 6.1. *Let $T > 0$, $m \in \mathbb{R}$, $s \in \mathbb{R}$, $f \in C([0, T], H_x^s)$, with $f(t, x) = \sum_{n \in \mathbb{Z}} f_n(t) e^{inx}$. Let A be the linear operator defined by $Af := v$ where v is the solution of*

$$\begin{cases} \partial_t v + m\partial_{xxx} v = f & \forall (t, x) \in [0, T] \times \mathbb{T}, \\ v(0, x) = 0. \end{cases} \tag{6.1}$$

Then

$$Af(t,x) = \sum_{n\in\mathbb{Z}} (Af)_n(t) e^{inx}, \quad (Af)_n(t) = \int_0^t e^{imn^3(\tau - t)} f_n(\tau)\, d\tau, \tag{6.2}$$

Af belongs to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x)$, and

$$\|Af\|_{T,s} \leq T \|f\|_{T,s}\,. \tag{6.3}$$

Proof. Formula (6.2) simply comes from variation of constants. By Hölder's inequality,

$$|(Af)_n(t)| \leq \sqrt{t} \Big(\int_0^t |f_n(\tau)|^2\, d\tau \Big)^{\frac{1}{2}} \quad \forall t \in [0,T]$$

and therefore, for each $t \in [0,T]$,

$$\begin{aligned} \|Af(t)\|^2_{H^s_x} &= \sum_{n\in\mathbb{Z}} |(Af)_n(t)|^2 \langle n \rangle^{2s} \leq \sum_{n\in\mathbb{Z}} t \int_0^t |f_n(\tau)|^2\, d\tau \langle n \rangle^{2s} \\ &\leq t \int_0^t \sum_{n\in\mathbb{Z}} |f_n(\tau)|^2 \langle n \rangle^{2s}\, d\tau = t \int_0^t \|f(\tau)\|^2_{H^s_x}\, d\tau \leq t^2 \|f\|^2_{C([0,t],H^s_x)}. \end{aligned}$$

Taking the sup over $t \in [0,T]$ we get the thesis. ☐

We remark that for $s \leq 3$ the operator A is well-defined in the sense of distributions. We also recall that $\mathcal{L}(H^s_x)$ is the space of linear bounded operators of H^s_x into itself, with operator norm $\|L\|_{\mathcal{L}(H^s_x)} := \sup\{\|Lh\|_s : h \in H^s_x, \|h\|_s = 1\}$.

Lemma 6.2. *(i)* (LWP). *Let $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x))$, and let*

$$r_s := \|\mathcal{R}\|_{C([0,T],\mathcal{L}(H^s_x))} = \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{\mathcal{L}(H^s_x)}, \qquad \mathcal{L}_5 := \partial_t + m\partial_{xxx} + \mathcal{R}\ . \tag{6.4}$$

Let $\alpha \in H^s_x$ and $f \in C([0,T], H^s_x)$. If $T\, r_s \leq 1/2$, then the Cauchy problem

$$\begin{cases} \mathcal{L}_5 u = f \\ u(0,x) = \alpha(x) \end{cases} \tag{6.5}$$

has a unique solution $u \in C([0,T], H^s_x)$. The solution u satisfies

$$\|u\|_{T,s} \leq (1 + 2Tr_s)\|\alpha\|_s + 2T\|f\|_{T,s} \leq 2(\|\alpha\|_s + T\|f\|_{T,s}). \tag{6.6}$$

(ii) (Tame LWP). *Let $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ with $s \geq s_1$, and let $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x)) \cap C([0,T], \mathcal{L}(H^{s_1}_x))$. Assume that*

$$\|\mathcal{R}(t)h\|_s \leq c_1\|h\|_s + c_s\|h\|_{s_1}\,, \quad \|\mathcal{R}(t)h\|_{s_1} \leq c_1\|h\|_{s_1} \quad \forall h \in H^s_x\,, \tag{6.7}$$

for all $t \in [0,T]$, where c_1, c_s are positive constants. Let $\alpha \in H^s_x$. If

$$Tc_1 \leq 1/2, \tag{6.8}$$

then the solution $u \in C([0,T], H^{s_1}_x)$ of the Cauchy problem (6.5) given in (i) belongs to $C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \leq 2T\|f\|_{T,s} + (1 + 2Tc_1)\|\alpha\|_s + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{6.9}$$

(iii) (GWP). *Let* $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x))$, *and let* r_s *be defined in* (6.4). *Let* $\alpha \in H^s_x$. *Then the Cauchy problem* (6.5) *has a unique global solution* $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \leq 2^{4Tr_s}(\|\alpha\|_s + 4T\|f\|_{T,s}). \tag{6.10}$$

(iv) (Tame GWP). *Let* $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *with* $s \geq s_1$, *and let* $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x)) \cap C([0,T], \mathcal{L}(H^{s_1}_x))$. *Assume that* (6.7) *holds for all* $t \in [0,T]$, *where* c_1, c_s *are positive constants. Let* $\alpha \in H^s_x$. *Then the global solution* $u \in C([0,T], H^s_x)$ *of the Cauchy problem* (6.5) *given in* (iii) *satisfies*

$$\|u\|_{T,s} \leq 2^{4Tc_1}(\|\alpha\|_s + 4Tc_s\|\alpha\|_{s_1} + 2T\|f\|_{T,s} + 4T^2 c_s\|f\|_{T,s_1}). \tag{6.11}$$

Proof. (i) Write $u = v + w$, where $v(t,x)$ is the solution of

$$\partial_t v + m\partial_{xxx} v = 0, \quad v(0,x) = \alpha(x). \tag{6.12}$$

Hence u solves (6.5) if and only if $w(t,x)$ solves

$$\partial_t w + m\partial_{xxx} w + \mathcal{R}w = -\mathcal{R}v + f, \quad w(0,x) = 0. \tag{6.13}$$

By Lemma 6.1, (6.13) is the fixed point problem

$$w = \Psi(w), \tag{6.14}$$

where $\Psi(w) := A[f - \mathcal{R}(v+w)]$. Let $B_\rho := \{w \in C([0,T], H^s_x) : \|u\|_{T,s} \leq \rho\}$, $\rho \geq 0$. Then

$$\|\Psi(w)\|_{T,s} \leq T\,(\|f\|_{T,s} + r_s\|\alpha\|_s + r_s\rho), \qquad \|\Psi(w_1) - \Psi(w_2)\|_{T,s} \leq T\,r_s\|w_1 - w_2\|_{T,s} \tag{6.15}$$

for all $w, w_1, w_2 \in B_\rho$. By assumption, $T\,r_s \leq 1/2$. Therefore, for any $\rho \geq 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$, Ψ is a contraction in B_ρ. In particular, we fix $\rho = \rho_0 := 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$. Hence there exists a fixed point $w \in B_{\rho_0}$ of Ψ, with $\|w\|_{T,s} \leq \rho_0 \leq 2T\|f\|_{T,s} + \|\alpha\|_s$. As a consequence, there exists a solution $u \in C([0,T], H^s_x)$ of (6.5) with $\|u\|_{T,s} \leq 2(T\|f\|_{T,s} + \|\alpha\|_s)$. By the contraction lemma, the solution u is unique in any ball B_ρ, $\rho \geq \rho_0$, and therefore it is unique in $C([0,T], H^s_x)$.

(ii) By assumption, $Tc_1 \leq 1/2$, and therefore, by (i), there exists a unique solution $u \in C([0,T], H^{s_1}_x)$. It remains to prove that u satisfies (6.9). By construction, $u = v + w$, where $v \in C([0,T], H^s_x)$ is the solution of (6.12), with $\|v(t)\|_s = \|\alpha\|_s$ for all $t \in [0,T]$, and $w \in C([0,T], H^{s_1}_x)$ solves (6.14). By the iterative scheme of the contraction lemma, w is the limit in $C([0,T], H^{s_1}_x)$ of the sequence (w_n), where $w_0 := 0$, and $w_{n+1} := \Psi(w_n)$ for all $n \in \mathbb{N}$. By (6.7) and (6.3), Ψ maps $C([0,T], H^s_x)$ into itself, therefore $w_n \in C([0,T], H^s_x)$ for all $n \geq 0$. Let $h_n := w_n - w_{n-1}$, $n \geq 1$, so that $w_n = \sum_{k=1}^n h_k$. One has $h_{n+1} = -A\mathcal{R}h_n$ for all $n \geq 1$, and

$$\|h_{n+1}\|_{T,s} \leq Tc_1\|h_n\|_{T,s} + Tc_s\|h_n\|_{T,s_1}, \quad \|h_{n+1}\|_{T,s_1} \leq Tc_1\|h_n\|_{T,s_1}, \quad \forall n \geq 1.$$

Hence, by induction, for all $n \geq 1$ we have

$$\begin{aligned} &\|h_n\|_{T,s} \leq (Tc_1)^{n-1}\|h_1\|_{T,s} + (n-1)(Tc_1)^{n-2}Tc_s\|h_1\|_{T,s_1}, \\ &\|h_n\|_{T,s_1} \leq (Tc_1)^{n-1}\|h_1\|_{T,s_1}. \end{aligned} \tag{6.16}$$

Also, $\|h_1\|_{T,s} \leq T\|f\|_{T,s} + Tc_1\|\alpha\|_s + Tc_s\|\alpha\|_{s_1}$ and $\|h_1\|_{T,s_1} \leq T\|f\|_{T,s_1} + Tc_1\|\alpha\|_{s_1}$. Therefore

$$\begin{aligned}
\|h_n\|_{T,s} &\leq (Tc_1)^{n-1}T\|f\|_{T,s} + (Tc_1)^n\|\alpha\|_s + (n-1)(Tc_1)^{n-2}Tc_sT\|f\|_{T,s_1} \\
&\quad + n(Tc_1)^{n-1}Tc_s\|\alpha\|_{s_1}, \\
\|h_n\|_{T,s_1} &\leq (Tc_1)^{n-1}T\|f\|_{T,s_1} + (Tc_1)^n\|\alpha\|_{s_1} \qquad \forall n \geq 1.
\end{aligned} \tag{6.17}$$

Since $Tc_1 \leq 1/2$, the sequence $w_n = \sum_{k=1}^n h_k$ converges in $C([0,T], H^s_x)$ to some limit $\tilde{w} \in C([0,T], H^s_x)$. Since w_n converges to w in $C([0,T], H^{s_1}_x)$, the two limits coincide, and $w \in C([0,T], H^s_x)$. Since $\|w\|_{T,s} \leq \sum_{k=1}^\infty \|h_k\|_{T,s}$, we get

$$\|w\|_{T,s} \leq 2T(\|f\|_{T,s} + c_1\|\alpha\|_s) + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{6.18}$$

Since $u = v + w$, we deduce (6.9).

(iii). If $Tr_s \leq 1/2$, the result is given by *(i)*. Let $Tr_s > 1/2$, and fix $N \in \mathbb{N}$ such that $2Tr_s \leq N \leq 4Tr_s$. Let $T_0 := T/N$, so that $1/4 \leq T_0 r_s \leq 1/2$. Divide the interval $[0,T]$ in the union $I_1 \cup \ldots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Applying *(i)* on the time interval $I_1 = [0, T_0]$ gives the solution $u_1 \in C(I_1, H^s_x)$, with $\|u_1\|_{C(I_1,H^s_x)} \leq b\|\alpha\|_s + 2T_0\|f\|_{T,s}$, where $b := 1 + 2T_0 r_s$. Now consider the Cauchy problem on I_2 with initial datum $u(T_0) = u_1(T_0)$. Applying *(i)* on I_2 gives the solution $u_2 \in C(I_2, H^s_x)$, with

$$\|u_2\|_{C(I_2,H^s_x)} \leq b\|u_1(T_0)\|_s + 2T_0\|f\|_{T,s} \leq b^2\|\alpha\|_s + (1+b)2T_0\|f\|_{T,s}.$$

We iterate the procedure N times. At the last step, we find the solution u_N defined on I_N, with $\|u_N\|_{C(I_N,H^s_x)} \leq b^N\|\alpha\|_s + (b^N - 1)\frac{1}{b-1}2T_0\|f\|_{T,s}$. We define $u(t) := u_n(t)$ for $t \in I_n$, and the thesis follows, using that $b \leq 2$.

(iv) If $Tc_1 \leq 1/2$, the result is given by *(ii)*. Let $Tc_1 > 1/2$, and fix $N \in \mathbb{N}$ such that $2Tc_1 \leq N \leq 4Tc_1$. Let $T_0 := T/N$, so that $1/4 \leq T_0 c_1 \leq 1/2$. Split $[0,T] = I_1 \cup \ldots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Perform the same procedure as above. Using (6.9), and $1 + 2T_0c_1 \leq 2$, by induction we get

$$\begin{aligned}
\|u_n\|_{C(I_n,H^s_x)} &\leq 2^n\|\alpha\|_s + (2^n - 1)2T_0\|f\|_{T,s} + n2^{n-1}4T_0c_s\|\alpha\|_{s_1} \\
&\quad + [2^n(n-1) + 1]4T_0c_sT_0\|f\|_{T,s_1}, \\
\|u_n\|_{C(I_n,H^{s_1}_x)} &\leq 2^n\|\alpha\|_{s_1} + (2^n - 1)2T_0\|f\|_{T,s_1}.
\end{aligned}$$

This implies (6.11), recalling that $T_0c_1 \leq 1/2$ and also $NT_0 = T$, $N \geq 1$. □

Lemma 6.3. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_{14}(t,x), a_{15}(t,x)$ be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$, and let $\mathcal{L}_4 := \partial_t + m\partial_{xxx} + a_{14}\partial_x + a_{15}$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0,T], H^s_x)$, $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_4 u = f, \quad u(0) = \alpha \tag{6.19}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \leq C_s \left\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\right\}. \tag{6.20}$$

Proof. Following the procedure given in Section 2.5, we define $\mathcal{S} := I + \gamma(t,x)\partial_x^{-1}$ (see (2.53)) with $\gamma(t,x) := -\frac{1}{3m}\partial_x^{-1} a_{14}(t,x)$. We have that u solves (6.19) if and only if $\widetilde{u} := \mathcal{S}^{-1}u$ satisfies

$$\mathcal{L}_5\widetilde{u} = \widetilde{f}, \quad \widetilde{u}(0) = \widetilde{\alpha}$$

where $\widetilde{f} := \mathcal{S}^{-1}f$, $\widetilde{\alpha} := \mathcal{S}^{-1}(0)\alpha$ and $\mathcal{L}_5 = \partial_t + m\partial_{xxx} + \mathcal{R}$, with $\mathcal{R} = \mathcal{S}^{-1}\{a_{15} + (a_{14}\gamma - (a_{14})_x)\pi_0 + (\mathcal{L}_4\gamma)\partial_x^{-1}\}$. Then the thesis follows by Lemmas 6.2 and 2.7. $\square$

Lemma 6.4. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_{12}(t,x), a_{13}(t,x)$ be two functions with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H_x^{s+\sigma})$, and let $\mathcal{L}_3 := \partial_t + m\partial_{xxx} + a_{12}\partial_x + a_{13}$. Let*

$$\delta(\mu) := \|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0,T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\mathcal{L}_3 u = f, \quad u(0) = \alpha \tag{6.21}$$

admits a unique solution $u \in C([0,T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \left\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\right\} . \tag{6.22}$$

Proof. Following the procedure given in Section 2.4, we define $\mathcal{T}h(t,x) := h(t, x + p(t))$ (see (2.46)) with $p(t) := -\frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} a_{12}(s,x)\, dxds$. We have that u solves (6.21) if and only if $\widetilde{u} := \mathcal{T}^{-1}u$ satisfies

$$\mathcal{L}_4\widetilde{u} = \widetilde{f}, \quad \widetilde{u}(0) = \alpha$$

(note that $\mathcal{T}(0)$ is the identity) where $\widetilde{f} := \mathcal{T}^{-1}f$, and $\mathcal{L}_4 = \partial_t + m\partial_{xxx} + a_{14}\partial_x + a_{15}$, with a_{14}, a_{15} given by formula (2.48). Then the thesis follows by Lemmas 6.3 and 2.6. $\square$

Lemma 6.5. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_8(t,x), a_9(t,x), a_{10}(t,x)$ be three functions with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_8(t,x)\, dx = 0$, and let $\mathcal{L}_2 := \partial_t + m\partial_{xxx} + a_8\partial_{xx} + a_9\partial_x + a_{10}$. Let*

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0,T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\mathcal{L}_2 u = f, \quad u(0) = \alpha \tag{6.23}$$

admits a unique solution $u \in C([0,T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \left\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\right\}. \tag{6.24}$$

Proof. Following the procedure given in Section 2.3, we define $\mathcal{M}h(t,x) := q(t,x)h(t,x)$ (see (2.37)) with $q(t,x) := \exp\{-\frac{1}{3m}(\partial_x^{-1}a_8)(t,x)\}$. We have that u solves (6.23) if and only if $\widetilde{u} := \mathcal{M}^{-1}u$ satisfies

$$\mathcal{L}_3\widetilde{u} = \widetilde{f}, \quad \widetilde{u}(0) = \widetilde{\alpha}$$

where $\widetilde{f} := \mathcal{M}^{-1}f$, $\widetilde{\alpha} := \mathcal{M}^{-1}(0)\alpha$, and $\mathcal{L}_3 = \partial_t + m\partial_{xxx} + a_{12}\partial_x + a_{13}$, with a_{12}, a_{13} given by formula (2.39). Then the thesis follows by Lemmas 6.4 and 2.5. $\square$

Lemma 6.6. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$ and let $a_4(t), a_5(t,x), a_6(t,x), a_7(t,x)$ be four functions with $a_4 \in C^1([0,T],\mathbb{R})$, $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_5(t,x)\, dx = 0$, and let $\mathcal{L}_1 := \partial_t + a_4 \partial_{xxx} + a_5 \partial_{xx} + a_6 \partial_x + a_7$. Let*

$$\delta(\mu) := \sup_{t\in[0,T]} |a_4(t) - 1| + \sup_{t\in(0,T)} |a_4'(t)| + \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{6.25}$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0,T], H^s_x)$, $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_1 u = f, \quad u(0) = \alpha \tag{6.26}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \leq C_s \left\{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \right\}. \tag{6.27}$$

Proof. Following the procedure given in Section 2.2, we define $\mathcal{B}h(t,x) := h(\psi(t), x)$ (see (2.25)) with $\psi(t) := \frac{1}{m}\int_0^t a_4(s)\, ds$, where $m := \frac{1}{T}\int_0^T a_4(t)\, dt$. We have that u solves (6.26) if and only if $\widetilde{u} := \mathcal{B}^{-1} u$ satisfies

$$\mathcal{L}_2 \widetilde{u} = \widetilde{f}, \quad \widetilde{u}(0) = \alpha$$

(note that $\mathcal{B}(0)$ is the identity) where $\widetilde{f} := \mathcal{B}^{-1} f$, and $\mathcal{L}_2 = \partial_t + m\partial_{xxx} + a_8 \partial_{xx} + a_9 \partial_x + a_{10}$, with a_8, a_9, a_{10} given by formula (2.32) (see also (2.26)). Then the thesis follows by Lemma 6.5 and 2.4. $\square$

Lemma 6.7. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$ and let $a_3(t,x), a_2(t,x), a_1(t,x), a_0(t,x)$ be four functions with a_3, $\partial_t a_3$, $\partial_{tt} a_3$, a_1, $\partial_t a_1$, $a_0 \in C([0,T], H^{s+\sigma}_x)$ and $a_2 = c\partial_x a_3$ for some $c \in \mathbb{R}$. Let*

$$\delta(\mu) := \|a_3, \partial_t a_3, \partial_{tt} a_3, a_1, \partial_t a_1, a_0\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{6.28}$$

Assume $\delta(0) \leq \delta_*$. Let $\mathcal{L}_0 := \partial_t + (1 + a_3)\partial_{xxx} + a_2 \partial_{xx} + a_1 \partial_x + a_0$. Let $f \in C([0,T], H^s_x)$, $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_0 u = f, \quad u(0) = \alpha \tag{6.29}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \leq C_s \left\{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \right\}. \tag{6.30}$$

Proof. Following the procedure given in Section 2.1, we define $(\mathcal{A}h)(t,x) := h(t, x+\beta(t,x))$ (see (2.9)) with $\beta(t,x) := (\partial_x^{-1}\rho_0)(t,x)$, where ρ_0 is defined in (2.16)-(2.17). We have that u solves (6.29) if and only if $\widetilde{u} := \mathcal{A}^{-1} u$ satisfies

$$\mathcal{L}_1 \widetilde{u} = \widetilde{f}, \quad \widetilde{u}(0) = \widetilde{\alpha}$$

where $\widetilde{f} := \mathcal{A}^{-1} f$, $\widetilde{\alpha} := \mathcal{A}^{-1}(0)\alpha$, and $\mathcal{L}_1 = \partial_t + a_4 \partial_{xxx} + a_5 \partial_{xx} + a_6 \partial_x + a_7$, with a_4 not depending on the space variable x and with a_4, a_5, a_6, a_7 given by formula (2.14). Then the thesis follows by Lemmas 6.6 and 2.3. $\square$

Remark 6.8. Consider the operators $\mathcal{L}_0, \ldots, \mathcal{L}_5$ defined in Lemmas 6.2-6.7. Define

$$\begin{aligned}
\mathcal{L}_0^* h &:= -\partial_t h - \partial_{xxx}[(1+a_3)h] + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h \\
\mathcal{L}_1^* h &:= -\partial_t h - a_4 \partial_{xxx} h + \partial_{xx}(a_5 h) - \partial_x(a_6 h) + a_7 h \\
\mathcal{L}_2^* h &:= -\partial_t h - m \partial_{xxx} h + \partial_{xx}(a_8 h) - \partial_x(a_9 h) + a_{10} h \\
\mathcal{L}_3^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{12} h) + a_{13} h \\
\mathcal{L}_4^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{14} h) + a_{15} h \\
\mathcal{L}_5^* h &:= -\partial_t h - m \partial_{xxx} h + \mathcal{R}^T h.
\end{aligned}$$

It is straightforward to check that Lemmas 6.2-6.7 also hold when the operator $\mathcal{L}_k$ ($k = 0, \ldots, 5$) is replaced by $\mathcal{L}_k^*$. The crucial observation is that for all $k = 0, \ldots, 5$ (see Remark 2.2 for the case $k = 0$) the operator $-\mathcal{L}_k^*$ has the same structure as $\mathcal{L}_k$ (one might need to worsen the constants σ since the coefficients of $-\mathcal{L}_k^*$ involve space derivatives of the coefficients of $\mathcal{L}_k$). It is also immediate to verify that the same estimates also hold for the backward Cauchy problems

$$\begin{cases} \mathcal{L}_k u = f \\ u(T) = \alpha \end{cases} \qquad \begin{cases} \mathcal{L}_k^* u = f \\ u(T) = \alpha \end{cases} \qquad k = 0, \ldots, 5. \tag{6.31}$$

7 Appendix B. Nash-Moser theorem

In this section we prove a Nash-Moser implicit function theorem that is a modified version of the theorem in Hörmander [28]. With respect to [28], here (Theorem 7.1) we assume slightly stronger hypotheses on the nonlinear operator Φ and its second derivative. These hypotheses are naturally verified in applications to PDEs. We use the iteration scheme of [27] (called *discrete Nash method* by Hörmander), which is neither the Newton scheme with smoothings used in [14], [15], [7], nor the scheme in [28] and [4]. The scheme of [27] is based on a telescoping series like in [28], but some corrections y_n (see (7.15)) are also introduced. In this way the scheme converges directly to a solution of the equation $\Phi(u) = \Phi(0) + g$, avoiding the intermediate step in [28] where Leray-Schauder theorem is applied. This makes it possible to remove two assumptions of Hörmander's theorem [28], which are the compact embeddings $F_b \hookrightarrow F_a$ in the codomain scale of Banach spaces $(F_a)_{a \geq 0}$, and the continuity of the approximate right inverse $\Psi(v)$ with respect to the approximate linearization point v. We point out that, unlike Theorem 2.2.2 of [27], our Theorem 7.1 also applies to the case of Sobolev spaces.

Let us begin with recalling the construction of "weak" spaces in [28].

Let E_a, $a \geq 0$, be a decreasing family of Banach spaces with injections $E_b \hookrightarrow E_a$ of norm ≤ 1 when $b \geq a$. Set $E_\infty = \cap_{a \geq 0} E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous. Assume that $S_\theta : E_0 \to E_\infty$ for $\theta \geq 1$ are linear operators such that, with constants C bounded when a and b are bounded,

$$\|S_\theta u\|_b \leq C \|u\|_a \qquad \text{if } \ b \leq a; \tag{7.1}$$

$$\|S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } \ a < b; \tag{7.2}$$

$$\|u - S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } \ a > b; \tag{7.3}$$

$$\Big\| \frac{d}{d\theta} S_\theta u \Big\|_b \leq C \theta^{b-a-1} \|u\|_a \,. \tag{7.4}$$

From (7.2)-(7.3) one can obtain the logarithmic convexity of the norms

$$\|u\|_{\lambda a+(1-\lambda)b} \leq C\|u\|_a^\lambda \|u\|_b^{1-\lambda} \ \text{ if } \ 0<\lambda<1. \tag{7.5}$$

Consider the sequence $\{\theta_j\}_{j\in\mathbb{N}}$, with $1=\theta_0<\theta_1<\ldots\to\infty$, such that $\frac{\theta_{j+1}}{\theta_j}$ is bounded. Set $\Delta_j := \theta_{j+1}-\theta_j$ and

$$R_0 u := \frac{S_{\theta_1}u}{\Delta_0}, \qquad R_j u := \frac{S_{\theta_{j+1}}u - S_{\theta_j}u}{\Delta_j}, \quad j\geq 1. \tag{7.6}$$

By (7.3) we deduce that, if $u\in E_b$ for some $b>a$, then

$$u = \sum_{j=0}^{\infty} \Delta_j R_j u \tag{7.7}$$

with convergence in E_a. Moreover, (7.4) implies that, for all b,

$$\|R_j u\|_b \leq C_{a,b}\theta_j^{b-a-1}\|u\|_a\,. \tag{7.8}$$

Conversely, assume that $a_1<a<a_2$, that $u_j\in E_{a_2}$ and that

$$\|u_j\|_b \leq M\theta_j^{b-a-1} \quad \text{if} \ \ b=a_1 \ \ \text{or} \ \ b=a_2. \tag{7.9}$$

By (7.5) this remains true with a constant factor on the right-hand side if $a_1<b<a_2$, so that $u=\sum \Delta_j u_j$ converges in E_b if $b<a$.

Let E'_a be the set of all sums $u=\sum\Delta_j u_j$ with u_j satisfying (7.9) and introduce the norm $\|u\|'_a$ as the infimum of M over all such decompositions. It follows that $\|\ \|'_a$ is stronger than $\|\ \|_b$ if $a>b$, while (7.7) and (7.8) show that $\|\ \|'_a$ is weaker than $\|\ \|_a$. Moreover (i) the space E'_a and, up to equivalence, its norm are independent of the choice of a_1 and a_2; (ii) E'_a is defined by (7.8) for any values of b to the left and to the right of a; (iii) E'_a does not depend on the smoothing operators; (iv) in (7.3) we can replace $\|u\|_a$ by $\|u\|'_a$, namely

$$\|u-S_\theta u\|_b \leq C'_{a,b}\theta^{b-a}\|u\|'_a \quad \text{if} \ \ a>b, \tag{7.10}$$

if we take another constant $C'_{a,b}$, which may tend to ∞ as b approaches a. All these four statements (i)-(iv) are proved in [28].

Now let us suppose that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators. Unlike [28], here we do not need to assume that the embedding $F_b\hookrightarrow F_a$ is compact for $b>a$.

Theorem 7.1. *Let $a_1,a_2,\alpha,\beta,a_0,\mu$ be real numbers with*

$$0\leq a_0\leq\mu\leq a_1, \quad a_1+\frac{\beta}{2}\leq\alpha<a_1+\beta\leq a_2, \quad 2\alpha<a_1+a_2. \tag{7.11}$$

Let V be a convex neighborhood of 0 in E_μ. Let Φ be a map from V to F_0 such that $\Phi: V\cap E_{a+\mu}\to F_a$ is of class C^2 for all $a\in[0,a_2-\mu]$, with

$$\|\Phi''(u)[v,w]\|_a \leq C\big(\|v\|_{a+\mu}\|w\|_{a_0}+\|v\|_{a_0}\|w\|_{a+\mu}+\|u\|_{a+\mu}\|v\|_{a_0}\|w\|_{a_0}\big) \tag{7.12}$$

for all $u \in V \cap E_{a+\mu}$, $v, w \in E_{a+\mu}$. Also assume that $\Phi'(v)$, for $v \in E_\infty \cap V$ belonging to some ball $\|v\|_{a_1} \le \delta_1$, has a right inverse $\Psi(v)$ mapping F_∞ to E_{a_2}, and that

$$\|\Psi(v)g\|_a \le C(\|g\|_{a+\beta-\alpha} + \|g\|_0 \|v\|_{a+\beta}) \quad \forall a \in [a_1, a_2]. \tag{7.13}$$

There exists $\delta > 0$ such that, for every $g \in F'_\beta$ in the ball $\|g\|'_\beta \le \delta$, there exists $u \in E'_\alpha$, with $\|u\|'_\alpha \le C\|g\|'_\beta$, solving $\Phi(u) = \Phi(0) + g$.

Proof. We follow the proof in [28] where possible, but we use a different iteration scheme. Let $\theta_j := j + 1$, so that $\Delta_j = 1$ for all j. Let $g \in F'_\beta$ and $g_j := R_j g$. Thus

$$g = \sum_{j=0}^{\infty} g_j, \quad \|g_j\|_b \le C_b \theta_j^{b-\beta-1} \|g\|'_\beta \quad \forall b \in [0, +\infty). \tag{7.14}$$

We claim that if $\|g\|'_\beta$ is small enough, then we can define a sequence $u_j \in V \cap E_{a_2}$ with $u_0 := 0$ by the recursion formula

$$u_{j+1} := u_j + h_j, \quad v_j := S_{\theta_j} u_j, \quad h_j := \Psi(v_j)(g_j + y_j) \quad \forall j \ge 0, \tag{7.15}$$

where $y_0 := 0$,

$$y_1 := -S_{\theta_1} e_0, \qquad y_j := -S_{\theta_j} e_{j-1} - R_{j-1} \sum_{i=0}^{j-2} e_i \quad \forall j \ge 2, \tag{7.16}$$

and $e_j := e'_j + e''_j$,

$$e'_j := \Phi(u_j + h_j) - \Phi(u_j) - \Phi'(u_j)h_j, \qquad e''_j := (\Phi'(u_j) - \Phi'(v_j))h_j. \tag{7.17}$$

We prove that for all $j \ge 0$

$$\|h_j\|_a \le K_1 \|g\|'_\beta \, \theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2], \tag{7.18}$$

$$\|v_j\|_a \le K_2 \|g\|'_\beta \, \theta_j^{a-\alpha} \quad \forall a \in [a_1 + \beta, a_2 + \beta], \tag{7.19}$$

$$\|u_j - v_j\|_a \le K_3 \|g\|'_\beta \, \theta_j^{a-\alpha} \quad \forall a \in [0, a_2]. \tag{7.20}$$

For $j = 0$, (7.19) and (7.20) are trivially satisfied, and (7.18) follows from (7.14) because $h_0 = \Psi(0)g_0$ and $\theta_0 = 1$.

Now assume that (7.18), (7.19), (7.20) hold for $j = 0, \dots, k$, for some $k \ge 0$. First we prove (7.20) for $j = k + 1$. Since $u_{k+1} = \sum_{j=0}^{k} h_j$, the definition of the norm of E'_α and (7.18) for $j = 0, \dots, k$ imply that $\|u_{k+1}\|'_\alpha \le K_1 \|g\|'_\beta$. By (7.10) one has

$$\|u_{k+1} - v_{k+1}\|_0 \le C K_1 \|g\|'_\beta \theta_{k+1}^{-\alpha} \tag{7.21}$$

where the constant C depends on α. From now until the end of this proof we denote by C any constant (possibly different from line to line) depending only on $a_1, a_2, \alpha, \beta, \mu, a_0$, which are fixed parameters. From (7.18) with $j = 0, \dots, k$ we get

$$\|u_{k+1}\|_a \le K_1 \|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2]. \tag{7.22}$$

We note that

$$\sum_{j=0}^{k} \theta_j^{p-1} \leq \frac{2}{p}\, \theta_{k+1}^{p} \quad \forall p > 0. \tag{7.23}$$

For $a = a_2$, by (7.1) one gets $\|v_{k+1}\|_{a_2} \leq C\|u_{k+1}\|_{a_2}$. Thus, using (7.23) at $p = a_2 - \alpha$,

$$\|u_{k+1} - v_{k+1}\|_{a_2} \leq C\|u_{k+1}\|_{a_2} \leq CK_1\|g\|'_\beta \theta_{k+1}^{a_2-\alpha}. \tag{7.24}$$

Using (7.5) to interpolate between (7.21) and (7.24), we get (7.20) for $j = k+1$, for all $a \in [0, a_2]$, provided that $K_3 \geq CK_1$.

To prove (7.19) for $j = k+1$, we use (7.2), (7.22) and (7.23) and we get

$$\|v_{k+1}\|_a \leq C\theta_{k+1}^{a-a_1-\beta}\|u_{k+1}\|_{a_1+\beta} \leq C\theta_{k+1}^{a-a_1-\beta} K_1\|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a_1+\beta-\alpha-1} \leq CK_1\|g\|'_\beta\, \theta_{k+1}^{a-\alpha}$$

for all $a \in [a_1+\beta, a_2+\beta]$. This gives (7.19) for $j = k+1$ provided that $K_2 \geq CK_1$.

To prove (7.18) for $j = k+1$, we begin with proving that

$$\|y_{k+1}\|_b \leq CK_1(K_1 + K_3)\|g\|'^2_\beta\, \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2 + \beta - \alpha]. \tag{7.25}$$

Since $u_j, v_j, u_j + h_j$ belong to V for all $j = 0, \ldots, k$, we use Taylor formula and (7.12) to deduce that, for $j = 0, \ldots, k$ and $a \in [0, a_2 - \mu]$,

$$\begin{aligned} \|e_j\|_a \leq C(&\|h_j\|_{a_0}\|h_j\|_{a+\mu} + \|u_j\|_{a+\mu}\|h_j\|^2_{a_0} + \|h_j\|_{a_0}\|v_j - u_j\|_{a+\mu} \\ &+ \|h_j\|_{a+\mu}\|v_j - u_j\|_{a_0} + \|u_j\|_{a+\mu}\|h_j\|_{a_0}\|v_j - u_j\|_{a_0}). \end{aligned} \tag{7.26}$$

Hence at $j = k$, using (7.2) and then (7.26), we have

$$\begin{aligned} \|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} &\leq C\theta_{k+1}^p \|e_k\|_{a_2+\beta-\alpha-p} \\ &\leq C\theta_{k+1}^p(\|h_k\|_{a_0}\|h_k\|_q + \|u_k\|_q\|h_k\|^2_{a_0} + \|h_k\|_{a_0}\|v_k - u_k\|_q \\ &\qquad + \|h_k\|_q\|v_k - u_k\|_{a_0} + \|u_k\|_q\|h_k\|_{a_0}\|v_k - u_k\|_{a_0}) \end{aligned} \tag{7.27}$$

where $p := \max\{0, \beta - \alpha + \mu\}$ and $q := a_2 + \beta - \alpha - p + \mu$. Note that $a_2 + \beta - \alpha - p \geq 0$ because $a_2 \geq \mu$. Since $q \leq a_2$, using also (7.23) we have

$$\|u_k\|_q \leq \|u_k\|_{a_2} \leq \sum_{j=0}^{k-1} \|h_j\|_{a_2} \leq K_1\|g\|'_\beta \sum_{j=0}^{k-1} \theta_j^{a_2-\alpha-1} \leq CK_1\|g\|'_\beta\, \theta_k^{a_2-\alpha}. \tag{7.28}$$

By (7.28), (7.18), (7.20), and since $a_0 \leq a_1$, the bound (7.27) implies that

$$\|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} \leq CK_1(K_1 + K_3)\|g\|'^2_\beta\, \theta_{k+1}^p(\theta_k^{a_1+q-2\alpha-1} + \theta_k^{a_2+2a_1-3\alpha-1})$$

provided that $K_1\|g\|'_\beta \leq 1$. We assume that

$$K_1\|g\|'_\beta \leq 1. \tag{7.29}$$

Both the exponents $(a_1 + q - 2\alpha - 1)$ and $(a_2 + 2a_1 - 3\alpha - 1)$ are $\leq (a_2 - \alpha - 1 - p)$ because $a_1 < \alpha$ and $a_1 + \beta + \mu \leq 2\alpha$. Thus

$$\|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} \leq CK_1(K_1 + K_3)\|g\|'^2_\beta\, \theta_{k+1}^{a_2-\alpha-1}. \tag{7.30}$$

Now we estimate $\|S_{\theta_{k+1}} e_k\|_0$. Since $a_0, \mu \le a_1$, by (7.1) and (7.26) we get

$$\|S_{\theta_{k+1}} e_k\|_0 \le C\|e_k\|_0 \le C(1 + \|u_k\|_\mu)(\|h_k\|_{a_1}^2 + \|h_k\|_{a_1} \|v_k - u_k\|_{a_1}). \tag{7.31}$$

By (7.18) and (7.29),

$$\|u_k\|_\mu \le \|u_k\|_{a_1} \le \sum_{j=0}^{k-1} \|h_j\|_{a_1} \le K_1 \|g\|'_\beta \sum_{j=0}^{\infty} \theta_j^{a_1-\alpha-1} = CK_1 \|g\|'_\beta \le C. \tag{7.32}$$

We use (7.18), (7.20) and (7.32) in (7.31), and the bound $\theta_{k+1}^{2a_1-2\alpha-1} \le \theta_{k+1}^{-\beta-1}$, to deduce that

$$\|S_{\theta_{k+1}} e_k\|_0 \le CK_1(K_1 + K_3)\|g\|'^2_\beta \, \theta_{k+1}^{-\beta-1}. \tag{7.33}$$

Using (7.5) to interpolate between (7.30) and (7.33) we obtain

$$\|S_{\theta_{k+1}} e_k\|_b \le CK_1(K_1 + K_3)\|g\|'^2_\beta \, \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2 + \beta - \alpha]. \tag{7.34}$$

Now we estimate the other terms in y_{k+1} (see (7.16)). By (7.8), (7.26), (7.18), (7.20) and (7.23),

$$\begin{aligned} \sum_{i=0}^{k-1} \|R_k e_i\|_b &\le \sum_{i=0}^{k-1} C\theta_k^{b-a_2+\mu-1} \|e_i\|_{a_2-\mu} \\ &\le CK_1(K_1+K_3)\|g\|'^2_\beta \, \theta_k^{b-a_2+\mu-1} \sum_{i=0}^{k-1} \theta_i^{a_1+a_2-2\alpha-1} \end{aligned} \tag{7.35}$$

for all $b \in [0, a_2 + \beta - \alpha]$. Since $a_1 + a_2 - 2\alpha > 0$, we apply (7.23) to the last sum in (7.35). Then, recalling that $\theta_k/\theta_{k+1} \in [\frac{1}{2}, 1]$, and using the bound $a_1 + \beta + \mu \le 2\alpha$, we deduce that

$$\sum_{i=0}^{k-1} \|R_k e_i\|_b \le CK_1(K_1 + K_3)\|g\|'^2_\beta \, \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2 + \beta - \alpha]. \tag{7.36}$$

The sum of (7.34) and (7.36) completes the proof of (7.25).

Now we are ready to prove (7.18) at $j = k+1$. By (7.1) and (7.22) we have $\|v_{k+1}\|_{a_1} \le C\|u_{k+1}\|_{a_1} \le CK_1\|g\|'_\beta$, and we assume that $CK_1\|g\|'_\beta \le \delta_1$, so that $\Psi(v_{k+1})$ is defined. By (7.15), (7.13), (7.14), (7.25), (7.19) one has, for all $a \in [a_1, a_2]$,

$$\|h_{k+1}\|_a \le C\|g\|'_\beta \{1 + (K_1 + K_3)K_1\|g\|'_\beta\} \, \theta_{k+1}^{a-\alpha-1} \tag{7.37}$$

provided that $K_2\|g\|'_\beta \le 1$. Bound (7.37) implies (7.18) provided that $C\{1 + (K_1 + K_3)K_1\|g\|'_\beta\} \le K_1$.

The induction proof of (7.18), (7.19), (7.20) is complete if $K_1, K_2, K_3, \|g\|'_\beta$ satisfy

$$K_3 \ge C_0K_1, \;\; K_2 \ge C_0K_1, \;\; C_0K_1\|g\|'_\beta \le 1, \;\; K_2\|g\|'_\beta \le 1, \;\; C_0\{1+(K_1+K_3)K_1\|g\|'_\beta\} \le K_1$$

where C_0 is the largest of the constants appearing above. First we fix $K_1 \ge 2C_0$. Then we fix K_2 and K_3 larger than C_0K_1, and finally we fix $\delta_0 > 0$ such that the last three inequalities hold for all $\|g\|'_\beta \le \delta_0$. This completes the proof of (7.18), (7.19), (7.20).

Bound (7.18) implies that the sequence (u_k) converges in E_a for all $a \in [0, \alpha)$. We call u its limit. Since $u = \sum_{j=0}^{\infty} h_j$ and each term h_j satisfies (7.18), it follows that $u \in E'_\alpha$ and $\|u\|'_\alpha \leq K_1 \|g\|'_\beta$ by the definition of the norm in E'_α.

Finally, we prove the convergence of the Nash-Moser scheme. By (7.16) and (7.6) one proves by induction that

$$\sum_{j=0}^{k} (e_j + y_j) = e_k + r_k, \quad \text{where } r_k := (I - S_{\theta_k}) \sum_{j=0}^{k-1} e_j, \quad \forall k \geq 1.$$

Hence, by (7.15) and (7.17), recalling that $\Phi'(v_j)\Psi(v_j)$ is the identity map, one has

$$\Phi(u_{k+1}) - \Phi(u_0) = \sum_{j=0}^{k} [\Phi(u_{j+1}) - \Phi(u_j)] = \sum_{j=0}^{k} (e_j + g_j + y_j) = G_k + e_k + r_k$$

where $G_k := \sum_{j=0}^{k} g_j$. By (7.14), $\|G_k - g\|_b \to 0$ as $k \to \infty$, for all $b \in [0, \beta)$. Let $a \in [a_1 - \mu, \alpha - \mu)$. By (7.22) and (7.29) we get $\|u_j\|_{a+\mu} \leq C$. By (7.26), (7.18) and (7.20) we deduce that

$$\|e_j\|_a \leq C K_1 (K_1 + K_3) \|g\|'^2_\beta \, \theta_j^{a_1 + a + \mu - 2\alpha - 1}. \tag{7.38}$$

Hence $\|e_k\|_a \to 0$ as $k \to \infty$ because $a_1 + a + \mu - 2\alpha < 0$, and, moreover, $\sum_{j=0}^{\infty} \|e_j\|_a$ converges. By (7.3) and (7.38), for all $\rho \in [0, a)$ we have

$$\|r_k\|_\rho \leq \sum_{j=0}^{k-1} \|(I - S_{\theta_k}) e_j\|_\rho \leq C \sum_{j=0}^{k-1} \theta_k^{\rho - a} \|e_j\|_a \leq C \theta_k^{\rho - a}, \tag{7.39}$$

so that $\|r_k\|_\rho \to 0$ as $k \to \infty$. We have proved that $\|\Phi(u_k) - \Phi(u_0) - g\|_\rho \to 0$ as $k \to \infty$ for all ρ in the interval $0 \leq \rho < \min\{\alpha - \mu, \beta\}$. Since $u_k \to u$ in E_a for all $a \in [0, \alpha)$, it follows that $\Phi(u_k) \to \Phi(u)$ in F_b for all $b \in [0, \alpha - \mu)$. The theorem is proved. □

8 Appendix C. Tame estimates

In this appendix we recall classical tame estimates for products, compositions of functions and changes of variables which are repeatedly used in the paper. Recall the notation (1.6) for functions $u(x)$, $x \in \mathbb{T}$, in the Sobolev space $H^s := H^s(\mathbb{T}, \mathbb{R})$.

Lemma 8.1. *Let s_0, s_1, s_2, s denote nonnegative real numbers, with $s_0 > 1/2$. There exist positive constants C_s, $s \geq s_0$, with the following properties.*
(Embedding and algebra) *For all $u, v \in H^{s_0}$,*

$$\|u\|_{L^\infty} \leq C_{s_0} \|u\|_{s_0}, \quad \|uv\|_{s_0} \leq C_{s_0} \|u\|_{s_0} \|v\|_{s_0}. \tag{8.1}$$

(Interpolation) *For $0 \leq s_1 \leq s \leq s_2$ and $s = \lambda s_1 + (1 - \lambda) s_2$, for all $u \in H^{s_2}$,*

$$\|u\|_s \leq \|u\|_{s_1}^\lambda \|u\|_{s_2}^{1-\lambda}. \tag{8.2}$$

(Tame product) *For $s \geq s_0$, for all $u, v \in H^s$,*

$$\|uv\|_s \leq C_{s_0} \|u\|_s \|v\|_{s_0} + C_s \|u\|_{s_0} \|v\|_s, \tag{8.3}$$

and for $s \in [0, s_0]$, for all $u \in H^{s_0}$, $v \in H^s$,

$$\|uv\|_s \leq C_{s_0} \|u\|_{s_0} \|v\|_s. \tag{8.4}$$

Proof. The lemma can be proved by using Fourier series and Hölder inequality. Otherwise, for (8.2) see, e.g., [4] (page 82) or [41] (p. 269); for (8.3) adapt [14] (appendix) or [4] (p. 84). For (8.4) use the bound $\sum_{j\in\mathbb{Z}}\langle n\rangle^{2s}\langle j\rangle^{-2s}\langle n-j\rangle^{-2s_0} \leq C_{s_0}$ for all $n \in \mathbb{Z}$, all $0 \leq s \leq s_0$, which can be proved by splitting the two cases $2|j| \leq |n|$ and $2|j| > |n|$. ☐

A function $f : \mathbb{T} \times B \to \mathbb{R}$, where $B := \{y \in \mathbb{R}^{p+1} : |y| < R\}$, induces the composition operator

$$\tilde{f}(u)(x) := f(x, u(x), u'(x), u''(x), \dots, u^{(p)}(x)) \tag{8.5}$$

where $u^{(k)}(x)$ denotes the k-th derivative of $u(x)$. Let B_p be a ball in $W^{p,\infty}(\mathbb{T},\mathbb{R})$ such that, if $u \in B_p$, then the vector $(u(x), u'(x), \dots, u^{(p)}(x))$ belongs to B for all $x \in \mathbb{T}$.

Lemma 8.2 (Composition of functions)**.** *Assume $f \in C^r(\mathbb{T} \times B)$. Then, for all $u \in H^{s+p} \cap B_p$, $s \in [0, r]$, the composition operator* (8.5) *is well defined and*

$$\|\tilde{f}(u)\|_s \leq C\|f\|_{C^r}(\|u\|_{s+p} + 1)$$

where C depends on r, p. If, in addition, $f \in C^{r+2}$, then, for $u, h \in H^{s+p}$ with $u, u + h \in B_p$, one has

$$\big\|\tilde{f}(u+h) - \tilde{f}(u)\big\|_s \leq C\|f\|_{C^{r+1}}\left(\|h\|_{s+p} + \|h\|_{W^{p,\infty}}\|u\|_{s+p}\right),$$
$$\big\|\tilde{f}(u+h) - \tilde{f}(u) - \tilde{f}'(u)[h]\big\|_s \leq C\|f\|_{C^{r+2}}\|h\|_{W^{p,\infty}}(\|h\|_{s+p} + \|h\|_{W^{p,\infty}}\|u\|_{s+p}).$$

Proof. For $s \in \mathbb{N}$ see [41] (p. 272–275) and [43] (Lemma 7, p. 202–203). For $s \notin \mathbb{N}$ see [4] (Proposition 2.2, p. 87). ☐

Lemma 8.3 (Change of variable)**.** *Let $p \in W^{s,\infty}(\mathbb{T},\mathbb{R})$, $s \geq 1$, with $\|p\|_{W^{1,\infty}} \leq 1/2$. Let $f(x) = x + p(x)$. Then f is invertible, its inverse is $f^{-1}(y) = g(y) = y + q(y)$ where q is 2π-periodic, $q \in W^{s,\infty}(\mathbb{T},\mathbb{R})$, and $\|q\|_{W^{s,\infty}} \leq C\|p\|_{W^{s,\infty}}$, where C depends on d, s.*

Moreover, if $u \in H^s(\mathbb{T},\mathbb{R})$, then $u \circ f(x) = u(x + p(x))$ also belongs to H^s, and

$$\|u \circ f\|_s + \|u \circ g\|_s \leq C(\|u\|_s + \|p\|_{W^{s,\infty}}\|u\|_1). \tag{8.6}$$

Proof. For $s \in \mathbb{N}$ see, e.g., [5] (Lemma B.4 in the appendix), where this lemma is proved by adapting [25] (Lemma 2.3.6, p. 149). For $s \notin \mathbb{N}$ the lemma can be proved by studying the conjugate of the pseudo-differential operator $|D_x|^s$ by a change of variable, either by Egorov's Theorem, see [47] (ch. VIII, sec. 1, p. 150) and [3] (appendix C, sec. C.1), or by asymptotic formula, see [4] (Proposition 7.1, p. 37). ☐

Remark 8.4. For time-dependent functions $u(t, x)$, $u \in C([0, T], H^s(\mathbb{T},\mathbb{R}))$, all the estimates of the present appendix hold with $\|u\|_s$ replaced by $\|u\|_{T,s} := \sup_{t\in[0,T]}\|u(t)\|_s$. ☐

References

[1] F. Alabau-Boussouira, J.-M. Coron, G. Olive, *Internal controllability of first order quasilinear hyperbolic systems with a reduced number of controls.* (2015) https://hal.archives-ouvertes.fr/hal-01139980.

[2] T. Alazard, *Boundary observability of gravity water waves.* Preprint 2015 (arXiv:1506.08520).

[3] T. Alazard, P. Baldi, D. Han-Kwan, *Control of water waves.* J. Eur. Math. Soc. (JEMS), accepted (preprint arXiv:1501.06366).

[4] S. Alinhac, P. Gérard, *Pseudo-differential operators and the Nash-Moser theorem.* Grad. Studies in Math. 82. Amer. Math. Soc., Providence, RI, 2007.

[5] P. Baldi, *Periodic solutions of fully nonlinear autonomous equations of Benjamin-Ono type*, Ann. Inst. H. Poincaré (C) Anal. Non Linéaire 30 (2013), 33-77.

[6] P. Baldi, M. Berti, R. Montalto, *KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation*. Math. Annalen 359 (2014), 471-536.
[7] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of KdV*. Ann. Inst. H. Poincaré Anal. Non Linéaire, in press (available online august 2015).
[8] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of mKdV*. Boll. Unione Mat. Ital. 9 (2016), 143-188.
[9] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*. SIAM J. Control Optim. 30 (1992), no. 5, 1024-1065.
[10] K. Beauchard, *Local controllability of a 1-D Schrodinger equation*. J. Math. Pures Appl. (9) 84 (2005), no. 7, 851-956.
[11] K. Beauchard, *Local controllability of a one-dimensional beam equation*. SIAM J. Control Optim. 47 (2008), no. 3, 1219-1273.
[12] K. Beauchard, J.-M. Coron, *Controllability of a quantum particle in a moving potential well*. J. Funct. Anal. 232 (2006), no. 2, 328-389.
[13] K. Beauchard, C. Laurent, *Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control*. J. Math. Pures Appl. (9) 94 (2010), no. 5, 520-554.
[14] M. Berti, Ph. Bolle, M. Procesi, *An abstract Nash-Moser theorem with parameters and applications to PDEs*. Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 1, 377-399.
[15] M. Berti, L. Corsi, M. Procesi, *An abstract Nash-Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds*. Comm. Math. Phys. 334 (2015), no. 3, 1413-1454.
[16] N. Burq, P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 7, 749-752.
[17] N. Burq, M. Zworski, *Control for Schrödinger operators on tori*. Math. Res. Lett. 19 (2012), no. 2, 309-324.
[18] J.-M. Coron, *Control and Nonlinearity*. Math. Surveys Monogr. 136, AMS, Providence, RI, 2007.
[19] J.-M. Coron, O. Glass, Z. Wang, *Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed*. SIAM J. Control Optim. 48 (2009/10), no. 5, 3105-3122.
[20] B. Dehman, G. Lebeau, *Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time*, SIAM J. Control Optim. 48 (2009), no. 2, 521-550.
[21] I. Ekeland, *An inverse function theorem in Fréchet spaces*. Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011), no. 1, 91-105.
[22] I. Ekeland, E. Séré, *An implicit function theorem for non-smooth maps between Fréchet spaces*. Preprint 2015 (arXiv:1502.01561).
[23] M.L. Gromov, *Smoothing and inversion of differential operators*. Mat. Sb. (N.S.) 88(130) (1972), 382-441.
[24] H. Guan, S. Kuksin, *The KdV equation under periodic boundary conditions and its perturbations*. Nonlinearity 27 (2014), no. 9, R61-R88.
[25] R.S. Hamilton, *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65-222.
[26] A. Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire*. J. Math. Pures Appl. (9) 68 (1989), no. 4, 457-465 (1990).
[27] L. Hörmander, *The boundary problems of physical geodesy*. Arch. Rational Mech. Anal. 62 (1976), no. 1, 1-52.
[28] L. Hörmander, *On the Nash-Moser implicit function theorem*. Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 255-259.
[29] L. Hörmander, *The Nash-Moser theorem and paradifferential operators*. Analysis, et cetera, 429-449, Academic Press, Boston, MA, 1990.
[30] J.-P. Kahane, *Pseudo-périodicité et séries de Fourier lacunaires*. Ann. Sci. École Norm. Sup., 79 (1962), 93-150.
[31] T. Kappeler, J. Pöschel, *KdV & KAM*, Springer-Verlag, Berlin, 2003.
[32] V. Komornik, *Exact controllability and stabilization* RAM: Research in Applied Mathematics, The multiplier method, Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
[33] V. Komornik, P. Loreti. *Fourier series in control theory*. Springer Monog. in Math, Springer-Verlag, New York, 2005.
[34] C. Laurent, *Global controllability and stabilization for the nonlinear Schrödinger equation on an interval*. ESAIM-COCV 16 (2010), 356-379.
[35] C. Laurent, L. Rosier, B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain*. Comm. Partial Differential Equations 35 (2010), no. 4, 707-744.

[36] T. Li, B. Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, SIAM J. Control Optim. 41 (2003), no. 6, 1748-1755.

[37] T. Li, B.-Y. Zhang, *Global exact controllability of a class of quasilinear hyperbolic systems*. J. Math. Anal. Appl. 225 (1998), no. 1, 289-311.

[38] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*. SIAM Rev. 30 (1988), no. 1, 1-68.

[39] S. Micu, E. Zuazua, *An introduction to the controllability of partial differential equations*. Quelques questions de théorie du contrôle. In: Sari T. (ed.) Collection Travaux en Cours (2004), 69-157.

[40] J. Moser, *A new technique for the construction of solutions of nonlinear differential equations*. Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1824-1831.

[41] J. Moser, *A rapidly convergent iteration method and non-linear differential equations. I.* Ann. Scuola Norm. Sup. Pisa (3) 20 (1966), 265-315.

[42] J. Nash, *The imbedding problem for Riemannian manifolds*. Ann. of Math. (2) 63 (1956), 20-63.

[43] P. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations. I.* Comm. Pure Appl. Math. 20 (1967), 145-205.

[44] L. Rosier, B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: recent progresses*. J. Syst. Sci. Complex. 22 (2009), no. 4, 647-682.

[45] D.L. Russell, *Computational study of the Korteweg-de Vries equation with localized control action*. Distributed Parameter Control Systems: New Trends and Applications, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, 1991, 128: 195-203.

[46] D.L. Russell, B.-Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*. SIAM J. Control Optim. 31 (1993), no. 3, 659-676.

[47] M.E. Taylor, *Pseudodifferential operators*. Princeton Mathematical Series, 34. Princeton University Press, Princeton, N.J., 1981.

[48] E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems. I-II.* Comm. Pure Appl. Math. 28 (1975), 91-140; and 29 (1976), 49-111.

[49] B.-Y. Zhang, *Some results of nonlinear dispersive wave equations with applications to control*, Ph.D. thesis, University of Wisconsin-Madison, 1990.

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