General theory for plane extensible elastica with arbitrary undeformed shape
SUPPLEMENTARY INFORMATION

(Dated: July 25, 2023)

I. DERIVATION OF THE STRAIN ENERGY FROM A GENERIC UNDEFORMED CONFIGURATION

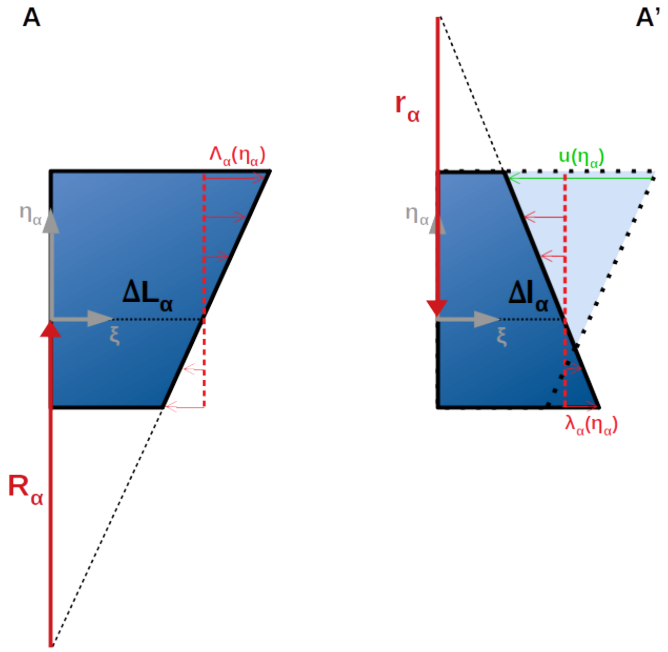

Figure 1. Strain energy from a generic undeformed configuration. **A**: The block undeformed configuration is such that the representative material line has a natural extension equal to ΔL_α. Correspondingly, the size of any other fiber is expressible through the relation (1), where the quantity $\Lambda(\eta_\alpha)$ are represented by orizontal thin red arrows. The spontaneous radius of curvature relative to the representative fiber R_α is shown by a vertical thick red arrow. **A** ′: In the deformed condition, the size of the representative fiber varies from ΔL_α to Δl_α and its radius of curvature becomes r_α. In the reference frame integral with the block, any fiber undergoes a displacement $u(\eta_\alpha)$ (green arrows). At the same time, the size of any deformed material fiber can be expressed as $\Delta l(\eta_\alpha) = \Delta l_\alpha + \lambda_\alpha(\eta_\alpha)$ (red arrows).

We consider the elementary block depicted in Fig. 1. The undeformed condition is represented in panel A, whereas its deformed state is shown in panel A′. The length of the representative fiber in the natural state is ΔL_α, and that of a generic fiber at an height η_α is

$$\Delta L(\eta_\alpha) = \Delta L_\alpha + \Lambda_\alpha(\eta_\alpha). \tag{1}$$

The corresponding deformed quantities are Δl_α and $\Delta l(\eta_\alpha)$ are defined by

$$\Delta l(\eta_\alpha) = \Delta l_\alpha + \lambda_\alpha(\eta_\alpha). \tag{2}$$

Hence, the deformation that a generic material segment experiences passing from the state A to the state A′ is expressed as

$$u(\eta_\alpha) = \Delta l_\alpha - \Delta L_\alpha + \Lambda_\alpha(\eta_\alpha) - \lambda_\alpha(\eta_\alpha). \tag{3}$$

The geometry of the undeformed shape satisfies the following equality [1, 2]

$$\frac{\Delta L_\alpha}{R_\alpha} = \frac{\Lambda_\alpha(\eta_\alpha)}{\eta_\alpha}, \tag{4}$$

while for the deformed configuration we have

$$\frac{\Delta l_\alpha}{r_\alpha} = \frac{\lambda_\alpha(\eta_\alpha)}{\eta_\alpha} . \tag{5}$$

Thanks to Eqs. (4) and (5), the displacement (3) attains the final form

$$u(\eta_\alpha) = \Delta l_\alpha - \Delta L_\alpha + \eta_\alpha \left(\frac{\Delta L_\alpha}{R_\alpha} - \frac{\Delta l_\alpha}{r_\alpha} \right) . \tag{6}$$

By substitution of Eqs. (1) and (6) into

$$\Delta E_\alpha = \frac{bY}{2} \int_{-\alpha h}^{(1-\alpha)h} \left[\frac{u(\eta_\alpha)}{\Delta L(\eta_\alpha)} \right]^2 \Delta L(\eta_\alpha)\ d\eta_\alpha, \tag{7}$$

we obtain the expression:

$$\Delta E_\alpha = \frac{Y}{2\,\Delta L_\alpha} \left[F_\alpha \left(\Delta l_\alpha - \Delta L_\alpha\right)^2 + 2 S_\alpha \left(\Delta l_\alpha - \Delta L_\alpha\right) \left(\frac{\Delta l_\alpha}{r_\alpha} - \frac{\Delta L_\alpha}{R_\alpha} \right) + I_\alpha \left(\frac{\Delta l_\alpha}{r_\alpha} - \frac{\Delta L_\alpha}{R_\alpha} \right)^2 \right] , \tag{8}$$

where

$$F_\alpha = \int_{-\alpha h}^{(1-\alpha)h} d\eta_\alpha \frac{b}{1 + \frac{\eta_\alpha}{R_\alpha}} , \tag{9}$$

$$S_\alpha = \int_{-\alpha h}^{(1-\alpha)h} d\eta_\alpha \frac{b\eta_\alpha}{1 + \frac{\eta_\alpha}{R_\alpha}} \tag{10}$$

and

$$I_\alpha = \int_{-\alpha h}^{(1-\alpha)h} d\eta_\alpha \frac{b\eta_\alpha^2}{1 + \frac{\eta_\alpha}{R_\alpha}} \tag{11}$$

These three factors have different functional forms according to whether R_α is positive or negative (see Fig. 2). The quantity R_α can be positive or negative, as required by the consistency of the Eq.(4). $|R_\alpha|$ becomes the radius of the osculating circle which locally approximates the reference segment in the continuum limit, and the sign of R_α is assigned in the following way. It is clear that the intersection C between the sidelines containing the block's sections lies on the axis $\xi = 0$ of the local reference system ξ-0-η_α. If C lies below the bottom fiber $\alpha = 0$, then R_α is positive (Fig.2A). If conversely C is above the upper fiber $\alpha = 1$, then R_α is negative (Fig.2A$'$). It follows that C has coordinates $(0, -R_\alpha)$ in the local reference system. The same prescription for the sign applies to r_α. If $R_\alpha > 0$ the solutions of (9), (10) and (11) read

$$F_\alpha = bR_\alpha \ln \left(1 + \frac{h}{R_0} \right) , \tag{12}$$

$$S_\alpha = bR_\alpha \left[h - R_\alpha \ln \left(1 + \frac{h}{R_0} \right) \right] \tag{13}$$

and

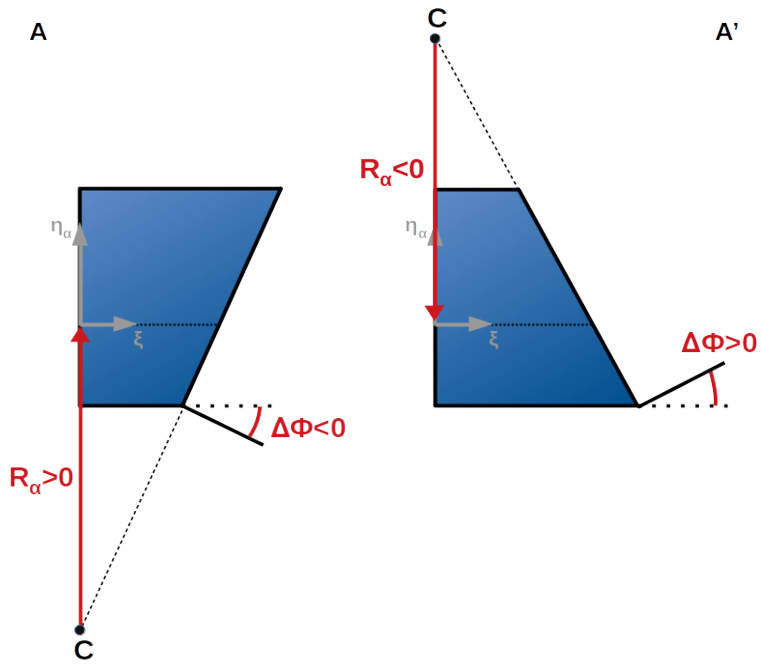

Figure 2. Sign of the radius of curvature. The radius of curvature has a sign assigned whether its direction, connecting the intersection C of the sidelines containing the block's sections with the beginning of the reference segment, coincides with that of the η_α axis of the frame integral with the block (panel **A**), or opposite to it (panel $\mathbf{A}'$).

$$I_\alpha = bR_\alpha \left[\left(\frac{1}{2} - \alpha \right) h^2 - hR_\alpha + R_\alpha^2 \ln \left(1 + \frac{h}{R_0} \right) \right], \tag{14}$$

with $R_\alpha = R_0 + \alpha h$. When we consider the case $R_\alpha < 0$, the three integrals 10-11 can be solved yielding

$$F_\alpha = -bR_\alpha \ln \left(1 - \frac{h}{R_1} \right), \tag{15}$$

$$S_\alpha = bR_\alpha \left[h + R_\alpha \ln \left(1 - \frac{h}{R_1} \right) \right] \tag{16}$$

and

$$I_\alpha = bR_\alpha \left[\left(\frac{1}{2} - \alpha \right) h^2 - hR_\alpha - R_\alpha^2 \ln \left(1 - \frac{h}{R_1} \right) \right], \tag{17}$$

where $R_\alpha = R_1 - (1-\alpha)h$. The two opposite bending states in Fig.2 are expressible in the compact forms provided in the main text recalling that, if $R_\alpha > 0$, hence necessarily $R_0 < R_1$ ($K_0 > K_1$), whilst, for $R_\alpha < 0$, therefore $|R_1| < |R_0|$ and $K_1 > K_0$. We stress the fact that the expressions of F_α, S_α and I_α are fully established once the value of α and $\max[K_0, K_1]$ are furnished. In particular the following identity holds if $K_0 > K_1$

$$K_\alpha = \frac{K_0}{1 + \alpha h K_0}, \tag{18}$$

while for $K_1 > K_0$

$$K_\alpha = \frac{K_1}{1 + (1-\alpha) h K_1}. \tag{19}$$

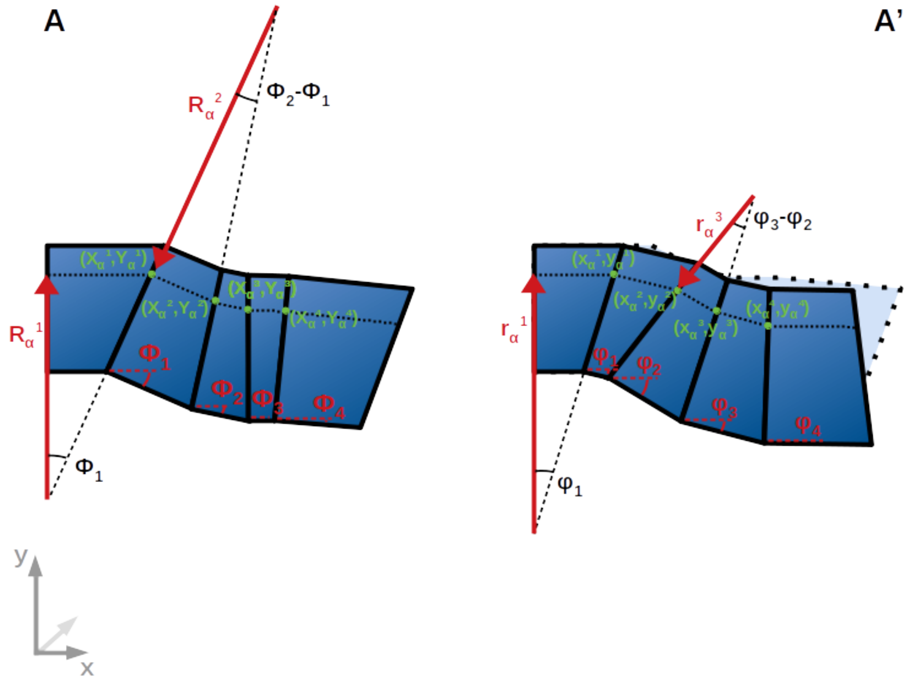

Figure 3. Beam discrete strain energy from a generic undeformed configuration. **A**: The polygonal, connecting the vertices of each undeformed representative segments, is represented by a dotted black thick line. In the laboratory frame, the vertices have planar coordinates $\left(X_\alpha^{(i)}, Y_\alpha^{(i)}\right)$ shown as green dots. The spontaneous radius of curvature can be positive (as R_α^1) or negative (as R_α^2). The connection between the size of representative segment, $\Delta L_\alpha^{(i)}$, and its spontaneous radius of curvature $R_\alpha^{(i)}$, is given by the relation (23), where the bending angles $\Phi^{(i)}$ define the deflection of the undeformed blocks from the external x axis. $\mathbf{A'}$: each block composing the beam undergoes a deformation, such that the deformed polygonal chains l_α is constructed by the sequence of the representative segments whose vertices are $\left(x_\alpha^{(i)}, y_\alpha^{(i)}\right)$ (green dots).

According to the integral expressions (9) and (11), it is always $F_\alpha > 0$ and $I_\alpha > 0$ because the integrand functions are strictly positive in the integration interval. To the contrary, the sign of S_α can vary according to the value of α and to $\max[K_0, K_1]$ ($\min[R_0, R_1]$). For the sake of clarity, $S_\alpha > 0$ for $\alpha < \alpha_U$, and $S_\alpha < 0$ for $\alpha > \alpha_U$. The choice of α_U ensuring the stretching-bending uncoupling, guarantees that

$$F_{\alpha_U} = bh, \tag{20}$$

$$S_{\alpha_U} = 0 \tag{21}$$

and

$$I_{\alpha_U} = \frac{b}{K_{\alpha_U}} \left(\frac{1}{2} - \alpha_U \right) h^2 = \frac{\operatorname{sgn}(K_0 - K_1)\, b h^3}{\ln\left(1 + h \max[K_0, K_1]\right)} \left[\frac{1}{2} - \frac{1}{\ln\left(1 + h \max[K_0, K_1]\right)} + \frac{1}{h \max[K_0, K_1]} \right] \tag{22}$$

The finite difference scheme, outlined so far, requires the evaluation of the discrete strain energy $E_\alpha = \sum_{i=1}^{N} \Delta E_\alpha^{(i)}$ needed to deform the elastica in Fig. 3 from A to A$'$. Moving to the laboratory frame we find that the relation

$$\frac{\Delta L_\alpha^{(i)}}{R_\alpha^{(i)}} = -\tan \Delta\Phi^{(i)}. \tag{23}$$

is always satisfied [3], where $\Delta\Phi^{(i)} = \Phi^{(i)} - \Phi^{(i-1)}$ and $\Phi^{(i)}$ is the i-th cross-sectional bending angle with respect the x axis. Since we assume the limit of small deflections, we can approximate $\tan \Delta\Phi^{(i)} \simeq \Delta\Phi^{(i)}$. The reference undeformed polygonal chain L_α is specified by the series of points $\mathbf{L}_\alpha^{(i)} = (X_\alpha^{(i)}, Y_\alpha^{(i)})$, with line segments $\Delta L_\alpha^{(i)} = \left|\mathbf{L}_\alpha^{(i)} - \mathbf{L}_\alpha^{(i-1)}\right|$ (Fig.3A).
In the deformed state (Fig.3A$'$) we have

$$\frac{\Delta l_\alpha^{(i)}}{r_\alpha^{(i)}} = -\tan \Delta\varphi^{(i)} \tag{24}$$

where $\Delta\varphi^{(i)} = \varphi^{(i)} - \varphi^{(i-1)}$, and $\varphi^{(i)}$ corresponds to the bending angle between the $i-$th block and the x axis. Again we assume the small deflection limit, i.e. $\tan \Delta\varphi^{(i)} \simeq \Delta\varphi^{(i)}$.The deformed chain l_α has the points $\mathbf{l}_\alpha^{(i)} = (x_\alpha^{(i)}, y_\alpha^{(i)})$ as vertices, with $\Delta l_\alpha^{(i)} = \left|\mathbf{l}_\alpha^{(i)} - \mathbf{l}_\alpha^{(i-1)}\right|$. We introduce the strain measure as $\varepsilon_\alpha^{(i)} = \frac{\Delta l_\alpha^{(i)} - \Delta L_\alpha^{(i)}}{\Delta L_\alpha^{(i)}}$, while the bending strain measure can be obtained by two different definitions: the first is due to Kammel [4]

$$\mu_\alpha^{(i)} = \frac{\Delta\Phi^{(i)} - \Delta\varphi^{(i)}}{\Delta L_\alpha^{(i)}}, \tag{25}$$

and the second to Antman [5]

$$\mu_\alpha^{(i)} = \frac{\Delta l_\alpha}{\Delta L_\alpha} \frac{1}{r_\alpha} - \frac{1}{R_\alpha}. \tag{26}$$

One can easily see that they are equivalent by the construction in Fig.6. Upon summation of the terms in Eq. (8), using the definition (26), we obtain the energy

$$E_\alpha = \frac{Y}{2} \sum_{i=1}^{N} \Delta L_\alpha^{(i)} \left\{ F_\alpha^{(i)} \, {\varepsilon_\alpha^{(i)}}^2 + 2 S_\alpha^{(i)} \, \varepsilon_\alpha^{(i)} \mu_\alpha^{(i)} + I_\alpha^{(i)} \, {\mu_\alpha^{(i)}}^2 \right\} \; . \tag{27}$$

The expression of F_α, S_α and I_α in terms of $\Delta\Phi$, ΔL_α and α, is achieved by inserting the relation (23) into the expressions (12)–(14) and (15)–(17):

$$F_\alpha^{(i)} = \frac{\Delta L_\alpha^{(i)}}{\left|\Delta\Phi^{(i)}\right|} \ln \left(1 + h \frac{\left|\Delta\Phi^{(i)}\right|}{\min[\Delta L_0^{(i)}, \Delta L_1^{(i)}]} \right), \tag{28}$$

$$S_\alpha^{(i)} = -\frac{\Delta L_\alpha^{(i)}}{\Delta\Phi^{(i)}} \left(h - F_\alpha^{(i)} \right) \tag{29}$$

and

$$I_\alpha^{(i)} = -\frac{\Delta L_\alpha^{(i)}}{\Delta\Phi^{(i)}} \left[\left(\frac{1}{2} - \alpha \right) h^2 - S_\alpha^{(i)} \right]. \tag{30}$$

We recall that it is convenient to take $\Delta L_\alpha = \Delta L_0 - \alpha h \Delta\Phi$ if $\min[\Delta L_0, \Delta L_1] = \Delta L_0$, and $\Delta L_\alpha = \Delta L_1 + (1-\alpha) h \Delta\Phi$ if $\min[\Delta L_0, \Delta L_1] = \Delta L_1$.

The differential strain energy $\mathscr{E}_\alpha$ is derived by firstly introducing two parametric expressions for the undeformed and deformed reference material curves as $\mathscr{L}_\alpha : [s_m, s_M] \to \mathbb{R}^2$ and $\ell_\alpha : [s_m, s_M] \to \mathbb{R}^2$ respectively. Secondly we take an arbitrary partition $s_m = s_0 < s_1 < s_2 < \cdots < s_N = s_M$ to which we connect the two polygonal chains L_α and l_a, in such a way that the vertices satisfy $\mathbf{L}_\alpha(s_i) \equiv \mathbf{L}_\alpha^{(i)}$ and $\mathbf{l}_\alpha(s_i) \equiv \mathbf{l}_\alpha^{(i)}$. Moreover we define the applications $\Phi : [s_m, s_M] \to \mathbb{R}$ and $\varphi : [s_m, s_M] \to \mathbb{R}$ with the properties $\Phi(s_i) \equiv \Phi^{(i)}$ and $\varphi(s_i) \equiv \varphi^{(i)}$. From the definitions of the two strain measures $\varepsilon_\alpha^{(i)}$ and $\mu_\alpha^{(i)}$, it follows

$$\varepsilon_\alpha(s_i) = \frac{\frac{|\Delta \mathbf{l}_\alpha(s_i)|}{\Delta s_i} - \frac{|\Delta \mathbf{L}_\alpha(s_i)|}{\Delta s_i}}{\frac{|\Delta \mathbf{L}_\alpha(s_i)|}{\Delta s_i}}. \tag{31}$$

$$\mu_\alpha(s_i) = \frac{\frac{\Delta\Phi(s_i)}{\Delta s_i} - \frac{\Delta\varphi(s_i)}{\Delta s_i}}{\frac{|\Delta \mathbf{L}_\alpha(s_i)|}{\Delta s_i}}. \tag{32}$$

In the continuum limit, N is increased until the lengths of the polygonal chains L_α and l_α equal those of the curves $\mathscr{L}_\alpha$ and ℓ_α. This condition is mathematically enforced by the limiting relations

$$\frac{|\Delta \mathbf{L}_\alpha(s_i)|}{\Delta s_i} \to |\mathscr{L}'_\alpha(s)|\,, \quad \frac{|\Delta \mathbf{l}_\alpha(s_i)|}{\Delta s_i} \to |\ell'_\alpha(s)| \tag{33}$$

as $\Delta s_i \to 0$. Correspondingly, the two tangents to the curves are defined as $\mathbf{T}_\alpha(s) = \frac{d\mathbf{L}_\alpha}{ds}$ and $\mathbf{t}_\alpha(s) = \frac{d\mathbf{l}_\alpha}{ds}$. Yet, the limit $\Delta s_i \to 0$ entails

$$\frac{\Delta\Phi(s_i)}{\Delta s_i} \to \Phi'(s), \quad \frac{\Delta\varphi(s_i)}{\Delta s_i} \to \varphi'(s). \tag{34}$$

The differential strain measures follow from the limit of Eq.s (31) and (32):

$$\varepsilon_\alpha(s) = \frac{|\mathbf{t}_\alpha(s)| - |\mathbf{T}_\alpha(s)|}{|\mathbf{T}_\alpha(s)|}. \tag{35}$$

$$\mu_\alpha(s) = \frac{\Phi'(s) - \varphi'(s)}{|\mathbf{T}_\alpha(s)|}. \tag{36}$$

Now, if $\sum_{i=1}^{N} \Delta L \to \int_0^L ds$, plugging the definitions of $\mathbf{T}_\alpha(s)$, (35) and (36) into Eq.(27) we recover the energy reported in the main text:

$$\mathscr{E}_\alpha = \frac{Y}{2} \int_{s_m}^{s_M} ds \left\{ \frac{F_\alpha(s)}{|\mathbf{T}_\alpha(s)|} \left[|\mathbf{t}_\alpha(s)| - |\mathbf{T}_\alpha(s)|\right]^2 - \frac{2S_\alpha(s)}{|\mathbf{T}_\alpha(s)|} \left[|\mathbf{t}_\alpha(s)| - |\mathbf{T}_\alpha(s)|\right] \left[\varphi'(s) - \Phi'(s)\right] + \frac{I_\alpha(s)}{|\mathbf{T}_\alpha(s)|} \left[\varphi'(s) - \Phi'(s)\right]^2 \right\}, \tag{37}$$

The differential formula for the three factors $F_\alpha(s)$, $S_\alpha(s)$ and $I_\alpha(s)$ are obtainable from the Eq.s (28), (29) and (30):

$$F_\alpha(s) = \left|\frac{\mathbf{T}_\alpha(s)}{\Phi'(s)}\right| \ln\left(1 + h \frac{|\Phi'(s)|}{\min[|\mathbf{T}_0(s)|\,, |\mathbf{T}_1(s)|]}\right), \tag{38}$$

$$S_\alpha(s) = -\frac{|\mathbf{T}_\alpha(s)|}{\Phi'(s)} \left(h - F_\alpha(s)\right) \tag{39}$$

and

$$I_\alpha(s) = -\frac{|\mathbf{T}_\alpha(s)|}{\Phi'(s)}\left[\left(\frac{1}{2}-\alpha\right)h^2 - S_\alpha(s)\right]. \tag{40}$$

It is clear that, when $\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|] = |\mathbf{T}_0(s)|$, we can express $|\mathbf{T}_\alpha(s)| = |\mathbf{T}_0(s)| - \alpha h\Phi'(s)$; conversely, when $\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|] = |\mathbf{T}_1(s)|$, then $|\mathbf{T}_\alpha(s)| = |\mathbf{T}_1(s)| + (1-\alpha)h\Phi'(s)$. Thus, the functional forms of $F_\alpha(s)$, $S_\alpha(s)$ and $I_\alpha(s)$ in Eqs.(38), (39) and (40) highlight the local character of these quantities and the fact that they are fully established by the values of $(\alpha, \Phi', \min[|\mathbf{T}_0|, |\mathbf{T}_1|])$.
To uncouple the bending and stretching contributions in the continuum energy expression $\mathscr{E}_\alpha$, the conditions $S_\alpha(s) = 0$ in (39) requires that

$$\alpha_U(s) = \frac{|\mathbf{T}_0(s)|}{h\Phi'(s)} - \frac{|\Phi'(s)|}{\Phi'(s)}\frac{1}{\ln\left(1 + h\frac{|\Phi'(s)|}{\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|]}\right)}, \tag{41}$$

which is also expressible as

$$\alpha_U(s) = 1 - \left[\frac{|\Phi'(s)|}{\Phi'(s)}\frac{1}{\ln\left(1 + h\frac{|\Phi'(s)|}{\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|]}\right)} - \frac{|\mathbf{T}_1(s)|}{h\Phi'(s)}\right]. \tag{42}$$

The two formulae (41) and (42) are equivalent and can be obtained recalling that $|\mathbf{T}_1(s)| = |\mathbf{T}_0(s)| - h\Phi'(s)$. Therefore the uncoupling condition is a local property of the elastica and, for either choice of $\alpha_U(s)$, the three factors (38)-(40) reduce to

$$F_{\alpha_U}(s) = bh, \tag{43}$$

$$S_{\alpha_U}(s) = 0 \tag{44}$$

and

$$I_{\alpha_U}(s) = -\frac{|\mathbf{T}_{\alpha_U}(s)|}{\Phi'(s)}\left[\frac{1}{2} - \alpha_U(s)\right]bh^2 \tag{45}$$

$$= -bh^3\frac{|\Phi'(s)|}{\Phi'(s)}\frac{1}{\ln\left(1 + h\frac{|\Phi'(s)|}{\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|]}\right)}\left[\frac{1}{2} - \frac{|\mathbf{T}_0(s)|}{h\Phi'(s)} + \frac{|\Phi'(s)|}{\Phi'(s)}\frac{1}{\ln\left(1 + h\frac{|\Phi'(s)|}{\min[|\mathbf{T}_0(s)|, |\mathbf{T}_1(s)|]}\right)}\right]. \tag{46}$$

The *neutral* arc-length parametrization requires that

$$|\mathbf{T}_{\alpha_U}(s)| = 1. \tag{47}$$

Therefore, adopting this parametrization and the neutral curve as representative of the whole elastica we have that for a generic transformation the energy is expressible as

$$\mathscr{E}_{\alpha_U} = \frac{bY}{2}\int_{s_m}^{s_M} ds\left\{h\left[|\mathbf{t}_{\alpha_\mathbf{U}}(s)| - 1\right]^2 - \frac{h^2}{\Phi'(s)}\left[\frac{1}{2} - \alpha_U(s)\right]\left[\varphi'(s) - \Phi'(s)\right]^2\right\}. \tag{48}$$

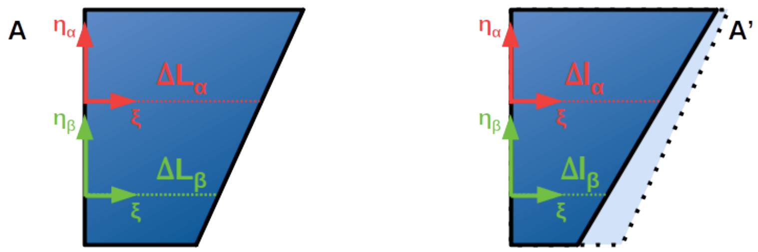

Figure 4. Block's energy invariance under change of reference material line. **A:** In red is represented the reference frame ξ-0-η_α integral with the block, when the reference material segment is placed at an height αh from the block's bottom surface. The length of the reference segment is ΔL_α. When the reference material segment is placed at a different height βh, the corresponding frame ξ-0-η_β is depicted in green and its size is ΔL_β. **A′:** The block undergoes a deformation from its natural shape (light blue): the size of the reference fiber changes to Δl_α or Δl_β. The energy cost associated to this deformation is the same whether the reference segment is placed at αh or βh.

II. ENERGY INVARIANCE UNDER CHANGE OF REFERENCE FRAME

Consider the Fig. 4. The undeformed elementary block is represented on the left A, and its shape upon deformation is displayed on the right A′. Let us assign to the block an integral planar reference system, where the ξ and η_α axes define respectively the block's longitudinal and transverse directions. The origin of such reference system is placed at an height αh $(0 \le \alpha \le 1)$ from the block's bottom surface, and on the left lateral block boundary. The quantity $\Delta L(\eta_\alpha)$ corresponds to the length of a generic material line placed at the height η_α, with $-\alpha h \le \eta_\alpha \le (1-\alpha)h$. By definition, the value of $\Delta L(\eta_\alpha = 0)$ is the representative material line length ΔL_α. Now, a translation of the reference system along the axis $\xi = 0$ is equivalent to a linear change of variables:

$$\eta_\beta = \eta_\alpha + (\alpha - \beta)h \tag{49}$$

where η_β is the new axis pointing along the block transverse direction. However, the material line lengths $\Delta L(\eta_\alpha)$ have to be invariant under the transformation (49):

$$\Delta L\,(\eta_\beta) = \Delta L\,(\eta_\alpha = \eta_\beta - (\alpha - \beta)h)\,. \tag{50}$$

It is also clear that in the reference system ξ-0-η_β, the representative material line has length $\Delta L_\beta = \Delta L\,(\eta_\beta = 0) = \Delta L\,(\eta_\alpha = (\beta - \alpha)h)$ and $-\beta h \le \eta_\beta \le (1-\beta)h$.

Let us turn to the deformed configuration A′. The deformed longitudinal length Δl follows the same law (50) under the shift of the reference system, i.e.

$$\Delta l\,(\eta_\beta) = \Delta l\,(\eta_\alpha = \eta_\beta - (\alpha - \beta)h)\,. \tag{51}$$

Therefore, since the extension is defined as $u = \Delta l - \Delta L$, thanks to Eqs.(50) and (51) we have that the following equality holds

$$u\,(\eta_\beta) = u\,(\eta_\alpha = \eta_\beta - (\alpha - \beta)h)\,. \tag{52}$$

The strain energy of the block calculated in the ξ-0-η_α reference frame is

$$\Delta E_\alpha = \frac{bY}{2} \int_{-\alpha h}^{(1-\alpha)h} \frac{u(\eta_\alpha)^2}{\Delta L(\eta_\alpha)}\, d\eta_\alpha. \tag{53}$$

By applying the change of variables (49), the equality of the integral expression (53) $\Delta E_\alpha = \Delta E_\beta$ follows from the relations (50) and (52).

III. DERIVATION OF THE STRAIN ENERGY FROM A FLAT UNDEFORMED CONFIGURATION

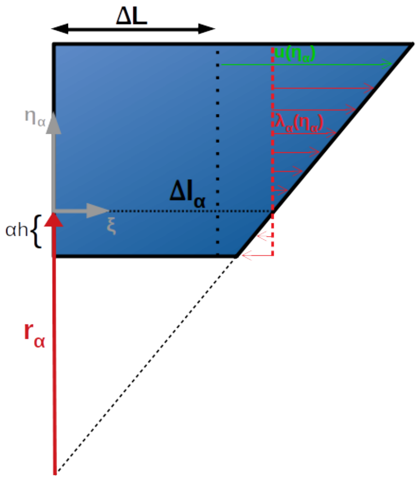

Figure 5. Strain energy from a flat undeformed configuration. The block in its undeformed configuration has a longitudinal size of ΔL (dotted black vertical line). When deformed, the size of the representative fiber varies from ΔL to Δl_α. In the reference frame integral with the block, any fiber undergoes a displacement $u(\eta_\alpha)$ (green arrows). At the same time, the size of any deformed material fiber can be expressed as $\Delta l(\eta_\alpha) = \Delta l_\alpha + \lambda_\alpha(\eta_\alpha)$, where $\lambda_\alpha(\eta_\alpha)$ are represented by red arrows.

Let us consider the deformation of the elementary block presented in Fig.5. The undeformed flat condition is depicted by a dotted black line, and it has the peculiarity that the longitudinal length is equal to ΔL for any choice of the representative segment. The plane integral reference frame is identified by the ξ and η_α axes, pointing respectively towards the block's longitudinal and transverse directions. The origin is placed at an height αh $(0 \le \alpha \le 1)$ from the block's bottom surface, and on the left lateral block boundary. Any fiber placed at an height η_α attains a length $\Delta l(\eta_\alpha)$ upon deformation, with $\Delta l(\eta_\alpha = 0) = \Delta l_\alpha$. According to the geometrical construction in Fig.5, the Eq.(2), is equivalent to

$$\Delta l(\eta_\alpha) = \Delta L + u(\eta_\alpha). \tag{54}$$

From Eqs.(54), (2) and (5) the elongation of any fiber can be expressed as

$$u(\eta_\alpha) = \Delta l_\alpha - \Delta L + \eta_\alpha \frac{\Delta l_\alpha}{r_\alpha}. \tag{55}$$

In this condition, the strain energy of the block takes the form

$$\Delta E_\alpha = \frac{bY}{2} \int_{-\alpha h}^{(1-\alpha)h} \left[\frac{\Delta l_\alpha - \Delta L}{\Delta L} + \frac{\eta_\alpha}{r_\alpha} \frac{\Delta l_\alpha}{\Delta L} \right]^2 \Delta L \, d\eta_\alpha, \tag{56}$$

where we hve inserted the relation (55). Solving the integral and defining the strain as $\varepsilon_\alpha = \frac{\Delta l_\alpha - \Delta L}{\Delta L}$, we arrive at the expression

$$\Delta E_\alpha = \frac{bY}{2} \left[h \, \Delta L \, \varepsilon_\alpha^2 + h^2 (1 - 2\alpha) \, \varepsilon_\alpha \frac{\Delta l_\alpha}{r_\alpha} + h^3 \left(\frac{1}{3} - \alpha + \alpha^2 \right) \frac{1}{\Delta L} \left(\frac{\Delta l_\alpha}{r_\alpha} \right)^2 \right] . \tag{57}$$

It is clear that the value of α which guarantees the axial-bending uncoupling is the line of the centroids $\alpha_U = 1/2$.

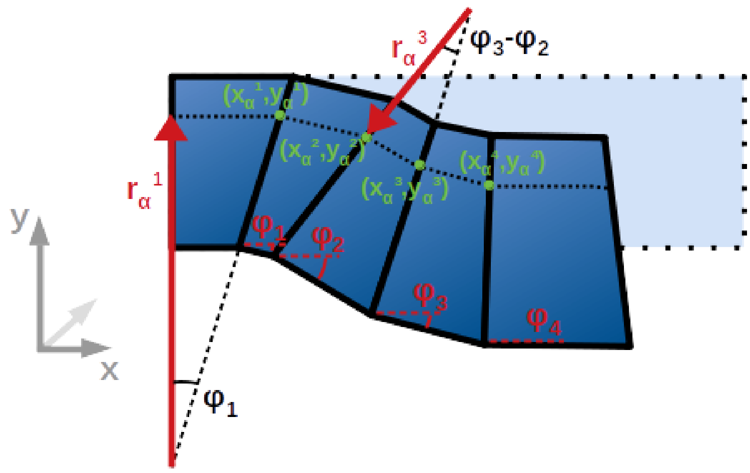

Figure 6. Beam discrete strain energy from a flat undeformed configuration. The polygonal, connecting the vertices of each deformed representative segments, is represented by a dotted black thick line. In the laboratory frame, the vertices have planar coordinates $\left(x_\alpha^{(i)}, y_\alpha^{(i)}\right)$ shown as green dots. The morphological change of each block is determined by the radius of curvature that can be positive (as r_α^1) or negative (as r_α^3). The connection between local longitudinal deformation of the representative fiber, $\Delta l_\alpha^{(i)}$, and its radius of curvature $r_\alpha^{(i)}$, is encapsulated in the relation (24), where the bending angles $\varphi^{(i)}$ define the deflection of the deformed blocks from the external x axis.

The strain energy of the whole elastica is given by the sum over the block contributions, formally by $E_\alpha = \sum_{i=1}^{N} \Delta E_\alpha^{(i)}$. We aim at furnishing, however, its analytical expression in the laboratory frame (Fig. 6). The single block's representative segment size $\Delta l_\alpha^{(i)}$ is a positive quantity, being $\Delta l_\alpha^{(i)} = \left|\mathbf{l}_\alpha^{(i)} - \mathbf{l}_\alpha^{(i-1)}\right|$. $\mathbf{l}_\alpha^{(i)} \equiv \left(x_\alpha^{(i)}, y_\alpha^{(i)}\right)$ are the vertices of the reference polygonal curve l_α in the lab reference system (Fig. 6). The polygonal curve is defined as the ordered sequence of the representative segments $\Delta l_\alpha^{(i)}$. The quantity $r_\alpha^{(i)}$, on the other side, can be positive or negative. This is required by the consistency of the Eq.(5). As a matter of fact, $\left|r_\alpha^{(i)}\right|$ becomes the radius of the osculating circle which locally approximates the reference segment in the continuum limit, and the sign of $r_\alpha^{(i)}$ is assigned in the following way. It is clear that the intersection between the sidelines containing the block's sections lies on the axis $\xi = 0$ of the local reference system integral with any block (see Fig. 2). Hence, in this reference system the coordinate of the circle's center are defined as $\left(0, -r_\alpha^{(i)}\right)$: this establishes uniquely the sign of $r_\alpha^{(i)}$. Now, moving to the laboratory frame we find that the relation (24) is always satisfied, with $\Delta\varphi^{(i)} = \varphi^{(i)} - \varphi^{(i-1)}$, and $\varphi^{(i)}$ corresponds to the bending angle between the $i-$th block and the x axis. In the small deflection limit, i.e. $\tan \Delta\varphi^{(i)} \simeq \Delta\varphi^{(i)}$, the discrete strain energy for the entire slender beam is therefore framed as

$$E_\alpha = \frac{bY}{2} \sum_{i=1}^{N} \left[h \Delta L \, {\varepsilon_\alpha^{(i)}}^2 - h^2 (1 - 2\alpha) \, \varepsilon_\alpha^{(i)} \Delta\varphi^{(i)} + h^3 \left(\frac{1}{3} - \alpha + \alpha^2 \right) \frac{{\Delta\varphi^{(i)}}^2}{\Delta L} \right]. \tag{58}$$

The bending measure is defined as

$$\mu^{(i)} = \frac{-\Delta\varphi^{(i)}}{\Delta L} \tag{59}$$

or, thanks to (24), as

$$\mu^{(i)} = \frac{\Delta l_\alpha^{(i)}}{\Delta L}\frac{1}{r_\alpha^{(i)}}. \tag{60}$$

Using these definitions, the energy (61) takes the following form

$$E_\alpha = \frac{bY\Delta L}{2}\sum_{i=1}^{N}\left[h\,{\varepsilon_\alpha^{(i)}}^2 + h^2(1-2\alpha)\varepsilon_\alpha^{(i)}\mu^{(i)} + h^3\left(\frac{1}{3}-\alpha+\alpha^2\right)\mu^{(i)2}\right] \tag{61}$$

Let us introduce two parametric expressions for the undeformed and deformed reference plane curves as $\mathscr{L}_\alpha : [0,L] \to \mathbb{R}^2$ and $\ell_\alpha : [0,L] \to \mathbb{R}^2$ respectively. As a consequence, the Cartesian coordinates of the undeformed reference curve in the laboratory frame are

$$\mathbf{L}_\alpha(s) = \begin{cases} X_\alpha(s) &= s \\ Y_\alpha(s) &= 0, \end{cases} \tag{62}$$

and those of the deformed curve $\ell_\alpha(s)$ are $\mathbf{l}_\alpha(s) \equiv (x_\alpha(s), y_\alpha(s))$. Taking an uniform partition of $[0,L]$, i.e. $0 = s_0 < s_1 < s_2 < \cdots < s_N = L$ such that $s_i - s_{i-1} = \Delta s \equiv \Delta L$ for any i, we obtain that the polygonal vertices are $\mathbf{l}_\alpha(s_i) = \mathbf{l}_\alpha^{(i)}$ and the local longitudinal strain $\varepsilon_\alpha^{(i)}$ is given by

$$\varepsilon_\alpha(s_i) = \frac{|\Delta\mathbf{l}_\alpha(s_i)|}{\Delta s} - 1. \tag{63}$$

Analogously, the bending angles at the polygonal vertices are $\varphi(s_i) = \varphi^{(i)}$. The continuum limit is taken by increasing N until the length of the polygonal chain l_α approaches from below that of the curve ℓ_α, i.e. $\frac{|\Delta\mathbf{l}_\alpha(s_i)|}{\Delta s} \to |\ell'_\alpha(s)|$ as $\Delta s \to 0$. The curve derivative is defined as $\ell'_\alpha(s) \equiv \mathbf{t}_\alpha(s)$, where we have introduced the tangent of the curve $\mathbf{t}_\alpha(s) = \frac{d\mathbf{l}_\alpha(s)}{ds}$. Finally, if the continuum limit entails that $\frac{\Delta\varphi(s_i)}{\Delta s} \to \varphi'(s)$ and $\sum_{i=1}^{N}\Delta L \to \int_0^L ds$, by substitution of $\varepsilon_\alpha(s) = |\mathbf{t}_\alpha(s)| - 1$ the energy (61) takes the form

$$\mathscr{E}_\alpha = \frac{bY}{2}\int_0^L ds\left\{h\,\left[|\mathbf{t}_\alpha(s)|-1\right]^2 - h^2(1-2\alpha)\,\left[|\mathbf{t}_\alpha(s)|-1\right]\varphi'(s) + h^3\left(\frac{1}{3}-\alpha+\alpha^2\right)\varphi'(s)^2\right\}. \tag{64}$$

IV. STRAIN ENERGY LIMITING CASES: REGAINING THE FLAT UNDEFORMED CONDITION

In the present section we show how to recover the straight beam strain energy (57), from the energy (8) calculated from a generic undeformed configuration. To this aim, it will be sufficient to study the behaviour of F_α, S_α and I_α in the limit of $\frac{h}{\min[|R_0|,|R_1|]} \to 0$.
Let us firstly express the relations (9)-(11) as

$$F_\alpha = |R_\alpha|\ln\left(1+\frac{h}{\min[|R_0|,|R_1|]}\right), \tag{65}$$

$$S_\alpha = R_\alpha\left[h - |R_\alpha|\ln\left(1+\frac{h}{\min[|R_0|,|R_1|]}\right)\right] \tag{66}$$

and

$$I_\alpha = R_\alpha\left[\left(\frac{1}{2}-\alpha\right)h^2 - hR_\alpha + R_\alpha^2\ln\left(1+\frac{h}{\min[|R_0|,|R_1|]}\right)\right], \tag{67}$$

Then we consider the condition $\frac{h}{\min[|R_0|,|R_1|]} \ll 1$ and expand the logarithm to the third order:

$$\ln\left(1+\frac{h}{\min[|R_0|,|R_1|]}\right) \simeq \frac{h}{\min[|R_0|,|R_1|]} - \frac{h^2}{2\min[|R_0|,|R_1|]^2} + \frac{h^3}{3\min[|R_0|,|R_1|]^3}. \tag{68}$$

Hence we get

$$F_\alpha = \begin{cases} h + \frac{h^2}{R_0}\left(\alpha-\frac{1}{2}\right) + \frac{h^3}{R_0^2}\left(\frac{1}{3}-\frac{\alpha}{2}\right) & \min[|R_0|,|R_1|] = |R_0| \\ h + \frac{h^2}{R_1}\left(\alpha-\frac{1}{2}\right) + \frac{h^3}{R_1^2}\left[\frac{1}{3}-\frac{(\alpha-1)}{2}\right] & \min[|R_0|,|R_1|] = |R_1| \end{cases}, \tag{69}$$

$$S_\alpha = \begin{cases} h^2\left(\alpha-\frac{1}{2}\right) - \frac{h^3}{R_0}\left(\frac{1}{3}-\alpha+\alpha^2\right) & \min[|R_0|,|R_1|] = |R_0| \\ h^2\left(\alpha-\frac{1}{2}\right) - \frac{h^3}{R_1}\left(\frac{1}{3}-\alpha+\alpha^2\right) & \min[|R_0|,|R_1|] = |R_1|, \end{cases} \tag{70}$$

$$I_\alpha = h^3\left(\frac{1}{3}-\alpha+\alpha^2\right). \tag{71}$$

By substitution of the former relations into (8), the Eq. (57) is correctly reestablished.

V. MACROSCOPIC CONSTITUTIVE EQUATIONS UNDER CHANGE OF MATERIAL CURVE

When the natural state is flat, the strain energy function is defined as $W_\alpha = \frac{\Delta E_\alpha}{\Delta L}$, where ΔE_α is given in Eq. (57):

$$W_\alpha = \frac{bY}{2}\left[h\,\varepsilon_\alpha^2 + h^2(1-2\alpha)\varepsilon_\alpha\mu + h^3\left(\frac{1}{3}-\alpha+\alpha^2\right)\mu^2\right]. \tag{72}$$

The usual choice of the middle fiber as the representative medium ($\alpha = 1/2$) yields the expression commonly used in several contexts [5–8]. However, for a generic choice of the representative fiber, the constitutive equations for the axial force and the bending moment are readily obtained:

$$\begin{cases} N_\alpha = \frac{\partial W_\alpha}{\partial \varepsilon_\alpha} = bY\left[h\varepsilon_\alpha + h^2\mu\left(\frac{1}{2}-\alpha\right)\right] \\ M_\alpha = \frac{\partial W_\alpha}{\partial \mu} = bY\left[h^2\varepsilon_\alpha\left(\frac{1}{2}-\alpha\right) + h^3\mu\left(\frac{1}{3}-\alpha+\alpha^2\right)\right]. \end{cases} \tag{73}$$

the axial force exerted on a material line α in (73) can be transformed into N_β by applying the change of material line

$$\varepsilon_\alpha = \varepsilon_\beta + h(\alpha-\beta)\mu. \tag{74}$$

It results immediately $N_\alpha = N_\beta$. The bending moment can be recast as

$$M_\alpha = bY\left[h^2\varepsilon_\alpha\left(\frac{1}{2}-\alpha\right) + h^3\frac{\mu}{12} + h^3\mu\left(\frac{1}{2}-\alpha\right)^2\right]. \tag{75}$$

Therefore, from the expression of the axial force we have

$$M_\alpha = M_{1/2} - h\left(\alpha - \frac{1}{2}\right) N_\alpha. \tag{76}$$

Hence by subtracting the expression for M_β from (76) and recalling the axial force invariance we have

$$M_\alpha = M_\beta - h\,(\alpha - \beta)\, N_\beta. \tag{77}$$

The case of a general undeformed condition can be deterimined as follows. First we notice how the strain stransorms under cange of material line

$$\varepsilon_\alpha = \frac{\Delta L_\beta}{\Delta L_\alpha}\left[\varepsilon_\beta + h(\alpha - \beta)\mu_\beta\right]. \tag{78}$$

Moreover, the reduced area, the reduced axial-bending coupling moments and the reduced moment of inertia change under material line transformation as

$$F_\alpha = \frac{1}{\Delta L_\beta}\left[\Delta L_\beta - (\alpha - \beta)h\Delta\Phi\right] F_\beta \tag{79}$$

$$S_\alpha = \frac{1}{\Delta L_\beta}\left[\Delta L_\beta - (\alpha - \beta)h\Delta\Phi\right]\left[S_\beta - (\alpha - \beta)hF_\beta\right] \tag{80}$$

$$I_\alpha = \frac{1}{\Delta L_\beta}\left[\Delta L_\beta - (\alpha - \beta)h\Delta\Phi\right]\left[I_\beta - 2(\alpha - \beta)hS_\beta + (\alpha - \beta)^2 h^2 F_\beta\right]. \tag{81}$$

Inserting the previous relations into

$$\begin{cases} N_\alpha = \frac{\partial W_\alpha}{\partial \varepsilon_\alpha} = Y\left(F_\alpha \varepsilon_\alpha + S_\alpha \mu_\alpha\right) \\ \\ M_\alpha = \frac{\partial W_\alpha}{\partial \mu_\alpha} = Y\left(S_\alpha \varepsilon_\alpha + I_\alpha \mu_\alpha\right), \end{cases} \tag{82}$$

it easily turns out that the equality $N_\alpha = N_\beta$ holds also in this case. Moreover, plugging the same transformations into the second of Eqs.(82), we recover the Eq. (77) also in the case of generic initial conditions.

VI. THE NEUTRAL FIBER: FROM FLAT TO RING AND VICEVERSA

Let us consider the deformation depicted in Fig.7A-A′, where a slender beam of length L is deformed into a circle. Let us take as the representative fiber the curve placed at an height αh from the bottom surface, so that the equation representative of the elastica undeformed configuration is

$$\mathbf{L}_\alpha(s) = \begin{cases} X_\alpha(s) = s \\ \\ Y_\alpha(s) = \alpha h, \end{cases} \tag{83}$$

with $s \in [0, L]$. The tangent is expressed as

$$\mathbf{T}_\alpha(s) = \begin{cases} \frac{dX_\alpha(s)}{ds} = 1 \\ \\ \frac{dY_\alpha(s)}{ds} = 0, \end{cases} \tag{84}$$

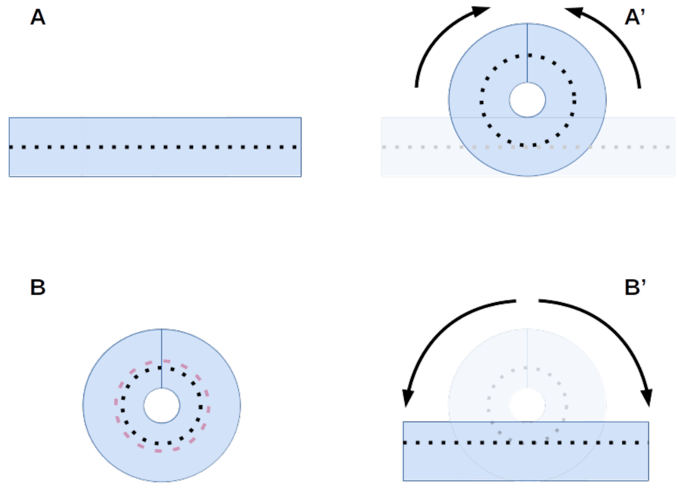

Figure 7.

and $\Phi(s) = 0$. By deforming the representative fiber into a circle of radius r_α, we easily obtain

$$\mathbf{l}_\alpha(s) = \begin{cases} x_\alpha(s) = r_\alpha \cos\left(\frac{2\pi s}{L}\right) \\ \\ y_\alpha(s) = r_\alpha \sin\left(\frac{2\pi s}{L}\right), \end{cases} \tag{85}$$

$$\mathbf{t}_\alpha(s) = \begin{cases} \frac{dx_\alpha(s)}{ds} = -\frac{2\pi r_\alpha}{L} \sin\left(\frac{2\pi s}{L}\right) \\ \\ \frac{dy_\alpha(s)}{ds} = \frac{2\pi r_\alpha}{L} \cos\left(\frac{2\pi s}{L}\right), \end{cases} \tag{86}$$

and $\varphi(s) = \frac{\pi}{2} - \frac{2\pi s}{L}$. The energy necessary for the complete bending of the beam into the circle is given by (64)

$$\mathscr{E}_\alpha(L; r_\alpha) = \frac{bY}{2L}\left[h\,(2\pi r_\alpha - L)^2 + 2\pi h^2(1-2\alpha)\,(2\pi r_\alpha - L) + 4\pi^2 h^3\left(\frac{1}{3} - \alpha + \alpha^2\right)\right]. \tag{87}$$

Without loss of generality, let us adopt the line of centroid as the representative material line, namely $\alpha = 1/2$. We know that this choice has the only advantage of yielding the axial-bending uncoupling in Eq.(87):

$$\mathscr{E}_{1/2}(L; r_{1/2}) = \frac{bY}{2L}\left[h\,\left(2\pi r_{1/2} - L\right)^2 + \frac{\pi^2 h^3}{3}\right]. \tag{88}$$

Nonetheless, if the transformation is such that $r_{1/2} = \frac{L}{2\pi}$, i.e. the middle fiber maintains its length constant (zero strain condition), the energy has a minimum. In other words, among all the possible deformations that transform a bar into a circle, that one which leaves unvaried the middle fiber (the neutral fiber) costs the minimum amount of work:

$$\mathscr{E}_{1/2}\left(L; r_{1/2} = \frac{L}{2\pi}\right) = \frac{bY\pi^2 h^3}{6L}. \tag{89}$$

This minimum principle can be seen as the straightforward application of Parent's principle $N_{1/2} = 0$.

Now let us consider the opposite situation, where a naturally curved beam is flattened into a bar as in Fig.7B-B′. The undeformed configuration is given by

$$\mathbf{L}_\alpha(\theta) = \begin{cases} X_\alpha(\theta) = R_\alpha \cos(\theta) \\ \\ Y_\alpha(\theta) = R_\alpha \sin(\theta), \end{cases} \tag{90}$$

with $\theta \in [0, 2\pi)$ being the internal parameter which is now adimensional, rather than having the dimension of an internal length.

$$\mathbf{T}_\alpha(\theta) = \begin{cases} \frac{dX_\alpha(\theta)}{d\theta} = -R_\alpha \sin(\theta) \\ \\ \frac{dY_\alpha(\theta)}{d\theta} = R_\alpha \cos(\theta), \end{cases} \tag{91}$$

so that $\Phi(\theta) = \frac{\pi}{2} - \theta$. On the other side the equation for the deformed bar of length l is

$$\mathbf{l}_\alpha(\theta) = \begin{cases} x_\alpha(\theta) = \frac{l\theta}{2\pi} \\ \\ y_\alpha(\theta) = \alpha h, \end{cases} \tag{92}$$

$$\mathbf{t}_\alpha(\theta) = \begin{cases} \frac{dx_\alpha(\theta)}{d\theta} = \frac{l}{2\pi} \\ \\ \frac{dy_\alpha(\theta)}{ds} = 0, \end{cases} \tag{93}$$

and $\varphi(\theta) = 0$. Hence, the energy cost connected to such a transformation is

$$\begin{aligned} \mathscr{E}_\alpha(R_\alpha; l) = \pi b Y \Bigg\{ \left(\frac{l}{2\pi} - R_\alpha\right)^2 \ln\left(1 + \frac{h}{R_0}\right) - 2\left[h - R_\alpha \ln\left(1 + \frac{h}{R_0}\right)\right]\left(\frac{l}{2\pi} - R_\alpha\right) + \\ + \left[\left(\frac{1}{2} - \alpha\right) h^2 - hR_\alpha + R_\alpha^2 \ln\left(1 + \frac{h}{R_0}\right)\right]\Bigg\}. \end{aligned} \tag{94}$$

In analogy to the previous case, we choose the value of α which entails the axial-bending uncoupling, namely, according to Eq.(41),

$$\alpha_U = \frac{1}{\ln\left(1 + \frac{h}{R_0}\right)} - \frac{R_0}{h}. \tag{95}$$

Thanks to the fact that $R_\alpha = R_0 + \alpha h$, from (95) it results $R_{\alpha_U} = \frac{h}{\ln\left(1+\frac{h}{R_0}\right)}$. Hence the Eq.(94) becomes

$$\mathscr{E}_{\alpha_U}(R_0; l) = \pi b Y \left\{ \left(\frac{l}{2\pi} - \frac{h}{\ln\left(1 + \frac{h}{R_0}\right)} \right)^2 \ln\left(1 + \frac{h}{R_0}\right) + \left[\frac{1}{2} - \frac{1}{\ln\left(1 + \frac{h}{R_0}\right)}\right] h^2 + hR_0 right\} \right. \tag{96}$$

Thus, it is possible to see that the minimum of energy necessary to flatten the ring is achieved only if the chosen uncoupling representative fiber keeps its length constant , i.e. $l = \frac{2\pi h}{\ln\left(1+\frac{h}{R_0}\right)} = 2\pi R_{\alpha_U}$. Such amount of energy turns out to be

$$\mathscr{E}_{\alpha_U}\left(R_0; l = \frac{2\pi h}{\ln\left(1+\frac{h}{R_0}\right)}\right) = \pi b Y \left\{R_0 h + \left[\frac{1}{2} - \frac{1}{\ln\left(1+\frac{h}{R_0}\right)}\right] h^2\right\}. \tag{97}$$

Again, the minimum of the energy is consistently required by the validity of Parent's principle.

So far we have considered the generic situation where circle in panel B of Fig.7 and that in panel A$'$ are different. If we set the same dimensions for both of them, we have $R_0 = \frac{L}{2\pi} - \frac{h}{2}$ that, inserted into the energy expression (96) yields

$$\mathscr{E}_{\alpha_U}(L; l) = \pi b Y \left\{\frac{l^2}{4\pi^4} \ln\left(\frac{L+\pi h}{L-\pi h}\right) + \frac{h}{\pi}\left(\frac{L}{2} - l\right)\right\}. \tag{98}$$

The amount of work needed to stretch the ring out, keeping constant the length of the neutral fiber, is

$$\mathscr{E}_{\alpha_U}\left(L; l = \frac{2\pi h}{\ln\left(\frac{L+\pi h}{L-\pi h}\right)}\right) = \pi b Y h \left\{\frac{L}{2\pi} - \frac{h}{\ln\left(\frac{L+\pi h}{L-\pi h}\right)}\right\}. \tag{99}$$

Conversely if we want to stretch the ring keeping constant the line of the centroids, therefore it is sufficient to replace $l = L$ into the expression (98):

$$\mathscr{E}_{\alpha_U}(L; l = L) = \pi b Y \left\{\frac{L^2}{4\pi^4} \ln\left(\frac{L+\pi h}{L-\pi h}\right) - \frac{hL}{2\pi}\right\}. \tag{100}$$

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[1] Leonhard Euler. *The Rational Mechanics of Flexible Or Elastic Bodies 1638-1788: Introduction to Vol. X and XI.* Springer Science & Business Media, 1960.

[2] Stephen Timoshenko. *History of strength of materials: with a brief account of the history of theory of elasticity and theory of structures.* Courier Corporation, 1983.

[3] James M Greenberg. On the equilibrium configurations of compressible slender bars. *Archive for Rational Mechanics and Analysis*, 27(3):181–194, 1967.

[4] G Kammel. Der einfluiss der liingsdehnung auf die elastische stabilitat geschlossener kreisringe. *Acta mechanica*, 1966.

[5] Stuart Antman. General solutions for plane extensible elasticae having nonlinear stress-strain laws. *Quarterly of Applied Mathematics*, 26(1):35–47, 1968.

[6] Anders Magnusson, Matti Ristinmaa, and Christer Ljung. Behaviour of the extensible elastica solution. *International Journal of Solids and Structures*, 38(46-47):8441–8457, 2001.

[7] Oz Oshri and Haim Diamant. Properties of compressible elastica from relativistic analogy. *Soft matter*, 12(3):664–668, 2016.

[8] TM Atanackovic. Buckling of a compressible elastic ring. *Acta mechanica*, 127(1-4):121–134, 1998.